



LINEAR ALGEBRA AND ITS APPLICATIONS

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MATRICES AND GAUSSIAN ELIMINATION

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Course Content: Gaussian Elimination

❖ **Rank Of A Matrix:** A Square matrix A of order n is said to have rank r if

- At least one minor of order r does not vanish ($\neq 0$).
- Every minor of order (r+1) vanishes ($= 0$).

Rank of matrix A is denoted by r i.e. rank(A)=r.

❖ If $A = [a_{ij}]_{m \times n}$ is a rectangular matrix, then **Rank of the Matrix** is defined as the number of non-zero rows in the Echelon form of A. It is also defined as the maximum number of Linearly Independent Rows or Columns of the Matrix A.

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GAUSSIAN ELIMINATION:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 2 & 2 & -3 \\ -2 & -2 & 3 \\ 4 & 4 & 6 \end{pmatrix}; \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 3 \end{pmatrix}; \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 10 \end{pmatrix}; \begin{pmatrix} 2 & -1 & 3 & 5 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$

$2 \qquad \qquad 1 \qquad \qquad 2 \qquad \qquad 2 \qquad \qquad 2$

Ex: Find the conditions on a and b so that the Matrix $\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix}$ has rank 1, 2, 3.

$$\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix} \xrightarrow{R_3 - \left(\frac{1}{a}\right)R_1} \begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 3 - (1/a) & 6 - (2/a) \end{pmatrix}$$

- (i) For no values of a and b this matrix will have rank 1.
- (ii) If $a=1/3$ and $b=4$, rank of the matrix is 2.
- (iii) If $a \neq 1/3$ and $b \neq 4$, rank of the matrix is 3.

GAUSSIAN ELIMINATION:

❖ Relation between Rank, Consistency and Solution:

❖ If $\text{rank}(A)=r$, then the following hold good:

- (i) If $\text{rank}(A)=\text{rank}[A:b]=r$, system $Ax=b$ is **consistent** and has **a solution**.
- (ii) If $\text{rank}(A)=\text{rank}[A:b]=r=n$, system $Ax=b$ is **consistent** and has **a unique solution**.
- (iii) If $\text{rank}(A)=\text{rank}[A:b]=r<n$, system $Ax=b$ is **consistent** and has **infinite number of solutions**.
- (iv) If $\text{rank}(A) \neq \text{rank}[A:b]$, system $Ax=b$ is **inconsistent** and has **no solution**.

Gaussian Elimination

Gaussian Elimination is used to check for Consistency and solve a System of linear equations.

For a given system of equations $Ax=b$ apply Elementary row transformations to the Augmented Matrix $[A:b]$ and reduce it to $[U:c]$ where U is an **Upper Triangular Matrix** so that we get an equivalent system $Ux=c$ which can be solved by **Backward Substitution**.

Here A and U are Equivalent Matrices and hence solution of $Ax=b$ is same as $Ux=c$.

❖ **The following steps are to be followed while performing Elementary Row Transformations in Gaussian Elimination:**

- **No exchange** of rows.
- First row should be **retained** as it is (**not altered**).
- The first non-zero element in every non-zero row is called **Pivot**.
- The original system $Ax=b$ and new system obtained $Ux=c$ have the **same solution**.

GAUSSIAN ELIMINATION:

Gaussian Elimination is illustrated below for a system of 3 equations with 3 variables.

Consider a system of 3 equations in 3 variables

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3\end{aligned}$$

$$\begin{aligned}[A : b] &= \begin{pmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ a_{21} & a_{22} & a_{23} : b_2 \\ a_{31} & a_{32} & a_{33} : b_3 \end{pmatrix} \xrightarrow[R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1]{R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ 0 & d_{22} & d_{23} : c_2 \\ 0 & d_{32} & d_{33} : c_3 \end{pmatrix} \\ &\xrightarrow{R_3 - \left(\frac{d_{32}}{d_{22}}\right)R_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ 0 & d_{22} & d_{23} : c_2 \\ 0 & 0 & e_{33} : c_4 \end{pmatrix} = [U : c] \\ &\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ d_{22}x_2 + d_{23}x_3 = c_2 \\ e_{33}x_3 = c_4 \end{cases}\end{aligned}$$

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GAUSSIAN ELIMINATION:

Check for consistency and solve the following system of equations if consistent:

$$(i) \quad \begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 &= 1 \end{aligned} \quad [A:b] = \begin{pmatrix} 1 & 1 & -2 & 4:5 \\ 2 & 2 & -3 & 1:3 \\ 3 & 3 & -4 & -2:1 \end{pmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 2 & -14:-14 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 0 & 0:0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ x_3 - 7x_4 = -7 \end{cases} \quad \begin{array}{l} r(A)=2 = r[A:b] < n(=4) \\ \text{System is consistent and has infinitely many solutions.} \end{array}$$

Solution is $(x_1, x_2, x_3, x_4) = (10k_1 - k_2 - 9, k_2, 7k_1 - 7, k_1)$

Depending upon values of k_1 and k_2 we get infinity of solutions.



THANK YOU

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