LINEAR ALGEBRA AND ITS APPLICATIONS

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MATRICES AND
GAUSSIAN ELIMINATION

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GAUSSIAN ELIMINATION:

Course Content: Gaussian Elimination

- **Rank Of A Matrix:** A Square matrix A of order n is said to have rank r if

  - At least one minor of order r does not vanish $(\neq 0)$.
  - Every minor of order (r+1) vanishes $(=0)$.

  Rank of matrix A is denoted by r i.e.rank(A)=r.

- If $A=\left[a_{ij}\right]_{mxn}$ is a rectangular matrix, then **Rank of the Matrix** is defined as the number of non-zero rows in the Echelon form of A. It is also defined as the maximum number of Linearly Independent Rows or Columns of the Matrix A.

GAUSSIAN ELIMINATION:

1) $R_2 \to R_2 + R_1$
2) $R_3 \to R_3 (R_2)$

$$\begin{pmatrix} 2 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix} \to 2; \quad \begin{pmatrix} 2 & 2 & -3 \\ -2 & -2 & 3 \\ 4 & 4 & 6 \end{pmatrix} \to 1; \quad \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 3 \end{pmatrix} \to 2; \quad \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 10 \end{pmatrix} \to 2; \quad \begin{pmatrix} 2 & -1 & 3 & 5 \\ 3 & 2 & 1 & 2 \end{pmatrix} \to 2$$

Ex: Find the conditions on a and b so that the Matrix $\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix}$ has rank 1, 2, 3.

$$\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix} \xrightarrow{R_3 - \left(\frac{1}{a}\right) R_1} \begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 3-(1/a) & 6-(2/a) \end{pmatrix}$$

(i) For no values of a and b this matrix will have rank 1.

(ii) If $a=1/3$ and $b=4$, rank of the matrix is 2.

(iii) If $a \neq 1/3$ and $b \neq 4$, rank of the matrix is 3.

GAUSSIAN ELIMINATION:

❖ **Relation between Rank, Consistency and Solution:**

❖ If rank(A)=r, then the following hold good:

(i) If **rank(A)=rank[A:b}=r**, system Ax=b is **consistent** and has **a solution**.

(ii) If **rank(A)=rank[A:b}=r=n**, system Ax=b is **consistent** and has **a unique solution**.

(iii) If **rank(A)=rank[A:b}=r<n**, system Ax=b is **consistent** and has **infinite number of solutions**.

(iv) If **rank(A) $\neq$ rank[A:b]** , system Ax=b is **inconsistent** and has **no solution**.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

GAUSSIAN ELIMINATION:

Gaussian Elimination

Gaussian Elimination is used to check for Consistency and solve a System of linear equations.

For a given system of equations Ax=b apply Elementary row transformations to the Augmented Matrix [A:b] and reduce it to [U:c] where U is an **Upper Triangular Matrix** so that we get an equivalent system Ux=c which can be solved by **Backward Substitution**.

Here A and U are Equivalent Matrices and hence solution of Ax=b is same as Ux=c.

- **The following steps are to be followed while performing Elementary Row Transformations in Gaussian Elimination:**

- **No exchange** of rows.
- First row should be **retained** as it is (not altered).
- The first non-zero element in every non-zero row is called **Pivot**.
- The original system Ax=b and new system obtained Ux=c have the **same solution**.

GAUSSIAN ELIMINATION:

Gaussian Elimination is illustrated below for a system of 3 equations with 3 variables.

Consider a system of 3 equations in 3 variables

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$[A:b] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ a_{21} & a_{22} & a_{23} & : b_2 \\ a_{31} & a_{32} & a_{33} & : b_3 \end{pmatrix} \xrightarrow[R_3-\left(\frac{a_{31}}{a_{11}}\right)R_1]{R_2-\left(\frac{a_{21}}{a_{11}}\right)R_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & d_{22} & d_{23} & : c_2 \\ 0 & d_{32} & d_{33} & : c_3 \end{pmatrix}$$

$$\xrightarrow{R_3-\left(\frac{d_{32}}{d_{22}}\right)R_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & d_{22} & d_{23} & : c_2 \\ 0 & 0 & e_{33} & : c_4 \end{pmatrix} = [U:c]$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ \qquad d_{22}x_2 + d_{23}x_3 = c_2 \\ \qquad\qquad e_{33}x_3 = c_4 \end{cases}$$

GAUSSIAN ELIMINATION:

Check for consistency and solve the following system of equations if consistent:

(i) $$\begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 &= 1 \end{aligned} \qquad [A:b] = \begin{pmatrix} 1 & 1 & -2 & 4:5 \\ 2 & 2 & -3 & 1:3 \\ 3 & 3 & -4 & -2:1 \end{pmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 2 & -14:-14 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 0 & 0:0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ x_3 - 7x_4 = -7 \end{cases}$$

r(A)=2 = r[A:b]<n(=4)

System is **consistent** and has **infinitely many solutions**.

Solution is $(x_1, x_2, x_3, x_4) = (10k_1 - k_2 - 9,\ k_2,\ 7k_1 - 7,\ k_1)$

Depending upon values of k_1 and k_2 we get infinity of solutions.

THANK YOU

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