Representation theory of algebras

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Abstract. The notes correspond to the master course **Representation Theory of Algebras** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences.

Contents

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Introduction

The notes correspond to the master course **Representation theory of algebras** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve two-hour lectures.

Most of the material is based on standard results of the representation theory of finite groups. Basic texts on representation theory are [**2**] and [**23**].

The notes include many exercises, some with full detailed solutions. Mandatory exercises have a green background, while optional ones (bonus exercises) have a yellow background.

The notes also include some additional comments. While these are entirely optional, I hope they offer further insight. They are highlighted with a pink background.

The notes include Magma code, which we use to verify examples and offer alternative solutions to certain exercises. Magma [**5**] is a powerful software tool designed for working with algebraic structures. There is a free online version of Magma available.

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1. Lecture: Week 1

§ 1.1. The Artin–Wedderburn theorem. We first review the basic definitions concerning finite-dimensional semisimple algebras. Proofs can be found in the notes to the course **Associative Algebras** (see Lectures 1, 2 and 3).

Our base field will be the field $\mathbb{C}$ of complex numbers.

A (complex) **algebra** A is a (complex) vector space with an associative multiplication $A \times A \to A$ such that

$$a(\lambda b + \mu c) = \lambda(ab) + \mu(ac), \quad (\lambda a + \mu b)c = \lambda(ac) + \mu(bc)$$

for all $a, b, c \in A$. If A contains an element $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$, then A is a unitary algebra. Our algebras will be unitary.

Our algebras will also be finite-dimensional. Clearly, $\mathbb{C}$ is an algebra. Other examples of algebras are $\mathbb{C}[X]$ and $M_n(\mathbb{C})$.

A (left) **module** M (over a unitary algebra A) is an abelian group M together with a map $A \times M \to M$, $(a, m) \mapsto am$, such that $1_A m = m$ for all $m \in M$ and $a(bm) = (ab)m$ and $a(m + m_1) = am + am_1$ for all $a, b \in A$ and $m, m_1 \in M$. A **submodule** N of M is a subgroup N such that $an \in N$ for all $a \in A$ and $n \in N$.

1.1. Exercise. Let A be a finite-dimensional algebra. If M is an A-module, then M is a vector space with $\lambda m = (\lambda 1_A)m$ for $\lambda \in \mathbb{C}$ and $m \in M$. Moreover, M is finitely generated (as an A-module) if and only if M is finite-dimensional.

A module M is said to be **simple** if $M \neq \{0\}$ and $\{0\}$ and M are the only submodules of M. A finite-dimensional module M is said to be **semisimple** if M is a direct sum of finitely many simple submodules. Clearly, simple modules are semisimple. Moreover, any finite direct sum of semisimples is semisimple.

A finite-dimensional algebra A is said to be **semisimple** if every finitely-generated A-module is semisimple.

1.2. Theorem (Artin–Wedderburn). *Let A be a complex finite-dimensional semisimple algebra, say with k isomorphism classes of simple modules. Then*

$$A \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$$

for some $n_1, \dots, n_k \in \mathbb{Z}_{>0}$.

The unique simple module of the algebra $M_{n_j}(\mathbb{C})$ is the column space $\mathbb{C}^{n_j}$. This means that the simple component of dimension n_j^2 has a simple module of dimension n_j.

We also give some basic facts on the Jacobson radical of finite-dimensional algebras. If A is a finite-dimensional algebra, the **Jacobson radical** is defined as

$$J(A) = \bigcap\{M : M \text{ is a maximal left ideal of } A\}.$$

It turns out that $J(A)$ is an ideal of A. If A is unitary, then Zorn's lemma implies that there a maximal left ideal of A and hence $J(A) \neq A$.

An ideal I of A is said to be **nilpotent** if $I^m = \{0\}$ for some m, that is $x_1 \cdots x_m = 0$ for all $x_1, \dots, x_m \in I$. One proves that the Jacobson radical of A contains every nilpotent ideal of A. An important fact is that

$$\begin{aligned} A \text{ is semisimple } &\iff J(A)=\{0\}\\ &\iff A \text{ has no non-zero nilpotent ideals.}\end{aligned}$$

§ 1.2. Group algebras. Let G be a finite group. The (complex) **group algebra** $\mathbb{C}[G]$ is the $\mathbb{C}$-vector space with basis $\{g : g \in G\}$ and multiplication

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{h\in G}\mu_h h\right)=\sum_{g,h\in G}\lambda_g\mu_h(gh).$$

Clearly, $\dim\mathbb{C}[G]=|G|$. Moreover, $\mathbb{C}[G]$ is commutative if and only if G is abelian.

If G is non-trivial, then $\mathbb{C}[G]$ contains proper non-trivial ideals. For example, the **augmentation ideal**

$$I(G)=\left\{\sum_{g\in G}\lambda_g g\in\mathbb{C}[G]:\sum_{g\in G}\lambda_g=0\right\}$$

is a non-zero proper ideal of $\mathbb{C}[G]$.

1.3. Exercise. Let G be a finite non-trivial group. Prove that $\mathbb{C}[G]$ has zero divisors.

For $n\in\mathbb{Z}_{\geq 2}$, we write C_n to denote the (multiplicative) cyclic group of order n.

1.4. Exercise. Prove that $\mathbb{C}[C_n]\simeq\mathbb{C}[X]/(X^n-1)$.

1.5. Exercise. Let G be a finite group. The set

$$\operatorname{Fun}(G,\mathbb{C})=\{\alpha\colon G\to\mathbb{C}\}$$

is a complex vector space with the operations

$$(\alpha+\beta)(x)=\alpha(x)+\beta(x),\quad (\lambda\alpha)(x)=\lambda\alpha(x),$$

for all $\alpha,\beta\in\operatorname{Fun}(G,\mathbb{C})$, $x\in G$ and $\lambda\in\mathbb{C}$. It is an algebra with the **convolution product**

$$(\alpha*\beta)(x)=\sum_{y\in G}\alpha(xy^{-1})\beta(y).$$

Let

$$\delta_x(y)=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

Prove the following statements:

1) The set $\{\delta_x : x\in G\}$ is a basis of $\operatorname{Fun}(G,\mathbb{C})$.

2) The map $\mathbb{C}[G]\to\operatorname{Fun}(G,\mathbb{C})$, $g\mapsto\delta_g$, extends linearly to an algebra isomorphism.

Recall that a finite-dimensional module M is semisimple if and only if for every submodule S of M there is a submodule T of M such that $M=S\oplus T$.

1.6. Theorem (Maschke). *Let G be a finite group and M be a finite-dimensional $\mathbb{C}[G]$-module. Then M is semisimple.*

Proof. We must show that every submodule S of M admits a complement. Since S is a subspace of M, there exists a subspace T_0 of M such that $M = S \oplus T_0$ (as vector spaces). We use T_0 to construct a submodule T of M that complements S. Since $M = S \oplus T_0$, every $m \in M$ can be written uniquely as $m = s + t_0$ for some $s \in S$ and $t_0 \in T$. Let

$$p_0 \colon M \to S, \quad p_0(m) = s,$$

where $m = s + t_0$ with $s \in S$ and $t_0 \in T$. If $s \in S$, then $p_0(s) = s$. In particular, $p_0^2 = p_0$, as $p_0(m) \in S$.

Generally, p_0 is not a $\mathbb{C}[G]$-modules homomorphism. Let

$$p \colon M \to S, \quad p(m) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot p_0(g \cdot m).$$

We claim that p is a homomorphism of $\mathbb{C}[G]$-modules. For that purpose, we need to show that $p(g \cdot m) = g \cdot p(m)$ for all $g \in G$ and $m \in M$. In fact,

$$p(g \cdot m) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \cdot p_0(h \cdot (g \cdot m)) = \frac{1}{|G|} \sum_{h \in G} (gh^{-1}) \cdot p_0(h \cdot m) = g \cdot p(m).$$

We now claim that $p(M) = S$. The inclusion $\subseteq$ is trivial to prove, as S is a submodule of M and $p_0(M) \subseteq S$. Conversely, if $s \in S$, then $g \cdot s \in S$, as S is a submodule. Thus $s = g^{-1} \cdot (g \cdot s) = g^{-1} \cdot p_0(g \cdot s)$ and hence

$$s = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot (g \cdot s) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot (p_0(g \cdot s)) = p(s).$$

Since $p(m) \in S$ for all $m \in M$, it follows that $p^2(m) = p(m)$, so p is a projector onto S. Hence S admits a complement in M, that is $M = S \oplus \ker(p)$. $\square$

1.7. Exercise. Let $G = \langle g \rangle$ be the cyclic group of order four and $\rho_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $M = \mathbb{C}^{2\times 1}$ as an $\mathbb{C}[G]$-module with

$$g \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ u \end{pmatrix}.$$

Prove that M is a semisimple non-simple $\mathbb{C}[G]$-module.

1.8. Exercise. Let $G = \langle g \rangle$ be the cyclic group of order four and $\rho_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $M = \mathbb{R}^{2\times 1}$ as an $\mathbb{R}[G]$-module with

$$g \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ u \end{pmatrix}.$$

Prove that M is a simple $\mathbb{R}[G]$-module.

If G is a finite group, then $\mathbb{C}[G]$ is semisimple. By Artin–Wedderburn theorem,

$$\mathbb{C}[G] \simeq \prod_{i=1}^{r} M_{n_i}(\mathbb{C}),$$

where r is the number of isomorphism classes of simple modules of $\mathbb{C}[G]$. Moreover,

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{r} n_i^2,$$

and the integers $n_1, n_2, \dots, n_r$ are the dimensions of the non-isomorphic simple modules of the complex group algebra $\mathbb{C}[G]$.

1.9. Theorem. *Let G be a finite group. The number of simple modules of $\mathbb{C}[G]$ coincides with the number of conjugacy classes of G.*

Proof. By Artin–Wedderburn theorem, $\mathbb{C}[G] \simeq \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$. Thus

$$Z(\mathbb{C}[G]) \simeq \prod_{i=1}^{r} Z(M_{n_i}(\mathbb{C})) \simeq \mathbb{C}^r.$$

In particular, $\dim Z(\mathbb{C}[G]) = r$. If $\alpha = \sum_{g \in G} \lambda_g g \in Z(\mathbb{C}[G])$, then $h^{-1}\alpha h = \alpha$ for all $h \in G$. Thus

$$\sum_{g \in G} \lambda_{hgh^{-1}} g = \sum_{g \in G} \lambda_g h^{-1} g h = \sum_{g \in G} \lambda_g g$$

and hence $\lambda_g = \lambda_{hgh^{-1}}$ for all $g, h \in G$. A basis for $Z(\mathbb{C}[G])$ is given by elements of the form

$$\sum_{g \in K} g,$$

where K is a conjugacy class of G. Therefore $\dim Z(\mathbb{C}[G])$ equals the number of conjugacy classes of G. □

If G is a finite group, then

$$\mathbb{C}[G] \simeq \prod_{i=1}^{k} M_{n_i}(\mathbb{C}),$$

where k is the number of conjugacy classes of G. In particular,

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{k} n_i^2.$$

1.10. Exercise. Prove that $\mathbb{C}[C_4] \simeq \mathbb{C}^4$.

For $n \geq 1$, let $\mathbb{S}_n$ denote the symmetric group in n letters.

1.11. Example. The group $\mathbb{S}_3$ has three conjugacy classes: $\{\mathrm{id}\}$, $\{(12), (13), (23)\}$ and $\{(123), (132)\}$. Since $6 = a^2 + b^2 + c^2$, it follows that $\mathbb{C}[G] \simeq \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

There is a multiplicative version of Maschke's theorem. A group G **acts by automorphisms** on A if there is a group homomorphism $\lambda \colon G \to \mathrm{Aut}(A)$. In this case, a subgroup B of A is said to be G-invariant if $\lambda_g(B) \subseteq B$ for all $g \in G$.

1.12. Bonus exercise. Let K be a finite group of order m. Assume that K acts by automorphisms on $V = U \times W$, where U and W are subgroups of V and U is abelian and K-invariant. Prove that if the map $U \to U$, $u \mapsto u^m$, is bijective, there exists a normal K-invariant subgroup N of V such that $V = U \times N$.

1.13. Bonus exercise. Let p be a prime number and K be a finite group with order not divisible by p. Let V be a p-elementary abelian group. Assume that K acts by automorphism on V. Prove that if U be a K-invariant subgroup of V, there exists a K-invariant subgroup N of V such that $V = U \times N$.

§ 1.3. Representations. Unless we state differently, we will always work with finite groups. All our vector spaces will be complex vector spaces.

1.14. Definition. Let G be a finite group. A **representation** of G is a group homomorphism $\rho\colon G \to \mathbf{GL}(V)$, where V is a finite-dimensional vector space. The **degree** (or dimension) of the representation is the integer $\deg\rho = \dim V$.

Let $G \to \mathbf{GL}(V)$ be a representation. If we fix a basis of V, then we obtain a **matrix representation** of G, that is a group homomorphism

$$\rho\colon G \to \mathbf{GL}(V) \simeq \mathbf{GL}_n(\mathbb{C}), \quad g \mapsto \rho_g,$$

where $n = \dim V$.

1.15. Example. Since $\mathbb{S}_3 = \langle (12), (123) \rangle$, the map $\rho\colon \mathbb{S}_3 \to \mathbf{GL}_3(\mathbb{C})$,

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is a representation of $\mathbb{S}_3$.

1.16. Example. Let $G = \langle g \rangle$ be cyclic of order six. The map $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$,

$$g \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is a representation of G.

1.17. Example. Let $G = \langle g \rangle$ be cyclic of order four. The map $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$,

$$g \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a representation of G.

1.18. Example. Let $G = \langle a, b : a^2 = b^3 = (ab)^3 = 1 \rangle$. The map

$$a \mapsto \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

defines a representation $G \to \mathbf{GL}_3(\mathbb{C})$.

1.19. Example. Let $Q_8 = \{-1, 1, i, -i, j, -j, k, -k\}$ be the quaternion group. Recall that

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

The group Q_8 is generated by $\{i, j\}$ and the map $\rho\colon Q_8 \to \mathbf{GL}_2(\mathbb{C})$,

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is a representation.

1.20. Example. Let G be a finite group that acts on a finite set X. Let $V=\mathbb{C}X$ the complex vector space with basis $\{x : x\in X\}$. The map

$$\rho\colon G\to\mathbf{GL}(V),\quad \rho_g\left(\sum_{x\in X}\lambda_x x\right)=\sum_{x\in X}\lambda_x\rho_g(x)=\sum_{x\in X}\lambda_{g^{-1}\cdot x}x,$$

is a representation of degree $|X|$.

1.21. Example. The map $\rho\colon G\to\mathbb{C}^\times$, $g\mapsto 1$, is a representation, that is $\mathbb{C}$ is a $\mathbb{C}[G]$-module with $g\cdot\lambda=\lambda$ for all $g\in G$ and $\lambda\in\mathbb{C}^\times$. This representation is known as the **trivial representation**.

1.22. Example. The map $\operatorname{sign}\colon\mathbb{S}_n\to\mathbf{GL}_1(\mathbb{C})=\mathbb{C}^\times$ is a representation of $\mathbb{S}_n$.

An important fact is that there exists a bijective correspondence between representations of a finite group G and finite-dimensional modules over $\mathbb{C}[G]$. The correspondence is given as follows. If $\rho\colon G\to\mathbf{GL}(V)$ is a representation, then V is a $\mathbb{C}[G]$-module with

$$\left(\sum_{g\in G}\lambda_g g\right)\cdot v=\sum_{g\in G}\lambda_g\rho_g(v).$$

Conversely, if V is a $\mathbb{C}[G]$-module, then $\rho\colon G\to\mathbf{GL}(V)$, $\rho_g\colon V\to V$, $v\mapsto g\cdot v$, is a representation.

This bijection between representations of groups and modules over group algebras allows us to construct a dictionary between concepts in the language of representations and that of modules. Both languages are useful, so depending on our convenience, we will use one or the other.

1.23. Exercise. Let G be a finite group and $\rho\colon G\to\mathbf{GL}(V)$ be a representation. Prove that each ρ_g is diagonalizable.

The previous exercise uses properties of the minimal polynomial. We will see a different proof later.

1.24. Definition. Let G be a group and $\phi\colon G\to\mathbf{GL}(V)$ and $\psi\colon G\to\mathbf{GL}(W)$ be representations of G. We say that ϕ and ψ are **equivalent** if there exists a linear isomorphism $T\colon V\to W$ such that

$$\psi_g T=T\phi_g$$

for all $g\in G$. In this case, we write $\phi\simeq\psi$.

Note that $\phi\simeq\psi$ if and only if V and W are isomorphic as $\mathbb{C}[G]$-modules.

1.25. Example. The representation

$$\phi\colon\mathbb{Z}/n\to\mathbf{GL}_2(\mathbb{C}),\quad \phi(m)=\begin{pmatrix}\cos(2\pi m/n) & -\sin(2\pi m/n)\\ \sin(2\pi m/n) & \cos(2\pi m/n)\end{pmatrix},$$

is equivalent to the representation

$$\psi\colon\mathbb{Z}/n\to\mathbf{GL}_2(\mathbb{C}),\quad \psi(m)=\begin{pmatrix}e^{2\pi i m/n} & 0\\ 0 & e^{-2\pi i m/n}\end{pmatrix}.$$

The equivalence is obtained with the matrix $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, as a direct calculation shows that $\phi_m T = T\psi_m$ for all m.

1.26. Exercise. Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation. Fix a basis of V and consider the corresponding matrix representation ϕ of ρ. Prove that ρ and ϕ are equivalent.

1.27. Definition. Let $\phi\colon G \to \mathbf{GL}(V)$ be a representation. A subspace $W \subseteq V$ is said to be **G-invariant** if $\phi_g(W) \subseteq W$ for all $g \in G$.

Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation. If W is a G-invariant subspace of V, then the restriction $\rho|_W\colon G \to \mathbf{GL}(W)$ is a representation. In particular, W is a submodule (over $\mathbb{C}[G]$) of V.

1.28. Definition. A non-zero representation $\rho\colon G \to \mathbf{GL}(V)$ is said to be **irreducible** if $\{0\}$ and V are the only G-invariant subspaces of V.

Note that a representation $\rho\colon G \to \mathbf{GL}(V)$ is irreducible if and only if V is simple.

1.29. Example. Degree-one representations are irreducible.

1.30. Exercise. Let G be a finite group. Prove that there exists a bijective correspondence between degree-one representations of G and degree-one representations of $G/[G,G]$.

1.31. Exercise. Let G be a finite group of order n with k conjugacy classes. Let $m = (G : [G,G])$. Prove that $n + 3m \geq 4k$.

In the following example, we work over the real numbers.

1.32. Example. Let $G = \langle g \rangle$ be the cyclic group of three elements and

$$\rho\colon G \to \mathbf{GL}(\mathbb{R}^3), \quad \rho_g(x,y,z) = (y,z,x).$$

The set

$$N = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\}$$

is a G-invariant subspace of $\mathbb{R}^3$.

We claim that N is irreducible. If N contains a non-zero G-invariant subspace S, let $(x_0,y_0,z_0) \in S \setminus \{(0,0,0)\}$. Since S is G-invariant,

$$(y_0,z_0,x_0) = g \cdot (x_0,y_0,z_0) \in S.$$

We claim that $\{(x_0,y_0,z_0),(y_0,z_0,x_0)\}$ is linearly independent. If there exists $\lambda \in \mathbb{R}$ such that $\lambda(x_0,y_0,z_0) = (y_0,z_0,x_0)$, then $x_0 = \lambda^3 x_0$. Since $x_0 = 0$ implies $y_0 = z_0 = 0$, it follows that $\lambda = 1$. In particular, $x_0 = y_0 = z_0$, a contradiction, as $x_0+y_0+z_0 = 0$. Hence $\dim S = 2$ and therefore $S = N$.

What happens in the previous example if we consider complex numbers?

1.33. Exercise. Let $\phi\colon G \to \mathbf{GL}(V)$, $g \mapsto \phi_g$, be a degree-two representation. Prove that ϕ is irreducible if and only if there is no common eigenvector for all the ϕ_g.

1.34. Example. Recall that $\mathbb{S}_3$ is generated by (12) and (23). The map

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

defines a representation ϕ of $\mathbb{S}_3$. Exercise 1.33 shows that ϕ is irreducible.

2. Lecture: Week 2

We now describe some crucial examples of representations.

2.1. Example. Let $\rho\colon G\to\mathbf{GL}(V)$ and $\psi\colon G\to\mathbf{GL}(W)$ be representations. The **direct sum** $\rho\oplus\psi\colon G\to\mathbf{GL}(V\oplus W)$, $g\mapsto(\rho_g,\psi_g)$, is a representation. This is equivalent to say that the vector space $V\oplus W$ is a $\mathbb{C}[G]$-module with

$$g\cdot(v,w)=(g\cdot v,g\cdot w),\quad g\in G,\ v\in v,\ w\in W.$$

Let V be a vector space with basis $\{v_1,\dots,v_k\}$ and W be a vector space with basis $\{w_1,\dots,w_l\}$. A **tensor product** of V and W is a vector space X with together with a bilinear map

$$V\times W\to X,\quad (v,w)\mapsto v\otimes w,$$

such that $\{v_i\otimes w_j:1\leq i\leq k,\ 1\leq j\leq l\}$ is a basis of X. The tensor product of V and W is unique up to isomorphism and it is denoted by $V\otimes W$. Note that

$$\dim(V\otimes W)=(\dim V)(\dim W).$$

2.2. Example. Let V and W be $\mathbb{C}[G]$-modules. The **tensor product** $V\otimes W$ is a $\mathbb{C}[G]$-module with

$$g\cdot v\otimes w=g\cdot v\otimes g\cdot w,\quad g\in G,\ v\in V,\ w\in W.$$

Let $\rho\colon G\to\mathbf{GL}(V)$ and $\psi\colon G\to\mathbf{GL}(W)$ be representations. The **tensor product** of ρ and ψ is the representation of G given by

$$\rho\otimes\psi\colon G\to\mathbf{GL}(V\otimes W),\quad g\mapsto(\rho\otimes\psi)_g,$$

where

$$(\rho\otimes\psi)_g(v\otimes w)=\rho_g(v)\otimes\psi_g(w)$$

for $v\in V$ and $w\in W$.

2.3. Exercise. Let G be a finite group and V be a $\mathbb{C}[G]$-module. Prove that the dual V^* is a $\mathbb{C}[G]$-module with

$$(g\cdot f)(v)=f(g^{-1}v),\quad f\in V^*,\ v\in V,\ g\in G.$$

2.4. Exercise. Let G be a finite group and V and W be $\mathbb{C}[G]$-modules. Prove that the set $\operatorname{Hom}(V,W)$ of complex linear maps $V\to W$ is a $\mathbb{C}[G]$-module with

$$(g\cdot f)(v)=gf(g^{-1}v),\quad f\in\operatorname{Hom}(V,W),\ v\in V,\ g\in G.$$

If, moreover, V and W are finite-dimensional, then

$$V^*\otimes W\simeq\operatorname{Hom}(V,W)$$

as $\mathbb{C}[G]$-modules.

2.5. Definition. A representation $\rho\colon G\to\mathbf{GL}(V)$ is said to be **completely reducible** if ρ can be decomposed as $\rho=\rho_1\oplus\cdots\oplus\rho_n$ for some irreducible representations $\rho_1,\dots,\rho_n$ of G.

Note that if $\rho\colon G\to\mathbf{GL}(V)$ is completely reducible and $\rho=\rho_1\oplus\cdots\oplus\rho_n$ for some irreducible representations $\rho_i\colon G\to\mathbf{GL}(V_i)$, $i\in\{1,\dots,n\}$, then each V_i is an invariant subspace of V and $V=V_1\oplus\cdots\oplus V_n$. Moreover, in some basis of V, the matrix ρ_g can be written as

$$\rho_g=\begin{pmatrix}(\rho_1)_g & & & \\ & (\rho_2)_g & & \\ & & \ddots & \\ & & & (\rho_n)_g\end{pmatrix}.$$

2.6. Definition. A representation $\rho\colon G\to\mathbf{GL}(V)$ is **decomposable** if V can be decomposed as $V=S\oplus T$ where S and T are non-zero invariant subspaces of V.

A representation is **indecomposable** if it is not decomposable.

2.7. Exercise. Let $\rho\colon G\to\mathbf{GL}(V)$ and $\psi\colon G\to\mathbf{GL}(W)$ be equivalent representations. Prove the following facts:

1) If ρ is irreducible, then ψ is irreducible.
2) If ρ is decomposable, then ψ is decomposable.
3) If ρ is completely reducible, then ψ is completely reducible.

§ 2.1. Characters. Fix a finite group G and consider (matrix) representations of G. We use linear algebra to study these representations.

2.8. Definition. Let $\rho\colon G\to\mathbf{GL}(V)$ be a representation. The **character** of ρ is the map $\chi_\rho\colon G\to\mathbb{C}$, $g\mapsto\operatorname{trace}\rho_g$.

If a representation ρ is irreducible, its character is said to be an **irreducible character**. The **degree** of a character is the degree of the affording representation.

2.9. Example. We can compute the character of the representation

$$(12)\mapsto\begin{pmatrix}-1&1\\0&1\end{pmatrix},\quad(23)\mapsto\begin{pmatrix}1&0\\1&-1\end{pmatrix},$$

of Example 1.34. Since

$$\rho_{(132)}=\rho_{(23)(12)}=\rho_{(23)}\rho_{(12)}=\begin{pmatrix}-1&1\\-1&0\end{pmatrix},$$

we conclude that $\rho_{(132)}=-1$. Similar calculations show that

$$\chi_{\mathrm{id}}=2,\quad\chi_{(12)}=\chi_{(13)}=\chi_{(23)}=0,\quad\chi_{(123)}=\chi_{(132)}=-1.$$

2.10. Proposition. *Let $\rho\colon G\to\mathbf{GL}(V)$ be a representation, χ be its character and $g\in G$. The following statements hold:*

1) $\chi(1)=\dim V$.
2) $\chi(g)=\chi(hgh^{-1})$ *for all* $h\in G$.
3) $\chi(g)$ *is the sum of* $\chi(1)$ *roots of one of order* $|g|$.
4) $\chi(g^{-1})=\overline{\chi(g)}$.
5) $|\chi(g)|\leq\chi(1)$.

PROOF. The first statement is trivial.

To prove 2) note that

$$\chi(hgh^{-1}) = \operatorname{trace}(\rho_{hgh^{-1}}) = \operatorname{trace}(\rho_h\rho_g\rho_h^{-1}) = \operatorname{trace}\rho_g = \chi(g).$$

Statement 3) follows from the fact that the trace of ρ_g is the sum of the eigenvalues of ρ_g and these numbers are roots of the polynomial $X^{|g|}-1 \in \mathbb{C}[X]$. To prove 4) write $\chi(g) = \lambda_1 + \cdots + \lambda_k$, where the λ_j are roots of one. Then

$$\overline{\chi(g)} = \sum_{j=1}^{k} \overline{\lambda_j} = \sum_{j=1}^{k} \lambda_j^{-1} = \operatorname{trace}(\rho_g^{-1}) = \operatorname{trace}(\rho_{g^{-1}}) = \chi(g^{-1}).$$

Finally, we prove 5). We use 3) to write $\chi(g)$ as the sum of $\chi(1)$ roots of one, say $\chi(g) = \lambda_1 + \cdots + \lambda_k$ for $k = \chi(1)$. Then

$$|\chi(g)| = |\lambda_1 + \cdots + \lambda_k| \leq |\lambda_1| + \cdots + |\lambda_k| = \underbrace{1 + \cdots + 1}_{k\text{-times}} = k. \qquad \square$$

If two representations are equivalent, their characters are equal.

2.11. DEFINITION. Let G be a group and $f\colon G \to \mathbb{C}$ be a map. Then f is a **class function** if $f(g) = f(hgh^{-1})$ for all $g, h \in G$.

Characters are class functions. If G is a finite group, we write

$$\operatorname{ClassFun}(G) = \{f\colon G \to \mathbb{C} : f \text{ is a class function}\}.$$

One proves that $\operatorname{ClassFun}(G)$ is a complex vector space.

2.12. EXERCISE. Let G be a finite group. For a conjugacy class K of G let

$$\delta_K\colon G \to \mathbb{C}, \quad \delta_K(g) = \begin{cases} 1 & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\{\delta_K : K \text{ is a conjugacy class of } G\}$ is a basis of $\operatorname{ClassFun}(G)$. In particular, $\dim \operatorname{ClassFun}(G)$ is the number of conjugacy classes of G.

2.13. PROPOSITION. *If $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ are representations, then $\chi_{\rho\oplus\psi} = \chi_\rho + \chi_\psi$.*

PROOF. For $g \in G$, it follows that $(\rho \oplus \psi)_g = \begin{pmatrix} \rho_g & 0 \\ 0 & \psi_g \end{pmatrix}$. Thus

$$\chi_{\rho\oplus\psi}(g) = \operatorname{trace}((\rho \oplus \phi)_g) = \operatorname{trace}(\rho_g) + \operatorname{trace}(\psi_g) = \chi_\rho(g) + \chi_\psi(g). \qquad \square$$

2.14. PROPOSITION. *If $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ are representations, then*

$$\chi_{\rho\otimes\psi} = \chi_\rho\chi_\psi.$$

PROOF. For each $g \in G$, the map ρ_g is diagonalizable. Let $\{v_1, \dots, v_n\}$ be a basis of eigenvectors of ρ_g and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be such that $\rho_g(v_i) = \lambda_i v_i$ for all $i \in \{1, \dots, n\}$. Similarly, let $\{w_1, \dots, w_m\}$ be a basis of eigenvectors of ψ_g and $\mu_1, \dots, \mu_m \in \mathbb{C}$ be such that $\psi_g(w_j) = \mu_j w_j$ for all $j \in \{1, \dots, m\}$. Each $v_i \otimes w_j$ is eigenvector of $(\rho \otimes \psi)_g$ with eigenvalue $\lambda_i\mu_j$, as

$$(\rho \otimes \psi)_g(v_i \otimes w_j) = \rho_g v_i \otimes \psi_g w_j = \lambda_i v_i \otimes \mu_j v_j = (\lambda_i\mu_j) v_i \otimes w_j.$$

Thus $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of eigenvectors and the $\lambda_i\mu_j$ are the eigenvalues of $(\rho \otimes \psi)_g$. It follows that

$$\chi_{\rho\otimes\psi}(g) = \sum_{i,j} \lambda_i\mu_j = \left(\sum_i \lambda_i\right)\left(\sum_j \mu_j\right) = \chi_\rho(g)\chi_\psi(g). \qquad \square$$

We know that it is also possible to define the dual $\rho^* \colon G \to \mathbf{GL}(V^*)$ of a representation $\rho \colon G \to \mathbf{GL}(V)$ by the formula

$$(\rho_g^* f)(v) = f(\rho_g^{-1}v), \quad g \in G,\ f \in V^* \text{ and } v \in V.$$

We claim that the character of the dual representation is then $\overline{\chi_\rho}$. Let $\{v_1, \dots, v_n\}$ be a basis of V and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be such that $\rho_g v_i = \lambda_i v_i$ for all $i \in \{1, \dots, n\}$. If $\{f_1, \dots, f_n\}$ is the dual basis of $\{v_1, \dots, v_n\}$, then

$$(\rho_g^* f_i)(v_j) = f_i(\rho_g^{-1}v_j) = \overline{\lambda_j} f_i(v_j) = \overline{\lambda_j}\delta_{ij}$$

and the claim follows.

Let G be a finite group. If $\chi, \psi \colon G \to \mathbb{C}$ are characters of G and $\lambda \in \mathbb{C}$, we define

$$(\chi + \psi)(g) = \chi(g) + \psi(g), \quad (\chi\psi)(g) = \chi(g)\psi(g), \quad (\lambda\chi)(g) = \lambda\chi(g).$$

We can then form linear combinations of characters. These functions, of course, are not necesarilly characters.

2.15. Theorem. *Let G be a finite group. Then irreducible characters of G are linearly independent.*

Proof. Let $S_1, \dots, S_k$ be a complete set of representatives of classes of simple $\mathbb{C}[G]$-modules. Let $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. By Artin–Wedderburn theorem, there is an algebra isomorphism $f \colon \mathbb{C}[G] \to M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$, where $\dim S_j = n_j$ for all j. Moreover,

$$M_{n_j}(\mathbb{C}) \simeq \underbrace{S_j \oplus \cdots \oplus S_j}_{n_j-\text{times}}$$

for all j. For each j let $e_j = f^{-1}(I_j)$, where I_j is the identity matrix of $M_{n_j}(\mathbb{C})$. We claim that

$$\chi_i(e_j) = \begin{cases} \dim S_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $\chi_i(g)$ is the trace of the action of g on S_j. Since $e_ie_j = 0$ if $i \neq j$, it follows that $\chi_i(e_j) = 0$ if $i \neq j$. Moreover, e_j acts as the identity on S_j, thus $\chi_j(e_j) = \dim S_j$.

Now if $\sum \lambda_i\chi_i = 0$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, then

$$(\dim S_j)\lambda_j = \sum \lambda_i\chi_i(e_j) = 0$$

and hence $\lambda_j = 0$, as $\dim S_j \neq 0$. $\square$

2.16. Theorem. *Let G be a finite group and $S_1, \dots, S_k$ be the simple $\mathbb{C}[G]$-modules (up to isomorphism). If $V = \oplus_{i=1}^k a_j S_j$, then $\chi_V = \sum a_i\chi_i$, where $\chi_i = \chi_{S_i}$ for all i. Moreover, if U and V are $\mathbb{C}[G]$-modules,*

$$U \simeq V \iff \chi_U = \chi_V.$$

PROOF. The first part is left as an exercise.

It is also an exercise to prove that $U \simeq V$ implies $\chi_U = \chi_V$. Let us prove the converse. Assume that $\chi_U = \chi_V$. Since $\mathbb{C}[G]$ is semisimple, $U \simeq \oplus_{i=1}^k a_i S_i$ and $V \simeq \oplus_{i=1}^k b_i S_i$ for some integers $a_1, \dots, a_k \geq 0$ and $b_1, \dots, b_k \geq 0$. Since

$$0 = \chi_U - \chi_V = \sum_{i=1}^k (a_i - b_i)\chi_i$$

and the χ_i are linearly independent, it follows that $a_i = b_i$ for all i. Hence $U \simeq V$. □

2.17. EXERCISE. Let G be a finite group and U be a $\mathbb{C}[G]$-module. Prove $\chi_{U^*} = \overline{\chi_U}$.

We will use the following exercise later:

2.18. EXERCISE. Prove that if G is a finite group and U and V are $\mathbb{C}[G]$-modules, then

$$\chi_{\mathrm{Hom}(U,V)} = \overline{\chi_U}\chi_V.$$

For a finite group G we write $\mathrm{Irr}(G)$ to denote the complete set of isomorphism classes of characters of irreducible representations of G.

2.19. EXERCISE. Let G be a finite group. Prove that the set $\mathrm{Irr}(G)$ is a basis of $\mathrm{ClassFun}(G)$.

Let G be a finite group and U be a $\mathbb{C}[G]$-module. Let

$$U^G = \{u \in U : g \cdot u = u \text{ for all } g \in G\}.$$

Then U^G is a subspace of U. The following lemma is important:

2.20. LEMMA. $\dim U^G = \frac{1}{|G|}\sum_{x\in G} \chi_U(x)$

PROOF. Let ρ be the representation associated with U and let

$$\alpha = \frac{1}{|G|}\sum_{x\in G} \rho_x \colon U \to U.$$

We claim that $\alpha^2 = \alpha$. Let $g \in G$. Then

$$\rho_g(\alpha) = \frac{1}{|G|}\sum_{x\in G} \rho_g\rho_x = \frac{1}{|G|}\sum_{x\in G} \rho_{gx} = \alpha.$$

Thus

$$\alpha(\alpha(u)) = \frac{1}{|G|}\sum_{x\in G} \rho_x(\alpha(u)) = \alpha(u)$$

for all $u \in U$. This implies that α has eigenvalues 0 and 1. In fact, if $u \in U$ is an eigenvector of α of eigenvalue $\lambda \in \mathbb{C}$, then

$$\lambda u = \alpha(u) = \alpha(\alpha(u)) = \alpha(\lambda u) = \lambda\alpha(u) = \lambda^2 u.$$

Thus $\lambda(\lambda - 1) = 0$.

Let V be the eigenspace of eigenvalue 1. We now claim that $V = U^G$. Let us first prove that $V \subseteq U^G$. For that purpose, let $v \in V$ and $g \in G$. Then

$$\begin{aligned} g \cdot v &= \rho_g(v) = \rho_g(\alpha(v)) \\ &= \frac{1}{|G|} \sum_{x \in G} \rho_g \rho_x(v) = \frac{1}{|G|} \sum_{y \in G} \rho_y(v) = \alpha(v) = v. \end{aligned}$$

Now we prove that $V \supseteq U^G$. Let $u \in U^G$, so $\rho_g(u) = u$ for all $g \in G$. Then

$$\alpha(u) = \frac{1}{|G|} \sum_{x \in G} \rho_x(u) = \frac{1}{|G|} \sum_{x \in G} u = u.$$

Thus

$$\dim U^G = \dim V = \operatorname{trace} \alpha = \frac{1}{|G|} \sum_{x \in G} \operatorname{trace} \rho_x = \frac{1}{|G|} \sum_{x \in G} \chi_U(x).$$ □

One proves that the operation

$$\langle \chi_U, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \overline{\chi_V(g)}$$

defines an inner product.

2.21. Theorem. *Let G be a finite group and U and V be $\mathbb{C}[G]$-modules. Then*

$$\langle \chi_U, \chi_V \rangle = \dim \operatorname{Hom}_G(U, V).$$

Proof. We claim that

$$\operatorname{Hom}_G(U, V) = \operatorname{Hom}(U, V)^G.$$

Let us first prove that $\operatorname{Hom}_G(U, V) \subseteq \operatorname{Hom}(U, V)^G$. Let $f \in \operatorname{Hom}_G(U, V)$ and $g \in G$. Then

$$(g \cdot f)(u) = g \cdot f(g^{-1} \cdot u) = g \cdot (g^{-1} \cdot f(u)) = f(u)$$

for all $u \in U$. Now we prove that $\operatorname{Hom}_G(U, V) \supseteq \operatorname{Hom}(U, V)^G$. Let $f \in \operatorname{Hom}(U, V)^G$. Then $f \colon U \to V$ is a linear such that $g \cdot f = f$ for all $g \in G$. Then we compute

$$\begin{aligned} (g \cdot f)(u) = f(u) &\implies g \cdot f(g^{-1} \cdot u) = f(u) \\ &\implies f(g^{-1} \cdot u) = g^{-1} \cdot f(u) \quad \text{for all } g \in G \text{ and } u \in U \end{aligned}$$

This means that one has

$$f(g \cdot u) = g \cdot f(u)$$

for all $g \in G$ and $u \in U$.

Using Exercise 2.18,

$$\begin{aligned} \dim \operatorname{Hom}_G(U, V) &= \dim \operatorname{Hom}(U, V)^G \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{Hom}(U,V)}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_U(g)} \chi_V(g) \\ &= \langle \chi_V, \chi_U \rangle. \end{aligned}$$

Since $\dim \operatorname{Hom}_G(U,V) \in \mathbb{R}$, one has

$$\langle \chi_U, \chi_V \rangle = \overline{\langle \chi_V, \chi_U \rangle} = \langle \chi_V, \chi_U \rangle$$

and the claim follows. □

Let G be a finite group and $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. Note that k is the number of conjugacy classes of G. Let $g_1, \dots, g_k$ be representatives of conjugacy classes of G. The **matrix of characters** of G is $X = (X_{ij})$, where

$$X_{ij} = \chi_i(g_j)$$

for $i, j \in \{1, \dots, k\}$.

2.22. Example. Let $G = \mathbb{S}_3$. The group G has three conjugacy classes, so $|\operatorname{Irr}(G)| = 3$. Let $g_1 = \mathrm{id}$, $g_2 = (12)$ and $g_3 = (123)$. We know that $6 = n_1^2 + n_2^2 + n_3^2$. We know two degree-one (irreducible) representations of G, the trivial one and the sign. This implies that $n_1 = n_2 = 1$ and $n_3 = 2$. The matrix of characters is then

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	?	?

Two entries of the character table of Example 2.22 are still unknown. As we further develop the theory of characters, we will discover several tricks that can be used to find these missing entries.

3. Lecture: Week 3

§ 3.1. Schur's orthogonality relations. We start with a crucial exercise. It is known as Schur's lemma.

3.1. Exercise. If G is a group and U and V are simple $\mathbb{C}[G]$-modules, then a non-zero module homomorphism $U \to V$ is an isomorphism.

We now discuss a handy application of Schur's lemma. Let G be a finite group and S be a simple $\mathbb{C}[G]$-module. We claim that $\mathrm{Hom}_G(S,S) \simeq \mathbb{C}$. Let $f \in \mathrm{Hom}_G(S,S)$ and $\lambda \in \mathbb{C}$ be an eigenvalue of f. Then $f - \lambda\,\mathrm{id}\colon S \to S$ is not invertible. By Schur's lemma, $f - \lambda\,\mathrm{id} = 0$ and hence $f = \lambda\,\mathrm{id}$.

3.2. Definition. Let G be a group. A representation $\rho\colon G \to \mathbf{GL}(V)$ is said to be **faithful** if ρ is injective.

3.3. Exercise. Let G be a finite group that admits a faithful irreducible representation. Prove that $Z(G)$ is cyclic.

To solve Exercise 3.3 one needs to use the following elementary fact: A finite subgroup of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is cyclic.

3.4. Theorem (Schur). *Let G be a finite group and $\chi, \psi \in \mathrm{Irr}(G)$. Then*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $S_1, \dots, S_k$ be the simples of $\mathbb{C}[G]$. For each j, let χ_j be the irreducible character of S_j. Then

$$\langle \chi_i, \chi_j \rangle = \dim \mathrm{Hom}_G(S_i, S_j) = \begin{cases} 1 & \text{if } S_i \simeq S_j, \\ 0 & \text{otherwise.} \end{cases}$$

But we know that $S_i \simeq S_j$ if and only if $\chi_i = \chi_j$. □

With the theorem, one can construct the character table of $\mathbb{S}_3$. For example, this can be done using that $\langle \chi_3, \chi_3 \rangle = 1$ and that $\langle \chi_1, \chi_3 \rangle = 0$. As an exercise, verify that the character table of $\mathbb{S}_3$ is given in Table 3.1.

Table 1. The character table of $\mathbb{S}_3$.

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

3.5. Exercise. Let G be a finite group. Prove that $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathrm{ClassFun}(G)$.

The previous exercise has some consequences. Let G be a finite group and assume that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. If $\alpha = \sum a_i \chi_i$, then $\alpha = \sum \langle \alpha, \chi_i \rangle \chi_i$.

3.6. THEOREM. *Let G be a finite group and $S_1,\dots,S_k$ be the simples of $\mathbb{C}[G]$. Then the left regular $\mathbb{C}[G]$-module decomposes as*

$$\mathbb{C}[G]\simeq\bigoplus_{i=1}^{k}(\dim S_i)S_i.$$

PROOF. Let $n=|G|$. Assume that $G=\{g_1,\dots,g_n\}$. Decompose the $\mathbb{C}[G]$-module corresponding to the left regular representation as

$$\mathbb{C}[G]\simeq a_1S_1\oplus\cdots\oplus a_kS_k$$

for some integers $a_1,\dots,a_k\geq 0$. Let $L\colon G\to\mathbb{S}_G$, $g\mapsto L_g$, where $L_g(g_j)=gg_j$ for all j. Since the matrix of L_g in the basis $\{g_1,\dots,g_n\}$ is

$$(L_g)_{ij}=\begin{cases}1 & \text{if } g_i=gg_j,\\ 0 & \text{otherwise,}\end{cases}$$

one obtains that

$$\chi_L(g)=\begin{cases}|G| & \text{if } g=1,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover,

$$\chi_L=\sum_{i=1}^{k}a_i\chi_i=\sum_{i=1}^{k}\langle\chi_L,\chi_i\rangle\chi_i$$

and

$$a_i=\langle\chi_L,\chi_i\rangle=\frac{1}{|G|}\sum_{g\in G}\chi_L(g)\overline{\chi_i(g)}=\frac{1}{|G|}|G|\overline{\chi_i(1)}=\dim S_i.$$

Thus $\mathbb{C}[G]\simeq\bigoplus_{i=1}^{k}(\dim S_i)S_i$. □

If G is a finite group, let $\mathrm{Char}(G)$ be the set of characters of G.

3.7. EXERCISE. Let $n\in\{1,2,3\}$. Let G be a finite group and $\alpha\in\mathrm{Char}(G)$. Prove that α is the sum of n irreducible characters if and only if $\langle\alpha,\alpha\rangle=n$.

We now prove **Schur's second orthogonality relation**.

3.8. THEOREM (Schur). *Let G be a finite group and $g,h\in G$. Then*

$$\sum_{\chi\in\mathrm{Irr}(G)}\chi(g)\overline{\chi(h)}=\begin{cases}|C_G(g)| & \textit{if } g \textit{ and } h \textit{ are conjugate,}\\ 0 & \textit{otherwise.}\end{cases}$$

PROOF. Let $g_1,\dots,g_r$ be the representatives of the conjugacy classes of G. Assume that $\mathrm{Irr}(G)=\{\chi_1,\dots,\chi_r\}$. For each $k\in\{1,\dots,r\}$, let $c_k=(G:C_G(g_k))$ denote the size of the conjugacy class of g_k. Then

$$\langle\chi_i,\chi_j\rangle=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)}=\frac{1}{|G|}\sum_{k=1}^{r}c_k\chi_i(g_k)\overline{\chi_j(g_k)}.$$

We write this as $I = \frac{1}{|G|}XDX^*$, where I denotes the identity matrix, $X_{ij} = \chi_i(g_j)$, $X^* = \overline{X}^T$ and

$$D = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_r \end{pmatrix}.$$

Since, in matrices, $AB = I$ implies $BA = I$, it follows that $I = \frac{1}{|G|}X^*XD$. Thus, using that $|G| = c_k|C_G(g_k)|$ holds for all k,

$$(|G|D^{-1})_{ij} = (X^*X)_{ij} = \sum_{k=1}^{r} \overline{\chi_k(g_i)}\chi_k(g_j) = \begin{cases} |C_G(g_j)| & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$ □

3.9. Exercise. Let G be a finite group and $g_1, \dots, g_r$ be the representatives of the conjugacy classes of G. Assume that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$. Compute the determinant of the matrix $X = (\chi_j(g_i))_{1 \le i,j \le r}$.

3.10. Theorem (Solomon). *Let G be a finite group and $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$. If $g_1, \dots, g_r$ are the representatives of the conjugacy classes of G and $i \in \{1, \dots, r\}$, then*

$$\sum_{j=1}^{r} \chi_i(g_j) \in \mathbb{Z}_{\ge 0}.$$

Proof. Let $n = |G|$. Assume that $G = \{g_1, g_2, \dots, g_r, g_{r+1}, \dots, g_n\}$. Let V be the complex vector space with basis $\{g_1, \dots, g_n\}$. The action of G on G by conjugation induces a group homomorphism $\rho\colon G \to \mathbf{GL}(V)$, $g \mapsto \rho_g$, where $\rho_g(h) = ghg^{-1}$. The matrix of ρ_g in the basis $\{g_1, \dots, g_n\}$ is

$$(\rho_g)_{ij} = \begin{cases} 1 & \text{if } g_jg = gg_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\chi_\rho(g) = \operatorname{trace} \rho_g = \sum_{k=1}^{|G|} (\rho_g)_{kk} = |\{k : g_kg = gg_k\}| = |C_G(g)|.$$

Write $\chi_\rho = \sum_{i=1}^{r} m_i\chi_i$ for $m_1, \dots, m_r \ge 0$. For each j let $c_j = (G : C_G(g_j))$. Then

$$\begin{aligned} m_i = \langle \chi_\rho, \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)\overline{\chi_i(g)} \\ &= \frac{1}{|G|} \sum_{j=1}^{r} c_j|C_G(g_j)|\overline{\chi_i(g_j)} = \sum_{j=1}^{r} \overline{\chi_i(g_j)}. \end{aligned}$$ □

3.11. Exercise (Solomon). Let G be a finite group and $g_1, \dots, g_r$ be the representatives of the conjugacy classes of G. Assume that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$. Prove that

$$|G| \ge \sum_{i=1}^{r} \sum_{j=1}^{r} \chi_i(C_j) \in \mathbb{Z}_{\ge 1}.$$

Moreover, if $|G| = \sum_{i=1}^{r} \sum_{j=1}^{r} \chi_i(C_j)$, then $G/Z(G)$ is abelian.

§ 3.2. Algebraic integers and characters.

3.12. Definition. Let $\alpha \in \mathbb{C}$. We say that α is **algebraic integer** if $f(\alpha)=0$ for some monic polynomial $f \in \mathbb{Z}[X]$.

Let $\mathbb{A}$ be the set of algebraic integers. Note that $\mathbb{Z} \subseteq \mathbb{A}$.

3.13. Example. Every root of one is an algebraic integer.

3.14. Proposition. *$\mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$.*

Proof. Let $m/n \in \mathbb{Q}$ with $\gcd(m,n)=1$ and $n>0$. If $f(m/n)=0$ for some

$$f = X^k + a_{k-1}X^{k-1} + \cdots + a_1X + a_0 \in \mathbb{Z}[X]$$

of degree $k \geq 1$, then

$$0 = n^k f(m/n) = m^k + a_{k-1}m^{k-1}n + \cdots + a_1mn^{k-1} + a_0n^k.$$

This implies that

$$m^k = -n\left(a_{k-1}m^{k-1} + \cdots + a_1mn^{k-2} + a_0n^{k-1}\right)$$

and hence n divides m^k. Thus $n=1$ and therefore $m/n \in \mathbb{Z}$. □

3.15. Proposition. *Let $x \in \mathbb{C}$. Then $x \in \mathbb{A}$ if and only if x is an eigenvalue of an integer matrix.*

Proof. Let us prove the non-trivial implication. Let

$$f = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X]$$

be such that $f(x)=0$. Then x is an eigenvalue of the companion matrix of f, that is the matrix

$$C(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in \mathbb{Z}^{n\times n}.$$ □

3.16. Theorem. *$\mathbb{A}$ is a subring of $\mathbb{C}$.*

Proof. Let $\alpha, \beta \in \mathbb{A}$. By the previous proposition, α is an eigenvalue of an integer matrix $A \in \mathbb{Z}^{n\times n}$, say $Av = \alpha v$ for some $v \neq 0$, β is an eigenvalue of an integer matrix $B \in \mathbb{Z}^{m\times m}$, say $Bw = \beta w$ for some $w \neq 0$. Then

$$(A \otimes I_{m\times m} + I_{n\times n} \otimes B)(v \otimes w) = (\alpha+\beta)(v \otimes w),$$

where $I_{k\times k}$ denotes the $(k \times k)$ identity matrix, and

$$(A \otimes B)(v \otimes w) = (\alpha\beta)v \otimes w.$$

This implies that $\alpha+\beta \in \mathbb{A}$ and $\alpha\beta \in \mathbb{A}$, again by the previous proposition. □

3.17. Theorem. *Let G be a finite group. If $\chi \in \mathrm{Char}(G)$ and $g \in G$, then $\chi(g) \in \mathbb{A}$.*

Proof. Let φ be a representation of G such that $\chi_\varphi = \chi$. Since φ_g is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_k \in \mathbb{A}$ (because G is finite and the λ_j are roots of one),

$$\chi(g) = \operatorname{trace}\varphi_g = \sum_{i=1}^{k} \lambda_i \in \mathbb{A}.$$ □

4. Lecture: Week 4

We will use the following notation: if χ is a character of a group G and C is a conjugacy class of G, then $\chi(g) = \chi(xgx^{-1})$ for all $x \in G$. We write $\chi(C)$ to denote the value $\chi(g)$ for any $g \in C$.

4.1. Theorem. *Let G be a finite group, $\chi \in \mathrm{Irr}(G)$ and K be a conjugacy class of G. Then*

$$\frac{\chi(K)}{\chi(1)}|K| \in \mathbb{A}.$$

We need a lemma.

4.2. Lemma. *Let $x \in \mathbb{C}$. Then $x \in \mathbb{A}$ if and only if there exist $z_1, \dots, z_k \in \mathbb{C}$ not all zero such that $xz_i = \sum_{j=1}^k a_{ij} z_j$ for some $a_{ij} \in \mathbb{Z}$ and all $i \in \{1, \dots, k\}$.*

Proof. Let us first prove $\implies$. Let $f = X^k + a_{k-1}X^{k-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$ be such that $f(x) = 0$. For $i \in \{1, \dots, k\}$ let $z_i = x^{i-1}$. Then $xz_i = x^i = z_{i+1}$ for all $i \in \{1, \dots, k-1\}$. Moreover, $xz_k = x^k = -a_0 - a_1 x - \cdots - a_{k-1}x^{k-1}$.

We now prove $\impliedby$. Let $A = (a_{ij}) \in \mathbb{Z}^{k \times k}$ and Z be the column vector $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$. Note that Z is non-zero. Moreover, $AZ = xZ$, as

$$(AZ)_i = \sum_{j=1}^k a_{ij} z_j = xz_i = (xZ)_i$$

for all i. Thus x is an eigenvalue of $A \in \mathbb{Z}^{k \times k}$ and hence $x \in \mathbb{A}$. □

The previous lemma could be used to give an alternative proof of the fact that the algebraic integers form a ring.

Proof of Theorem 4.1. Let φ be a representation of G and χ be its character. Note that φ is irreducible. Let $C_1, \dots, C_r$ be the conjugacy classes of G and for every $i \in \{1, \dots, r\}$ let

$$T_i = \sum_{x \in C_i} \varphi_x.$$

Claim. $T_i = \left(\frac{|C_i|}{\chi(1)}\chi(C_i)\right) \mathrm{id}$.

We proceed in several steps. First, we prove that $T_i = \lambda \,\mathrm{id}$ for some $\lambda \in \mathbb{C}$. We prove that T_i is a morphism of representations:

$$\varphi_g T_i \varphi_g^{-1} = \sum_{x \in C_i} \varphi_g \varphi_x \varphi_g^{-1} = \sum_{x \in C_i} \varphi_{gxg^{-1}} = \sum_{y \in C_i} \varphi_y = T_i.$$

Now Schur's lemma implies that $T_i = \lambda \,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.

We now prove that

$$\lambda = \frac{|C_i|\chi(C_i)}{\chi(1)}.$$

To prove this we compute λ:

$$\lambda\chi(1) = \operatorname{trace}(\lambda\,\mathrm{id}) = \operatorname{trace} T_i = \sum_{x\in C_i} \operatorname{trace}\varphi_x = \sum_{x\in C_i}\chi(x) = |C_i|\chi(C_i).$$

Then the claim follows.

Now we claim that

$$T_iT_j = \sum_{k=1}^{r} a_{ijk}T_k$$

for some $a_{ijk}\in\mathbb{Z}_{\geq 0}$. In fact,

$$T_iT_j = \sum_{x\in C_i}\sum_{y\in C_j}\varphi_x\varphi_y = \sum_{x\in C_i}\sum_{y\in C_j}\varphi_{xy} = \sum_{g\in G} a_{ijg}\varphi_g,$$

where a_{ijg} is the number of elements $(x,y)\in C_i\times C_j$ such that $g = xy$.

Claim. Once i and j are fixed, a_{ijg} depends only on the conjugacy class of g.

Let $X_g = \{(x,y)\in C_i\times C_j : g = xy\}$. If $h = kgk^{-1}$, the map

$$X_g\to X_h,\quad (x,y)\mapsto (kxk^{-1}, kyk^{-1}),$$

is well-defined. It is bijective with inverse

$$X_h\to X_g,\quad (a,b)\mapsto (k^{-1}ak, k^{-1}bk).$$

Hence $|X_g| = |X_h|$.

Let a_{ijk} be the number of elements $(x,y)\in C_i\times C_j$ such that $xy = g$ for some $g\in C_k$. Then

$$T_iT_j = \sum_{g\in G} a_{ijg}\varphi_g = \sum_{k=1}^{r}\sum_{g\in C_k} a_{ijg}\varphi_g = \sum_{k=1}^{r} a_{ijk}\sum_{g\in C_k}\varphi_g = \sum_{k=1}^{r} a_{ijk}T_k.$$

Therefore

$$\left(\frac{|C_i|}{\chi(1)}\chi(C_i)\right)\left(\frac{|C_j|}{\chi(1)}\chi(C_j)\right) = \sum_{k=1}^{r} a_{ijk}\left(\frac{|C_k|}{\chi(1)}\chi(C_k)\right). \tag{4.1}$$

By the previous lemma, $x = \frac{|C_j|}{\chi(1)}\chi(C_j)\in\mathbb{A}$. □

§ 4.1. Frobenius' divisibility theorem.

4.3. Theorem (Frobenius). *Let G be a finite group and $\chi\in\operatorname{Irr}(G)$. Then $\chi(1)$ divides $|G|$.*

Proof. Let φ be an irreducible representation with character χ. Since $\langle\chi,\chi\rangle = 1$,

$$\frac{|G|}{\chi(1)} = \frac{|G|}{\chi(1)}\langle\chi,\chi\rangle = \sum_{g\in G}\frac{\chi(g)}{\chi(1)}\overline{\chi(g)}.$$

Note that this is a rational number. Let $C_1,\dots,C_r$ be the conjugacy classes of G. Then

$$\frac{|G|}{\chi(1)} = \sum_{i=1}^{r}\sum_{g\in C_i}\frac{\chi(g)}{\chi(1)}\overline{\chi(g)} = \sum_{i=1}^{r}\left(\frac{|C_i|}{\chi(1)}\chi(C_i)\right)\overline{\chi(C_i)}\in\mathbb{A}\cap\mathbb{Q} = \mathbb{Z},$$

as $\overline{\chi(C_i)}\in\mathbb{A}$. This implies that $\chi(1)$ divides $|G|$. □

The character table gives information on the structure of the group. For example, with the previous result, one can easily prove that groups of order p^2 (where p is a prime number) are abelian.

4.4. Exercise. Let p and q be prime numbers such that $p < q$. If $q \not\equiv 1 \bmod p$, then a group of order pq is abelian.

Another application:

4.5. Theorem. *Let G be a finite simple group. Then $\chi(1) \neq 2$ for all $\chi \in \mathrm{Irr}(G)$.*

Proof. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) = 2$. Let $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$ be an irreducible representation of G with character χ. Since G is simple, $\ker\rho = \{1\}$. Since $\chi(1) = 2$, G is non-abelian and hence $[G,G] = G$. Since G has $(G : [G,G]) = 1$ degree-one characters, it follows that G has only one degree-one character, the trivial one. The composition

$$G \xhookrightarrow{\rho} \mathbf{GL}_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times$$

is a degree-one representation, which means that $\det\rho_g = 1$ for all $g \in G$. By Frobenius' theorem, $|G|$ is even (because $2 = \chi(1)$ divides $|G|$). Let $x \in G$ be such that $|x| = 2$ (Cauchy's theorem). Then $|\rho_x| = 2$, as ρ is injective. Since ρ_x is diagonalizable, there exists $C \in \mathbf{GL}_2(\mathbb{C})$ such that

$$C\rho_x C^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

for some $\lambda, \mu \in \{-1, 1\}$. Since $1 = \det\rho_x = \lambda\mu$ and ρ_x is not the identity matrix, $\lambda = \mu = -1$. In particular, $C\rho_x C^{-1}$ is central and hence ρ_x is central. Since ρ is injective, x is central and thus $Z(G) \neq \{1\}$, a contradiction. $\square$

§ 4.2. Schur's divisibility theorem.

4.6. Theorem (Schur). *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then $\chi(1)$ divides $(G : Z(G))$.*

Let G and G_1 be groups. If V is a $\mathbb{C}[G]$-module and V_1 is a $\mathbb{C}[G_1]$-module, then $V \otimes V_1$ is a $\mathbb{C}[G \times G_1]$-module with

$$(g, g_1) \cdot v \otimes v_1 = (g \cdot v) \otimes (g_1 \cdot v_1)$$

for $(g, g_1) \in G \times G_1$, $v \in V$ and $v_1 \in V_1$.

4.7. Lemma. *Let G and G_1 be finite groups. If ρ is an irreducible representation of G and ρ_1 is an irreducible representation of G_1, then $\rho \otimes \rho_1$ is an irreducible representation of $G \times G_1$.*

Proof. Write $\chi=\chi_\rho$ and $\chi_1=\chi_{\rho_1}$. Since χ is irreducible, $\langle\chi,\chi\rangle=1$. Similarly, $\langle\chi_1,\chi_1\rangle=1$. Now $\rho\otimes\rho_1$ is irreducible, as

$$\begin{aligned}\langle\chi\chi_1,\chi\chi_1\rangle&=\frac{1}{|G\times G_1|}\sum_{(g,g_1)\in G\times G_1}(\chi\chi_1)(g,g_1)\overline{(\chi\chi_1)(g,g_1)}\\&=\frac{1}{|G||G_1|}\sum_{g\in G}\sum_{g_1\in G}\chi(g)\chi_1(g_1)\overline{\chi(g)\chi_1(g_1)}\\&=\frac{1}{|G||G_1|}\sum_{g\in G}\chi(g)\overline{\chi(g)}\sum_{g_1\in G}\chi_1(g_1)\overline{\chi_1(g_1)}\\&=\langle\chi,\chi\rangle\langle\chi_1,\chi_1\rangle=1.\end{aligned}$$

□

4.8. Exercise. Let G and G_1 be finite groups. Prove that irreducible characters of $G\times G_1$ are of the form $\chi\chi_1$ for $\chi\in\operatorname{Irr}(G)$ and $\chi_1\in\operatorname{Irr}(G_1)$.

We now prove Schur's theorem. The proof goes back to Tate; it uses the **tensor power trick**. See Tao's blog for other applications of this powerful trick.

Proof of Theorem 4.6. Let $\rho\colon G\to\mathbf{GL}(V)$ be an irreducible representation with character χ. Let $z\in Z(G)$. Then ρ_z commutes with ρ_g for all $g\in G$. By Schur's lemma, $\rho_z(v)=\lambda(z)v$ for all $v\in V$. Note that $\lambda\colon Z(G)\to\mathbb{C}^\times$, $z\mapsto\lambda(z)$, is a well-defined group homomorphism, as

$$\lambda(z_1z_2)v=\rho_{z_1z_2}(v)=\rho_{z_1}\rho_{z_2}(v)=\lambda(z_2)\rho_{z_1}(v)=\lambda(z_1)\lambda(z_2)v$$

for all $v\in V$ and $z_1,z_2\in Z(G)$.

Let $n\in\mathbb{Z}_{\geq1}$. Write $G^n=G\times\cdots\times G$ (n-times). Let

$$\sigma\colon G^n\to\mathbf{GL}(V^{\otimes n}),\quad (g_1,\dots,g_n)\mapsto\rho_{g_1}\otimes\cdots\otimes\rho_{g_n}.$$

Then σ is a representation. The character of σ is χ^n. By the previous lemma, σ is irreducible. For $z_1,\dots,z_n\in Z(G)$, we compute

$$\begin{aligned}\sigma(z_1,\dots,z_n)(v_1\otimes\cdots\otimes v_n)&=\rho_{z_1}v_1\otimes\cdots\otimes\rho_{z_n}v_n\\&=\lambda(z_1)\cdots\lambda(z_n)v_1\otimes\cdots\otimes v_n\\&=\lambda(z_1\cdots z_n)v_1\otimes\cdots\otimes v_n.\end{aligned}$$

Let

$$H=\{(z_1,\dots,z_n)\in Z(G)^n:z_1\cdots z_n=1\}.$$

Then H is a central subgroup of G^n. Moreover, H acts trivially on $V^{\otimes n}$, so there exists a group homomorphism σ that makes the diagram

$$\begin{tikzcd} G^n \arrow[r, "\sigma"] \arrow[d] & \mathbf{GL}(V^{\otimes n}) \\ G^n/H \arrow[ur, dashed, "\tau"'] & \end{tikzcd}$$

commutative. Thus

$$\tau\colon G^n/H\to\mathbf{GL}(V^{\otimes n}),$$

is a representation of degree $\chi(1)^n$: Since σ is irreducible, so is τ. By Frobenius' theorem, $\chi(1)$ divides $|G|$ and $\chi(1)^n$ divides $|G^n/H| = \frac{|G|^n}{|Z(G)|^{n-1}}$. Write

$$|G| = \chi(1)s \quad \text{and} \quad |G|(G:Z(G))^{n-1} = \chi(1)^n r$$

for some $r, s \in \mathbb{Z}$. Let a and b be such that $\gcd(a,b) = 1$ and $\frac{a}{b} = \frac{(G:Z(G))}{\chi(1)}$. Then

$$s\left(\frac{a}{b}\right)^{n-1} = s\frac{(G:Z(G))^{n-1}}{\chi(1)^{n-1}} = \frac{|G|}{\chi(1)}\frac{(G:Z(G))^{n-1}}{\chi(1)^{n-1}} = r \in \mathbb{Z}.$$

Thus b^{n-1} divides s and hence $b = 1$ (because n is arbitrary). □

The following conjecture was proposed in 2018:

4.9. Conjecture (Harada). Let G be a finite group and let $K_1, \dots, K_r$ be its conjugacy classes. Then $\prod_{\chi \in \mathrm{Irr}(G)} \chi(1)$ divides $\prod_{i=1}^{r} |K_i|$.

§ 4.3. Examples of character tables. Let G be a finite group and $\chi_1, \dots, \chi_r$ be the irreducible characters of G. Without loss of generality, we may assume that χ_1 is the trivial character, i.e., $\chi_1(g) = 1$ for all $g \in G$. Recall that r is the number of conjugacy classes of G. Each χ_j is constant on conjugacy classes. The **character table** of G is presented as follows, arranging group elements and character values in a tabular format:

	1	k_2	$\cdots$	k_r
	1	g_2	$\cdots$	g_r
χ_1	1	1	$\cdots$	1
χ_2	n_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	n_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

Here, the numbers n_j represent the degrees of the irreducible representations of G, and each k_j denotes the size of the conjugacy class of the element g_j. By convention, the character table contains only the values of the irreducible characters of the group.

4.10. Example. For $n \geq 2$, let $G = \langle g \rangle$ be the cyclic group of order n. Let λ be a primitive n-th root of one. For each i, let V_i be a (complex) one-dimensional vector space with basis $\{v\}$. Each V_i is a $\mathbb{C}[G]$-module with

$$g \cdot v = \lambda^{i-1} v.$$

Moreover, each V_i is simple, as $\dim V_i = 1$. The character χ_i associated with V_i is given by $\chi_i(g^m) = \lambda^{m(i-1)}$ for all $m \in \{1, \dots, n\}$. Since the $\chi_1, \dots, \chi_n$ are all different and G admits n irreducible representations, it follows that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_n\}$. The character table of G is shown in Table 2.

4.11. Example. Let $G = \langle g : g^4 = 1 \rangle$ be the cyclic group of order four. The character table of G is given by Table 3. Let us see how to see this calculation on the computer:

Table 2. The character table of the cyclic group C_n of order n.

	1	1	1	$\cdots$	1
	1	g	g^2	$\cdots$	g^{n-1}
χ_1	1	1	1	$\cdots$	1
χ_2	1	λ	λ^2	$\cdots$	λ^{n-1}
χ_3	1	λ^2	λ^4	$\cdots$	λ^{n-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_n	1	λ^{n-1}	λ^{n-2}	$\cdots$	λ

```
> C4 := CyclicGroup(4);
> T := CharacterTable(C4);
> T;


Character Table of Group C4
---------------------------


--------------------
Class |   1  2  3  4
Size  |   1  1  1  1
Order |   1  2  4  4
--------------------
p  =  2   1  1  2  2
--------------------
X.1   +   1  1  1  1
X.2   +   1  1 -1 -1
X.3   0   1 -1  I -I
X.4   0   1 -1 -I  I


Explanation of Character Value Symbols
--------------------------------------

I = RootOfUnity(4)
```

Table 3. The character table of the cyclic group C_4 of order four.

	1	1	1	1
	1	g	g^2	g^3
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	i	$-i$
χ_4	1	-1	$-i$	i

Some remarks:

1) The symbol `I` denotes a primitive fourth root of 1.
2) The function `CharacterTable` computes more than just the character table of the group; it also provides additional information.

```
> T[1];
( 1, 1, 1, 1 )
> Degree(T[1]);
1
> Degree(T[2]);
1
> Degree(T[3]);
1
> Degree(T[4]);
1
```

4.12. Example. The character table of the group $C_2 \times C_2 = \{1, a, b, ab\}$ is shown in Table 4.

Table 4. The character table of $C_2 \times C_2$.

	1	1	1	1
	1	a	b	ab
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Let us do this by computer:

```
> C2xC2 := AbelianGroup([2,2]);
> T := CharacterTable(C2xC2);
> T;


Character Table of Group C2xC2
------------------------------


--------------------
Class |   1  2  3  4
Size  |   1  1  1  1
Order |   1  2  2  2
--------------------
p  =  2   1  1  1  1
--------------------
X.1   +   1  1  1  1
X.2   +   1 -1  1 -1
X.3   +   1  1 -1 -1
X.4   +   1 -1 -1  1
```

4.13. Example. The character table of $\mathbb{S}_3$ was computed on Table 3.1; see page 18. Let us recall briefly one possible way to compute this table. Degree-one characters were easy to compute. To compute the third row of the table, one possible approach is to use the irreducible representation

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$\chi_3\left((12)\right) = \operatorname{trace}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = 0,$$
$$\chi_3\left((123)\right) = \chi_3\left((12)(23)\right) = \operatorname{trace}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = -1.$$

We should remark that the irreducible representation mentioned is not needed to compute the third row of the character table.

```
> S3 := Sym(3);
> T := CharacterTable(S3);
> T;


Character Table of Group S3
---------------------------


-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

4.14. Example. Let us compute the character table of $\mathbb{S}_4$. We know that $|\mathbb{S}_4| = 24$ and that $\mathbb{S}_4$ has five conjugacy classes:

Representative	id	(12)	(12)(34)	(123)	(1234)
Size	1	6	3	8	6

Thus $\operatorname{Irr}(\mathbb{S}_4) = \{\chi_1, \chi_2, \dots, \chi_5\}$. We may assume that χ_1 is the trivial character and that χ_2 is the sign. Since $[\mathbb{S}_4, \mathbb{S}_4] \simeq \mathbb{A}_4$, the quotient $\mathbb{S}_4/[\mathbb{S}_4, \mathbb{S}_4]$ has order two and hence $\mathbb{S}_4$ admits exactly two degree-one irreducible representations. Hence we know two rows of the character table of $\mathbb{S}_4$:

	id	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1

There exist $n_3, n_4, n_5 \in \{2,3,4\}$ such that $24 = 1+1+n_3^2+n_4^2+n_5^2$. A direct calculation shows that $(n_3,n_4,n_5) = (2,3,3)$ is the only solution with $n_3 \leq n_4 \leq n_5$.

To find the other characters, it is useful to use the action of $\mathbb{S}_4$ on the vector space

$$V = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1+x_2+x_3+x_4 = 0\},$$

given by

$$g \cdot (x_1,x_2,x_3,x_4) = (x_{g^{-1}(1)}, x_{g^{-1}(2)}, x_{g^{-1}(3)}, x_{g^{-1}(4)}).$$

Let

$$v_1 = (1,0,0,-1), \quad v_2 = (0,1,0,-1), \quad v_3 = (0,0,1,-1).$$

Then $\{v_1,v_2,v_3\}$ is a basis of V and

$$\begin{aligned} &(12)\cdot v_1 = v_2, && (12)\cdot v_2 = v_1, && (12)\cdot v_3 = v_3,\\ &(1432)\cdot v_1 = -v_3, && (1432)\cdot v_2 = v_1 - v_3, && (1432)\cdot v_3 = v_2 - v_3. \end{aligned}$$

Since $\mathbb{S}_4 = \langle (12), (1432)\rangle$, this is enough to know how any element $g \in \mathbb{S}_4$ acts on any $v \in V$. This action yields a representation $\rho\colon \mathbb{S}_4 \to \mathbf{GL}(V)$:

$$\rho_{(12)} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \rho_{(1432)} = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -1 & -1\end{pmatrix}.$$

Let χ be the character of ρ. Then $\chi(\mathrm{id}) = 3$, $\chi((12)) = 1$, $\chi((1234)) = -1$. How to compute the value of χ on 3-cycles? Here is the trick:

$$\chi((234)) = \chi((12)(1234)) = \operatorname{trace}(\rho_{(12)}\rho_{(1234)}) = \operatorname{trace}\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & -1 & -1\end{pmatrix} = 0.$$

Similarly, to compute χ on products of two transpositions, we note that

$$\chi((13)(24)) = \chi((1234)(1234)) = \operatorname{trace}(\rho_{(1234)}^2) = \operatorname{trace}\begin{pmatrix} 0 & 0 & 1\\ 1 & -1 & -1\\ -1 & 2 & 0\end{pmatrix} = -1.$$

Now is an easy exercise to check that this χ is irreducible:

$$\langle \chi, \chi\rangle = \frac{1}{24}(3^2 + 6 + 0 + 6 + 3) = 1.$$

Moreover, $\operatorname{sign}\otimes\chi$ is also an irreducible representation:

$$\langle \operatorname{sign}\otimes\chi, \operatorname{sign}\otimes\chi\rangle = \frac{1}{24}(3^2 + (-1)^2 6 + (-1)^2 3 + 6) = 1.$$

With the trivial representation χ_1, the sign representation χ_2 and these two new characters, namely $\chi_3 = \chi$ and $\chi_4 = \operatorname{sign}\otimes\chi$, we are almost done. Only one irreducible character is missing. Let us call this character χ_5. This character can be determined using the left regular representation L:

$$
\begin{aligned}
0 &= \chi_L((12)) = 1 + (-1) + 3 + 3(-1) + 2\chi_5((12)),\\
0 &= \chi_L((12)(34)) = 1 + 1 + 3(-1) + 3(-1) + 2\chi_5((12)(34)),\\
0 &= \chi_L((123)) = 1 + 1 + 0 + 0 + 2\chi_5((123)),\\
0 &= \chi_L((1234)) = 1 + (-1) + 3(-1) + 3 + 2\chi_5((1234)) = 0,
\end{aligned}
$$

Now we are ready to compute the character table of $\mathbb{S}_4$:

Table 5. The character table of $\mathbb{S}_4$.

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
sign	1	-1	1	1	-1
χ	3	1	-1	0	-1
sign $\otimes \chi$	3	-1	-1	0	1
χ_5	2	0	2	-1	0

4.15. Exercise. Compute the character table of $\mathbb{A}_4$.

4.16. Exercise. Compute the character table of a non-abelian group of order eight.

There are two non-isomorphic non-abelian groups of order eight: the dihedral group $\mathbb{D}_4$ and the quaternion group Q_8. One does not need to use this information to solve Exercise 4.16.

5. Lecture: Week 5

§ 5.1. McKay's conjecture. Let G be a finite group and let p be a prime number dividing $|G|$. Write $\mathrm{Syl}_p(G)$ to denote the (non-empty) set of Sylow p-subgroups of G. Recall that the **normalizer** of P is the subgroup

$$N_G(P) = \{g \in G : gPg^{-1} = P\}.$$

McKay made the following conjecture for the prime $p = 2$ and simple groups and later generalized by Alperin in [**1**] and independently by Isaacs in [**22**].

5.1. Conjecture (McKay). Let p be a prime. If G is a finite group and $P \in \mathrm{Syl}_p(G)$, then

$$|\{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}| = |\{\psi \in \mathrm{Irr}(N_G(P)) : p \nmid \psi(1)\}|.$$

Isaacs proved the conjecture for solvable groups; see [**22, 25**]. Using [**26**] and the classification of finite simple groups, Malle and Späth proved the conjecture for p = 2 in [**33**].

In full generality, McKay's conjecture was proved in 2024.

5.2. Theorem (Cabanes–Späth). *If G is finite, p a prime number and $P \in \mathrm{Syl}_p(G)$, then*

$$|\{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}| = |\{\psi \in \mathrm{Irr}(N_G(P)) : p \nmid \psi(1)\}|.$$

We cannot prove Theorem 5.2 here. However, we can use the computer to prove some particular cases with the following function:

```
> McKay := function(G, p)
function> local N, n, m;
function> N := Normalizer(G, SylowSubgroup(G, p));
function> degG := CharacterDegrees(G);
function> degN := CharacterDegrees(N);
function> n := &+[ d[2] : d in degG | d[1] mod p ne 0 ];
function> m := &+[ d[2] : d in degN | d[1] mod p ne 0 ];
function> return n eq m;
function> end function;
```

As a concrete example, let us verify McKay's conjecture for the Mathieu simple group M_{11} of order 7920.

```
> M11 := sub<Sym(11) | (1,10)(2,8)(3,11)(5,7), (1,4,7,6)(2,11,10,9)>;
> McKay(M11,2);
true
> McKay(M11,3);
true
> McKay(M11,5);
true
> McKay(M11,11);
true
```

5.3. Bonus exercise. Verify the McKay's conjecture for all sporadic simple groups.

The following conjecture refines McKay's conjecture. It was formulated by Isaacs and Navarro:

5.4. Conjecture (Isaacs–Navarro). Let p be a prime and $k \in \mathbb{Z}$. If G is a finite group and $P \in \mathrm{Syl}_p(G)$, then

$$\begin{aligned}|\{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1) \text{ and } \chi(1) \equiv \pm k \bmod p\}|\\ = |\{\psi \in \mathrm{Irr}(N_G(P)) : p \nmid \psi(1) \text{ and } \psi(1) \equiv \pm k \bmod p\}|.\end{aligned}$$

The Isaacs–Navarro conjecture is still open. However, it is known to be true for solvable groups, sporadic simple groups and symmetric groups, see [27].

5.5. Bonus exercise. Verify the Isaacs–Navarro conjecture in some small groups, such as the Mathieu simple group M_{11}.

§ 5.2. Commutators. Let G be a finite group with conjugacy classes $C_1, \dots, C_s$. For $i \in \{1, \dots, s\}$ and $\chi \in \mathrm{Irr}(G)$ let

$$\omega_\chi(C_i) = \frac{|C_i|\chi(C_i)}{\chi(1)} \in \mathbb{A}.$$

In the proof of Theorem 4.1, Equality (4.1), we obtained that

$$\omega_\chi(C_i)\omega_\chi(C_j) = \sum_{k=1}^{s} a_{ijk}\omega_\chi(C_k), \tag{5.1}$$

where a_{ijk} is the number of solutions of $xy = z$ with $x \in C_i$, $y \in C_j$ and $z \in C_k$.

5.6. Theorem (Burnside). *Let G be a finite group with conjugacy classes $C_1, \dots, C_s$. Then*

$$a_{ijk} = \frac{|C_i||C_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(C_i)\chi(C_j)\overline{\chi(C_k)}}{\chi(1)}.$$

Proof. By (5.1),

$$\frac{|C_i||C_j|}{\chi(1)}\chi(C_i)\chi(C_j) = \sum_{k=1}^{s} a_{ijk}|C_k|\chi(C_k).$$

Multiply by $\overline{\chi(C_l)}$ and sum over all $\chi \in \mathrm{Irr}(G)$ to obtain

$$\begin{aligned}|C_i||C_j| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\overline{\chi(C_l)}}{\chi(1)}\chi(C_i)\chi(C_j) &= \sum_{\chi \in \mathrm{Irr}(G)} \sum_{k=1}^{s} a_{ijk}|C_k|\chi(C_k)\overline{\chi(C_l)}\\ &= \sum_{k=1}^{s} a_{ijk}|C_k| \sum_{\chi \in \mathrm{Irr}(G)} \chi(C_k)\overline{\chi(C_l)}\\ &= a_{ijl}|G|,\end{aligned}$$

because

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(C_k)\overline{\chi(C_l)} = \begin{cases} \frac{|G|}{|C_l|} & \text{if } k = l,\\ 0 & \text{otherwise.}\end{cases}$$ □

5.7. Theorem (Burnside). *Let G be a finite group and $g, x \in G$. Then g and $[x, y]$ are conjugate for some $y \in G$ if and only if*

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(x)|^2 \chi(g)}{\chi(1)} > 0.$$

Proof. Let $C_1, \dots, C_s$ be the conjugacy classes of G. Assume that $x \in C_i$ and $g \in C_k$ for some i and k. Then $C_i^{-1} = \{z^{-1} : z \in C_i\} = C_j$ for some j. By Burnside's theorem,

$$a_{ijk} = \frac{|C_i|^2}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(C_i)|^2 \overline{\chi(C_k)}}{\chi(1)}.$$

We first prove $\Longleftarrow$. Since $a_{ijk} > 0$, there exist $u \in C_i$ and $v \in C_j$ such that $g = uv$ (since $zgz^{-1} = u_1 v_1$ for some $u_1 \in C_i$ and $v_1 \in C_j$, it follows that $g = (z^{-1} u_1 z)(z^{-1} v_1 z)$, so take $u = z^{-1} u_1 z \in C_i$ and $v = z^{-1} v_1 z \in C_j$). If x and u are conjugate, say $u = zxz^{-1}$ for some z, then x^{-1} and v are conjugate, as

$$zxz^{-1} = u \implies zx^{-1}z^{-1} = u^{-1} \in C_i^{-1} = C_j.$$

Let $z_2 \in G$ be such that $z_2 x^{-1} z_2^{-1} = v$. If $y = z^{-1} z_2$, then g and $[x, y]$ are conjugate, as

$$g = uv = (zxz^{-1})(z_2 x^{-1} z_2^{-1}) = (zxyx^{-1}y^{-1}) y z_2^{-1} = z[x, y]z^{-1}.$$

We now prove $\Longrightarrow$. Let $y \in G$ be such that g and $[x, y]$ are conjugate, say $g = z[x, y]z^{-1}$ for some $z \in G$. Let $v = yxy^{-1}$. Then g and $xv^{-1} = xyx^{-1}y^{-1} = [x, y]$ are conjugate. In particular, since $g \in C_i C_j$, $a_{ijk} > 0$. □

5.8. Exercise. Let G be a finite group, $g \in G$ and $\chi \in \mathrm{Irr}(G)$. Prove that

$$\sum_{h \in G} \chi([h, g]) = \frac{|G|}{\chi(1)} |\chi(g)|^2.$$

Prove also that

$$\chi(g)\chi(h) = \frac{\chi(1)}{|G|} \sum_{z \in G} \chi(zgz^{-1}h)$$

holds for all $h \in G$.

We now prove a theorem of Frobenius that uses character tables to recognize commutators. For that purpose, let

$$\tau(g) = |\{(x, y) \in G \times G : [x, y] = g\}|.$$

5.9. Theorem (Frobenius). *Let G be a finite group. Then*

$$\tau(g) = |G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}.$$

Proof. Let $\chi \in \mathrm{Irr}(G)$. Since χ is irreducible,

$$1 = \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{z \in G} \chi(z)\overline{\chi(z)} = \frac{1}{|G|} \sum_{C} |C| \chi(C) \overline{\chi(C)},$$

where the last sum is taken over all conjugacy classes of G. Let C be the conjugacy class of g. The equation $xu^{-1}=g$ with $x\in C$ and $u\in C$ has

$$\frac{|C||C^{-1}|}{|G|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(C)\chi(C^{-1})\chi(g^{-1})}{\chi(1)}$$

solutions. If (x,u) is a solution of $xu^{-1}=g$, then there are exactly $|C_G(x)|$ elements y such that $yxy^{-1}=u$. In fact, since x and u are conjugate, there exists y such that $yxy^{-1}=u$. And if $u=y_1xy_1^{-1}$ for some y_1, then $y_1^{-1}y\in C_G(x)$ which implies that $y_1=y\xi$ for some $\xi\in C_G(x)$. Now $[x,y]=x(yx^{-1}y^{-1})=g$ has

$$|C|\sum_{\chi}\frac{\chi(C)\chi(C^{-1})\chi(g^{-1})}{\chi(1)}$$

solutions, where the sum is taken over all irreducible characters of G. Now we sum over all conjugacy classes C of G:

$$\begin{aligned}\sum_C\sum_\chi |C|\frac{\chi(C)\chi(C^{-1})\chi(g^{-1})}{\chi(1)} &= \sum_\chi \frac{\chi(g^{-1})}{\chi(1)}\left(\sum_C |C|\chi(C)\chi(C^{-1})\right)\\ &= |G|\sum_\chi\frac{\chi(g^{-1})}{\chi(1)}.\end{aligned}$$

From this, the formula follows. □

Application:

5.10. Corollary. *Let G be a finite group and $g\in G$. Then g is a commutator if and only if*

$$\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(g)}{\chi(1)}\neq 0.$$

§ 5.3. Ore's conjecture. In 1951 Ore and independently Itô proved that every element of any alternating simple group is a commutator. Ore also mentioned that "it is possible that a similar theorem holds for any simple group of finite order, but it seems that at present we do not have the necessary methods to investigate the question".

5.11. Conjecture (Ore). Let G be a finite simple non-abelian group. Then every element of G is a commutator.

Ore's conjecture was proved in 2010:

5.12. Theorem (Liebeck–O'Brien–Shalev–Tiep). *Every element of a non-abelian finite simple group is a commutator.*

The proof appears in [**31**]. It needs about 70 pages and uses the classification of finite simple groups (CFSG) and character theory. See [**32**] for more information on Ore's conjecture and its proof.

Although the proof of Ore's conjecture is too complicated for this course, we can use the computer to prove the conjecture in some particular cases:

5.13. Bonus exercise. Verify Ore's conjecture for sporadic simple groups.

See [30] for other applications of character theory.

§ 5.4. The Cauchy–Frobenius–Burnside theorem. The result we will now see is often called Burnside's lemma. Burnside proved this lemma in his book on finite groups, attributing it to Frobenius. However, the formula was already known to Cauchy in 1845. Because of this, the result is sometimes referred to as the lemma that is not Burnside's; see [34].

5.14. Theorem (Cauchy–Frobenius–Burnside). *Let G be a finite group that acts on a finite set X. If m is the number of orbits, then*

$$m = \frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)|,$$

where $\operatorname{Fix}(g) = \{x \in X : g\cdot x = x\}$.

Proof. Let $X = \{x_1,\dots,x_n\}$ and V be the complex vector space with basis $\{x_1,\dots,x_n\}$. Let $\rho\colon G \to \mathbf{GL}_n(\mathbb{C})$, $g \mapsto \rho_g$, be the representation

$$(\rho_g)_{ij} = \begin{cases} 1 & \text{if } g\cdot x_j = x_i,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $(\rho_g)_{ii} = 1$ if $x_i \in \operatorname{Fix}(g)$ and $(\rho_g)_{ii} = 0$ if $x_i \notin \operatorname{Fix}(g)$. Thus

$$\chi_\rho(g) = \operatorname{trace}\rho_g = \sum_{i=1}^{n} (\rho_g)_{ii} = |\operatorname{Fix}(g)|.$$

Recall that

$$V^G = \{v \in V : g\cdot v = v \text{ for all } g \in G\}$$

and that

$$\dim V^G = \frac{1}{|G|}\sum_{z\in G}\chi_\rho(z) = \langle \chi_\rho, \chi_1\rangle$$

where χ_1 is the trivial character of G (see Lemma 2.20).

We can assume that, after a possible re-enumeration, $x_1,\dots,x_m$ are the representatives of the orbits of G on X. For $i \in \{1,\dots,m\}$, let $v_i = \sum_{x\in G\cdot x_i} x$.

Claim. $\{v_1,\dots,v_m\}$ is a basis of V^G.

If $g \in G$, then $g\cdot v_i = \sum_{x\in G\cdot x_i} g\cdot x = \sum_{y\in G\cdot x_i} y = v_i$. Hence $\{v_1,\dots,v_m\} \subseteq V^G$. Moreover, $\{v_1,\dots,v_m\}$ is linearly independent because the v_j are orthogonal and non-zero:

$$\langle v_i, v_j\rangle = \begin{cases} |G\cdot x_i| & \text{if } i = j,\\ 0 & \text{otherwise.}\end{cases}$$

We now prove that $V^G = \langle v_1,\dots,v_m\rangle$. Let $v \in V^G$. Then $v = \sum_{x\in X}\lambda_x x$ for some coefficients $\lambda_x \in \mathbb{C}$. If $g \in G$, then $g\cdot v = v$. Since

$$\sum_{x\in X}\lambda_x x = v = g\cdot v = \sum_{x\in X}\lambda_x (g\cdot x) = \sum_{x\in X}\lambda_{g^{-1}\cdot x} x,$$

it follows that $\lambda_x = \lambda_{g^{-1}\cdot x}$ for all $x \in X$ and $g \in G$. This means that if $y, z \in X$ and $g \in G$ is such that $g \cdot y = z$, then $\lambda_y = \lambda_z$. Thus

$$v = \sum_{x \in X} \lambda_x x = \sum_{i=1}^{m} \lambda_{x_i} \sum_{y \in G \cdot x_i} y = \sum_{i=1}^{m} \lambda_{x_i} v_i.$$

Hence

$$m = \dim V^G = \langle \chi_\rho, \chi_1 \rangle = \frac{1}{|G|} \sum_{z \in G} \chi_\rho(z) = \frac{1}{|G|} \sum_{z \in G} |\operatorname{Fix}(z)|. \qquad \square$$

It is possible to give an alternative short proof of the theorem. For example, for transitive actions (i.e., $m = 1$), we proceed as follows:

$$\sum_{g \in G} |\operatorname{Fix}(g)| = \sum_{g \in G} \sum_{\substack{x \in X \\ g \cdot x = x}} 1 = \sum_{x \in X} \sum_{\substack{g \in G \\ g \cdot x = x}} 1 = \sum_{x \in X} |G_x| = |G_{x_0}||X| = |G|,$$

where $x_0 \in X$ is any fixed element of X. I learned about this analytic number theory-style proof in Serre's paper [**36**].

5.15. Exercise. Use the previous idea to prove Theorem 5.14.

A probabilistic proof of Theorem 5.14 is presented in [**44**].

Let G act on a finite set X. Then G acts on $X \times X$ by

$$g \cdot (x, y) = (g \cdot x, g \cdot y). \tag{5.2}$$

The orbits of this action are called the **orbitals** of G on X. The **rank** of G on X is the number of orbitals.

5.16. Proposition. *Let G be a group that acts on a finite set X. The rank of G on X is*

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|^2.$$

Proof. The action (5.2) has $\operatorname{Fix}(g) \times \operatorname{Fix}(g)$ as fixed points, as

$$\begin{aligned} g \cdot (x, y) = (x, y) &\iff (g \cdot x, g \cdot y) = (x, y) \\ &\iff g \cdot x = x \text{ and } g \cdot y = y \iff (x, y) \in \operatorname{Fix}(g) \times \operatorname{Fix}(g). \end{aligned}$$

Now the claim follows from Cauchy–Frobenius–Burnside theorem. $\square$

5.17. Definition. Let G act on a finite set X. We say that G is **2-transitive** on X if given $x, y \in X$ with $x \neq y$ and $x_1, y_1 \in X$ with $x_1 \neq y_1$ there exists $g \in G$ such that $g \cdot x = x_1$ and $g \cdot y = y_1$.

The symmetric group $\mathbb{S}_n$ acts 2-transitively on $\{1, \dots, n\}$.

5.18. Proposition. *If G is 2-transitive on X, then the rank of G on X is two.*

Proof. The set $\Delta = \{(x, x) : x \in X\}$ is an orbital. The complement $X \times X \setminus \Delta$ is another orbital: if $x, x_1, y, y_1 \in X$ are such that $x \neq y$ and $x_1 \neq y_1$, then there exists $g \in G$ such that $g \cdot x = x_1$ and $g \cdot y = y_1$, so $g \cdot (x, y) = (x_1, y_1)$. $\square$

6. Lecture: Week 6

The Cauchy–Frobenius–Burnside theorem is helpful to find characters.

6.1. Proposition. *Let G be 2-transitive on X with character $\chi(g) = |\operatorname{Fix}(g)|$. Then $\chi - \chi_1$ is an irreducible character.*

Proof. By assumption, G is 2-transitive on X. In particular, G is transitive on X. Let $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_k\}$, where χ_1 is the trivial character. Since χ_1 is irreducible, $\langle \chi_1, \chi_1 \rangle = 1$. By the Cauchy–Frobenius–Burnside theorem, the rank of G on X is

$$2 = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|^2 = \langle \chi, \chi \rangle.$$

Moreover, again by the Cauchy–Frobenius–Burnside theorem,

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) = 1,$$

since the action of G on X is, in particular, transitive. Thus

$$\langle \chi - \chi_1, \chi - \chi_1 \rangle = \langle \chi, \chi \rangle - \langle \chi, \chi_1 \rangle - \langle \chi_1, \chi \rangle + \langle \chi_1, \chi_1 \rangle = 2 - \frac{2}{|G|} \sum_{g \in G} \chi(g) + 1 = 1.$$

Now write $\chi - \chi_1 = \sum_{i=1}^{k} a_i \chi_i$ for some integers $a_1, \dots, a_k \in \mathbb{Z}$. Since $a_1 = \langle \chi, \chi_1 \rangle$, it follows that

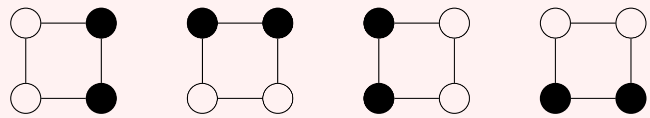
$$1 = \left\langle \sum_{i=1} a_i \chi_i, \sum_{j=1} a_j \chi_j \right\rangle = \sum_{i=2}^{k} a_i^2.$$

Since χ is a character, $\chi - \chi_1$ is an integer linear combination of the irreducible characters of G. Then there exists a unique $i \in \{2, \dots, k\}$ such that $a_i \in \{-1, 1\}$ and $a_j = 0$ for all $j \neq i$. Hence $\chi - \chi_1 = \pm \chi_i$. Since $(\chi - \chi_1)(1) = |X| - 1 \geq 0$, it follows that $\chi - \chi_1 = \chi_i$. □

6.2. Example. The symmetric group $\mathbb{S}_n$ is 2-transitive on $\{1, \dots, n\}$. The alternating group $\mathbb{A}_n$ is 2-transitive on $\{1, \dots, n\}$ if $n \geq 4$. These groups then have an irreducible character χ given by $\chi(g) = |\operatorname{Fix}(g)| - 1$.

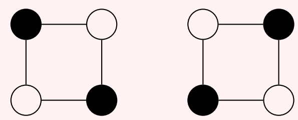
6.3. Example. Let p be a prime number and let $q = p^m$. Let V be the vector space of dimension 2 over the finite field of q elements. The group $G = \mathbf{GL}_2(q)$ acts 2-transitively on the set X of one-dimensional subspaces of V. In fact, if $\langle v \rangle \neq \langle v_1 \rangle$ and $\langle w \rangle \neq \langle w_1 \rangle$, then $\{v, v_1\}$ and $\{w, w_1\}$ are bases of V. The matrix g that corresponds to the linear map $v \mapsto w$, $v_1 \mapsto w_1$, is invertible. Thus $g \in \mathbf{GL}_2(q)$. The previous proposition produces the irreducible character $\chi(g) = |\operatorname{Fix}(g)| - 1$.

6.4. Example. In how many ways can we color (in black and white) the vertices of a square? We will count colorings up to symmetric. This means that, for example, the colorings

(6.1) 

will be considered equivalent. Let $G = \langle g \rangle$ the cyclic group of order four. Let X be the set of colorings of the square. Then $|X| = 16$.

Let g act on X by anti-clockwise rotations of 90°. All the colorings of (6.1) belong to the same orbit. Another orbit of X is

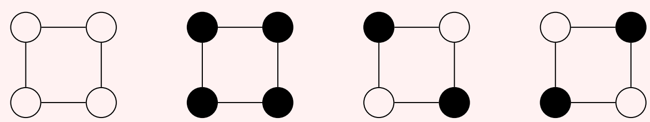


Cauchy–Frobenius–Burnside theorem states that there are

$$\frac{1}{|G|}\sum_{x\in G} |\operatorname{Fix}(x)|$$

orbits.

For each $x \in G = \{1, g, g^2, g^3\}$ we compute $\operatorname{Fix}(x)$. The identity fixes the 16 elements of X, both g and g^3 fix only two elements of X and g^2 fixes four elements of X. For example, the elements of X fixed by g^2 are



Thus X is the union of

$$\frac{1}{|G|}\sum_{x\in G} |\operatorname{Fix}(x)| = \frac{1}{4}(16+2+4+2) = 6$$

orbits.

6.5. Bonus exercise. In how many ways (up to symmetry) can you arrange eight non-attacking rooks on a chessboard? Symmetries are given by the dihedral group $\mathbb{D}_4$ of eight elements.

There are 5282 ways (up to symmetry) to arrange eight non-attacking rooks on a chessboard.

§ 6.1. Commuting probability. For a finite group G, let $\operatorname{cp}(G)$ be the probability that two random elements of G commute. This number is also known as the **commutativity** of G. As an application of Cauchy–Frobenius–Burnside theorem, we prove that $\operatorname{cp}(G) = k/|G|$, where k is the number of conjugacy classes of G. Let

$$C = \{(x, y) \in G \times G : xy = yx\}.$$

We claim that

$$\operatorname{cp}(G)=\frac{|C|}{|G|^2}=\frac{k}{|G|}.$$

Let G act on G by conjugation. By Cauchy–Frobenius–Burnside theorem,

$$k=\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}(g)|=\frac{1}{|G|}\sum_{g\in G}|C_G(g)|=\frac{|C|}{|G|},$$

as $\operatorname{Fix}(g)=\{x\in G:gxg^{-1}=x\}=C_G(g)$ and $\sum_{g\in G}|C_G(g)|=|C|$. Alternatively, using Theorem 5.9 with $g=1$,

$$\operatorname{cp}(G)=\frac{\tau(1)}{|G|^2}=\frac{1}{|G|}\sum_{\chi\in\operatorname{Irr}(G)}1=\frac{k}{|G|},$$

as $k=|\operatorname{Irr}(G)|$.

6.6. Theorem. *If G is a non-abelian finite group, then* $\operatorname{cp}(G)\leq 5/8$.

Proof. Let $y_1,\dots,y_m$ the representatives of conjugacy classes of G of size ≥ 2. By the class equation,

$$|G|=|Z(G)|+\sum_{i=1}^{m}(G:C_G(y_i))\geq|Z(G)|+2m.$$

Thus $m\leq(1/2)(|G|-|Z(G)|)$ and hence

$$k=|Z(G)|+m\leq|Z(G)|+\frac{1}{2}(|G|-|Z(G)|)=\frac{1}{2}(|Z(G)|+|G|).$$

Since G is non-abelian, $G/Z(G)$ is not cyclic. In particular, $(G:Z(G))\geq 4$. Therefore

$$k\leq\frac{1}{2}(|Z(G)|+|G|)\leq\frac{1}{2}\left(\frac{1}{4}+1\right)|G|,$$

that is $k/|G|\leq 5/8$. □

Theorem 6.6 is known as the 5/8 **Theorem**.

6.7. Exercise.

1) Prove that $\operatorname{cp}(Q_8)=5/8$.
2) Prove that $\operatorname{cp}(\mathbb{A}_5)=1/12$.

6.8. Exercise. Let G be a finite non-abelian group and p be the smallest prime number dividing $|G|$. Prove that $\operatorname{cp}(G)\leq(p^2+p-1)/p^3$.

6.9. Bonus exercise. Let G be a finite group and H be a subgroup of G.

1) $\operatorname{cp}(G)\leq\operatorname{cp}(H)$.
2) If H is normal in G, then $\operatorname{cp}(G)\leq\operatorname{cp}(G/H)\operatorname{cp}(H)$.

For the next proposition, which provides a lower bound for the commuting probability, we will use the **Cauchy–Schwarz inequality**:

$$x_1,\dots,x_n\in\mathbb{R}\implies\sum x_i^2\geq\frac{1}{n}(\sum x_i)^2.$$

6.10. Proposition. *If G is a finite group, then*

$$\operatorname{cp}(G) \geq \left(\frac{\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)}{|G|}\right)^2.$$

Proof. Let k be the number of conjugacy classes of G. By the Cauchy–Schwarz inequality,

$$\left(\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)\right)^2 \leq \left(\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)^2\right)\left(\sum_{\chi\in\operatorname{Irr}(G)}1\right) = |G|k.$$

From this, the claim follows. $\square$

Using basic facts about irreducible characters, we derive a generalization of Theorem 6.6.

6.11. Theorem. *Let G be a finite group. Then*

$$\operatorname{cp}(G) \leq \frac{1}{4}\left(1+\frac{3}{|[G,G]|}\right).$$

Proof. For $n\in\mathbb{Z}_{>0}$, let ρ_n be the number of irreducible characters of degree n. Then the number of conjugacy classes of G is $k=\sum_{i\geq1}\rho_i$ and $|G|=\sum_{i\geq1}i^2\rho_i$. It follows that

$$|G|-\rho_1=\sum_{i\geq2}i^2\rho_i \geq 4\sum_{i\geq2}\rho_i = 4(k-\rho_1)=4(|G|\operatorname{cp}(G)-\rho_1).$$

Since $\rho_1=(G:[G,G])$,

$$\operatorname{cp}(G)\leq\frac{1}{4}+\frac{3}{4}\frac{\rho_1}{|G|}=\frac{1}{4}+\frac{3}{4|[G,G]|}.$$ $\square$

6.12. Exercise. Use Theorem 6.11 to prove Theorem 6.6.

Theorem 6.11 can also be used to prove similar statements.

6.13. Exercise. Let G be a finite group. Prove the following statements:

1) If $\operatorname{cp}(G)>1/2$, then G is nilpotent.
2) If $\operatorname{cp}(G)>21/80$, then G is solvable.

In the following exercise, we will discuss the notion of isoclinic groups. We first need a preliminary result:

6.14. Exercise. Let G be a group. Prove that the commutator map

$$c_G\colon G/Z(G)\times G/Z(G)\to[G,G],\quad c_G(xZ(G),yZ(G))=[x,y],$$

is well-defined.

The idea is that two groups are said to be isoclinic if their commutator functions are somewhat equal.

6.15. Exercise. Let G and H be groups. A pair (σ,τ) of maps is an **isoclinism** between G and H if $\sigma\colon G/Z(G)\to H/Z(H)$ and $\tau\colon [G,G]\to[H,H]$ are group isomorphisms and the diagram

$$
\begin{tikzcd}
G/Z(G)\times G/Z(G) \arrow[r, "\sigma\times\sigma"] \arrow[d, "c_G"'] & H/Z(H)\times H/Z(H) \arrow[d, "c_H"] \\
{[G,G]} \arrow[r, "\tau"'] & {[H,H]}
\end{tikzcd}
\tag{6.2}
$$

commutes. We write $G\sim H$ when there exists an isoclinism between G and H.

Prove the following statements:

1) If $G\simeq H$, then $G\sim H$.

2) If G and H are finite groups such that $G\sim H$, then $\operatorname{cp}(G)=\operatorname{cp}(H)$.

6.16. Exercise. Let S be a non-abelian simple group and G be a group such that $G\sim S$. Prove that $G\simeq S\times A$ for some abelian group A.

6.17. Exercise. Let H be a subgroup of G. If $G=HZ(G)$, then $G\sim H$. Conversely, if $G\sim H$ and H is finite, then $G=HZ(G)$.

The following theorem appeared in 1970 as a problem in volume 13 of the *Canadian Math. Bulletin*. The solution appeared in 1973. Iván Sadosfchi Costa found in 2018 the proof we present here.

6.18. Theorem (Dixon). *The commuting probability of every finite non-abelian simple group is at most* $1/12$.

Sketch of the proof. Let G be a finite non-abelian simple group. We claim that $\operatorname{cp}(G)\leq 1/12$. We assume that $\operatorname{cp}(G)>1/12$. Since G is a non-abelian simple group, the identity of G is the only central element of G.

Let us assume first that there is a conjugacy class of G of size m, where m is such that $1<m\leq 12$. Then G is a transitive subgroup of $\mathbb{S}_m$. For these groups, the problem is easy: we show that there are no non-abelian simple groups that act transitively on sets of size $m\in\{2,\dots,12\}$ with commuting probability $>1/12$. To do this, we list these transitive groups and their commuting probabilities and verify that all commuting probabilities are $\leq 1/12$. This is left as an exercise.

Now assume that all non-trivial conjugacy classes of G have at least 13 elements. Let k be the number of conjugacy classes of G. Then the class equation implies that

$$|G|\geq 1+(k-1)13=13k-12.$$

Since $\operatorname{cp}(G)=k/|G|>1/12$, $k>|G|/12$. Thus

$$|G|>\frac{13}{12}|G|-12$$

and therefore $|G|<144$. Thus one needs to check what happens with groups of order <144. But we know that the only non-abelian simple group of size <144 is the alternating simple group $\mathbb{A}_5$. This completes the proof. $\square$

6.19. Bonus exercise. Provide the details of the proof of Theorem 6.18.

The alternating group $\mathbb{A}_5$ is important in this setting:

6.20. Theorem (Guralnick–Robinson). *If G is a finite non-solvable group such that $\operatorname{cp}(G) > 3/40$, then $G \simeq \mathbb{A}_5 \times T$ for some abelian group T and $\operatorname{cp}(G) = 1/12$.*

The proof appears in [**16**].

Results on probability of commuting elements generalize in other directions. In [**40, 41, 42, 43**], Thompson proved the following result:

6.21. Theorem (Thompson). *If G is a finite group such that every pair of elements of G generate a solvable group, then G is solvable.*

The proof uses the classification of finite simple groups (CFSG). A simpler proof independent of the CFSG appears in [**10**].

There is a probabilistic version of Thompson's theorem:

6.22. Theorem (Guralnick–Wilson). *Let G be a finite group.*

1) *If the probability that two random elements of G generate a solvable group is $> 11/30$, then G is solvable.*
2) *If the probability that two random elements of G generate a nilpotent group is $> 1/2$, then G is nilpotent.*
3) *If the probability that two random elements of G generate a group of odd order is $> 11/30$, then G has odd order.*

The proof uses the CFSG and appears in [**17**].

§ 6.2. Jordan's theorem and applications. We now follow [**36**] to present other applications.

6.23. Theorem (Jordan). *Let G be a non-trivial finite group. If G acts transitively on a finite set X and $|X| > 1$, then there exists $g \in G$ with no fixed points.*

Proof. The Cauchy–Frobenius–Burnside theorem implies that

$$1 = \frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)| = \frac{1}{|G|}\left(|X| + \sum_{g\neq 1} |\operatorname{Fix}(g)|\right).$$

If every $g \in G \setminus \{1\}$ contains at least one fixed-point, then

$$1 = \frac{1}{|G|}\left(|X| + \sum_{g\neq 1} |\operatorname{Fix}(g)|\right) \geq \frac{1}{|G|}(|X| + |G| - 1) = 1 + \frac{|X|-1}{|G|}$$

and thus $|X| \leq 1$, a contradiction. □

6.24. Corollary. *Let G be a finite group and H be a proper subgroup of G. Then $G \neq \bigcup_{g\in G} gHg^{-1}$.*

Proof. The group G acts transitively by left multiplication on $X = G/H$. The stabilizer of xH is

$$G_{xH} = \{g \in G : gxH = xH\} = xHx^{-1}.$$

Since $H \neq G$, it follows that $|X| = |G/H| > 1$. Jordan's theorem now implies that there exists $g \in G$ with no fixed-points, that is there is an element $g \in G$ such that $g \notin \bigcup_{x \in G} xHx^{-1}$. □

Let G be a finite group. We say that the conjugacy classes C and D **commute** if there exist $c \in C$ and $d \in D$ such that $[c, d] = 1$. Note that C and D commute if and only if for all $c \in C$ there exists $d \in D$ such that $[c, d] = 1$.

6.25. Corollary (Wildon). *Let G be a finite group and C be a conjugacy class of G. Then $|C| = 1$ if and only if C commutes with every conjugacy class of G.*

Proof. We prove $\Longleftarrow$. Assume that C commutes with every conjugacy class of G. Let $c \in C$ and $H = C_G(c)$. Then $H \cap D \neq \emptyset$ for every conjugacy class D. We claim that $G = \bigcup_{g \in G} gHg^{-1}$. In fact, let $x \in G$. Then $x \in D$ for some conjugacy class D. Let $h \in H \cap D$. There exists $y \in G$ such that $h = yxy^{-1}$, that is $x = y^{-1}hy \in \bigcup_{g \in G} gHg^{-1}$. By Corollary 6.24, $H = G$. Thus c is central and hence $C = \{c\}$.

We now prove $\Longrightarrow$. If $C = \{c\}$, then $c \in Z(G)$ and C commute with every conjugacy class of G. □

There is a theorem similar to Jordan's.

6.26. Theorem (Fein–Kantor–Schacher). *Let G be a non-trivial finite group. If G acts transitively on a finite set X and $|X| > 1$, then there exist a prime number p and an element $g \in G$ with no fixed-points with order a power of p.*

The proof appears in [**8**] and depends on the CFSG. It is unknown whether a proof of Theorem 6.26 without relying on the CFSG exists.

7. Lecture: Week 7

§ 7.1. The Brauer–Fowler theorem. Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation with character χ. The $\mathbb{C}[G]$-module $V \otimes V$ has character χ^2. Let $\{v_1, \dots, v_n\}$ be a basis of V and

$$T\colon V \otimes V \to V \otimes V, \quad v_i \otimes v_j \mapsto v_j \otimes v_i.$$

It is an exercise to check that

$$T(v \otimes w) = w \otimes v$$

for all $v, w \in V$. (Thus T does not depend on the basis $\{v_1, \dots, v_n\}$.) Note that T is a homomorphism of $\mathbb{C}[G]$-modules, as

$$T(g \cdot (v \otimes w)) = T((g \cdot v) \otimes (g \cdot w)) = (g \cdot w) \otimes (g \cdot v) = g \cdot T(v \otimes w)$$

for all $g \in G$ and $v, w \in V$. In particular, the **symmetric part**

$$S(V \otimes V) = \{x \in V \otimes V : T(x) = x\}$$

and the **antisymmetric** part

$$A(V \otimes V) = \{x \in V \otimes V : T(x) = -x\}$$

of $V \otimes V$ are both $\mathbb{C}[G]$-submodules of $V \otimes V$. The terminology is motivated by the following fact:

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V).$$

In fact, $S(V \otimes V) \cap A(V \otimes V) = \{0\}$, as $x \in S(V \otimes V) \cap A(V \otimes V)$ implies $x = T(x)$ and $x = -T(x)$. Hence $x = 0$. Moreover, $V \otimes V = S(V \otimes V) + A(V \otimes V)$, as every $x \in V \otimes V$ can be written as

$$x = \frac{1}{2}(x + T(x)) + \frac{1}{2}(x - T(x))$$

with $\frac{1}{2}(x + T(x)) \in S(V \otimes V)$ and $\frac{1}{2}(x - T(x)) \in A(V \otimes V)$.

We claim that

$$\{v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$$

is a basis of $S(V \otimes V)$, and that

$$\{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$$

is a basis of $A(V \otimes V)$. Since both sets are linearly independent,

$$\dim S(V \otimes V) \geq n(n+1)/2 \text{ and } \dim A(V \otimes V) \geq n(n-1)/2.$$

Moreover,

$$n^2 = \dim(V \otimes V) = \dim S(V \otimes V) + \dim A(V \otimes V),$$

so it follows that $\dim S(V \otimes V) = n(n+1)/2$ and $\dim A(V \otimes V) = n(n-1)/2$.

7.1. Proposition. *Let G be a finite group and V be a finite-dimensional $\mathbb{C}[G]$-module with character χ. If $S(V \otimes V)$ has character χ_S and $A(V \otimes V)$ has character χ_A, then*

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)) \qquad \textit{and} \qquad \chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)).$$

Proof. Let $g \in G$ and $\rho\colon G \to \mathbf{GL}(V)$ be the representation associated with V, that is $\rho(g)(v) = \rho_g(v) = g \cdot v$. Since ρ_g is diagonalizable, let $\{e_1, \dots, e_n\}$ be a basis of eigenvectors of ρ_g, say $g \cdot e_i = \lambda_i e_i$ with $\lambda_i \in \mathbb{C}$ for all $i \in \{1, \dots, n\}$. In particular, $\chi(g) = \sum_{i=1}^n \lambda_i$.

Since $\{e_i \otimes e_j - e_j \otimes e_i : 1 \leq i < j \leq n\}$ is a basis of $A(V \otimes V)$ and

$$g \cdot (e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i),$$

it follows that

$$\chi_A(g) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j.$$

On the other hand, $g^2 \cdot e_i = \lambda_i^2 e_i$ for all i, $\chi(g^2) = \sum_{i=1}^n \lambda_i^2$. Thus

$$\chi^2(g) = \chi(g)^2 = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j = 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j + \sum_{i=1}^n \lambda_i^2 = 2\chi_A(g) + \chi(g^2).$$

Since $V \otimes V = S(V \otimes V) \oplus A(V \otimes V)$, it follows that $\chi^2(g) = \chi_S(g) + \chi_A(g)$, that is $\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$. □

An **involution** of a group is an element $x \neq 1$ such that $x^2 = 1$. It is possible to use the character table to count the number of involutions.

7.2. Proposition. *If G is a finite group with t involutions, then*

$$1 + t = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_S - \chi_A, \chi_1 \rangle \chi(1),$$

where χ_1 is the trivial character of G.

Proof. Assume that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. For $x \in G$ let

$$\theta(x) = |\{y \in G : y^2 = x\}|.$$

Since θ is a class function, θ is a linear combination of the χ_j's, say

$$\theta = \sum_{\chi \in \mathrm{Irr}(G)} \langle \theta, \chi \rangle \chi.$$

For every $\chi \in \mathrm{Irr}(G)$ we compute:

$$\begin{aligned} \langle \chi_S - \chi_A, \chi_1 \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g^2) \\ &= \frac{1}{|G|} \sum_{x \in G} \sum_{\substack{g \in G \\ g^2 = x}} \chi(g^2) = \frac{1}{|G|} \sum_{x \in G} \theta(x) \chi(x) = \langle \theta, \chi \rangle. \end{aligned}$$

Thus $\theta = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_S - \chi_A, \chi_1 \rangle \chi$. Now the claim follows after evaluating this expression in $x = 1$. □

7.3. Example. We know that $\mathbb{S}_3$ has three involutions, namely (12), (23) and (13). Thus $t = 3$. Let us use Proposition 7.2 to verify this. We have already computed the character table of $\mathbb{S}_3$:

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

A direct calculation shows that

$$(\chi_1)_S = (\chi_2)_S = \chi_1 \quad \text{and} \quad (\chi_1)_A = (\chi_2)_A = 0.$$

Moreover, the values of $(\chi_3)_S$ and $(\chi_3)_A$ are given by the following table:

	1	(12)	(123)
$(\chi_3)_S$	3	1	0
$(\chi_3)_A$	1	-1	1

Let t be the number of elements of order two of $\mathbb{S}_3$. Since

$$\langle \chi_S - \chi_A, \chi_1 \rangle = 1$$

for all $\chi \in \{\chi_1, \chi_2\}$ and

$$\langle (\chi_3)_S - (\chi_3)_A, \chi_1 \rangle = \frac{1}{6}(12 + 6 - 2) = \frac{1}{6}(2 + 6 - 2) = 1,$$

Proposition 7.2 yields

$$\begin{aligned} 1 + t &= \langle (\chi_1)_S - (\chi_1)_A, \chi_1 \rangle \chi_1(1) + \langle (\chi_2)_S - (\chi_2)_A, \chi_1 \rangle \chi_2(1) + \langle (\chi_3)_S - (\chi_3)_A, \chi_1 \rangle \chi_3(1) \\ &= 1 + 1 + 2. \end{aligned}$$

Thus $t = 3$.

Before proving the Brauer–Fowler theorem, we need a lemma.

7.4. Lemma. *Let G be a finite group with k conjugacy classes. If t is the number of involutions of G, then $t^2 \leq (k-1)(|G|-1)$.*

Proof. Assume that $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_k\}$, where χ_1 is the trivial character of G. If $\chi \in \operatorname{Irr}(G)$, then

$$\langle \chi^2, \chi_1 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g) = \langle \chi, \overline{\chi} \rangle = \begin{cases} 1 & \text{if } \chi = \overline{\chi}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi^2 = \chi_S + \chi_A$, if $\langle \chi^2, \chi_1 \rangle = 1$, then the trivial character is an irreducible component either of χ_S or χ_A, but not both. Thus

$$\langle \chi_S - \chi_A, \chi_1 \rangle \in \{-1, 1, 0\}.$$

We claim that $t \leq \sum_{i=2}^{k} \chi_i(1)$. In fact, since $|\langle \chi_S - \chi_A, \chi_1 \rangle| \leq 1$,

$$\begin{aligned} 1 + t = \theta(1) &= \left| \sum_{\chi \in \operatorname{Irr}(G)} \langle \chi_S - \chi_A, \chi_1 \rangle \chi(1) \right| \\ &\leq \sum_{\chi \in \operatorname{Irr}(G)} |\langle \chi_S - \chi_A, \chi_1 \rangle| \chi(1) \leq \sum_{\chi \in \operatorname{Irr}(G)} \chi(1). \end{aligned}$$

It follows that $t \leq \sum_{i=2}^{k} \chi_i(1)$. By the Cauchy–Schwarz inequality,

$$t^2 \leq \left(\sum_{i=2}^{k} \chi_i(1)\right)^2 \leq (k-1)\sum_{i=2}^{k} \chi_i(1)^2 = (k-1)(|G|-1). \qquad \square$$

Now we prove the Brauer–Fowler theorem.

7.5. Theorem (Brauer–Fowler). *Let G be a finite simple group and x be an involution of G. If $|C_G(x)| = n$, then $|G| \leq (n^2)!$*

Proof. If G is abelian, the claim is trivial. Let G be a finite non-abelian simple group. We first assume the existence of a proper subgroup H of G such that

$$(G : H) \leq n^2.$$

Let G act on G/H by left multiplication, and let $\rho\colon G \to \mathbb{S}_{n^2}$ be the corresponding group homomorphism. Since G is simple, either $\ker\rho = \{1\}$ or $\ker\rho = G$. If $\ker\rho = G$, then $\rho(g)(yH) = yH$ for all $g \in G$ and $y \in G$. Hence $H = G$, a contradiction. Therefore ρ is injective and hence G is isomorphic to a subgroup of $\mathbb{S}_{n^2}$. In particular, $|G|$ divides $(n^2)!$.

Let $m = (|G|-1)/t$, where t is the number of involutions of G. Since $|C_G(x)| = n$, the group G has at least $|G|/n$ involutions (because the conjugacy class of x has size $|G|/n$ and all its elements are involutions), that is $t \geq |G|/n$. Hence

$$m = (|G|-1)/t < n.$$

It is enough to show that G contains a subgroup of index $\leq m^2$.

Let $C_1, \dots, C_k$ be the conjugacy classes of G, where $C_1 = \{1\}$. Since G is simple and non-abelian, $|C_i| > 1$ for all $i \in \{2, \dots, k\}$. By the previous lemma,

$$t^2 \leq (k-1)(|G|-1) \implies |G|-1 = \frac{mt^2}{t} \leq \frac{(k-1)(|G|-1)^2}{t^2} = (k-1)m^2.$$

If $|C_i| > m^2$ for all $i \in \{2, \dots, k\}$, then

$$|G|-1 = \sum_{i=2}^{k} |C_i| > (k-1)m^2,$$

a contradiction. Thus there exists a non-trivial conjugacy class C of G such that $|C| \leq m^2$. If $g \in C$, then $C_G(g)$ is a proper subgroup of G of index $|C| \leq m^2$. $\square$

The bound of the Brauer–Fowler theorem is not essential. What matters is the following consequence:

7.6. Corollary. *Let $n \geq 1$ be an integer. There are at most finitely many finite simple groups with an involution with a centralizer of order n.*

As an exercise, a simple application:

7.7. Exercise. If G is a finite simple group and x is an involution with centralizer of order two, then $G \simeq \mathbb{Z}/2$.

§ 7.2. The character table of $\mathbb{S}_5$. Let $G = \mathbb{S}_5$. The conjugacy classes of G are given in the following table:

Representative	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
Size	1	10	20	15	30	20	24

Thus there are seven irreducible characters. The trivial character χ_1 and the sign χ_2 are degree-one (hence irreducible) characters.

	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
sign	1	-1	1	1	-1	-1	1

Since $[G,G] = \mathbb{A}_5$ and $|G/[G,G]| = 2$, it follows from Exercise 1.30 that χ_1 and sign are the only degree-one characters.

Since G acts 2-transitively on $\{1,\dots,5\}$, Proposition 6.1 implies that $\varsigma(g) = |\operatorname{Fix}(g)| - 1$ is an irreducible character. A direct calculation yields the values of ς:

	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
ς	4	2	1	0	0	-1	-1

The values of the product $\operatorname{sign}\varsigma$ are easily computed:

	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
$\operatorname{sign}\varsigma$	4	-2	1	0	0	1	-1

Since

$$\begin{aligned}\langle \operatorname{sign}\varsigma, \operatorname{sign}\varsigma\rangle &= \frac{1}{120}(4^2 + 10(-2)^2 + 20 + 15\cdot 0 + 30\cdot 0 + 20 + 24)\\ &= \frac{1}{120}(16+40+20+20+24) = 1,\end{aligned}$$

it follows that $\operatorname{sign}\varsigma \in \operatorname{Irr}(G)$.

We now consider the characters

$$\psi(g) = \frac{1}{2}(\varsigma^2(g) + \varsigma(g^2)) \quad \text{and} \quad \eta(g) = \frac{1}{2}(\varsigma^2(g) - \varsigma(g^2)),$$

where $\varsigma^2(g) = \varsigma(g)\varsigma(g) = \varsigma(g)^2$ (see Proposition 7.1). A straightforward calculation shows that

	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
ψ	10	4	1	2	0	1	0
η	6	0	0	-2	0	0	1

Since

$$\langle \eta, \eta \rangle = \frac{1}{120}(6^2 + 15(-2)^2 + 24) = 1,$$

it follows that $\eta \in \mathrm{Irr}(G)$. On the other hand,

$$\langle \psi, \psi \rangle = \frac{1}{120}(10^2 + 10 \cdot 16 + 20 + 15 \cdot 4 + 20) = 3.$$

Thus ψ is the sum of three irreducible characters (see Exercise 3.7). Since

$$\langle \psi, \chi_1 \rangle = \frac{1}{120}(10 + 10 \cdot 4 + 20 + 15 \cdot 2 + 20) = 1,$$
$$\langle \psi, \varsigma \rangle = \frac{1}{120}(10 \cdot 4 + 10 \cdot 4 \cdot 2 + 20 - 20) = 1,$$

it follows that $\psi = \chi_1 + \varsigma + \chi$ for some $\chi \in \mathrm{Irr}(G)$. Thus we can compute χ:

	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ	5	1	-1	1	-1	1	0

We are missing one irreducible character. Let n be the degree of this character. Since $120 = 1 + 1 + 16 + 16 + 36 + 25 + n^2$, it follows that $n = 5$. Since we need a degree-five irreducible character, we can try with $\xi = \mathrm{sign}\,\chi$:

	id	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
ξ	5	-1	-1	1	1	-1	0

Since

$$\langle \xi, \xi \rangle = \frac{1}{120}(25 + 10(-1)^2 + 20(-1)^2 + 15 + 30 + 20(-1)^2) = 1,$$

it follows that $\xi \in \mathrm{Irr}(G)$. We have found the character table of G.

Table 6. The character table of $\mathbb{S}_5$.

	1 id	10 (12)	20 (123)	15 (12)(34)	30 (1234)	20 (123)(45)	24 (12345)
χ_1	1	1	1	1	1	1	1
sign	1	-1	1	1	-1	-1	1
ς	4	2	1	0	0	-1	-1
$\mathrm{sign}\,\varsigma$	4	-2	1	0	0	1	-1
η	6	0	0	-2	0	0	1
χ	5	1	-1	1	-1	1	0
ξ	5	-1	-1	1	1	-1	0

§ 7.3. An elementary proof of the Brauer–Fowler theorem. We need to find a subgroup of index $\leq 2n^2$. Let X be the conjugacy class of x. For $g \in G$ let
$$J(g) = \{z \in X : zgz^{-1} = g^{-1}\}.$$
We claim that $|J(g)| \leq |C_G(g)|$. The map $J(g) \to C_G(g)$, $z \mapsto gz$, is well-defined, as
$$(gz)g(gz)^{-1} = g(zgz^{-1})g^{-1} = g^{-1} \in C_G(g).$$
It is injective, as $gz = gz_1$ implies $z = z_1$.

Let $J = \{(g, z) \in G \times X : zgz^{-1} = g^{-1}\}$. Since $X \times X \to J$, $(y, z) \mapsto (yz, z)$, is well-defined (since $z(yz)z^{-1} = zy = (yz)^{-1}$) and it is trivially injective,
$$|X|^2 \leq |J| = \sum_{(g,z)\in J} 1 \leq \sum_{g\in G} |J(g)| \leq \sum_{g\in G} |C_G(g)| = k|G|,$$
where k is the number of conjugacy classes of G, as $(g, z) \in J$ if and only if $z \in J(g)$. Thus $|G| \leq kn^2$, as
$$\left(\frac{|G|}{|C_G(x)|}\right)^2 = |X|^2 = \frac{|G|^2}{n^2} \leq k|G|.$$

Claim. There exists a non-trivial conjugacy class with $\leq 2n^2$ elements.

Assume that the claim is not true. Let $C_1, \dots, C_k$ be the conjugacy classes of G, where $C_1 = \{1\}$ and $|C_i| > 2n^2$ for all $i \in \{2, \dots, k\}$. Then
$$|G| = 1 + \sum_{i=2}^{k} |C_i| > 1 + \sum_{i=2}^{k} 2n^2 = 1 + (k-1)2n^2 \geq |G|,$$
a contradiction.

Claim. There exists a subgroup H of G such that $(G : H) \leq 2n^2$.

Let C be a conjugacy class of G such that $|C| \leq 2n^2$. Let $g \in C$. Then $H = C_G(g)$ is a subgroup of G such that $(G : H) \leq 2n^2$.

§ 7.4. Restriction of characters.

7.8. Definition. Let G be a finite group and $f\colon G \to \mathbb{C}$ be a map. For a subgroup H of G, the **restriction** of f to H is the map $\mathrm{Res}_H^G = f|_H\colon H \to \mathbb{C}$, $h \mapsto f(h)$.

7.9. Exercise. Let G be a finite group. Prove that the map
$$\mathrm{Res}_H^G\colon \mathrm{ClassFun}(G) \to \mathrm{ClassFun}(H), \quad f \mapsto \mathrm{Res}_H^G f,$$
is a well-defined linear map.

One important property is the following:

7.10. Exercise. Let G be a finite group, H a subgroup of G and $\chi \in \mathrm{Char}(G)$. Prove that $\mathrm{Res}_H^G \chi \in \mathrm{Char}(H)$.

The restriction of an irreducible representations does not need to be irreducible:

7.11. Example. Let $G = \mathbb{D}_4 = \langle r, s : r^4 = s^2 = 1,\ srs = r^{-1}\rangle$ the dihedral group of eight elements. Let V be a complex vector space with basis $\{v_1, v_2\}$. Then V is a $\mathbb{C}[G]$-module with

$$r \cdot v_1 = v_2, \quad r \cdot v_2 = -v_1, \quad s \cdot v_1 = v_1, \quad s \cdot v_2 = -v_2.$$

The character of V is

$$\chi(g) = \begin{cases} 2 & \text{if } g = 1, \\ -2 & \text{if } g = r^2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\langle \chi, \chi\rangle = 1$, the character χ is irreducible. Thus V is simple as a $\mathbb{C}[G]$-module.

Let $H = \langle r^2, s\rangle = \{1, r^2, s, r^2 s\}$. Then $\operatorname{Res}^G_H V$ is V as a vector space, with the $\mathbb{C}[H]$-module structure given by

$$r^2 \cdot v_1 = -v_1, \quad r^2 \cdot v_2 = -v_1, \quad s \cdot v_1 = -v_1, \quad s \cdot v_2 = -v_2.$$

The character of $\operatorname{Res}^G_H V$ is

$$\operatorname{Res}^G_H \chi(h) = \begin{cases} 2 & \text{if } h = 1, \\ -2 & \text{if } h = r^2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\langle \operatorname{Res}^G_H \chi, \operatorname{Res}^G_H \chi\rangle = 0$, the character of $\operatorname{Res}^G_H \chi$ is not irreducible. Thus V is not simple as a $\mathbb{C}[H]$-module.

7.12. Exercise. Let H be a subgroup of G and $\phi \in \operatorname{Char}(H)$. Prove that there exists $\chi \in \operatorname{Irr}(G)$ such that $\langle \operatorname{Res}^G_H \chi, \phi\rangle \neq 0$.

7.13. Exercise. Let H be a subgroup of G, $\operatorname{Irr}(H) = \{\phi_1, \dots, \phi_l\}$, and $\chi \in \operatorname{Irr}(G)$. Prove that

$$\operatorname{Res}^G_H \chi = \sum_{i=1}^{l} d_i \phi_i,$$

where $\sum_{i=1}^{l} d_i^2 \leq (G : H)$. Moreover, $\sum_{i=1}^{l} d_i^2 = (G : H)$ if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.

§ 7.5. Induction of characters. We now define the induction of class functions. Let G be a finite group and H be a subgroup of G. If $f\colon H \to \mathbb{C}$ is a map, then

$$f^0(x) = \begin{cases} f(x) & \text{if } x \in H, \\ 0 & \text{otherwise.} \end{cases}$$

It is an exercise to prove that the map $f \mapsto f^0$ is linear.

7.14. Definition. Let G be a finite group and H be a subgroup of G. Let $f\colon H \to \mathbb{C}$ be a map. The **induction** of f to G is the map

$$g \mapsto \operatorname{Ind}^G_H f(g) = \frac{1}{|H|} \sum_{x \in G} f^0(x^{-1} g x).$$

7.15. Exercise. Let G be a finite group. Prove that the map

$$\operatorname{Ind}_H^G\colon \operatorname{ClassFun}(H)\to\operatorname{ClassFun}(G),\quad f\mapsto \operatorname{Ind}_H^G(f),$$

is a well-defined linear map.

For a finite group G, $\mathbf{1}_G$ denotes the trivial character of G.

7.16. Exercise. Let G be a finite group and $H=\{1\}$. Compute $\operatorname{Ind}_H^G \mathbf{1}_H$.

7.17. Exercise. Let G be a finite group, H be a subgroup of G. Prove that $\operatorname{Ind}_H^G \mathbf{1}_H$ is the character of the representation induced by the action of G on G/H by left multiplication.

The following exercise shows that the induction of a character yields a character; in fact, it demonstrates even more.

7.18. Exercise. Let H be a subgroup of G and $t_1,\dots,t_m$ be a left transversal of H in G. For a representation $\rho\colon H\to\mathbf{GL}_n(\mathbb{C})$ with character χ and $x\in G$, let

$$\rho_x^0=\begin{cases}\rho_x & \text{if } x\in H,\\ 0_{n\times n} & \text{otherwise,}\end{cases}$$

where $0_{n\times n}$ represents the zero $n\times n$ matrix. Prove that

$$\operatorname{Ind}_H^G\rho\colon G\to\mathbf{GL}_{mn}(\mathbb{C}),\quad g\mapsto\left(\rho^0_{t_i^{-1}gt_j}\right)_{1\le i,j\le m}\in\mathbb{C}^{nm\times nm}$$

is a representation with character $\operatorname{Ind}_H^G\chi$.

Let us show an easy application of Exercise 7.18.

7.19. Example. Let $G=\{\pm1,\pm i,\pm j,\pm k\}$ the quaternion group. Then $[G,G]=\{1,-1\}$ and $G/[G,G]\simeq C_2\times C_2$ (since G is non-abelian, $G/[G,G]$ cannot be cyclic). Thus there are four degree-one representations of G (see Exercise 1.30). Let $H=\langle i\rangle=\{1,-1,i,-i\}$. Thus $|H|=4$ and $(G:H)=2$. Let $t_1=1$ and $t_2=j$. Then $\{t_1,t_2\}$ is a left transversal of H in G. Let

$$\rho\colon H\to\mathbb{C}^\times,\quad i\mapsto\sqrt{-1}.$$

Then ρ is a representation. Let us compute $\operatorname{Ind}_H^G\rho$:

$$(\operatorname{Ind}_H^G\rho)(\pm1)=\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad (\operatorname{Ind}_H^G\rho)(\pm i)=\pm\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix},$$

$$(\operatorname{Ind}_H^G\rho)(\pm j)=\pm\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad (\operatorname{Ind}_H^G\rho)(\pm k)=\pm\begin{pmatrix}0&-\sqrt{-1}\\-\sqrt{-1}&0\end{pmatrix}.$$

As an exercise, the reader is invited to verify that the representation $\operatorname{Ind}_H^G\rho$ is irreducible.

§ 7.6. Frobenius' reciprocity. Before proving that the induction of a character is a character, we mention the following crucial property:

7.20. Theorem (Frobenius' reciprocity). *Let G be a finite group and H be a subgroup of G. If $a \in \mathrm{ClassFun}(H)$ and $b \in \mathrm{ClassFun}(G)$, then*

$$\langle \mathrm{Ind}_H^G a, b\rangle = \langle a, \mathrm{Res}_H^G b\rangle \quad \textit{and} \quad \langle \mathrm{Res}_H^G a, b\rangle = \langle a, \mathrm{Ind}_H^G b\rangle.$$

Proof. We only need to prove the first equality. We compute

$$\langle \mathrm{Ind}_H^G a, b\rangle = \frac{1}{|G|}\sum_{x\in G}\mathrm{Ind}_H^G a(x)\overline{b(x)} = \frac{1}{|G|}\frac{1}{|H|}\sum_{x,y\in G} a^0(y^{-1}xy)\overline{b(x)}. \tag{7.1}$$

Setting $h = y^{-1}xy$, we can write (7.1) as

$$\begin{aligned}\langle \mathrm{Ind}_H^G a, b\rangle &= \frac{1}{|G|}\frac{1}{|H|}\sum_{y\in G}\sum_{h\in H} a(h)\overline{b(yhy^{-1})}\\ &= \frac{1}{|G|}\frac{1}{|H|}\sum_{y\in G}\sum_{h\in H} a(h)\overline{b(h)}\\ &= \frac{1}{|G|}\sum_{y\in G}\langle a, \mathrm{Res}_H^G b\rangle.\end{aligned}$$

□

7.21. Corollary. *Let G be a finite group and H be a subgroup of G. Let $\chi \in \mathrm{Char}(H)$ be such that $\chi(1) = n$. Then $\mathrm{Ind}_H^G \chi \in \mathrm{Char}(G)$ and has degree $n(G:H)$.*

Proof. It is enough to show that each $m_\psi = \langle \mathrm{Ind}_H^G \chi, \psi\rangle \in \mathbb{Z}_{\geq 0}$ for all $\psi \in \mathrm{Irr}(G)$. Let $\psi \in \mathrm{Irr}(G)$. By Frobenius' reciprocity theorem,

$$m_\psi = \langle \mathrm{Ind}_H^G \chi, \psi\rangle = \langle \chi, \mathrm{Res}_H^G \psi\rangle \in \mathbb{Z}_{\geq 0}$$

because both χ and $\mathrm{Res}_H^G \psi$ are characters of H. To prove this, let $\mathrm{Irr}(H) = \{\theta_1, \dots, \theta_k\}$. Since $\chi \in \mathrm{Char}(H)$ and $\mathrm{Res}_H^G \psi \in \mathrm{Char}(H)$, there are non-negative integers $a_1, \dots, a_k$ and $b_1, \dots, b_k$ such that $\chi = \sum_{i=1}^k a_i\theta_i$ and $\mathrm{Res}_H^G \psi = \sum_{j=1}^k b_j\theta_j$. Then

$$\langle \chi, \mathrm{Res}_H^G \psi\rangle = \sum_{i=1}^k\sum_{j=1}^k a_i b_j\langle\theta_i, \theta_j\rangle = \sum_{i=1}^k a_i b_i \in \mathbb{Z}_{\geq 0}.$$

Therefore

$$\mathrm{Ind}_H^G \chi = \sum_{\psi\in\mathrm{Irr}(G)} m_\psi \psi \in \mathrm{Char}(G).$$

In particular,

$$\left(\mathrm{Ind}_H^G \chi\right)(1) = \frac{1}{|H|}\sum_{x\in G}\chi^0(1) = \frac{1}{|H|}|G|\chi(1) = \chi(1)(G:H).$$

□

7.22. Exercise. Let $G = \mathbb{S}_3$ and $H = \langle (12)\rangle$. Let $\varphi = \mathrm{sign}\,|_H$ be the restriction sign homomorphism to the subgroup H. Compute $\mathrm{Ind}_H^G \varphi$.

There are some useful properties that are easy to show.

7.23. Exercise. Let G be a finite group, H be a subgroup of G, $a \in \mathrm{ClassFun}(H)$ and $b \in \mathrm{ClassFun}(G)$. Prove that

$$\mathrm{Ind}_H^G((\mathrm{Res}_H^G b)a) = b(\mathrm{Ind}_H^G a).$$

7.24. Exercise (Transitivity of induction). Let G be a finite group, $H \subseteq K$ be subgroups of G and $a \in \operatorname{ClassFun}(H)$. Prove that

$$\operatorname{Ind}_K^G \operatorname{Ind}_H^K a = \operatorname{Ind}_H^G a.$$

7.25. Exercise. Let G be a finite group, H be a subgroup of G and $t_1, \dots, t_k$ be a transversal of H in G. Prove that

$$(\operatorname{Ind}_H^G \alpha)(g) = \sum_{i=1}^{k} \alpha^0(t_i^{-1} g t_i)$$

for all $\alpha \in \operatorname{ClassFun}(H)$.

8. Lecture: Week 8

§ 8.1. The correspondence theorem. Let N be a normal subgroup of G and

$$\pi\colon G \to G/N, \quad g \mapsto gN,$$

be the canonical map. If $\rho\colon G/N \to \mathbf{GL}(V)$, $gN \mapsto \rho_{gN}$, is a representation of G/N with character χ, the map

$$\operatorname{Inf}_{G/N}^{G} \rho\colon G \to \mathbf{GL}(V), \quad g \mapsto \rho_{gN},$$

is a representation of G. This representation $\operatorname{Inf}_{G/N}^{G} \rho$ of G is called the **inflation** of the representation ρ. It follows that the character $\operatorname{Inf}_{G/N}^{G} \chi$ of $\operatorname{Inf}_{G/N}^{G} \rho$ is

$$(\operatorname{Inf}_{G/N}^{G} \chi)(g) = \operatorname{trace}((\operatorname{Inf}_{G/N}^{G} \rho)_g) = \operatorname{trace} \rho_{gN} = \chi(gN).$$

In particular, $\chi(1) = (\operatorname{Inf}_{G/N}^{G} \chi)(1)$. The character $\operatorname{Inf}_{G/N}^{G} \chi$ is also called the **lifting** to G of the character χ of G/N.

8.1. Definition. Let $\chi \in \operatorname{Char}(G)$. The **kernel** of χ is the subset

$$\ker \chi = \{g \in G : \chi(g) = \chi(1)\}.$$

8.2. Proposition. *Let $\rho\colon G \to \mathbf{GL}_n(\mathbb{C})$ be a representation with character χ. Then $\ker \chi = \ker \rho$. In particular, $\ker \chi$ is a normal subgroup of G.*

Proof. Note that $\ker \rho \subseteq \ker \chi$, as $\rho_g = \mathrm{id}$ implies $\chi(g) = \operatorname{trace}(\rho_g) = n = \chi(1)$.

We now prove that $\ker \chi \subseteq \ker \rho$. If $g \in G$ is such that $\chi(g) = \chi(1)$, since ρ_g is diagonalizable, there exist eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

$$n = \chi(1) = \chi(g) = \sum_{i=1}^{n} \lambda_i.$$

Since each λ_i is a root of one, $\lambda_1 = \cdots = \lambda_n = 1$. Hence $\rho_g = \mathrm{id}$. $\square$

8.3. Theorem (Correspondence theorem). *Let N be a normal subgroup of a finite group G. There exists a bijective correspondence*

$$\operatorname{Char}(G/N) \longleftrightarrow \{\psi \in \operatorname{Char}(G) : N \subseteq \ker \psi\}$$

that maps irreducible characters to irreducible characters.

Proof. For $\chi \in \operatorname{Char}(G/N)$, let $\psi = \operatorname{Inf}_{G/N}^{G} \chi$. Let $n \in N$. Then

$$\psi(n) = \chi(nN) = \chi(N) = \psi(1)$$

and thus $N \subseteq \ker \psi$. Thus we have constructed a well-defined map

$$\sigma\colon \operatorname{Char}(G/N) \to \{\psi \in \operatorname{Char}(G) : N \subseteq \ker \psi\}, \quad \chi \mapsto \operatorname{Inf}_{G/N}^{G} \chi.$$

If $\psi \in \operatorname{Char}(G)$ is such that $N \subseteq \ker \psi$, let $\rho\colon G \to \mathbf{GL}(V)$, $g \mapsto \rho(g)$, be a representation with character ψ. Let $\widetilde{\rho}\colon G/N \to \mathbf{GL}(V)$, $gN \mapsto \rho(g)$. We claim that $\widetilde{\rho}$ is well-defined:

$$gN = hN \iff h^{-1}g \in N \implies \rho(h^{-1}g) = \mathrm{id} \iff \rho(h) = \rho(g).$$

Moreover, $\widetilde{\rho}$ is a representation, as

$$\widetilde{\rho}((gN)(hN)) = \widetilde{\rho}(ghN) = \rho(gh) = \rho(g)\rho(h) = \widetilde{\rho}(gN)\widetilde{\rho}(hN).$$

If $\widetilde{\psi}$ is the character of $\widetilde{\rho}$, then $\widetilde{\psi}(gN) = \psi(g)$. Thus we have constructed a well-defined map

$$\tau\colon \{\psi \in \operatorname{Char}(G) : N \subseteq \ker \psi\} \to \operatorname{Char}(G/N), \quad \psi \mapsto \widetilde{\psi}.$$

If $\chi \in \mathrm{Char}(G/N)$ and $g \in G$, then a direct calculation shows that $\tau(\sigma(\chi))(gN) = \chi(gN)$. Similarly, if $\psi \in \mathrm{Char}(G)$ is such that $N \subseteq \ker\psi$ and $g \in G$, then

$$\sigma(\tau(\psi))(g) = \sigma(\widetilde{\psi})(g) = (\mathrm{Inf}_{G/N}^{G}\widetilde{\psi})(g) = \widetilde{\psi}(gN) = \psi(g).$$

We now prove that χ is irreducible if and only if $\widetilde{\chi}$ is irreducible. If U is a subspace of V, then

$$\begin{aligned} U \text{ is } G\text{-invariant} &\iff \rho(g)(U) \subseteq U \text{ for all } g \in G \\ &\iff \widetilde{\rho}(gN)(U) \subseteq U \text{ for all } g \in G. \end{aligned}$$

Thus

$$\begin{aligned} \chi \text{ is irreducible } &\iff \rho \text{ is irreducible} \\ &\iff \widetilde{\rho} \text{ is irreducible } \iff \widetilde{\chi} \text{ is irreducible .} \end{aligned}$$

□

8.4. Example. Let $G = \mathbb{S}_4$ and $N = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. We know that N is normal in G and that $G/N = \langle a, b\rangle \simeq \mathbb{S}_3$, where $a = (123)N$ and $b = (12)N$. The character table of G/N is

	N	$(12)N$	$(123)N$
$\widetilde{\chi}_1$	1	1	1
$\widetilde{\chi}_2$	1	-1	1
$\widetilde{\chi}_3$	2	0	-1

For each $i \in \{1, 2, 3\}$ we compute the lifting χ_i to G of the character $\widetilde{\chi}_i$ of G/N. Since $(12)(34) \in N$ and $(13)(1234) = (12)(34) \in N$,

$$\chi((12)(34)) = \widetilde{\chi}(N), \quad \chi((1234)) = \widetilde{\chi}((13)N) = \widetilde{\chi}((12)N).$$

Since the characters $\widetilde{\chi}_i$ are irreducibles, the liftings χ_i are also irreducibles. With this process we obtain the following irreducible characters of G:

	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0

Theorem 8.3 has some unexpected applications. For example, the following exercise is elementary but tricky. A simpler solution uses the second orthogonality relation and Theorem 8.3.

8.5. Exercise. Let G be a finite group, $g \in G$ and N be a normal subgroup of G. Prove that $|C_{G/N}(gN)| \leq |C_G(g)|$.

The character table of a group can be used to find the lattice of normal subgroups. In particular, the character table detects simple groups.

8.6. Lemma. *Let G be a finite group and let $g, h \in G$. Then g and h are conjugate if and only if $\chi(g) = \chi(h)$ for all $\chi \in \mathrm{Char}(G)$.*

Proof. If g and h are conjugate, then $\chi(g) = \chi(h)$, as characters are class functions of G. Conversely, if $\chi(g) = \chi(h)$ for all $\chi \in \mathrm{Char}(G)$, then $f(g) = f(h)$ for all class function f

of G, as characters G generate the space of class functions of G. In particular, $\delta(g) = \delta(h)$, where

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ and } g \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

This implies that g and h are conjugate. □

As a consequence, we get that

$$\bigcap_{\chi \in \mathrm{Irr}(G)} \ker \chi = \{1\}. \tag{8.1}$$

Indeed, if $g \in \ker \chi$ for all $\chi \in \mathrm{Irr}(G)$, then $g = 1$ since the lemma implies that g and 1 are conjugate because $\chi(g) = \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

8.7. Proposition. *Let G be a finite group. If N is a normal subgroup of G, then there exist characters $\chi_1, \dots, \chi_k \in \mathrm{Irr}(G)$ such that*

$$N = \bigcap_{i=1}^{k} \ker \chi_i.$$

Proof. Apply the previous remark to the group G/N to obtain that

$$\bigcap_{\widetilde{\chi} \in \mathrm{Irr}(G/N)} \ker \widetilde{\chi} = \{N\}.$$

Assume that $\mathrm{Irr}(G/N) = \{\widetilde{\chi}_1, \dots, \widetilde{\chi}_k\}$. We lift the irreducible characters of G/N to G to obtain (some) irreducible characters $\chi_1, \dots, \chi_k$ of G such that

$$N \subseteq \ker \chi_1 \cap \cdots \cap \ker \chi_k.$$

If $g \in \ker \chi_i$ for all $i \in \{1, \dots, k\}$, then

$$\widetilde{\chi}_i(N) = \chi_i(1) = \chi_i(g) = \widetilde{\chi}_i(gN)$$

for all $i \in \{1, \dots, k\}$. This implies that

$$gN \in \bigcap_{i=1}^{k} \ker \widetilde{\chi}_i = \{N\},$$

that is $g \in N$. □

Recall that a non-trivial group is **simple** if it contains no non-trivial normal proper subgroups. Examples of simple groups are cyclic groups of prime order and the alternating groups $\mathbb{A}_n$ for $n \geq 5$. As a corollary of Proposition 8.7, we can use the character table to detect simple groups.

8.8. Proposition. *Let G be a finite group. Then G is not simple if and only if there exists a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ for some $g \in G \setminus \{1\}$.*

Proof. If G is not simple, there exists a normal subgroup N of G such that $N \neq G$ and $N \neq \{1\}$. By Proposition 8.7, there exist characters $\chi_1, \dots, \chi_k \in \mathrm{Irr}(G)$ such that $N = \ker \chi_1 \cap \cdots \cap \ker \chi_k$. In particular, there exists a non-trivial character χ_i such that $\ker \chi_i \neq \{1\}$. Thus there exists $g \in G \setminus \{1\}$ such that $\chi_i(g) = \chi_i(1)$.

Assume now that there exists a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ for some $g \in G \setminus \{1\}$. In particular, $g \in \ker\chi$ and hence $\ker\chi \neq \{1\}$. Since χ is non-trivial, $\ker\chi \neq G$. Thus $\ker\chi$ is a proper non-trivial normal subgroup of G. $\square$

8.9. Example. If there exists a group G with a character table of the form

χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	1	1	-1	-1	1
χ_5	2	-2	2	0	0	0
χ_6	8	0	-1	0	0	0

then G cannot be simple. Note that such a group G would have order $\sum_{i=1}^{6}\chi_i(1)^2 = 72$. Mathieu's group M_9 has precisely this character table!

8.10. Example. Let $\alpha = \frac{1}{2}(-1+\sqrt{7}i)$. If there exists a group G with a character table of the form

χ_1	1	1	1	1	1	1
χ_2	7	-1	-1	1	0	0
χ_3	8	0	0	-1	1	1
χ_4	3	-1	1	0	α	$\overline{\alpha}$
χ_5	3	-1	1	0	$\overline{\alpha}$	α
χ_6	6	2	0	0	0	0

then G is simple. Note that such a group G would have order $\sum_{i=1}^{6}\chi_i(1)^2 = 168$. The group

$$\mathbf{PSL}_2(7) = \mathbf{SL}_2(7)/Z(\mathbf{SL}_2(7))$$

is a simple group that has precisely this character table!

§ 8.2. Frobenius' groups. If p is a prime number, then the units $(\mathbb{Z}/p)^\times$ of $\mathbb{Z}/p$ form a multiplicative group. Moreover, $(\mathbb{Z}/p)^\times$ is cyclic of order $p-1$.

Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \in (\mathbb{Z}/p)^\times,\ y \in \mathbb{Z}/p \right\}.$$

Then G is a group with the usual matrix multiplication and $|G| = p(p-1)$. Let p and q be prime numbers such that q divides $p-1$, $z \in \mathbb{Z}$ be an element of multiplicative order q modulo p and

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} z & 1 \\ 0 & 1 \end{pmatrix}, \quad H = \langle a, b\rangle.$$

A direct calculation shows that

$$a^p = b^q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad bab^{-1} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = a^z. \tag{8.2}$$

Every element of H is of the form a^ib^j for $i \in \{0,\dots,p-1\}$ and $j \in \{0,\dots,q-1\}$. Thus $|H| = pq$. Using (8.2) we can compute the multiplication table of G.

8.11. Exercise. Let p and q be prime numbers such that q divides $p-1$. Let $u, v \in \mathbb{Z}$ be elements of order q modulo p. Prove that

$$\langle a, b : a^p = b^q = 1, bab^{-1} = a^u\rangle \simeq \langle a, b : a^p = b^q = 1, bab^{-1} = a^v\rangle.$$

The group

$$\langle a, b : a^p = b^q = 1, bab^{-1} = a^u \rangle,$$

where $u \in \mathbb{Z}$ has order q modulo p, is a particular case of a **Frobenius group**.

8.12. Exercise. Let p and q be prime numbers such that $p > q$. Let G be a group of order pq. Then either G is abelian or q divides $p-1$ and

$$G \simeq \langle a, b : a^p = b^q = 1, bab^{-1} = a^u \rangle$$

for some $u \in \mathbb{Z}$ of order q modulo p.

Using Exercise 8.12, we can prove, for example, that every group of order 15 is abelian.

8.13. Definition. We say that a finite group G is a **Frobenius group** if G has a non-trivial proper subgroup H such that $H \cap xHx^{-1} = \{1\}$ for all $x \in G \setminus H$. In this case, the subgroup H is called a **Frobenius complement**.

A subgroup H such that $gHg^{-1} \cap H = \{1\}$ for all $g \notin H$ is called a **malnormal** subgroup. Note that if H is malnormal, then $N_G(H) = H$.

8.14. Exercise. Let G be a group and H be a subgroup of G. Prove that the following statements are equivalent:

1) H is malnormal.
2) The action of H on $G/H \setminus \{H\}$ by left multiplication is free.
3) Any $g \in G \setminus \{1\}$ has zero or one fixed point on G/H.

For any group G, the subgroups $\{1\}$ and G are malnormal in G. Moreover, they are the only subgroups of G that are both normal and malnormal

8.15. Exercise. Let G be a group. Prove the following statements:

1) If H is malnormal in G, then gHg^{-1} is malnormal in G for all $g \in G$.
2) If H is malnormal in G and K is malnormal in H, then K is malnormal in G.
3) The intersection of malnormal subgroups is malnormal.
4) If H is malnormal in G and S is a subgroup of G, then $H \cap S$ is malnormal in S.

8.16. Example. Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times \right\} \subseteq G.$$

Let $g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in G \setminus H$. Then $y \neq 0$. Since

$$g \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} a & -ay + y \\ 0 & 1 \end{pmatrix},$$

it follows that the subgroup H is malnormal in G.

8.17. Exercise. Let G be a group and H be a non-trivial subgroup of G. Prove that if $Z(G) \neq \{1\}$, then H is not malnormal in G.

8.18. Bonus exercise. Let G be a group with no 2-torsion that contains a normal infinite cyclic group. Prove that G cannot contain a non-trivial proper malnormal subgroup.

8.19. Example. Let G be a finite group and $P \in \mathrm{Syl}_p(G)$ be such that $|P| = p$ and $N_G(P) = P$. Then G is a Frobenius group with complement P.

The previous example shows that $\mathbb{A}_4$ is a Frobenius group with complement $\langle(123)\rangle$. Another situation where the example applies is the dihedral group

$$\mathbb{D}_{2n+1} = \langle r, s : r^{2n+1} = s^2 = 1, srs = r^{-1}\rangle$$

of order $2(2n+1)$. It follows that $\mathbb{D}_{2n+1}$ is a Frobenius group with complement $\langle s\rangle$.

8.20. Theorem (Frobenius). *Let G be a Frobenius group with complement H. Then*

$$N = \left(G \setminus \bigcup_{x\in G} xHx^{-1}\right) \cup \{1\}$$

is a normal subgroup of G.

Proof. Let 1_H and 1_G be the trivial characters of H and G, respectively. For each $\chi \in \mathrm{Irr}(H)$, $\chi \neq 1_H$, let $\alpha = \chi - \chi(1)1_H \in \mathrm{ClassFun}(H)$, where 1_H denotes the trivial character of H.

We claim that $\mathrm{Res}_H^G \mathrm{Ind}_H^G \alpha = \alpha$. First, $\mathrm{Ind}_H^G \alpha(1) = \alpha(1) = 0$. If $h \in H \setminus \{1\}$, then

$$\mathrm{Ind}_H^G \alpha(h) = \frac{1}{|H|} \sum_{\substack{x\in G\\ x^{-1}hx\in H}} \alpha(x^{-1}hx) = \frac{1}{|H|} \sum_{x\in H} \alpha(h) = \alpha(h),$$

since, if $x \notin H$, then $x^{-1}hx \in H$ implies that $h \in H \cap xHx^{-1} = \{1\}$.

By Frobenius' reciprocity and the definition of α,

$$\langle \mathrm{Ind}_H^G \alpha, \mathrm{Ind}_H^G \alpha\rangle = \langle \alpha, \mathrm{Res}_H^G \mathrm{Ind}_H^G \alpha\rangle = \langle \alpha, \alpha\rangle = 1 + \chi(1)^2. \tag{8.3}$$

Again, by Frobenius' reciprocity,

$$\langle \mathrm{Ind}_H^G \alpha, 1_G\rangle = \langle \alpha, \mathrm{Res}_H^G 1_G\rangle = \langle \alpha, 1_H\rangle = \langle \chi - \chi(1)1_H, 1_H\rangle = -\chi(1),$$

where 1_G is the trivial character of G. If we write

$$\mathrm{Ind}_H^G \alpha = \sum_{\eta\in\mathrm{Irr}(G)} \langle \mathrm{Ind}_H^G \alpha, \eta\rangle\eta = \langle \mathrm{Ind}_H^G \alpha, 1_G\rangle 1_G + \underbrace{\sum_{\substack{1_G\neq\eta\\ \eta\in\mathrm{Irr}(G)}} \langle \mathrm{Ind}_H^G \alpha, \eta\rangle\eta}_{\phi},$$

then $\mathrm{Ind}_H^G \alpha = -\chi(1)1_G + \phi$, where ϕ is a linear combination of non-trivial irreducible characters of G. We compute

$$1 + \chi(1)^2 = \langle \mathrm{Ind}_H^G \alpha, \mathrm{Ind}_H^G \alpha\rangle = \langle \phi - \chi(1)1_G, \phi - \chi(1)1_G\rangle = \langle \phi, \phi\rangle + \chi(1)^2$$

and hence $\langle \phi, \phi\rangle = 1$.

Claim. If $\eta \in \mathrm{Irr}(G)$ is such that $\eta \neq 1_G$, then $\langle \mathrm{Ind}_H^G \alpha, \eta\rangle \in \mathbb{Z}$.

By Frobenius' reciprocity, $\langle \operatorname{Ind}_H^G \alpha, \eta\rangle = \langle \alpha, \operatorname{Res}_H^G \eta\rangle$. If we decompose $\operatorname{Res}_H^G \eta$ into irreducibles of H, say

$$\operatorname{Res}_H^G \eta = m_1 1_H + m_2\chi + m_3\theta_3 + \cdots + m_t\theta_t$$

for some $m_1, m_2, \dots, m_t \geq 0$, then, since

$$\langle \alpha, 1_H\rangle = \langle \chi - \chi(1)1_H, 1_H\rangle = -\chi(1), \qquad \langle \alpha, \chi\rangle = \langle \chi - \chi(1)1_H, \chi\rangle = 1,$$

and

$$\langle \alpha, \theta_j\rangle = \langle \chi - \chi(1)1_H, \theta_j\rangle = 0$$

for all $j \in \{3, \dots, t\}$, we conclude that

$$\langle \operatorname{Ind}_H^G \alpha, \eta\rangle = -m_1\chi(1) + m_2 \in \mathbb{Z}.$$

Claim. $\phi \in \operatorname{Irr}(G)$.

Since $\langle \operatorname{Ind}_H^G \alpha, \eta\rangle \in \mathbb{Z}$ for all $\eta \in \operatorname{Irr}(G)$ such that $\eta \neq 1_G$ and

$$1 = \langle \phi, \phi\rangle = \sum_{\substack{\eta,\theta\in\operatorname{Irr}(G)\\ \eta,\theta\neq 1_G}} \langle \operatorname{Ind}_H^G \alpha, \eta\rangle\langle \operatorname{Ind}_H^G \alpha, \theta\rangle\langle \eta, \theta\rangle = \sum_{\substack{\eta\neq 1_G\\ \eta\in\operatorname{Irr}(G)}} \langle \operatorname{Ind}_H^G \alpha, \eta\rangle^2,$$

there is a unique $\eta \in \operatorname{Irr}(G)$ such that $\langle \operatorname{Ind}_H^G \alpha, \eta\rangle^2 = 1$ and all the other products are zero, that is $\phi = \pm\eta$ for some $\eta \in \operatorname{Irr}(G)$. Since

$$\chi - \chi(1)1_H = \alpha = \operatorname{Res}_H^G \operatorname{Ind}_H^G \alpha = \operatorname{Res}_H^G(\phi - \chi(1)1_G) = \operatorname{Res}_H^G \phi - \chi(1)1_H,$$

it follows that $\phi(1) = \operatorname{Res}_H^G \phi(1) = \chi(1) \in \mathbb{Z}_{\geq 1}$. Thus $\phi \in \operatorname{Irr}(G)$.

We have proved that if $\chi \in \operatorname{Irr}(H)$ is such that $\chi \neq 1_H$, then there exists $\phi_\chi \in \operatorname{Irr}(G)$ such that $\operatorname{Res}_H^G(\phi_\chi) = \chi$.

We prove that N is equal to

$$M = \bigcap_{\substack{\chi\in\operatorname{Irr}(H)\\ \chi\neq 1_H}} \ker \phi_\chi.$$

We first prove that $N \subseteq M$. Let $n \in N \setminus \{1\}$ and $\chi \in \operatorname{Irr}(H) \setminus \{1_H\}$. Since n does not belong to a conjugate of H,

$$\operatorname{Ind}_H^G \alpha(n) = \frac{1}{|H|} \sum_{\substack{x\in G\\ x^{-1}nx\in H}} \alpha(x^{-1}nx) = 0,$$

as $n \in N$ implies that the set $\{x \in G : x^{-1}nx \in H\}$ is empty. Since

$$0 = \operatorname{Ind}_H^G \alpha(n) = \phi_\chi(n) - \chi(1) = \phi_\chi(n) - \phi_\chi(1),$$

we conclude that $n \in \ker \phi_\chi$.

We now prove that $M \subseteq N$. Let $h \in M \cap H$ and $\chi \in \operatorname{Irr}(H) \setminus \{1_H\}$. Then

$$\phi_\chi(h) - \chi(1) = \operatorname{Ind}_H^G \alpha(h) = \alpha(h) = \chi(h) - \chi(1),$$

and $h \in \ker \chi$, as

$$\chi(h) = \phi_\chi(h) = \phi_\chi(1) = \chi(1).$$

Therefore

$$h \in \bigcap_{\chi\in\operatorname{Irr}(H)} \ker \chi = \{1\}.$$

By (8.1), the kernels of irreducible characters have trivial intersection. We now prove that $M \cap xHx^{-1} = \{1\}$ for all $x \in G$. Let $x \in G$ and $m \in M \cap xHx^{-1}$. Since $m = xhx^{-1}$ for some $h \in H$, $x^{-1}mx \in H \cap M = \{1\}$. This implies that $m = 1$. $\square$

There is no proof of Frobenius' theorem that is independent of character theory. Purely group-theoretic proofs exist in cases where the Frobenius complement has even order or is solvable; see [**21**, Remark 16.2]. The Feit–Thompson theorem (which relies heavily on character theory and is significantly more difficult than Frobenius' theorem) implies that these two cases cover all possibilities.

In 2013, Terence Tao discovered an alternative Fourier-analytic proof of Frobenius' theorem, though it resembles the original character-theoretic approach.

9. Lecture: Week 9

9.1. Definition. Let G be a Frobenius group. The normal subgroup N of Frobenius' theorem is called the **Frobenius kernel**.

9.2. Proposition. *Let G be a Frobenius group with complement H. Then there exists a normal subgroup N of G such that $G = HN$ and $H \cap N = \{1\}$.*

Proof. Frobenius' theorem yields the subgroup N. Since $H \cap gHg^{-1} = \{1\}$ for all $g \in G \setminus H$, it follows that $N_G(H) = H$. It follows that H has $(G : H)$ conjugates. Let

$$N = \left(G \setminus \bigcup_{x \in G} xHx^{-1} \right) \cup \{1\}.$$

Then $|N| = |G| - (G : H)(|H| - 1) = (G : H)$. Since, moreover, $N \cap H = \{1\}$, we conclude that

$$|HN| = |N||H|/|H \cap N| = |N||H| = |G|.$$

Therefore $G = NH$. $\square$

In his doctoral thesis Thompson proved the following result, conjectured by Frobenius.

9.3. Theorem (Thompson). *Let G be a Frobenius group. If N is the Frobenius kernel, then N is nilpotent.*

See [24, Theorem 6.24] for the proof.

§ 9.1. The Cameron–Cohen theorem (again). In this section, we use Frobenius' theorem to strengthen the Cameron–Cohen theorem on derangements (Theorem 9.9). To do so, we first require an alternative version of Frobenius' theorem.

9.4. Corollary (Frobenius). *Let G be a group acting transitively on a finite set X. Assume that each $g \in G \setminus \{1\}$ fixes at most one element of X. The set N formed by the identity and the derangements of G is a normal subgroup of G.*

Proof. Let $x \in X$ and $H = G_x$. We claim that if $g \in G \setminus H$, then $H \cap gHg^{-1} = \{1\}$. If $h \in H \cap gHg^{-1}$, then $h \cdot x = x$ and $(g^{-1}hg) \cdot x = x$. Since $g \cdot x \neq x$, h fixes two elements of X. Thus $h = 1$, as every non-trivial element fixes at most one element of X.

By Theorem 8.20,

$$N = \left(G \setminus \bigcup_{g \in G} gHg^{-1} \right) \cup \{1\}$$

is a subgroup of G. Let us compute the elements of N. If $h \in \bigcup_{g \in G} gHg^{-1}$, then there exists $g \in G$ such that $g^{-1}hg \in H$, that is $(g^{-1}hg) \cdot x = x$; equivalently, $h \in G_{g \cdot x}$. Therefore, the non-identity elements of N are the elements of G moving every element of X. $\square$

9.5. Example. Let F be a finite field and G be the group of maps $f \colon F \to F$ of the form $f(x) = ax + b$, $a, b \in F$ with $a \neq 0$. The group G acts on F and every $f \neq \mathrm{id}$ fixes at most one element of F, as

$$x = f(x) = ax + b \implies a \neq 1 \text{ and } x = b/(1 - a).$$

In this case, $N = \{f : f(x) = x + b \, , b \in F\}$ is a subgroup of G.

9.6. Exercise. Prove that Theorem 8.20 can be obtained from Corollary 9.4.

§ 9.2. Derangements: The Cameron–Cohen theorem. Let G be a finite group that acts faithfully and transitively on a finite set X, say $G \leq \mathbb{S}_n$, where $X = \{1, 2, \dots, n\}$. Let G_0 be the set of elements $g \in G$ with no fixed-points, that is $g(x) \neq x$ for all $x \in X$. Such permutations are known as **derangements**.

9.7. Example. Let $G = \mathbb{S}_3$. Then $G_0 = \{(123), (132)\}$.

9.8. Example. Let $G = \mathbb{S}_4$. Then

$$G_0 = \{(12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432)\}.$$

We want to estimate the number of derangements. For this purpose, let $c_0 = |G_0|/|G|$.

9.9. Theorem (Cameron–Cohen). *If G is a subgroup of $\mathbb{S}_n$ that acts transitively on $\{1, \dots, n\}$, then $c_0 \geq \frac{1}{n}$. Moreover, if n is not the power of a prime number, then $c_0 > \frac{1}{n}$.*

Proof. Let $X = \{1, \dots, n\}$. By definition, the rank of G is the number of orbitals of G on X. It follows that the rank is ≥ 2, as $X \times X$ decomposes as

$$X \times X = \Delta \cup ((X \times X) \setminus \Delta)$$

Let $\chi(g) = |\operatorname{Fix}(g)|$ and $G_0 = \{g \in G : \chi(g) = 0\}$. If $g \notin G_0$, then $1 \leq \chi(g) \leq n$. Since $(\chi(g) - 1)(\chi(g) - n) \leq 0$,

$$\frac{1}{|G|} \sum_{g \in G \setminus G_0} (\chi(g) - 1)(\chi(g) - n) \leq 0.$$

On the one hand,

$$\begin{aligned}
&\frac{1}{|G|} \sum_{g \in G} (\chi(g) - 1)(\chi(g) - n) \\
&\qquad = \frac{1}{|G|} \left\{ \sum_{g \in G_0} + \sum_{g \in G \setminus G_0} \right\} (\chi(g) - 1)(\chi(g) - n) \\
&\qquad = \frac{1}{|G|} \sum_{g \in G_0} (\chi(g)^2 - (n+1)\chi(g) + n) + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus G_0} (\chi(g) - 1)(\chi(g) - n)}_{\leq 0} \\
&\qquad \leq n \frac{|G_0|}{|G|} = nc_0.
\end{aligned}$$

On the other hand, since the rank of G is ≥ 2,

$$\begin{aligned}
\frac{1}{|G|} \sum_{g \in G} (\chi(g) - 1)(\chi(g) - n) &= \frac{1}{|G|} \sum_{g \in G} (\chi(g)^2 - (n+1)\chi(g) + n) \\
&\geq 2 - \frac{n+1}{|G|} \sum_{g \in G} \chi(g) + n
\end{aligned}$$

Since G is transitive on X, the Cauchy–Frobenius–Burnside theorem implies that

$$\sum_{g\in G}\chi(g)=|G|.$$

Thus $2-(n+1)+n\leq nc_0$ and hence $1/n\leq c_0$.

Assume now that $c_0=1/n$. Then

$$\frac{1}{|G|}\sum_{g\in G}(\chi(g)^2-(n+1)\chi(g)+n)=1$$

and hence $\frac{1}{|G|}\sum_{g\in G}\chi(g)^2=2$. Moreover, since

$$\frac{1}{|G|}\sum_{g\in G_0}(\chi(g)-1)(\chi(g)-n)+\frac{1}{|G|}\sum_{g\in G\setminus G_0}(\chi(g)-1)(\chi(g)-n)=1,$$

it follows that

$$\sum_{g\in G\setminus G_0}(\chi(g)-1)(\chi(g)-n)=0.$$

Hence $(\chi(g)-1)(\chi(g)-n)=0$ for all $g\in G\setminus G_0$.

By Corollary 9.4, the subset $N=G_0\cup\{\mathrm{id}\}$ is a normal subgroup of G. Moreover, $G=N\rtimes H$ for some subgroup H of G of order n. Since $n=|H|=|N|-1$, H acts freely and transitively on $N\setminus\{1\}$.

We claim that N is a p-group for some prime number p. Let $n,m\in N\setminus\{1\}$. Since H is transitive on $N\setminus\{1\}$, there exists $h\in H$ such that $h\cdot n=m$. Then

$$|n|=|h\cdot n|=|m|,$$

since for each $h\in H$, the map $x\mapsto h\cdot x$ is an automorphism of N. Thus every two elements of $N\setminus\{1\}$ have the same order. Let p be a prime divisor of $|N|$. By Cauchy's theorem, there exists $n\in N$ such that $|n|=p$. Since all non-trivial elements of N have the same order, N is a p-group. Therefore $n=|N|$ is a power of a prime. $\square$

In some cases, the bound in the Cameron–Cohen theorem can be improved:

9.10. Theorem (Guralnick–Wan). *Let G be a finite transitive group of degree $n\geq 2$. If n is not a power of a prime number and $G\neq\mathbb{S}_n$ for $n\in\{2,4,5\}$, then $c_0\geq 2/n$.*

The proof appears in [**14**] and uses the classification of finite 2-transitive groups, which depends on the CFSG.

9.11. Exercise. Let G be the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{Z}/5$ and $a \neq 0$. Then $|G| = 20$. Let

$$h = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

A direct calculation shows that $h^4 = 1$, $k^5 = 1$ and $hkh^{-1} = k^2$. Let $H = \langle h \rangle$ and $K = \langle k \rangle$. Prove the following statements:

1) Prove that $G = K \rtimes H$.

2) Find the conjugacy classes of G:

Size	1	4	5	5	5
Representative	1	k	h	h^2	h^3

3) Prove that G/K is cyclic of order four.

4) Prove that $[G, G] = K$.

5) Use Theorem 8.3 on G/K to find the degree-one characters of G.

6) Let $\chi \in \mathrm{Irr}(K)$ be such that $\chi(k) = \exp(2\pi i/5)$. Prove that $\mathrm{Ind}_K^G \chi \in \mathrm{Irr}(G)$.

The previous exercise demonstrates that the character table of the Frobenius group of order 20 corresponds to that of Table 7.

Table 7. Character table of the Frobenius group $F_{5,4}$ of order 20.

	1	k	h	h^2	h^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

§ 9.3. Burnside's theorem on real characters. For $n \geq 1$ let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{C}^n$. The **natural representation** of $\mathbb{S}_n$ is $\rho\colon \mathbb{S}_n \to \mathbf{GL}_n(\mathbb{C})$, $\sigma \mapsto \rho_\sigma$, where $\rho_\sigma(e_j) = e_{\sigma(j)}$ for all $j \in \{1, \dots, n\}$. The matrix of ρ_σ in the standard basis is

$$(\rho_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases} \tag{9.1}$$

9.12. Lemma. *For $n \geq 1$ let $\rho\colon \mathbb{S}_n \to \mathbf{GL}_n(C)$ be the natural representation of the symmetric group. If $A \in \mathbb{C}^{n\times n}$ and $\sigma \in \mathbb{S}_n$, then*

$$A_{ij} = (\rho_\sigma A)_{\sigma(i)j} = (A\rho_\sigma)_{i\sigma^{-1}(j)}$$

for all $i, j \in \{1, \dots, n\}$.

Proof. With (9.1) we compute:

$$(A\rho_\sigma)_{ij} = \sum_{k=1}^{n} A_{ik}(\rho_\sigma)_{kj} = A_{i\sigma(j)}, \quad (\rho_\sigma A)_{ij} = \sum_{k=1}^{n} (\rho_\sigma)_{ik}A_{kj} = A_{\sigma^{-1}(i)j}. \qquad \square$$

9.13. Definition. Let G be a finite group. A character χ of G is said to be **real** if $\chi = \overline{\chi}$, that is $\chi(g) \in \mathbb{R}$ for all $g \in G$.

9.14. Exercise. Let G be a finite group. If $\chi \in \operatorname{Irr}(G)$, then $\overline{\chi}$ is irreducible.

9.15. Definition. Let G be a group. A conjugacy class C of G is said to be **real** if for every $g \in C$ one has $g^{-1} \in C$.

We use the following notation: if G is a group and $C = \{xgx^{-1} : x \in G\}$ is a conjugacy class of G, then $C^{-1} = \{xg^{-1}x^{-1} : x \in G\}$.

9.16. Theorem (Burnside). *Let G be a finite group. The number of real conjugacy classes equals the number of real irreducible characters.*

Proof. Let $C_1, \dots, C_r$ be the conjugacy classes of G and let $\chi_1, \dots, \chi_r$ be the irreducible characters of G. Let $\alpha, \beta \in \mathbb{S}_r$ be such that $\overline{\chi_i} = \chi_{\alpha(i)}$ and $C_i^{-1} = C_{\beta(i)}$ for all $i \in \{1, \dots, r\}$. Note that χ_i is real if and only if $\alpha(i) = i$ and that C_i is real if and only if $\beta(i) = i$. The number n of fixed points of α is equal to the number of real irreducible characters of G, and the number m of fixed points of β is equal to the number of real classes. Let $\rho\colon \mathbb{S}_r \to \mathbf{GL}_r(\mathbb{C})$ be the natural representation of $\mathbb{S}_r$, with character χ_ρ. Then $\chi_\rho(\alpha) = n$ and $\chi_\rho(\beta) = m$.

We claim that $\operatorname{trace} \rho_\alpha = \operatorname{trace} \rho_\beta$. Let $X = (\chi_i(C_j)) \in \mathbf{GL}_r(\mathbb{C})$ be the character matrix of G. Then

$$\rho_\alpha X = \overline{X} = X \rho_\beta.$$

For example, using Lemma 9.12,

$$\overline{X_{ij}} = \overline{\chi_i(C_j)} = \chi_i(C_j^{-1}) = \chi_i(C_{\beta(j)} = X_{i\beta(j)} = (X\rho_\beta)_{ij}.$$

Since X is invertible, $\rho_\alpha = X\rho_\beta X^{-1}$. Thus

$$n = \chi_\rho(\alpha) = \operatorname{trace} \rho_\alpha = \operatorname{trace} \rho_\beta = \chi_\rho(\beta) = m. \qquad \square$$

9.17. Corollary. *Let G be a finite group. Then $|G|$ is odd if and only if the only real $\chi \in \operatorname{Irr}(G)$ is the trivial character.*

Proof. If $|G|$ is even, there exists $g \in G$ of order two (Cauchy's theorem). The conjugacy class of g is real.

Conversely, assume that G has a non-trivial real conjugacy class C. Let $g \in C$. We claim that G has an element of even order. Let $h \in G$ be such that $hgh^{-1} = g^{-1}$. Then $h^2 \in C_G(g)$, as $h^2gh^{-2} = g$. If $h \in \langle h^2 \rangle \subseteq C_G(g)$, then g has even order, as $g^{-1} = g$. If $h \notin \langle h^2 \rangle$, then h^2 does not generate $\langle h \rangle$. Hence h has even order, as $|h| \neq |h^2| = |h|/\gcd(|h|, 2)$, so $\gcd(|h|, 2) \neq 1$. $\square$

9.18. Theorem (Burnside). *Let G be a finite group of odd order with r conjugacy classes. Then $r \equiv |G| \bmod 16$.*

Proof. Since $|G|$ is odd, every non-trivial $\chi \in \operatorname{Irr}(G)$ is not real by the previous corollary. The irreducible characters of G are

$$\chi_1, \chi_2, \overline{\chi_2}, \dots, \chi_k, \overline{\chi_k}, \quad r = 1 + 2(k-1),$$

where χ_1 denotes the trivial character. For every $j \in \{2, \dots, k\}$ let $d_j = \chi_j(1)$. Since each d_j divides $|G|$ by Frobenius' theorem and $|G|$ is odd, every d_j is an odd number, say

$d_j = 1 + 2m_j$. Thus

$$\begin{aligned}|G| &= 1 + \sum_{j=2}^{k} 2d_j^2 = 1 + \sum_{j=2}^{k} 2(2m_j + 1)^2 \\ &= 1 + \sum_{j=2}^{k} 2(4m_j^2 + 4m_j + 1) = 1 + 2(k-1) + 8\sum_{j=2}^{k} m_j(m_j + 1).\end{aligned}$$

Hence $|G| \equiv r \bmod 16$, as $r = 1 + 2k$ and every $m_j(m_j + 1)$ is even. $\square$

As an immediate consequence of Theorem 9.18, we conclude that every group of order 15 is abelian.

10. Lecture: Week 10

§ 10.1. The character table of $\mathbb{A}_5$. Let $G = \mathbb{A}_5$. The group G is a non-abelian simple group of order 60. It has five conjugacy classes, namely

Representative	id	(12)(34)	(123)	(12345)	(12354)
Size	1	15	20	12	12

One can easily get the conjugacy classes of $\mathbb{A}_5$ with Magma:

```
> A5 := Alt(5);
> ConjugacyClasses(A5);
Conjugacy Classes of group A5
-----------------------------
[1]     Order 1       Length 1
        Id(A5)

[2]     Order 2       Length 15
        (1, 2)(3, 4)

[3]     Order 3       Length 20
        (1, 2, 3)

[4]     Order 5       Length 12
        (1, 2, 3, 4, 5)

[5]     Order 5       Length 12
        (1, 3, 4, 5, 2)
```

Let us see how to obtain all conjugacy classes of $\mathbb{A}_5$ without computers. Let $\sigma \in \mathbb{A}_5$ and C be its conjugacy class in $\mathbb{S}_5$. Thus $|C| = (\mathbb{S}_5 : C_{\mathbb{S}_5}(\sigma))$. There are two cases to consider

Assume first that $C_{\mathbb{S}_5}(\sigma) \not\subseteq \mathbb{A}_5$. Since $\mathbb{A}_5$ is a maximal subgroup of $\mathbb{S}_5$, it follows that $\mathbb{A}_5 C_{\mathbb{S}_5}(\sigma) = \mathbb{S}_5$. Using the isomorphism theorems,

$$\mathbb{S}_5/\mathbb{A}_5 = \mathbb{A}_5 C_{\mathbb{S}_5(\sigma)}/\mathbb{A}_5 \simeq C_{\mathbb{S}_5}(\sigma)/(C_{\mathbb{S}_5}(\sigma) \cap \mathbb{A}_5) = C_{\mathbb{S}_5}(\sigma)/C_{\mathbb{A}_5}(\sigma).$$

Hence

$$(\mathbb{A}_5 : C_{\mathbb{A}_5}(\sigma)) = \frac{(\mathbb{S}_5 : C_{\mathbb{A}_5}(\sigma))}{(\mathbb{S}_5 : \mathbb{A}_5)} = \frac{(\mathbb{S}_5 : C_{\mathbb{A}_5}(\sigma))}{(C_{\mathbb{S}_5}(\sigma) : C_{\mathbb{A}_5}(\sigma))} = (\mathbb{S}_5 : C_{\mathbb{S}_5}(\sigma)) = |C|.$$

Therefore C is the class of σ in $\mathbb{A}_5$.

Assume now that $C_{\mathbb{S}_5}(\sigma) \subseteq \mathbb{A}_5$. Then $C_{\mathbb{A}_5}(\sigma) = C_{\mathbb{S}_5}(\sigma) \cap \mathbb{A}_5 = C_{\mathbb{S}_5}(\sigma)$ and therefore

$$(\mathbb{A}_5 : C_{\mathbb{A}_5}(\sigma)) = (\mathbb{A}_5 : C_{\mathbb{S}_5}(\sigma)) = \frac{1}{2}(\mathbb{S}_5 : C_{\mathbb{S}_5}(\sigma)) = \frac{1}{2}|C|.$$

Thus C splits into two conjugacy classes of $\mathbb{A}_5$ of equal size.

The identity permutation is central. The even permutations $(12)(34)$ and (123) both commutes with some odd permutation in $\mathbb{S}_5$ (e.g. $[(12)(34),(34)] = [(123),(45)] = \mathrm{id}$). Thus these classes do not split in $\mathbb{A}_5$. There are twenty-four 5-cycles in $\mathbb{S}_5$. Since 24 does not divide $|\mathbb{A}_5| = 60$, it follows that the class of 5-cycles splits in $\mathbb{A}_5$. As representatives of these classes we can take (12345) and (12354).

Since $\mathbb{A}_5$ has five conjugacy classes, $|\operatorname{Irr}(G)| = 5$. We already know one irreducible character of G, namely the trivial character $\mathbf{1}_G$.

Let $H = \mathbb{A}_4$. We compute $\operatorname{Ind}_H^G \mathbf{1}_H$, where $\mathbf{1}_H$ is the trivial character of H. By Corollary 7.21,

$$\left(\operatorname{Ind}_H^G \mathbf{1}_H\right)(\mathrm{id}) = 5.$$

And a direct calculation shows

$$\begin{aligned}
\left(\operatorname{Ind}_H^G \mathbf{1}_H\right)((12)(34)) &= 1,\\
\left(\operatorname{Ind}_H^G \mathbf{1}_H\right)((123)) &= 2,\\
\left(\operatorname{Ind}_H^G \mathbf{1}_H\right)((12345)) &= 0\\
\left(\operatorname{Ind}_H^G \mathbf{1}_H\right)((12354)) &= 0.
\end{aligned}$$

By Frobenius' reciprocity,

$$\langle \operatorname{Ind}_H^G \mathbf{1}_H, \mathbf{1}_G\rangle = \langle \mathbf{1}_H, \operatorname{Res}_H^G \mathbf{1}_G\rangle = \langle \mathbf{1}_H, \mathbf{1}_H\rangle = 1.$$

Let $\chi_2 = \operatorname{Ind}_H^G \mathbf{1}_H - \mathbf{1}_G$. Since

$$\langle \operatorname{Ind}_H^G \mathbf{1}_H - \mathbf{1}_G, \operatorname{Ind}_H^G \mathbf{1}_H - \mathbf{1}_G\rangle = 1,$$

it follows that $\chi_2 \in \operatorname{Irr}(G)$.

10.1. Exercise. Use Proposition 6.1 to derive (once again) the values of χ_2.

So far we have the following table:

	id	(12)(34)	(123)	(12345)	(12354)
$\mathbf{1}_G$	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	n_3	·	·	·	·
χ_4	n_4	·	·	·	·
χ_5	n_5	·	·	·	·

As G is simple non-abelian, $|G/[G,G]| = 1$. It follows that $\mathbf{1}_G$ is the only linear character of G. Moreover, $\chi_j(1) \geq 3$ by Theorem 4.5. Since

$$60 = 1 + 16 + n_3^2 + n_4^2 + n_5^2$$

and each n_j divides $|G| = 60$ (see Theorem 4.3), it follows that $n_j \in \{3,4,5,6\}$. If some $n_j = 6$, say without loss of generality $n_3 = 6$, then

$$7 = 43 - 36 = n_2^2 + n_3^2,$$

a contradiction. Thus $n_j \in \{3,4,5\}$ for all $j \in \{3,4,5\}$. Without loss of generality, we may assume that $n_3 = n_4 = 3$ and $n_5 = 5$.

	id	(12)(34)	(123)	(12345)	(12354)
$\mathbf{1}_G$	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	3	·	·	·	·
χ_4	3	·	·	·	·
χ_5	5	·	·	·	·

The group $\mathbb{A}_5$ acts on the set Y of subsets of $\{1,2,\dots,5\}$ of two elements, namely

$$g\cdot\{a,b\}=\{g\cdot a,g\cdot b\}.$$

Note that $|Y|=\binom{5}{2}=10$. Moreover, this action is transitive. Let us compute the character ψ of the corresponding $\mathbb{C}\mathbb{A}_5$-module and the difference $\psi-\mathbf{1}_G$ (We know ψ counts fixed points.)

	id	(12)(34)	(123)	(12345)	(12354)
ψ	10	2	1	0	0
$\psi-\mathbf{1}_G$	9	1	0	-1	-1

The identity, of course, fixes all the ten elements of Y. The permutation (12)(34) fixed two two-elements subsets, namely $\{1,2\}$ and $\{3,4\}$. The permutation (123) fixes only one two-elements subset, namely $\{4,5\}$. Finally, (12345) and (12354) fix no two-element subsets.

Now we compute

$$\langle\psi-\mathbf{1}_G,\psi-\mathbf{1}_G\rangle=2$$

and hence $\psi-\mathbf{1}_G$ is the sum of two irreducible characters (see Exercise 3.7). Since

$$\langle\psi-\mathbf{1}_G,\chi_2\rangle=1,$$

it follows that $\psi-\mathbf{1}_G-\chi_2\in\operatorname{Irr}(G)$. Let $\chi_5=\psi-\mathbf{1}_G-\chi_2$. Then

	id	(12)(34)	(123)	(12345)	(12354)
$\mathbf{1}_G$	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	3	·	·	·	·
χ_4	3	·	·	·	·
χ_5	5	1	-1	0	0

Let $K=\langle(12345)\rangle$ and $\eta\in\operatorname{Irr}(K)$ be such that $\eta((12345))=\zeta$, where $\zeta=\exp(2\pi i/5)$ is a primitive 5-th root of one. We can then compute $\operatorname{Ind}_K^G\eta$.

	id	(12)(34)	(123)	(12345)	(12354)
$\operatorname{Ind}_K^G\eta$	12	0	0	$\zeta^2+\zeta^3$	$\zeta+\zeta^4$

Since $\langle\operatorname{Ind}_K^G\eta,\chi_2\rangle=1=\langle\operatorname{Ind}_H^G\eta,\chi_5\rangle$, it follows that

	id	(12)(34)	(123)	(12345)	(12354)
$\operatorname{Ind}_K^G\eta-\chi_2-\chi_5$	3	-1	0	$-\zeta-\zeta^4$	$-\zeta^2-\zeta^3$

Let $\chi_3=\operatorname{Ind}_K^G\eta-\chi_2-\chi_5$. Then $\chi_3\in\operatorname{Irr}(G)$, because it is not the sum of three copies of the trivial character. Thus this is how our character table looks like:

	id	(12)(34)	(123)	(12345)	(12354)
$\mathbf{1}_G$	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	3	-1	0	$-\zeta-\zeta^4$	$-\zeta^2-\zeta^3$
χ_4	3	·	·	·	·
χ_5	5	1	-1	0	0

10.2. Exercise. Use the orthogonality relations to compute the missing row of the character table of $\mathbb{A}_5$.

The previous exercise finishes the calculation of the character table of $\mathbb{A}_5$; see Table 8.

Table 8. The character table of $\mathbb{A}_5$.

	1	15	20	12	12
	id	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	3	-1	0	$-\zeta-\zeta^4$	$-\zeta^2-\zeta^3$
χ_4	3	-1	0	$-\zeta^2-\zeta^3$	$-\zeta-\zeta^4$
χ_5	5	1	-1	0	0

One last observation: Since $\zeta=\exp(2\pi i/5)$, it follows that

$$-\zeta-\zeta^4=\frac{1-\sqrt{5}}{2},\quad -\zeta^2-\zeta^3=\frac{1+\sqrt{5}}{2}.$$

§ 10.2. Mackey's theorem. We begin with two routine exercises.

10.3. Exercise. Let G be a finite group and N be a normal subgroup of G. Prove that G acts on $\operatorname{Irr}(N)$ via

$$(g\cdot\theta)(n)=\theta(g^{-1}ng),\quad g\in G,\theta\in\operatorname{Irr}(N),n\in N.$$

10.4. Exercise. Let G be a finite group and N be a normal subgroup of G. Let $\chi\in\operatorname{ClassFun}(G)$, $\theta\in\operatorname{ClassFun}(N)$ and $g\in G$. Prove that

$$\langle\operatorname{Res}_N^G\chi,g\cdot\theta\rangle=\langle\operatorname{Res}_N^G\chi,\theta\rangle.$$

10.5. Definition. Two representations of a finite group are said to be **disjoint** if they have no common irreducible constituent.

10.6. Exercise. Prove that two representations are disjoint if and only if their characters are orthogonal.

Let H and K be subgroups of a group G. The group $H\times K$ acts on G via $(h,k)\cdot g=hgk^{-1}$. The orbit of g under this action is the **double coset**

$$HgK=\{(h,k)\cdot g:h\in H,k\in K\}=\{hgk^{-1}:h\in H,k\in K\}$$

with representative g.

10.7. THEOREM (Mackey). *Let G be a finite group and H and K be subgroups of G. Let S be a complete set of representatives of double (H,K)-cosets. If $\alpha \in \mathrm{ClassFun}(K)$, then*

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G \alpha = \sum_{s\in S} \mathrm{Ind}_{H\cap sKs^{-1}}^H \mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(s\cdot f).$$

PROOF. For $s \in S$, let $X(s)$ be a left transversal for $H\cap sKs^{-1}$ on H. Then

$$H = \bigcup_{x\in X(s)} x(H\cap sKs^{-1}),$$

where the union is disjoint.

CLAIM. $HsK = \bigcup_{x\in X(s)} xsK$, where the union is disjoint.

Let $z \in HsK$. Then $z = hsk$ for some $h \in H$ and $k \in K$. Since $h \in x(H\cap sKs^{-1})$ for some $x \in X(s)$,

$$z = hsk \in x(H\cap sKs^{-1})sK \subseteq xsK.$$

Conversely, let $z \in xsK$ for some $x \in X(s) \subseteq H$. Then $z \in xsK \subseteq HsK$. To see that the union is disjoint, suppose that $xsK = x_1sK$ for some $x, x_1 \in X(s)$. Then $x_1^{-1}x \in sKs^{-1}\cap H$. Thus $x(sKs^{-1}\cap H) = x_1(sKs^{-1}\cap H)$ and hence $x = x_1$, because $X(s)$ is a left transversal for $sKs^{-1}\cap H$ in H.

Let $T(s) = \{xs : x \in X(s)\}$ and

$$T = \bigcup_{s\in S} T(s)$$

To see that the union is disjoint, we proceed as follows. Let $xs = x_1s_1$ for some $s, s_1 \in S$, $x \in X(s)$ and $x_1 \in X(s_1)$. Since $x^{-1}x_1 \in H$ and $HsK = Hx^{-1}x_1s_1K = Hs_1K$, $s = s_1$ and hence $x = x_1$.

Then

$$G = \bigcup_{s\in S} HsK = \bigcup_{s\in S}\bigcup_{x\in X(s)} xsK = \bigcup_{s\in S}\bigcup_{t\in T(s)} tK = \bigcup_{t\in T} tK.$$

Since the unions are disjoint, it follows that T is a left transversal of K in G.

For $h \in H$,

$$\begin{aligned}
(\operatorname{Ind}_K^G \alpha)(h) &= \sum_{t\in T} \alpha^0(t^{-1}ht)\\
&= \sum_{s\in S}\sum_{t\in T(s)} \alpha^0(t^{-1}ht)\\
&= \sum_{s\in S}\sum_{x\in X(s)} \alpha^0(s^{-1}x^{-1}hxs)\\
&= \sum_{s\in S}\sum_{\substack{x\in X(s)\\ x^{-1}hx\in sKs^{-1}}} (s\cdot\alpha)(x^{-1}hx)\\
&= \sum_{s\in S}\sum_{\substack{x\in X(s)\\ x^{-1}hx\in H\cap sKs^{-1}}} \operatorname{Res}^{sKs^{-1}}_{H\cap sKs^{-1}}(s\cdot\alpha)(\underbrace{x^{-1}hx}_{\in H\cap sKs^{-1}})\\
&= \sum_{s\in S}\operatorname{Ind}^H_{H\cap sKs^{-1}}\operatorname{Res}^{sKs^{-1}}_{H\cap sKs^{-1}}(s\cdot\alpha)(h).
\end{aligned}$$

$\square$

10.8. Theorem (Mackey's irreducibility criterion). *Let H be a subgroup of a finite group G and $\chi \in \operatorname{Char}(H)$. Then $\operatorname{Ind}_H^G \chi \in \operatorname{Irr}(G)$ if and only if $\chi \in \operatorname{Irr}(H)$ and $\operatorname{Res}^H_{H\cap sHs^{-1}}\chi$ and $\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi)$ are disjoint for all $s \notin H$.*

Proof. Let S be a complete set of representatives of (H,H)-double cosets. Without loss of generality, we may assume that $1 \in S$. Note that if $s = 1$, then $H \cap sHs^{-1} = H$ and $s\cdot\chi = \chi$. By Mackey's theorem,

$$\begin{aligned}
\operatorname{Res}^G_H\operatorname{Ind}^G_H\chi &= \sum_{s\in S}\operatorname{Ind}^H_{H\cap sHs^{-1}}\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi)\\
&= \chi + \sum_{1\neq s\in S}\operatorname{Ind}^H_{H\cap sHs^{-1}}\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi).
\end{aligned}$$

By Frobenius' reciprocity,

$$\begin{aligned}
\langle\operatorname{Ind}^G_H\chi,\operatorname{Ind}^G_H\chi\rangle &= \langle\operatorname{Res}^G_H\operatorname{Ind}^G_H\chi,\chi\rangle\\
&= \underbrace{\langle\chi,\chi\rangle}_{\geq 1} + \sum_{1\neq s\in S}\underbrace{\langle\operatorname{Ind}^H_{H\cap sHs^{-1}}\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi),\chi\rangle}_{\geq 0}.
\end{aligned}$$

If $\chi \in \operatorname{Irr}(H)$ and $\operatorname{Res}^H_{H\cap sHs^{-1}}\chi$ and $\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi)$ are disjoint for all $s \notin H$, then $\langle\operatorname{Ind}^G_H\chi,\operatorname{Ind}^G_H\chi\rangle = 1$ and hence $\operatorname{Ind}^G_H\chi \in \operatorname{Irr}(G)$.

Conversely, if $\operatorname{Ind}^G_H\chi \in \operatorname{Irr}(G)$, then $\langle\operatorname{Ind}^G_H\chi,\operatorname{Ind}^G_H\chi\rangle = 1$. Thus $\langle\chi,\chi\rangle = 1$ and

$$\langle\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi),\operatorname{Res}^H_{H\cap sHs^{-1}}\chi\rangle = \langle\operatorname{Ind}^H_{H\cap sHs^{-1}}\operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}}(s\cdot\chi),\chi\rangle = 0$$

for all $s \in S\setminus\{1\}$. As every element $s \notin H$ could serve as a representative of an (H,H)-double coset, the claim follows. $\square$

Theorem 10.8 takes a particularly elegant form when the subgroup is normal.

10.9. Exercise. Let H be a normal subgroup of a finite group G and $\chi \in \operatorname{Char}(H)$. Then $\operatorname{Ind}^G_H\chi \in \operatorname{Irr}(G)$ if and only if $\chi \in \operatorname{Irr}(H)$ and $\chi \neq s\cdot\chi$ for all $s \notin H$.

10.10. Example. For a prime number $p \geq 3$, let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : 0 \neq a \in \mathbb{Z}/p,\, b \in \mathbb{Z}/p \right\} \text{ and } H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}/p \right\}.$$

Then $|G| = p(p-1)$, H is a normal subgroup of G and $|G/H| = p-1$. Let

$$\chi\colon H \to \mathbb{C}^\times, \qquad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mapsto \exp(2\pi i b/p).$$

Then χ is a group homomorphism. For each $a \in \mathbb{Z}/p \setminus \{0,1\}$, let $s(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(s(a)\cdot\chi)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \chi\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} = \exp(2\pi i a^{-1}b/p) \neq \exp(2\pi i b/p).$$

Hence $s(a)\cdot\chi \neq \chi$ for all $a \in \mathbb{Z}/p \setminus \{0,1\}$. By Exercise 10.9, $\operatorname{Ind}_H^G \chi \in \operatorname{Irr}(G)$ and

$$\deg \operatorname{Ind}_H^G \chi = (\operatorname{Ind}_H^G \chi)(1) = (G:H)\chi(1) = p-1.$$

Since $|G| - (p-1)^2 = p-1$, we still need additional irreducible characters to fully determine $\operatorname{Irr}(G)$. The group G/H is cyclic of order $p-1$, so it has $p-1$ irreducible characters, all of degree one. These irreducible characters lift to irreducible characters of G (see Theorem 8.3).

10.11. Bonus exercise. Find the character table of the group of Example 10.10.

11. Lecture: Week 11

§ 11.1. Clifford theory. Recall that every character χ of a finite group is uniquely a sum of irreducible characters. These are called the **irreducible constituents** of χ. The set of irreducible constituents of χ is the set

$$\{\eta \in \mathrm{Irr}(G) : \langle \chi, \eta \rangle > 0\}.$$

Let G be a finite group, N be a normal subgroup and $\theta \in \mathrm{Irr}(N)$. The set

$$I_G(\theta) = \{g \in G : g \cdot \theta = \theta\}$$

is a subgroup of G and is called **inertia subgroup** of θ in G. Note that the inertia subgroup is the stabilizer of the action of G on characters by conjugation (see of Exercise 10.3). In particular, θ has $(G : I_G(\theta))$ conjugates.

11.1. Theorem (Clifford). *Let G be a finite group and N be a normal subgroup of G. Let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of $\mathrm{Res}_N^G \chi$. Then*

$$\mathrm{Res}_N^G \chi = e(\theta_1 + \cdots + \theta_t),$$

where $\theta = \theta_1, \dots, \theta_t$ are the conjugates of θ in G, and e is a positive integer. In particular, all the constituents of $\mathrm{Res}_N^G \chi$ have the same degree.

Proof. Let $G \cdot \theta = \{\theta_1, \dots, \theta_t\}$ be the orbit of θ. For each $i \in \{1, \dots, t\}$,

$$\langle \mathrm{Res}_N^G \chi, \theta_i \rangle = \langle \mathrm{Res}_N^G \chi, \theta \rangle > 0$$

since by assumption θ is an irreducible constituent of $\mathrm{Res}_N^G \chi$. Let $e = \langle \mathrm{Res}_N^G \chi, \theta \rangle$. Then

$$\mathrm{Res}_N^G \chi = e(\theta_1 + \cdots + \theta_t) + \eta$$

for some $\eta = 0$ or $\eta \in \mathrm{Char}(N)$, we prove that $\eta = 0$ to conclude the theorem. Since

$$e = \langle \mathrm{Res}_N^G \chi, \theta \rangle = \langle \mathrm{Res}_N^G \chi, \theta_j \rangle = \langle e(\theta_1 + \cdots + \theta_t) + \eta, \theta_j \rangle = e \sum_{i=1}^{t} \langle \theta_i, \theta_j \rangle + \langle \eta, \theta_j \rangle = e + \langle \eta, \theta_j \rangle,$$

it follows that $\langle \eta, \theta_j \rangle = 0$ for all $j \in \{1, \dots, t\}$. By Frobenius' reciprocity,

$$\langle \chi, \mathrm{Ind}_N^G \theta \rangle = \langle \mathrm{Res}_N^G \chi, \theta \rangle = e.$$

Thus $\mathrm{Ind}_N^G \theta = e\chi + \lambda$ for some $\lambda = 0$ or $\lambda \in \mathrm{Char}(G)$. Since

$$e = \langle \chi, \mathrm{Ind}_N^G \theta \rangle = \langle \chi, e\chi + \lambda \rangle = e\langle \chi, \chi \rangle + \langle \chi, \lambda \rangle = e + \langle \chi, \lambda \rangle,$$

it follows that $\langle \chi, \lambda \rangle = 0$.

Claim. $\mathrm{Res}_N^G \mathrm{Ind}_N^G \theta = (I_G(G) : N) \sum_{i=1}^{t} \theta_i$.

Let $n \in N$. For $i \in \{1,\dots,t\}$ let $x_i \in G$ be such that $x_i \cdot \theta = \theta_i$. Then $x_1,\dots,x_t$ is a transversal of $I_G(\theta)$ in G. In particular, $G = \bigcup_{i=1}^t x_i I_G(\theta)$. Now

$$\begin{aligned}
(\operatorname{Ind}_N^G \theta)(n) &= \frac{1}{|N|}\sum_{x\in G} \theta^0(x^{-1}nx)\\
&= \frac{1}{|N|}\sum_{x\in G}(x\cdot\theta)(n)\\
&= \frac{1}{|N|}\sum_{i=1}^t \sum_{g\in I_G(\theta)} (x_i\cdot(g\cdot\theta))(n)\\
&= \frac{1}{|N|}\sum_{i=1}^t |I_G(G)|(x_i\cdot\theta)(n)\\
&= (I_G(G):N)\sum_{i=1}^t \theta_i(n),
\end{aligned}$$

where we have used that $n \in N$ and N is normal in G (because $x^{-1}nx \in N$ if and only if $n \in xNx^{-1} = N$).

Therefore

$$\begin{aligned}
(I_G(G):N)(\theta_1+\cdots+\theta_t) &= \operatorname{Res}_N^G \operatorname{Ind}_N^G \theta\\
&= \operatorname{Res}_N^G(e\chi+\lambda)\\
&= e\operatorname{Res}_N^G \chi + \operatorname{Res}_N^G \lambda\\
&= e^2(\theta_1+\cdots+\theta_t) + e\eta + \operatorname{Res}_N^G \lambda.
\end{aligned}$$

Taking inner product against η,

$$(I_G(G):N)\sum_{i=1}^t \langle\theta_i,\eta\rangle = e^2\sum_{i=1}^t\langle\theta_i,\eta\rangle + e\langle\eta,\eta\rangle + \langle\operatorname{Res}_N^G\lambda,\eta\rangle.$$

Since $\langle\theta_i,\eta\rangle = 0$ for all $i \in \{1,\dots,t\}$,

$$0 = e\langle\eta,\eta\rangle + \langle\operatorname{Res}_N^G\lambda,\eta\rangle. \tag{11.1}$$

We know that $e > 0$. Moreover, each term of the right hand side of (11.1) is non-negative, that is $\langle\eta,\eta\rangle \geq 0$ and $\langle\operatorname{Res}_N^G\lambda,\eta\rangle \geq 0$. Therefore $\langle\eta,\eta\rangle = 0$ and hence $\eta = 0$. □

The integer e in Theorem 11.1 is known as the **ramification index** of χ on N. In general, the number e is not easy to control.

11.2. Exercise. Let G be a finite group and N be a normal subgroup of G. Let $\chi \in \operatorname{Irr}(G)$ and θ be an irreducible constituent of $\operatorname{Res}_N^G \chi$. Prove that $\theta(1)$ divides $\chi(1)$.

11.3. Theorem (Clifford correspondence). *Let G be a finite group and N be a normal subgroup of G. Let $\theta \in \operatorname{Irr}(N)$ and $I = I_G(\theta)$. Then the map*

$$\{\psi \in \operatorname{Irr}(I) : \langle \operatorname{Res}_N^I \psi, \theta\rangle > 0\} \to \{\chi \in \operatorname{Irr}(G) : \langle \operatorname{Res}_N^G \chi,\theta\rangle > 0\},\quad \psi \mapsto \operatorname{Ind}_I^G \psi,$$

is bijective. Moreover, if ψ is a constituent of $\operatorname{Res}_N^I \theta$, then $\langle \operatorname{Res}_N^I \psi,\theta\rangle = \langle \operatorname{Res}_N^G \chi,\theta\rangle$.

Proof. There are several things to prove.

Claim. The map is $\psi \mapsto \operatorname{Ind}_I^G \psi$ well-defined.

Let $\psi \in \operatorname{Irr}(I)$ be such that $e = \langle \operatorname{Res}_N^I \psi, \theta \rangle > 0$ and let $\chi \in \operatorname{Irr}(G)$ be a constituent of $\operatorname{Ind}_I^G \psi$. By Frobenius' reciprocity,

$$\langle \psi, \operatorname{Res}_I^G \chi \rangle = \langle \operatorname{Ind}_I^G \psi, \chi \rangle > 0. \tag{11.2}$$

Thus ψ is a constituent of $\operatorname{Res}_I^G \chi$, that is

$$\operatorname{Res}_I^G \chi = \psi + \lambda$$

for some $\lambda = 0$ or $\lambda \in \operatorname{Char}(I)$. Thus

$$\operatorname{Res}_N^G \chi = \operatorname{Res}_N^I \operatorname{Res}_I^G \chi = \operatorname{Res}_N^I (\psi + \lambda) = \operatorname{Res}_N^I \psi + \operatorname{Res}_N^I \lambda,$$

that is $\operatorname{Res}_N^I \psi$ is part of $\operatorname{Res}_N^G \chi$. As χ is a constituent of $\operatorname{Ind}_I^G \psi$,

$$\chi(1) \leq (\operatorname{Ind}_I^G \psi)(1) = (G : I)\psi(1).$$

Let $f = \langle \operatorname{Res}_N^G \chi, \theta \rangle$. Then

$$f = \langle \operatorname{Res}_N^G \chi, \theta \rangle \geq \langle \operatorname{Res}_N^I \psi, \theta \rangle = e > 0.$$

Since $\operatorname{Res}_N^G \chi = f(\theta_1 + \cdots + \theta_t)$, where $G \cdot \theta = \{\theta_1, \dots, \theta_t\}$ is the orbit of θ under the action of G and $t = (G : I)$,

$$ft\theta(1) = (\operatorname{Res}_N^G \chi)(1).$$

Using the result from above we conclude:

$$ft\theta(1) = (\operatorname{Res}_N^G \chi)(1) = \chi(1) \leq (\operatorname{Ind}_I^G \psi)(1) = t\psi(1) = et\theta(1) \leq ft\theta(1), \tag{11.3}$$

where the last equality follows since $\operatorname{Res}_N^I \psi = e\theta$ by Clifford's theorem. Therefore $e = f$. Moreover, Equality (11.3) implies that $(\operatorname{Ind}_I^G \psi)(1) = \chi(1)$. Equality (11.2) implies that $\operatorname{Ind}_I^G \psi = \chi + \xi$ for some map $\xi \colon G \to \mathbb{C}$ that is either zero or a character of G. Since $\operatorname{Ind}_I^G \psi$ and χ have the same degree, it follows that $\operatorname{Ind}_I^G \psi = \chi$.

Claim. The map is $\psi \mapsto \operatorname{Ind}_I^G \psi$ is injective.

Let $\psi_1 \in \operatorname{Irr}(I)$ and $\psi_2 \in \operatorname{Irr}(I)$ be such that $\langle \operatorname{Res}_N^I \psi_i, \theta \rangle > 0$ for all $i \in \{1, 2\}$ and $\chi = \operatorname{Ind}_I^G \psi_1 = \operatorname{Ind}_I^G \psi_2$. In the first claim, we proved that $\chi \in \operatorname{Irr}(G)$. We want to prove that $\psi_1 = \psi_2$.

Suppose $\psi_1 \neq \psi_2$. We know that ψ_1 and ψ_2 from the first claim that are constituents of $\operatorname{Res}_I^G \chi$, that is

$$\operatorname{Res}_I^G \chi = \psi_1 + \psi_2 + \xi$$

for some map $\xi \colon I \to \mathbb{C}$. (The map ξ is either zero or a character of I.) Then both $\operatorname{Res}_N^I \psi_1$ and $\operatorname{Res}_N^I \psi_2$ are part of $\operatorname{Res}_N^G \chi$, as

$$\begin{aligned} \operatorname{Res}_N^G \chi &= \operatorname{Res}_N^I \operatorname{Res}_I^G \chi \\ &= \operatorname{Res}_N^I (\psi_1 + \psi_2 + \xi) \\ &= \operatorname{Res}_N^I \psi_1 + \operatorname{Res}_N^I \psi_2 + \operatorname{Res}_N^I \xi. \end{aligned}$$

Moreover,

$$\begin{aligned}\langle \operatorname{Res}_N^G \chi, \theta\rangle &= \langle \operatorname{Res}_N^I \psi_1 + \operatorname{Res}_N^I \psi_2 + \operatorname{Res}_N^I \xi, \theta\rangle \\ &= \langle \operatorname{Res}_N^I \psi_1, \theta\rangle + \langle \operatorname{Res}_N^I \psi_2, \theta\rangle + \langle \operatorname{Res}_N^I \xi, \theta\rangle \\ &\geq \langle \operatorname{Res}_N^I \psi_1, \theta\rangle + \langle \operatorname{Res}_N^I \psi_2, \theta\rangle \\ &= \langle \operatorname{Res}_N^G \chi, \theta\rangle + \langle \operatorname{Res}_N^G \chi, \theta\rangle,\end{aligned}$$

where the last equality holds because we proved in the previous claim that

$$\langle \operatorname{Res}_N^I \psi_i, \theta\rangle = \langle \operatorname{Res}_N^G \chi, \theta\rangle$$

for all $i \in \{1, 2\}$. This implies that $\langle \operatorname{Res}_N^G \chi, \theta\rangle = 0$, a contradiction.

Claim. The map is $\psi \mapsto \operatorname{Ind}_I^G \psi$ is surjective.

Let $\chi \in \operatorname{Irr}(G)$ be such that $e = \langle \operatorname{Res}_N^G \chi, \theta\rangle > 0$. Since

$$\operatorname{Res}_I^G \chi = \sum_{\psi \in \operatorname{Irr}(I)} \langle \operatorname{Res}_I^G \chi, \psi\rangle \psi,$$

it follows that

$$\operatorname{Res}_N^G \chi = \operatorname{Res}_N^I \operatorname{Res}_I^G \chi = \sum_{\psi \in \operatorname{Irr}(I)} \langle \operatorname{Res}_I^G \chi, \psi\rangle \operatorname{Res}_N^I \psi.$$

Since

$$e = \langle \operatorname{Res}_N^G \chi, \theta\rangle = \sum_{\psi \in \operatorname{Irr}(I)} \langle \operatorname{Res}_I^G \chi, \psi\rangle \langle \operatorname{Res}_N^I \psi, \theta\rangle$$

is a positive number, there exists some $\psi \in \operatorname{Irr}(I)$ such that $\langle \operatorname{Res}_I^G \chi, \psi\rangle \langle \operatorname{Res}_N^I \psi, \theta\rangle > 0$. In particular, $\langle \operatorname{Res}_N^I \psi, \theta\rangle > 0$ and $\langle \chi, \operatorname{Ind}_I^G \psi\rangle = \langle \operatorname{Res}_I^G \chi, \psi\rangle > 0$. Hence $\chi = \operatorname{Ind}_I^G \psi$. □

§ 11.2. Itô's theorem. We now present a result that is stronger than Schur's Theorem 4.6. To that end, we introduce some exercises on basic properties of the center of characters.

11.4. Definition. Let G be a finite group and $\chi \in \operatorname{Char}(G)$. The **center** of χ is

$$Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}.$$

11.5. Exercise. Let G be a finite group and $\rho\colon G \to \mathbf{GL}_n(\mathbb{C})$ be a representation with character χ. Prove the following statements:

1) $Z(\chi) = \{g \in G : \rho_g \text{ is a multiple of the identity matrix}\}$.
2) $Z(\chi)$ is a normal subgroup of G.
3) $Z(\chi)/\ker\chi$ is cyclic.

11.6. Exercise. Let G be a finite group and $\chi \in \operatorname{Irr}(G)$. Prove that

$$Z(\chi)/\ker\chi = Z(G/\ker\chi).$$

11.7. Exercise. Let G be a finite group. Prove that

$$Z(G) = \bigcap \{Z(\chi) : \chi \in \operatorname{Irr}(G)\}.$$

The previous exercise shows that the center of a finite group can be determined from its character table. It follows that the character table detects nilpotency. To do this, one computes $Z(G)$ from the character table of G, then the character table of $G/Z(G)$, and by iterating this process, one obtains the upper central series of the group G.

11.8. LEMMA. *Let G be a finite group and $\chi \in \operatorname{Irr}(G)$. Then $\chi(1)$ divides $(G : Z(\chi))$.*

PROOF. Let $Q = G/\ker\chi$. By Theorem 8.3, χ corresponds to $\eta \in \operatorname{Irr}(Q)$. By Schur's theorem 4.6, $\chi(1) = \eta(1)$ divides $(Q : Z(Q))$. By Exercise 11.6, $(Q : Z(Q)) = (G : Z(\chi))$. $\square$

11.9. THEOREM (Itô). *Let G be a finite group and $\chi \in \operatorname{Irr}(G)$. Then $\chi(1)$ divides $(G : A)$ for all normal abelian subgroup A of G.*

PROOF. Let A be a normal abelian subgroup of G and $\theta \in \operatorname{Irr}(A)$ be an irreducible constituent of $\operatorname{Res}^G_A \chi$, that is $\langle \operatorname{Res}^G_A \chi, \theta\rangle > 0$. Let $I = I_G(\theta)$. By Clifford correspondence (Theorem 11.3), $\chi = \operatorname{Ind}^G_I(\psi)$ for some $\psi \in \operatorname{Irr}(I)$ such that $\langle \operatorname{Res}^I_A \psi, \theta\rangle > 0$. By Clifford's theorem, since I acts trivially on θ, $\operatorname{Res}^I_A \psi = e\theta$, where $e = \langle \operatorname{Res}^I_A \psi, \theta\rangle > 0$. Since A is abelian and $\theta \in \operatorname{Irr}(A)$, $\theta(1) = 1$.

We claim that $A \subseteq Z(\psi)$. In fact, if $a \in A$, then

$$|\psi(a)| = |\operatorname{Res}^I_A \psi(a)| = |e\theta(a)| = e|\theta(a)| = e1 = e = \psi(1).$$

By Lagrange's theorem, $|A|$ divides $|Z(\psi)|$. Thus $(I : Z(\psi))$ divides $(I : A)$.

By Lemma 11.8, $\psi(1)$ divides $(I : Z(\psi))$. Then $\psi(1)$ divides $(I : A)$. Now

$$\chi(1) = (\operatorname{Ind}^G_I \psi)(1) = (G : I)\psi(1)$$

divides $(G : I)(I : A) = (G : A)$. $\square$

11.10. BONUS EXERCISE. Prove that Itô's theorem remains valid under the assumption that A is subnormal in G.

12. Lecture: Week 12

§ 12.1. Kronecker's theorem. We begin with a classical theorem of Kronecker on algebraic integers. Recall that $\alpha \in \mathbb{C}$ is an **algebraic integer** if there is a monic polynomial $f \in \mathbb{Z}[X]$ such that $f(\alpha) = 0$ (see Definition 3.12). Let $\mathbb{A}$ be the set of algebraic integers.

12.1. Exercise. Let $\alpha \in \mathbb{A}$. Prove that there exists a monic polynomial $f \in \mathbb{Z}[X]$, irreducible $f \in \mathbb{Q}[X]$ such that $f(\alpha) = 0$.

The polynomial of Exercise 12.1 is called the **minimal polynomial** of α.

12.2. Exercise. Let $\alpha \in \mathbb{A}$. Prove that the roots of the minimal polynomial of α are pairwise distinct.

The **conjugates** of α are the roots of the minimal polynomial of α.

Recall that for an $n \times n$ matrix $A = (a_{ij})$, its **norm** (more precisely, **infinity-norm**) is defined as the maximum absolute row sum of the matrix, that is

$$\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

For $A, B \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$, the following properties hold:

1) $\|A\| \geq 0$.
2) $\|A\| = 0$ if and only if A is the $n \times n$ zero matrix.
3) $\|\lambda A\| = |\lambda|\|A\|$.
4) $\|A + B\| \leq \|A\| + \|B\|$.
5) $\|AB\| \leq \|A\|\|B\|$.

For our purposes, the choice of norm is not important at all (and any other norm could have been chosen). Nevertheless, we provide an example. Let

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & -7 \\ 11 & 2 & -3 \end{pmatrix}.$$

Then $\|A\| = \max\{6, 12, 16\} = 16$.

12.3. Theorem (Kronecker). *Let $\alpha \in \mathbb{A}$. Assume that all the conjugates of α have absolute value at most one. Then either $\alpha = 0$ or α is a root of one.*

Proof. Assume that $\alpha \neq 0$. Let $f \in \mathbb{Z}[X]$ be the minimal polynomial of α, say

$$f = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$$

for integers $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$. Then $f(0) \neq 0$ because f is irreducible in $\mathbb{Q}[X]$ (see Exercise 12.1). Let

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in \mathbb{Z}^{n\times n}$$

be the **companion matrix** of f. The characteristic polynomial and the minimal polynomial of the matrix F are equal to f. Moreover, the roots of f are the eigenvalues of F. Since all the roots of f are distinct, all the eigenvalues of F are different. Thus F is diagonalizable, so there exists $P \in \mathbf{GL}_n(\mathbb{C})$ such that $F = PDP^{-1}$, where D is the $n \times n$ diagonal matrix with diagonal entries $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n$, the roots of f (i.e., the conjugates of α), so all with absolute value at most one. Thus $\|D\| \leq 1$. Since $0 \notin \{\alpha_1, \dots, \alpha_n\}$, the matrix F is invertible. Moreover,

$$F^k = (PDP^{-1})^k = PD^kP^{-1}$$

for all $k \geq 1$. Note that the set $X = \{F^k : k \geq 1\} \subseteq M_n(\mathbb{Z})$ is bounded in $M_n(\mathbb{C})$, as

$$\|F^k\| = \|PD^kP^{-1}\| \leq \|P\|\|D\|^k\|P^{-1}\| \leq \underbrace{\|P\|\|P^{-1}\|}_{\text{This is independent of } k}.$$

Thus X is finite. In particular, there are integers $i < j$ such that $F^i = F^j$. Since F is invertible, F^{j-i} is the $n \times n$ identity matrix. Since α is an eigenvalue of F, it follows that $\alpha^{j-i} = 1$. □

The proof of the theorem presented here goes back to Greiter [13]. Kronecker's original proof is somewhat similar, relying on Vieta's formulas and estimates involving binomial coefficients; see [7].

§ 12.2. Solvable groups and Burnside's theorem. For a group G let $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. The **derived series** of G is the sequence

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

Each $G^{(i)}$ is a characteristic subgroup of G. We say that G is **solvable** if $G^{(n)} = \{1\}$ for some n.

12.4. Example. Abelian groups are solvable.

12.5. Example. The group $\mathbf{SL}_2(3)$ is solvable. Let us see what the computer says:

```
> G := SL(2,3);;
> IsSolvable(G);
true
> [GroupName(x) : x in DerivedSeries(G)];
[ SL(2,3), Q8, C2, C1 ]
```

12.6. Example. Non-abelian simple groups cannot be solvable.

For $n \geq 5$, the group $\mathbb{A}_n$ is not solvable.

12.7. Exercise. Let G be a group. Prove the following statements:

1) A subgroup H of G is solvable, when G is solvable.
2) Let K be a normal subgroup of G. Then G is solvable if and only if K and G/K are solvable.

For $n \geq 5$, the group $\mathbb{S}_5$ is not solvable.

12.8. Exercise. Let p be a prime number. Prove that finite p-groups are solvable.

Exercises 12.7 and 12.8 may be omitted if the reader is already familiar with solvable groups.

12.9. Exercise. Prove that the conjugates of a number of the form $\alpha+\beta$ are of the form $\alpha_1+\beta_1$, where α_1 and β_1 are conjugate of α and β, respectively.

For this exercise, one can use either Galois theory or the ideas used in the proof of Theorem 3.16.

12.10. Theorem (Burnside). *Let G be a finite group. If $\phi\colon G\to\mathbf{GL}_n(\mathbb{C})$ is a representation with character χ and C is a conjugacy class of G such that $\gcd(|C|,n)=1$, then for every $g\in C$ either $\chi(g)=0$ or ϕ_g is a scalar matrix.*

Proof. Let $\epsilon_1,\dots,\epsilon_n$ be the eigenvalues of ϕ_g. Then $\epsilon_1,\dots,\epsilon_n$ are roots of one. Since $\gcd(|C|,n)=1$, there exist $a,b\in\mathbb{Z}$ such that $a|C|+bn=1$. As $|C|\chi(g)/n\in\mathbb{A}$, after multiplying by $\chi(g)/n$, we obtain that

$$a|C|\frac{\chi(g)}{n}+b\chi(g)=\frac{\chi(g)}{n}=\frac{1}{n}(\epsilon_1+\cdots+\epsilon_n)\in\mathbb{A}.$$

Let $\alpha_1=\chi(g)/n\in\mathbb{A}$ and $\alpha_2,\dots,\alpha_m$ be its conjugates. Since $|\alpha_1|\leq 1$ and $\alpha_2,\dots,\alpha_m$ are conjugates of α_1, it follows that $|\alpha_j|\leq 1$ for all $j\in\{1,\dots,n\}$. Why? (See Exercise 12.9.)

By Kronecker's theorem, either $\alpha_1=0$ or α_1 is a root of one. If $\alpha_1=0$, then $\chi(g)=0$. If α_1 is a root of one, then

$$1=|\alpha_1|=\frac{|\chi(g)|}{n}.$$

Thus $|\chi(g)|=n=\chi(1)$. This means that $g\in\mathbb{Z}(\chi)$. By Exercise 11.5, ϕ_g is a scalar matrix. □

12.11. Theorem (Burnside). *Let p be a prime number. If G is a finite group and C is a conjugacy class of G with $p^k>1$ elements, then G is not simple.*

Proof. Let $g\in C\setminus\{1\}$. Schur's second orthogonality relation (Theorem 3.8) implies that

$$\begin{aligned}0&=\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)\chi(g)\\&=\sum_{p\mid\chi(1)}\chi(1)\chi(g)+\sum_{p\nmid\chi(1):\chi\neq\mathbf{1}_G}\chi(1)\chi(g)+1,\end{aligned}\tag{12.1}$$

where $\mathbf{1}_G$ denotes the trivial representation of G.

If $\chi(g)=0$ for all $\chi\in\mathrm{Irr}(G)$, looking at the equality modulo p, we get $0\equiv 1 \bmod p$, a contradiction. Thus there exists an irreducible non-trivial representation ϕ with character χ such that p does not divide $\chi(1)$ and $\chi(g)\neq 0$. By the previous theorem, ϕ_g is a scalar matrix.

Since G is simple, there are two cases to consider: either $\ker\phi=\{1\}$ or $\ker\phi=G$. If $\ker\phi=\{1\}$, then g is a non-trivial central element, a contradiction since $|C|>1$. If $\ker\phi=G$, then ϕ_g is the identity matrix for all $g\in G$. In particular, $\chi(g)=\operatorname{trace}\phi_g=n$. Then

$$\langle\chi,\mathbf{1}_G\rangle=\frac{1}{|G|}\sum_{g\in G}\chi(g)=n.$$

Hence $\chi = n\mathbf{1}_G$, a contradiction. □

12.12. THEOREM (Burnside). *Let p and q be prime numbers. If G has order $p^a q^b$, then G is solvable.*

PROOF. If G is abelian, then it is solvable. Suppose now G is non-abelian. Let us assume that the theorem is not true. Let G be a group of minimal order $p^a q^b$ that is not solvable. Then $a \geq 1$ and $b \geq 1$. Since $|G|$ is minimal, G is a non-abelian simple group. By the previous theorem, G has no conjugacy classes of size p^k nor conjugacy classes of size q^l with $k, l \geq 1$. The size of every conjugacy class of G is one or divisible by pq. Note that, since G is a non-abelian simple group, the center of G is trivial. Thus there is only one conjugacy class of size one. By the class equation,

$$|G| = 1 + \sum_{C:|C|>1} |C| \equiv 1 \bmod pq,$$

where the sum is taken over all conjugacy classes of G with more than one element, a contradiction. □

There are several interesting theorems that generalize Burnside's theorem. We briefly mention two of them.

12.13. THEOREM (Kegel–Wielandt). *If G is a finite group and there are nilpotent subgroups A and B of G such that $G = AB$, then G is solvable.*

See [**3**, Theorem 2.4.3] for the proof.

12.14. EXERCISE. Prove that Theorem 12.13 implies Theorem 12.12.

There is an interesting variation on the Kegel–Wielandt theorem, but with an extra condition on the order of the finite group.

12.15. THEOREM (Syskin). *Let G be a finite group of order coprime with three. If $G = AB$ for solvable subgroups A and B, then G is solvable.*

For the proof, see [**39**].

Another generalization of Burnside's theorem is based on word maps.

A **word map** of a group G is a map

$$G^k \to G, \quad (x_1, \dots, x_k) \mapsto w(x_1, \dots, x_k)$$

for some word $w(x_1, \dots, x_k)$ of the free group F_k of rank k. Some word maps are surjective in certain families of groups. For example, Ore's conjecture is precisely the surjectivity of the word map $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ in every finite non-abelian simple group.

12.16. Theorem (Guralnick–Liebeck–O'Brien–Shalev–Tiep). *Let $a, b \geq 0$, p and q be prime numbers and $N = p^a q^b$. The map $(x, y) \mapsto x^N y^N$ is surjective in every finite simple group.*

The proof appears in [**15**].

12.17. Exercise. Prove that Theorem 12.16 implies Theorem 12.12.

§ 12.3. The Feit–Thompson theorem.

12.18. Theorem (Feit–Thompson). *Groups of odd order are solvable.*

The proof of Feit–Thompson theorem is extremely hard. It occupies a full volume of the **Pacific Journal of Mathematics** [**9**]. A formal verification of the proof (based on the computer software `Coq`) was announced in [**12**].

Back in the day it was believed that if a certain divisibility conjecture is true, the proof of Feit–Thompson theorem could be simplified.

12.19. Conjecture (Feit–Thompson). There are no prime numbers p and q such that $\frac{p^q-1}{p-1}$ divides $\frac{q^p-1}{q-1}$.

The conjecture remains open. However, now we know that proving the conjecture will not simplify further the proof of Feit–Thompson theorem.

In 2012 Le proved that the conjecture is true for $q = 3$, see [**29**].

In [**38**] Stephens proved that a certain stronger version of the conjecture does not hold, as the integers $\frac{p^q-1}{p-1}$ and $\frac{q^p-1}{q-1}$ could have common factors. In fact, if $p = 17$ and $q = 3313$, then

$$\gcd\left(\frac{p^q - 1}{p - 1}, \frac{q^p - 1}{q - 1}\right) = 112643.$$

Nowadays, this can be easily checked on almost any desktop computer. As far as I know, no other counterexamples to Stephen's stronger version of the conjecture have been found.

13. Project: Irreducible characters of dihedral groups

Let $n \geq 3$. Recall that the **dihedral group** of order $2n$ is the group

$$\mathbb{D}_n = \langle r, s : r^n = s^2 = 1, srs = r^{-1} \rangle.$$

Every element of $\mathbb{D}_n$ is of the form $s^i r^j$ for some $i \in \{0,1\}$ and $j \in \{0, \dots, n-1\}$.

Our goal is the construct the character table of $\mathbb{D}_n$.

13.1. Proposition. *Let $n \geq 3$. If n is odd, then*

$$\{1\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}, \cdots, \{r^{(n-1)/2}, r^{(1-n)/2}\}, \{s, sr, sr^2, \dots, sr^{n-1}\}$$

are the conjugacy classes of $\mathbb{D}_n$. If n is even, then

$$\{1\}, \{r, r^{-1}\}, \{r^2, r^{-2}\}, \cdots, \{r^{n/2-1}, r^{1-n/2}\},$$
$$\{r^{n/2}\}, \{s, sr^2, sr^4, \dots, sr^{n-2}\}, \{sr, sr^3, \dots, sr^{n-1}\}$$

are the conjugacy classes of $\mathbb{D}_n$.

Proof. Recall that $sr^j = r^{-j}s$ for all j. Let $g = s^i r^j \in \mathbb{D}_n$ and $x = s^k r^l \in \mathbb{D}_n$. Let us compute xgx^{-1}. We split the proof into several steps.

Assume first that $i = 0$, that is $g = r^j$. Then

$$xgx^{-1} = (s^k r^l) r^j (r^{-l} s^{-k}) = s^k r^j s^{-k} = \begin{cases} r^j & \text{if } k = 0, \\ r^{-j} & \text{if } k = 1. \end{cases}$$

Hence the conjugacy class of $g = r^j$ is $\{r^j, r^{-j}\}$.

Now assume that $i = 1$, that is $g = sr^j$. Since $k \in \{0,1\}$, a direct calculation using the fact that $r^l s = sr^{-l}$ yields

$$xgx^{-1} = \begin{cases} sr^{-2l+j} & \text{if } k = 0, \\ sr^{2l-j} & \text{if } k = 1. \end{cases}$$

Hence the conjugacy class of $g = sr^j$ is $\{sr^{2l-j}, sr^{-2l+j} : 0 \leq l \leq n-1\}$.

Assume that n is odd. We have determined the conjugacy classes

$$\{1\}, \{b, b^{-1}\}, \{b^2, b^{-2}\}, \dots, \{b^{(n-1)/2}, b^{(1-n)/2}\}$$

which together cover all the elements of the subgroup $\langle b \rangle = \{1, b, b^2, \dots, b^{n-1}\}$. Since n is odd, for every integer m there exists an integer x such that $2x \equiv m \bmod n$. Thus the conjugacy class of s is $\{s, sr, sr^2, \dots, sr^{n-1}\}$. These classes together cover all the elements of $\mathbb{D}_n$.

Now assume that n is even. We have determined the conjugacy classes

$$\{1\}, \{b, b^{-1}\}, \{b^2, b^{-2}\}, \dots, \{b^{n/2-1}, b^{1-n/2}\}, \{b^{n/2}\}$$

which together cover all the elements of the subgroup $\langle b \rangle = \{1, b, b^2, \dots, b^{n-1}\}$. The class of s is $\{s, sr^2, sr^4, \dots, sr^{n-2}\}$ and the class of sr is $\{sr, sr^3, \dots, sr^{n-1}\}$. These classes together cover all the elements of $\mathbb{D}_n$. $\square$

The previous proposition gives the number of conjugacy classes of the dihedral group $\mathbb{D}_n$, namely

$$\frac{2n + 9 + (-1)^n 3}{4} = \begin{cases} \frac{n+6}{2} & \text{if } n \text{ is even,} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

This number is precisely the number of irreducible representations of $\mathbb{D}_n$.

13.2. Exercise. Compute $Z(\mathbb{D}_n)$.

13.3. Exercise. Prove that $\lim_{n\to\infty} \operatorname{cp}(\mathbb{D}_n) = 1/4$.

To determine the number of degree-one representations of our group, we need the index of the commutator subgroup.

13.4. Exercise. Prove that $[\mathbb{D}_n, \mathbb{D}_n] = \langle r^2 \rangle$. Moreover,

$$(G : [G,G]) = \begin{cases} 2 & \text{if } n \text{ is odd.} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

§ 13.1. n odd. By Proposition 13.1, the representatives of the conjugacy classes of $\mathbb{D}_n$ are $1, r, r^2, \dots, r^{(n-1)/2}, s$. By Exercise 13.4, there are two degree-one characters, namely the trivial character and the character η such that $r \mapsto 1$ and $s \mapsto -1$.

	1	r	r^2	$\cdots$	$r^{(n-1)/2}$	s
trivial	1	1	1	$\cdots$	1	1
η	1	1	1	$\cdots$	1	-1

Assume that $n = 2k - 1$. We need $\frac{n-1}{2} = k - 1$ additional irreducible characters. For $m \in \{1, \dots, k-1\}$, let $\omega_m = e^{2\pi i m/k}$ and

$$\rho_m \colon \mathbb{D}_n \to \mathbf{GL}_2(\mathbb{C}), \quad r \mapsto \begin{pmatrix} \omega_m & 0 \\ 0 & \omega_m^{-1} \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

13.5. Exercise. Prove that each ρ_m is a group homomorphism.

13.6. Exercise. Let $G = \mathbb{D}_n$ and $H = \langle r \rangle$. For each $m \in \{1, \dots, k-1\}$, let

$$\chi_m \colon H \to \mathbb{C}^\times, \quad r \mapsto \omega_m.$$

Prove that $\operatorname{Ind}_H^G \chi_m = \rho_m$.

A direct calculation produces the values of the character χ_m of ρ_m.

	1	r	r^2	$\cdots$	$r^{(n-1)/2}$	s
χ_m	1	$\omega_m + \omega_m^{-1}$	$\omega_m^2 + \omega_m^{-2}$	$\cdots$	$\omega_m^{(n-1)/2} + \omega_m^{(1-n)/2}$	0

13.7. Exercise. Let $i, j \in \{1, \dots, k-1\}$. Prove that $\chi_i \neq \chi_j$ whenever $i \neq j$.

13.8. Exercise. Prove that each χ_m is irreducible.

It remains only to note that we have constructed $\frac{n+3}{2}$ irreducible characters of $\mathbb{D}_n$, so the character table of $\mathbb{D}_n$ for odd n is complete!

§ 13.2. n even. In this case, by Exercise 13.4, there are four degree-one representations. These are the group homomorphisms defined as follows: For $j \in \{1,2,3,4\}$, let $\eta_j \colon \mathbb{D}_n \to \mathbb{C}$ be given by

$$\begin{aligned} &\eta_1(r) = 1, && \eta_2(r) = 1, && \eta_3(r) = -1, && \eta_4(r) = -1, \\ &\eta_1(s) = 1, && \eta_2(s) = -1, && \eta_3(s) = 1, && \eta_4(s) = -1. \end{aligned}$$

Of course, η_1 is the trivial character of $\mathbb{D}_n$. By a direct calculation, we compute the values of the other characters:

	1	r	r^2	$\cdots$	$r^{n/2}$	s	sr
η_1	1	1	1	$\cdots$	1	1	1
η_2	1	1	1	$\cdots$	1	-1	-1
η_3	1	-1	1	$\cdots$	$(-1)^{n/2}$	1	$(-1)^{n/2}$
η_4	1	-1	1	$\cdots$	$(-1)^{n/2}$	-1	$(-1)^{n/2+1}$

Assume now that $n = 2k$. We need $\frac{n-1}{2} = k-1$ additional irreducible characters. For $m \in \{1,\dots,k-1\}$, let $\omega_m = e^{2\pi i m/k}$ and

$$\rho_m \colon \mathbb{D}_n \to \mathbf{GL}_2(\mathbb{C}), \quad r \mapsto \begin{pmatrix} \omega_m & 0 \\ 0 & \omega_m^{-1} \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Each ρ_m is a group homomorphism (see Exercise 13.5). A direct calculation produces the values of the character χ_m of ρ_m.

	1	r	r^2	$\cdots$	$r^{n/2}$	s	sr
χ_m	1	$\omega_m + \omega_m^{-1}$	$\omega_m^2 + \omega_m^{-2}$	$\cdots$	$\omega_m^{n/2} + \omega_m^{-n/2}$	0	0

In the same way that we constructed the character table when n is odd, we now need to verify that we have constructed $\frac{n+6}{2}$ irreducible characters of $\mathbb{D}_n$.

13.9. Exercise. Prove that we have constructed $\frac{n+6}{2}$ irreducible characters of $\mathbb{D}_n$.

14. Project: Hurwitz' theorem

We know that $x^2y^2=(xy)^2$ holds for all $x,y\in\mathbb{R}$. Fibonacci found the identity

$$(x_1^2+x_2^2)(y_1^2+y_2^2)=(x_1y_1-x_2y_2)^2+(x_1y_2+x_2y_1)^2. \tag{14.1}$$

Euler and Hamilton independently found a similar identity:

$$(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)=z_1^2+z_2^2+z_3^2+z_4^2,$$

where

$$\begin{aligned} z_1&=x_1y_1-x_2y_2-x_3y_3-x_4y_4,\\ z_2&=x_1y_2+x_2y_1+x_3y_4-x_4y_3,\\ z_3&=x_1y_3-x_2y_4+x_3y_1+x_4y_2,\\ z_4&=x_1y_4+x_2y_3-x_3y_2+x_4y_1. \end{aligned} \tag{14.2}$$

Cayley found a similar identity for sums of eight squares. Are there other identities of this type? Hurwitz proved that this is not the case.

The question can be reformulated as follows. For which n does there exist a bilinear map $\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $(x,y)\mapsto xy$, such that

$$\|xy\|=\|x\|\|y\|$$

for all $x,y\in\mathbb{R}^n$? Here, of course, we use the standard notation

$$\|(x_1,\dots,x_n)\|=\sqrt{x_1^2+\cdots+x_n^2}.$$

14.1. Lemma. *Let $n>2$ be an even number. If there exists a group G with generators $\epsilon,x_1,\dots,x_{n-1}$ and relations*

$$x_1^2=\cdots=x_{n-1}^2=\epsilon\neq 1,\quad \epsilon^2=1,\quad [x_i,x_j]=\epsilon\quad \textit{if}\quad i\neq j,$$

then the following statements hold:

- **1)** $|G|=2^n$.
- **2)** $[G,G]=\{1,\epsilon\}$. *In particular, G has exactly 2^{n-1} degree-one representations.*
- **3)** *If $g\notin Z(G)$, then the conjugacy class of g is $\{g,\epsilon g\}$.*
- **4)** $Z(G)=\{1,\epsilon,x_1\cdots x_{n-1},\epsilon x_1\cdots x_{n-1}\}$.
- **5)** *G has $2^{n-1}+2$ conjugacy classes.*
- **6)** *G has two irreducible representations of degree $2^{\frac{n-2}{2}}>1$.*

Proof. Let us prove 1) and 2). Note that $\epsilon\in Z(G)$, as $\epsilon=x_i^2$ for all $i\in\{1,\dots,n-1\}$. Since $n-1>2$, $[x_1,x_2]=\epsilon$. Hence $\epsilon\in[G,G]$. Moreover, $G/\langle\epsilon\rangle$ is abelian. Thus $[G,G]=\langle\epsilon\rangle$. Since $G/[G,G]$ is elementary abelian of order 2^{n-1}, it follows that $|G|=2^n$.

We now prove 3). Let $g\in G\setminus Z(G)$ and $x\in G$ be such that $[x,g]\neq 1$. Then $[x,g]=\epsilon$ and $xgx^{-1}=\epsilon g$.

To prove 4) let $g\in G$. Write

$$g=\epsilon^{a_0}x_1^{a_1}\cdots x_{n-1}^{a_{n-1}},$$

where $a_j\in\{0,1\}$ for all $j\in\{1,\dots,n-1\}$. If $g\in Z(G)$, then $gx_i=x_ig$ for all i. Hence $g\in Z(G)$ if and only if

$$\epsilon^{a_0}x_1^{a_1}\cdots x_{n-1}^{a_{n-1}}=x_i(\epsilon^{a_0}x_1^{a_1}\cdots x_{n-1}^{a_{n-1}})x_i^{-1}.$$

Since $x_i x_j^{a_j} x_i^{-1} = \epsilon^{a_j} x_j^{a_j}$ whenever $i \neq j$ and $\epsilon \in Z(G)$, the element g is central if and only if

$$\sum_{\substack{j=1\\ j\neq i}}^{n-1} a_j \equiv 0 \bmod 2$$

for all $i \in \{1, \dots, n-1\}$. In particular,

$$\sum_{j\neq i} a_j \equiv \sum_{j\neq k} a_j$$

for all $k \neq i$. Therefore $a_i \equiv a_k \bmod 2$ for all $i, k \in \{1, \dots, n-1\}$. Thus $a_1 = \cdots = a_{n-1}$ and $Z(G) = \{1, x_1 \cdots x_{n-1}, \epsilon, \epsilon x_1 \cdots x_{n-1}\}$.

To prove 5) we use the class equation:

$$2^n = |G| = |Z(G)| + \sum_{i=1}^{N} 2 = 4 + 2N.$$

It follows that G has $N + 4 = 2^{n-1} + 2$ conjugacy classes.

Finally we prove 6). Since G has exactly 2^{n-1} degree-one representations (because $|G/[G,G]| = 2^{n-1}$) and has $2^{n-1} + 2$ conjugacy classes, it follows from

$$2^n = |G| = \underbrace{1 + \cdots + 1}_{2^{n-1}} + f_1^2 + f_2^2 = 2^{n-1} + f_1^2 + f_2^2,$$

that G has two irreducible representations of degrees $f_1 = f_2 = 2^{\frac{n-2}{2}} > 1$. □

14.2. Example. The formulas (14.2) give a representation for the group G of the previous lemma. Write each z_i as

$$z_i = \sum_{k=1}^{4} a_{ik}(x_1, \dots, x_4) y_k.$$

Let A be a matrix such that $A_{ij} = a_{ij}(x_1, \dots, x_4)$, that is

$$A = \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix}.$$

The matrix A can be written as $A = A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4$, where

$$A_1 = \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} & -1 && \\ 1 &&& \\ &&& -1 \\ && 1 & \end{pmatrix}, \quad A_3 = \begin{pmatrix} && -1 & \\ &&& 1 \\ 1 &&& \\ & -1 && \end{pmatrix}, \quad A_4 = \begin{pmatrix} &&& -1 \\ && -1 & \\ & 1 && \\ 1 &&& \end{pmatrix}.$$

For $i \in \{1, \dots, 4\}$, let $B_i = A_4^T A_i$. Then $B_i = -B_i^T$ and $B_i^2 = -I$ for all $i \in \{1, 2, 3\}$. Moreover, $B_i B_j = -B_j B_i$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. The group generated by $\{B_1, B_2, B_3\}$ has 2^4 elements, all of them of the form

$$\pm B_1^{k_1} B_2^{k_2} B_3^{k_3}$$

for $k_j \in \{0, 1\}$. This group provides an example of the group G of Lemma 14.1.

14.3. Theorem (Hurwitz). *If there is an identity of the form*

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_n^2, \tag{14.3}$$

where the x_j's and the y_j's are real numbers and each z_k is a bilinear function in the x_j's and the y_j's, then $n \in \{1, 2, 4, 8\}$.

Proof. Without loss of generality, we may assume that $n > 2$. For $i \in \{1, \dots, n\}$ let

$$z_i = \sum_{k=1}^{n} a_{ki}(x_1, \dots, x_n) y_k,$$

where the a_{ik}'s are linear functions. Then

$$z_i^2 = \sum_{k,l=1}^{n} a_{ki}(x_1, \dots, x_n) a_{li}(x_1, \dots, x_n) y_k y_l$$

for all $i \in \{1, \dots, n\}$. Using these expressions for each z_i in (14.3) and comparing coefficients,

$$\sum_{i=1}^{n} a_{ki}(x_1, \dots, x_n) a_{li}(x_1, \dots, x_n) = \delta_{k,l}(x_1^2 + \cdots + x_n^2), \tag{14.4}$$

where $\delta_{k,l}$ is the usual Kronecker's map. Let A be the $n \times n$ matrix given by

$$A_{ij} = a_{ij}(x_1, \dots, x_n).$$

Then

$$AA^T = (x_1^2 + \cdots + x_n^2) I, \tag{14.5}$$

where I denotes the $n \times n$ identity matrix, as

$$(AA^T)_{kl} = \sum_{i=1}^{n} a_{ki}(x_1, \dots, x_n) a_{li}(x_1, \dots, x_n) = \delta_{kl}(x_1^2 + \cdots + x_n^2)$$

by (14.4). Since each $a_{ki}(x_1, \dots, x_n)$ is a linear function, there exist $\alpha_{ij1}, \dots, a_{ijn} \in \mathbb{C}$ such that

$$a_{ij}(x_1, \dots, x_n) = \alpha_{ij1} x_1 + \cdots + \alpha_{ijn} x_n.$$

Write

$$A = A_1 x_1 + \cdots + A_n x_n,$$

where each A_k is the matrix $(A_k)_{ij} = \alpha_{ijk}$. The formula (14.5) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j^T x_i x_j = (x_1^2 + \cdots + x_n^2) I.$$

Thus

$$A_i A_j^T + A_j A_i^T = 0 \quad i \neq j, \quad A_i A_i^T = I. \tag{14.6}$$

We need n complex square matrices of size $n \times n$ satisfying (14.6). For $i \in \{1, \dots, n\}$ let $B_i = A_n^T A_i$. Then (14.6) turn into

$$B_i B_j^T + B_j B_i^T = 0 \quad i \neq j, \quad B_i B_i^T = I, \quad B_n = I.$$

Set $j=n$ in the first family of equations to obtain $B_i=-B_i^T$ for all $i\in\{1,\dots,n-1\}$. It follows that

$$(14.7)\qquad \begin{aligned} &B_i^2=-I && \text{for all } i\in\{1,\dots,n-1\},\\ &[B_i,B_j]=-I && \text{for all } i,j\in\{1,\dots,n-1\}.\end{aligned}$$

Claim. n is even.

Computing the determinant of $B_iB_j=-B_jB_i$ we obtain that

$$\det(B_iB_j)=(-1)^n\det(B_jB_i).$$

Thus $1=(-1)^n$ and hence n is even.

Claim. The group G of the lemma admits a faithful representation $\rho\colon G\to\mathbf{GL}_n(\mathbb{C})$.

By (14.7), there is a well-defined group homomorphism ρ such that $x_i\mapsto B_i$ for all $i\in\{1,\dots,n-1\}$ and $\epsilon\mapsto -I$. This homomorphism is injective, as its image has $|G|=2^n$ different elements.

Claim. $2^{\frac{n-2}{2}}$ divides n.

Since $\epsilon\in[G,G]$ by Lemma 14.1, every one-dimensional representation satisfies $\epsilon\mapsto 1$. This implies that ρ cannot have degree-one sub representations. In fact, if $W=\langle w\rangle$ is G-invariant subspace of $\mathbb{C}^n$, then $\psi=\rho|_W\colon G\to\mathbf{GL}(W)\simeq\mathbb{C}^\times$ is a representation. In particular,

$$-w=-Iw=\psi_\epsilon(w)=\psi_{[x_i,x_j]}(w)=\psi_{x_i}\psi_{x_j}\psi_{x_i}^{-1}\psi_{x_j}^{-1}(w)=w,$$

a contradiction.

This means that the $\mathbb{C}[G]$-module $\mathbb{C}^n$ decomposes as $\mathbb{C}^n\simeq aS\oplus bT$, where a and b are integers and S and T are simple $\mathbb{C}[G]$-modules of dimension $2^{\frac{n-2}{2}}$. In particular,

$$n=\dim V=\dim(aS\oplus bT)=(a+b)2^{\frac{n-2}{2}}.$$

To finish the proof of the theorem, write $n=2^ab$ for $a\geq 1$ and b an odd integer. Since $\frac{n-2}{2}$ divides n,

$$2^{\frac{n}{2}-1}=2^{\frac{n-2}{2}}\leq n=2^ab.$$

Thus $\frac{n}{2}-1\leq a$ and hence $2^a\leq n\leq 2(a+1)$. It follows that $n\in\{4,8\}$. □

We now present an application; see [46] for more information.

14.4. Theorem. *Let V be a real vector space (with an inner product) of dimension $n\geq 3$. If there exists a bilinear function $V\times V\to V$, $(v,w)\mapsto v\times w$, such that $v\times w$ is orthogonal both to v and w and*

$$\|v\times w\|^2=\|v\|^2\|w\|^2-\langle v,w\rangle^2,$$

where $\|v\|^2=\langle v,v\rangle$, then $n\in\{3,7\}$.

Proof. Let $W=V\oplus\mathbb{R}$ with the inner product

$$\langle (v_1,r_1),(v_2,r_2)\rangle=\langle v_1,v_2\rangle+r_1r_2.$$

Note that

$$\begin{aligned}\langle v_1\times v_2+r_1v_2+r_2v_1,\, &v_1\times v_2+r_1v_2+r_2v_1\rangle\\ &=\|v_1\times v_2\|^2+r_1^2\|v_2\|^2+2r_1r_2\langle v_1,v_2\rangle+r_2^2\|v_1\|^2.\end{aligned}$$

Thus

$$\begin{aligned}(\|v_1\|^2 + r_1^2)(\|v_2\|^2 + r_2) \\ &= \|v_1\|^2\|v_2\|^2 + r_2^2\|v_1\|^2 + r_1^2\|v_2\|^2 + r_1^2 r_2^2 \\ &= \|v_1 \times v_2 + r_1 v_1 + r_2 v_2\|^2 - 2r_1 r_2 \langle v_1, v_2\rangle + \langle v_1, v_2\rangle^2 + r_1^2 r_2^2 \\ &= \|v_1 \times v_2 + r_1 v_1 + r_2 v_2\|^2 + (\langle v_1, v_2\rangle - r_1 r_2)^2 \\ &= z_1^2 + \cdots + z_{n+1}^2,\end{aligned}$$

where the z_k's are bilinear functions in (v_1, r_1) and (v_2, r_2). By Hurwitz's theorem, we conclude that $n+1 \in \{4, 8\}$. Hence $n \in \{3, 7\}$. $\square$

In the theorem, if $\dim V = 3$, we obtain the usual cross product. If $\dim V = 7$, let

$$W = \{(v, k, w) : v, w \in V, k \in \mathbb{R}\}$$

with the inner product

$$\langle (v_1, k_1, w_1), (v_2, k_2, w_2)\rangle = \langle v_1, v_2\rangle + k_1 k_2 + \langle w_1, w_2\rangle.$$

It is an exercise to show that

$$\begin{aligned}(v_1, k_1, w_1) \times (v_2, k_2, w_2) \\ &= (k_1 w_2 - k_2 w_1 + v_1 \times v_2 - w_1 \times w_2, -\langle v_1, w_2\rangle + \langle v_2, w_1\rangle, \\ &\qquad k_2 v_1 - k_1 v_2 - v_1 \times w_2 - w_1 \times v_2)\end{aligned}$$

satisfies the properties of the theorem.

15. Project: Induced representations

15.1. Definition. Let R and S be rings. An abelian group M is called a **(R,S)-bimodule** if M is a left R-module, M is a right S-module, and

$$r\cdot(m\cdot s)=(r\cdot m)\cdot s$$

holds for all $r\in R$, $s\in S$ and $m\in M$.

Note that every left R-module is an $(R,\mathbb{Z})$-bimodule. Similarly, every right S-module is an $(\mathbb{Z},S)$-bimodule. Every ring R is an (R,R)-bimodule.

15.2. Example. If M is an (R,S)-bimodule and N is a left R-module, then the set $\operatorname{Hom}_R(M,N)$ of left R-module homomorphisms $M\to N$ is a left S-module with

$$(s\cdot\varphi)(m)=\varphi(m\cdot s),\quad s\in S,\ \varphi\in\operatorname{Hom}_R(M,N),\ m\in M.$$

Let M be an (R,S)-bimodule, N be an S-module and U be a R-module. We say that a map $f\colon M\times N\to U$ is **balanced** if

$$\begin{aligned}
&f(m_1+m_2,n)=f(m_1,n)+f(m_2,n),\\
&f(m,n_1+n_2)=f(m,n_1)+f(m,n_2),\\
&f(m\cdot s,n)=f(m,s\cdot n),\\
&f(r\cdot m,n)=r\cdot f(m,n)
\end{aligned}$$

for all $m,m_1,m_2\in M$, $n,n_1,n_2\in N$, $r\in R$ and $s\in S$.

15.3. Example. If M is an R-module, the map $f\colon R\times M\to M$, $(r,m)\mapsto r\cdot m$, is balanced.

Let M be an (R,S)-bimodule, N be an S-module and U be an R-module. A **tensor product** $M\otimes_S N$ is an R-module with a balanced map $\eta\colon M\times N\to M\otimes_S N$ satisfying the following universal property:

> If $f\colon M\times N\to U$ is a balanced map, then there exists a unique R-module homomorphism $\alpha\colon M\otimes_S N\to U$ such that $f=\alpha\circ\eta$.

Notation: $m\otimes n=\eta(m,n)$ for $m\in M$ and $n\in N$. The tensor product of bimodules exists and one can show it is unique up to isomorphism. More precisely, $M\otimes_S N$ is the R-module generated by the set $\{m\otimes n: m\in M,\ n\in N\}$, where the elements $m\otimes n$ satisfy the following properties:

$$(m+m_1)\otimes n=m\otimes n+m_1\otimes n \qquad m,m_1\in M,\ n\in N, \tag{15.1}$$

$$m\otimes(n+n_1)=m\otimes n+m\otimes n_1 \qquad m\in M,\ n,n_1\in N, \tag{15.2}$$

$$(ms)\otimes n=m\otimes(sn) \qquad m\in M,\ n\in N,\ s\in S, \tag{15.3}$$

$$(rm)\otimes n=r(m\otimes n) \qquad m\in M,\ n\in N,\ r\in R. \tag{15.4}$$

An arbitrary element of $M\otimes_S N$ is a finite sum of the form $\sum_{i=1}^k m_i\otimes n_i$, where $m_1,\dots,m_k\in M$ and $n_1,\dots,n_k\in N$, and not necessarily an element of the form $m\otimes n$.

15.4. Example. $M\simeq R\otimes_R M$ as R-modules. Since the map $R\times M\to M$, $(r,m)\mapsto r\cdot m$, is balanced, it induces an isomomorphism $R\otimes_R M\to M$, $r\otimes m\mapsto r\cdot m$ with inverse $M\to R\otimes_R M$, $m\mapsto 1\otimes m$.

15.5. Example. If $M_1,\dots,M_k$ are (R,S)-bimodules and N is an S-module, then

$$(M_1\oplus\cdots\oplus M_k)\otimes_S N\simeq (M_1\otimes_S N)\oplus\cdots\oplus(M_k\otimes_S N).$$

Some exercises:

15.6. Exercise. Prove that $M\otimes_R N\simeq N\otimes_{R^{\mathrm{op}}} M$.

15.7. Exercise. Prove that $\mathbb{Z}/n\otimes_{\mathbb{Z}}\mathbb{Q}=\{0\}$.

15.8. Exercise. Let M be an (R,S)-bimodule and N be an (S,T)-bimodule. Prove that $M\otimes_S N$ is an (R,T)-bimodule with $r(m\otimes n)t=(rm)\otimes(nt)$, where $m\in M$, $n\in N$, $r\in R$, $t\in T$.

15.9. Exercise. Prove that $(M\otimes_R N)\otimes_R T\simeq M\otimes_R(N\otimes_R T)$.

15.10. Exercise. State and prove the associativity of tensor product of bimodules.

If G is a finite group, H is a subgroup of G and V is a $\mathbb{C}[H]$-module, then $\mathbb{C}[G]$ is a $(\mathbb{C}[G],\mathbb{C}[H])$-bimodule.

15.11. Definition. Let G be a finite group and H be a subgroup of G. If V is a $\mathbb{C}[H]$-module of G, we define the **induced** $\mathbb{C}[G]$-module of V as

$$\operatorname{Ind}_H^G V=\mathbb{C}[G]\otimes_{\mathbb{C}[H]}V.$$

15.12. Example. Let $G=\mathbb{S}_3$ and $H=\{\mathrm{id},(12)\}$. Then $T=\{\mathrm{id},(123),(23)\}$ is a transversal of H in G. We can decompose G as

$$G=\{\mathrm{id},(12)\}\cup\{(123),(13)\}\cup\{(132),(23)\}=\bigcup_{t\in T}tH.$$

Each $g\in G$ can be written uniquely as $g=th$ for some $t\in T$ and $h\in H$. We can define define a linear transformation $\varphi\colon\mathbb{C}[G]\to\mathbb{C}[H]\oplus\mathbb{C}[H]\oplus\mathbb{C}[H]=|T|\mathbb{C}[H]$, such that for each $g=th$ returns h in the position corresponding to $t\in T$, namely

$$\begin{aligned}\mathrm{id}&\mapsto(\mathrm{id},0,0), & (12)&\mapsto((12),0,0), & (123)&\mapsto(0,\mathrm{id},0),\\ (23)&\mapsto(0,0,\mathrm{id}), & (13)&\mapsto(0,(12),0), & (132)&\mapsto(0,0,(12)).\end{aligned}$$

For example,

$$\varphi(5(12)-3(123)+7\,\mathrm{id})=(7\,\mathrm{id}+5(12),-3\,\mathrm{id},0).$$

Note that φ is an isomorphism of right $\mathbb{C}[H]$-modules.

The previous example is helpful to understand the following result:

15.13. Proposition. *Let G be a finite group and H be a subgroup of G. If V is a $\mathbb{C}[H]$-module, then*

$$\operatorname{Ind}_H^G(V)=\bigoplus_{t\in T}t\otimes V,$$

where T is a transversal of H in G and $t\otimes V=\{t\otimes v:v\in V\}$. In particular,

$$\dim\operatorname{Ind}_H^G V=(G:H)\dim V.$$

Proof. Decompose G into H-cosets with the transversal T, that is

$$G = \bigcup_{t \in T} tH.$$

Each $g \in G$ can be written uniquely as $g = th$ for some $t \in T$ and $h \in H$. As we did in the previous example, this produces an isomorphism $\varphi \colon \mathbb{C}[G] \to |T|\mathbb{C}[H]$ of right $\mathbb{C}[H]$-modules, where $\varphi(g)$ is h in the summand corresponding to $t \in T$ and is zero in the rest of the summands. Hence

$$\operatorname{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \simeq (|T|\mathbb{C}[H]) \otimes_{\mathbb{C}[H]} V \simeq |T|(\mathbb{C}[H] \otimes_{\mathbb{C}[H]} V) \simeq |T|V$$

as $\mathbb{C}[H]$-modules. In particular, $\dim \operatorname{Ind}_H^G V = |T| \dim V$.

Write $g = th$ with $t \in T$ and $h \in H$. Then $g \otimes v = (th) \otimes v = t \otimes h \cdot v \in t \otimes V$. Hence $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \subseteq \oplus_{t \in T} t \otimes V$. The other inclusion is trivial. By definition, the sum over $t \in T$ of the $t \otimes V$'s is direct. □

15.14. Theorem (Frobenius' reciprocity). *Let G be a finite group and H be a subgroup of G. If U is a $\mathbb{C}[G]$-module and V is a $\mathbb{C}[H]$-module, then*

$$\operatorname{Hom}_{\mathbb{C}[H]}(V, \operatorname{Res}_H^G U) \simeq \operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G V, U)$$

as vector spaces

Proof. For $\varphi \in \operatorname{Hom}_{\mathbb{C}[H]}(V, \operatorname{Res}_H^G U)$, let

$$f_\varphi \colon \mathbb{C}[G] \times V \to U, \quad (g, v) \mapsto g \cdot \varphi(v).$$

We claim that f_φ is balanced. A direct calculation shows that

$$f_\varphi(g + g_1, v) = f_\varphi(g, v) + f_\varphi(g_1, v), \qquad f_\varphi(g, v + w) = f_\varphi(g, v) + f_\varphi(g, w).$$

Since φ is a $\mathbb{C}[H]$-module homomorphism,

$$f_\varphi(gh, v) = (gh) \cdot \varphi(v) = g \cdot (h \cdot \varphi(v)) = g \cdot (h \cdot \varphi(v)) = g \cdot \varphi(h \cdot v) = f_\varphi(g, h \cdot v)$$

for all $g \in G$, $h \in H$ and $v \in V$. Moreover,

$$f_\varphi(gg_1, v) = (gg_1) \cdot \varphi(v) = g \cdot (g_1 \cdot \varphi(v)) = g \cdot f_\varphi(g_1, v)$$

for all $g, g_1 \in G$ and $v \in V$.

For every $\varphi \in \operatorname{Hom}_{\mathbb{C}[H]}(V, \operatorname{Res}_H^G U)$, there exists $\Gamma(\varphi) \in \operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G V, U)$ such that $\Gamma(\varphi)(g \otimes v) = g \cdot \varphi(v)$. We have defined a map

$$\Gamma \colon \operatorname{Hom}_{\mathbb{C}[H]}(V, \operatorname{Res}_H^G U) \to \operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G V, U), \quad \varphi \mapsto \Gamma(\varphi).$$

The map Γ is linear and injective, something that is quite easy to verify.

The map is surjective: if $\theta \in \operatorname{Hom}_{\mathbb{C}[H]}(\operatorname{Ind}_H^G V, U)$, then the map $\varphi(v) = \theta(1 \otimes v)$ is such that $\varphi \in \operatorname{Hom}_{\mathbb{C}[H]}(V, \operatorname{Res}_H^G U)$ and satisfies

$$\Gamma(\varphi)(g \otimes v) = g \cdot \varphi(v) = g \cdot \theta(1 \otimes v) = \theta(g \otimes v).$$ □

Let H be a subgroup of G. If U is a $\mathbb{C}[G]$-module with character χ, the character $\operatorname{Res}_H^G U$ will be denoted by $\chi|_H$. Then $\chi|_H(1) = \chi(1)$. Note that

$$\langle \phi, \chi|_H \rangle = \dim \operatorname{Hom}_{\mathbb{C}[H]}(V, \operatorname{Res}_H^G U) = \dim \operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G V, U) = \langle \phi^G, \chi \rangle,$$

15.15. Definition. If $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$ and $\mathrm{Irr}(H) = \{\phi_1, \dots, \phi_l\}$, we define the **induction-restriction** matrix as $(c_{ij}) \in \mathbb{C}^{l\times k}$, where

$$c_{ij} = \langle \phi_i^G, \chi_j \rangle = \langle \phi_i, \chi_j|_H \rangle.$$

The i-th row of the induction-restriction matrix gives the multiplicity of the character χ_j in the decomposition of ϕ_i^G. The j-th column is the multiplicity of the character ϕ_i in the decomposition of $\chi_j|H$.

15.16. Example. Let $G = \mathbb{S}_3$. The character table of G is

	1 1	3 (12)	2 (123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The character table of the subgroup $H = \{\mathrm{id}, (12)\}$ is

	1 id	1 (12)
ϕ_1	1	1
ϕ_2	1	-1

By inspection, we see that $\chi_1|_H = \phi_1$, $\chi_2|_H = \phi_2$ and $\chi_3|_H = \phi_1 + \phi_2$. The induction-restriction matrix is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Moreover, $\phi_1^G = \chi_1 + \chi_3$ and $\phi_2^G = \chi_2 + \chi_3$.

Let us see (again) how to compute induced characters.

15.17. Proposition. *Let H be a subgroup of G and V a $\mathbb{C}[H]$-module with character χ. If T is a transversal of H in G, then*

$$\chi^G(g) = \sum_{\substack{t\in T\\ t^{-1}gt\in H}} \chi(t^{-1}gt)$$

for all $g \in G$.

Proof. We know that $\mathrm{Ind}_H^G V = \oplus_{t\in T} t \otimes V$. Assume that $T = \{t_1, \dots, t_m\}$ and let $\{v_1, \dots, v_n\}$ be a basis of V. Then $\{t_i \otimes v_k : 1 \leq i \leq m,\ 1 \leq k \leq n\}$ is a basis of $\mathrm{Ind}_H^G V$. The action of g on $\mathrm{Ind}_H^G V$ is given by

$$\rho^G(g) = \begin{cases} \rho(t_j^{-1}gt_i) & \text{if } t_j^{-1}gt_i \in H, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, if $gt_i = t_jh$ for $h \in H$ and certain i, j, then

$$g \cdot (t_i \otimes v_k) = gt_i \otimes v_k = t_jh \otimes v_k = t_j \otimes h \cdot v_k.$$

Moreover, $gt_i = t_jh$ if and only if $t_j^{-1}gt_i = h \in H$. We conclude that g acts like $t^{-1}gt$ on V if $t^{-1}gt \in H$ and as zero otherwise. $\square$

15.18. Corollary. *Let H be a subgroup of G and V a $\mathbb{C}[H]$-module with character χ. If $g\in G$, then*

$$\chi^G(g)=\frac{1}{|H|}\sum_{\substack{x\in G\\ x^{-1}gx\in H}}\chi(x^{-1}gx).$$

Proof. Let T be a transversal of H in G. If $x\in G$, write $x=th$ for some $t\in T$ and $h\in H$. Como $x^{-1}gx=h^{-1}(t^{-1}gt)h$, it follows that $x^{-1}gx\in H\iff t^{-1}gt\in H$. Moreover, $\chi(x^{-1}gx)=\chi(t^{-1}gt)$, as χ is a class function. This implies that there are $|H|$ elements $x\in G$ such that $x^{-1}gx\in H$. For those x, $\chi(x^{-1}gx)=\chi(t^{-1}gt)$. □

16. Project: A theorem of Herstein

We present a theorem of [18]. The proof uses Frobenius' theorem (Theorem 8.20). Recall that a proper subgroup M of G is said to be **maximal** if $M \subseteq H$ for some subgroup H of G implies that $M = H$ or $H = G$.

16.1. Theorem (Herstein). *Let G be a finite group and A be an abelian subgroup of G. If A is maximal, then G is solvable.*

We start with two lemmas.

16.2. Lemma. *Let G be a finite group and $\phi \in \operatorname{Aut}(G)$ be such that $\phi(x) = x$ implies $x = 1$. Then $G = \{x^{-1}\phi(x) : x \in G\}$.*

Proof. Let us consider the map $x \mapsto x^{-1}\phi(x)$. This map is injective, as

$$x^{-1}\phi(x) = y^{-1}\phi(y) \implies yx^{-1} = \phi(yx^{-1}) \implies yx^{-1} = 1 \implies x = y.$$

Thus $|G| \leq |\{x^{-1}\phi(x) : x \in G\}| \leq |G|$ and hence $G = \{x^{-1}\phi(x) : x \in G\}$. □

16.3. Lemma. *Let G be a finite group and p be a prime divisor of $|G|$. Let A be an abelian subgroup of $\operatorname{Aut}(G)$ such that if $\phi \in A$ and $\phi(x) = x$, then $x = 1$. Then there exists a unique $P \in \operatorname{Syl}_p(G)$ such that $\phi(P) = P$ for all $\phi \in A$.*

Proof. Let $\phi \in A$. By Lemma 16.2, $G = \{x^{-1}\phi(x) : x \in G\}$.

We claim that there exists $P \in \operatorname{Syl}_p(G)$ such that $\phi(P) = P$. In fact, let $Q \in \operatorname{Syl}_p(G)$. Then $\phi(Q) \in \operatorname{Syl}_p(G)$. Thus there exists $y \in G$ such that $yQy^{-1} = \phi(Q)$. Let $x \in G$ be such that $y^{-1} = x^{-1}\phi(x)$. Then

$$\begin{aligned}\phi(xQx^{-1}) &= \phi(x)\phi(Q)\phi(x)^{-1}\\ &= \phi(x)yQy^{-1}\phi(x)^{-1}\\ &= \phi(x)\phi(x)^{-1}xQx^{-1}\phi(x)\phi(x)^{-1}\\ &= xQx^{-1}.\end{aligned}$$

Thus $P = xQx^{-1} \in \operatorname{Syl}_p(G)$ is such that $\phi(P) = P$.

We claim that $\phi(N_G(P)) = N_G(P)$. If $y \in \phi(N_G(P))$, then $y = \phi(x)$ for some $x \in N_G(P)$. Then

$$yPy^{-1} = \sigma(x)P\sigma(x)^{-1} = \sigma(x)\sigma(P)\sigma(x)^{-1} = \sigma(xPx^{-1}) = \sigma(P) = P.$$

Thus $y \in N_G(P)$. Conversely, if $x \in N_G(P)$, then $x = \sigma(y)$ for some $y \in G$. Since

$$P = xPx^{-1} = \sigma(y)P\sigma(y)^{-1} = \sigma(yPy_{-1}),$$

$yPy^{-1} = \sigma^{-1}(P) = P$ and hence $y \in N_G(P)$.

We now claim that P is the only Sylow p-subgroup of G such that $\phi(P) = P$. Suppose that $P_1 \in \operatorname{Syl}_p(G)$ is such that $\phi(P_1) = P_1$. Since P and P_1 are conjugate, $P_1 = xPx^{-1}$ for some $x \in G$. Since

$$xPx^{-1} = P_1 = \phi(P_1) = \phi(xPx^{-1}) = \phi(x)\phi(P)\phi(x)^{-1} = \phi(x)P\phi(x)^{-1},$$

it follows that $x^{-1}\phi(x) \in N_G(P)$. Note that the restriction $\phi|_{N_G(P)}$ of ϕ to the subgroup $N_G(P)$ is an isomorphism that only fixes the identity element. By Lemma 16.2 applied to the group $N_G(P)$ and the automorphism $\phi|_{N_G(P)}$, there exists $y \in N_G(P)$ such that $x^{-1}\phi(x) = y^{-1}\phi(y)$. Then $x = y \in N_G(P)$ and therefore $P_1 = P$.

Let $\psi \in A$. Since A is abelian,

$$\psi(P) = \psi(\phi(P)) = (\psi\phi)(P) = (\phi\psi)(P) = \phi(\psi(P)).$$

Thus $\psi(P) \in \mathrm{Syl}_p(G)$ is fixed by ϕ. By the previous claim, $P = \psi(P)$. □

Now we are ready to prove the theorem.

Proof of Theorem 16.1. We proceed by induction on $|G|$. There are two cases to consider.

Assume first that $N_G(A) \neq A$. Then $A \subseteq N_G(A) \subseteq G$. Since A is a maximal subgroup, $N_G(A) = G$. Thus A is normal in G. By the correspondence theorem, G/A has no non-trivial proper subgroups. Thus G/A is cyclic (of prime order). In particular, G is solvable as both G/A and A are solvable.

Assume now that $N_G(A) = A$. Let $x \notin A$ and $B = xAx^{-1} \cap A$. If $B \neq \{1\}$, let $b \in B$ be such that $b \neq 1$. Since A is abelian, $A \subseteq C_G(b)$. Moreover, since $b \in xAx^{-1}$, $A \neq xAx^{-1} \subseteq C_G(b)$. Hence $C_G(b) \neq A$. By the maximality of A, $C_G(b) = G$. In particular, $b \in Z(G)$. The subgroup $\langle b\rangle$ is central (so normal in G). Let $\pi\colon G \to G/\langle b\rangle$ be the canonical map. By the correspondence theorem, $\pi(A)$ is a maximal subgroup of $G/\langle b\rangle$ and $\pi(A)$ is abelian. Since $|G/\langle b\rangle| < |G|$, the inductive hypothesis implies that $G/\langle b\rangle$ is solvable. Thus G is solvable and both $G/\langle b\rangle$ and $\langle b\rangle$ are solvable.

If $B = \{1\}$, then A is a Frobenius group. Let T be the complement of A in G. Then T is a normal subgroup of G. In particular, $aTa^{-1} = T$ for all $a \in A$. Each $a \in A$ induces an automorphism $\gamma_a\colon T \to T$, $t \mapsto ata^{-1}$. We claim that if $1 \neq a \in A$ and $t \in T$ are such that $\gamma_a(t) = t$, then $t = 1$. In fact, since $t \notin A$,

$$ata^{-1} = \gamma_a(t) = t \implies a = t^{-1}at \in A \cap t^{-1}At = \{1\}.$$

By the lemma, there exists $P \in \mathrm{Syl}_p(T)$ such that $\gamma_a(P) = P$ for all $a \in A$. Thus $A \subseteq N_G(P)$. Moreover, $P \subseteq N_G(P)$. But $P \not\subseteq A$, as $P \subseteq T$ and $T \cap A = \{1\}$. Thus $N_G(P) = G$ since A is a maximal subgroup of G. Hence P is normal in G and AP is a subgroup of G. Since $A \subsetneq AP \subseteq G$, the maximality of A implies that $AP = G$. If $t \in T$, write $t = ax$ for some $a \in A$ and $x \in P \subseteq T$. Thus $a = tx^{-1} \in A \cap T = \{1\}$, that is $t = x \in P$. We have proved that $T = P$. Since both $G/T \simeq A$ and A are solvable, it follows that G is solvable. □

The following result goes back to Kaplansky and Herstein.

16.4. Corollary. *Let $G = ABA$ be a finite group where A is an abelian subgroup and B is a cyclic subgroup of prime order. Then G is solvable.*

Proof. By Herstein's theorem, it is enough to show that A is a maximal subgroup of G. Assume that $B = \langle b\rangle$. Suppose that A is not maximal. Then there exists a proper subgroup M of G such that $A \subsetneq M$. Let $x \in M \setminus A$. Then $x = a_1 b^k a_2$ for some $a_1, a_2 \in A$ and $k \in \mathbb{Z}$ such that $b^k \neq 1$. Since $A \subseteq M$,

$$b^k = a_1^{-1} x a_2^{-1} \in M.$$

Since B is cyclic of prime order, $b \in M$. Thus $A \cup B \subseteq M$ and hence $ABA \subseteq M$, a contradiction. □

17. Project: Wall's theorem

§ 17.1. Ore's theorem.

17.1. Lemma. *Let G be a finite group and M be a maximal subgroup with* $\operatorname{Core}_G M = \{1\}$*. If H is a non-trivial nilpotent normal subgroup of G, then $G = HM$ and $H \cap M = \{1\}$.*

Proof. Since H is normal in G, then HM is a subgroup of G. Since M is maximal and $M \subseteq HM$, either $HM = M$ or $HM = G$. In the first case, $H \subseteq HM = M$ and for every $x \in G$,

$$H = xHx^{-1} \subseteq xMx^{-1}.$$

Thus $\{1\} \neq H \subseteq \bigcap_{x\in G} xMx^{-1} = \operatorname{Core}_G M = \{1\}$, a contradiction. Hence $HM = G$.

Since H is normal in G, $M \subseteq N_G(M \cap H)$. The maximality of M implies that either $N_G(M \cap H) = M$ or $N_G(M \cap H) = G$. If $N_G(M \cap H) = M$, then

$$N_H(M \cap H) = H \cap N_G(M \cap H) = M \cap H,$$

a contradiction to the nilpotency of H. Thus $N_G(M \cap H) = G$ and $M \cap H$ is normal in G. For each $x \in G$,

$$M \cap H = x(M \cap H)x^{-1} \subseteq xMx^{-1}.$$

Thus $M \cap H \subseteq \operatorname{Core}_G M = \{1\}$ and therefore $M \cap H = \{1\}$. □

17.2. Lemma. *Let G be a finite solvable group and M be a maximal subgroup with* $\operatorname{Core}_G M = \{1\}$*. There exists a unique non-trivial nilpotent normal subgroup N such that $NM = G$ and $N \cap M = \{1\}$.*

Proof. Let N be a minimal normal subgroup of G. Since G is solvable, N is an elementary abelian p-group for some prime number p. In particular, N is nilpotent. By Lemma 17.2, $G = NM$ and $N \cap M = \{1\}$.

To prove the uniqueness of N, let N_1 be a normal subgroup of G such that $G = N_1M$ and $N_1 \cap M = \{1\}$. Then

$$N = N \cap G = N \cap (N_1M) = N_1(N \cap M) = N_1$$

by Dedekind's lemma. □

17.3. Lemma. *Let G be a solvable group and M_1 and M_2 be core-free maximal subgroups of G. If N is a minimal normal subgroup of G, then M_1 and M_2 are conjugate.*

Proof. Let $i \in \{1, 2\}$. By Lemma 17.2, $G = NM_i$ and $N \cap M_i = \{1\}$. If $G = N$, then $M_1 = M_2 = \{1\}$ and the lemma is proved. Suppose then that $G \neq N$. Since G is solvable, N is an elementary abelian p-group for some prime number p, say $|N| = p^\alpha$. Let $\pi\colon G \to G/N$ be the canonical map. Let A be a normal subgroup of G containing N such that $\pi(A)$ is minimal normal in G/N. Since G/N is solvable, $\pi(A)$ is an elementary abelian q-group for some prime number q, say $|\pi(A)| = q^\beta$, By the correspondence theorem,

$$\frac{|G|}{|N|q^\beta} = (\pi(G) : \pi(A)) = (G : A) = \frac{|G|}{|A|} = \frac{|G|}{p^\alpha}. \tag{17.1}$$

We claim that $p \neq q$. In fact, if $p = q$, then (17.1) implies that $|A| = p^{\alpha+\beta}$. In particular, A is non-trivial nilpotent normal subgroup of G. By Lemma 17.1, $G = AM_i$ and $A \cap M_i = \{1\}$. Hence $A = N$ by Lemma 17.2, a contradiction. Therefore $p \neq q$ and $|A| = p^\alpha q^\beta$ by (17.1).

Since M_i is maximal and $M_i \subseteq M_iA$, either $M_iA = M_i$ or $M_iA = G$. As M_i is core-free, $G = AM_i$ (otherwise, $A \subseteq \operatorname{Core}_G M = \{1\}$). Since $|G| = |NM_i| = |N||M_i| = q^\beta |M_i|$ and

$$p^\alpha |M_i| = |G| = |AM_i| = \frac{|A||M_i|}{|A \cap M_i|} = \frac{p^\alpha q^\beta |M_i|}{|A \cap M_i|},$$

it follows that $|A \cap M_i| = q^\beta$. Thus $A \cap M_i \in \operatorname{Syl}_q(A)$. By the second Sylow's theorem, there exists $a \in A$ such that

$$a(A \cap M_1)a^{-1} = A \cap M_2.$$

We claim that $aM_1a^{-1} = M_2$. If $aM_1a^{-1} \neq M_2$, then $G = \langle M_2, aM_1a^{-1} \rangle$ by the maximality of M_2. Note that $A \cap M_i$ is normal in M_i (because A is normal in G). It follows that $A \cap M_2$ is a nilpotent non-trivial normal subgroup of G, since for example

$$(am_1a^{-1})(A \cap M_2)(am_1a^{-1})^{-1} = am_1(A \cap M_1)m_1^{-1}a^{-1} = a(A \cap M_1)a^{-1} = A \cap M_2$$

for all $a \in A$, $m_1 \in M_1$ and $x \in A \cap M_2$. By Lemma 17.2, $N = A \cap M_2 \subseteq M_2$, a contradiction to $\operatorname{Core}_G M_2 = \{1\}$. Hence $aM_1a^{-1} = M_2$. □

Now we are ready to state and prove the theorem.

17.4. Theorem (Ore). *Let G be a finite solvable group. If M_1 and M_2 are two maximal subgroups of G, then $M_1M_2 = G$ or M_1 and M_2 are conjugate.*

Proof. We proceed by induction on $|G|$. We divide the proof in three cases.

Assume first that there exists a minimal normal subgroup N of G contained in $M_1 \cap M_2$. Let $\pi\colon G \to G/N$ be the canonical map. Since M_1 and M_2 are both maximal subgroups of G containing N, the subgroups $\pi(M_1)$ and $\pi(M_2)$ are maximal in the solvable group G/N. Since $|G/N| < |G|$, the inductive hypothesis implies that either $G/N = \pi(M_1)\pi(M_2)$ or $\pi(M_1)$ and $\pi(M_2)$ are conjugate. If $G/N = \pi(M_1)\pi(M_2)$, then

$$G = M_1M_2N \subseteq M_1M_2$$

because $N \subseteq M_1 \cap M_2$. If $\pi(M_1)$ and $\pi(M_2)$ are conjugate, then

$$\pi(xM_1x^{-1}) = \pi(x)\pi(M_1)\pi(x)^{-1} = \pi(M_2)$$

for some $x \in G$. Thus $xM_1x^{-1} \subseteq M_2N = M_2$ (again, because $N \subseteq M_2$). By the maximality of M_2, $xM_1x^{-1} = M_2$.

Assume now that there exists a minimal normal subgroup N such that $N \subseteq M_1$ and $N \not\subseteq M_2$. Since G is solvable, N is nilpotent. By Lemma 17.1, $G = NM_2$ and $N \cap M_2 = \{1\}$. By Dedekind's lemma,

$$M_1 = M_1 \cap G = M_1 \cap (NM_2) = N(M_1 \cap M_2).$$

Thus

$$M_1M_2 = N(M_1 \cap M_2)M_2 = NM_2(M_1 \cap M_2) = G.$$

Finally, if neither M_1 nor M_2 contain a normal subgroup of G, then M_1 and M_2 are conjugate by Lemma 17.3. □

We now present a character-theoretic proof of a theorem of Wall [**45**], which bounds the number of maximal subgroups of a finite solvable group.

17.5. Exercise. Let G be a finite group and M be a maximal subgroup of G. Prove that M has exactly $(G : M)$ conjugates.

17.6. Lemma. *Let G be a finite solvable group and M be a maximal subgroup of G. If M is core-free, then every irreducible constituent of $\operatorname{Ind}_M^G \mathbf{1}_M$ has multiplicity one.*

Proof. Let N be a minimal normal subgroup of G. Then $G = NM$ and $N \cap M = \{1\}$ (see By Lemma 17.2). In particular, $|N| = (G : M)$.

We claim that $\operatorname{Res}_N^G \operatorname{Ind}_M^G \mathbf{1}_M$ is the regular character of N. For $n \in N$,

$$(\operatorname{Ind}_M^G \mathbf{1}_M)(n) = \frac{1}{|M|} \sum_{x \in G} \mathbf{1}_M^0(x^{-1}nx) = \begin{cases} (G : M) & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.6, $\operatorname{Res}_N^G \operatorname{Ind}_M^G \mathbf{1}_M$ is the regular character of N and hence

$$\operatorname{Res}_N^G \operatorname{Ind}_M^G \mathbf{1}_M = \sum_{\lambda \in \operatorname{Irr}(N)} \lambda(1)\lambda.$$

Since N is abelian, $\lambda(1) = 1$ for all $\lambda \in \operatorname{Irr}(N)$. Now assume that some irreducible constituent $\psi \in \operatorname{Irr}(G)$ of $\operatorname{Ind}_M^G \mathbf{1}_M$ has multiplicity $m \geq 2$, say $\operatorname{Ind}_M^G \mathbf{1}_M = m\psi + \xi$, where $\langle \psi, \xi \rangle = 0$. Then

$$\sum_{\lambda \in \operatorname{Irr}(N)} \lambda = \operatorname{Res}_N^G \operatorname{Ind}_M^G \mathbf{1}_M = m \operatorname{Res}_N^G \psi + \operatorname{Res}_N^G \xi,$$

a contradiction. □

17.7. Lemma. *Let G be a finite solvable group and M be a maximal subgroup of G. If M is core-free, then*

$$\ker \operatorname{Ind}_M^G \mathbf{1}_M = \bigcap \{\ker \chi : \chi \in \operatorname{Irr}(G) \text{ such that } \langle \operatorname{Ind}_M^G \mathbf{1}_M, \chi \rangle \neq 0\}.$$

Thus $\ker \operatorname{Ind}_M^G \mathbf{1}_M$ is the intersection of the kernels of the irreducible constituents of $\operatorname{Ind}_M^G \mathbf{1}_M$.

Proof. By Lemma 17.6, every irreducible constituent of $\operatorname{Ind}_M^G \mathbf{1}_M$ appears with multiplicity one, that is

$$\operatorname{Ind}_M^G \mathbf{1}_M = \sum_{j=1}^k \chi_j$$

for some subset $\{\chi_1, \dots, \chi_k\} \subseteq \operatorname{Irr}(G)$. If $g \in \ker \chi_1 \cap \cdots \cap \ker \chi_k$, then $\chi_j(g) = \chi_j(1)$ for all $j \in \{1, \dots, k\}$. Thus

$$\sum_{j=1}^k \chi_j(g) = (\operatorname{Ind}_M^G \mathbf{1}_M)(g) = (\operatorname{Ind}_M^G \mathbf{1}_M)(1) = \sum_{j=1}^k \chi_j(1).$$

Conversely, let $g \in \ker \operatorname{Ind}_M^G \mathbf{1}_M$. Then

$$\sum_{j=1}^k \chi_j(g) = (\operatorname{Ind}_M^G \mathbf{1}_M)(g) = (\operatorname{Ind}_M^G \mathbf{1}_M)(1) = \sum_{j=1}^k \chi_j(1).$$

Assume that there exists $i \in \{1, \dots, k\}$ such that $g \notin \ker \chi_i$. Then $|\chi_i(g)| < \chi_i(1)$ and hence

$$\sum_{j=1}^k \chi_j(1) = \left| \sum_{j=1}^k \chi(g) \right| \leq \sum_{j=1}^k |\chi_j(g)| < \sum_{j=1}^k \chi_j(1),$$

a contradiction. □

17.8. Lemma. *Let G be a finite solvable group and M be a maximal subgroup of G. If M is not normal in G, then $\mathbf{1}_G$ is the only degree-one constituent of* $\operatorname{Ind}_M^G \mathbf{1}_M$.

Proof. Let $\psi \in \operatorname{Irr}(G)$ be a degree-one constituent of $\operatorname{Ind}_M^G \mathbf{1}_M$. By Frobenius' reciprocity, $0 \neq \langle \operatorname{Ind}_M^G \mathbf{1}_M, \psi\rangle = \langle \mathbf{1}_M, \operatorname{Res}_M^G \psi\rangle$. Since $\operatorname{Res}_M^G \psi$ is a degree-one character, it follows the irreducibility that $\mathbf{1}_M = \operatorname{Res}_M^G \psi$. In particular,

$$M = \ker \mathbf{1}_M = \ker(\operatorname{Res}_M^G \psi) = M \cap \ker\psi \subseteq \ker\psi.$$

Assume now that $\psi \neq \mathbf{1}_G$. Then

$$M \subseteq \ker\psi = \operatorname{Core}_G M \subsetneq M,$$

a contradiction. Hence $\psi = \mathbf{1}_G$. $\square$

17.9. Theorem (Wall). *Let G be a finite solvable group. Then the number of maximal subgroups of G is at most $|G|-1$.*

Proof. For a maximal subgroup M of G, let $C(M)$ be the set of non-trivial constituents of $\operatorname{Ind}_M^G \mathbf{1}_M$, that is

$$C(M) = \{\chi \in \operatorname{Irr}(G) : \chi \neq \mathbf{1}_G \text{ and } \langle \eta, \operatorname{Ind}_M^G \mathbf{1}_M\rangle \neq 0\}.$$

Let M_1 and M_2 be maximal subgroups of G. We claim that $C(M_1) \cap C(M_2) = \emptyset$ if M_1 and M_2 are not conjugate.

Assume that M_1 and M_2 are not conjugate. By Ore's theorem 17.4, $G = M_1M_2$. In particular, there is only one double (M_1, M_2)-coset, with representative 1. Thus

$$\operatorname{Res}_{M_1}^G \operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2} = \operatorname{Ind}_{M_1\cap M_2}^{M_1} \operatorname{Res}_{M_1\cap M_2}^{M_2} \mathbf{1}_{M_2} = \operatorname{Ind}_{M_1\cap M_2}^{M_1} \mathbf{1}_{M_1\cap M_2}.$$

Using Frobenius' reciprocity (Theorem 7.20) and Mackey's theorem 10.7,

$$\begin{aligned}
\langle \operatorname{Ind}_{M_1}^G \mathbf{1}_{M_1}, \operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2}\rangle &= \langle \mathbf{1}_{M_1}, \operatorname{Res}_{M_1}^G \operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2}\rangle\\
&= \langle \mathbf{1}_{M_1}, \operatorname{Ind}_{M_1\cap M_2}^{M_1} \mathbf{1}_{M_1\cap M_2}\rangle\\
&= \langle \operatorname{Res}_{M_1\cap M_2}^{M_1} \mathbf{1}_{M_1}, \mathbf{1}_{M_1\cap M_2}\rangle\\
&= \langle \mathbf{1}_{M_1\cap M_2}, \mathbf{1}_{M_1\cap M_2}\rangle\\
&= 1.
\end{aligned}$$

For $i \in \{1,2\}$, by Frobenius' reciprocity,

$$\langle \operatorname{Ind}_{M_i}^G \mathbf{1}_{M_i}, \mathbf{1}_G\rangle = \langle \mathbf{1}_{M_i}, \operatorname{Res}_{M_i}^G \mathbf{1}_G\rangle = \langle \mathbf{1}_{M_i}, \mathbf{1}_{M_i}\rangle = 1.$$

Thus $\mathbf{1}_G$ is an irreducible constituent of both $\operatorname{Ind}_{M_1}^G \mathbf{1}_{M_1}$ and $\operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2}$. Since

$$\langle \operatorname{Ind}_{M_1}^G \mathbf{1}_{M_1}, \operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2}\rangle = 1,$$

the set of non-trivial irreducible constituents of $\operatorname{Ind}_{M_1}^G \mathbf{1}_{M_1}$ and $\operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2}$ are disjoint, that is

$$\operatorname{Ind}_{M_1}^G \mathbf{1}_{M_1} = \mathbf{1}_G + \eta_1, \quad \operatorname{Ind}_{M_2}^G \mathbf{1}_{M_2} = \mathbf{1}_G + \eta_2, \quad \langle \eta_1, \eta_2\rangle = 0.$$

Hence $C(M_1) \cap C(M_2) = \emptyset$.

Let X be the set of maximal subgroups of G that are normal in G, and let Y be the set of representatives of non-normal maximal subgroups of G. By Exercise 17.5, the number of maximal subgroups of G is

$$m = |X| + \sum_{M\in Y} (G:M).$$

For every maximal subgroup M of G such that M is normal in G, we know that $\operatorname{Ind}_M^G \mathbf{1}_M$ decomposes as $\operatorname{Ind}_M^G \mathbf{1}_M = \mathbf{1}_G + \eta_1 + \cdots + \eta_k$ for some $\eta_1, \dots, \eta_k \in \operatorname{Irr}(G) \setminus \{\mathbf{1}_G\}$ such that $\eta_j(1) = 1$ for all $j \in \{1, \dots, k\}$. Then

$$(G : M) - 1 = \eta_1(1) + \cdots + \eta_k(1) = \sum_{i=1}^{k} \eta_i(1)^2$$

since $\eta_j(1) = 1$ for all j. Since p is the smallest prime divisor of $|G|$, it follows that

$$\sum_{\eta \in C(M)} \eta(1)^2 = (G : M) - 1 \geq p - 1.$$

For every maximal subgroup M of G such that M is not normal in G, $\operatorname{Ind}_M^G \mathbf{1}_M$ decomposes as $\operatorname{Ind}_M^G \mathbf{1}_M = \mathbf{1}_G + \xi_1 + \cdots + \xi_l$ for some distinct characters $\xi_1, \dots, \xi_k \in \operatorname{Irr}(G)$ such that $\xi_j(1) \geq p$ for all $j \in \{1, \dots, k\}$. Then

$$\sum_{\xi \in C(M)} \xi(1)^2 \geq p \sum_{\xi \in C(M)} \xi(1) = p((G : M) - 1) \geq (p - 1)(G : M).$$

Now

$$\begin{aligned} |G| - 1 = \sum_{\mathbf{1}_G \neq \chi \in \operatorname{Irr}(G)} \chi(1)^2 &\geq \sum_{M \in X} \sum_{\eta \in C(X)} \eta(1)^2 + \sum_{M \in Y} \sum_{\xi \in C(X)} \xi(1)^2 \\ &\geq (p - 1)|X| + (p - 1) \sum_{M \in Y} (G : M) \\ &= (p - 1) \left(|X| + \sum_{M \in Y} (G : M) \right) = (p - 1)m. \end{aligned}$$

□

17.10. Exercise. Prove that a finite solvable group has exactly $|G| - 1$ maximal subgroups if and only if it is an elementary abelian 2-group.

17.11. Example. Let $G = \mathbb{S}_3$. Recall from Table 3.1 that $\operatorname{Irr}(G) = \{\mathbf{1}_G, \operatorname{sign}, \chi\}$, where sign is the sign representation and $\chi \colon G \to \mathbb{C}^\times$ is given by

$$\chi(g) = \begin{cases} 2 & \text{if } g = \mathrm{id}, \\ 0 & \text{if } g \in \{(12), (13), (23)\}, \\ -1 & \text{if } g \in \{(123), (132)\}. \end{cases}$$

The group G has two conjugacy classes of maximal subgroups, namely

$$\{\langle(123)\rangle\} \text{ and } \{\langle(12)\rangle, \langle(23)\rangle, \langle(13)\rangle\}.$$

As the group G is rather small, this can be easily verified by a direct calculation. In any case, here is the Magma code:

```
> S3 := Sym(3);
> max := MaximalSubgroups(S3);
> max;
Conjugacy classes of subgroups
------------------------------

```

```
[1]     Order 2             Length 3
        Permutation group M acting on a set of cardinality 3
        Order = 2
            (2, 3)
[2]     Order 3             Length 1
        Permutation group N acting on a set of cardinality 3
        Order = 3
            (1, 2, 3)
```

Let $M = \langle(23)\rangle$ and $N = \langle(123)\rangle$. Thus $X = \{N\}$ and $Y = \{M\}$.

Let us compute $C(N)$. For that purpose, let $t_1 = 1$ and $t_2 = (12)$ be a transversal of N in G. Then

$$(\operatorname{Ind}_N^G \mathbf{1}_N)(g) = \mathbf{1}_N^0(g) + \mathbf{1}_N^0((12)g(12)) = \begin{cases} 2 & \text{if } g = \mathrm{id}, \\ 0 & \text{if } g \in \{(12),(23),(13)\}, \\ 2 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{Ind}_N^G \mathbf{1}_N = \mathbf{1}_G + \mathrm{sign}$ and $C(N) = \{\mathrm{sign}\}$. Here is the Magma code:

```
> N := max[2]`subgroup;
> g := Character(TrivialRepresentation(N));
> ind_N := Induction(g, S3);
> ind_N;
( 2, 0, 2 )
```

To decompose our induced character, with Magma we proceed as follows:

```
> T := CharacterTable(S3);
> Decomposition(T, ind_N);
[
    1,
    1,
    0
]
( 0, 0, 0 )
> InnerProduct(T[3], ind_N);
0
> InnerProduct(T[2], ind_N);
1
> InnerProduct(T[1], ind_N);
1
```

Let us now compute $C(M)$. A direct calculation shows that

$$(\operatorname{Ind}_M^G \mathbf{1}_M)(g) = \begin{cases} 3 & \text{if } g = \mathrm{id}, \\ 1 & \text{if } g \in \{(12),(23),(13)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{Ind}_M \mathbf{1}_M = \mathbf{1}_G + \chi$ and $C(M) = \{\chi\}$. We leave it as an exercise to verify these calculations, either by hand, with Magma, or perhaps both.

17.12. Example. Let $G = \mathbb{A}_4$. There are two conjugacy classes of maximal subgroups of G with representatives are $M = \langle (234) \rangle$ and $N = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Then $X = \{N\}$ and $Y = \{M\}$.

In Exercise 4.15, we asked for the construction of the character table of $\mathbb{A}_4$. The completed table is shown in Table 9.

Table 9. The Character table of $\mathbb{A}_4$.

	id	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$
χ_3	1	1	$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$
χ_4	3	-1	0	0

A direct calculation shows that $\operatorname{Ind}_N^G \mathbf{1}_N = \mathbf{1}_G + \chi_2 + \chi_3$ and $\operatorname{Ind}_M^G \mathbf{1}_M = \mathbf{1}_G + \chi_4$.

The ideas used here to prove Theorem 17.9 can be applied to obtain the following variant of Wall's theorem, established by Cook, Wiegold, and Williamson in [6].

17.13. Theorem (Cook–Wiegold–Williamson). *Let G be a finite solvable group and p the smallest prime divisor of $|G|$. Then the number of maximal subgroups of G is at most $\frac{|G|-1}{p-1}$. Equality holds if and only if G is an elementary p-group.*

17.14. Bonus exercise. Prove Theorem 17.13.

17.15. Exercise. Let H be a subgroup of a finite group G and $\chi \in \operatorname{Char}(H)$. Prove that $\ker \operatorname{Ind}_H^G \chi = \bigcap_{x\in G} x(\ker\chi)x^{-1}$.

17.15. Let $g \in \ker \operatorname{Ind}_H^G \chi$. Then

$$\sum_{x\in G} \chi^0(x^{-1}gx) = \sum_{x\in G} \chi(1).$$

Then $\chi^0(x^{-1}gx) = \chi(1)$ for all $x \in G$. (Otherwise, $|\chi^0(y^{-1}gy)| < \chi(1)$ for some $y \in G$ and hence

$$\sum_{x\in G} \chi(1) = \left| \sum_{x\in G} |\chi^0(x^{-1}gx) \right| \leq \sum_{x\in G} |\chi^0(x^{-1}gx)| < \sum_{x\in G} \chi(1),$$

a contradiction.) In particular, $x^{-1}gx \in \ker\chi \subseteq H$ for all $x \in G$, that is $g \in x(\ker\chi)x^{-1}$ for all $x \in G$.

Conversely, let $g \in \bigcap_{x\in G} x(\ker\chi)x^{-1}$. Then $g \in x(\ker\chi)x^{-1}$ for all $x \in G$. This implies that $x^{-1}gx \in \ker\chi \subseteq H$ for all $x \in G$. Thus

$$\chi^0(x^{-1}gx) = \chi(x^{-1}gx) = \chi(1)$$

for all $x \in G$. Summing over all $x \in G$ and dividing by $|H|$,

$$(\operatorname{Ind}_H^G \chi)(g) = \frac{1}{|H|} \sum_{x\in G} \chi^0(x^{-1}gx) = (G:H)\chi(1) = (\operatorname{Ind}_H^G \chi)(1).$$

17.16. Exercise. Let M be a maximal subgroup of a finite group G, and $\mathbf{1}_G \neq \chi \in \operatorname{Irr}(G)$ be a constituent of $\operatorname{Ind}_M^G \mathbf{1}_M$. Prove that $\ker\chi = \operatorname{Core}_G M$.

17.16. We first prove that $\ker\chi \subseteq \operatorname{Core}_G M$. If not, $G = (\ker\chi)M$ because M is a maximal subgroup of G.

We claim that $\operatorname{Res}_M^G \chi \in \operatorname{Irr}(M)$. Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation with character χ. Every $g \in G$ can be written as $g = xm$ for $x \in \ker\chi$ and $m \in M$. Thus $\rho_g = \rho_x\rho_m = \rho_m$ and a subspace W of V such that $\rho|_M(W) \subseteq W$ will also be such that $\rho(W) \subseteq W$.

By Frobenius' reciprocity,

$$0 \neq \langle \chi, \operatorname{Ind}_M^G \mathbf{1}_M \rangle = \langle \operatorname{Res}_M^G \chi, \mathbf{1}_M \rangle.$$

Hence $\operatorname{Res}_M^G \chi = \mathbf{1}_M$. This implies that $M \subseteq \ker\chi$ and hence $M = (\ker\chi)M = G$, a contradiction. Therefore $\ker\chi \subseteq M$ and hence $\ker\chi \subseteq \operatorname{Core}_G M$.

For the other inclusion, use Exercise 17.15.

18. Project: Fourier analysis on groups

§ 18.1. Fourier transform on abelian groups.

18.1. Definition. Let G be a finite group and $\alpha, \beta \in L(G)$. The **convolution** of α and β is the function

$$\alpha * \beta \colon G \to \mathbb{C}, \quad (\alpha * \beta)(x) = \sum_{y \in G} \alpha(xy^{-1})\beta(y).$$

18.2. Exercise. Let G be a finite group. For $x \in G$, let

$$\delta_x \colon G \to \mathbb{C}, \quad \delta_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\delta_{xy} = \delta_x * \delta_y$.

A direct calculation shows that $L(G)$ is a commutative ring with the operations

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x), \quad (\alpha * \beta)(x) = \sum_{y \in G} \alpha(xy^{-1})\beta(y).$$

18.3. Proposition. *Let G be a finite group and $f \in L(G)$. Then $f \in \mathrm{ClassFun}(G)$ if and only if $f * \alpha = \alpha * f$ for all $\alpha \in L(G)$. In particular, $\mathrm{ClassFun}(G) = Z(L(G))$.*

Proof. If $f \in \mathrm{ClassFun}(G)$, then

$$(\alpha * f)(x) = \sum_{y \in G} \alpha(xy^{-1})f(y) = \sum_{y \in G} \alpha(xy^{-1})f(xyx^{-1}).$$

Let $z = xy^{-1}$. Then

$$(\alpha * f)(x) = \sum_{z \in G} \alpha(z)f(xz^{-1}) = \sum_{z \in G} f(xz^{-1})\alpha(z) = (f * \alpha)(x).$$

Conversely, if $f * \alpha = \alpha * f$ for all α, then, in particular,

$$f(zx) = \sum_{y \in G} \delta_{z^{-1}}(xy^{-1})f(y) = (\delta_{z^{-1}} * f)(x) = (f * \delta_{z^{-1}})(x) = \sum_{y \in G} f(xy^{-1})\delta_{z^{-1}}(y)$$

for all $z \in G$. Thus $f(zxz^{-1}) = f(z^{-1}zx) = f(x)$ for all $x, z \in G$. $\square$

18.4. Exercise. Let G be a finite abelian group and $\widehat{G} = \mathrm{Irr}(G)$. Prove that $\widehat{G}$ with the operation

$$(\chi\theta)(g) = \chi(g)\theta(g), \quad g \in G,$$

is an abelian group of order $|G|$.

18.5. Example. Let $G = \mathbb{Z}/n$. Then

$$\widehat{G} = \{\chi \colon G \to \mathbb{C}^\times : \chi \text{ is a group homomorphism}\} = \{x \mapsto \exp(2\pi i a x/n) : a \in \mathbb{Z}\}.$$

18.6. Definition. Let G be a finite abelian group. The **Fourier transform** of $f \in L(G)$ is the function

$$\widehat{G} \to \mathbb{C}, \quad \chi \mapsto \widehat{f}(\chi) = |G|\langle f, \chi\rangle = \sum_{x \in G} f(x)\overline{\chi(x)}.$$

18.7. EXAMPLE. We compute the Fourier transform of the map $f\colon \mathbb{Z}/n \to \mathbb{C}$, $f(x) = 1$. If $\chi_j(x) = \exp(2\pi ijx/n)$, then

$$\widehat{f}(\chi_j) = n\langle f, \chi_j\rangle = \sum_{m\in\mathbb{Z}/n} f(m)\overline{\chi_j(m)} = \sum_{m\in\mathbb{Z}/n} \exp(-2\pi imj/n) = \begin{cases} 1 & \text{if } j=0,\\ 0 & \text{otherwise.}\end{cases}$$

18.8. EXERCISE. Let $f(x) = \frac{1}{2}(\delta_1(x) + \delta_{-1}(x))$. Prove that

$$\widehat{f}(y) = \cos(2\pi y/n).$$

18.9. EXERCISE. Let $f(x) = \frac{1}{3}(\delta_1(x) + \delta_0(x) + \delta_{-1}(x))$. Prove that

$$\widehat{f}(y) = \frac{1}{3}(1 + 2\cos(2\pi y/n)).$$

The following result is known as the **inversion formula**.

18.10. PROPOSITION. *Let G be a finite abelian group and $f \in L(G)$. Then*

$$f = \frac{1}{|G|}\sum_{\chi\in\widehat{G}} \widehat{f}(\chi)\chi.$$

In particular, the map $L(G) \to L(\widehat{G})$, $f \mapsto \widehat{f}$, is a linear isomorphism.

PROOF. Since G is abelian, $\widehat{G}$ is an orthonormal basis of $L(G)$. Thus

$$f = \sum_{\chi\in\widehat{G}} \langle f, \chi\rangle\chi = \frac{1}{|G|}\sum_{\chi\in\widehat{G}} |G|\langle f, \chi\rangle\chi = \frac{1}{|G|}\sum_{\chi\in\widehat{G}} \widehat{f}(\chi)\chi.$$

A direct calculation shows that $f \mapsto \widehat{f}$ is linear and injective. It is then bijective since $\dim L(G) = \dim L(\widehat{G})$. $\square$

There are two products that turn $L(G)$ into a commutative ring. One of these is pointwise multiplication; the other is convolution. These two ring structures are isomorphic, as the following result shows:

18.11. THEOREM. *Let $\alpha, \beta \in L(G)$. Then $\widehat{(\alpha * \beta)}(\chi) = (\alpha\beta)(\chi)$ for all $\chi \in \widehat{G}$.*

PROOF. Note that

$$\widehat{(\alpha * \beta)}(\chi) = |G|\langle \alpha * \beta, \chi\rangle = \sum_{x\in G} (\alpha * \beta)(x)\overline{\chi(x)} = \sum_{x\in G}\sum_{y\in G} \alpha(xy^{-1})\beta(y)\overline{\chi(x)}.$$

Letting $z = xy^{-1}$ and using that χ is a group homomorphism,

$$\begin{aligned}\widehat{(\alpha * \beta)}(\chi) &= \sum_{y\in G}\beta(y)\sum_{z\in G}\alpha(z)\overline{\chi(zy)}\\ &= \sum_{y\in G}\beta(y)\sum_{z\in G}\alpha(z)\overline{\chi(z)\chi(y)} = |G|\langle\alpha,\chi\rangle|G|\langle\beta,\chi\rangle = \widehat{\alpha}(\chi)\widehat{b}(\chi).\end{aligned}$$ $\square$

18.12. COROLLARY. *Let G be a finite group of order n. Then $L(G) \simeq \mathbb{C}^n$ as algebras.*

Proof. Assume that $\widehat{G}=\{\chi_1,\dots,\chi_n\}$. Let

$$T\colon L(G)\to\mathbb{C}^n,\quad f\mapsto(\widehat{f}(\chi_1),\dots,\widehat{f}(\chi_n)).$$

A routine calculation shows that T is linear. By Proposition 18.10, T is injective. Hence T is bijective since $\dim L(G)=n$. If $\alpha,\beta\in L(G)$, then Theorem 18.11 implies that

$$\begin{aligned}T(\alpha*\beta)&=(\widehat{(\alpha*\beta)}(\chi_1),\dots,\widehat{(\alpha*\beta)}(\chi_n))\\&=(\widehat{\alpha}(\chi_1)\widehat{\beta}(\chi_1),\dots,\widehat{\alpha}(\chi_n)\widehat{\beta}(\chi_n))\\&=(\widehat{\alpha}(\chi_1),\dots,\widehat{\alpha}(\chi_n))(\widehat{\beta}(\chi_1),\dots,\widehat{\beta}(\chi_n))\\&=T(\alpha)T(\beta).\end{aligned}$$

□

The following result is known as the **Plancherel formula**.

18.13. Exercise. Let G be a finite abelian group and $\alpha,\beta\in L(G)$. Prove that

$$|G|\langle\alpha,\beta\rangle=\langle\widehat{\alpha},\widehat{\beta}\rangle.$$

§ 18.2. Application: graph theory. Recall that a **graph** Γ is a pair (V,E), where E is a subset of the set of non-ordered pairs of V. The set $V=V(\Gamma)$ is the set of **vertices** of Γ and $E=E(\Gamma)$ is the set of **edges** of Γ. If V and E are finite, then the graph (V,E) is said to be **finite**. The **adjacency matrix** of a finite graph Γ with n vertices is the matrix $A=(A_{ij})_{1\le i,j\le n}$ given by

$$A_{ij}=\begin{cases}1 & \text{if the vertices } v_i \text{ and } v_j \text{ are connected},\\ 0 & \text{otherwise.}\end{cases}$$

The adjacency matrix of a finite graph is symmetric and hence diagonalizable with real eigenvalues. The **spectrum** of Γ is the set $\operatorname{Spec}(\Gamma)$ of eigenvalues of its adjacency matrix.

18.14. Lemma. *Let G be a finite abelian group and $\alpha\in L(G)$. Then $A\colon L(G)\to L(G)$, $\beta\mapsto\alpha*\beta$, is diagonalizable. Moreover, each $\chi\in\widehat{G}$ is an eigenvector of A with eigenvalue $\widehat{a}(\chi)$.*

Proof. Let $\chi,\theta\in\widehat{G}$. By Theorem 18.11,

$$\widehat{(\alpha*\chi)}(\theta)=\widehat{\alpha}(\theta)\widehat{\chi}(\theta)=\widehat{\alpha}(\theta)|G|\delta_\chi(\theta),\quad \delta_\chi(\theta)=\begin{cases}1 & \text{if } \chi=\theta,\\ 0 & \text{otherwise.}\end{cases}$$

By the inversion formula,

$$\alpha*\chi=\frac{1}{|G|}\sum_{\theta\in\widehat{G}}\widehat{(\alpha*\chi)}(\theta)\theta=\frac{1}{|G|}\sum_{\theta\in\widehat{G}}\widehat{\alpha}(\theta)|G|\delta_\chi(\theta)\theta=\widehat{\alpha}(\chi)\chi.$$

Thus each χ is an eigenvector of A with eigenvalue $\widehat{\alpha}(\chi)$. Since the $\chi\in\widehat{G}$ form an orthogonal basis of eigenvectors, A is diagonalizable. □

For a group G and a subset $S\subseteq G$, we say that S is a **symmetric subset** if $1\notin S$ and for each $s\in S$ one has $s^{-1}\in S$.

18.15. Definition. Let G be a finite group and $S \subseteq G$ be a symmetric subset. The **Cayley graph** of G with respect to S is the graph $X(G,S)$ with vertices G and edges of the form $\{g, sg\}$ for $g \in G$ and $s \in S$.

18.16. Exercise. Prove that $X(G,S)$ is connected if and only if $G = \langle S\rangle$.

18.17. Theorem. *Let $G = \{g_1, \dots, g_n\}$ be a finite abelian group and $S \subseteq G$ be a symmetric subset. Assume that $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_n\}$. If A is the adjacency matrix of $X(G,S)$, then*

$$v_i = \frac{1}{\sqrt{n}} \begin{pmatrix} \chi_i(g_1) \\ \vdots \\ \chi_i(g_n) \end{pmatrix}$$

is an orthonormal basis of eigenvectors of A, where

$$Av_i = \lambda_i v_i, \quad \lambda_i = \sum_{s \in S} \chi_i(s)$$

for all $i \in \{1, \dots, n\}$.

Proof. Let $\delta_S = \sum_{s\in S} \delta_s$ be the characteristic function of S and $F \colon L(G) \to L(G)$, $b \mapsto \delta_S * b$. By Lemma 18.14, F is diagonalizable with eigenvalues

$$\widehat{\delta_S}(\chi_i) = n\langle \delta_S, \chi_i\rangle = \sum_{x\in G} \delta_S(x)\overline{\chi_i(x)} = \sum_{s\in S} \overline{\chi_i(s)} = \sum_{s\in S} \chi_i(s^{-1}) = \sum_{s\in S} \chi_i(s),$$

since S is a symmetric subset.

The matrix $[F]$ of F in the basis $\{\delta_{g_1}, \dots, \delta_{g_n}\}$ has eigenvectors $v_1, \dots, v_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Moreover, $[F]$ is the adjacency matrix of $X(G,S)$, that is

$$[F]_{ij} = \begin{cases} 1 & \text{if } g_i = sg_j, \\ 0 & \text{otherwise,} \end{cases}$$

since

$$F(\delta_{g_j}) = \delta_S * \delta_{g_j} = \sum_{s\in S} \delta_s * \delta_{g_j} = \sum_{s\in S} \delta_{sg_j},$$

by Exercise 18.2. Since the χ_j are orthonormal, so are the v_j. □

18.18. Definition. A **circulant matrix** is a matrix $A = (A_{ij}) \in \mathbb{C}^{n\times n}$ such that there exists a map $f \colon \mathbb{Z}/n \to \mathbb{C}$ with $A_{ij} = f(j-i)$ for all $i, j \in \mathbb{Z}/n$, that is a matrix of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

18.19. Example. A 3×3 circulant matrix is of the form $\begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix}$.

18.20. Exercise. Let G be a finite group and S be a symmetric subset of G. Prove that the circulant matrix corresponding to the characteristic function of S is the adjacency matrix of $X(G,S)$.

18.21. Corollary. *Let A be a circulant matrix of size $n\times n$, that is the adjacency matrix of some $X(\mathbb{Z}/n,S)$. The eigenvalues of A are*

$$\lambda_k=\sum_{m\in S}\exp(2\pi ikm/n),\quad k\in\{0,\dots,n-1\}.$$

The corresponding orthonormal basis of eigenvectors is

$$v_k=\frac{1}{\sqrt{n}}(1,\exp(2\pi ik2/n),\dots,\exp(2\pi ik(n-1)/n))^T,\quad k\in\{0,\dots,n-1\}.$$

Proof. It follows immediately from Theorem 18.17 with $G=\mathbb{Z}/n$. □

§ 18.3. Application: elementary geometry. Let P be a polygon in $\mathbb{R}^2$ with vertices $z_0,z_1,\dots,z_{k-1}$. For each $j\geq 1$, let d_j be the midpoint between z_{j-1} and z_j, that is

$$d_j=\frac{1}{2}(z_{j-1}+z_j)$$

Let $z\colon\mathbb{Z}/k\to\mathbb{C}$, $z(j)=z_j$, and

$$D_z=\frac{1}{2}((\delta_0+\delta_1)*z)\in L(\mathbb{Z}/k).$$

18.22. Exercise. Prove that $D_z(j)=d_j$ for all $j\in\{1,\dots,k-1\}$.

Now we consider the polygon P' with vertices $D_z(1),\dots,D_z(k-1)$. By repeatedly taking midpoints, we obtain a sequence of polygons

$$P,P^{(1)},P^{(2)},P^{(3)},\dots$$

where $P^{(0)}=P$ and $P^{(n+1)}=(P^{(n)})'$ for $n\geq 0$.

Our goal is to show that as $n\to\infty$, the polygon $P^{(n)}$ converges to the **centroid** of P, defined by

$$\frac{1}{k}\sum_{j=0}^{k-1}z_j.$$

To aid in understanding the underlying ideas, we will slightly abuse notation in the formulation of this result.

18.23. Theorem. *Let P be a plane polygon with vertices $z_0,z_1,\dots,z_{k-1}$. Then*

$$\lim_{n\to\infty}P^{(n)}=\frac{1}{k}(z_0+\cdots+z_{k-1}).$$

Proof. Without loss of generality we may assume that P has its centroid at the origin, that is $\sum_{j=0}^{k-1}z(j)=0$. Let $d=\frac{1}{2}(\delta_0+\delta_1)$. By Exercise 18.22,

$$D_z=\frac{1}{2}(\delta_0+\delta_1)*z=d*z.$$

Identifying $\mathbb{Z}/k$ with $\widehat{\mathbb{Z}/k}$ via $j \mapsto \chi_j$, where $\chi_j(m) = \exp(2\pi i j m/k)$, we compute

$$\widehat{(d * z)}(j) = \widehat{d}(j)\widehat{z}(j) = \frac{1}{2}(1 + \exp(-2\pi i j/k)).$$

Hence

$$\lim_{n\to\infty} \widehat{d^{*n}}(j) = \begin{cases} 0 & \text{if } j \in \{1, \dots, k_1\}, \\ 1 & \text{if } j = 0, \end{cases}$$

since

$$\begin{aligned} \lim_{n\to\infty} \widehat{d^{*n}}(j) &= \lim_{n\to\infty} \left(\frac{1}{2}(1 + \exp(-2\pi i j/k)\right)^r \\ &= \lim_{n\to\infty} \left(\frac{1}{2}\exp(-\pi i j/k)(\exp(\pi i j/k) + \exp(-\pi i j/k)\right)^n \\ &= \lim_{n\to\infty} \frac{1}{2^n}\exp(-\pi i j n/k)(2\cos \pi j/k)^n \\ &= \lim_{n\to\infty} \exp(-\pi i j n/k)(\cos(\pi j/k)^n \end{aligned}$$

and

$$|\cos(-2\pi i j/k)| < 1$$

if $j \in \{1, \dots, k-1\}$.

Since $\widehat{z}(0) = |G|\langle z, 1\rangle = \sum_{m\in\mathbb{Z}/k} z(m) = 0$, it follows that

$$\lim_{n\to\infty} \widehat{(d^{*n} * z)}(j) = \lim_{n\to\infty} \widehat{d^{*n}}(j)\widehat{z}(j) = 0.$$

Applying the inverse of the Fourier transform, we conclude that $\lim_{n\to\infty} d^{*r} * z(j) = 0$. $\square$

§ 18.4. Application: Uncertainty principle.

18.24. Definition. Let G be a finite group and $f \in L(G)$. The **support** of f is the set $\operatorname{supp}(f) = \{x \in G : f(x) \neq 0\}$.

18.25. Theorem. *Let G be a finite abelian group. Then* $|G| \leq |\operatorname{supp}(f)||\operatorname{supp}(\widehat{f})|$.

Proof. Note that

$$\|f\|_2^2 = \langle f, f\rangle = \frac{1}{|G|}\sum_{x\in G}|f(x)|^2 = \frac{1}{|G|}\sum_{x\in\operatorname{supp}(f)}|f(x)|^2 \leq \frac{1}{|G|}|\operatorname{supp}(f)|\|f\|_\infty^2, \tag{18.1}$$

where $\|f\|_\infty = \max_{x\in G}|f(x)|$.

By the inversion formula of Proposition 18.10,

$$|f(x)| = \left|\frac{1}{|G|}\sum_{\chi\in\widehat{G}}\widehat{f}(\chi)\chi(x)\right| \leq \frac{1}{|G|}\sum_{\chi\in\widehat{G}}|\widehat{f}(\chi)||\chi(x)| \leq \frac{1}{|G|}\sum_{\chi\in\widehat{G}}|\widehat{f}(\chi)|$$

since $|\chi(x)|\leq 1$ for all $x\in G$. Hence

$$\begin{aligned}\|f\|_\infty^2 &\leq \frac{1}{|G|^2}\left(\sum_{\chi\in\operatorname{supp}(\widehat{f})}|\widehat{f}(\chi)|\right)^2\\ &\leq \frac{1}{|G|^2}\sum_{\chi\in\operatorname{supp}(\widehat{f})}|\widehat{f}(\chi)|^2\sum_{\chi\in\operatorname{supp}(\widehat{f})}1^2=\frac{|\operatorname{supp}(\widehat{f})|}{|G|}\langle\widehat{f},\widehat{f}\rangle=|\operatorname{supp}(\widehat{f})|\|f\|_2^2,\end{aligned}\tag{18.2}$$

where we have used the Cauchy–Schwarz inequality in the second inequality and Plancherel's formula in the last equality.

Combining (18.1) and (18.2),

$$\|f\|_\infty^2\leq|\operatorname{supp}\widehat{f}|\|f\|_2^2\leq\frac{1}{|G|}|\operatorname{supp}\widehat{f}||\operatorname{supp}f|\|f\|_\infty^2.$$

Since $f\neq 0$, the claim follows. □

18.26. Example. If $f=\delta_1$, then $\widehat{f}(\chi)=1$ for all $\chi\in\operatorname{Irr}(G)$. Here the inequality is optimal, as $\operatorname{supp}\widehat{f}=G$ and $\operatorname{supp}f=\{1\}$.

§ 18.5. Fourier transform on non-abelian groups.

18.27. Definition. Let G be a finite group. Let $\phi^1,\dots,\phi^s$ be the equivalence classes of irreducible representations of G. For $k\in\{1,\dots,s\}$, let $d_k=\deg\phi^k$. The **Fourier transform** is the map

$$T\colon L(G)\to M_{d_1}(\mathbb{C})\times\cdots\times M_{d_s}(\mathbb{C}),\quad f\mapsto(\widehat{f}(\phi^1),\dots,\widehat{f}(\phi^s)),$$

where

$$\widehat{f}(\phi^k)=\sum_{g\in G}f(g)\overline{\phi^k_g}.$$

The matrix $\widehat{f}(\phi^k)$ appearing in the Fourier transform is

$$\widehat{f}(\phi^k)_{ij}=|G|\langle f,\phi^k_{ij}\rangle=\sum_{g\in G}f(g)\overline{\phi^k(g)}_{ij}.$$

18.28. Exercise. Prove that T is a linear transformation.

18.29. Exercise. Prove that $\widehat{\delta_x}(\phi)=\overline{\phi_x}$ for every irreducible representation ϕ.

We now present the **inversion formula**.

18.30. Theorem. *Let G be a finite group and $f\in L(G)$. Then*

$$f(x)=\frac{1}{|G|}\sum_{\phi}(\deg\phi)\operatorname{trace}(\phi_x\widehat{f}(\phi)^T),$$

where the sum is taken over all irreducible representations of G.

Proof. As the expression we need to show is linear on f, it is enough to show that the formula is true when $f=\delta_y$. Note that for each unitary ϕ, $\widehat{\delta_y}(\phi)=\overline{\phi_y}$ by Exercise 18.29. Then

$$\begin{aligned}\frac{1}{|G|}\sum_{\phi}(\deg\phi)\operatorname{trace}(\phi_x\overline{\phi_y}^T) &= \frac{1}{|G|}\sum_{\phi}(\deg\phi)\operatorname{trace}(\phi_x\phi_{y^{-1}})\\ &= \frac{1}{|G|}\sum_{\phi}(\deg\phi)\operatorname{trace}(\phi_{xy^{-1}})\\ &= \frac{1}{|G|}\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)\chi(xy^{-1})\\ &= \delta_y(x),\end{aligned}$$

where the last equality follows from Theorem 3.8. $\square$

18.31. Exercise. Let G be a finite group and $f\in L(G)$. Prove that

$$f=\frac{1}{|G|}\sum_{i,j,k}d_k\widehat{f}(\phi^k)_{ij}\phi^k_{ij}.$$

18.32. Example. Let $G=\mathbb{S}_3$. The group G has three irreducible representations, namely the trivial one 1_G, the sign and a degree-two representation $\rho\colon G\to\mathbf{GL}_2(\mathbb{C})$ given by

$$\begin{aligned}&\mathrm{id}\mapsto\begin{pmatrix}1&0\\0&1\end{pmatrix} && (12)\mapsto\begin{pmatrix}-1&-1\\0&1\end{pmatrix}, && (123)\mapsto\begin{pmatrix}-1&-1\\1&0\end{pmatrix},\\ &(13)\mapsto\begin{pmatrix}1&0\\-1&-1\end{pmatrix} && (23)\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix}, && (132)\mapsto\begin{pmatrix}0&1\\-1&-1\end{pmatrix}.\end{aligned}$$

Let us compute the Fourier transform of $f\in L(G)$. We first compute

$$\widehat{f}(1_G)=\sum_{x\in G}f(x),\qquad \widehat{f}(\mathrm{sign})=\sum_{x\in G}\mathrm{sign}(x)f(x).$$

Now we compute $\widehat{f}(\rho)$. Since ρ has degree two,

$$\widehat{f}(\rho)=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{C}).$$

Now we compute

$$\begin{aligned}a=\widehat{f}(\rho)_{11}&=\sum_{x\in G}f(x)\overline{\rho(x)}_{11}=f(\mathrm{id})-f(12)+f(13)-f(123),\\ b=\widehat{f}(\rho)_{12}&=\sum_{x\in G}f(x)\overline{\rho(x)}_{12}=-f(12)-f(123)+f(23)+f(132),\\ c=\widehat{f}(\rho)_{21}&=\sum_{x\in G}f(x)\overline{\rho(x)}_{21}=f(123)-f(13)+f(23)-f(132),\\ d=\widehat{f}(\rho)_{22}&=\sum_{x\in G}f(x)\overline{\rho(x)}_{22}=f(\mathrm{id})+f(12)-f(13)-f(132).\end{aligned}$$

Let us see a concrete example. Let

$$f\colon G \to \mathbb{C}, \quad f(x) = \begin{cases} 1 & \text{if } |x| = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then $T(f) = (2, 2, -I)$, where I is the 2×2 identity matrix. By the inversion formula,

$$f(x) = \frac{1}{6}(2 + 2\operatorname{sign}(x) - 2\rho(x)_{11} - 2\rho(x)_{22}).$$

18.33. Theorem (Wedderburn). *The linear transformation T is an algebra isomorphism.*

Proof. The Fourier transform $L(G) \to M_{d_1}(\mathbb{C}) \times \cdots \times M_{d_s}(\mathbb{C})$ is linear (see Exercise 18.28). It is injective because of the inversion formula (see Exercise 18.31). Since $\dim L(G) = |G| = d_1^2 + \cdots + d_s^2$, T is an isomorphism of vector spaces. We now need to show that

$$\widehat{(\alpha * \beta)}(\phi^k) = \widehat{\alpha}(\phi^k)\widehat{\beta}(\phi^k)$$

for all $\alpha, \beta \in L(G)$ and $k \in \{1, \dots, s\}$. In fact,

$$\begin{aligned} \widehat{(\alpha * \beta)}(\phi^k) &= \sum_{x \in G} (\alpha * \beta)(x)\overline{\phi^k(x)} \\ &= \sum_{x,y \in G} \alpha(xy^{-1})\beta(y)\overline{\phi^k(x)} \\ &= \sum_{y \in G} \beta(y) \sum_{x \in G} \alpha(xy^{-1})\overline{\phi^k(x)} \\ &= \sum_{y \in G} \beta(y) \sum_{z \in G} \alpha(z)\overline{\phi^k(zy)} \\ &= \sum_{y \in G} \beta(y)\overline{\phi^k(z)} \sum_{z \in G} \alpha(z)\overline{\phi^k(y)} \\ &= \widehat{\alpha}(\phi^k)\widehat{\beta}(\phi^k). \end{aligned}$$

□

18.34. Theorem (Plancherel). *Let G be a finite group and $\alpha, \beta \in L(G)$. Then*

$$\langle \alpha, \beta \rangle = \frac{1}{|G|^2} \sum_{\phi} (\deg \phi) \operatorname{trace}\left(\widehat{\alpha}(\phi)\widehat{\beta}(\phi)^*\right),$$

where the sum runs over all irreducible representations of G.

Proof. It is enough to show the theorem in the case where $\alpha = \delta_y$. The equality we want to prove is equivalent to

$$\beta(y) = \frac{1}{|G|} \sum_{\phi} (\deg \phi) \operatorname{trace}\left(\phi_y \widehat{\beta}(\phi)^T\right),$$

which follows from Theorem 18.30. □

Some other topics for final projects

We collect here some topics for final presentations. Some topics can also be used as bachelor or master theses.

Kolchin's theorem. If V is a finite-dimensional complex vector space and G is a subgroup of $\mathbf{GL}(V)$ such that every element g of G is unipotent (i.e., $g-1$ is a nilpotent linear transformation), then there exists a basis of V in which all the element of G are represented by upper triangular matrices with ones on the diagonal. See my notes for Associative Algebra) or [**2**, Chapter 2].

Staircase groups. This topic describes a situation similar to that of Kolchin's theorem (see the course Associative Algebra), but more general. See [**2**, Chapter 5].

Solvable and nilpotent groups. The character table of a finite group detects solvability and nilpotency of groups, see [**2**, Chapter 6].

Characters of $\mathbf{GL}_2(q)$ and $\mathbf{SL}_2(q)$. One possible topic is the character table of $\mathbf{GL}_2(q)$, see [**37**, §5.2]. Alternatively, one can present the character table of the group $\mathbf{SL}_2(p)$ following Humphreys's paper [**20**]. The character theory of $\mathbf{SL}_2(q)$ appears in [**37**, §5.2], see [**4**, Chapter 20] for details.

Representations of the symmetric group. See for example [**37**, §10] and [**11**].

Random walks on finite groups. The goal is to construct the character table or the irreducible representations of the symmetric group. The topic has connections with combinatorics and applications to voting and card shuffling. See [**11**, 4] and [**37**, §11].

McKay's conjecture. Prove McKay's conjecture 5.1 for all sporadic simple groups. This was first proved by Wilson in [**47**]. Note that for some "small" sporadic simple groups this can be done with the script presented in §5.1. However, for several sporadic simple groups a different approach is needed. One needs to know the structure of normalizers.

Hirsh's theorem. In [**19**] Hirsch found a generalization of Burnside's Theorem 9.18. If G is a finite group and d is the greatest common divisor of all the numbers p^2-1, where the p's are prime divisors of $|G|$ and r the number of conjugate sets in G. Then

$$|G| \equiv \begin{cases} r \bmod 2d & \text{if } |G| \text{ odd},\\ r \bmod 3 & \text{if } |G| \text{ even and } \gcd(|G|,3)=1.\end{cases}$$

The proof is elementary and does not use character theory. Is it possible to prove Hirsch's theorem using characters?

Irreducible characters of groups of order pq. Let G be a non-abelian group of order pq, where p and q are prime numbers with $p>q$. Then $q \mid p-1$ and G is a Frobenius group (see Exercise 8.12). The character table of Frobenius groups of order pq can be found in [**28**, Chapter 25].

Irreducible characters of the simple group of order 168. The smallest non-abelian simple group is $\mathbb{A}_5$, of order 60. The next smallest is a certain group of order 168. The character table of this group can be found in [**28**, Chapter 27].

Irreducible characters of semidirect products. What can be said about irreducible characters of semidirect products? The case of semidirect products by abelian groups is treated in [**35**, Section 8.2].

Some solutions

1.12. Let $\theta\colon U \times W \to U$, $(u,w) \mapsto u$. Then θ is a group homomorphism such that $\theta(u) = u$ for all $u \in U$. Since U is K-invariant,

$$k^{-1} \cdot \theta(k \cdot v) \in U$$

for all $k \in K$ and $v \in V$. Since K is finite and U is abelian, the map

$$\varphi\colon V \to U, \quad v \mapsto \prod_{k \in K} k^{-1} \cdot \theta(k \cdot v),$$

is well-defined. We claim that φ is a group homomorphism. If $x, y \in V$, then

$$\begin{aligned}
\varphi(xy) &= \prod_{k \in K} k^{-1} \cdot \theta(k \cdot (xy)) \\
&= \prod_{k \in K} k^{-1} \cdot (\theta(k \cdot x)\theta(k \cdot y)) \\
&= \prod_{k \in K} k^{-1} \cdot \theta(k \cdot x) \prod_{k \in K} k^{-1} \cdot \theta(k \cdot y) = \varphi(x)\varphi(y),
\end{aligned}$$

since U is abelian and K acts by automorphisms on V.

We claim that $N = \ker \varphi$ is K-invariant. We need to show that $\varphi(l \cdot x) = l \cdot \varphi(x)$ for all $l \in K$ and $x \in V$. If $l \in K$ and $x \in V$, then

$$l^{-1} \cdot \varphi(l \cdot x) = l^{-1} \cdot \left(\prod_{k \in K} k^{-1} \cdot \theta(k \cdot (l \cdot x)) \right) = \prod_{k \in K} (kl)^{-1} \cdot \theta((kl) \cdot x) = \varphi(x),$$

since kl runs over all the elements of K whenever k runs over all the elements of K. In conclusion, $\ker \varphi$ is K-invariant.

It remains to show that V is the direct product of U and N. By assumption, U is normal in V. We first prove that $U \cap N = \{1\}$. If $u \in U$, then $k \cdot u \in U$ for all $k \in K$. This implies that $k^{-1} \cdot \theta(k \cdot u) = k^{-1} \cdot (k \cdot u) = u$. Hence $\varphi(u) = u^m$. Since this map is bijective by assumption,

$$U \cap N = U \cap \ker \varphi = \{1\}.$$

We now show that $V \subseteq UN$, as the other inclusion is trivial. Since $N = \ker \varphi$,

$$\varphi(V) \subseteq U = \varphi(U) = \varphi(U)\varphi(N) = \varphi(UN)$$

and hence $V \subseteq (UN)N = UN$. Therefore V is the direct product of U and N, as N is normal in V.

1.13. Let $m = |K|$. Since m and $|U|$ are coprime, the map $u \mapsto u^m$ is bijective in U. Since V is a vector space over the field $\mathbb{Z}/p$, it follows that $V = U \times W$ for some subgroup W of V. Now the claim follows from the previous theorem.

1.33. Assume that ϕ is not irreducible. There exists a proper non-zero G-invariant subspace W of V. Thus $\dim W = 1$. Let $w \in W \setminus \{0\}$. For each $g \in G$, $\phi_g(w) \in W$. Thus $\phi_g(w) = \lambda w$ for some λ. This means that w is a common eigenvector for all the ϕ_g. Conversely, if ϕ admits a common eigenvector $v \in V$, then the subspace generated by v is G-invariant.

3.3. Let G be a finite subgroup of $S^1 = \{z \in \mathbb{Z} : z^n = 1\}$. We claim that G is cyclic. Let $n = |G|$. It is enough to show that $G \subseteq \{\exp(2\pi ik/n) : 0 \leq k \leq n-1\}$, since $\{\exp(2\pi ik/n) : 0 \leq k \leq n-1\}$ is cyclic. Let $g \in G$. Since $g^n = 1$, g is a n-th root of one, say $g = \exp(2\pi ik/n)$ for some $k \in \{1, \dots, n-1\}$.

Let $\rho\colon G \to \mathbf{GL}(V)$ be a faithful irreducible representation. Let $z \in Z(G)$ and $g \in G$. The map $T\colon V \to V$, $v \mapsto z \cdot v$ is invariant, since

$$T(g \cdot v) = z \cdot (g \cdot v) = (zg) \cdot v = (gz) \cdot v = g \cdot (z \cdot v) = g \cdot T(v)$$

for all $g \in G$. By Schur's lemma, there exists $\lambda \in \mathbb{C}$ such that $T(v) = \lambda v$ for all $v \in V$. In particular, $\rho(Z(G))$ is isomorphic to a subgroup of S^1. Thus $\rho(Z(G))$ is cyclic. Since ρ is faithful, $Z(G)$ is cyclic.

3.11. All we need to do is carefully study the proof of Theorem 3.10. Let $n = |G|$. The action of G on itself by conjugation induces a group homomorphism $\rho\colon G \to \mathbf{GL}_n(\mathbb{C})$ with character $\chi_\rho = \sum_{i=1}^r m_i\chi_i$. In the proof of Solomon's theorem, we saw that each $m_i = \sum_{j=1}^r \chi_i(g_j)$ is a non-negative integer. Thus

$$\sum_{i=1}^r \sum_{j=1}^r \chi_i(g_j) = \sum_{i=1}^r m_i \in \mathbb{Z}_{\geq 1}.$$

To prove the other inequality,

$$n = \chi_\rho(1) = \sum_{i=1}^r m_i\chi_i(1) \geq \sum_{i=1}^k m_i.$$

Now assume that $\sum_{i=1}^k m_i = |G|$. Then $m_i = 0$ whenever $\chi_i(1) > 1$, that is ρ only have degree-one irreducible components. In particular, ρ is a degree-one representation. To see that $G/Z(G)$ is abelian, it is enough to see that $[G,G] \subseteq Z(G)$. Since ρ is then a degree-one representation, $\rho(G) \subseteq \mathbb{C}^\times$. Thus

$$\rho([G,G]) \subseteq [\rho(G), \rho(G)] = \{1\}.$$

Hence $[G,G] \subseteq \ker\rho = Z(G)$.

5.8. Let $C_1, \dots, C_t$ be the conjugacy classes of G. For each $i \in \{1, \dots, t\}$, let g_i be a representative of C_i. Assume that g_i is conjugate to g and g_j is conjugate to h. Let $\gamma \in G$. Then

$$\begin{aligned}\sum_{z\in G} \chi(zg_iz^{-1}g_j) &= \sum_{z\in G} \chi(\gamma zg_iz^{-1}g_j\gamma^{-1})\\ &= \sum_{z\in G} \chi(\gamma zg_iz^{-1}\gamma^{-1}\gamma g_j\gamma^{-1})\\ &= \sum_{y\in G} \chi(yg_iy^{-1}\gamma g_j\gamma^{-1}).\end{aligned}$$

Hence

$$\sum_{z\in G} \chi(zg_iz^{-1}g_j) = \frac{1}{|G|} \sum_{z,\gamma\in G} \chi(zg_iz^{-1}\gamma g_j\gamma^{-1}).$$

Now $z_1 g_i z_1^{-1} = z_2 g_i z_2^{-1}$ if and only if $z_2^{-1} z_1 \in C_G(g_i)$. Thus

$$\begin{aligned}\sum_{z\in G}\chi(zg_iz^{-1}g_j) &= \frac{1}{|G|}|C_G(g_i)||C_G(g_j)|\sum_{\substack{x\in C_i\\ y\in C_j}}\chi(xy)\\ &= \frac{|G|}{|C_i||C_j|}\sum_{\substack{x\in C_i\\ y\in C_j}}\chi(xy).\end{aligned}$$

Now

$$\omega_\chi(C_i)\omega_\chi(C_j) = \sum_{i=1}^{t} a_{ijk}\omega_\chi(C_k),$$

where

$$\omega_\chi(C_i) = \frac{|C_i|\chi(C_i)}{\chi(1)}$$

and a_{ijk} is the number of solutions of the equation $xy = z$ with $x \in C_i$, $y \in C_j$ and $z \in C_k$. Therefore

$$\begin{aligned}\frac{\chi(1)}{|G|}\sum_{z\in G}\chi(zg_iz^{-1}g_j) &= \frac{\chi(1)}{|C_i||C_j|}\sum_{\substack{x\in C_i\\ y\in C_j}}\chi(xy)\\ &= \frac{\chi(1)}{|C_i||C_j|}\sum_{k=1}^{t} a_{ijk}\chi(g_k)|C_k|\\ &= \frac{\chi(1)^2}{|C_i||C_j|}\sum_{k=1}^{t} a_{ijk}\omega_\chi(C_k)\\ &= \chi(g_i)\chi(g_j).\end{aligned}$$

To prove the second formula, set $h = g^{-1}$ in the first formula. Then

$$\begin{aligned}\chi(g)\chi(g^{-1}) = \frac{\chi(1)}{|G|}\sum_{z\in G}\chi(zgz^{-1}g^{-1}) &\iff \chi(g)\overline{\chi(g)} = \frac{\chi(1)}{|G|}\sum_{z\in G}\chi([z,g])\\ &\iff |\chi(g)|^2 = \frac{\chi(1)}{|G|}\sum_{z\in G}\chi([z,g])\\ &\iff \frac{|G|}{\chi(1)}|\chi(g)|^2 = \sum_{z\in G}\chi([z,g]).\end{aligned}$$

6.8. Let $g_1, \dots, g_m$ be the representatives of non-trivial conjugacy classes. Then $C_G(g_i)$ is non-tivial for all i. Since p is the smallest prime dividing the order of G, it follows that $(G : C_G(g_i)) \geq p$. Now use the class equation to get

$$|G| \geq |Z(G)| + pm,$$

which is equivalent to $m \leq \frac{1}{p}(|G| - |Z(G)|)$. Since G is non-abelian, $G/Z(G)$ is not cyclic. Thus $(G : Z(G)) \geq p^2$. Now

$$\frac{k(G)}{|G|} = \frac{|Z(G)| + m}{|G|} \leq \frac{(p-1)|Z(G)| + |G|}{p|G|} \leq \frac{p^2+p-1}{p^3}.$$

This bound is reached if and only if $(G : Z(G)) = p^2$.

6.12. If $\operatorname{cp}(G) > 5/8$, then $|[G,G]| < 2$. Thus $[G,G]$ is the trivial group and hence G is abelian.

6.13.

1) If $\operatorname{cp}(G) > 1/2$, then $|[G,G]| < 3$ by Theorem 6.11. If $|[G,G]| = 1$, then G is abelian and hence G is nilpotent. If $|[G,G]| = 2$, then $[G,G] \subseteq Z(G)$. It follows that $G/Z(G)$ is abelian (and hence nilpotent), so G is nilpotent.

2) If $\operatorname{cp}(G) < 21/80$, then $|[G,G]| < 60$. Thus $[G,G]$ is solvable, as groups of order < 60 are solvable. Hence G is solvable.

6.15.

1)

2) Using that σ and τ are automorphisms and the commutativity of the diagram (6.2), we compute

$$\begin{aligned}
(G : Z(G))^2 \operatorname{cp}(G) &= \frac{1}{|Z(G)|^2} |\{(x,y) \in G \times G : xy = yx\}| \\
&= \frac{1}{|Z(G)|^2} |\{(x,y) \in G \times G : [x,y] = 1\}| \\
&= \frac{1}{|Z(G)|^2} |\{(x,y) \in G \times G : c_G(x,y) = 1\}| \\
&= |\{(u,v) \in (G/Z(G))^2 : c_G(u,v) = 1\}| \\
&= |\{(u,v) \in (G/Z(G))^2 : \tau c_G(u,v) = 1\}| \\
&= |\{(u,v) \in (G/Z(G))^2 : c_G(\sigma u, \sigma v) = 1\}| \\
&= |\{(a,b) \in (H/Z(H))^2 : c_H(a,b) = 1\}|.
\end{aligned}$$

It follows that $(G : Z(G))^2 \operatorname{cp}(G) = (H : Z(G))^2 \operatorname{cp}(H)$.

7.12. Assume that $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. Let L be the regular representation of G. Then

$$\chi_L(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Write $\chi_L = \sum_{i=1}^k \chi_i(1)\chi_i$. Since

$$0 \neq \frac{|G|}{|H|}\phi(1) = \langle \operatorname{Res}_H^G \chi_L, \phi \rangle = \sum_{i=1}^k \chi_i(1) \langle \operatorname{Res}_H^G \chi_i, \phi \rangle,$$

there exists $i \in \{1, \dots, k\}$ such that $\langle \operatorname{Res}_H^G \chi_i|_H, \phi \rangle \neq 0$.

7.13. Note that

$$\sum_{i=1}^l d_i^2 = \langle \operatorname{Res}_H^G \chi, \operatorname{Res}_H^G \chi \rangle = \frac{1}{|H|} \sum_{h \in H} \chi(h)\overline{\chi(h)}.$$

Since χ is irreducible,

$$\begin{aligned}1=\langle\chi,\chi\rangle&=\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi(g)}\\&=\frac{1}{|G|}\sum_{h\in H}\chi(h)\overline{\chi(h)}+\frac{1}{|G|}\sum_{g\in G\setminus H}\chi(g)\overline{\chi(g)}\\&=\frac{|H|}{|G|}\sum_{i=1}^{l}d_i^2+\frac{1}{|G|}\sum_{g\in G\setminus H}\chi(g)\overline{\chi(g)}.\end{aligned}$$

Since $\sum_{g\in G\setminus H}\chi(g)\overline{\chi(g)}\geq 0$, we conclude that $\sum_{i=1}^{l}d_i^2\leq(G:H)$. Moreover, the equality holds if and only if $\sum_{g\in G\setminus H}\chi(g)\overline{\chi(g)}=0$, that is, if and only if $\chi(g)=0$ for all $g\in G\setminus H$.

7.16. Let $g\in G$. Then

$$(\operatorname{Ind}_H^G\mathbf{1}_H)(g)=\sum_{x\in G}\mathbf{1}_H^0(x^{-1}gx)=\begin{cases}|G|&\text{if } g=1,\\0&\text{otherwise,}\end{cases}$$

since $x^{-1gx}=1$ if and only if $g=1$. This $\operatorname{Ind}_H^G\mathbf{1}_H$ is equal to the character of the regular representation of G (see Theorem 3.6).

7.17. The action of G on G/H by left multiplication yields a group homomorphism $\rho\colon G\to\mathbb{S}_{G/H}$, $g\mapsto\rho_g$, where $\rho_g(xH)=gxH$. Now

$$xH\in\operatorname{Fix}(g)\iff gxH=xH\iff x^{-1}gx\in H\iff g\in xHx^{-1}.$$

Thus

$$|\operatorname{Fix}(g)|=\frac{1}{|H|}|\{x\in G:x^{-1}gx\in H\}|.$$

Since

$$\mathbf{1}_H^0(x^{-1}gx)=\begin{cases}1&\text{if } x^{-1}gx\in H,\\0&\text{otherwise,}\end{cases}$$

it follows that $(\operatorname{Ind}_H^G\mathbf{1}_H)(g)=|\operatorname{Fix}(g)|$, the character of the representation of G obtained from ρ.

7.18. Write $\psi=\operatorname{Ind}_H^G\rho$. For $x,y\in G$ and $i,j\in\{1,\dots,m\}$,

$$(\psi_x\psi_y)_{ij}=\sum_{k=1}^{m}(\psi_x)_{ik}(\psi_y)_{kj}=\sum_{k=1}^{m}\rho^0_{t_i^{-1}xt_k}\rho^0_{t_k^{-1}yt_j}.$$

If we want each $\rho^0_{t_k^{-1}yt_j}\neq 0$, we need $t_k^{-1}yt_j\in H$ for all k. This means $yt_j\in t_kH$ for all k. But there exists a unique $l\in\{1,\dots,m\}$ such that $yt_jH=t_lH$. Thus $k=l$ and then

$$(\psi_x\psi_y)_{ij}=\rho^0_{t_i^{-1}xt_l}\rho_{t_l^{-1}yt_j}=\rho_{t_i^{-1}xt_l}\rho_{t_l^{-1}yt_j}=\rho_{t_i^{-1}xyt_j}$$

whenever $t_i^{-1}xt_l\in H$ (or equivalently $xt_l\in t_iH$). Note that $t_i^{-1}xyt_j\in H$, since $yt_j\in t_lH$ implies

$$xyt_j\in xt_lH=t_iH.$$

It follows that

$$(\psi_x\psi_y)_{ij}=\rho^0_{t_i^{-1}xyt_j}=(\psi_{xy})_{ij}.$$

8.5. We use the second orthogonality relation and Theorem 8.3 to compute

$$|C_{G/N}(gN)| = \sum_{\chi\in\mathrm{Irr}(G/N)} |\chi(gN)|^2 = \sum_{\substack{\eta\in\mathrm{Irr}(G)\\ N\subseteq\ker\eta}} |\eta(g)|^2 \leq \sum_{\eta\in\mathrm{Irr}(G)} |\eta(g)|^2 = |C_G(g)|.$$

8.12. Assume that G is not abelian. By Sylow's theorems, q divides $p-1$ and there exists a unique Sylow p-subgroup P of G. Let $a, b \in G$ be such that $P = \langle a\rangle \simeq \mathbb{Z}/p$ and $G/P = \langle bP\rangle \simeq \mathbb{Z}/q$. By Lagrange's theorem, $G = \langle a, b\rangle$. We compute the order of b^q. Since G is not cyclic (because it is not abelian) and $b^q \in P$, we conclude that $|b^q| = 1$. Since P is normal in G, $bab^{-1} \in P$ and hence $bab^{-1} = a^z$ for some $z \in \mathbb{Z}$. Therefore $b^q ab^{-q} = a^{z^q}$. This implies that $z^q \equiv 1 \bmod p$. The order of z in $(\mathbb{Z}/p)^\times$ divides q and hence it is equal to q (otherwise, $z = 1$ and thus $bab^{-1} = a$, which implies that G is abelian). In conclusion, $G \simeq F_{p,q}$.

8.14. We first prove that 1) $\implies$ 2). Let $x \notin H$ and $h \in H$ be such that $h \cdot xH = xH$. Then $h \in xHx^{-1} \cap H = \{1\}$.

We now prove 2) $\implies$ 3). Let $x, y \in G$ be such that $xH \neq yH$, and $g \in G$ be an element that fixes both xH and yH, that is $g \cdot xH = xH$ and $g \cdot yH = yH$. Let $h = x^{-1}gx \in H$. Note that $x^{-1}yH \neq H$. Since

$$h \cdot x^{-1}yH = (x^{-1}gx) \cdot x^{-1}yH = x^{-1} \cdot yH = x^{-1}yH,$$

it follows that $h = 1$, which implies $g = 1$.

Finally, we prove that 3) $\implies$ 1). Let $x \notin H$ and $h \in H \cap xHx^{-1}$. Then $hx = xh_1$ for some $h_1 \in H$. Hence

$$h \cdot xH = hxH = xh_1H = xH, \quad h \cdot H = H.$$

Since h has at least two fixed points in G/H, $h = 1$.

8.18. Let $H \neq \{1\}$ be a malnormal subgroup of G and $N = \langle n\rangle \simeq \mathbb{Z}$ be a normal subgroup of G.

Assume first that $N \cap H \neq \{1\}$. Let $k > 0$ be minimal such that $n^k \in H$. If $k = 1$, then $H = N$ is normal in G. Since H is malnormal by assumption, $H = G$.

Assume now that $N \cap H = \{1\}$. Note that for every $h \in H \setminus \{1\}$, $hnh^{-1} \in \{n, n^{-1}\}$. If there exists $h \in H \setminus \{1\}$ such that $hnh^{-1} = n$, then $1 \neq h = nhn^{-1} \in H \cap nHn^{-1}$. Thus H is not malnormal in G, a contradiction.

Since G has no 2-torsion, there exist $h_1, h_2 \in H \setminus \{1\}$ with $h_1 \neq h_2$ and $h_1h_2 \neq 1$. If $h_jnh_j = n$ for some $j \in \{1, 2\}$, then the previous argument shows that H is not malnormal in G. Thus we may assume that $h_1nh_2 = n^{-1}$ and $h_2nh_2 = n^{-1}$. Then

$$(h_2h_1)n(h_2h_1)^{-1} = h_2n^{-1}h_2^{-1} = (h_2nh_2^{-1})^{-1} = n.$$

Hence, again by the previous argument, H is not malnormal in G, a contradiction.

8.11. Since $(\mathbb{Z}/p)^\times$ is cyclic, there exists m such that $u^m = v \bmod p$. As both u and v have order q modulo p, $\gcd(m, q) = 1$. Thus b^m has order q modulo p. Hence

$$b^m ab^{-m} = a^{u^m} = a^v.$$

Let $c = b^m$. Then

$$\begin{aligned}\langle a, b : a^p = b^q = 1, bab^{-1} = a^u\rangle &= \langle a, c : a^p = c^q, cac^{-1} = a^v\rangle \\ &\simeq \langle a, b : a^p = b^q = 1, bab^{-1} = a^v\rangle.\end{aligned}$$

8.12. Assume that G is not abelian. Then q divides $p-1$. By Sylow's theorem, $|\mathrm{Syl}_p(G)| = 1$. Let $P \in \mathrm{Syl}_p(G)$. Then P is normal in G and $|P| = p$. Both P and G/P are cyclic, say $P = \langle a\rangle$ and $G/P = \langle bP\rangle$. Thus $G = \langle a, b\rangle$. Since $b^q \in P$ and b does not have order pq, b has order q. Since P is normal in G, $bab^{-1} \in P = \langle a\rangle$. Thus $bab^{-1} = a^u$ for some u. Moreover,

$$a = b^q a b^q = (a^u)^q = a^{u^q}$$

and hence $u^q \equiv 1 \bmod p$. Thus the ortder of u in $(\mathbb{Z}/p)^\times$ divides q. If u has order one, then $bab^{-1} = a$ and G is abelian, a contradiction. So u has order q and the claim follows.

13.6. Since $|H| = n$, $(G : H) = 2$. Let $t_1 = 1$ and $t_2 = s$. Then $\{t_1, t_2\}$ is a transversal of H in G. We compute

$$\begin{aligned} t_1^{-1} r t_1 &= r, & t_1^{-1} s t_1 &= s, & t_1^{-1} r t_2 &= sr^{-1}, & t_1^{-1} s t_2 &= 1, \\ t_2^{-1} r t_1 &= sr, & t_2^{-1} s t_1 &= 1, & t_2^{-1} r t_2 &= r^{-1}, & t_2^{-1} s t_2 &= s. \end{aligned}$$

Thus

$$(\rho_m)(r) = \begin{pmatrix} \omega_m & 0 \\ 0 & \omega_m^{-1} \end{pmatrix}, \quad (\rho_m)(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

18.2. If $x, y, z \in G$, then

$$(\delta_x * \delta_y)(z) = \sum_{g\in G} \delta_x(zg^{-1})\delta_y(g) = \begin{cases} 1 & \text{if } xy = z, \\ 0 & \text{otherwise.} \end{cases}$$

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