Chapter 11

Special Functions—Gamma, Beta, Bessel and Legendre

INTRODUCTION

Algebraic function $f(x)$ is obtained by the algebraic operations of addition, subtraction, multiplication, division and square rooting of x polynomial and rational functions are such functions. Transcendental functions include trigonometric functions (sine, cosine, tan) exponential, logarithmic and hyperbolic functions.

Algebraic and transcendental functions together constitute the elementary functions. Special functions (or higher functions) are functions other than the elementary functions such as Gamma, Beta functions (expressed as integrals) Bessel's functions, Legendre polynomials (as solutions of ordinary differential equations). Special functions also include Laguerre, Hermite, Chebyshev polynomials, error function, sine integral, exponential integral, Fresnel integrals, etc.

Many integrals which can not be expressed in terms of elementary functions can be evaluated in terms of beta and gamma functions.

Heat equation, wave equation and Laplace's equation with cylindrical symmetry can be solved in terms of Bessel's functions, with spherical symmetry by Legendre polynomials. We consider Fourier-Legendre series and Fourier-Bessel series. Chebyshev-polynomials which are useful in approximation theory are also presented.

11.1 GAMMA FUNCTION

Gamma function denoted by $\Gamma(p)$ is defined by the improper integral which is dependent on the parameter p.

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt, \quad (p > 0) \tag{1}$$

Gamma function is also known as Euler's integral of the second kind.

Integrating by parts

$$\Gamma(p+1) = \int_0^\infty e^{-t} t^p\, dt$$

$$= -e^{-t} t^p \Big|_0^\infty + p \int_0^\infty e^{-t} t^{p-1}\, dt$$

$$= 0 + p\Gamma(p)$$

Thus $\quad \Gamma(p+1) = p\Gamma(p) \quad (2)$

(2) is known as the functional relation or reduction or recurrence formula for gamma function.

Result:
$\Gamma(n+a) = (n+a-1)(n+a-2)(n+a-3)\cdots a\cdot\Gamma(a)$, n is integer.

By definition

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = \frac{e^{-t}}{-1}\Big|_0^\infty = 1 \tag{3}$$

By the reduction formula (2),

$$\Gamma(2) = 1\cdot\Gamma(1) = 1$$

and $\quad \Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$

and in general when p is a positive integer n

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

Thus for $p = n$, positive integer

$$\Gamma(n+1) = n! \tag{4}$$

For this reason, Gamma function is regarded as the generalization of the elementary factorial function.
Gamma function for negative values of p i.e., $p < 0$: Rewrite (2) as left-marching recurrence formula,

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \tag{5}$$

As $p \to 0$, $\quad \Gamma(0) = \lim_{p\to 0} \frac{\Gamma(1)}{p} = \lim_{p\to 0} \frac{1}{p} \to \infty$

Thus $\Gamma(0)$ is undefined and it follows from (5) that $\Gamma(-1), \Gamma(-2), \Gamma(-3)$, etc. are all undefined.

Repeated application of (5) results in

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} = \frac{\Gamma(p+2)}{p(p+1)} = \cdots$$
$$= \frac{\Gamma(p+k+1)}{p(p+1)\cdots(p+k)} \tag{6}$$

Relation (6) is used to find gamma function for $p < 0$ (except at $p = 0, -1, -2, -3, \ldots$).
Hence gamma function is continuous for any $p > 0$ and is discontinuous at $p = 0, -1, -2, -3, \ldots$.
Thus $\Gamma(p)$ is defined for all p, except for zero and negative integers.

Standard Results

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

By definition $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt$, put $t = u^2$ then $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du$

$$\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right) = \left[2\int_0^\infty e^{-u^2} du\right]\left[2\int_0^\infty e^{-v^2} dv\right]$$
$$= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)} du\, dv$$

This double integral in the first quadrant is evaluated by changing to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, $J = r$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{\infty} d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} d\theta = 2\cdot\theta\Big|_0^{\frac{\pi}{2}} = 2\cdot\frac{\pi}{2} = \pi.$$

Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2. $\Gamma\left(\frac{p+1}{q}\right) = qa^{(p+1)/q}\int_0^\infty x^p e^{-ax^q} dx$; p, q, a are positive constants

Put $y = ax^q$ then $dy = aqx^{q-1}dx$

$$\int_0^\infty x^p e^{-ax^q} dx = \int_0^\infty \left[\left(\frac{y}{a}\right)^{\frac{1}{q}}\right]^p e^{-y}\cdot\frac{1}{aqx^{q-1}}dy$$
$$= \left[qa^{(p+1)/q}\right]^{-1}\int_0^\infty y^{(p+1-q)/q} e^{-y} dy$$
$$= \frac{\Gamma\left(\frac{p+1}{q}\right)}{qa^{\frac{(p+1)}{q}}}.$$

3. $\Gamma(n+1) = (m+1)^{n+1}(-1)^n\int_0^1 x^m(\ln x)^n dx$ where n is a positive integer and $m > -1$.

Put $x = e^{-y}$ then $dx = -e^{-y}dy = -x\,dy$

$$\int_0^1 x^m(\ln x)^n dx = \int_0^\infty e^{-my}\cdot(-y)^n e^{-y} dy$$
$$= (-1)^n\int_0^\infty y^n\cdot e^{-(m+1)y} dy,$$

Put $(m+1)y = u$

$$= (-1)^n\int_0^\infty \frac{u^n}{(m+1)^n}\cdot e^{-u}\cdot\frac{du}{m+1}$$
$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-u}\cdot u^n du = \frac{(-1)^n}{(m+1)^{n+1}}\cdot\Gamma(n+1)$$

11.2 BETA FUNCTION

Beta function $\beta(p,q)$ defined by

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \ (p>0, q>0) \tag{1}$$

is convergent for $p > 0$, $q > 0$. This function is also known as Euler's integral of the first kind.

Standard Results

(2)

1. **Symmetry:** $\beta(p,q) = \beta(q,p)$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx. \quad \text{Put } x = 1-y$$
$$= \int_1^0 (1-y)^{p-1}\cdot y^{q-1}(-dy)$$

$$= \int_1^0 y^{q-1}(1-y)^{p-1}dy = \beta(q,p).$$

2. Beta function in terms of trigonometric functions

$$\beta(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1} x \cdot \cos^{2q-1} x \, dx \quad (3)$$

Putting $x = \sin^2\theta$, $dx = 2\sin\theta \cdot \cos\theta \, d\theta$ and $1-x = \cos^2\theta$,

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2p-2}\theta \cdot \cos^{2q-2}\theta \cdot 2 \times \sin\theta \cdot \cos\theta \, d\theta$$

$$\beta(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta \cdot \cos^{2q-1}\theta \, d\theta$$

$$= 2 \cdot I_{2p-1,2q-1}$$

or $$I_{p,q} = \int_0^{\frac{\pi}{2}} \sin^p\theta \, \cos^q\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right), p > -1, q > -1 \quad (4)$$

3. Beta function expressed as an improper integral

$$\beta(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy$$

$$= \int_0^\infty \frac{y^{q-1}dy}{(1+y)^{p+q}} \quad (5)$$

Putting $x = \frac{y}{1+y}$ or $y = \frac{x}{1-x}$, limits for y are 0 to ∞.

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p-1}} \cdot \left(\frac{1}{1+y}\right)^{q-1} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy$$

$$= \int_0^\infty \frac{y^{q-1}}{(1+y)^{q+p}}dy.$$

The last integral follows from symmetry.

4. Relation between β and Γ functions

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (6)$$

By definition

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx.$$

Put $x = t^2$, $dx = 2t\,dt$

$$\Gamma(p) = \int_0^\infty t^{2p-2} \cdot e^{-t^2} \cdot 2t\,dt$$

$$= 2\int_0^\infty t^{2p-1} \cdot e^{-t^2}\,dt$$

Then $$\Gamma(p)\,\Gamma(q) = \left[2\int_0^\infty x^{2p-1}e^{-x^2}dx\right] \times \left[2\int_0^\infty y^{2q-1}e^{-y^2}dy\right]$$

Here t, x, y are dummy variables.

$$= 4\int_0^\infty \int_0^\infty x^{2p-1} \cdot y^{2q-1} \cdot e^{-(x^2+y^2)}dx\,dy.$$

Introduce polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

As x, y vary in the first quadrant (i.e., $0 < x < \infty$, $0 < y < \infty$), r varies from 0 to ∞ and θ from 0 to $\frac{\pi}{2}$. Jacobian: r

$$\Gamma(p)\,\Gamma(q) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty (r\cos\theta)^{2p-1} \cdot (r\sin\theta)^{2q-1} \times e^{-r^2} \cdot r\,dr\,d\theta$$

$$= 4\left[\int_0^{\frac{\pi}{2}} \sin^{2q-1}\theta \cdot \cos^{2p-1}\theta\,d\theta\right] \times \left[\int_0^\infty e^{-r^2} \cdot r^{2p+2q-1} \cdot dr\right]$$

Using result 2 above in this page and putting $r^2 = t$

$$\Gamma(p)\,\Gamma(q) = 4 \cdot \left[\frac{1}{2}\beta(p,q)\right]\left[\frac{1}{2}\int_0^\infty e^{-t} \cdot t^{p+q-1}dt\right]$$

$\Gamma(p)\,\Gamma(q) = \beta(p,q) \cdot \Gamma(p+q)$, hence the result.

5. $\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$, $0 < p < 1$

Put $q = 1-p$ in (6) and use (5)

$$\frac{\Gamma(p)\Gamma(1-p)}{\Gamma(p+1-p)} = \beta(p, 1-p)$$

$$= \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$$

(which follows from residue theorem) and since $\Gamma(1) = 1$

6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Put $p = q = \frac{1}{2}$ in (6) and use (3)

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^\circ\theta \cos^\circ\theta \, d\theta$$

$$= 2 \cdot \frac{\pi}{2} = \pi$$

Since $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) = \pi$

7. $\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \frac{\sqrt{\pi}}{2}$

which follows from (4) with $p = n$ and $q = 0$

Similarly with $p = 0$, $q = n$ from (4), we get

8. $\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{n+1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \frac{\sqrt{\pi}}{2}$

9. Legendre's duplication formula for Γ function:

$$\sqrt{\pi}\Gamma(2p) = 2^{2p-1} \cdot \Gamma(p)\Gamma\left(p + \frac{1}{2}\right)$$

Putting $p = q$ in (3), we get

$$\beta(p, p) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta \cdot \cos^{2p-1}\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta \cdot \cos\theta)^{2p-1} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2p-1} d\theta$$

$$= \frac{2}{2^{2p-1}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2p-1} d\theta$$

Put $2\theta = t$,

$$\beta(p, p) = \frac{1}{2^{2p-1}} \cdot \int_0^{\pi} \sin^{2p-1} t \, dt$$

$$= \frac{1}{2^{2p-1}} \left[2\int_0^{\frac{\pi}{2}} \sin^{2p-1} t \, dt\right]$$

$$\beta(p, p) = \frac{1}{2^{2p-1}} \cdot \beta\left(p, \frac{1}{2}\right)$$

since from (3) with $q = \frac{1}{2}$,

$$\beta(p, \frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1} x \, dx.$$

Using (6), express β in terms of Γ functions

$$2^{2p-1} \cdot \frac{\Gamma(p)\Gamma(p)}{\Gamma(p+p)} = \frac{\Gamma(p)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p+\frac{1}{2}\right)}$$

or $$2^{2p-1} \cdot \Gamma(p)\Gamma\left(p+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\Gamma(2p)$$

$$= \sqrt{\pi}\Gamma(2p)$$

10. $I = \int_0^{\frac{\pi}{2}} \sin^p\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^p\theta \, d\theta =$

a. $\frac{1\cdot 3\cdot 5\cdots(p-1)}{2\cdot 4\cdot 6\cdots p} \cdot \frac{\pi}{2}$ if p is an even positive integer and

b. I $\frac{2\cdot 4\cdot 6\cdots(p-1)}{1\cdot 3\cdot 5\cdots p}$ if p is an odd positive integer.

a. $\int_0^{\frac{\pi}{2}} \sin^p\theta \, d\theta = \frac{\Gamma\left(\frac{1}{2}(p+1)\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}(p+2)\right)}$ from result 7.

If $p = 2r$,

$$I = \frac{\Gamma\left(r+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(r+1)}$$

$$= \frac{\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{2\cdot r!}$$

$$I = \frac{(2r-1)(2r-3)\cdots 3\cdot 1}{2r\cdot(2r-2)\cdot(2r-4)\cdots 2} \cdot \frac{\pi}{2}$$

b. If $p = 2r + 1$,

$$I = \frac{\Gamma(r+1)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(r+\frac{3}{2}\right)}$$

$$= \frac{r!\sqrt{\pi}}{2\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{2^r \cdot r!}{(2r+1)(2r-1)\cdots 5\cdot 3\cdot 1}$$

$$= \frac{2\cdot 4\cdot 6\cdots(2r-2)\cdot 2r}{1\cdot 3\cdot 5\cdots(2r-1)(2r+1)}$$

11. $\beta(p,q) = \beta(p+1,q) + \beta(p,q+1)$

By definition

$$\beta(p+1,q) + \beta(p,q+1)$$

$$= \int_0^1 x^p(1-x)^{q-1}dx + \int_0^1 x^{p-1}(1-x)^q\,dx$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}\Big[x + (1-x)\Big]dx$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}dx = \beta(p,q).$$

12. $\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$ for m,n positive integers

From the above result 11

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$$

Expressing in Γ functions using (6)

$$\beta(m,n) = \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)}$$

Since m and n are positive integers

$$= \frac{m!(n-1)! + (m-1)!n!}{(m+n)!}$$

$$= \frac{(m-1)!(n-1)![m+n]}{(m+n)!} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

13. $\int_0^1 x^p(1-x^q)^r dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$

Put $x^q = y$, $qx^{q-1}dx = dy$ or $qy^{\left(\frac{q-1}{q}\right)}dx = dy$

$$\int_0^1 x^p(1-x^q)^r dx = \int_0^1 y^{\left(\frac{p}{q}\right)}(1-y)^r \frac{1}{qy^{(q-1)/q}}\cdot dy$$

$$= \frac{1}{q}\int_0^1 y^{(p-q+1)/q}(1-y)^r dy$$

$$= \frac{1}{q}\beta\left(\frac{p-q+1}{q} + 1, r+1\right)$$

$$= \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$$

Note: When $q = 1$, $\int_0^1 x^p(1-x)^r\,dx = \beta(p+1, r+1)$.

14. $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}dx = \frac{1}{b^n(b+c)^m}\cdot\beta(m,n)$

Put $y = \frac{x(1+a)}{b+cx}$ then $x = \frac{yb}{1+a-cy}$,

$$1-x = \frac{(1+a)-y(c+b)}{(1+a-cy)}, \quad b+cx = \frac{b(1+a)}{(1+a-cy)}$$

and $dx = \frac{(b+cx)dy}{(1+a-cy)} = \frac{b(1+a)dy}{(1+a-cy)^2}$.

Now y varies from 0 to $\frac{1+a}{b+c} = e$ say

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}dx$$

$$= \int_0^e \left(\frac{yb}{1+a-cy}\right)^{m-1}\left[\frac{(1+a)-y(c+b)}{1+a-cy}\right]^{n-1} \times \frac{(1+a-cy)^{m+n}}{\big[b(1+a)\big]^{m+n}}\,\frac{b(1+a)\,dy}{(1+a-cy)^2}$$

$$= \int_0^e \frac{1}{b^n(1+a)^{m+n-1}}y^{m-1} \times \Big[(1+a) - y(c+b)\Big]^{n-1}dy$$

$$= \frac{1}{b^n}\frac{1}{(1+a)^m}\int_0^e y^{m-1}\left(1-\frac{1}{e}y\right)^{n-1}dy$$

Put $\frac{1}{e}y = t$

$$= \frac{1}{b^n}\frac{1}{(1+a)^m}\int_0^1 (et)^{m-1}(1-t)^{n-1}\cdot e\,dt$$

$$= \frac{1}{b^n}\frac{1}{(1+a)^m}\cdot\left(\frac{1+a}{b+c}\right)^m\cdot\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= \frac{1}{b^n}\frac{1}{(b+c)^m}\cdot\beta(m,n).$$

15. When $c = 1, b = a$, from above result 14,

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}dx = \frac{1}{a^n(1+a)^m}\beta(m,n).$$

Worked Out Examples

Gamma and Beta functions

Example 1: Compute (a) $\Gamma(4.5)$ (b) $\Gamma(-3.5)$ (c) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (d) $\beta\left(\frac{5}{2},\frac{3}{2}\right)$ (e) $\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$.

Solution:

a. Using $\Gamma(p+1) = p\Gamma(p)$

$$\Gamma(4.5) = \Gamma(3.5+1) = 3.5\Gamma(3.5) = (3.5)(2.5)\Gamma(2.5)$$
$$= (3.5)(2.5)(1.5)(.5)\Gamma(.5)$$
$$= 6.5625\sqrt{\pi} = 11.62875$$

b. Using $\Gamma(p) = \frac{\Gamma(p+1)}{p}$

$$\Gamma(-3.5) = \frac{\Gamma(-3.5+1)}{-3.5} = \frac{\Gamma(-2.5)}{-3.5} = \frac{\Gamma(-2.5+1)}{(-3.5)(-2.5)}$$
$$= \frac{\Gamma(-1.5)}{(3.5)(2.5)} = \frac{\Gamma(-1.5+1)}{-(3.5)(2.5)(1.5)}$$
$$= \frac{\Gamma(.5)}{(3.5)(2.5)(1.5)(.5)} = \frac{\sqrt{\pi}}{(3.5)(2.5)(1.5)(.5)}$$
$$= .270019$$

c. Using $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$
$$= \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\pi = 4.444$$

d. $$\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{3!}$$
$$= \frac{1}{4}\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{1}{4}\cdot\frac{1}{4}\cdot\pi = \cdot 1964$$

e. $$\Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}-1\right)\left(n+\frac{1}{2}-2\right)$$
$$\times\left(n+\frac{1}{2}-3\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{(2n-1)}{2}\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)$$
$$\times\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$
$$= \frac{[(2n-1)(2n-3)(2n-5)\cdots 1\cdot\sqrt{\pi}]}{2^n}$$

Since $\Gamma(n+1) = n!$ thus

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{2^n\cdot n!}\sqrt{\pi}.$$

Example 2: Evaluate $I = \int_0^\infty x^4 e^{-x^4} dx$.

Solution: Put $x^4 = t$, $4x^3 dx = dt$, $dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

$$I = \int_0^\infty t\cdot e^{-t}\cdot\frac{t^{-\frac{3}{4}}}{4}\cdot dt = \frac{1}{4}\int_0^\infty e^{-t}\cdot t^{\frac{1}{4}}dt$$
$$= \frac{1}{4}\Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{5}{4}\right).$$

Example 3: Evaluate $I = \int_0^1 \sqrt[3]{x\ln\left(\frac{1}{x}\right)}\,dx$.

Solution: Put $\ln\left(\frac{1}{x}\right) = t$, $x = e^{-t}$, $dx = -e^{-t}dt$

$$I = \int_\infty^0 (e^{-t}\cdot t)^{\frac{1}{3}}(-e^{-t})dt$$
$$= \int_0^\infty t^{\frac{1}{3}} e^{-\frac{4t}{3}}dt, \quad \text{Put } \frac{4t}{3} = y$$
$$= \int_0^\infty \left(\frac{3}{4}\right)^{\frac{4}{3}} e^{-y} y^{\frac{1}{3}} dy = \left(\frac{3}{4}\right)^{\frac{4}{3}}\Gamma\left(\frac{1}{3}+1\right)$$
$$= \left(\frac{3}{4}\right)^{\frac{4}{3}}\Gamma\left(\frac{4}{3}\right).$$

Example 4: Evaluate

a. $\int_0^{\frac{\pi}{2}} \sin^{10}\theta\, d\theta$

b. $\int_0^{\frac{\pi}{2}} \cos^9\theta\, d\theta$

c. $\int_0^{\frac{\pi}{2}} \sin^6\theta\cdot\cos^7\theta\, d\theta$

d. $\int_0^{\frac{\pi}{2}} \left(\frac{\sqrt[3]{\sin 8x}}{\sqrt{\cos x}}\right) dx$

Solution:

a. $\int_0^{\frac{\pi}{2}} \sin^{10}\theta\, d\theta = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\frac{\pi}{2} = \frac{63}{256}\pi$, since $p = 8$ is even

b. $\int_0^{\frac{\pi}{2}} \cos^9\theta\, d\theta = \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7\cdot 9}$, $\quad p = 9$ is odd

c. $$\int_0^{\frac{\pi}{2}} \sin^6\theta\cdot\cos^7\theta\, d\theta$$
$$= \frac{1}{2}\beta\left(\frac{6+1}{2}, \frac{7+1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{\Gamma\left(\frac{15}{2}\right)}$$
$$= \frac{1}{2}\frac{\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\left(\frac{7}{2}-3\right)\cdot 3!}{\left(\frac{15}{2}-1\right)\left(\frac{15}{2}-2\right)\left(\frac{15}{2}-3\right)\left(\frac{15}{2}-4\right)\left(\frac{15}{2}-5\right)\left(\frac{15}{2}-6\right)\left(\frac{15}{2}-7\right)}$$
$$= \frac{2^4}{3\cdot 7\cdot 11\cdot 13}$$

d. $\int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\sin 8x}}{\sqrt{\cos x}}dx$

$$=\int_0^{\frac{\pi}{2}} \sin^{\frac{8}{3}} x \cdot \cos^{-\frac{1}{2}} x\, dx = \frac{1}{2}\beta\left(\frac{\frac{8}{3}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$=\frac{1}{2}\beta\left(\frac{11}{6}, \frac{1}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{11}{6}+\frac{1}{4}\right)}$$

$$=\frac{1}{2}\cdot\frac{\left(\frac{5}{6}\right)\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\frac{13}{12}\cdot\frac{1}{12}\Gamma\left(\frac{1}{12}\right)} = \frac{60}{13}\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{12}\right)}.$$

Example 5: Evaluate $I = \int_0^1 \left(\frac{x}{1-x^3}\right)^{\frac{1}{2}} dx.$

Solution: $I = \int_0^1 x^{\frac{3}{2}}(1-x^3)^{-\frac{1}{2}}dx.$ put $x^3 = t$

$$I=\int_0^1 t^{\frac{1}{2}}\cdot(1-t)^{-\frac{1}{2}}\cdot\frac{1}{3}t^{-\frac{2}{3}}dt$$

$$=\frac{1}{3}\int_0^1 t^{-\frac{1}{6}}\cdot(1-t)^{-\frac{1}{2}}dt = \frac{1}{2}\beta\left(1-\frac{1}{6}, 1-\frac{1}{2}\right)$$

$$=\frac{1}{3}\beta\left(\frac{5}{6}, \frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\left(\frac{5}{6}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)}$$

$$=\frac{1}{3}\frac{\Gamma\left(\frac{5}{6}\right)\sqrt{\pi}}{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{3}\right)}\cdot\Gamma\left(\frac{1}{3}+\frac{1}{2}\right).$$

using duplication form $\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{2}{3}\right)}{2^{\frac{2}{3}-1}}$

Also $\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}}.$

Substituting

$$=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{3}\right)}\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\cdot 2^{\frac{1}{3}}}$$

$$=\frac{\pi}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2\cdot 2^{\frac{1}{3}}}\cdot\frac{\pi}{\Gamma\left(\frac{1}{3}\right)\cdot\sin\left(\frac{\pi}{3}\right)}$$

$$I=\frac{\pi^2 2^{-\frac{1}{3}}}{\left[\Gamma\left(\frac{1}{3}\right)\right]^3\cdot\sin\frac{\pi}{3}}.$$

Example 6: Evaluate $I = \int_0^\infty a^{-bx^2}\, dx.$

Solution: Put $a^{-bx^2} = e^{-t}$, $-bx^2 \ln a = -t$

$2bx \ln a\, dx = dt$, also $x = \left(\frac{t}{b\ln a}\right)^{\frac{1}{2}}$

So
$$dx = \frac{t^{-\frac{1}{2}}dt}{(2b\ln a)^{\frac{1}{2}}}$$

$$I = \int_0^\infty \frac{e^{-t}\cdot t^{-\frac{1}{2}}dt}{(2b\ln a)^{\frac{1}{2}}} = \frac{1}{(2b\ln a)^{\frac{1}{2}}}\cdot\Gamma\left(1-\frac{1}{2}\right)$$

$$=\frac{\sqrt{\pi}}{(2b\ln a)^{\frac{1}{2}}}.$$

Example 7: Evaluate

$$I = \left[\int_0^\infty x\, e^{-x^8}dx\right]\times\left[\int_0^\infty x^2\, e^{-x^4}dx\right].$$

Solution: $I = I_1 \times I_2.$

Put $x^8 = t$ in I_1, $x = t^{\frac{1}{8}}$, $dx = \frac{1}{8}t^{-\frac{7}{8}}dt$

$$I_1 = \int_0^\infty x\, e^{-x^8}\, dx = \int_0^\infty t^{\frac{1}{8}}\cdot e^{-t}\cdot\frac{1}{8}t^{-\frac{7}{8}}dt$$

$$I_1 = \frac{1}{8}\int_0^\infty t^{-\frac{3}{4}}e^{-t}dt = \frac{1}{8}\Gamma\left(1-\frac{3}{4}\right) = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)$$

Put $x^4 = t$ in I_2, so $x = t^{\frac{1}{4}}$, $dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

$$I_2 = \int_0^\infty x^2 e^{-x^4}dx = \int_0^\infty t^{\frac{1}{2}}\cdot e^{-t}\cdot\frac{1}{4}t^{-\frac{3}{4}}dt$$

$$=\frac{1}{4}\int_0^\infty t^{-\frac{1}{4}}e^{-t}dt = \frac{1}{4}\Gamma(1-\frac{1}{4}) = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

Thus

$$I = I_1\cdot I_2 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$=\frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{32}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}\pi}{32}$$

Since $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}.$

Example 8: Show that

$$\int_0^\infty \cos\,(bz^{\frac{1}{n}})\, dz = \frac{\Gamma(n+1)\cdot\cos\frac{n\pi}{2}}{b^n}.$$

Solution: Put $z = x^n$, $x = z^{\frac{1}{n}}$, $dz = nx^{n-1}dx$

$$\int_0^\infty \cos\,(bz^{\frac{1}{n}})dz = \int_0^\infty nx^{n-1}\cdot\cos\,(bx)dx$$

$$= \text{Real part of}\left\{\int_0^\infty n\,x^{n-1}\cdot e^{-ibx}dx\right\}$$

But $\int_0^\infty x^{n-1}e^{-ibx}dx$

$$= \int_0^\infty \left(\frac{t}{ib}\right)^{n-1} \cdot e^{-t} \cdot \frac{dt}{ib} = \frac{1}{(ib)^n}\Gamma(n)$$

where $t = ibx$.

$$\int_0^\infty \cos(bz^{\frac{1}{n}})dz = \mathrm{Re}\left\{\frac{n \cdot \Gamma(n)}{b^n}(i)^{-n}\right\}$$

$$= \mathrm{Re}\left\{\frac{\Gamma(n+1)}{b^n}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-n}\right\}$$

$$= \mathrm{Re}\left\{\frac{\Gamma(n+1)}{b^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)\right\}$$

$$= \frac{\Gamma(n+1)}{b^n} \cdot \cos\frac{n\pi}{2}.$$

Example 9: Prove that $\int_{-1}^{1}(1-t^2)^n dt = \frac{2^{n+1} \cdot n!}{1\cdot 3\cdot 5\cdots(2n+1)}$ for $n = 0, 1, 2, \ldots$

Solution: Put $t = \sin\theta$, $1 - t^2 = 1 - \sin^2\theta = \cos^2\theta$, $dt = \cos\theta d\theta$

Limits for θ are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Thus

$$\int_{-1}^{1}(1-t^2)^n dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}\theta \cdot \cos\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\, d\theta$$

$$= 2 \cdot \frac{2\cdot 4\cdot 6\cdot 8\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)} = \frac{2\cdot 2^n \cdot 1\cdot 2\cdot 3\cdots n}{1\cdot 3\cdot 5\cdots(2n+1)}$$

$$= \frac{2^{n+1}\cdot n!}{1\cdot 3\cdot 5\cdots(2n+1)}$$

Example 10: Show that $\int_0^\infty \frac{x^2 dx}{(1+x^4)^3} = \frac{5\pi\sqrt{2}}{128}$.

Solution: Put $x = \sqrt{\tan\theta}$, $dx = \frac{1}{2}\frac{1}{\sqrt{\tan\theta}} \cdot \sec^2\theta\, d\theta$

$$\int_0^\infty \frac{x^2 dx}{(1+x^4)^3}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan\theta \cdot \frac{1}{2}(\tan\theta)^{-\frac{1}{2}} \cdot \sec^2\theta\, d\theta}{(1+\tan^2\theta)^3}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\tan\theta)^{\frac{1}{2}} \cdot \sec^{-4}\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cdot \cos^{\frac{7}{2}}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\beta\left(\frac{1+\frac{1}{2}}{2}, \frac{1+\frac{7}{2}}{2}\right) = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{9}{4}\right)$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{9}{4}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\frac{5}{4}\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{\Gamma(3)}$$

$$= \frac{1}{4}\cdot\frac{5}{16}\cdot\frac{1}{2!}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{5}{128}\cdot\pi\sqrt{2}$$

since $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$.

Example 11: Prove that

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \cdot d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}$$

Solution: Put $\tan\theta = t$, $d\theta = \cos^2\theta\, dt$, $\sin\theta = t\cos\theta$

$$\int_0^\infty \frac{\cos^{2m-1}\theta \cdot \sin^{2n-1}\theta\, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}$$

$$= \int_0^\infty \frac{\cos^{2m-1}\theta \cdot t^{2n-1} \cdot \cos^{2n-1}\theta \cdot \cos^2\theta\, dt}{(a\cos^2\theta + bt^2\cos^2\theta)^{m+n}}$$

$$= \int_0^\infty \frac{\cos^{2m+2n}\theta \cdot t^{2n-1}dt}{\cos^{2m+2n}\theta(a+bt^2)^{m+n}}, \quad \text{Put } \sqrt{b}t = \sqrt{ay}$$

$$= \int_0^\infty \frac{\left(\frac{ay}{b}\right)^{\left(\frac{2n-1}{2}\right)} \cdot \frac{a}{b}\frac{1}{2\sqrt{y}}dy}{(a+ay)^{m+n}}$$

$$= \frac{a^n}{2b^n a^{m+n}}\int_0^\infty \frac{y^{n-1}dy}{(1+y)^{m+n}} = \frac{1}{2a^m b^n}\cdot\beta(m,n).$$

Example 12: Evaluate

$$I = \int_0^1 x^{\frac{3}{2}}(1-x^2)^{\frac{5}{2}}dx.$$

Solution: From result 13 Page 5 with $p = \frac{3}{2}, q = 2$, $r = \frac{5}{2}$

$$I = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right) = \frac{1}{2}\beta\left(\frac{\frac{3}{2}+1}{2}, \frac{5}{2}+1\right)$$

$$= \frac{1}{2}\beta\left(\frac{5}{4}, \frac{7}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{4}+\frac{7}{2}\right)}$$

$$= \frac{1}{2}\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{\Gamma\left(\frac{19}{4}\right)}$$

Since $\Gamma\left(\frac{19}{4}\right) = \frac{15}{4}\cdot\frac{11}{4}\cdot\frac{7}{4}\cdot\frac{3}{4}\cdot\Gamma\left(\frac{3}{4}\right)$.

$$I = \frac{4\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\left[121\Gamma\left(\frac{3}{4}\right)\right]}.$$

EXERCISE

Gamma and Beta functions

1. Compute (a) $\frac{\Gamma(6)}{2\Gamma(3)}$ (b) $\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$ (c) $\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)}$ (d) $\Gamma\left(-\frac{5}{2}\right)$.

Ans. (a) 30 (b) $\frac{3}{4}$ (c) $\frac{16}{315}$ (d) $-\frac{8\sqrt{\pi}}{15}$

2. Evaluate (a) $\int_0^\infty \sqrt{y}e^{-y^2}dy$ (b) $\int_0^\infty 3^{-4z^2}dz$ (c) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$.

Ans. (a) $\frac{\sqrt{\pi}}{3}$ (b) $\frac{\sqrt{\pi}}{(4\sqrt{\ln 3})}$ (c) $\sqrt{\pi}$

3. Evaluate (a) $\int_0^1 x^4(1-x)^3dx$ (b) $\int_0^2 \frac{x^2dx}{\sqrt{2-x}}$ (c) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$.

Ans. (a) $\frac{1}{280}$ (b) $\frac{64\sqrt{2}}{15}$ (c) $\frac{\pi a^6}{32}$.

4. Evaluate (a) $\int_0^{2\pi}\sin^8\theta\,d\theta$ (b) $\int_0^{\frac{\pi}{2}}\cos^6\theta\,d\theta$ (c) $\int_0^{\frac{\pi}{2}}\sin^4\theta\cdot\cos^5\theta\,d\theta$.

Ans. (a) $I = 4\int_0^{\frac{\pi}{2}}\sin^8\theta d\theta = \frac{4\cdot1\cdot3\cdot5\cdot7}{(2\cdot4\cdot6\cdot8)}\frac{\pi}{2} = \frac{35\pi}{64}$ (b) $\frac{5\pi}{32}$ (c) $\frac{8}{315}$

5. Show that $\int_0^2 x\sqrt[3]{8-x^3}dx = \frac{16\pi}{(9\sqrt{3})}$

6. Find $\int_0^{\frac{\pi}{2}}\sqrt{\cot\theta}\,d\theta$.

Ans. $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{2}\pi\sqrt{2}$

7. Show that $\int_0^{\frac{\pi}{2}}(\sqrt{\tan\theta}+\sqrt{\sec\theta})\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right)+\frac{\sqrt{(\pi)}}{\Gamma\left(\frac{3}{4}\right)}\right\}$.

8. Prove that $\int_0^1 x^4\left[\ln\left(\frac{1}{x}\right)\right]^3dx = \frac{6}{625}$.

9. Show that $\left[\int_0^1 x^2(1-x^4)^{-\frac{1}{2}}dx\right]\times\left[\int_0^1(1+x^4)^{-\frac{1}{2}}dx\right] = \frac{\pi}{4\sqrt{2}}$.

10. Prove that $\left[\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}d\theta\right]\left[\int_0^{\frac{\pi}{2}}(\sin\theta)^{-\frac{1}{2}}d\theta\right] = \pi$.

11. Show that $\int_0^{\frac{\pi}{2}}\sin^7\theta\cdot\cos^7\theta\,d\theta = \frac{1}{280}$.

12. Evaluate $\int_0^a x^3(a^3-x^3)^5dx$.

Ans. $\frac{a^{19}\cdot3^5}{19\cdot16\cdot13\cdot7}$.

13. Prove that $\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is a positive integer and $m > -1$.

14. Prove that $\int_0^\infty \frac{t^2dt}{1+t^4} = \frac{\pi}{\sqrt{2}}$.
Hint: Put $t = \sqrt{\tan\theta}$.

15. Show that the area under the normal curve $y = \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2\sigma^2}}$ and x- axis is unity.

16. Show that $\frac{\beta(p,q)}{p+q} = \frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p}$.

17. Prove that $\beta(m,n) = \frac{1}{2}\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}$.
Hint: Use symmetry property of β function.

18. $\int_0^\infty x^{-\frac{3}{2}}(1-e^{-x})dx$.

Ans. $2\sqrt{\pi}$.

19. Show that $\int_b^a(x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}\cdot\beta(m,n)$
Hint: Put $x = \frac{(t-b)}{(a-b)}$.

20. Prove that $\int_0^\infty e^{-x^4}dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$.

21. Evaluate $\int_0^\infty \frac{x^a}{a^x}dx$.

Ans. $\frac{\Gamma(a+1)}{(\ln a)^{a+1}}$

22. Show that $\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx = \beta(p,q)$
Hint: From (5) $\beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dx = \int_0^1+\int_0^\infty$. Put $y = \frac{1}{z}$ in 2nd integral.

23. Show that $\int_{-1}^1\sqrt{\frac{1+t}{1-t}}dt = \pi$.

24. Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$.

Ans. 0

25. Prove that $\int_0^\infty e^{-ax}\cdot x^{n-1}dx = \frac{\Gamma(n)}{a^n}$ where a and n are positive.

11.3 BESSEL'S FUNCTIONS

The boundary value problems (such as the one-dimensional heat equation) with cylindrical symmetry (independent of θ) reduces to two ordinary differential equations by the separation of variables technique. One of them is the most important differential equation known as **Bessel's* differential equation**

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-p^2)y = 0$$

$$x^2y''+xy'+(x^2-p^2)y = 0 \tag{1}$$

Here p, which is a given constant (not necessarily an integer) is known as the order of the Bessel's equation.

* Friedrich Wilhelm Bessel (1784–1846) German mathematician.

Bessel's Functions (Cylindrical functions)

Bessel's functions (Cylindrical functions) are series solution of the Bessel's differential Equation (1) obtained by Frobenius method.

Assume that p is real and non-negative. Assume the series solution of (1) as

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{2}$$

To determine the unknown coefficients a_m and power (exponent) r, substitute (2) in (1), we get

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - p^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

Now equate the sum of the coefficients of x^{s+r} to zero. For $s=0$ and $s=1$, the contribution comes from first, second and fourth series (not from third series because it starts with x^{r+2}). For $s \geq 2$, all the four terms contribute. Thus sum of the coefficients of powers of r, $r+1$ and $s+r$ are respectively given by

$$r(r-1)a_0 + ra_0 - p^2 a_0 = 0 \qquad (s=0) \tag{4}$$

$$(r+1)ra_1 + (r+1)a_1 - p^2 a_1 = 0 \qquad (s=1) \tag{5}$$

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - p^2 a_s = 0 \quad (s = 2, 3 \cdots) \tag{6}$$

Solving (4), we get the indicial equation

$$(r+p)(r-p) = 0 \tag{7}$$

Solutions of (7) are $r_1 = p(\geq 0)$ and $r_2 = -p$.

Case 1: $r_1 = p$

With $r_1 = p$, Equation (5) becomes $(2p+1)a_1 = 0$ so $a_1 = 0$

Rewrite (6) as

$$(s+r+p)(s+r-p)a_s + a_{s-2} = 0$$

Substituting $r = p$, this becomes

$$s(s+2p)a_s + a_{s-2} = 0 \tag{8}$$

or $$a_s = -\frac{a_{s-2}}{s(s+2p)}$$

For $$s = 3, \quad a_3 = -\frac{a_1}{3(3+2p)}$$

Since $a_1 = 0$ and $p \geq 0$, then $a_3 = 0$. Thus from (8) it follows that

$$a_3 = 0, \quad a_5 = 0, \quad a_7 = 0 \text{ etc.}$$

i.e., all coefficients with odd subscripts are zero. Rewriting (8) with $s = 2m$, we have

$$2m(2m+2p)a_{2m} + a_{2n-2} = 0$$

Solving

$$a_{2m} = -\frac{1}{2^2 m(m+p)} \cdot a_{2m-2}, \quad m = 1, 2, \ldots$$

Thus $$a_2 = -\frac{a_0}{2^2(1+p)}$$

$$a_4 = -\frac{a_2}{2^2 \cdot 2(2+p)} = \frac{a_0}{2^4 2!(p+1)(p+2)}$$

In general

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} \cdot m!(p+1)(p+2)\cdots(p+m)}, \quad m = 1, 2, \ldots \tag{9}$$

a_0 which is arbitrary may be taken as

$$a_0 = \frac{1}{2^p \Gamma(p+1)}$$

Then $$a_2 = -\frac{a_0}{2^2(p+1)} = -\frac{1}{2^2 \cdot 2^p(p+1)\Gamma(p+1)} = \frac{-1}{2^{2+p}\Gamma(p+2)}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Similarly,

$$a_4 = \frac{-a_2}{2^2 \cdot 2 \cdot (p+2)} = \frac{1}{2^2 \cdot 2 \cdot 2^{2+p} \cdot (p+2)\Gamma(p+2)} = \frac{1}{2^{4+p} \cdot 2!\Gamma(p+3)}$$

In general

$$a_{2m} = \frac{(-1)^m}{2^{2m+p} \cdot m!\Gamma(p+m+1)} \quad \text{for } m = 1, 2, \cdots \tag{10}$$

By substituting these coefficients from (10) in (2) and observing that $a_1 = a_3 = a_5 = \cdots = 0$, a particular solution of the Bessel's Equation (1) is obtained as

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p} \cdot m!\Gamma(p+m+1)} \tag{11}$$

(11) is known as the Bessel's function of the first kind of order p, which converges for all x (by ratio test).

Case 2: For $r_2 = -p$

By replacing p by $-p$ in (11), we get a second linearly independent solution of (1) as

$$J_{-p}(x) = x^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-p} m! \Gamma(m-p+1)} \tag{12}$$

Hence the general solution of Bessel's Equation (1) for all $x \neq 0$ is

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x) \tag{13}$$

provided p is not an integer.

Linear Dependence of Bessel's Functions: J_n and J_{-n}

Assume that $p = n$ where n is an integer.

Then from (11), we get

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \cdot m! \Gamma(n+m+1)}$$

Since $\Gamma(n+1) = n!$, we have $\Gamma(n+m+1) = (n+m)!$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \cdot m!(m+n)!} \tag{14}$$

Book Work: Prove that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{for } n = 1, 2, 3, \ldots$$

Proof: Replacing p by $-n$ in (11), we get

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} \cdot m! \Gamma(m-n+1)} \tag{15}$$

When $m - n + 1 \leq 0$ or $m \leq (n-1)$, the gamma function of zero or negative integers is infinite. Therefore for $m = 0$ to $n-1$, the coefficients in (15) become zero. So m starts at n. Thus

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} \cdot m!(m-n)!}$$

since $\Gamma(m-n+1) = (m-n)!$

Put $m - n = s$ then s varies from 0 to ∞.

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} x^{2(s+n)-n}}{2^{2(s+n)-n}(s+n)!s!}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+n}}{2^{2s+n} \cdot s!(s+n)!}$$

$$J_{-n}(x) = (-1)^n J_n(x). \tag{16}$$

Generating Function

Generating function of a sequence of functions $f_n(x)$ is

$$G(u, x) = \sum_{n=-\infty}^{\infty} f_n(x) \cdot u^n$$

which generates $f_n(x)$ i.e., $f_n(x)$ appear as coefficients of powers of u.

Theorem: *Prove that the generating function for Bessel's functions of integral order is*

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} \tag{17}$$

Proof: If $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ is the generating function of Bessel function then the coefficients of different powers of t in the expansion of (17) are the Bessel's functions of different integral orders.

Consider

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}}$$

Expanding in series, we get

$$= \left[1 + \frac{xt}{2} + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right] \times$$

$$\times \left[1 - \frac{x}{2t} + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 \cdots\right] \tag{18}$$

Case 1: $n = 0$.

The coefficient of $t^0 = 1$ in the expansion (18) is

$$1 - \left(\frac{x}{2}\right)^2 + \left(\frac{1}{2!}\right)^2 \left(\frac{x}{2}\right)^4$$

$$- \left(\frac{1}{3!}\right)^2 \left(\frac{x}{2}\right)^6 + \left(\frac{1}{4!}\right)^2 \left(\frac{x}{2}\right)^8 - \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m} = J_0(x). \tag{19}$$

Case 2: Positive powers of $t : t^n$
The coefficient of t^n in the above expansion (18) is

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!}\frac{1}{(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$= J_n(x). \qquad (20)$$

Case 3: Negative powers of $t : t^{-n}$
The coefficient of t^{-n} in the expansion (18) is

$$\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \left(\frac{x}{2}\right)\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+1} + \frac{1}{2!}\left(\frac{x}{2}\right)^2\frac{(-1)^{n+2}}{(n+2)!}\left(\frac{x}{2}\right)^{n+2} + \cdots$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$= (-1)^n J_n(x) = J_{-n}(x) \qquad (21)$$

Thus from (19), (20) and (21), we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

Equation Reducible to Bessel's Equation

The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - p^2)y = 0 \qquad (22)$$

where λ is a parameter, can be reduced Bessel's differential equation of order p in t,

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - p^2)y = 0 \qquad (23)$$

where $t = \lambda x$ $\left(\text{so } \frac{dy}{dx} = \lambda\frac{dy}{dt}, \frac{d^2y}{dx^2} = \lambda^2\frac{d^2y}{dt^2}\right)$.

For p non-integral, the general solution of Equation (23) is

$$y = c_1 J_n(t) + c_2 J_{-n}(t).$$

Thus the general solution of Equation (22) is

$$y(x) = c_1 J_n(\lambda x) + c_2 J_{-n}(\lambda x)$$

when p is non-integral.

Orthogonality of Bessel's Functions

Prove that

$$\int_0^a x J_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{a^2}{2}J_{n+1}^2(a\alpha), & \text{if } \alpha = \beta \end{cases}$$

where α and β are roots of $J_n(ax) = 0$.

Proof: Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively be the solutions of the equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad (1)$$

and

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad (2)$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ and subtracting

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

or

$$\frac{d}{dx}\left\{x(u'v - uv')\right\} = (\beta^2 - \alpha^2)xuv \qquad (3)$$

Integrating both sides of (3) from $x = 0$ to a

$$(\beta^2 - \alpha^2)\int_0^a xuv\,dx = \left[x(u'v - uv')\right]\Big|_0^a$$

$$= a\left[u'(a)v(a) - u(a)v'(a)\right] \qquad (4)$$

where $'$ denotes differentiation w.r.t., x.

Now $\quad u' = \frac{d}{dx}u = \frac{d}{dx}J_n(\alpha x) = \alpha J_n'(\alpha x) \qquad (5)$

Similarly, $\quad v' = \frac{dv}{dx} = \frac{d}{dx}J_n(\beta x) = \beta J_n'(\beta x) \qquad (6)$

Substituting u' and v' from (5) and (6) in (4), we get

$$\int_0^a x J_n(\alpha x)J_n(\beta x)dx = \frac{a}{\beta^2 - \alpha^2}\left[\alpha J_n'(\alpha a)J_n(\beta a) - \beta J_n(\alpha a)J_n'(\beta a)\right] \qquad (7)$$

Case 1: Suppose α and β are two distinct roots of $J_n(ax) = 0$ then $J_n(a\alpha) = J_n(a\beta) = 0$.

Thus for $\alpha \neq \beta$

$$\int_0^a x J_n(\alpha x)J_n(\beta x)dx = 0 \qquad (8)$$

(8) is known as the orthogonality relation for Bessel's functions.

Case 2: Suppose $\beta = \alpha$; then R.H.S. of (4) is $\frac{0}{0}$ form. Assuming α as a root of $J_n(ax) = 0$, evaluate

R.H.S. of (4) as $\beta \to \alpha$

$$\lim_{\beta\to\alpha}\int_0^a x\, J_n(\alpha x) J_n(\beta x)dx$$

$$= \lim_{\beta\to\alpha}\left(\frac{a}{\beta^2-\alpha^2}\right)\left[\alpha J_n'(a\alpha)J_n(a\beta) - 0\right]$$

Since $J_n(a\alpha) = 0$.

Now applying L'Hospital's rule (differentiating w.r.t., β), we get

$$= \lim_{\beta\to\alpha}\frac{a}{2\beta}\left[\alpha\, J_n'(a\alpha)\cdot a\, J_n'(a\beta)\right]$$

$$= \frac{a^2}{2}\left[J_n'(a\alpha)\right]^2$$

In the recurrence relation IV on Page 11.14

$$J_{n+1}(x) = \frac{n}{x}J_n(x) - J_n'(x)$$

Put $x = a\alpha$, then $J_{n+1}(a\alpha) = \frac{n}{a\alpha}J_n(a\alpha) - J_n'(a\alpha)$. Since α is a root, $J_n(a\alpha) = 0$. Then

$$J_n'(a\alpha) = -J_{n+1}(a\alpha)$$

Thus for $\alpha \neq \beta$,

$$\int_0^a x\, J_n(\alpha x)J_n(\beta x)dx = \frac{a^2}{2}\left[J_n'(a\alpha)\right]^2$$

$$= \frac{a^2}{2}\left[J_{n+1}(a\alpha)\right]^2$$

Note: Put $x = a\alpha$ in the recurrence relation VI on Page 11.14

$$J_{n-1}(a\alpha) + J_{n+1}(a\alpha) = \frac{2n}{a\alpha}J_n(a\alpha).$$

Since $J_n(a\alpha) = 0$, $J_{n-1}(a\alpha) = -J_{n+1}(a\alpha)$.

Thus

$$\int_0^a x\, J_n(\alpha x)J_n(\beta x)dx = \frac{a^2}{2}\left[J_{n-1}(a\alpha)\right]^2$$

Recurrence Relations (or identities) for Bessel's Functions

Valid for any p.

Prove that

I. $\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x)$

Proof: From (11)

$$J_p(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+p}}{2^{2m+p}\cdot m!\Gamma(m+p+1)}$$

So

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = \frac{d}{dx}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+2p}}{2^{2m+p}\cdot m!\Gamma(m+p+1)}\right\}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2m+2p)x^{2m+2p-1}}{2^{2m+p}\cdot m!(m+p)\Gamma(m+p)}$$

$$= x^p\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+(p-1)}}{2^{2m+(p-1)}\cdot m!\Gamma(m+(p-1)+1)}$$

$$= x^p J_{p-1}(x)$$

II. $\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = -x^{-p} J_{p+1}(x)$.

Proof: Multiplying (11) by x^{-p} and differentiating

$$\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = \frac{d}{dx}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m\cdot x^{2m}}{2^{2m+p}\cdot m!\Gamma(m+p+1)}\right\}$$

$$= \sum_{m=1}^{\infty}\frac{(-1)^m\cdot 2m\cdot x^{2m-1}}{2^{2m+p}\cdot m!\Gamma(m+p+1)}$$

since for $m = 0$, the first term in R.H.S. is zero.

$$= \sum_{m=1}^{\infty}\frac{(-1)^m\cdot x^{2m-1}}{2^{2m+p-1}\cdot(m-1)!\Gamma(m+p+1)}$$

Put $s = m-1$ or $m = s+1$ then

$$= \sum_{s=0}^{\infty}\frac{(-1)^{s+1}\cdot x^{2(s+1)-1}}{2^{2(s+1)+p-1}\cdot s!\Gamma(s+1+p+1)}$$

$$= -x^{-p}\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s+(p+1)}}{2^{2s+(p+1)}s!\Gamma((s+1)+p+1)}$$

$$= -x^{-p}\cdot J_{p+1}(x).$$

III. $\frac{d}{dx}\left\{J_p(x)\right\} = J_{p-1}(x) - \frac{p}{x}J_p(x)$

or $\quad xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x)$

Proof: From recurrence relation (I)

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x)$$

Performing the differentiation in the L.H.S.,

$$x^p \cdot \frac{d}{dx}\left\{J_p(x)\right\} + px^{p-1} \cdot J_p(x) = x^p J_{p-1}(x)$$

or $$J'_p(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$

or $$J'_p(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$$

IV. $J'_p(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$

Proof: From recurrence relation (II)

$$\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = -x^{-p} J_{p+1}(x)$$

Performing the differentiation in the L.H.S.,

$$x^{-p} \cdot \frac{d}{dx} J_p(x) - px^{-p-1} J_p(x) = -x^{-p} J_{p+1}(x)$$

or $$J'_p(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

or $$J'_p(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$$

V. $J'_p(x) = \frac{1}{2}\left\{J_{p-1}(x) - J_{p+1}(x)\right\}$ is obtained by adding recurrence relations (III) and (IV)

VI. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$ is obtained by subtracting (IV) from (III).

Elementary Bessel's Functions

Bessel's functions J_p of orders $p = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are elementary and can be expressed in terms of sine and cosines and powers of x.

Result 1: $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x.$

Proof: With $p = \frac{1}{2}$, (11) reduces to

$$J_{\frac{1}{2}}(x) = \sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}} \cdot m!\Gamma\left(m+\frac{3}{2}\right)}$$

Now

$$\Gamma\left(m+\frac{3}{2}\right) = \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots \times\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2m+1)(2m-1)(2m-3)\cdots 3 \cdot 1 \cdot \sqrt{\pi}}{2^{m+1}}$$

Also

$$\begin{aligned} 2^{2m+1} \cdot m! &= 2^{m+1} \cdot 2^m \cdot m! \\ &= 2^{m+1} \cdot 2^m (m)(m-1)\cdots 2 \cdot 1 \\ &= 2^{m+1} \cdot (2m)(2m-2)\cdots 4 \cdot 2. \end{aligned}$$

Thus

$$\begin{aligned} &2^{2m+1} \cdot m! \cdot \Gamma\left(m+\frac{3}{2}\right) \\ &= \left[2^{m+1} \cdot 2m \cdot (2m-2)\cdots 4 \cdot 2\right] \\ &\times\left[(2m+1)(2m-1)\cdots 3 \cdot 1\right] \cdot 2^{-(m+1)} \cdot \sqrt{\pi} \\ &= (2m+1)!\sqrt{\pi} \end{aligned}$$

Then

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+1} \cdot m!\Gamma\left(m+\frac{3}{2}\right)} \\ &= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1}}{(2m+1)!\sqrt{\pi}} \\ &= \sqrt{\frac{2}{\pi x}} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1}}{(2m+1)!} \\ &= \sqrt{\frac{2}{\pi x}} \cdot \sin x. \end{aligned}$$

Result 2: In the recurrence relation I, put $p = \frac{1}{2}$ then

$$\frac{d}{dx}\left\{\sqrt{x} J_{\frac{1}{2}}(x)\right\} = \sqrt{x} J_{-\frac{1}{2}}(x)$$

$$\frac{d}{dx}\left\{\sqrt{x}\sqrt{\frac{2}{\pi x}} \cdot \sin x\right\} = \sqrt{x} J_{-\frac{1}{2}}(x)$$

$$\sqrt{\frac{2}{\pi}} \cos x = \sqrt{x} J_{-\frac{1}{2}}(x)$$

or $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Similarly with $p = \frac{1}{2}$, we get from recurrence relation VI.

Result 3:

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x)$$

or $$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

Using result (1) and (2) for $J_{\frac{1}{2}}$ and $J_{-\frac{1}{2}}$, we get

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

Similarly with $p = -\frac{1}{2}$ in recurrence relation VI

Result 4:
$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$
$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right).$$

Integrals of Bessel's Functions

Integrating the recurrence relation

$$\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x), \quad \text{we get}$$

$$\int x^p J_{p-1}(x)dx = x^p J_p(x) + c \tag{1}$$

For $p = 1$,
$$\int x J_0(x)\,dx = x J_1(x) + c \tag{2}$$

Integrating the recurrence relation

$$\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = -x^{-p} J_{p+1}(x), \quad \text{we get}$$

$$\int x^{-p} J_{p+1}(x)dx = -x^{-p} J_p(x) + c \tag{3}$$

For $p = 0$,
$$\int J_1(x)\,dx = -J_0(x) + c \tag{4}$$

In general $\int x^m J_n(x)dx$ for m and n integers with $m+n \geq 0$ can be integrated by parts completely if $m+n$ is odd. But when $m+n$ is even, the integral depends on the residual integral $\int J_0(x)dx$ which has been tabulated.

Integrating
$$J'_p(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right]$$

$$2J_p(x) = \int J_{p-1}(x)\,dx - \int J_{p+1}(x)dx$$

or
$$\int J_{p+1}(x)dx = \int J_{p-1}(x)dx - 2J_p(x).$$

Bessel's Function of Second Kind of Order n or Neumann Function

When n is integral, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and do not constitute the solution.

Let $y = u(x)\,J_n(x)$ be a solution of (1). Substituting in (1),

$$x^2(u''J_n + 2u'J'_n + uJ''_n) + x(u'J_n + uJ'_n) + (x^2 - n^2)uJ_n = 0$$

or
$$u\left\{x^2J''_n + xJ'_n + (x^2 - n^2)J_n\right\} + x^2u''J_n + 2x^2u'J'_n + xu'J_n = 0$$

Since J_n is a solution of (1), the first term is zero. Dividing throughout by $x^2u'J_n$, we get

$$\frac{u''}{u} + 2\frac{J'_n}{J_n} + \frac{1}{x} = 0$$

Integrating $\ln(u'J_n^2 \cdot x) = \ln B$ or $xu'J_n^2 = B$. Thus

$$u' = \frac{B}{xJ_n^2}$$

Integrating

$$u = B\int \frac{dx}{xJ_n^2} + c$$

Hence $y = AJ_n(x) + BY_n(x)$ is the complete solution of (1) where

$$Y_n(x) = J_n(x) \cdot \int \frac{dx}{x[J_n(x)]^2}$$

$Y_n(x)$ is known as Bessel's function of second kind of order n or Neumann function.

Worked Out Examples

Example 1: Find $J_0(x)$ and $J_1(x)$.

Solution: Put $n = 0$ in

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} \cdot m!(m+n)!}$$

Then
$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \left(\frac{1}{3!}\right)^2\left(\frac{x}{2}\right)^6 + \cdots$$

For $n = 1$,

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+n)!}$$

or
$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^6 + \cdots\right].$$

Example 2: Show that $J_n(x)$ is an even function when n is even and odd function when n is odd.

Solution: Suppose n is even.

$$J_n(-x) = (-x)^n \sum_{m=0}^{\infty} \frac{(-1)^m (-x)^{2m}}{2^{2m+n} \cdot m!(m+n)!}$$

For n even $(-1)^n = 1$ and $(-1)^{2m} = 1$

$$J_n(-x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \cdot m!(m+n)!} = J_n(x)$$

Thus $J_n(x)$ is an even function.

Suppose n is odd. Then $(-1)^n = -1$, so

$$J_n(-x) = -x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(m+n)!} = -J_n(x).$$

Thus $J_n(x)$ is an odd function.

Example 3: Express $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Solution: Rewriting the recurrence relation (VI)

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{1}$$

Put $n = 1, 2, 3, 4, 5$ in (1), we get (suppressing the argument x)

$$J_2 = \frac{2}{x} J_1 - J_0 \tag{2}$$

$$J_3 = \frac{4}{x} J_2 - J_1 \tag{3}$$

$$J_4 = \frac{6}{x} J_3 - J_2 \tag{4}$$

$$J_5 = \frac{8}{x} J_4 - J_3 \tag{5}$$

$$J_6 = \frac{10}{x} J_5 - J_4 \tag{6}$$

Substituting (3) in (2),

$$J_3 = \left\{\frac{8}{x^2} - 1\right\} J_1 - \frac{4}{x} J_0 \tag{7}$$

Substituting (7) and (2) in (4),

$$J_4 = \left[\frac{48}{x^3} - \frac{8}{x}\right] J_1 + \left(1 - \frac{24}{x^2}\right) J_0 \tag{8}$$

Substituting (8), and (3) in (5)

$$J_5 = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right) J_1 + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0 \tag{9}$$

Substituting (9) and (8) in (6), we get

$$J_6 = \frac{10}{x}\left\{\left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right) J_1 + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0\right\} - \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1 - \left(1 - \frac{24}{x^2}\right) J_0$$

$$J_6(x) = \left(\frac{3840}{x^4} - \frac{768}{x^3} - \frac{2}{x}\right) J_1(x) + \left(\frac{144}{x^2} - 1 - \frac{1920}{x^4}\right) J_0(x).$$

Example 4: Express $J_{\frac{7}{2}}(x)$ in terms of sine and cosine functions.

Solution: Put $n = \frac{5}{2}$ in the recurrence relation

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{1}$$

so
$$J_{\frac{7}{2}}(x) = \frac{5}{x} J_{\frac{5}{2}}(x) - J_{\frac{3}{2}}(x) \tag{2}$$

Put $n = \frac{3}{2}$ in (1) then

$$J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) \tag{3}$$

Put $n = \frac{1}{2}$ in (1) then

$$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

Since
$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right) \tag{4}$$

Substituting (4) and $J_{\frac{1}{2}}(x)$ in (3)

$$J_{\frac{5}{2}}(x) = \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3 - x^2}{x^2}\right) \sin x - \frac{3}{x} \cos x\right] \tag{5}$$

Substituting (5) and (4) in (2), we get

$$J_{\frac{5}{2}}(x) = \frac{5}{x}\cdot\sqrt{\frac{2}{\pi x}}\left[\left(\frac{3-x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$$

$$-\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$=\sqrt{\frac{2}{\pi x}}\left[\left(\frac{15-6x^2}{x^3}\right)\sin x + \left(\frac{15}{x^2}-1\right)\cos x\right].$$

Example 5: Show that $J_1''(x) = -J_1(x) + \frac{1}{x}J_2(x)$.

Solution: Put $n = 1$ in the recurrence relation III

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \quad (1)$$

Then

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x} \quad (2)$$

Differentiating (2) w.r.t., 'x'

$$J_1''(x) = J_0'(x) + \frac{1}{x^2}J_1(x) - \frac{1}{x}J_1'(x) \quad (3)$$

Put $n = 0$, in (1) then

$$J_0'(x) = J_{-1}(x) - 0 \quad (4)$$

But $\quad J_{-n}(x) = (-1)^n J_n(x)$

So with $\quad n = +1,\ J_{-1}(x) = -J_1(x) \quad (5)$

Substituting (5) in (4),

$$J_0'(x) = J_{-1}(x) = -J_1(x) \quad (6)$$

Put (6) and (2) in (3), we get

$$J_1'' = -J_1(x) + \frac{1}{x^2}J_1(x) - \frac{1}{x}\left\{J_0(x) - \frac{J_1(x)}{x}\right\} \quad (7)$$

Put $n = 1$ in recurrence relation (VI)

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Then

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \quad (8)$$

or

$$J_0(x) = \frac{2}{x}J_1(x) - J_2(x) \quad (9)$$

Using (9) eliminate J_0 from (7) then

$$J_1'' = -J_1(x) + \frac{1}{x^2}J_1(x) - \frac{1}{x}\left\{\frac{2}{x}J_1(x) - J_2(x)\right\} + \frac{J_1(x)}{x^2}$$

$$J_1''(x) = -J_1(x) + \frac{1}{x}J_2(x).$$

Example 6: Prove that

$$\frac{d}{dx}\left\{J_n^2(x)\right\} = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

Solution: $\frac{d}{dx}\left\{J_n^2(x)\right\} = 2\cdot J_n(x)J_n'(x)$.

Using recurrence relation (V)

$$J_n'(x) = \frac{1}{2}\left\{J_{n-1} - J_{n+1}\right\}$$

$$\frac{d}{dx}\{J_n^2\} = 2\cdot J_n(x)\cdot\left[\frac{1}{2}\left(J_{n-1} - J_{n+1}\right)\right]$$

From recurrence relation VI

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\frac{d}{dx}\left\{J_n^2(x)\right\} = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\times\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right].$$

Example 7: Show that

$$J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x).$$

Solution: In recurrence relation

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

Put $n = 2$, $\quad J_2'(x) = J_1(x) - \frac{2}{x}J_2(x) \quad (1)$

Since $\quad J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x)$

for $n = 1$, $\quad J_2(x) + J_0(x) = \frac{2}{x}J_1(x) \quad (2)$

Substituting J_2 from (2) in (1)

$$J_2'(x) = J_1(x) - \frac{2}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x).$$

Example 8: Evaluate $\int J_5(x)dx$.

Solution: Putting $n = 4$ in

$$\int J_{p+1}(x)dx = \int J_{p-1}(x)dx - 2J_p(x),$$

we get

$$\int J_5(x)dx = \int J_3(x)dx - 2J_4(x) \qquad (1)$$

Again with $p = 2$

$$\int J_3(x)dx = \int J_1(x)dx - 2J_2(x) \qquad (2)$$

Also we know that

$$\int J_1(x)dx = -J_0(x) + c \qquad (3)$$

Substituting (2) and (3) in (1)

$$\int J_5(x)dx = \left[-J_0(x) + c\right] - 2J_2(x) - 2J_4(x)$$

Example 9: Evaluate $\int x^2 J_1(x)dx$.

Solution: Put $p = 2$ in

$$\int x^p J_{p-1}(x)dx = x^p J_p(x) + c$$

Then $$\int x^2 J_1(x)dx = x^2 J_2(x) + c$$

But $$J_2(x) = \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$\int x^2 J_1(x)dx = x^2\left[\frac{2}{x}J_1(x) - J_0(x)\right] + c$$
$$= 2xJ_1(x) - x^2 J_0(x) + c.$$

Example 10: Evaluate $\int x^3 J_3(x)dx$.

Solution: Integrating by parts (suppressing argument x)

$$\int x^3 J_3 dx = \int x^5[x^{-2}J_3]dx = x^5[x^{-2}J_2]$$
$$-\int -x^{-2}J_2 \cdot 5x^4 dx$$
$$= -x^3 J_2 + 5\int x^2 J_2 dx$$

Now $$\int x^2 J_2 dx = \int x^3[x^{-1}J_2]dx = x^3[-x^{-1}J_1]$$
$$-\int -x^{-1}J_1 3x^2 dx$$
$$= -x^2 J_1 + 3\int xJ_1\, dx$$

But $$\int xJ_1(x)dx = -\int xJ_0'(x)dx$$
$$= -\left[xJ_0 - \int J_0(x)dx\right]$$

Substituting

$$\int x^3 J_3(x)dx = -x^3 J_2(x) + 5\Big\{-x^2 J_1(x)$$
$$+3\left[-xJ_0(x) + \int J_0 dx\right]\Big\}$$
$$= -x^3 J_2(x) - 5x^2 J_1(x) - 15xJ_0(x)$$
$$+15\int J_0(x)dx.$$

Note: $\int J_0(x)dx$ can not be integrated but its values are tabulated.

EXERCISE

1. Show that **a.** $J_0'(x) = -J_1(x)$
   b. $\frac{d}{dx}(xJ_1) = xJ_0$.
2. Express $J_{\frac{3}{2}}, J_{-\frac{3}{2}}$ in terms of sin and cos.

Ans. $J_{\frac{3}{2}} = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$;
$J_{-\frac{3}{2}} = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right)$

3. Express $J_{\frac{5}{2}}, J_{-\frac{5}{2}}$ in terms of sin and cos.

Ans. $J_{\frac{5}{2}} = \sqrt{\frac{2}{\pi x}}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$
$J_{-\frac{5}{2}} = \sqrt{\frac{2}{\pi x}}\left\{\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right\}$

4. Express $J_4(x)$ in terms of J_0 and J_1.

Ans. $J_4 = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x)$

5. Express $J_5(x)$ in terms of J_0 and J_1.

Ans. $J_5 = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) +$
$\left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x)$

6. Prove that $J_n'' = \frac{1}{4}[J_{n-2} - 2J_n + J_{n+2}]$.
7. Show that $\frac{d}{dx}\{xJ_n \cdot J_{n+1}\} = x[J_n^2 - J_{n+1}^2]$.

8. Prove that $J_0'' = \frac{1}{2}[J_2 - J_0]$.
9. Show that $\frac{d}{dx}[J_n^2+J_{n+1}^2]=2[\frac{n}{x}J_n^2-\frac{n+1}{x}J_{n+1}^2]$.
10. Prove that $J_n'''=\frac{1}{8}[J_{n-3}-3J_{n-1}+3J_{n+1}-J_{n+3}]$.
11. Show that $2J_0''(x) = J_2(x) - J_0(x)$.
12. Evaluate $\int J_3(x)dx$.

Ans. $-2J_1 - 2J_2 + c + \int J_0(x)dx$.

13. Evaluate: $\int x^3 J_0(x)dx$

Ans. $x^3J_1(x) - 2x^2J_2(x) + c$

14. Evaluate: $\int x^4 J_1(x)dx$

Ans. $(8x^2 - x^4)J_0(x) + (4x^3 - 16x)J_1(x)$

15. Establish the Jacobi series:
   a. $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots$
   b. $\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \cdots]$.

Hint: $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0 + \left(t - \frac{1}{t}\right) J_1 + \left(t^2 - \frac{1}{t^2}\right) J_2 + \left(t^3 - \frac{1}{t^3}\right) J_3 + \cdots$ obtained from generating function using $J_{-n}(x) = (-1)^n J_n(x)$. Put $t = \cos\theta + i\sin\theta$, $\frac{1}{t} = \cos\theta - i\sin\theta$, then $t^p + \frac{1}{t^p} = 2\cos p\theta$, $t^p - \frac{1}{t^p} = 2i\sin p\theta$, $t - \frac{1}{t} = 2i\sin\theta$. Equate real and imaginary parts. Replace θ by $\frac{\pi}{2} - \theta$.

11.4 DIFFERENTIAL EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

Various differential equations which are not Bessel's equations can be reduced to Bessel's equation by changing the dependent or and independent variable. The differential equation

$$x^2y''+x(a+2bx^p)y'+[c+dx^{2q}+b(a+p-1)x^p + b^2x^{2p}]y = 0 \quad (1)$$

can be transformed to Bessel's equation in the new variables X and Y where

$$y = x^{(1-a)/2}e^{-(b/p)x^p}Y \text{ and } x = \left(\frac{qX}{\sqrt{|d|}}\right)^{1/q}.$$

Then the general solution of (1) is given by

$$\boxed{y(x) = x^{\alpha}e^{-\beta x^p}[c_1J_v(\lambda x^q) + c_2Y_v(\lambda x^q)]} \quad (2)$$

Here $\boxed{\alpha = \frac{1-a}{2}, \beta = \frac{b}{p}, \lambda = \frac{\sqrt{|d|}}{q}, v = \frac{\sqrt{(1-a)^2-4c}}{2q}}$ (3)

We assume that $d \neq 0$, $p \neq 0$, $q \neq 0$ and $(1-a^2) \geq 4c$. Also if $d < 0$, replace J_v and Y_v by I_v and k_v respectively. When v is not an integer, replace Y_v and k_v by J_{-v} and I_{-v} respectively.

Corollary: The general solution of

$$x^ry'' + rx^{r-1}y' + (ax^s + bx^{r-2})y = 0 \quad (4)$$

is $y = x^{\alpha}[c_1J_v(\lambda x^{\gamma}) + c_2Y_v(\lambda x^{\gamma})]$ (5)

where $\alpha = \frac{1-r}{2}$, $\gamma = \frac{2-r+s}{2}$, $\lambda = \frac{2\sqrt{|a|}}{2-r+s}$,

$$v = \frac{\sqrt{(1-r)^2 - 4b}}{2 - r + s} \quad (6)$$

we assume that $2 - r + s \neq 0$ and $(1-r)^2 \geq 4b$. If $a < 0$, replace J_v and y_v by I_v and k_v respectively.

WORKED OUT EXAMPLES

Reduce the given differential equation to the Bessel's equation and solve.

Example 1: $y'' + \left(\varepsilon^2 - \frac{4n^2-1}{4x^2}\right)y = 0$

Solution: Rewriting the given D.E.

$$x^2y'' + \left(\varepsilon^2x^2 - \frac{4n^2-1}{4}\right)y = 0.$$

Comparing this with D.E. (1), we have $a = 0$, $b = 0$, $c = -\frac{(4n^2-1)}{4}$, $d = \varepsilon^2$, $q = 1$. So from relations (3), $\alpha = \frac{1-0}{2} = \frac{1}{2}$, $\beta = 0$, $\lambda = \frac{\varepsilon}{1}$, $v = \frac{\sqrt{1+4n^2-1}}{1} = 2n$.

Then the general solution of the given D.E. is $y(x) = x^{1/2}[c_1J_{2n}(\varepsilon x) + c_2Y_{2n}(\varepsilon x)]$

Example 2: $y'' + 2y' + \left(x^2 + 1 - \frac{2}{x^2}\right)y = 0.$

Solution: Rewriting

$$x^2y'' + 2x^2y' + (x^4 + x^2 - 2)y = 0$$

Comparing with D.E. (1), we have $a = 0$, $b = 1$, $p = 1, c = -2, d = 1, q = 2$. Then from (3) $\alpha = \frac{1}{2}$, $\beta = \frac{1}{1} = 1, \lambda = \frac{1}{2}, \upsilon = \frac{\sqrt{(1-0)^2-4(-2)}}{4} = \frac{3}{4}$. Thus the general solution of given D.E. is

$$y = x^{1/2}e^{-x}\left[c_1 J_{3/4}\left(\frac{1}{2}x^2\right) + c_2 Y_{\frac{3}{4}}\left(\frac{1}{2}x^2\right)\right]$$

since $\upsilon = \frac{3}{4}$ is not an integer, we can replace $Y_{\frac{3}{4}}$ by $J_{-3/4}$, thus the general solution takes the form

$$y = \sqrt{x}e^{-x}\left[c_1 J_{3/4}\left(\frac{x^2}{2}\right) + c_2 J_{-3/4}\left(\frac{x^2}{2}\right)\right]$$

Example 3: $(x^5y')' = y$

Solution: Rewriting $x^5y'' + 5x^4y' - y = 0$. Comparing this equation with (4), we have $r = 5$, $a = -1, s = 0, b = 0$ then from relations (5), $\alpha = \frac{1-5}{2} = -2$, $\gamma = \frac{2-5+0}{2} = -\frac{3}{2}$, $\lambda = \frac{2}{2-5+0} = \frac{-2}{3}$ and $\upsilon = \frac{\sqrt{(1-5)^2-0}}{2-5+0} = \frac{4}{-3}$. Thus the general solution from (6) is $y = x^{-2}\left[c_1 I_{4/3}\left(-\frac{2}{3}x^{-3/2}\right) + c_2 I_{-4/3}\left(-\frac{2}{3}x^{-3/2}\right)\right]$ since $a = -1 < 0$

Example 4: Show that the general solution of $y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$ is $\sqrt{x} \cdot y = c_1 \sin x + c_2 \cos x$.

Solution: Comparing with D.E. (1), we have $a = 1$, $b = 0$, $c = -\frac{1}{4}$, $d = 1$, $q = 1$. So from (3) $\alpha = 0$, $\beta = 0$, $\lambda = 1$, $\upsilon = \frac{1}{2}$. Thus the general solution is $y(x) = c_1\ J_{\frac{1}{2}}(x) + c_2 Y_{\frac{1}{2}}(x)$. Since $\upsilon = \frac{1}{2}$ is not an integer,

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

$$= c_1\sqrt{\frac{2}{\pi x}}\sin x + c_2\frac{\sqrt{2}}{\pi x}\cos x$$

$$\text{or} \quad \sqrt{x}\, y(x) = c_1^* \sin x + c_2^* \cos x$$

EXERCISE

Reduce the given differential equations to the Bessel's equation and solve.

1. $y'' + \frac{1}{2x}y' + \frac{1}{16}\left(x^{-3/2} + \frac{15}{16}x^{-2}\right)y = 0$

Ans. $y = x^{1/4}(c_1 J_{1/4}(x^{1/4}) + c_2 J_{-1/4}(x^{1/4}))$

Hint: $a = \frac{1}{2}$, $b = 0$, $c = \frac{15}{256}$, $d = \frac{1}{16}$, $q = \frac{1}{4}$, $\alpha = \frac{1}{4}$, $\beta = 0$, $\lambda = 1$, $\upsilon = \frac{1}{4}$

2. $81x^2y'' + 27xy' + (9x^{2/3} + 8)y = 0$

Ans. $y(x) = x^{1/3}[c_1 J_{1/3}(x^{1/3}) + c_2 J_{-1/3}(x^{1/3})]$

Hint: $a = \frac{1}{3}$, $b = 0$, $c = \frac{8}{81}$, $d = \frac{1}{9}$, $q = \frac{1}{3}$, $\alpha = \frac{1}{3}$, $\beta = 0$, $\lambda = 1$, $\upsilon = \frac{1}{3}$

3. $y'' + 3\sqrt{x}y = 0$

Ans. $y(x) = \sqrt{x}\left[c_1 J_{2/5}\left(\frac{4}{5}\sqrt{3}x^{5/4}\right) + c_2 Y_{2/5}\left(\frac{4}{5}\sqrt{3x^{5/4}}\right)\right]$

Hint: $a = 0, b = 0, c = 0, d = 3, q = \frac{5}{4}, \alpha = \frac{1}{2}, \beta = 0, \lambda = \frac{\sqrt{3}}{(5/4)}, \upsilon = \frac{2}{5}$.

4. $xy'' + 3y' + y = 0$

Ans. $y(x) = x^{-1}\left[c_1 J_2(2\sqrt{x}) + c_2 Y_2\left(2\sqrt{x}\right)\right]$

Hint: $a = 3, b = 0, c = 0, d = 1, q = \frac{1}{2}, \alpha = -1, \beta = 0, \lambda = 2, \upsilon = 2$

5. $(xy')' - 5x^3y = 0$

Ans. $y(x) = c_1\ I_0\left(\frac{\sqrt{5}}{2}x^2\right) + c_2 k_0\left(\frac{\sqrt{5}}{2}x^2\right)$

Hint: $a = 1$, $b = 0$, $c = 0$, $d = -5$, $q = 2$, $\alpha = 0$, $\beta = 0$, $\lambda = \frac{\sqrt{5}}{2}$, $\upsilon = 0$

6. $y'' + \frac{y'}{x} + \left(1 - \frac{1}{9x^2}\right)y = 0$

Ans. $y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$

Hint: $a = 1$, $b = 0$, $c = -\frac{1}{9}$, $d = 1$, $q = 1$, $\alpha = 0$, $\beta = 0$, $\lambda = 1$, $\upsilon = \frac{1}{3}$

7. $x^2y'' + xy' + \left(x^2 - \frac{1}{6.25}\right)y = 0$

Ans. $y(x) = c_1 J_{2/5}(x) + c_2 J_{-2/5}(x)$

Hint: $a = 1, b = 0, c = -\frac{1}{6.25}, d = 1, q = 1$, $\alpha = 0$, $\beta = 0$, $\upsilon = \frac{2}{5}$

8. $x^2y'' + x(4x^4 - 3)y' + (4x^8 - 5x^2 + 3)y = 0$

Ans. $y(x) = x^2 e^{-x^{4/2}}[c_1 I_1(\sqrt{5}x) + c_2 k_1 \sqrt{5}x)]$

Hint: $a = -3, b = 2, p = 4, c = 3, d = -5, q = 1, \alpha = 2, \beta = \frac{1}{2}, \lambda = \sqrt{5}, v = 1$

9. $\left(\frac{1}{x}y'\right)' + \left(\frac{1}{x^2} + \frac{1}{x^3}\right) y = 0$

Ans. $y = x[c_1 J_0(2\sqrt{x}) + c_2 Y_0(2\sqrt{x})]$

Hint: $r = -1,\ s = -2,\ a = b = 1,\ \alpha = 1, v = \frac{1}{2}, \lambda = 2, v = 0$

10. $x^2y'' - 2xy' + (2 - x^3)y = 0$

Ans. $y = x^{3/2}\left[c_1 I_{1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2 I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)\right]$

Hint: $a = -2, b = 0, c = 2, d = -1, q = \frac{3}{2}, \alpha = \frac{3}{2}, \beta = 0, \lambda = 2/3, v = 1/3$

11. $x^2y'' + xy' + 8x^2 = y$

Ans. $y = c_1 J_1(2\sqrt{2}x) + c_2 Y_1(2\sqrt{2}x)$

Hint: $a = 1,\ b = 0,\ c = -1,\ d = 8,\ q = 1, \alpha = 0, \beta = 0, \lambda = 2\sqrt{2}, v = 1$

12. $4y'' + 9xy = 0$

Ans. $y = \sqrt{x}\left[c_1 J_{1/3}(x^{3/2}) + c_2 J_{-1/3}(x^{3/2})\right]$

Hint: $a = 0, b = 0, c = 0, d = \frac{9}{4}, q = \frac{3}{2}, \alpha = \frac{1}{2}, \beta = 0, \lambda = 1, v = \frac{1}{3}$

13. $9x^2y'' + 9xy' + (36x^4 - 16)y = 0$

Ans. $y(x) = c_1 J_{2/3}(x^2) + c_2 J_{-2/3}(x^2)$

Hint: $a = 1,\ b = 0,\ c = -\frac{16}{9},\ d = 4,\ q = 2, \alpha = 0, \beta = 0, \lambda = 1, v = \frac{2}{3}$

14. $y'' + k^2x^4y = 0$

Ans. $y = \sqrt{x}\left[c_1 J_{1/4}\left(\frac{1}{2}kx^2\right) + c_2 Y_{1/4}\left(\frac{1}{2}kx^2\right)\right]$

Hint: $a = 0,\ b = 0,\ c = 0,\ d = k^2,\ q = 2, \alpha = \frac{1}{2}, \beta = 0, \lambda = \frac{k}{2}, v = \frac{1}{4}$

15. $2xy'' + 4y' + xy = 0$

Ans. $y = [c_1 J_{1/2}(x/\sqrt{2}) + c_2 J_{-1/2}(x/\sqrt{2})]x^{-1/2}$

Hint: $a = 2,\ b = 0,\ c = 0,\ d = \frac{1}{2},\ q = +1, \alpha = -\frac{1}{2}\ \beta = 0, \lambda = \frac{1}{\sqrt{2}}, v = \frac{1}{2}$.

11.5 LEGENDRE* FUNCTIONS

The boundary value problems with spherical symmetry (independent of θ) by the application of separation of variables reduces two ordinary differential equations. One of them is the very important differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{1}$$

known as the Legendre's differential equation.

The parameter n is given integer, (although it could be a real number).

The solution of Legendre's Equation (1) is known as Legendre's function of order n.

Assume a power series solution of (1) as

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{2}$$

Substitute (2) and its derivatives in (1), then

$$(1 - x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

where $k = n(n+1)$. Rewriting

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0 \tag{3}$$

(3) is an identity since (2) is a solution of (1). So equate the sum of the coefficients of each power of x to zero.

Coefficient of x^0 arise from 1st and fourth series in (3). Thus

$$2a_2 + n(n+1)a_0 = 0 \tag{4}$$

coefficient of x^1 arise from 1st, 3rd and 4th series in (3). So

$$6a_3 + [-2 + n(n+1)]a_1 = 0 \tag{5}$$

All the four series in (3) contribute coefficients of x^s for $s \geq 2$. Thus

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0 \tag{6}$$

Solving (6)

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \quad \text{for} \quad s = 0, 1, 2, \ldots \tag{7}$$

* Adrien Marie Legendre (1752–1833), French mathematician.