Exercise

Gamma and Beta functions

1. Compute (a) $\frac{\Gamma(6)}{2\Gamma(3)}$ (b) $\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$ (c) $\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)}$ (d) $\Gamma\left(-\frac{5}{2}\right)$.

Ans. (a) 30 (b) $\frac{3}{4}$ (c) $\frac{16}{315}$ (d) $-\frac{8\sqrt{\pi}}{15}$

2. Evaluate (a) $\int_0^\infty \sqrt{y}e^{-y^2}dy$ (b) $\int_0^\infty 3^{-4z^2}dz$ (c) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$.

Ans. (a) [illegible] (b) $\frac{\sqrt{\pi}}{(4\sqrt{\ln 3})}$ (c) $\sqrt{\pi}$

3. Evaluate (a) $\int_0^1 x^4(1-x)^3dx$ (b) $\int_0^2 \frac{x^2dx}{\sqrt{2-x}}$ (c) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$.

Ans. (a) $\frac{1}{280}$ (b) $\frac{64\sqrt{2}}{15}$ (c) $\frac{\pi a^6}{32}$.

4. Evaluate (a) $\int_0^{2\pi} \sin^8\theta\, d\theta$ (b) $\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$ (c) $\int_0^{\frac{\pi}{2}} \sin^4\theta\cdot\cos^5\theta\, d\theta$.

Ans. (a) $I = 4\int_0^{\frac{\pi}{2}} \sin^8\theta d\theta = \frac{4\cdot 1\cdot 3\cdot 5\cdot 7}{(2\cdot 4\cdot 6\cdot 8)}\frac{\pi}{2} = \frac{35\pi}{64}$

(b) $\frac{5\pi}{32}$ (c) $\frac{8}{315}$

5. Show that $\int_0^2 x\sqrt[3]{8-x^3}dx = \frac{16\pi}{(9\sqrt{3})}$

6. Find $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$.

Ans. $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{2}\pi\sqrt{2}$

7. Show that $\int_0^{\frac{\pi}{2}}(\sqrt{\tan\theta} + \sqrt{\sec\theta)}\, d\theta =$ $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{(\pi)}}{\Gamma\left(\frac{3}{4}\right)}\right\}$.

8. Prove that $\int_0^1 x^4\left[\ln\left(\frac{1}{x}\right)\right]^3 dx = \frac{6}{625}$.

9. Show that $\left[\int_0^1 x^2(1-x^4)^{-\frac{1}{2}}dx\right] \times \left[\int_0^1 (1+x^4)^{-\frac{1}{2}}dx\right] = \frac{\pi}{4\sqrt{2}}$.

10. Prove that $\left[\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}d\theta\right]\left[\int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{1}{2}}d\theta\right] = \pi$.

11. Show that $\int_0^{\frac{\pi}{2}} \sin^7\theta\cdot\cos^7\theta\, d\theta = \frac{1}{280}$.

12. Evaluate $\int_0^a x^3(a^3-x^3)^5dx$.

Ans. $\frac{a^{19}\cdot 3^5}{19\cdot 16\cdot 13\cdot 7}$.

13. Prove that $\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is a positive integer and $m > -1$.

14. Prove that $\int_0^\infty \frac{t^2dt}{1+t^4} = \frac{\pi}{\sqrt{2}}$.

Hint: Put $t = \sqrt{\tan\theta}$.

15. Show that the area under the normal curve $y = \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2\sigma^2}}$ and x- axis is unity.

16. Show that $\frac{\beta(p,q)}{p+q} = \frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p}$.

17. Prove that $\beta(m,n) = \frac{1}{2}\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}$.

Hint: Use symmetry property of β function.

18. $\int_0^\infty x^{-\frac{3}{2}}(1-e^{-x})dx$.

Ans. $2\sqrt{\pi}$.

19. Show that $\int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}\cdot\beta(m,n)$

Hint: Put $x = \frac{(t-b)}{(a-b)}$.

20. Prove that $\int_0^\infty e^{-x^4}dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$.

21. Evaluate $\int_0^\infty \frac{x^a}{a^x}dx$.

Ans. $\frac{\Gamma(a+1)}{(\ln a)^{a+1}}$

22. Show that $\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx = \beta(p,q)$

Hint: From (5) $\beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dx = \int_0^1 + \int_0^\infty$. Put $y = \frac{1}{z}$ in 2nd integral.

23. Show that $\int_{-1}^1 \sqrt{\frac{1+t}{1-t}}dt = \pi$.

24. Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$.

Ans. 0

25. Prove that $\int_0^\infty e^{-ax}\cdot x^{n-1}dx = \frac{\Gamma(n)}{a^n}$ where a and n are positive.

11.3 BESSEL'S FUNCTIONS

The boundary value problems (such as the one-dimensional heat equation) with cylindrical symmetry (independent of θ) reduces to two ordinary differential equations by the separation of variables technique. One of them is the most important differential equation known as **Bessel's* differential equation**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \tag{1}$$

Here p, which is a given constant (not necessarily an integer) is known as the order of the Bessel's equation.

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* Friedrich Wilhelm Bessel (1784–1846) German mathematician.

Bessel's Functions (Cylindrical functions)

Bessel's functions (Cylindrical functions) are series solution of the Bessel's differential Equation (1) obtained by Frobenius method.

Assume that p is real and non-negative. Assume the series solution of (1) as

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{2}$$

To determine the unknown coefficients a_m and power (exponent) r, substitute (2) in (1), we get

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$
$$+\sum_{m=0}^{\infty} a_m x^{m+r+2} - p^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

Now equate the sum of the coefficients of x^{s+r} to zero. For $s = 0$ and $s = 1$, the contribution comes from first, second and fourth series (not from third series because it starts with x^{r+2}). For $s \geq 2$, all the four terms contribute. Thus sum of the coefficients of powers of r, $r+1$ and $s+r$ are respectively given by

$$r(r-1)a_0 + ra_0 - p^2 a_0 = 0 \qquad (s=0) \tag{4}$$
$$(r+1)ra_1 + (r+1)a_1 - p^2 a_1 = 0 \qquad (s=1) \tag{5}$$
$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - p^2 a_s = 0 \quad (s = 2, 3\cdots) \tag{6}$$

Solving (4), we get the indicial equation

$$(r+p)(r-p) = 0 \tag{7}$$

Solutions of (7) are $r_1 = p(\geq 0)$ and $r_2 = -p$.

Case 1: $r_1 = p$

With $r_1 = p$, Equation (5) becomes $(2p+1)a_1 = 0$ so $a_1 = 0$

Rewrite (6) as

$$(s+r+p)(s+r-p)a_s + a_{s-2} = 0$$

Substituting $r = p$, this becomes

$$s(s+2p)a_s + a_{s-2} = 0 \tag{8}$$

or
$$a_s = -\frac{a_{s-2}}{s(s+2p)}$$

For
$$s = 3, \quad a_3 = -\frac{a_1}{3(3+2p)}$$

Since $a_1 = 0$ and $p \geq 0$, then $a_3 = 0$. Thus from ... it follows that

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0 \text{ etc.}$$

i.e., all coefficients with odd subscripts are ... Rewriting (8) with $s = 2m$, we have

$$2m(2m+2p)a_{2m} + a_{2m-2} = 0$$

Solving

$$a_{2m} = -\frac{1}{2^2 m(m+p)} \cdot a_{2m-2}, \quad m = 1, 2, \ldots$$

Thus
$$a_2 = -\frac{a_0}{2^2(1+p)}$$
$$a_4 = -\frac{a_2}{2^2 \cdot 2(2+p)} = \frac{a_0}{2^4 2!(p+1)(p+2)}$$

In general

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} \cdot m!(p+1)(p+2)\cdots(p+m)}$$
$$m = 1, 2, \ldots$$

a_0 which is arbitrary may be taken as

$$a_0 = \frac{1}{2^p \Gamma(p+1)}$$

Then
$$a_2 = -\frac{a_0}{2^2(p+1)} = -\frac{1}{2^2 \cdot 2^p (p+1)\Gamma(p+1)}$$
$$= \frac{-1}{2^{2+p}\Gamma(p+2)}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Similarly,

$$a_4 = \frac{-a_2}{2^2 \cdot 2 \cdot (p+2)} = \frac{1}{2^2 \cdot 2 \cdot 2^{2+p} \cdot (p+2)\Gamma(p+2)}$$
$$= \frac{1}{2^{4+p} \cdot 2!\Gamma(p+3)}$$

In general

$$a_{2m} = \frac{(-1)^m}{2^{2m+p} \cdot m!\Gamma(p+m+1)} \quad \text{for } m = 1, 2, \ldots \tag{10}$$

By substituting these coefficients from (10) in (2) and observing that $a_1 = a_3 = a_5 = \cdots = 0$, a particular solution of the Bessel's Equation (1) is obtained as

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p} \cdot m!\Gamma(p+m+1)} \tag{11}$$

is known as the Bessel's function of the first kind of order p, which converges for all x (by ratio test).

For $r_2 = -p$: Replacing p by $-p$ in (11), we get a second linearly independent solution of (1) as

$$J_{-p}(x) = x^{-p}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-p}\, m!\,\Gamma(m-p+1)} \qquad (12)$$

Hence the general solution of Bessel's Equation (1) for $r_1 - r_2 \neq 0$ is

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x) \qquad (13)$$

provided p is not an integer.

Linear Dependence of Bessel's Functions: J_n and J_{-n}

Assume that $p = n$ where n is an integer. Then from (11), we get

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\cdot m!\,\Gamma(n+m+1)}$$

Since $\Gamma(n+1) = n!$, we have $\Gamma(n+m+1) = (n+m)!$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\cdot m!(m+n)!} \qquad (14)$$

Work: Prove that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{for } n = 1, 2, 3, \ldots$$

Proof: Replacing p by $-n$ in (11), we get

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n}\cdot m!\,\Gamma(m-n+1)} \qquad (15)$$

When $m-n+1 \leq 0$ or $m \leq (n-1)$, the gamma function of zero or negative integers is infinite. Therefore for $m = 0$ to $n-1$, the coefficients in (15) become zero. So m starts at n. Thus

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n}\cdot m!(m-n)!}$$

since $\Gamma(m-n+1) = (m-n)!$

Put $m - n = s$ then s varies from 0 to ∞.

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} x^{2(s+n)-n}}{2^{2(s+n)-n}(s+n)!\,s!}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+n}}{2^{2s+n}\cdot s!(s+n)!}$$

$$J_{-n}(x) = (-1)^n J_n(x). \qquad (16)$$

Generating Function

Generating function of a sequence of functions $f_n(x)$ is

$$G(u, x) = \sum_{n=-\infty}^{\infty} f_n(x)\cdot u^n$$

which generates $f_n(x)$ i.e., $f_n(x)$ appear as coefficients of powers of u.

Theorem: *Prove that the generating function for Bessel's functions of integral order is*

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} \qquad (17)$$

Proof: If $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ is the generating function of Bessel function then the coefficients of different powers of t in the expansion of (17) are the Bessel's functions of different integral orders.

Consider

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}}\cdot e^{-\frac{x}{2t}}$$

Expanding in series, we get

$$= \left[1 + \frac{xt}{2} + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right]\times$$

$$\times\left[1 - \frac{x}{2t} + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 \cdots\right] \qquad (18)$$

Case 1: $n = 0$.

The coefficient of $t^0 = 1$ in the expansion (18) is

$$1 - \left(\frac{x}{2}\right)^2 + \left(\frac{1}{2!}\right)^2\left(\frac{x}{2}\right)^4 - \left(\frac{1}{3!}\right)^2\left(\frac{x}{2}\right)^6 + \left(\frac{1}{4!}\right)^2\left(\frac{x}{2}\right)^8 - \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m} = J_0(x). \qquad (19)$$

Case 2: Positive powers of $t : t^n$

The coefficient of t^n in the above expansion (18) is

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!}\frac{1}{(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$= J_n(x). \tag{20}$$

Case 3: Negative powers of $t : t^{-n}$

The coefficient of t^{-n} in the expansion (18) is

$$\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \left(\frac{x}{2}\right)\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+1} + \frac{1}{2!}\left(\frac{x}{2}\right)^2\frac{(-1)^{n+2}}{(n+2)!}\left(\frac{x}{2}\right)^{n+2} + \cdots$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$= (-1)^n J_n(x) = J_{-n}(x) \tag{21}$$

Thus from (19), (20) and (21), we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

Equation Reducible to Bessel's Equation

The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - p^2)y = 0 \tag{22}$$

where λ is a parameter, can be reduced Bessel's differential equation of order p in t,

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - p^2)y = 0 \tag{23}$$

where $t = \lambda x$ $\left(\text{so } \frac{dy}{dx} = \lambda\frac{dy}{dt},\ \frac{d^2y}{dx^2} = \lambda^2\frac{d^2y}{dt^2}\right)$.

For p non-integral, the general solution of Equation (23) is

$$y = c_1J_n(t) + c_2J_{-n}(t).$$

Thus the general solution of Equation (22) is

$$y(x) = c_1J_n(\lambda x) + c_2J_{-n}(\lambda x)$$

when p is non-integral.

Orthogonality of Bessel's Functions

Prove that

$$\int_0^a xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \alpha \ne \beta \\ \frac{a^2}{2}J_{n+1}^2(\alpha a), & \alpha = \beta \end{cases}$$

where α and β are roots of $J_n(ax) = 0$.

Proof: Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively be the solutions of the equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0$$

and
$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ and subtracting

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

or
$$\frac{d}{dx}\left[x(u'v - uv')\right] = (\beta^2 - \alpha^2)xuv$$

Integrating both sides of (3) from $x = 0$ to a

$$(\beta^2 - \alpha^2)\int_0^a xuv\,dx = \left[x(u'v - uv')\right]_0^a$$

$$= a\left[u'(a)v(a) - u(a)v'(a)\right]$$

where $'$ denotes differentiation w.r.t. x.

Now
$$u' = \frac{d}{dx}u \equiv \frac{d}{dx}J_n(\alpha x) = \alpha J_n'(\alpha x)$$

Similarly,
$$v' = \frac{dv}{dx} = \frac{d}{dx}J_n(\beta x) = \beta J_n'(\beta x)$$

Substituting u' and v' from (5) and (6) in (4), we

$$\int_0^a xJ_n(\alpha x)J_n(\beta x)dx$$

$$= \frac{a}{\beta^2 - \alpha^2}\left[\alpha J_n'(\alpha a)J_n(\beta a) - \beta J_n(\alpha a)J_n'(\beta a)\right]$$

Case 1: Suppose α and β are two distinct roots of $J_n(ax) = 0$ then $J_n(a\alpha) = J_n(a\beta) = 0$.

Thus for $\alpha \ne \beta$

$$\int_0^a xJ_n(\alpha x)J_n(\beta x)dx = 0$$

(8) is known as the orthogonality relation for Bessel functions.

Case 2: Suppose $\beta = \alpha$; then R.H.S. of (4) is of indeterminate form. Assuming α as a root of $J_n(ax) = 0$ ...

R.H.S. of (4) as $\beta \to$ ...

$\lim_{\beta\to\alpha}\int_0^a xJ_n(\alpha x)$ ...

$= \lim_{\beta\to\alpha}$ ...

Since $J_n(a\alpha) = 0$. Now applying ... w.r.t. β), we get

$= \lim_{\beta\to}$ ...

$= \frac{a^2}{2}$ ...

In the recurrence ...

Put $x = a\alpha$, ...

Since α is a r...

Thus for α ...

Note: ... Page 11. ...

Since ... Thus ...

Rec... Bes...

Vali...

1. ...

... as $\beta \to \alpha$

$$\int_0^a x J_n(\alpha x) J_n(\beta x)\, dx$$

$$= \lim_{\beta \to \alpha} \left(\frac{a}{\beta^2 - \alpha^2}\right)\left[\alpha J_n'(a\alpha) J_n(a\beta) - 0\right]$$

... $J_n(a\alpha) = 0$. ... applying L'Hospital's rule (differentiating ...), we get

$$= \lim_{\beta \to \alpha} \frac{a}{2\beta}\left[\alpha J_n'(a\alpha) \cdot a J_n'(a\beta)\right]$$

$$= \frac{a^2}{2}\left[J_n'(a\alpha)\right]^2$$

... recurrence relation IV on Page 11.14

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x)$$

... $x = a\alpha$, then $J_{n+1}(a\alpha) = \frac{n}{a\alpha} J_n(a\alpha) - J_n'(a\alpha)$.

... $a\alpha$ is a root, $J_n(a\alpha) = 0$. Then

$$J_n'(a\alpha) = -J_{n+1}(a\alpha)$$

... for $\alpha \neq \beta$.

$$\int_0^a x J_n(\alpha x) J_n(\beta x)\, dx = \frac{a^2}{2}\left[J_n'(a\alpha)\right]^2$$

$$= \frac{a^2}{2}\left[J_{n+1}(a\alpha)\right]^2$$

...: Put $x = a\alpha$ in the recurrence relation VI on Page 11.14

$$J_{n-1}(a\alpha) + J_{n+1}(a\alpha) = \frac{2n}{a\alpha} J_n(a\alpha).$$

... $J_n(a\alpha) = 0$, $J_{n-1}(a\alpha) = -J_{n+1}(a\alpha)$.

Thus

$$\int_0^a x J_n(\alpha x) J_n(\beta x)\, dx = \frac{a^2}{2}\left[J_{n-1}(a\alpha)\right]^2$$

Recurrence Relations (or identities) for Bessel's Functions

Valid for any p.

I. Prove that

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x)$$

Proof: From (11)

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m+p} \cdot m!\,\Gamma(m+p+1)}$$

So

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = \frac{d}{dx}\left\{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2p}}{2^{2m+p} \cdot m!\,\Gamma(m+p+1)}\right\}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2m+2p) x^{2m+2p-1}}{2^{2m+p} \cdot m!(m+p)\Gamma(m+p)}$$

$$= x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+(p-1)}}{2^{2m+(p-1)} \cdot m!\,\Gamma(m+(p-1)+1)}$$

$$= x^p J_{p-1}(x)$$

II. $\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = -x^{-p} J_{p+1}(x)$.

Proof: Multiplying (11) by x^{-p} and differentiating

$$\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = \frac{d}{dx}\left\{\sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m}}{2^{2m+p} \cdot m!\,\Gamma(m+p+1)}\right\}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m \cdot 2m \cdot x^{2m-1}}{2^{2m+p} \cdot m!\,\Gamma(m+p+1)}$$

since for $m = 0$, the first term in R.H.S. is zero.

$$= \sum_{m=1}^{\infty} \frac{(-1)^m \cdot x^{2m-1}}{2^{2m+p-1} \cdot (m-1)!\,\Gamma(m+p+1)}$$

Put $s = m - 1$ or $m = s + 1$ then

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+1} \cdot x^{2(s+1)-1}}{2^{2(s+1)+p-1} \cdot s!\,\Gamma(s+1+p+1)}$$

$$= -x^{-p} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+(p+1)}}{2^{2s+(p+1)} s!\,\Gamma((s+1)+p+1)}$$

$$= -x^{-p} \cdot J_{p+1}(x).$$

III. $\frac{d}{dx}\left\{J_p(x)\right\} = J_{p-1}(x) - \frac{p}{x} J_p(x)$

or $\quad x J_p'(x) = x J_{p-1}(x) - p J_p(x)$

Proof: From recurrence relation (I)

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x)$$

Performing the differentiation in the L.H.S.,

$$x^p \cdot \frac{d}{dx}\left\{J_p(x)\right\} + px^{p-1}\cdot J_p(x) = x^p J_{p-1}(x)$$

or $$J'_p(x) + \frac{p}{x}J_p(x) = J_{p-1}(x)$$

or $$J'_p(x) = J_{p-1}(x) - \frac{p}{x}J_p(x)$$

IV. $J'_p(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$

Proof: From recurrence relation (II)

$$\frac{d}{dx}\left\{x^{-p}J_p(x)\right\} = -x^{-p}J_{p+1}(x)$$

Performing the differentiation in the L.H.S.,

$$x^{-p}\cdot\frac{d}{dx}J_p(x) - px^{-p-1}J_p(x) = -x^{-p}J_{p+1}(x)$$

or $$J'_p(x) - \frac{p}{x}J_p(x) = -J_{p+1}(x)$$

or $$J'_p(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$$

V. $J'_p(x) = \frac{1}{2}\left\{J_{p-1}(x) - J_{p+1}(x)\right\}$ is obtained by adding recurrence relations (III) and (IV)

VI. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x)$ is obtained by subtracting (IV) from (III).

Elementary Bessel's Functions

Bessel's functions J_p of orders $p = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are elementary and can be expressed in terms of sine and cosines and powers of x.

Result 1: $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cdot\sin x.$

Proof: With $p = \frac{1}{2}$, (11) reduces to

$$J_{\frac{1}{2}}(x) = \sqrt{x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}}\cdot m!\Gamma\left(m+\frac{3}{2}\right)}$$

Now

$$\Gamma\left(m+\frac{3}{2}\right) = \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots \times\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2m+1)(2m-1)(2m-3)\cdots 3\cdot 1\cdot\sqrt{\pi}}{2^{m+1}}$$

Also

$$2^{2m+1}\cdot m! = 2^{m+1}\cdot 2^m\cdot m! = 2^{m+1}\cdot 2^m(m)(m-1)\cdots 2\cdot 1 = 2^{m+1}\cdot(2m)(2m-2)\cdots 4\cdot 2$$

Thus

$$2^{2m+1}\cdot m!\cdot\Gamma\left(m+\frac{3}{2}\right) = \left[2^{m+1}\cdot 2m\cdot(2m-2)\cdots 4\cdot 2\right]\times\left[(2m+1)(2m-1)\cdots 3\cdot 1\right]\cdot 2^{-(m+1)}\sqrt{\pi}$$

$$= (2m+1)!\sqrt{\pi}$$

Then

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m+1}\cdot m!\Gamma\left(m+\frac{3}{2}\right)}$$

$$= \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m-1}}{(2m+1)!\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m-1}}{(2m+1)!}$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\sin x.$$

Result 2: In the recurrence relation I, put $p =$ then

$$\frac{d}{dx}\left\{\sqrt{x}J_{\frac{1}{2}}(x)\right\} = \sqrt{x}J_{-\frac{1}{2}}(x)$$

$$\frac{d}{dx}\left\{\sqrt{x}\sqrt{\frac{2}{\pi x}}\cdot\sin x\right\} = \sqrt{x}J_{-\frac{1}{2}}(x)$$

$$\sqrt{\frac{2}{\pi}}\cos x = \sqrt{x}J_{-\frac{1}{2}}(x)$$

or $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

Similarly with $p = \frac{1}{2}$, we get from recurrence relation VI.

Result 3:

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$

or $$J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

...ing result (1) and (2) for $J_{\frac{1}{2}}$ and $J_{-\frac{1}{2}}$, we get

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

...larly with $p = -\frac{1}{2}$ in recurrence relation VI

...ult 4:

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right).$$

...grals of Bessel's Functions

...grating the recurrence relation

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x), \quad \text{we get}$$

$$\int x^p J_{p-1}(x)dx = x^p J_p(x) + c \qquad (1)$$

...$p = 1$.

$$\int xJ_0(x)\,dx = xJ_1(x) + c \qquad (2)$$

...grating the recurrence relation

$$\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = -x^{-p} J_{p+1}(x), \quad \text{we get}$$

$$\int x^{-p} J_{p+1}(x)dx = -x^{-p} J_p(x) + c \qquad (3)$$

...$p = 0$,

$$\int J_1(x)\,dx = -J_0(x) + c \qquad (4)$$

...general $\int x^m J_n(x)dx$ for m and n integers with ...$-n \ge 0$ can be integrated by parts completely if ...$-n$ is odd. But when $m + n$ is even, the integral ...ends on the residual integral $\int J_0(x)dx$ which has been tabulated.

...grating $J'_p(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right]$

$$2J_p(x) = \int J_{p-1}(x)\,dx - \int J_{p+1}(x)dx$$

$$\int J_{p+1}(x)dx = \int J_{p-1}(x)dx - 2J_p(x).$$

Bessel's Function of Second Kind of Order ...or Neumann Function

When n is integral, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and do not constitute the solution.

Let $y = u(x)\,J_n(x)$ be a solution of (1). Substituting in (1),

$$x^2(u''J_n + 2u'J'_n + uJ''_n) + x(u'J_n + uJ'_n) + (x^2 - n^2)uJ_n = 0$$

or

$$u\left\{x^2J''_n + xJ'_n + (x^2 - n^2)J_n\right\} + x^2u''J_n + 2x^2u'J'_n + xu'J_n = 0$$

Since J_n is a solution of (1), the first term is zero. Dividing throughout by $x^2u'J_n$, we get

$$\frac{u''}{u'} + 2\frac{J'_n}{J_n} + \frac{1}{x} = 0$$

Integrating $\ln(u'J_n^2 \cdot x) = \ln B$ or $xu'J_n^2 = B$.

Thus

$$u' = \frac{B}{xJ_n^2}$$

Integrating

$$u = B\int\frac{dx}{xJ_n^2} + c$$

Hence $y = AJ_n(x) + BY_n(x)$ is the complete solution of (1) where

$$Y_n(x) = J_n(x)\cdot\int\frac{dx}{x[J_n(x)]^2}$$

$Y_n(x)$ is known as Bessel's function of second kind of order n or Neumann function.

Worked Out Examples

Example 1: Find $J_0(x)$ and $J_1(x)$.

Solution: Put $n = 0$ in

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+n}}{2^{2m+n}\cdot m!(m+n)!}$$

Then

$$J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \left(\frac{1}{3!}\right)^2\left(\frac{x}{2}\right)^6 + \cdots$$