Quantum Computing for the very curious, summary

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In this document I was originally to wrap up the Strangeworks' paced repetition document named 'Quantum computing for the very curious'. Eventually ended up on wrapping also information available in Qiskit's and Azure's documentation. Items incorporated are ones that I felt needing as available supplementary material along the road. Exspecially on the first few hundred meters, where composing this document was a very good practical excersize to learn the stuff as well.
Hopefully this document is helpfull fo other people as well.

1 Beneficial matrix fundamentals

Dirac notation

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_0,\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_1,\ \langle 0| = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \langle 1| = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$|\psi\rangle^\dagger = \langle\psi|$$

Matrix addition

$$\frac{1+i}{2}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle = \begin{bmatrix} \frac{1+i}{2} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

Unitarity

$$U^\dagger U = I, \quad U^\dagger = (U^T)^*, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ d^* & d^* \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Tensor product

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -1 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

2 Gates

2.1 Pauli gates: X(/NOT), Y and Z

$$NOT|0\rangle = |1\rangle \quad NOT|1\rangle = |0\rangle, \quad NOT(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle,$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$XX|\psi\rangle = \psi$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Y|0\rangle = i|1\rangle,\ Y|1\rangle = -i|0\rangle$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$$

2.2 Hadamard Gate

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$H(\alpha|0\rangle + \beta|1\rangle) = \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ H = H^\dagger,\ HH = HH^\dagger = I$$

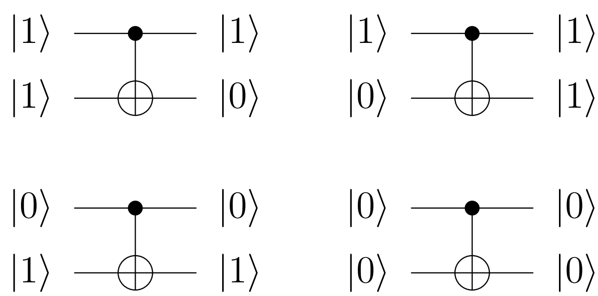
$$\Rightarrow HH|0\rangle = |0\rangle,\ HH|1\rangle = |1\rangle,\ HH|\psi\rangle = |\psi\rangle$$

Applying Hadamard to a qubit in Hadamard state, gives back the basis state Hadamard was originally applied to

$$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \ \text{—}\boxed{H}\text{—}\boxed{\mathbf{M}}\ = |0\rangle$$

$$\frac{|0\rangle-|1\rangle}{\sqrt{2}} \ \text{—}\boxed{H}\text{—}\boxed{\mathbf{M}}\ = |1\rangle$$

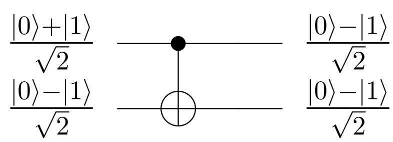
2.3 CNOT Gate



In general superposition, doesn't seem to modify control bit

$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \rightarrow \alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle$$

But in this specific Hadamard case, seemingly does it



2.4 Rotation

$$R_y(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

3 Multiple qubits

Two-qubit state is given by the tensor (or Kronecker) product

$$|0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}0\\1\end{bmatrix}\\0\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} \otimes \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} =$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} & \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\\\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} & -\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

E.g. for below two qubit Hadamard system

$$|H|0\rangle|H|1\rangle\rangle = H|0\rangle \otimes H|1\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} \otimes \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} * -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} * -\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

In other words

$$H \otimes H(|01\rangle) = H^{\otimes 2}\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$