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Title: Code Demonstration: Approximation of nearly-periodic symplectic maps via structure-preserving neural networks

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Intended for: Demonstration of our neural network implementation through a notebook and as a presentation.

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Code Demonstration:
Approximation of nearly-periodic symplectic maps via structure-preserving neural networks

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This notebook presents a simplified version of the code used to generate some of the numerical results presented in our paper [Duruisseaux et al., 2022].

The notebook was performed on jupyter.nersc.gov using NERSC's CPU nodes

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Introduction

In this notebook, we use the symplectic gyroceptron architecture from [Duruisseaux et al., 2022] to learn a surrogate map for the nearly-periodic symplectic flow map associated to a nearly-periodic Hamiltonian system composed of two nonlinearly coupled oscillators, where one of them oscillates significantly faster than the other:

$$\begin{cases} \dot{q}_1 = p_1 & \dot{p}_1 = -q_1 - \varepsilon \partial_{q_1} U(q_1, q_2) \\ \dot{q}_2 = \varepsilon p_2 & \dot{p}_2 = -\varepsilon q_2 - \varepsilon \partial_{q_2} U(q_1, q_2) \end{cases}$$

These equations of motion are the Hamilton's equations associated to the Hamiltonian

$$H_\varepsilon(q_1, q_2, p_1, p_2) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}\varepsilon(q_2^2 + p_2^2) + \varepsilon U(q_1, q_2)$$

1 Imports

[1]:
```
import os
import time
import numpy as np
import matplotlib.pyplot as plt
os.environ['TF_CPP_MIN_LOG_LEVEL'] = '2'

import tensorflow as tf
from tensorflow import keras
from tensorflow.keras import Model
from tensorflow.keras import layers
from tensorflow.keras.layers import Dense
tf.keras.backend.set_floatx('float32')
```

2 Parameters

2.1 Problem Parameters

[2]:
```
# Near-periodicity parameter
epsilon = 0.01

# Step size
rk4_step = 0.0005

# Number of steps per iteration
n_steps_per_rk4_eval = 100
```

2.2 Symplectic Gyroceptron Parameters

[3]:
```
# Number and shape of layers in Henon Network psi
HenonLayersShape = []
for i in range(10):
    HenonLayersShape.append(8)

# Number and shape of layers in Near-Identity Henon Network I_epsilon
NIHenonLayersShape = []
for i in range(8):
    NIHenonLayersShape.append(6)
```

2.3 Training Parameters

[4]:
```
# Loss Function (Weigthed MSE)
def custom_loss(y_true, y_pred):
    diff = (y_true - y_pred)**2
    Oscillator1_Loss =  epsilon * (tf.math.reduce_sum(diff[:,0]) + tf.math.reduce_sum(diff[:,2]))
    Oscillator2_Loss = tf.math.reduce_sum(diff[:,1]) + tf.math.reduce_sum(diff[:,3])
    return  Oscillator1_Loss + Oscillator2_Loss


# Optimizer
Optim = keras.optimizers.Adam(learning_rate = 0.1, clipnorm = 0.1)


# Either train from scratch, or upload weights from previous training
train = True

if train:

    initial_epoch = 0

    # Number of data points to learn from
        # In general Window
    n_general_points = 20000
        # In small window near orbits on the right
    n_points_small_right_orbits = 20000
        # In small window near orbits on the left
    n_points_small_left_orbits = 20000

    number_data_points = n_general_points + n_points_small_right_orbits + n_points_small_left_orbits
```

```
    # Batch Size
    my_batch_size = 512

    # Number of epochs
    n_epochs = 5600

    # Save every how many epochs ?
    n_epochs_before_save = n_epochs

    # Define scheduler
    scheduler = tf.keras.callbacks.ReduceLROnPlateau(monitor='loss', factor=0.8, patience=8, mode='min')

else:
    # Location of saved weights
    SavedWeights = "./TrainingWeights/my_saved_weights.ckpt"
```

2.4 Prediction Parameters

[5]:
```
# Number of iterations computed in the final predictions
n_pred_steps = 40000

# Number of trajectories to evolve in the final predictions
n_ics = 7

# Initial conditions
q1_ic = 1.5*np.ones((n_ics,1))
p1_ic = np.zeros((n_ics,1))
q2_ic = np.array([0.0, 0.15, 0.3, 0.4, 1.0, 1.45, 1.55]).reshape([n_ics,1])
p2_ic = np.zeros((n_ics,1))

z_ic = np.hstack([q1_ic,q2_ic,p1_ic,p2_ic])
```

3 Potential and Vector Field

Here the potential is taken to be

$$U(q_1, q_2) = q_1 q_2 \sin\left(2q_1 + 2q_2\right)$$

and the equations of motion are

$$\begin{cases} \dot{q}_1 = p_1 & \dot{p}_1 = -q_1 - \varepsilon \partial_{q_1} U(q_1, q_2) \\ \dot{q}_2 = \varepsilon p_2 & \dot{p}_2 = -\varepsilon q_2 - \varepsilon \partial_{q_2} U(q_1, q_2) \end{cases}$$

[6]:
```
# Potential U and its partial derivatives

def Potential(q1,q2):
    return  q1*q2*tf.math.sin(2*q1+2*q2)


def dUdq1(q1,q2):
    return q2*tf.math.sin(2*q1+2*q2) + 2*tf.math.cos(2*q1+2*q2)*q1*q2


def dUdq2(q1,q2):
    return q1*tf.math.sin(2*q1+2*q2) + 2*tf.math.cos(2*q1+2*q2)*q1*q2
```

```python
# Takes set of z=(q1,p1,q2,p2) points, and returns array of corresponding values of zdot
@tf.function
def zdot(z,epsilon):
    n = z.shape[0]
    q1 = z[:,0]
    q2 = z[:,1]
    p1 = z[:,2]
    p2 = z[:,3]

    q1dot = tf.reshape(p1, [n,1])
    q2dot = tf.reshape(epsilon*p2, [n,1])
    p1dot = tf.reshape(-q1-epsilon*dUdq1(q1,q2), [n,1])
    p2dot = tf.reshape(- epsilon*q2 - epsilon*dUdq2(q1,q2) , [n,1])

    return tf.concat([q1dot, q2dot, p1dot, p2dot], 1)
```

To simulate the reference trajectories and create the dataset, we use the classical Runge-Kutta 4 method with very small time-steps

[7]:
```python
# RK4 integration for n_steps
def rk4(z0,time_step,epsilon,n_steps):
    h = time_step
    z = z0
    for i in range(n_steps):
        k1 = zdot(z,epsilon)
        k2 = zdot(z + 0.5*h*k1,epsilon)
        k3 = zdot(z + 0.5*h*k2,epsilon)
        k4 = zdot(z + h * k3,epsilon)
        z = z + (1.0/6.0) * h * (k1 + 2*k2 + 2*k3 + k4)
    return z
```

4 Generate Data and Labels for Training

[8]:
```
if train:

    # Points in full [-0.9,1.6]x[-1.1,1.1] window for oscillator 2
    q1_data1 = -1.6 + 3.2*tf.random.uniform((n_general_points,1), dtype=tf.dtypes.float32)
    q2_data1 = -0.9 + 2.5*tf.random.uniform((n_general_points,1), dtype=tf.dtypes.float32)
    p1_data1 = -1.6 + 3.2*tf.random.uniform((n_general_points,1), dtype=tf.dtypes.float32)
    p2_data1 = -1.1 + 2.2*tf.random.uniform((n_general_points,1), dtype=tf.dtypes.float32)

    # More points for smaller right orbits in [0.6,1.5]x[-0.5,0.5] for oscillator 2
    q1_data2 = -1.6 + 3.2*tf.random.uniform((n_points_small_right_orbits,1), dtype=tf.dtypes.float32)
    q2_data2 =  0.6 + 0.9*tf.random.uniform((n_points_small_right_orbits,1), dtype=tf.dtypes.float32)
    p1_data2 = -1.6 + 3.2*tf.random.uniform((n_points_small_right_orbits,1), dtype=tf.dtypes.float32)
    p2_data2 = -0.5 + 1.0*tf.random.uniform((n_points_small_right_orbits,1), dtype=tf.dtypes.float32)

    # More points for smaller left orbits in [-0.75,0.75]x[-0.75,0.75] for oscillator 2
    q1_data3 = -1.6 + 3.2*tf.random.uniform((n_points_small_left_orbits,1), dtype=tf.dtypes.float32)
    p1_data3 = -1.6 + 3.2*tf.random.uniform((n_points_small_left_orbits,1), dtype=tf.dtypes.float32)
    q2_data3 = -0.8 + 1.6*tf.random.uniform((n_points_small_left_orbits,1), dtype=tf.dtypes.float32)
    p2_data3 = -0.8 + 1.6*tf.random.uniform((n_points_small_left_orbits,1), dtype=tf.dtypes.float32)

    # Concatenate all the data
    data = tf.concat([tf.concat([q1_data1, q1_data2, q1_data3], 0),
                      tf.concat([q2_data1, q2_data2, q2_data3], 0),
                      tf.concat([p1_data1, p1_data2, p1_data3], 0),
                      tf.concat([p2_data1, p2_data2, p2_data3], 0)], 1)

    # Evolve the data using RK4 to generate the labels
    labels = rk4(data,rk4_step,epsilon, n_steps_per_rk4_eval)
```

5 Neural Network Architecture

5.1 Henon map, Henon layer, and HenonNet

We first define the near-identity Henon map

$$H_\varepsilon[V,\eta]\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y+\eta\\-x+\varepsilon\nabla V(y)\end{pmatrix}$$

which reduces to the regular Henon Map when $\varepsilon = 1$.

In our network, we will take

$$V(y) = W_{out}^\top \tanh\left(W_{in}^\top y + b_{in}\right)$$

for some weight matrices W_{in}, W_{out} and bias b_{in}

[9]:
```
def HenonMap(X,Y,W_in,W_out,b_in,eta,epsilon=1):
    with tf.GradientTape() as tape:
        tape.watch(Y)
        V = tf.linalg.matmul(tf.math.tanh(tf.linalg.matmul(Y, W_in) + b_in),W_out)
    X_out= Y + eta
    Y_out= -X + epsilon*tape.gradient(V,Y)
    return X_out, Y_out
```

We then define a Henon layer as the composition of 4 copies of a Henon map

$$L_\varepsilon[V,\eta] = H_\varepsilon[V,\eta]^4$$

[10]:
```
class HenonLayer(layers.Layer):
    def __init__(self,ni,epsilon =1):
        super(HenonLayer, self).__init__()
        self.dim = 2   # half of the dimension of the full space
        self.epsilon = epsilon

        # Initialize weights
        init = tf.initializers.GlorotNormal()
        init_zero = tf.zeros_initializer()
        W_in_init = init(shape=[self.dim,ni], dtype = tf.float32)
        W_out_init = init(shape=[ni,1], dtype = tf.float32)
        b_in_init = init(shape=[1,ni], dtype = tf.float32)
        eta_init = init(shape=[1,self.dim], dtype = tf.float32)

        self.W_in = tf.Variable(W_in_init, dtype = tf.float32)
        self.W_out = tf.Variable(W_out_init, dtype = tf.float32)
        self.b_in = tf.Variable(b_in_init, dtype = tf.float32)
        self.eta = tf.Variable(eta_init, dtype = tf.float32)

    def call(self,z):
        xnext,ynext=HenonMap(z[:,0:self.dim],z[:,self.dim:],
                             self.W_in,self.W_out,self.b_in,self.eta,self.epsilon)
        xnext,ynext=HenonMap(xnext,ynext,self.W_in,self.W_out,self.b_in,self.eta,self.epsilon)
        xnext,ynext=HenonMap(xnext,ynext,self.W_in,self.W_out,self.b_in,self.eta,self.epsilon)
        xnext,ynext=HenonMap(xnext,ynext,self.W_in,self.W_out,self.b_in,self.eta,self.epsilon)
        return tf.concat([xnext,ynext], axis =1)
```

5.2 Inverse Henon Map and Henon Layer

Next we define the inverse of the Henon map

$$H_\varepsilon[V,\eta]\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y+\eta\\-x+\varepsilon\nabla V(y)\end{pmatrix}$$

which is given by

$$H_\varepsilon^{-1}[V,\eta]\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\varepsilon\nabla V(x-\eta)-y\\x-\eta\end{pmatrix}$$

[11]:
```
def InverseHenonMap(X,Y,W_in,W_out,b_in,eta,epsilon=1.):

    Y_out = X - eta

    with tf.GradientTape() as tape:
        tape.watch(Y_out)
        V = tf.linalg.matmul(tf.math.tanh(tf.linalg.matmul(Y_out, W_in) + b_in),W_out)

    X_out =  epsilon*tape.gradient(V,Y_out) - Y
    return X_out, Y_out
```

The inverse Henon layer is then the composition of 4 copies of the inverse Henon map

[11]:
```python
def InverseHenonLayer(z, original_layer):
    # The original Henon layer is an input

    # Get parameter values and weights of the original Henon layer
    epsilon = original_layer.epsilon
    dim = original_layer.dim
    W_in = original_layer.W_in
    W_out = original_layer.W_out
    b_in = original_layer.b_in
    eta = original_layer.eta

    # Evaluate the inverse Henon layer Map
    xnext,ynext=InverseHenonMap(z[:,0:dim],z[:,dim:],W_in,W_out,b_in,eta,epsilon)
    xnext,ynext=InverseHenonMap(xnext,ynext,W_in,W_out,b_in,eta,epsilon)
    xnext,ynext=InverseHenonMap(xnext,ynext,W_in,W_out,b_in,eta,epsilon)
    xnext,ynext=InverseHenonMap(xnext,ynext,W_in,W_out,b_in,eta,epsilon)
    return tf.concat([xnext,ynext], axis =1)
```

5.3 Circle Action Layer

The circle action is

$$\Phi_\theta(q_1, q_2, p_1, p_2) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}$$

This corresponds to clockwise circular rotation of the the components (q_1, p_1) in phase space

[12]:
```
class CircleActionLayer(layers.Layer):
    def __init__(self):
        super(CircleActionLayer, self).__init__()
        init = tf.initializers.GlorotNormal()
        theta0_init = init(shape=[1,1], dtype = tf.float32)
        self.theta0 = tf.Variable(theta0_init, dtype = tf.float32)

    def call(self,z):
        theta0 = self.theta0
        q1 = z[:,0:1]
        q2 = z[:,1:2]
        p1 = z[:,2:3]
        p2 = z[:,3:]
        # Last columns for p2 values
        r_out = p2
        # Add 3rd column for new values of p1
        r_out = tf.concat([tf.cos(theta0)*p1-tf.sin(theta0)*q1, r_out], 1)
        # Add 2nd column for new values of q2
        r_out = tf.concat([q2, r_out], 1)
        # Add 1st column for new values of q1
        r_out = tf.concat([tf.cos(theta0)*q1+tf.sin(theta0)*p1, r_out], 1)
        return r_out
```

5.4 Symplectic Gyroceptron

The symplectic gyroceptron is defined as the composition

$$I_\varepsilon \circ \psi \circ \Phi_\theta \circ \psi^{-1}$$

where

- I_ε is a neural network capable of learning near-identity symplectic maps
- ψ is a neural network capable of learning symplectic maps
- Φ_θ is a circle action.

Here we choose I_ε to be a near-identity Henon network and ψ to be a Henon network.

These networks are defined as compositions of near-identity Henon layers and Henon layers

[13]:
```
class SymplecticGyroceptron(Model):

    def __init__(self,NIHenon_unit_list,Henon_unit_list,eps):
        super(SymplecticGyroceptron, self).__init__()

        self.N_Henon = len(Henon_unit_list)
        self.N_NI_Henon = len(NIHenon_unit_list)
        self.psi_layers = []
        self.NI_layers = []
        self.hlayers = []
        self.epsilon = eps
```

```python
# Create Circle Action
hl = CircleActionLayer()
self.hlayers.append(hl)


# HenonNet psi
for i in range(self.N_Henon):
    # Get the number of neurons in i-th Henon layer
    ni = Henon_unit_list[i]

    # Create i-th Henon layer
    hl = HenonLayer(ni,epsilon=1.0)

    # Add i-th Henon layer
    self.psi_layers.append(hl)
    self.hlayers.append(hl)


# Near_Identity_Henon_Net
for i in range(self.N_NI_Henon):
    # Get the number of neurons in i-th Henon layer
    ni = NIHenon_unit_list[i]

    # Create and add the i-th Henon layer
    hl = HenonLayer(ni,epsilon=self.epsilon)
    self.hlayers.append(hl)
    self.NI_layers.append(hl)
```

```
    @tf.function
    def call(self, r):
        # Evaluate the HenonNet by passing the input r through each layer
        r_out = r

        # First pass through psi inverse
        NumberPsiLayers = self.psi_layers.__len__()
        for i in range(NumberPsiLayers):
            r_out = InverseHenonLayer(r_out, self.psi_layers[NumberPsiLayers-i-1])

        # Then pass through the other layers
        NumberLayers = self.hlayers.__len__()
        for i in range(NumberLayers):
            r_out = self.hlayers[i](r_out)
        return r_out
```

6 Build and Train Model

6.1 Define Model

[14]:
```
# Build model
SG = SymplecticGyroceptron(NIHenonLayersShape,HenonLayersShape, epsilon)

# Upload weights
if train == False:
    SG.load_weights(SavedWeights)

# Compile model
SG.compile(optimizer = Optim, loss = custom_loss)
```

[14]:

```python
# Evaluate Model Briefly to make sure it works and print its summary
SG.evaluate(tf.random.normal([10,4]),tf.random.normal([10,4]), batch_size=5)
SG.summary()
```

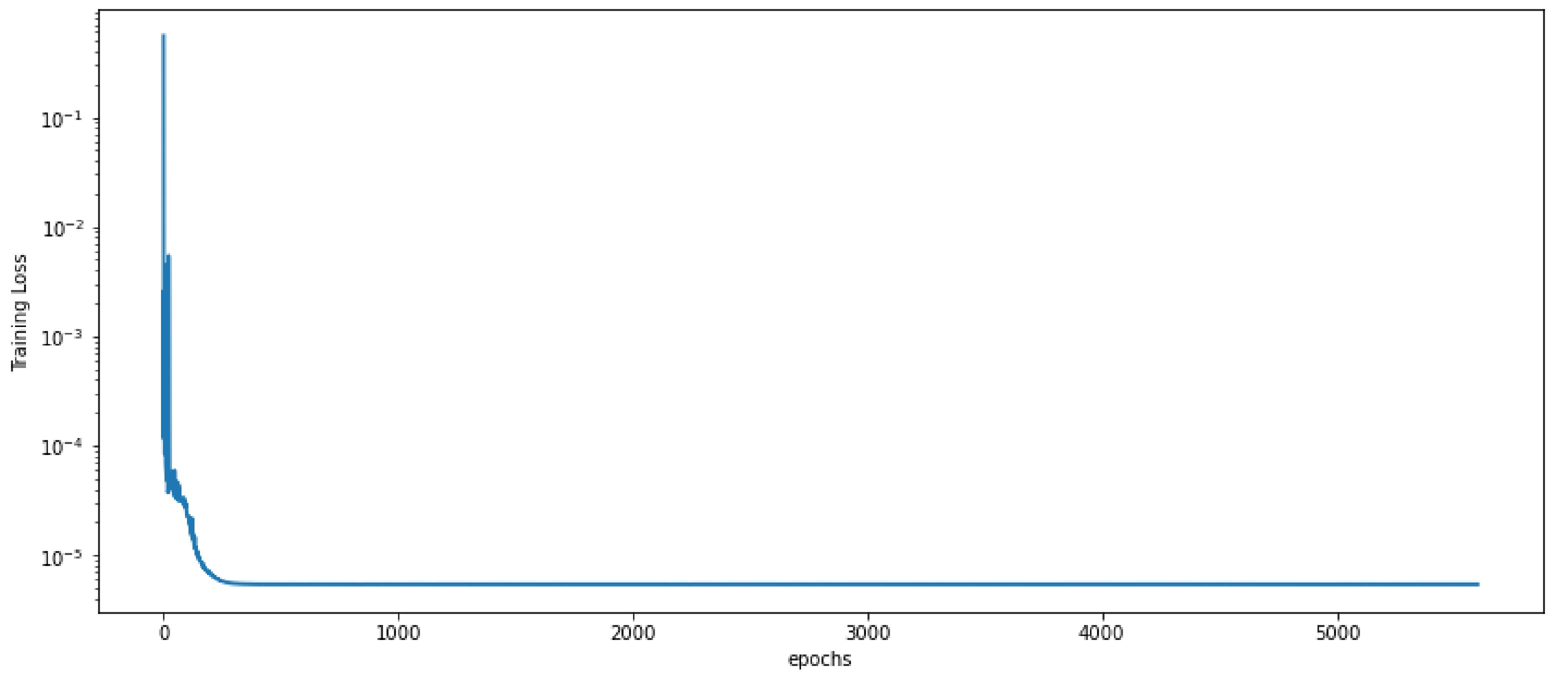
```
2/2 [==============================] - 2s 6ms/step - loss: 15.8110
Model: "symplectic_gyroceptron"
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
henon_layer (HenonLayer)     multiple                  34
_________________________________________________________________
henon_layer_1 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_2 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_3 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_4 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_5 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_6 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_7 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_8 (HenonLayer)   multiple                  34
_________________________________________________________________
henon_layer_9 (HenonLayer)   multiple                  34
```

```
_________________________________________________________________
henon_layer_10 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_11 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_12 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_13 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_14 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_15 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_16 (HenonLayer)  multiple                  26

_________________________________________________________________
henon_layer_17 (HenonLayer)  multiple                  26

_________________________________________________________________
circle_action_layer (CircleA multiple                  1
=================================================================
Total params: 549
Trainable params: 549
Non-trainable params: 0

_________________________________________________________________
```

6.2 Train Model

[15]:
```
if train:

    # Callback to Save Weights

    checkpoint_path = "./TrainingWeights/epoch_{epoch:05d}.ckpt"

    saving_frequency = n_epochs_before_save*(int(number_data_points/my_batch_size)+1)

    callback_SaveWeights = tf.keras.callbacks.ModelCheckpoint(filepath=checkpoint_path,
                                                              save_weights_only=True,
                                                              save_freq=saving_frequency,
                                                              verbose=0)



    # Training
    h = SG.fit(data, labels, batch_size=my_batch_size,
               epochs=n_epochs, initial_epoch=initial_epoch,
               callbacks=[scheduler, callback_SaveWeights],
               verbose=1)



    plt.subplots(figsize=(14,6))
    plt.semilogy(h.history['loss'], linewidth=2)
    plt.xlabel("epochs")
    plt.ylabel("Training Loss")
```

```
Epoch 1/5600
118/118 [==============================] - 12s 24ms/step - loss: 0.5554
Epoch 2/5600
118/118 [==============================] - 3s 22ms/step - loss: 1.1546e-04
Epoch 3/5600
118/118 [==============================] - 3s 22ms/step - loss: 0.0026
Epoch 4/5600
118/118 [==============================] - 3s 22ms/step - loss: 1.1331e-04
Epoch 5/5600
118/118 [==============================] - 3s 22ms/step - loss: 8.3917e-05
........
```



7 Prediction

7.1 Symplectic Gyroceptron Prediction

[16]:
```
# Initialize model
NN_state_model = tf.convert_to_tensor(z_ic, dtype = tf.float32)

# Initialize arrays collecting predictions
NN_model = np.zeros([n_ics,4,n_pred_steps+1])

start = time.time()
for i in range(n_pred_steps+1):
    # Collect Results
    NN_model[:,:, i] = NN_state_model.numpy()[:,:]
    # Evolve solutions
    NN_state_model = SG(NN_state_model)
end = time.time()

print('Symplectic Gyroceptron Prediction took {} seconds'.format(end-start))
```

```
Symplectic Gyroceptron Prediction took 48.89636421203613 seconds
```

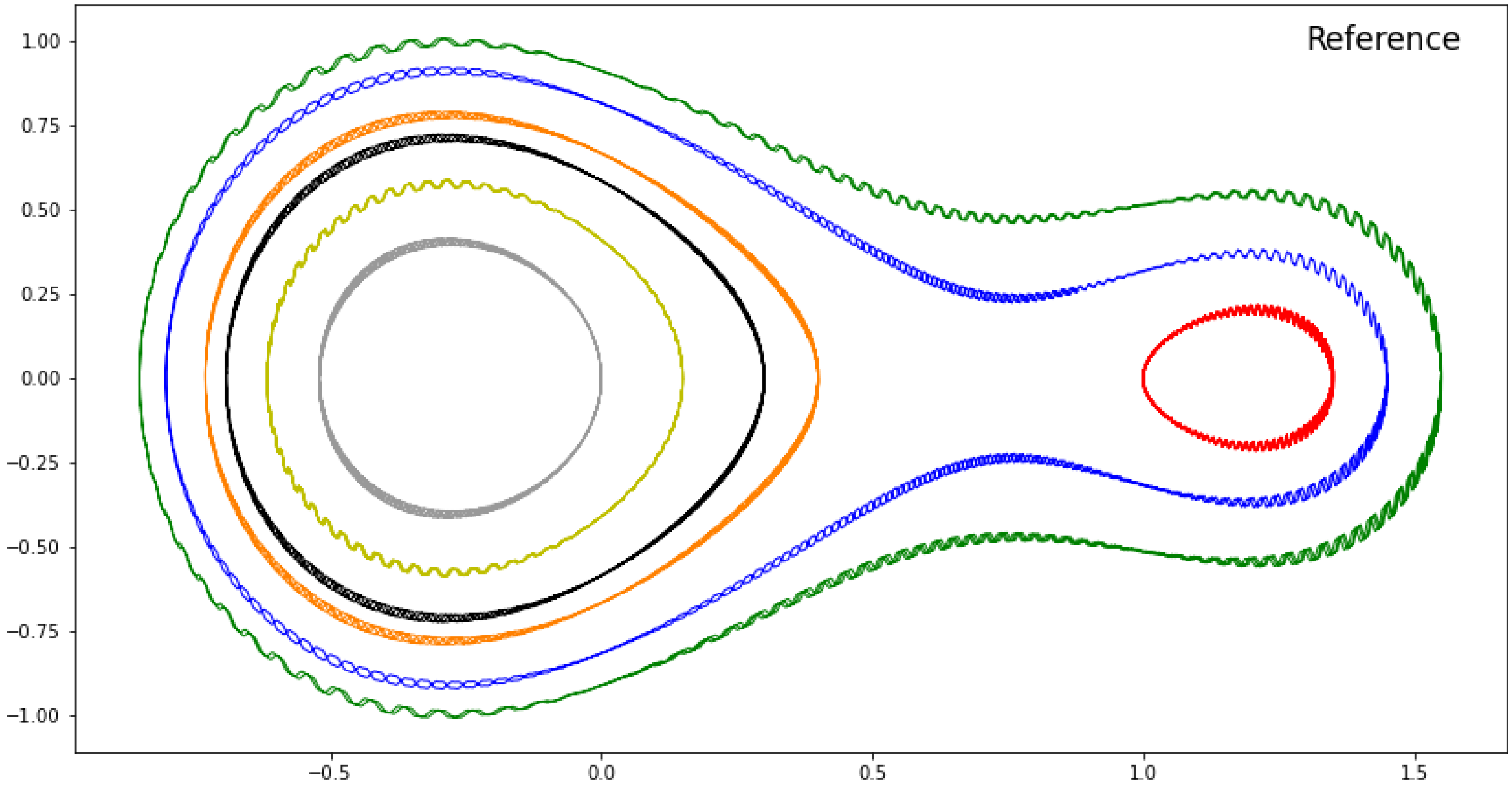
7.2 RK4 Reference Prediction

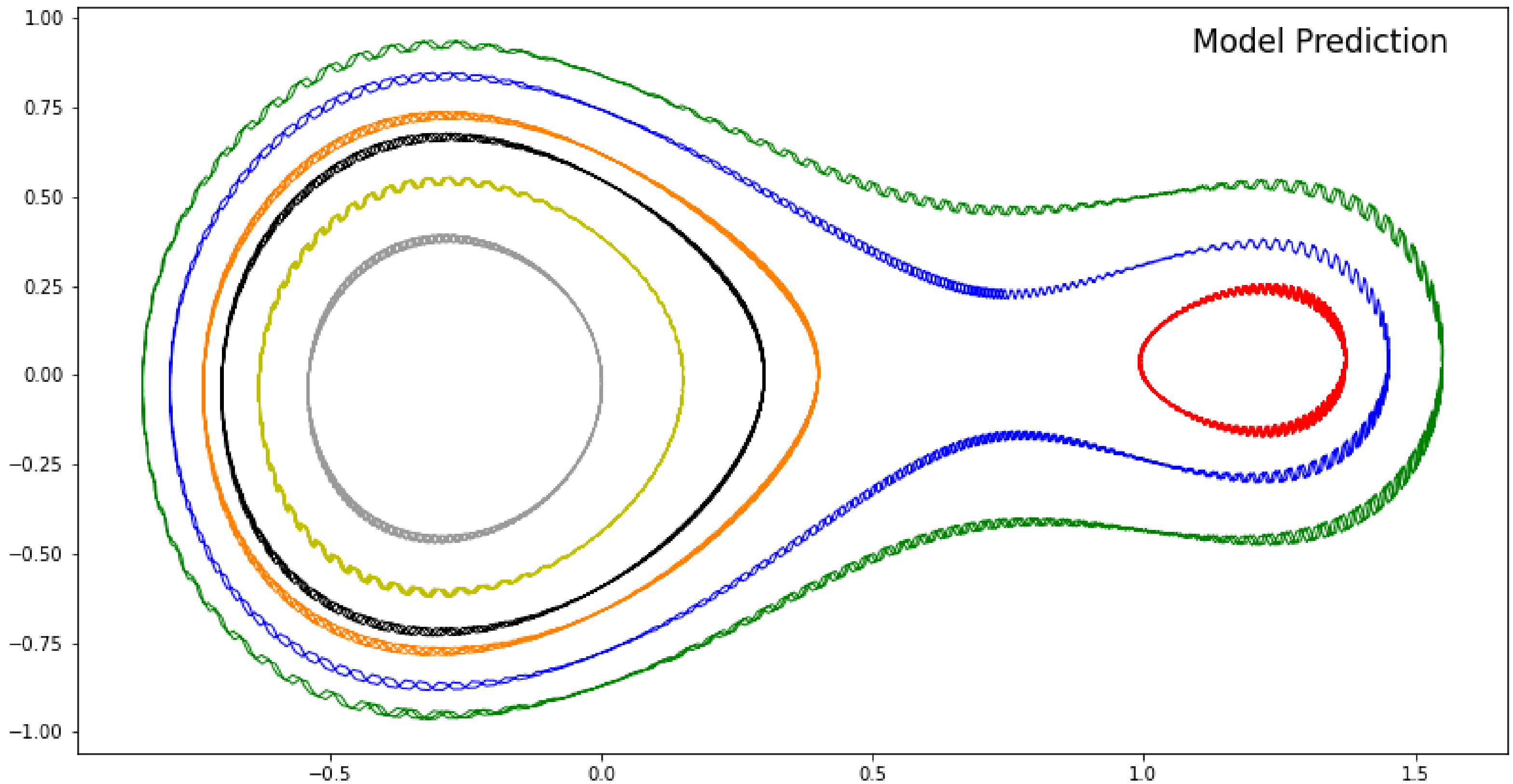
[17]:
```
# Initialize Model state and RK state
rk_state_model = tf.convert_to_tensor(z_ic, dtype = tf.float32)
# Initialize arrays collecting predictions
rk_model = np.zeros([n_ics,4,n_pred_steps+1])

start = time.time()
# Evolve the systems
for i in range(n_pred_steps+1):
    # Collect Results
    rk_model[:,:, i] = rk_state_model.numpy()[:,:]
    # Evolve solutions
    rk_state_model = rk4(rk_state_model,rk4_step,epsilon, n_steps_per_rk4_eval)
end = time.time()

print('RK4 Prediction took {} seconds'.format(end-start))
```

```
RK4 Prediction took 2616.064284801483 seconds
```

7.3 Plotting Predictions

[18]:
```
# RK4 Reference Plot for Oscillator 2

fig3, ax3 = plt.subplots(figsize=(14,7.5))
line3,=ax3.plot(rk_model[4,1,:],rk_model[4,3,:],'r',linewidth = 1)
line3,=ax3.plot(rk_model[3,1,:],rk_model[3,3,:],color =[1, 0.5, 0],linewidth = 1)
line3,=ax3.plot(rk_model[2,1,:],rk_model[2,3,:],'k',linewidth = 1)
line3,=ax3.plot(rk_model[1,1,:],rk_model[1,3,:], 'y' ,linewidth = 1)
line3,=ax3.plot(rk_model[0,1,:],rk_model[0,3,:],color=[0.6, 0.6, 0.6],linewidth = 1)
line3,=ax3.plot(rk_model[5,1,:],rk_model[5,3,:],'b',linewidth = 1)
line3,=ax3.plot(rk_model[6,1,:],rk_model[6,3,:],'g',linewidth = 1)
textstr = "Reference"
ax3.text(0.86,0.94,textstr,transform=ax3.transAxes,fontsize = 17)
Filename = "Reference.png"
plt.savefig(Filename,bbox_inches = 'tight', dpi = 500)



# Symplectic Gyroceptron Plot for Oscillator 2

fig4, ax4 = plt.subplots(figsize=(14,7.5))
line4,=ax4.plot(NN_model[4,1,:],NN_model[4,3,:],'r',linewidth = 1)
line4,=ax4.plot(NN_model[3,1,:],NN_model[3,3,:],color =[1, 0.5, 0],linewidth = 1)
line4,=ax4.plot(NN_model[2,1,:],NN_model[2,3,:],'k',linewidth = 1)
line4,=ax4.plot(NN_model[1,1,:],NN_model[1,3,:], 'y' ,linewidth = 1)
line4,=ax4.plot(NN_model[0,1,:],NN_model[0,3,:],color=[0.6, 0.6, 0.6],linewidth = 1)
line4,=ax4.plot(NN_model[5,1,:],NN_model[5,3,:],'b',linewidth = 1)
line4,=ax4.plot(NN_model[6,1,:],NN_model[6,3,:],'g',linewidth = 1)
textstr = "Model Prediction"
ax4.text(0.78,0.94,textstr,transform=ax4.transAxes,fontsize = 17)
Filename = "Model_Prediction.png"
plt.savefig(Filename,bbox_inches = 'tight', dpi = 500)
```


Reference
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−0.5
0.0
0.5
1.0
1.5



References

[Duruisseaux et al., 2022] Duruisseaux, V., Burby, J. W., and Tang, Q. (2022). Approximation of nearly-periodic symplectic maps via structure-preserving neural networks.