



DESIGN AND ANALYSIS OF ALGORITHMS

The Knapsack Problem

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Department of Computer Science and Engineering

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THE KNAPSACK PROBLEM

UNIT 5: Limitations of Algorithmic Power and Coping with the Limitations



- Dynamic Programming
 - ▶ Computing a Binomial Coefficient
 - ▶ **The Knapsack Problem**
 - ▶ Memory Functions
 - ▶ Warshall's and Floyd's Algorithms
- Limitations of Algorithmic Power
 - ▶ Lower-Bound Arguments
 - ▶ Decision Trees
 - ▶ P, NP, and NP-Complete, NP-Hard Problems
- Coping with the Limitations
 - ▶ Backtracking
 - ▶ Branch-and-Bound. Architecture (microprocessor instruction set)

Concepts covered

- The Knapsack Problem
 - ▶ Introduction
 - ▶ Recurrence
 - ▶ Example

- Given
 - ▶ n items of integer weights : $w_1 \quad w_2 \quad \dots \quad w_n$
values : $v_1 \quad v_2 \quad \dots \quad v_n$
 - ▶ knapsack of capacity W (integer $W > 0$)
- Find the most valuable subset of items such that sum of their weights does not exceed W

THE KNAPSACK PROBLEM

Knapsack Recurrence

- To design a dynamic programming algorithm, we need to derive a recurrence relation that expresses a solution to an instance of the knapsack problem in terms of solutions to its smaller subinstances
- Consider the smaller knapsack problem where number of items is i ($i \leq n$) and the knapsack capacity is j ($j \leq W$)
- Then

$$F(i, j) = \begin{cases} \max(F(i-1, j), v_i + F(i-1, j - w_i)) & \text{if } j - w_i \geq 0 \\ F(i-1, j) & \text{if } j - w_i < 0 \end{cases}$$

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Dynamic Programming Example

item i	weight w_i	value v_i
1	2	12
2	1	10
3	3	20
4	2	15

What is the maximum value that can be stored in a knapsack of capacity 6?

THE KNAPSACK PROBLEM

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Dynamic Programming Example

item i	weight w_i	value v_i
1	2	12

	capacity j				
i	1	2	3	4	5
1					
2					
3					
4					

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Dynamic Programming Example

item i	weight w_i	value v_i
1	2	12

	capacity j				
i	1	2	3	4	5
1	0				
2					
3					
4					

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1	0	12			
2					
3					
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	capacity j				
i	1	2	3	4	5
1	0	12	12		
2					
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i	1	2	3	4	5
1	0	12	12	12	12
2	10	12	22		
3					
4					

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2	10	12	22	22	22
3	10	12			
4					

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2	10	12	22	22	22
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4					

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4	10				

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2	10	12	22	22	22
3	10	12	22	30	32
4	10	15	25		

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1	0	12	12	12	12
2	10	12	22	22	22
3	10	12	22	30	32
4	10	15	25	30	37

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i	1	2	3	4	5
1	0	12	12	12	12
2	10	12	22	22	22
3	10	12	22	30	32
4	10	15	25	30	37

Given above 6 items, maximum value that can be stored in a knapsack of capacity 5 is

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Complexity

- Space complexity: $\Theta(nW)$
- Time complexity: $\Theta(nW)$
- Time to compose optimal solution: $O(n)$

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Think About It



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- Write pseudocode of the bottom-up dynamic programming algorithm for the knapsack problem

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Think About It

- Write pseudocode of the bottom-up dynamic programming algorithm for the knapsack problem
- True or False:
 - ① A sequence of values in a row of the dynamic programming table for the knapsack problem is always nondecreasing?

THE KNAPSACK PROBLEM

Think About It

- Write pseudocode of the bottom-up dynamic programming algorithm for the knapsack problem
- True or False:
 - ① A sequence of values in a row of the dynamic programming table for the knapsack problem is always nondecreasing?
 - ② A sequence of values in a column of the dynamic programming table for the knapsack problem is always nondecreasing?