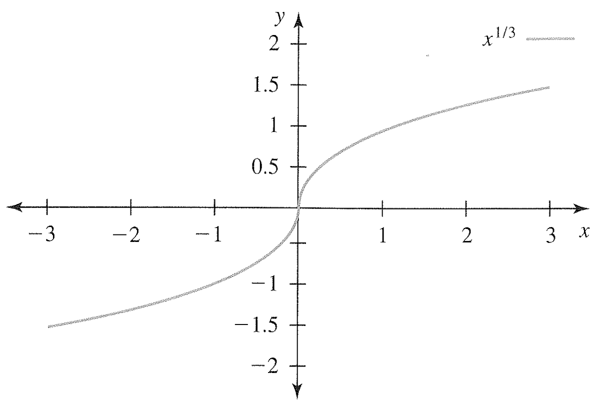

Figure 4.19 The function $f(x) = x^{1/3}$ has a vertical tangent line at $x = 0$. It is therefore not differentiable at $x = 0$.

Section 4.1 Problems

4.1.1

In Problems 1–8, find the derivative at the indicated point from the graph of each function.

1. $f(x) = 5; x = 1$ **2.** $f(x) = -3x; x = -2$
3. $f(x) = 4x - 3; x = -1$ **4.** $f(x) = -5x + 1; x = 0$
5. $f(x) = 2x^2; x = 0$ **6.** $f(x) = (x+2)^2; x = 1$
7. $f(x) = \cos x; x = 0$ **8.** $f(x) = \sin x; x = \frac{\pi}{2}$

In Problems 9–16, find c so that $f'(c) = 0$.

9. $f(x) = -3x^2 + 1$ **10.** $f(x) = -x^2 + 4$
11. $f(x) = (x-2)^2$ **12.** $f(x) = (x+3)^2$
13. $f(x) = x^2 - 6x + 9$ **14.** $f(x) = x^2 + 4x + 4$
15. $f(x) = \sin\left(\frac{\pi}{2}x\right)$ **16.** $\cos(\pi - x)$

In Problems 17–20, compute $f(c+h) - f(c)$ at the indicated point.

17. $f(x) = -2x + 1; c = 2$ **18.** $f(x) = 3x^2; c = 1$
19. $f(x) = \sqrt{x}; c = 4$ **20.** $f(x) = \frac{1}{x}; c = -2$

21. (a) Use the formal definition of the derivative to find the derivative of $y = 5x^2$ at $x = -1$.

(b) Show that the point $(-1, 5)$ is on the graph of $y = 5x^2$, and find the equation of the tangent line at the point $(-1, 5)$.

(c) Graph $y = 5x^2$ and the tangent line at the point $(-1, 5)$ in the same coordinate system.

22. (a) Use the formal definition to find the derivative of $y = -2x^2$ at $x = 1$.

(b) Show that the point $(1, -2)$ is on the graph of $y = -2x^2$, and find the equation of the tangent line at the point $(1, -2)$.

(c) Graph $y = -2x^2$ and the tangent line at the point $(1, -2)$ in the same coordinate system.

23. (a) Use the formal definition to find the derivative of $y = 1 - x^3$ at $x = 2$.

(b) Show that the point $(2, -7)$ is on the graph of $y = 1 - x^3$, and find the equation of the normal line at the point $(2, -7)$.

(c) Graph $y = 1 - x^3$ and the tangent line at the point $(2, -7)$ in the same coordinate system.

24. (a) Use the formal definition to find the derivative of $y = \frac{1}{x}$ at $x = 2$.

(b) Show that the point $(2, \frac{1}{2})$ is on the graph of $y = \frac{1}{x}$, and find the equation of the normal line at the point $(2, \frac{1}{2})$.

(c) Graph $y = \frac{1}{x}$ and the tangent line at the point $(2, \frac{1}{2})$ in the same coordinate system.

25. Use the formal definition to find the derivative of

$$y = \sqrt{x}$$

for $x > 0$.

26. Use the formal definition to find the derivative of

$$f(x) = \frac{1}{x+1}$$

for $x \neq -1$.

27. Find the equation of the tangent line to the curve $y = 3x^2$ at the point $(1, 3)$.

28. Find the equation of the tangent line to the curve $y = 2/x$ at the point $(2, 1)$.

29. Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(4, 2)$.

30. Find the equation of the tangent line to the curve $y = x^2 - 3x + 1$ at the point $(2, -1)$.

31. Find the equation of the normal line to the curve $y = -3x^2$ at the point $(-1, -3)$.

32. Find the equation of the normal line to the curve $y = 4/x$ at the point $(-1, -4)$.

33. Find the equation of the normal line to the curve $y = 2x^2 - 1$ at the point $(1, 1)$.

34. Find the equation of the normal line to the curve $y = \sqrt{x-1}$ at the point $(5, 2)$.

35. The following limit represents the derivative of a function f at the point $(a, f(a))$:

$$\lim_{h \to 0} \frac{2(a+h)^2 - 2a^2}{h}$$

Find $f(x)$.

exists and is equal to $f'(c)$. To show that f is continuous at $x = c$, we must show that

$$\lim_{x\to c} f(x) = f(c), \qquad \text{or} \qquad \lim_{x\to c}[f(x) - f(c)] = 0$$

First, note that f is defined at $x = c$. [Otherwise, we could not have computed the difference quotient $\frac{f(x)-f(c)}{x-c}$.] Now,

$$\lim_{x\to c}[f(x) - f(c)] = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}(x - c)$$

Given that

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

[this is Equation (4.4)] exists and is equal to $f'(c)$, and that

$$\lim_{x\to c}(x - c)$$

exists (it is equal to 0), we can apply the product rule for limits and find that

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}(x - c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \lim_{x\to c}(x - c) = f'(c) \cdot 0 = 0$$

This set of equations shows that

$$\lim_{x\to c}[f(x) - f(c)] = 0$$

and consequently that f is continuous at $x = c$. ■

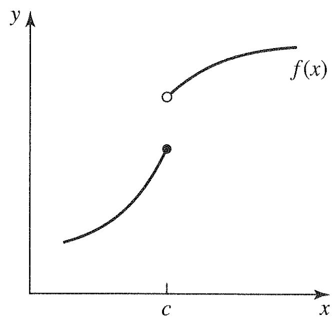

Figure 4.18 The function $y = f(x)$ is not differentiable at $x = c$.

It follows from the preceding theorem that if a function f is not continuous at $x = c$, then f is not differentiable at $x = c$. The function $y = f(x)$ in Figure 4.18 is discontinuous at $x = c$; we cannot draw a tangent line there.

Functions can have vertical tangent lines, but since the slope of a vertical line is not defined, the function would not be differentiable at any point where the tangent line is vertical. This situation is illustrated in the next example.

EXAMPLE 5 **Vertical Tangent Line** Show that

$$f(x) = x^{1/3}$$

is not differentiable at $x = 0$.

Solution We see from the graph of $f(x)$ in Figure 4.19 that $f(x)$ is continuous at $x = 0$. Using the formal definition, we find that

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty \qquad \text{does not exist} \end{aligned}$$

Since the limit does not exist, $f(x)$ is not differentiable at $x = 0$. We see from the graph that the tangent line at $x = 0$ is vertical. ■

36. The following limit represents the derivative of a function f at the point $(a, f(a))$:

$$\lim_{h\to 0}\frac{4(a+h)^3-4a^3}{h}$$

Find $f(x)$.

37. The following limit represents the derivative of a function f at the point $(a, f(a))$:

$$\lim_{h\to 0}\frac{\frac{1}{(2+h)^2+1}-\frac{1}{5}}{h}$$

Find f and a.

38. The following limit represents the derivative of a function f at the point $(a, f(a))$:

$$\lim_{h\to 0}\frac{\sin(\frac{\pi}{6}+h)-\sin\frac{\pi}{6}}{h}$$

Find f and a.

4.1.2

39. Velocity A car moves along a straight road. Its location at time t is given by

$$s(t) = 20t^2,\ 0 \le t \le 2$$

where t is measured in hours and $s(t)$ is measured in kilometers.

(a) Graph $s(t)$ for $0 \le t \le 2$.

(b) Find the average velocity of the car between $t = 0$ and $t = 2$. Illustrate the average velocity on the graph of $s(t)$.

(c) Use calculus to find the instantaneous velocity of the car at $t = 1$. Illustrate the instantaneous velocity on the graph of $s(t)$.

40. Velocity A train moves along a straight line. Its location at time t is given by

$$s(t) = \frac{100}{t},\quad 1 \le t \le 5$$

where t is measured in hours and $s(t)$ is measured in kilometers.

(a) Graph $s(t)$ for $1 \le t \le 5$.

(b) Find the average velocity of the train between $t = 1$ and $t = 5$. Where on the graph of $s(t)$ can you find the average velocity?

(c) Use calculus to find the instantaneous velocity of the train at $t = 2$. Where on the graph of $s(t)$ can you find the instantaneous velocity? What is the speed of the train at $t = 2$?

41. Velocity If $s(t)$ denotes the position of an object that moves along a straight line, then $\Delta s/\Delta t$, called the average velocity, is the average rate of change of $s(t)$, and $v(t) = ds/dt$, called the (instantaneous) velocity, is the instantaneous rate of change of $s(t)$. The speed of the object is the absolute value of the velocity, $|v(t)|$.

Suppose now that a car moves along a straight road. The location at time t is given by

$$s(t) = \frac{160}{3}t^2,\quad 0 \le t \le 1$$

where t is measured in hours and $s(t)$ is measured in kilometers.

(a) Where is the car at $t = 3/4$, and where is it at $t = 1$?

(b) Find the average velocity of the car between $t = 3/4$ and $t = 1$.

(c) Find the velocity and the speed of the car at $t = 3/4$.

42. Velocity Suppose a particle moves along a straight line. The position at time t is given by

$$s(t) = 3t - t^2,\quad t \ge 0$$

where t is measured in seconds and $s(t)$ is measured in meters.

(a) Graph $s(t)$ for $t \ge 0$.

(b) Use the graph in (a) to answer the following questions:

(i) Where is the particle at time 0?

(ii) Is there another time at which the particle visits the location where it was at time 0?

(iii) How far to the right on the straight line does the particle travel?

(iv) How far to the left on the straight line does the particle travel?

(v) Where is the velocity positive? where negative? equal to 0?

(c) Find the velocity of the particle.

(d) When is the velocity of the particle equal to 1 m/s?

43. Tilman's Resource Model In Subsection 4.1.2, we considered Tilman's resource model. Denote the biomass at time t by $B(t)$, and assume that

$$\frac{1}{B}\frac{dB}{dt} = f(R) - m$$

where R denotes the resource level,

$$f(R) = 200\frac{R}{5+R}$$

and $m = 40$. Use the graphical approach to find the value R^* at which $\frac{1}{B}\frac{dB}{dt} = 0$. Then compute R^* by solving $\frac{1}{B}\frac{dB}{dt} = 0$.

44. Exponential Growth Assume that $N(t)$ denotes the size of a population at time t and that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = rN$$

where r is a constant.

(a) Find the per capita growth rate.

(b) Assume that $r < 0$ and that $N(0) = 20$. Is the population size at time 1 greater than 20 or less than 20? Explain your answer.

45. Logistic Growth Assume that $N(t)$ denotes the size of a population at time t and that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = 3N\left(1-\frac{N}{20}\right)$$

Let $f(N) = 3N(1-\frac{N}{20})$ for $N \ge 0$. Graph $f(N)$ as a function of N and identify all equilibria (i.e., all points where $\frac{dN}{dt} = 0$).

46. Island Model Assume that a species lives in a habitat that consists of many islands close to a mainland. The species occupies both the mainland and the islands, but, although it is present on the mainland at all times, it frequently goes extinct on the islands. Islands can be recolonized by migrants from the mainland. The following model keeps track of the fraction of islands occupied: Denote the fraction of islands occupied at time t by $p(t)$. Assume that each island experiences a constant risk of extinction and that vacant islands (the fraction $1-p$) are colonized from the mainland at a constant rate. Then

$$\frac{dp}{dt} = c(1-p) - ep$$

where c and e are positive constants.

(a) The gain from colonization is $f(p) = c(1-p)$ and the loss from extinction is $g(p) = ep$. Graph $f(p)$ and $g(p)$ for $0 \le p \le 1$ in the same coordinate system. Explain why the two graphs intersect whenever e and c are both positive. Compute the point of intersection and interpret its biological meaning.

(b) The parameter c measures how quickly a vacant island becomes colonized from the mainland. The closer the islands, the larger is the value of c. Use your graph in (a) to explain what happens to the point of intersection of the two lines as c increases. Interpret your result in biological terms.

47. Chemical Reaction Consider the chemical reaction

$$A + B \longrightarrow AB$$

If $x(t)$ denotes the concentration of AB at time t, then

$$\frac{dx}{dt} = k(a-x)(b-x)$$

where k is a positive constant and a and b denote the concentrations of A and B, respectively, at time 0. Assume that $k = 3$, $a = 7$, and $b = 4$. For what values of x is $dx/dt = 0$? Interpret the meaning of $dx/dt = 0$.

48. Chemical Reaction Consider the autocatalytic reaction

$$A + X \longrightarrow X$$

which was introduced in Problem 30 of Section 1.2. Find a differential equation that describes the rate of change of the concentration of the product X.

49. Logistic Growth Suppose that the rate of change of the size of a population is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

where $N = N(t)$ denotes the size of the population at time t and r and K are positive constants. Find the equilibrium size of the population—that is, the size at which the rate of change is equal to 0. Use your answer to explain why K is called the carrying capacity.

50. Biotic Diversity (Adapted from Valentine, 1985.) Walker and Valentine (1984) suggested a model for species diversity which assumes that species extinction rates are independent of diversity but speciation rates are regulated by competition. Denoting the number of species at time t by $N(t)$, the speciation rate by b, and the extinction rate by a, they used the model

$$\frac{dN}{dt} = N\left[b\left(1 - \frac{N}{K}\right) - a\right]$$

where K denotes the number of "niches," or potential places for species in the ecosystem.

(a) Find possible equilibria under the condition $a < b$.

(b) Use your result in (a) to explain the following statement by Valentine (1985):

> In this situation, ecosystems are never "full," with all potential niches occupied by species so long as the extinction rate is above zero.

(c) What happens when $a \ge b$?

4.1.3

51. Which of the following statements is true?
(A) If $f(x)$ is continuous, then $f(x)$ is differentiable.
(B) If $f(x)$ is differentiable, then $f(x)$ is continuous.

52. Explain the relationship between continuity and differentiability.

53. Sketch the graph of a function that is continuous at all points in its domain and differentiable in the domain except at one point.

54. Sketch the graph of a periodic function defined on $\mathbf{R}$ that is continuous at all points in its domain and differentiable in the domain except at $c = k$, $k \in \mathbf{Z}$.

55. If $f(x)$ is differentiable for all $x \in \mathbf{R}$ except at $x = c$, is it true that $f(x)$ must be continuous at $x = c$? Justify your answer.

In Problems 56–69, graph each function and, on the basis of the graph, guess where the function is not differentiable. (Assume the largest possible domain.)

56. $y = |x - 2|$ **57.** $y = -|x + 5|$

58. $y = 2 - |x - 3|$ **59.** $y = |x + 2| - 1$

60. $y = \dfrac{1}{2 + x}$ **61.** $y = \dfrac{1}{x - 3}$

62. $y = \dfrac{3 - x}{3 + x}$ **63.** $y = \dfrac{x - 1}{x + 1}$

64. $y = |x^2 - 3|$ **65.** $y = |2x^2 - 1|$

66. $f(x) = \begin{cases} x & \text{for } x \le 0 \\ x + 1 & \text{for } x > 0 \end{cases}$

67. $f(x) = \begin{cases} 2x & \text{for } x \le 1 \\ x + 2 & \text{for } x > 1 \end{cases}$

68. $f(x) = \begin{cases} x^2 & \text{for } x \le -1 \\ 2 - x^2 & \text{for } x > -1 \end{cases}$

69. $f(x) = \begin{cases} x^2 + 1 & \text{for } x \le 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$

70. Suppose the function $f(x)$ is piecewise defined; that is, $f(x) = f_1(x)$ for $x \le a$ and $f(x) = f_2(x)$ for $x > a$. Assume that $f_1(x)$ is continuous and differentiable for $x < a$ and that $f_2(x)$ is continuous and differentiable for $x > a$. Sketch graphs of $f(x)$ for the following three cases:

(a) $f(x)$ is continuous and differentiable at $x = a$.
(b) $f(x)$ is continuous, but not differentiable, at $x = a$.
(c) $f(x)$ is neither continuous nor differentiable at $x = a$.

4.2 The Power Rule, the Basic Rules of Differentiation, and the Derivatives of Polynomials

In this section, we will begin a systematic treatment of the computation of derivatives. Knowing how to differentiate is fundamental to your understanding of the rest of the course. Although computer software is now available to compute derivatives of many functions (such as $y = cx^n$ or $y = e^{\sin x}$), it is nonetheless important that you master the techniques of differentiation.

EXAMPLE 4 **Tangent and Normal Lines** If $f(x) = 2x^3 - 3x + 1$, find the tangent and normal lines at $(-1, 2)$.

Solution The slope of the tangent line at $(-1, 2)$ is $f'(-1)$. We begin calculating this derivative as follows:

$$f'(x) = 6x^2 - 3$$

Evaluating $f'(x)$ at $x = -1$, we get

$$f'(-1) = 6(-1)^2 - 3 = 3$$

Therefore, the equation of the tangent line at $(-1, 2)$ is

$$y - 2 = 3(x - (-1)), \qquad \text{or} \qquad y = 3x + 5$$

To find the equation of the normal line, recall that the normal line is perpendicular to the tangent line; hence, the slope m of the normal line is given by

$$m = -\frac{1}{f'(-1)} = -\frac{1}{3}$$

The normal line goes through the point $(-1, 2)$ as well. The equation of the normal line is therefore

$$y - 2 = -\frac{1}{3}(x - (-1)), \qquad \text{or} \qquad y = -\frac{1}{3}x + \frac{5}{3}$$

The graph of $f(x)$, including the tangent and normal lines at $(-1, 2)$, is shown in Figure 4.22. ■

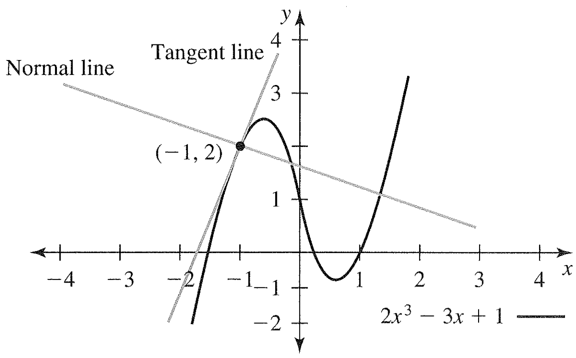

Figure 4.22 The graph of $f(x) = 2x^3 - 3x + 1$, together with the tangent and normal lines at $(-1, 2)$.

Look again at the last example: When we computed $f'(-1)$, we *first* computed $f'(x)$; the *second* step was to evaluate $f'(x)$ at $x = -1$. It makes no sense to plug -1 into $f(x)$ and then differentiate the result. Since $f(-1) = 2$ is a constant, the derivative would be 0, which is obviously not $f'(-1)$. Just look at Figure 4.22 to convince yourself. The notation $f'(-1)$ means that we evaluate the function $f'(x)$ at $x = -1$.

Section 4.2 Problems

Differentiate the functions given in Problems 1–22 with respect to the independent variable.

1. $f(x) = 4x^3 - 7x + 1$
2. $f(x) = -3x^4 + 5x^2$
3. $f(x) = -2x^5 + 7x - 4$
4. $f(x) = -3x^4 + 6x^2 - 2$
5. $f(x) = 3 - 4x - 5x^2$
6. $f(x) = -1 + 3x^2 - 2x^4$
7. $g(s) = 5s^7 + 2s^3 - 5s$
8. $g(s) = 3 - 4s^2 - 4s^3$
9. $h(t) = -\frac{1}{3}t^4 + 4t$
10. $h(t) = \frac{1}{2}t^2 - 3t + 2$
11. $f(x) = x^2 \sin\frac{\pi}{3} + \tan\frac{\pi}{4}$
12. $f(x) = 2x^3 \cos\frac{\pi}{3} + \cos\frac{\pi}{6}$

13. $f(x) = -3x^4 \tan\frac{\pi}{6} - \cot\frac{\pi}{6}$

14. $f(x) = x^2 \sec\frac{\pi}{6} + 3x\sec\frac{\pi}{4}$

15. $f(t) = t^3e^{-2} + t + e^{-1}$

16. $f(x) = \frac{1}{2}x^2e^3 - x^4$

17. $f(s) = s^3e^3 + 3e$

18. $f(x) = \frac{x}{e} + e^2x + e$

19. $f(x) = 20x^3 - 4x^6 + 9x^8$

20. $f(x) = \frac{x^3}{15} - \frac{x^4}{20} + \frac{2}{15}$

21. $f(x) = \pi x^3 - \frac{1}{\pi} + \frac{x}{\pi}$

22. $f(x) = \pi x e^2 - \frac{x^2\pi}{e}$

23. Differentiate
$$f(x) = ax^3$$
with respect to x. Assume that a is a constant.

24. Differentiate
$$f(x) = x^3 + a$$
with respect to x. Assume that a is a constant.

25. Differentiate
$$f(x) = ax^2 - 2a$$
with respect to x. Assume that a is a constant.

26. Differentiate
$$f(x) = a^2x^4 - 2ax^2$$
with respect to x. Assume that a is a constant.

27. Differentiate
$$h(s) = rs^2 - r$$
with respect to s. Assume that r is a constant.

28. Differentiate
$$f(r) = rs^2 - r$$
with respect to r. Assume that s is a constant.

29. Differentiate
$$f(x) = rs^2x^3 - rx + s$$
with respect to x. Assume that r and s are constants.

30. Differentiate
$$f(x) = \frac{r+x}{rs^2} - rsx + (r+s)x - rs$$
with respect to x. Assume that r and s are nonzero constants.

31. Differentiate
$$f(N) = (b-1)N^4 - \frac{N^2}{b}$$
with respect to N. Assume that b is a nonzero constant.

32. Differentiate
$$f(N) = \frac{bN^2 + N}{K + b}$$
with respect to N. Assume that b and K are positive constants.

33. Differentiate
$$g(t) = a^3t - at^3$$
with respect to t. Assume that a is a constant.

34. Differentiate
$$h(s) = a^4s^2 - as^4 + \frac{s^2}{a^4}$$
with respect to s. Assume that a is a positive constant.

35. Differentiate
$$V(t) = V_0(1 + \gamma t)$$
with respect to t. Assume that V_0 and γ are positive constants.

36. Differentiate
$$p(T) = \frac{NkT}{V}$$
with respect to T. Assume that N, k, and V are positive constants.

37. Differentiate
$$g(N) = N\left(1 - \frac{N}{K}\right)$$
with respect to N. Assume that K is a positive constant.

38. Differentiate
$$g(N) = rN\left(1 - \frac{N}{K}\right)$$
with respect to N. Assume that K and r are positive constants.

39. Differentiate
$$g(N) = rN^2\left(1 - \frac{N}{K}\right)$$
with respect to N. Assume that K and r are positive constants.

40. Differentiate
$$g(N) = rN(a - N)\left(1 - \frac{N}{K}\right)$$
with respect to N. Assume that r, a, and K are positive constants.

41. Differentiate
$$R(T) = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3}T^4$$
with respect to T. Assume that k, c, and h are positive constants.

In Problems 42–48, find the tangent line, in standard form, to $y = f(x)$ at the indicated point.

42. $y = 3x^2 - 4x + 7$, at $x = 2$

43. $y = 7x^3 + 2x - 1$, at $x = -3$

44. $y = -2x^3 - 3x + 1$, at $x = 1$

45. $y = 2x^4 - 5x$, at $x = 1$

46. $y = -x^3 - 2x^2$, at $x = 0$

47. $y = \frac{1}{\sqrt{2}}x^2 - \sqrt{2}$, at $x = 4$

48. $y = 3\pi x^5 - \frac{\pi}{2}x^3$, at $x = -1$

In Problems 49–54, find the normal line, in standard form, to $y = f(x)$ at the indicated point.

49. $y = 2 + x^2$, at $x = -1$

50. $y = 1 - 3x^2$, at $x = -2$

51. $y = \sqrt{3}x^4 - 2\sqrt{3}x^2$, at $x = -\sqrt{3}$

52. $y = -2x^2 - x$, at $x = 0$

53. $y = x^3 - 3$, at $x = 1$

54. $y = 1 - \pi x^2$, at $x = -1$

55. Find the tangent line to
$$f(x) = ax^2$$
at $x = 1$. Assume that a is a positive constant.

56. Find the tangent line to
$$f(x) = ax^3 - 2ax$$
at $x = -1$. Assume that a is a positive constant.

57. Find the tangent line to

$$f(x) = \frac{ax^2}{a^2+2}$$

at $x = 2$. Assume that a is a positive constant.

58. Find the tangent line to

$$f(x) = \frac{x^2}{a+1}$$

at $x = a$. Assume that a is a positive constant.

59. Find the normal line to

$$f(x) = ax^3$$

at $x = -1$. Assume that a is a positive constant.

60. Find the normal line to

$$f(x) = ax^2 - 3ax$$

at $x = 2$. Assume that a is a positive constant.

61. Find the normal line to

$$f(x) = \frac{ax^2}{a+1}$$

at $x = 2$. Assume that a is a positive constant.

62. Find the normal line to

$$f(x) = \frac{x^3}{a+1}$$

at $x = 2a$. Assume that a is a positive constant.

In Problems 63–70, find the coordinates of all of the points of the graph of $y = f(x)$ that have horizontal tangents.

63. $f(x) = x^2$

64. $f(x) = 2 - x^2$

65. $f(x) = 3x - x^2$

66. $f(x) = 4x + 2x^2$

67. $f(x) = 3x^3 - x^2$

68. $f(x) = -4x^4 + x^3$

69. $f(x) = \frac{1}{2}x^4 - \frac{7}{3}x^3 - 2x^2$

70. $f(x) = 3x^5 - \frac{3}{2}x^4$

71. Find a point on the curve

$$y = 4 - x^2$$

whose tangent line is parallel to the line $y = 2$. Is there more than one such point? If so, find all other points with this property.

72. Find a point on the curve

$$y = (4 - x)^2$$

whose tangent line is parallel to the line $y = -3$. Is there more than one such point? If so, find all other points with this property.

73. Find a point on the curve

$$y = 2x^2 - \frac{1}{2}$$

whose tangent line is parallel to the line $y = x$. Is there more than one such point? If so, find all other points with this property.

74. Find a point on the curve

$$y = 1 - 3x^3$$

whose tangent line is parallel to the line $y = -x$. Is there more than one such point? If so, find all other points with this property.

75. Find a point on the curve

$$y = x^3 + 2x + 2$$

whose tangent line is parallel to the line $3x - y = 2$. Is there more than one such point? If so, find all other points with this property.

76. Find a point on the curve

$$y = 2x^3 - 4x + 1$$

whose tangent line is parallel to the line $y - 2x = 1$. Is there more than one such point? If so, find all other points with this property.

77. Show that the tangent line to the curve

$$y = x^2$$

at the point $(1, 1)$ passes through the point $(0, -1)$.

78. Find all tangent lines to the curve

$$y = x^2$$

that pass through the point $(0, -1)$.

79. Find all tangent lines to the curve

$$y = x^2$$

that pass through the point $(0, -a^2)$, where a is a positive number.

80. How many tangent lines to the curve

$$y = x^2 + 2x$$

pass through the point $(-\frac{1}{2}, -3)$?

81. Suppose that $P(x)$ is a polynomial of degree 4. Is $P'(x)$ a polynomial as well? If yes, what is its degree?

82. Suppose that $P(x)$ is a polynomial of degree k. Is $P'(x)$ a polynomial as well? If yes, what is its degree?

4.3 The Product and Quotient Rules, and the Derivatives of Rational and Power Functions

4.3.1 The Product Rule

The derivative of a sum of differentiable functions is the sum of the derivatives of the functions. The rule for products is not so simple, as can be seen from the following

Section 4.3 Problems

4.3.1

In Problems 1–16, use the product rule to find the derivative with respect to the independent variable.

1. $f(x) = (x+5)(x^2-3)$

2. $f(x) = (2x^3-1)(3+2x^2)$

3. $f(x) = (3x^4-5)(2x-5x^3)$

4. $f(x) = (3x^4-x^2+1)(2x^2-5x^3)$

5. $f(x) = \left(\frac{1}{2}x^2-1\right)(2x+3x^2)$

6. $f(x) = 2(3x^2-2x^3)(1-5x^2)$

7. $f(x) = \frac{1}{5}(x^2-1)(x^2+1)$

8. $f(x) = 3(x^2+2)(4x^2-5x^4)-3$

9. $f(x) = (3x-1)^2$

10. $f(x) = (4-2x^2)^2$

11. $f(x) = 3(1-2x)^2$

12. $f(x) = \frac{(2x^2-3x+1)^2}{4}+2$

13. $g(s) = (2s^2-5s)^2$

14. $h(t) = 4(3t^2-1)(2t+1)$

15. $g(t) = 3(2t^2-5t^4)^2$

16. $h(s) = (4-3s^2+4s^3)^2$

In Problems 17–20, apply the product rule to find the tangent line, in slope–intercept form, of $y = f(x)$ at the specified point.

17. $f(x) = (3x^2-2)(x-1)$, at $x = 1$

18. $f(x) = (1-2x)(1+2x)$, at $x = 2$

19. $f(x) = 4(2x^4+3x)(4-2x^2)$, at $x = -1$

20. $f(x) = (3x^3-3)(2-2x^2)$, at $x = 0$

In Problems 21–24, apply the product rule to find the normal line, in slope–intercept form, of $y = f(x)$ at the specified point.

21. $f(x) = (1-x)(2-x^2)$, at $x = 2$

22. $f(x) = (2x+1)(3x^2-1)$, at $x = 1$

23. $f(x) = 5(1-2x)(x+1)-3$, at $x = 0$

24. $f(x) = \frac{(2-x)(3-x)}{4}$, at $x = -1$

In Problems 25–28, apply the product rule repeatedly to find the derivative of $y = f(x)$.

25. $f(x) = (2x-1)(3x+4)(1-x)$

26. $f(x) = (x-3)(2-3x)(5-x)$

27. $f(x) = (x-3)(2x^2+1)(1-x^2)$

28. $f(x) = (2x+1)(4-x^2)(1+x^2)$

29. Differentiate

$$f(x) = a(x-1)(2x-1)$$

with respect to x. Assume that a is a positive constant.

30. Differentiate

$$f(x) = (a-x)(a+x)$$

with respect to x. Assume that a is a positive constant.

31. Differentiate

$$f(x) = 2a(x^2-a)^2+a$$

with respect to x. Assume that a is a positive constant.

32. Differentiate

$$f(x) = \frac{3(x-1)^2}{2+a}$$

with respect to x. Assume that a is a positive constant.

33. Differentiate

$$g(t) = (at+1)^2$$

with respect to t. Assume that a is a positive constant.

34. Differentiate

$$h(t) = \sqrt{a}(t-a)+a$$

with respect to t. Assume that a is a positive constant.

35. Suppose that $f(2) = -4$, $g(2) = 3$, $f'(2) = 1$, and $g'(2) = -2$. Find

$$(fg)'(2)$$

36. Suppose that $f(2) = -4$, $g(2) = 3$, $f'(2) = 1$, and $g'(2) = -2$. Find

$$(f^2+g^2)'(2)$$

In Problems 37–40, assume that $f(x)$ is differentiable. Find an expression for the derivative of y at $x = 1$, assuming that $f(1) = 2$ and $f'(1) = -1$.

37. $y = 2xf(x)$

38. $y = 3x^2f(x)$

39. $y = -5x^3f(x)-2x$

40. $y = \frac{xf(x)}{2}$

In Problems 41–44, assume that $f(x)$ and $g(x)$ are differentiable at x. Find an expression for the derivative of y.

41. $y = 3f(x)g(x)$

42. $y = [f(x)-3]g(x)$

43. $y = [f(x)+2g(x)]g(x)$

44. $y = [-2f(x)-3g(x)]g(x)+\frac{2g(x)}{3}$

45. Let $B(t)$ denote the biomass at time t with specific growth rate $g(B)$. Show that the specific growth rate at $B = 0$ is given by the slope of the tangent line on the graph of the growth rate at $B = 0$.

46. Let $N(t)$ denote the size of a population at time t. Differentiate

$$f(N) = rN\left(1-\frac{N}{K}\right)$$

with respect to N, where r and K are positive constants.

47. Let $N(t)$ denote the size of a population at time t. Differentiate

$$f(N) = r\left(aN-N^2\right)\left(1-\frac{N}{K}\right)$$

with respect to N, where r, K, and a are positive constants.

48. Consider the chemical reaction

$$A + B \longrightarrow AB$$

If x denotes the concentration of AB at time t, then the reaction rate $R(x)$ is given by

$$R(x) = k(a-x)(b-x)$$

where k, a, and b are positive constants. Differentiate $R(x)$.

4.3.2

In Problems 49–70, differentiate with respect to the independent variable.

49. $f(x) = \frac{3x-1}{x+1}$

50. $f(x) = \frac{1-4x^3}{1-x}$

51. $f(x) = \frac{3x^2-2x+1}{2x+1}$

52. $f(x) = \frac{x^4+2x-1}{5x^2-2x+1}$

53. $f(x) = \frac{3-x^3}{1-x}$

54. $f(x) = \frac{1+2x^2-4x^4}{3x^3-5x^5}$

55. $h(t) = \dfrac{t^2 - 3t + 1}{t + 1}$

56. $h(t) = \dfrac{3 - t^2}{(t + 1)^2}$

57. $f(s) = \dfrac{4 - 2s^2}{1 - s}$

58. $f(s) = \dfrac{2s^3 - 4s^2 + 5s - 7}{(s^2 - 3)^2}$

59. $f(x) = \sqrt{x}(x - 1)$

60. $f(x) = \sqrt{x}(x^4 - 5x^2)$

61. $f(x) = \sqrt{3x}(x^2 - 1)$

62. $f(x) = \dfrac{\sqrt{5x}(1 + x^2)}{\sqrt{2}}$

63. $f(x) = x^3 - \dfrac{1}{x^3}$

64. $f(x) = x^5 - \dfrac{1}{x^5}$

65. $f(x) = 2x^2 - \dfrac{3x - 1}{x^3}$

66. $f(x) = -x^3 + \dfrac{2x^2 - 3}{4x^4}$

67. $g(s) = \dfrac{s^{1/3} - 1}{s^{2/3} - 1}$

68. $g(s) = \dfrac{s^{1/7} - s^{2/7}}{s^{3/7} + s^{4/7}}$

69. $f(x) = (1 - 2x)\left(\sqrt{2x} + \dfrac{2}{\sqrt{x}}\right)$

70. $f(x) = (x^3 - 3x^2 + 2)\left(\sqrt{x} + \dfrac{1}{\sqrt{x}} - 1\right)$

In Problems 71–74, find the tangent line, in slope–intercept form, of $y = f(x)$ at the specified point.

71. $f(x) = \dfrac{x^2 + 3}{x^3 + 5}$, at $x = -2$

72. $f(x) = \dfrac{3}{x} - \dfrac{4}{\sqrt{x}} + \dfrac{2}{x^2}$, at $x = 1$

73. $f(x) = \dfrac{2x - 5}{x^3}$, at $x = 2$

74. $f(x) = \sqrt{x}(x^3 - 1)$, at $x = 1$

75. Differentiate

$$f(x) = \frac{ax}{3 + x}$$

with respect to x. Assume that a is a positive constant.

76. Differentiate

$$f(x) = \frac{ax}{k + x}$$

with respect to x. Assume that a and k are positive constants.

77. Differentiate

$$f(x) = \frac{ax^2}{4 + x^2}$$

with respect to x. Assume that a is a positive constant.

78. Differentiate

$$f(x) = \frac{ax^2}{k^2 + x^2}$$

with respect to x. Assume that a and k are positive constants.

79. Differentiate

$$f(R) = \frac{R^n}{k^n + R^n}$$

with respect to R. Assume that k is a positive constant and n is a positive integer.

80. Differentiate

$$h(t) = \sqrt{at}(1 - a) + a$$

with respect to t. Assume that a is a positive constant.

81. Differentiate

$$h(t) = \sqrt{at}(t - a) + at$$

with respect to t. Assume that a is a positive constant.

82. Suppose that $f(2) = -4$, $g(2) = 3$, $f'(2) = 1$, and $g'(2) = -2$. Find

$$\left(\frac{1}{f}\right)'(2)$$

83. Suppose that $f(2) = -4$, $g(2) = 3$, $f'(2) = 1$, and $g'(2) = -2$. Find

$$\left(\frac{f}{2g}\right)'(2)$$

In Problems 84–87, assume that $f(x)$ is differentiable. Find an expression for the derivative of y at $x = 2$, assuming that $f(2) = -1$ and $f'(2) = 1$.

84. $y = \dfrac{f(x)}{x^2 + 1}$

85. $y = \dfrac{x^2 + 4f(x)}{f(x)}$

86. $y = [f(x)]^2 - \dfrac{x}{f(x)}$

87. $y = \dfrac{f(x)}{f(x) + x}$

In Problems 88–91, assume that $f(x)$ and $g(x)$ are differentiable at x. Find an expression for the derivative of y.

88. $y = \dfrac{2f(x) + 1}{3g(x)}$

89. $y = \dfrac{f(x)}{[g(x)]^2}$

90. $y = \dfrac{x^2}{f(x) - g(x)}$

91. $y = \sqrt{x}f(x)g(x)$

92. Assume that $f(x)$ is a differentiable function. Find the derivative of the reciprocal function $g(x) = 1/f(x)$ at those points x where $f(x) \neq 0$.

93. Find the tangent line to the hyperbola $yx = c$, where c is a positive constant, at the point (x_1, y_1) with $x_1 > 0$. Show that the tangent line intersects the x-axis at a point that does not depend on c.

94. *(Adapted from Roff, 1992)* The males in the frog species *Eleutherodactylus coqui* (found in Puerto Rico) take care of their brood. On the other hand, while they protect the eggs, they cannot find other mates and therefore cannot increase their number of offspring. On the other hand, if they do not spend enough time with their brood, then the offspring might not survive. The proportion $w(t)$ of offspring hatching per unit time is given as a function of (1) the probability $f(t)$ of hatching if time t is spent brooding, and (2) the cost C associated with the time spent searching for other mates:

$$w(t) = \frac{f(t)}{C + t}$$

Find the derivative of $w(t)$.

4.4 The Chain Rule and Higher Derivatives

4.4.1 The Chain Rule

In Section 1.2, we defined the composition of functions. To find the derivative of composite functions, we need the **chain rule**, the proof of which is given at the end of this section.

(b) To find the time it takes the object to hit the ground, we set $s(t) = 30$ m and solve for t:

$$30\,\text{m} = \frac{1}{2}(9.81)\frac{\text{m}}{\text{s}^2}t^2$$

This yields

$$t^2 = \frac{60}{9.81}\text{s}^2, \qquad \text{or} \qquad t = \sqrt{\frac{60}{9.81}}\,\text{s} \approx 2.47\,\text{s}$$

(We need consider only the positive solution.) The velocity at the time of impact is then

$$v(t) = gt = (9.81)\frac{\text{m}}{\text{s}^2}\sqrt{\frac{60}{9.81}}\,\text{s} \approx 24.3\frac{\text{m}}{\text{s}}$$

■

Section 4.4 Problems

4.4.1

In Problems 1–28, differentiate the functions with respect to the independent variable.

1. $f(x) = (x-3)^2$
2. $f(x) = (4x+5)^3$
3. $f(x) = (1-3x^2)^4$
4. $f(x) = (5x^2-3x)^3$
5. $f(x) = \sqrt{x^2+3}$
6. $f(x) = \sqrt{2x+7}$
7. $f(x) = \sqrt{3-x^3}$
8. $f(x) = \sqrt{5x+3x^4}$
9. $f(x) = \dfrac{1}{(x^3-2)^4}$
10. $f(x) = \dfrac{2}{(1-5x^2)^3}$
11. $f(x) = \dfrac{3x-1}{\sqrt{2x^2-1}}$
12. $f(x) = \dfrac{(1-2x^2)^3}{(3-x^2)^2}$
13. $f(x) = \dfrac{\sqrt{2x-1}}{(x-1)^2}$
14. $f(x) = \dfrac{\sqrt{x^2-1}}{2+\sqrt{x^2+1}}$
15. $f(s) = \sqrt{s+\sqrt{s}}$
16. $g(t) = \sqrt{t^2+\sqrt{t+1}}$
17. $g(t) = \left(\dfrac{t}{t-3}\right)^3$
18. $h(s) = \left(\dfrac{2s^2}{s+1}\right)^4$
19. $f(r) = (r^2-r)^3(r+3r^3)^{-4}$
20. $h(s) = \dfrac{2(3-s)^2}{s^2+(7s-1)^2}$
21. $h(x) = \sqrt[5]{3-x^4}$
22. $h(x) = \sqrt[3]{1-2x}$
23. $f(x) = \sqrt[7]{x^2-2x+1}$
24. $f(x) = \sqrt[4]{2-4x^2}$
25. $g(s) = (3s^7-7s)^{3/2}$
26. $h(t) = (t^4-5t)^{5/2}$
27. $h(t) = \left(3t+\dfrac{3}{t}\right)^{2/5}$
28. $h(t) = \left(4t^4+\dfrac{4}{t^4}\right)^{1/4}$

29. Differentiate

$$f(x) = (ax+1)^3$$

with respect to x. Assume that a is a positive constant.

30. Differentiate

$$f(x) = \sqrt{ax^2-2}$$

with respect to x. Assume that a is a positive constant.

31. Differentiate

$$g(N) = \frac{bN}{(k+N)^2}$$

with respect to N. Assume that b and k are positive constants.

32. Differentiate

$$g(N) = \frac{N}{(k+bN)^3}$$

with respect to N. Assume that b and k are positive constants.

33. Differentiate

$$g(T) = a(T_0-T)^3 - b$$

with respect to T. Assume that a, b, and T_0 are positive constants.

34. Suppose that $f'(x) = 2x+1$. Find the following:

(a) $\dfrac{d}{dx}f(x^2)$ at $x=-1$ **(b)** $\dfrac{d}{dx}f(\sqrt{x})$ at $x=4$

35. Suppose that $f'(x) = \frac{1}{x}$. Find the following:

(a) $\dfrac{d}{dx}f(x^2+3)$ **(b)** $\dfrac{d}{dx}f(\sqrt{x-1})$

In Problems 36–39, assume that $f(x)$ and $g(x)$ are differentiable.

36. Find $\dfrac{d}{dx}\sqrt{f(x)+g(x)}$.
37. Find $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}+1\right)^2$.
38. Find $\dfrac{d}{dx}f\left[\dfrac{1}{g(x)}\right]$.
39. Find $\dfrac{d}{dx}\dfrac{[f(x)]^2}{g(2x)+2x}$.

In Problems 40–46, find $\frac{dy}{dx}$ by applying the chain rule repeatedly.

40. $y = (\sqrt{1-2x^2}+1)^2$
41. $y = (\sqrt{x^3-3x}+3x)^4$
42. $y = \left(1+2(x+3)^4\right)^2$
43. $y = \left(1+(3x^2-1)^3\right)^2$
44. $y = \left(\dfrac{x}{2(x^2-1)^2-1}\right)^2$
45. $y = \left(\dfrac{2x+1}{3(x^3-1)^3-1}\right)^3$
46. $y = \left(\dfrac{(2x+1)^2-x}{(3x^3+1)^3-x}\right)^2$

4.4.2

In Problems 47–54, find $\frac{dy}{dx}$ by implicit differentiation.

47. $x^2+y^2=4$
48. $y = x^2+3yx$
49. $x^{3/4}+y^{3/4}=1$
50. $xy-y^3=1$
51. $\sqrt{xy} = x^2+1$
52. $\dfrac{1}{2xy} - y^3 = 4$
53. $\dfrac{x}{y} = \dfrac{y}{x}$
54. $\dfrac{x}{xy+1} = 2xy$

In Problems 55–57, find the lines that are (a) tangential and (b) normal to each curve at the given point.

55. $x^2+y^2=25$, $(4,-3)$ (circle)
56. $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$, $(1, \frac{3}{2}\sqrt{3})$ (ellipse)
57. $\dfrac{x^2}{25}-\dfrac{y^2}{9}=1$, $(\frac{25}{3}, 4)$ (hyperbola)

58. Lemniscate
(a) The curve with equation $y^2 = x^2 - x^4$ is shaped like the numeral eight. Find $\frac{dy}{dx}$ at $(\frac{1}{2}, \frac{1}{4}\sqrt{3})$.
(b) Use a graphing calculator to graph the curve in (a). If the calculator cannot graph implicit functions, graph the upper and the lower halves of the curve separately; that is, graph

$$y_1 = \sqrt{x^2 - x^4}$$
$$y_2 = -\sqrt{x^2 - x^4}$$

Choose the viewing rectangle $-2 \leq x \leq 2$, $-1 \leq y \leq 1$.

59. Astroid
(a) Consider the curve with equation $x^{2/3} + y^{2/3} = 4$. Find $\frac{dy}{dx}$ at $(-1, 3\sqrt{3})$.
(b) Use a graphing calculator to graph the curve in (a). If the calculator cannot graph implicit functions, graph the upper and the lower halves of the curve separately. To get the left half of the graph, make sure that your calculator evaluates $x^{2/3}$ in the order $(x^2)^{1/3}$. Choose the viewing rectangle $-10 \leq x \leq 10$, $-10 \leq y \leq 10$.

60. Kampyle of Eudoxus
(a) Consider the curve with equation $y^2 = 10x^4 - x^2$. Find $\frac{dy}{dx}$ at $(1, 3)$.
(b) Use a graphing calculator to graph the curve in (a). If the calculator cannot graph implicit functions, graph the upper and the lower halves of the curve separately. Choose the viewing rectangle $-3 \leq x \leq 3$, $-10 \leq y \leq 10$.

■ 4.4.3

61. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $x^2 + y^2 = 1$, $\frac{dx}{dt} = 2$ for $x = \frac{1}{2}$, and $y > 0$.

62. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $y^2 = x^2 - x^4$, $\frac{dx}{dt} = 1$ for $x = \frac{1}{2}$, and $y > 0$.

63. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $x^2y = 1$ and $\frac{dx}{dt} = 3$ for $x = 2$.

64. Assume that u and v are differentiable functions of t. Find $\frac{du}{dt}$ when $u^2 + v^3 = 12$, $\frac{dv}{dt} = 2$ for $v = 2$, and $u > 0$.

65. Assume that the side length x and the volume $V = x^3$ of a cube are differentiable functions of t. Express dV/dt in terms of dx/dt.

66. Assume that the radius r and the area $A = \pi r^2$ of a circle are differentiable functions of t. Express dA/dt in terms of dr/dt.

67. Assume that the radius r and the surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Express dS/dt in terms of dr/dt.

68. Assume that the radius r and the volume $V = \frac{4}{3}\pi r^3$ of a sphere are differentiable functions of t. Express dV/dt in terms of dr/dt.

69. Suppose that water is stored in a cylindrical tank of radius 5 m. If the height of the water in the tank is h, then the volume of the water is $V = \pi r^2 h = (25\text{m}^2)\pi h = 25\pi h\ \text{m}^2$. If we drain the water at a rate of 250 liters per minute, what is the rate at which the water level inside the tank drops? (Note that 1 cubic meter contains 1000 liters.)

70. Suppose that we pump water into an inverted right circular conical tank at the rate of 5 cubic feet per minute (i.e., the tank stands with its point facing downward). The tank has a height of 6 ft and the radius on top is 3 ft. What is the rate at which the water level is rising when the water is 2 ft deep? (Note that the volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

71. Two people start biking from the same point. One bikes east at 15 mph, the other south at 18 mph. What is the rate at which the distance between the two people is changing after 20 minutes and after 40 minutes?

72. Allometric equations describe the scaling relationship between two measurements, such as skull length versus body length. In vertebrates, we typically find that

$$[\text{skull length}] \propto [\text{body length}]^a$$

for $0 < a < 1$. Express the growth rate of the skull length in terms of the growth rate of the body length.

■ 4.4.4

In Problems 73–82, find the first and the second derivatives of each function.

73. $f(x) = x^3 - 3x^2 + 1$

74. $f(x) = (2x^2 + 4)^3$

75. $g(x) = \dfrac{x-1}{x+1}$

76. $h(s) = \dfrac{1}{s^2 + 2}$

77. $g(t) = \sqrt{3t^3 + 2t}$

78. $f(x) = \dfrac{1}{x^2} + x - x^3$

79. $f(s) = \sqrt{s^{3/2} - 1}$

80. $f(x) = \dfrac{2x}{x^2 + 1}$

81. $g(t) = t^{-5/2} - t^{1/2}$

82. $f(x) = x^3 - \dfrac{1}{x^3}$

83. Find the first 10 derivatives of $y = x^5$.

84. Find $f^{(n)}(x)$ and $f^{(n+1)}(x)$ of $f(x) = x^n$.

85. Find a second-degree polynomial $p(x) = ax^2 + bx + c$ with $p(0) = 3$, $p'(0) = 2$, and $p''(0) = 6$.

86. The position at time t of a particle that moves along a straight line is given by the function $s(t)$. The first derivative of $s(t)$ is called the velocity, denoted by $v(t)$; that is, the velocity is the rate of change of the position. The rate of change of the velocity is called **acceleration**, denoted by $a(t)$; that is,

$$\frac{d}{dt}v(t) = a(t)$$

Given that $v(t) = s'(t)$, it follows that

$$\frac{d^2}{dt^2}s(t) = a(t)$$

Find the velocity and the acceleration at time $t = 1$ for the following position functions:
(a) $s(t) = t^2 - 3t$ **(b)** $s(t) = \sqrt{t^2 + 1}$ **(c)** $s(t) = t^4 - 2t$

87. Neglecting air resistance, the height h (in meters) of an object thrown vertically from the ground with initial velocity v_0 is given by

$$h(t) = v_0 t - \frac{1}{2}gt^2$$

where $g = 9.81\ \text{m/s}^2$ is the earth's gravitational constant and t is the time (in seconds) elapsed since the object was released.
(a) Find the velocity and the acceleration of the object.
(b) Find the time when the velocity is equal to 0. In which direction is the object traveling right before this time? in which direction right after this time?

Section 4.5 Problems

In Problems 1–58, find the derivative with respect to the independent variable.

1. $f(x) = 2\sin x - \cos x$
2. $f(x) = 3\cos x - 2\sin x$
3. $f(x) = 3\sin x + 5\cos x - 2\sec x$
4. $f(x) = -\sin x + \cos x - 3\csc x$
5. $f(x) = \tan x - \cot x$
6. $f(x) = \sec x - \csc x$
7. $f(x) = \sin(3x)$
8. $f(x) = \cos(-5x)$
9. $f(x) = 2\sin(3x+1)$
10. $f(x) = -3\cos(1-2x)$
11. $f(x) = \tan(4x)$
12. $f(x) = \cot(2-3x)$
13. $f(x) = 2\sec(1+2x)$
14. $f(x) = -3\csc(3-5x)$
15. $f(x) = 3\sin(x^2)$
16. $f(x) = 2\cos(x^3-3x)$
17. $f(x) = \sin^3(x^2-3)$
18. $f(x) = \cos^2(x^2-1)$
19. $f(x) = 3\sin^2 x^2$
20. $f(x) = -\sin^2(2x^3-1)$
21. $f(x) = 4\cos x^2 - 2\cos^2 x$
22. $f(x) = -5\cos(2-x^3) + 2\cos^3(x-4)$
23. $f(x) = 4\cos^2 x + 2\cos x^4$
24. $f(x) = -3\cos^2(3x^2-4)$
25. $f(x) = 2\tan(1-x^2)$
26. $f(x) = -\cot(3x^3-4x)$
27. $f(x) = -2\tan^3(3x-1)$
28. $f(x) = \sqrt{\sin x} + \sin\sqrt{x}$
29. $f(x) = \sqrt{\sin(2x^2-1)}$
30. $g(s) = (\cos^2 s - 3s^2)^2$
31. $g(s) = \sqrt{\cos s} - \cos\sqrt{x}$
32. $g(t) = \dfrac{\sin(3t)}{\cos(5t)}$
33. $g(t) = \dfrac{\sin(2t)+1}{\cos(6t)-1}$
34. $f(x) = \dfrac{\cot(2x)}{\tan(4x)}$
35. $f(x) = \dfrac{\sec(x^2-1)}{\csc(x^2+1)}$
36. $f(x) = \sin x \cos x$
37. $f(x) = \sin(2x-1)\cos(3x+1)$
38. $f(x) = \tan x \cot x$
39. $f(x) = \tan(3x^2-1)\cot(3x^2+1)$
40. $f(x) = \sec x \cos x$
41. $f(x) = \sin x \sec x$
42. $f(x) = \dfrac{1}{\sin^2 x + \cos^2 x}$
43. $f(x) = \dfrac{1}{\tan^2 x - \sec^2 x}$
44. $g(x) = \dfrac{1}{\sin(3x)}$
45. $g(x) = \dfrac{1}{\sin(3x^2-1)}$
46. $g(x) = \dfrac{1}{\csc^2(5x)}$
47. $g(x) = \dfrac{1}{\csc^3(1-5x^2)}$
48. $h(x) = \cot(3x)\csc(3x)$
49. $h(x) = \dfrac{3}{\tan(2x)-x}$
50. $g(t) = \left(\dfrac{1}{\sin t^2}\right)^{3/2}$
51. $h(s) = \sin^3 s + \cos^3 s$
52. $f(x) = (2x^3-x)\cos(1-x^2)$
53. $f(x) = \dfrac{\sin(2x)}{1+x^2}$
54. $f(x) = \dfrac{1+\cos(3x)}{2x^3-x}$
55. $f(x) = \tan\dfrac{1}{x}$
56. $f(x) = \sec\dfrac{1}{1+x^2}$
57. $f(x) = \dfrac{\sec x^2}{\sec^2 x}$
58. $f(x) = \dfrac{\csc(3-x^2)}{1-x^2}$

59. Find the points on the curve $y = \sin(\frac{\pi}{3}x)$ that have a horizontal tangent.

60. Find the points on the curve $y = \cos^2 x$ that have a horizontal tangent.

61. Use the identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and the definition of the derivative to show that

$$\frac{d}{dx}\cos x = -\sin x$$

62. Use the quotient rule to show that

$$\frac{d}{dx}\cot x = -\csc^2 x$$

(*Hint*: Write $\cot x = \frac{\cos x}{\sin x}$.)

63. Use the quotient rule to show that

$$\frac{d}{dx}\sec x = \sec x \tan x$$

[*Hint*: Write $\sec x = (\cos x)^{-1}$.]

64. Use the quotient rule to show that

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

[*Hint*: Write $\csc x = (\sin x)^{-1}$.]

65. Find the derivative of

$$f(x) = \sin\sqrt{x^2+1}$$

66. Find the derivative of

$$f(x) = \cos\sqrt{x^2+1}$$

67. Find the derivative of

$$f(x) = \sin\sqrt{3x^3+3x}$$

68. Find the derivative of

$$f(x) = \cos\sqrt{1-4x^4}$$

69. Find the derivative of

$$f(x) = \sin^2(x^2-1)$$

70. Find the derivative of

$$f(x) = \cos^2(2x^2+3)$$

71. Find the derivative of

$$f(x) = \tan^3(3x^3-3)$$

72. Find the derivative of

$$f(x) = \sec^2(2x^2-2)$$

73. Suppose that the concentration of nitrogen in a lake exhibits periodic behavior. That is, if we denote the concentration of nitrogen at time t by $c(t)$, then we assume that

$$c(t) = 2 + \sin\left(\frac{\pi}{2}t\right)$$

(a) Find

$$\frac{dc}{dt}$$

(b) Use a graphing calculator to graph both $c(t)$ and $\frac{dc}{dt}$ in the same coordinate system.

(c) By inspecting the graph in (b), answer the following questions:
(i) When $c(t)$ reaches a maximum, what is the value of dc/dt?
(ii) When dc/dt is positive, is $c(t)$ increasing or decreasing?
(iii) What can you say about $c(t)$ when $dc/dt = 0$?

where W_0 is the amount of material at time 0 and λ is called the *radioactive decay rate*. Show that $W(t)$ satisfies the differential equation

$$\frac{dW}{dt} = -\lambda W(t)$$

Solution We use the chain rule to find the derivative of $W(t)$:

$$\frac{d}{dt}W(t) = \underbrace{W_0 e^{-\lambda t}}_{W(t)}(-\lambda)$$

That is,

$$\frac{dW}{dt} = -\lambda W(t)$$

In words, the rate of decay is proportional to the amount of material left. This equation should remind you of the subtangent problem; there, we wanted to find a function whose derivative is proportional to the function itself. That is exactly the situation we have in this example: The derivative of $W(t)$ is proportional to $W(t)$. ■

EXAMPLE 6 **Exponential Growth** Find the per capita growth rate of a population whose size $N(t)$ at time t follows the exponential growth function

$$N(t) = N(0)e^{rt}$$

where $N(0)$ is the population size at time 0 and r is a constant.

Solution We first find the derivative of $N(t)$:

$$\frac{dN}{dt} = N(0)re^{rt}$$

Since $N(0)e^{rt} = N(t)$, we can write

$$\frac{dN}{dt} = rN(t)$$

Thus, the per capita growth rate of an exponentially growing population is constant; that is,

$$\frac{1}{N}\frac{dN}{dt} = r$$

■

Section 4.6 Problems

Differentiate the functions in Problems 1–52 with respect to the independent variable.

1. $f(x) = e^{3x}$
2. $f(x) = e^{-2x}$
3. $f(x) = 4e^{1-3x}$
4. $f(x) = 3e^{2-5x}$
5. $f(x) = e^{-2x^2+3x-1}$
6. $f(x) = e^{4x^2-2x+1}$
7. $f(x) = e^{7(x^2+1)^2}$
8. $f(x) = e^{-3(x^3-1)^4}$
9. $f(x) = xe^x$
10. $f(x) = 2xe^{-3x}$
11. $f(x) = x^2e^{-x}$
12. $f(x) = (3x^2-1)e^{1-x^2}$
13. $f(x) = \frac{1+e^x}{1+x^2}$
14. $f(x) = \frac{x-e^{-x}}{1+xe^{-x}}$
15. $f(x) = \frac{e^x+e^{-x}}{2+e^x}$
16. $f(x) = \frac{x}{e^x+e^{-x}}$
17. $f(x) = e^{\sin(3x)}$
18. $f(x) = e^{\cos(4x)}$
19. $f(x) = e^{\sin(x^2-1)}$
20. $f(x) = e^{\cos(1-2x^3)}$
21. $f(x) = \sin(e^x)$
22. $f(x) = \cos(e^x)$
23. $f(x) = \sin(e^{2x}+x)$
24. $f(x) = \cos(3x - e^{x^2-1})$
25. $f(x) = \exp[x - \sin x]$
26. $f(x) = \exp[x^2 - 2\cos x]$
27. $g(s) = \exp[\sec s^2]$
28. $g(s) = \exp[\tan s^3]$
29. $f(x) = e^{x \sin x}$
30. $f(x) = e^{1-x\cos x}$
31. $f(x) = -3e^{x^2+\tan x}$
32. $f(x) = 2e^{-x\sec(3x)}$
33. $f(x) = 2^x$
34. $f(x) = 3^x$
35. $f(x) = 2^{x+1}$
36. $f(x) = 3^{x-1}$
37. $f(x) = 5^{\sqrt{2x-1}}$
38. $f(x) = 3^{\sqrt{1-3x}}$
39. $f(x) = 2^{x^2+1}$
40. $f(x) = 3^{x^3-1}$
41. $h(t) = 2^{t^2-1}$
42. $h(t) = 4^{2t^3-t}$
43. $f(x) = 2^{\sqrt{x}}$
44. $f(x) = 3^{\sqrt{x+1}}$
45. $f(x) = 2^{\sqrt{x^2-1}}$
46. $f(x) = 4^{\sqrt{1-2x^3}}$
47. $h(t) = 5^{\sqrt{t}}$
48. $h(t) = 6^{\sqrt{6t^6-6}}$

49. $g(x) = 2^{2\cos x}$

50. $g(r) = 2^{-3\sin r}$

51. $g(r) = 3^{r^{1/5}}$

52. $g(r) = 4^{r^{1/4}}$

Compute the limits in Problems 53–56.

53. $\lim_{h\to 0} \frac{e^{2h} - 1}{h}$

54. $\lim_{h\to 0} \frac{e^{5h} - 1}{3h}$

55. $\lim_{h\to 0} \frac{e^{h} - 1}{\sqrt{h}}$

56. $\lim_{h\to 0} \frac{2^{h} - 1}{h}$

57. Find the length of the subtangent to the curve $y = 2^x$ at the point $(1, 2)$.

58. Find the length of the subtangent to the curve $y = \exp[x^2]$ at the point $(2, e^4)$.

59. Population Growth Suppose that the population size at time t is

$$N(t) = e^{2t}, \quad t \geq 0$$

(a) What is the population size at time 0?

(b) Show that

$$\frac{dN}{dt} = 2N$$

60. Population Growth Suppose that the population size at time t is

$$N(t) = N_0 e^{rt}, \quad t \geq 0$$

where N_0 is a positive constant and r is a real number.

(a) What is the population size at time 0?

(b) Show that

$$\frac{dN}{dt} = rN$$

61. Bacterial Growth Suppose that a bacterial colony grows in such a way that at time t the population size is

$$N(t) = N(0)2^t$$

where $N(0)$ is the population size at time 0. Find the rate of growth dN/dt. Express your solution in terms of $N(t)$. Show that the growth rate of the population is proportional to the population size.

62. Bacterial Growth Suppose that a bacterial colony grows in such a way that at time t the population size is

$$N(t) = N(0)2^t$$

where $N(0)$ is the population size at time 0. Find the per capita growth rate.

63. Logistic Growth

(a) Find the derivative of the logistic growth curve (see Example 3 in Section 3.3)

$$N(t) = \frac{K}{1 + \left(\frac{K}{N(0)} - 1\right)e^{-rt}}$$

where r and K are positive constants and $N(0)$ is the population size at time 0.

(b) Show that $N(t)$ satisfies the equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

[*Hint*: Use the function $N(t)$ given in (a) for the right-hand side, and simplify until you obtain the derivative of $N(t)$ that you computed in (a).]

(c) Plot the per capita rate of growth $\frac{1}{N}\frac{dN}{dt}$ as a function of N, and note that it decreases with increasing population size.

64. Fish Recruitment Model The following model is used in the fisheries literature to describe the recruitment of fish as a function of the size of the parent stock: If we denote the number of recruits by R and the size of the parent stock by P, then

$$R(P) = \alpha P e^{-\beta P}, \quad P \geq 0$$

where α and β are positive constants.

(a) Sketch the graph of the function $R(P)$ when $\beta = 1$ and $\alpha = 2$.

(b) Differentiate $R(P)$ with respect to P.

(c) Find all the points on the curve that have a horizontal tangent.

65. Von Bertalanffy Growth Model The growth of fish can be described by the von Bertalanffy growth function

$$L(x) = L_\infty - (L_\infty - L_0)e^{-kx}$$

where x denotes the age of the fish and k, L_∞, and L_0 are positive constants.

(a) Set $L_0 = 1$ and $L_\infty = 10$. Graph $L(x)$ for $k = 1.0$ and $k = 0.1$.

(b) Interpret L_∞ and L_0.

(c) Compare the graphs for $k = 0.1$ and $k = 1.0$. According to which graph do fish reach $L = 5$ more quickly?

(d) Show that

$$\frac{d}{dx}L(x) = k(L_\infty - L(x))$$

That is, $dL/dx \propto L_\infty - L$. What does this proportionality say about how the rate of growth changes with age?

(e) The constant k is the proportionality constant in (d). What does the value of k tell you about how quickly a fish grows?

66. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t (measured in days). Assume that the radioactive decay rate of the material is 0.2/day. Find the differential equation for the radioactive decay function $W(t)$.

67. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t (measured in days). Assume that the radioactive decay rate of the material is 4/day. Find the differential equation for the radioactive decay function $W(t)$.

68. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t (measured in days). Assume that the half-life of the material is 3 days. Find the differential equation for the radioactive decay function $W(t)$.

69. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t (measured in days). Assume that the half-life of the material is 5 days. Find the differential equation for the radioactive decay function $W(t)$.

70. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t. Assume that $W(0) = 15$ and that

$$\frac{dW}{dt} = -2W(t)$$

(a) How much material is left at time $t = 2$?

(b) What is the half-life of this material?

71. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t. Assume that $W(0) = 6$ and that

$$\frac{dW}{dt} = -3W(t)$$

(a) How much material is left at time $t = 4$?

(b) What is the half-life of the material?

72. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t. Assume that $W(0) = 10$ and $W(1) = 8$.

(a) Find the differential equation that describes this situation.

(b) How much material is left at time $t = 5$?

(c) What is the half-life of the material?

73. Radioactive Decay Suppose $W(t)$ denotes the amount of a radioactive material left after time t. Assume that $W(0) = 5$ and $W(1) = 2$.

(a) Find the differential equation that describes this situation.

(b) How much material is left at time $t = 3$?

(c) What is the half-life of the material?

4.7 Derivatives of Inverse Functions, Logarithmic Functions, and the Inverse Tangent Function

Recall that the logarithmic function is the inverse of the exponential function. To find the derivative of the logarithmic function, we must therefore learn how to compute the derivative of an inverse function.

4.7.1 Derivatives of Inverse Functions

We begin with an example (Figure 4.32). Let $f(x) = x^2$, $x \geq 0$. We computed the inverse function of f in Subsection 1.2.6. First note that $f(x) = x^2$, $x \geq 0$, is one to one (use the horizontal line test from Subsection 1.2.6); hence, we can define its inverse. We repeat the steps from Subsection 1.2.6 to find an inverse function. [Recall that we obtain the graph of the inverse function by reflecting $y = f(x)$ about the line $y = x$.]

1. Write $y = f(x)$:
$$y = x^2$$
2. Solve for x:
$$x = \sqrt{y}$$
3. Interchange x and y:
$$y = \sqrt{x}$$

Since the range of $f(x)$, which is the interval $[0, \infty)$, becomes the domain for the inverse function, it follows that

$$f^{-1}(x) = \sqrt{x} \quad \text{for } x \geq 0$$

We already know the derivative of $\sqrt{x}$, namely, $1/(2\sqrt{x})$. But we will try to find the derivative in a different way that we can generalize to get a formula for finding the derivative of any inverse function. Let $g(x) = f^{-1}(x)$. Then

$$(f \circ g)(x) = f[g(x)] = (\sqrt{x})^2 = x, \quad x \geq 0$$

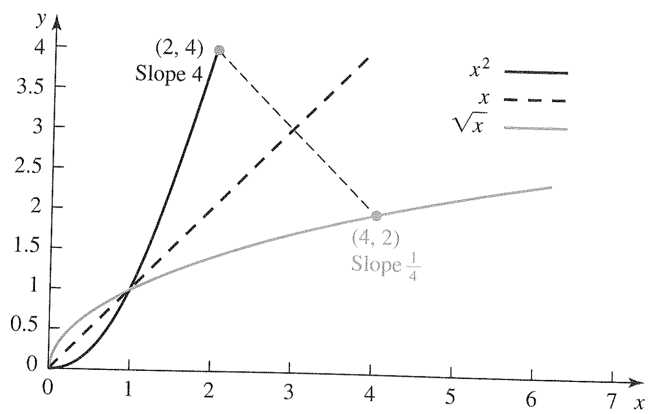

Figure 4.32 The function $y = x^2, x \geq 0$, and its inverse function $y = \sqrt{x}, x \geq 0$.

Proof We set $y = x^r$ and use logarithmic differentiation to obtain

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[\ln x^r]$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[r \ln x]$$
$$\frac{1}{y}\frac{dy}{dx} = r\frac{1}{x}$$

Solving for dy/dx yields

$$\frac{dy}{dx} = r\frac{1}{x}y = r\frac{1}{x}x^r = rx^{r-1}$$

■

Section 4.7 Problems

■ 4.7.1

In Problems 1–6, find the inverse of each function and differentiate each inverse in two ways: (i) Differentiate the inverse function directly, and (ii) use (4.12) *to find the derivative of the inverse.*

1. $f(x) = \sqrt{2x+1}, x \geq -\frac{1}{2}$ **2.** $f(x) = \sqrt{x-1}, x \geq 1$

3. $f(x) = 2x^2 - 1, x \geq 0$ **4.** $f(x) = 3x^2 + 2, x \geq 0$

5. $f(x) = 3 - 2x^3, x \geq 0$ **6.** $f(x) = \frac{2x^2-1}{x^2-1}, x > 1$

In Problems 7–22, use (4.12) *to find the derivative of the inverse at the indicated point.*

7. Let
$$f(x) = 2x^2 - 2, \quad x \geq 0$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=0}$. [Note that $f(1) = 0$.]

8. Let
$$f(x) = -x^3 + 7, \quad x \geq 0$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=-1}$. [Note that $f(2) = -1$.]

9. Let
$$f(x) = \sqrt{x+1}, \quad x \geq 0$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=2}$. [Note that $f(3) = 2$.]

10. Let
$$f(x) = \sqrt{2+x^2}, \quad x \geq 0$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=\sqrt{3}}$. [Note that $f(1) = \sqrt{3}$.]

11. Let
$$f(x) = x + e^x, \quad x \in \mathbf{R}$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=1}$. [Note that $f(0) = 1$.]

12. Let
$$f(x) = x + \ln(x+1), \quad x > -1$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=0}$. [Note that $f(0) = 0$.]

13. Let
$$f(x) = x - \sin x, \quad x \in \mathbf{R}$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=\pi}$. [Note that $f(\pi) = \pi$.]

14. Let
$$f(x) = x - \cos x, \quad x \in \mathbf{R}$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=-1}$. [Note that $f(0) = -1$.]

15. Let
$$f(x) = x^2 + \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=0}$. [Note that $f(0) = 0$.]

16. Let
$$f(x) = x^2 + \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
Find $\frac{d}{dx}f^{-1}(x)\big|_{x=\frac{\pi^2}{16}+1}$. [Note that $f(\frac{\pi}{4}) = \frac{\pi^2}{16} + 1$.]

17. Let $f(x) = \ln(\sin x)$, $0 < x < \pi/2$. Find $\frac{d}{dx}f^{-1}(x)$ at $x = -\ln 2$.

18. Let $f(x) = \ln(\tan x), 0 < x < \pi/2$. Find $\frac{d}{dx}f^{-1}(x)$ at $x = \frac{\ln 3}{2}$.

19. Let $f(x) = x^5 + x + 1, -1 < x < 1$. Find $\frac{d}{dx}f^{-1}(x)$ at $x = 1$.

20. Let $f(x) = e^{-x^2} + x$. Find $\frac{d}{dx}f^{-1}(x)$ at $x = 1$.

21. Let $f(x) = e^{-x^2/2} + 2x$. Find $\frac{d}{dx}f^{-1}(x)$ at $x = 1$.

22. Denote the inverse of $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, by $y = \arcsin x$, $-1 \leq x \leq 1$. Show that
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

■ 4.7.2

In Problems 23–60, differentiate the functions with respect to the independent variable. (Note that log *denotes the logarithm to base* 10.*)*

23. $f(x) = \ln(x+1)$ **24.** $f(x) = \ln(3x+4)$

25. $f(x) = \ln(1-2x)$ **26.** $f(x) = \ln(4-3x)$

27. $f(x) = \ln x^2$ **28.** $f(x) = \ln(1-x^2)$

29. $f(x) = \ln(2x^3 - x)$ **30.** $f(x) = \ln(1-x^3)$

31. $f(x) = (\ln x)^2$ **32.** $f(x) = (\ln x)^3$

33. $f(x) = (\ln x^2)^2$ **34.** $f(x) = (\ln(1-x^2))^3$

35. $f(x) = \ln\sqrt{x^2+1}$ **36.** $f(x) = \ln\sqrt{2x^2-x}$

37. $f(x) = \ln\frac{x}{x+1}$ **38.** $f(x) = \ln\frac{2x}{1+x^2}$

39. $f(x) = \ln\frac{1-x}{1+2x}$ **40.** $f(x) = \ln\frac{x^2-1}{x^3-1}$

41. $f(x) = \exp[x - \ln x]$ **42.** $g(s) = \exp[s^2 + \ln s]$

43. $f(x) = \ln(\sin x)$ **44.** $f(x) = \ln(\cos(1-x))$

45. $f(x) = \ln(\tan x^2)$ **46.** $g(s) = \ln(\sin^2(3s))$

47. $f(x) = x\ln x$ **48.** $f(x) = x^2\ln x^2$

49. $f(x) = \frac{\ln x}{x}$ **50.** $h(t) = \frac{\ln t}{1+t^2}$

51. $h(t) = \sin(\ln(3t))$ **52.** $h(s) = \ln(\ln s)$

53. $f(x) = \ln|x^2 - 3|$

54. $f(x) = \log(2x^2 - 1)$

55. $f(x) = \log(1 - x^2)$

56. $f(x) = \log(3x^3 - x + 2)$

57. $f(x) = \log(x^3 - 3x)$

58. $f(x) = \log(\sqrt[3]{\tan x^2})$

59. $f(u) = \log_3(3 + u^4)$

60. $g(s) = \log_5(3^s - 2)$

61. Let $f(x) = \ln x$. We know that $f'(x) = \frac{1}{x}$. We will use this fact and the definition of derivatives to show that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

(a) Use the definition of the derivative to show that

$$f'(1) = \lim_{h\to 0}\frac{\ln(1+h)}{h}$$

(b) Show that (a) implies that

$$\ln[\lim_{h\to 0}(1+h)^{1/h}] = 1$$

(c) Set $h = \frac{1}{n}$ in (b) and let $n \to \infty$. Show that this implies that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

62. Assume that $f(x)$ is differentiable with respect to x. Show that

$$\frac{d}{dx}\ln\left[\frac{f(x)}{x}\right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

4.7.3

In Problems 63–74, use logarithmic differentiation to find the first derivative of the given functions.

63. $f(x) = 2x^x$

64. $f(x) = (2x)^{2x}$

65. $f(x) = (\ln x)^x$

66. $f(x) = (\ln x)^{3x}$

67. $f(x) = x^{\ln x}$

68. $f(x) = x^{2\ln x}$

69. $f(x) = x^{1/x}$

70. $f(x) = x^{3/x}$

71. $y = x^{x^x}$

72. $y = (x^x)^x$

73. $y = x^{\cos x}$

74. $y = (\cos x)^x$

75. Differentiate

$$y = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

76. Differentiate

$$y = \frac{e^{x-1}\sin^2 x}{(x^2+5)^{2x}}$$

4.8 Linear Approximation and Error Propagation

Suppose we want to find an approximation to $\ln(1.05)$ without using a calculator. The method for solving this problem will be useful in many other applications. Let's look at the graph of $f(x) = \ln x$ (Figure 4.39). We know that $\ln 1 = 0$, and we see that 1.05 is quite close to 1—so close, in fact, that the curve connecting $(1, 0)$ to $(1.05, \ln 1.05)$ is close to a straight line. This suggests that we should approximate the curve by a straight line—but not just any straight line: We choose the tangent line to the graph of $f(x) = \ln x$ at $x = 1$ (Figure 4.39). We can find the equation of the tangent line without a calculator. We note that the slope of $f(x) = \ln x$ at $x = 1$ is $f'(1) = \frac{1}{x}\big|_{x=1} = 1$. This, together with the point $(1, 0)$, allows us to find the tangent line at $x = 1$:

$$L(x) = f(1) + f'(1)(x-1) = 0 + (1)(x-1) = x - 1$$

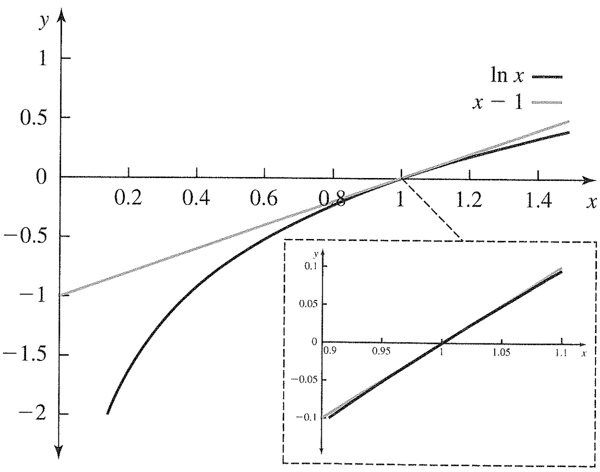

Figure 4.39 The tangent line approximation for $\ln x$ at $x = 1$ to approximate $\ln(1.05)$. When x is close to 1, the tangent line and the graph of $y = \ln x$ are close (see inset).