# Discrete Random Variable and Discrete Probability Distribution

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Course material created using various Internet resources and text book

# Descriptive and Inferential Statistics

Statistics can be broken into two basic types:

- Descriptive Statistics :

  We have already learnt this topic

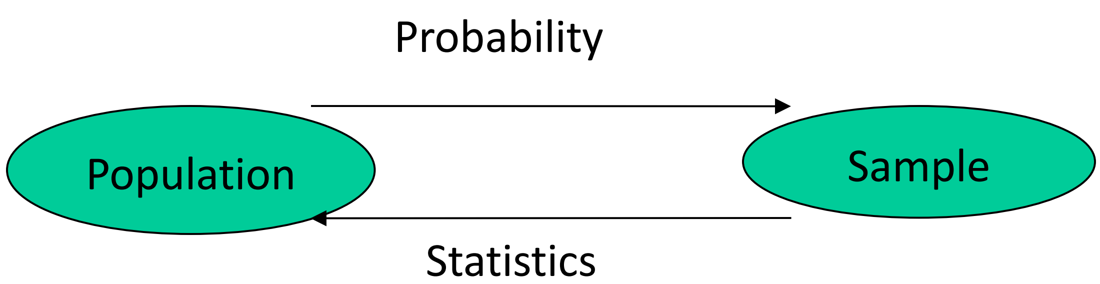
- Inferential Statistics

  Methods that making decisions or predictions about a population based on sampled data.

Probability

# Why Learn Probability?

- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes, from business to medicine to the weather
- A probability provides a quantitative description of the chances or likelihoods associated with various outcomes
- It provides a bridge between descriptive and inferential statistics



# Probabilistic vs Statistical Reasoning

- Suppose I know exactly the proportions of car makes in Bangalore. Then I can find the probability that the first car I see in the street is a Ford. This is probabilistic reasoning as I know the population and predict the sample
- Now suppose that I do not know the proportions of car makes in Bangalore, but would like to estimate them. I observe a random sample of cars in the street and then I have an estimate of the proportions of the population. This is statistical reasoning

# Key Concepts

## I. Experiments and the Sample Space

1. Experiments, events, mutually exclusive events, simple events
2. The sample space

## II. Probabilities

1. Relative frequency definition of probability
2. Properties of probabilities
    a. Each probability lies between 0 and 1.
    b. Sum of all simple-event probabilities equals 1.
3. $P(A)$, the sum of the probabilities for all simple events in $A$

# Key Concepts

## III. Counting Rules

1. *mn* Rule; extended *mn* Rule
2. Permutations: $P_r^n = \frac{n!}{(n-r)!}$
3. Combinations: $C_r^n = \frac{n!}{r!(n-r)!}$

## IV. Event Relations

1. Unions and intersections
2. Events

a. Disjoint or mutually exclusive: $P(A \cap B) = 0$

b. Complementary: $P(A) = 1 - P(A^C)$

# Key Concepts

3. Conditional probability: $P(A|B)=\frac{P(A\cap B)}{P(B)}$

4. Independent and dependent events

5. Additive Rule of Probability:

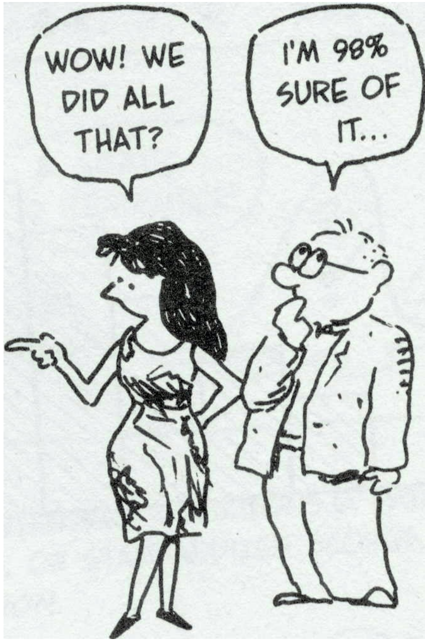
$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

6. Multiplicative Rule of Probability:

$$P(A\cap B)=P(A)P(B|A)$$

7. Law of Total Probability

8. Bayes' Rule



# Random Variables

- A quantitative variable $x$ is a **random variable** if the value that it assumes, corresponding to the outcome of an experiment is a chance or random event.
- Random variables can be **discrete** or **continuous.**

- **Examples:**
  - ✓ $x$ = SAT score for a randomly selected student
  - ✓ $x$ = number of people in a room at a randomly selected time of day
  - ✓ $x$ = number on the upper face of a randomly tossed die

# Random Variables

Definition: A **random variable** assigns a numerical value to each outcome in a sample space.

Definition: A random variable is **discrete** if its possible values form a discrete set.

This means that if the possible values are arranged in order, there is a gap between each value and the next one. The set of possible values may be infinite; for example, the set of all integers is a discrete set.

# Discrete Random variable

- **Takes on one of a finite (or at least countable) number of different values.**
- **X = 1 if heads, 0 if tails**
- **Y = 1 if male, 0 if female (phone survey)**
- **Z = # of spots on face of thrown die**

# Continuous Random variable

- **Takes on one in an infinite range of different values**
- **W = % GDP grows (shrinks?) this year**
- **V = hours until light bulb fails**

**What is the probability that a continuous r.v. takes on a specific value? E.g. Prob(X_light_bulb_fails = 3.14159265 hrs) = ??**

**0**

**However, ranges of values can have non-zero probability.**

**E.g. Prob(3 hrs <= X_light_bulb_fails <= 4 hrs) = 0.1**

- ***Ranges* of values have a probability**

# Probability Mass Function

- The description of the possible values of *X* and the probabilities of each has a name: the probability mass function.

Definition: The **probability mass function** (pmf) of a discrete random variable $X$ is the function $p(x) = P(X = x)$. The probability mass function is sometimes called the **probability distribution**.

# Probability Distribution

The probability distribution is a complete probabilistic description of a random variable.

All other statistical concepts (expectation, variance, etc) are derived from it.

Once we know the probability distribution of a random variable, we know everything we can learn about it from statistics.

# Probability Distributions for Discrete Random Variables

The **probability distribution for a discrete random variable *x*** resembles the relative frequency distributions . It is a graph, table or formula that gives the possible values of *x* and the probability *p(x)* associated with each value.

We must have

$$0 \le p(x) \le 1 \text{ and } \sum p(x) = 1$$

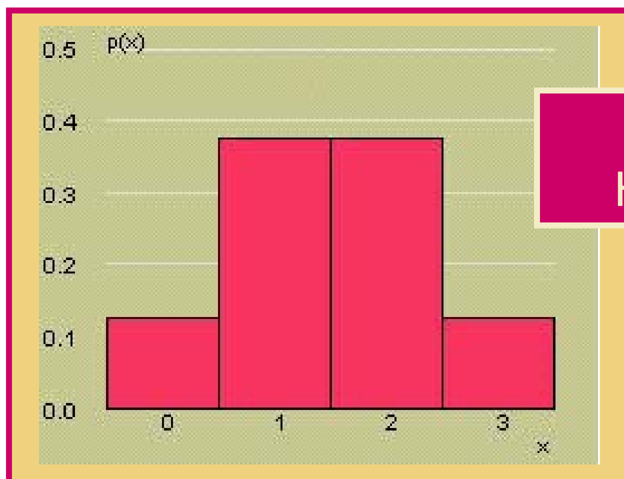
# Example

Toss a fair coin three times and define ***x* = number of heads.**

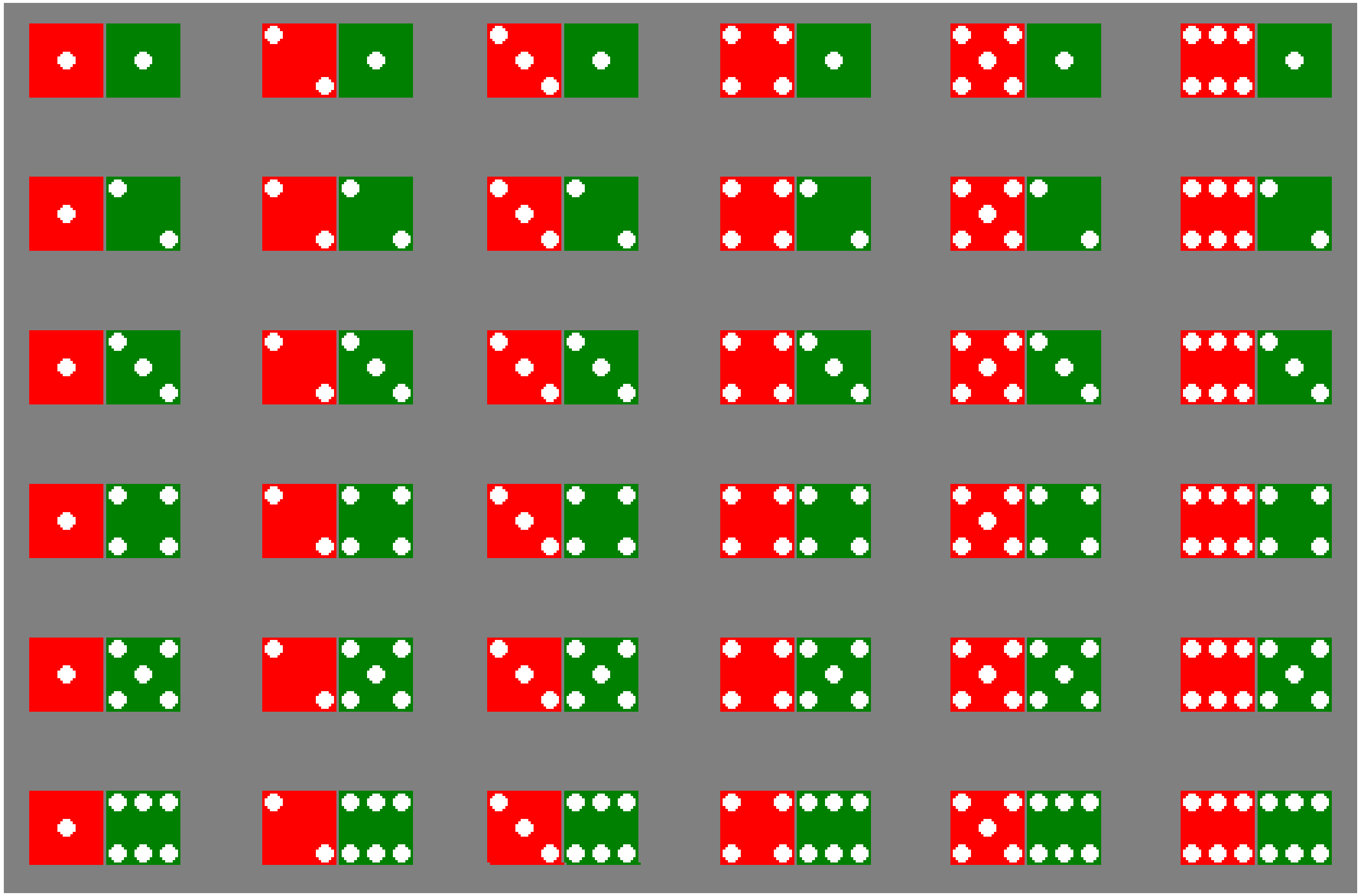
| | | *x* |
|---|---|---|
| HHH | 1/8 | 3 |
| HHT | 1/8 | 2 |
| HTH | 1/8 | 2 |
| THH | 1/8 | 2 |
| HTT | 1/8 | 1 |
| THT | 1/8 | 1 |
| TTH | 1/8 | 1 |
| TTT | 1/8 | 0 |

$P(x = 0) = 1/8$
$P(x = 1) = 3/8$
$P(x = 2) = 3/8$
$P(x = 3) = 1/8$

| $x$ | $p(x)$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |



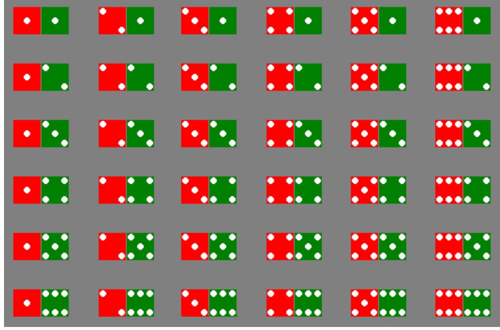

Probability Histogram for *x*

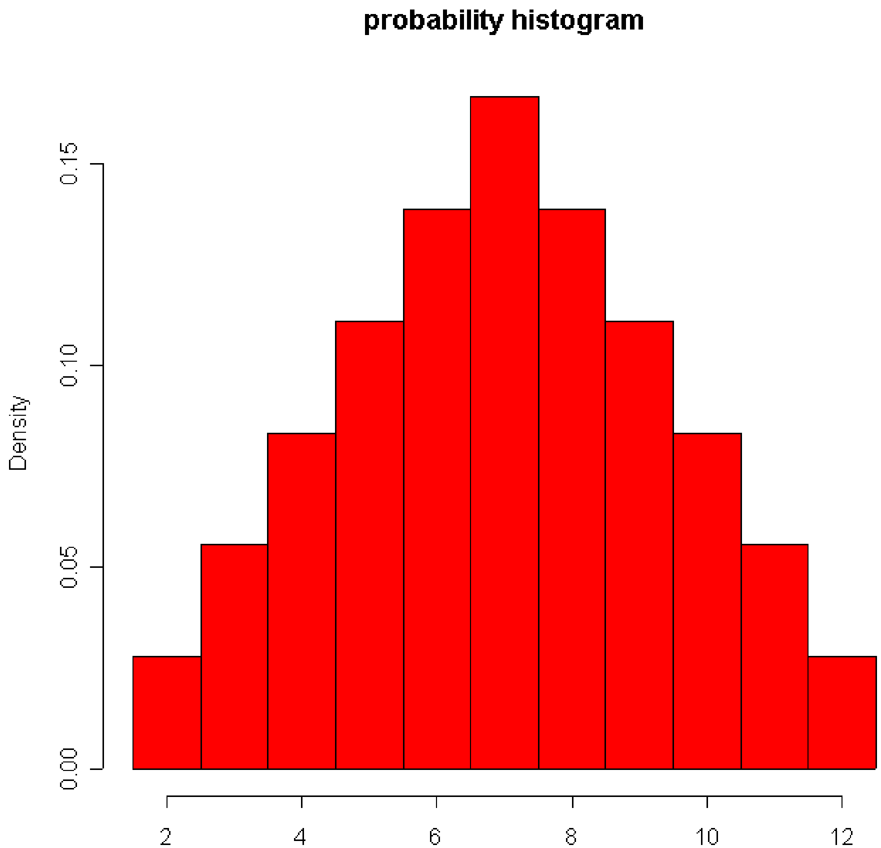


# Example

Toss two dice and define

***x* = sum of two dice.**






| *x* | *p(x)* |
|---|---|
| 2 | 1/36 |
| 3 | 2/36 |
| 4 | 3/36 |
| 5 | 4/36 |
| 6 | 5/36 |
| 7 | 6/36 |
| 8 | 5/36 |
| 9 | 4/36 |
| 10 | 3/36 |
| 11 | 2/36 |
| 12 | 1/36 |

# Probability Distributions

Probability distributions can be used to describe the population, just as we described samples .

- –**Shape:** Symmetric, skewed, mound-shaped…
- –**Outliers:** unusual or unlikely measurements
- –**Center and spread:** mean and standard deviation. A population mean is called **$\mu$** and a population standard deviation is called **$\sigma$**.

# Probability Distribution

The probability function

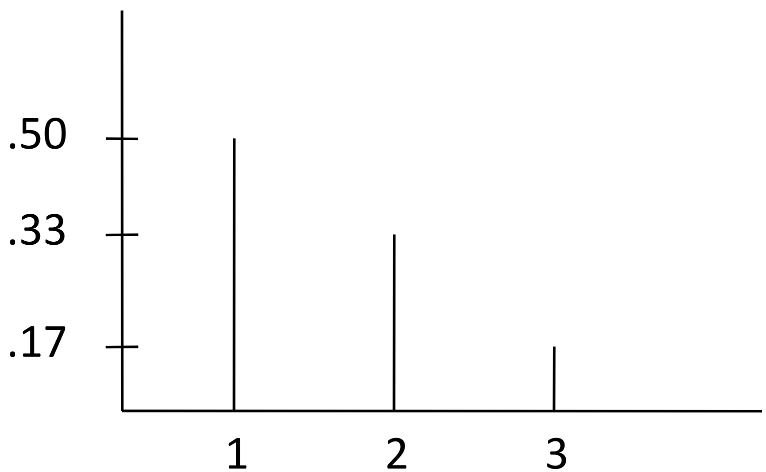
- May be tabular:

$$X = \begin{cases} 1 & w.p. \quad 1/2 \\ 2 & w.p. \quad 1/3 \\ 3 & w.p. \quad 1/6 \end{cases}$$

# Probability Distribution

## The probability function

- May be graphical:



# Probability Distribution

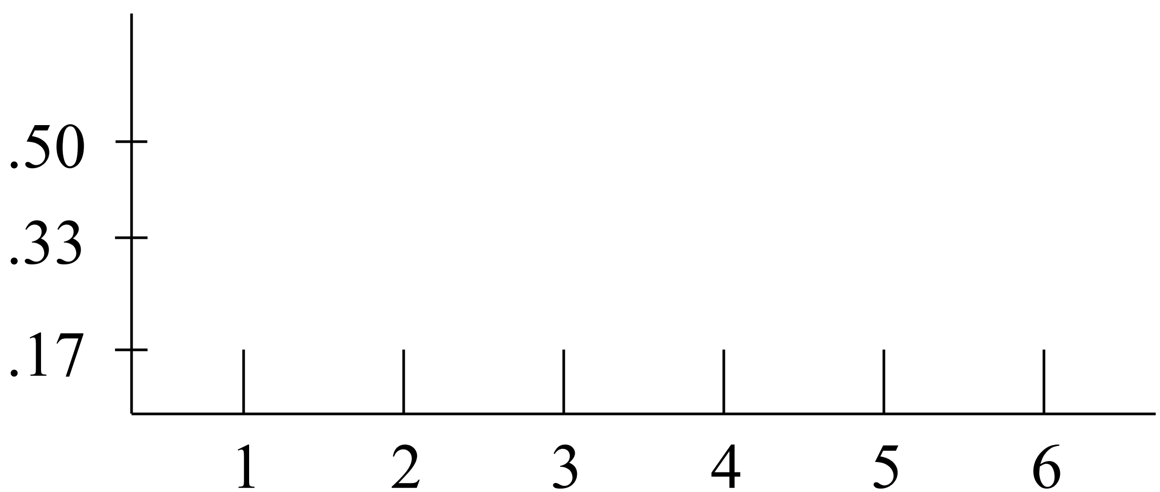
The probability function

– May be formulaic:

$$P\left(X=x\right)=\frac{4-x}{6}\quad \text{for x}=1,2,3$$

# Probability Distribution: Fair die

$$X = \begin{cases} 1 & w.p. \ 1/6 \\ 2 & w.p. \ 1/6 \\ 3 & w.p. \ 1/6 \\ 4 & w.p. \ 1/6 \\ 5 & w.p. \ 1/6 \\ 6 & w.p. \ 1/6 \end{cases}$$



# Probability Distribution

## The probability function, properties

$$P_X(x) \geq 0 \quad \text{for each } x$$

$$\sum_x P_X(x) = 1$$

# Cumulative Distribution Function

- The probability mass function specifies the probability that a random variable is equal to a given value.

- A function called the **cumulative distribution function** (cdf) specifies the probability that a random variable is less than or equal to a given value.

- The cumulative distribution function of the random variable $X$ is the function $F(x) = P(X \leq x)$.

# Cumulative Probability Distribution

– The relationship between the cdf and the probability function:

$$F_X(x) = P(X \leq x) = \sum_{y \leq x} P_X(X = y)$$

$$P_X(x) = P(X = x) = 1/6$$

# Cumulative Probability Distribution

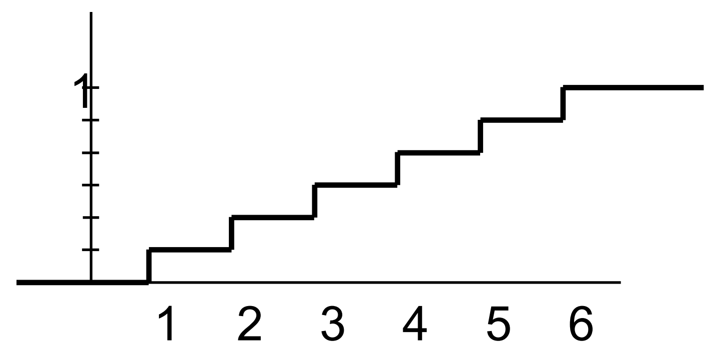
## Die-throwing

$$F_X(x) = P(X \le x) = \sum_{y \le x} P_X(X = y)$$

tabular

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \le x < 2 \\ 2/6 & 2 \le x < 3 \\ 3/6 & 3 \le x < 4 \\ 4/6 & 4 \le x < 5 \\ 5/6 & 5 \le x < 6 \\ 6/6 & x \ge 6 \end{cases}$$

graphical




# Cumulative Probability Distribution

- The cumulative distribution function
  - May be formulaic (die-throwing):

$$P(X \le x) = \frac{floor(\min(\max(x,0),6))}{6}$$

# Cumulative Probability Distribution

## The cdf, properties

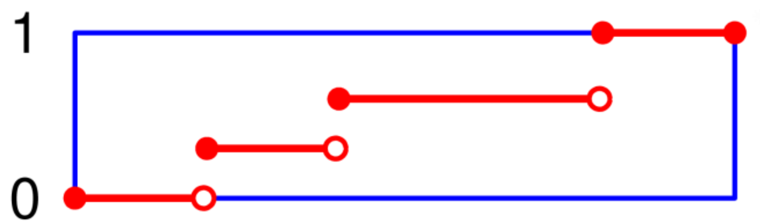
$$0 \le F_X(x) \le 1 \quad \text{for each } x$$

$$F_X(x) \text{ is non-decreasing}$$

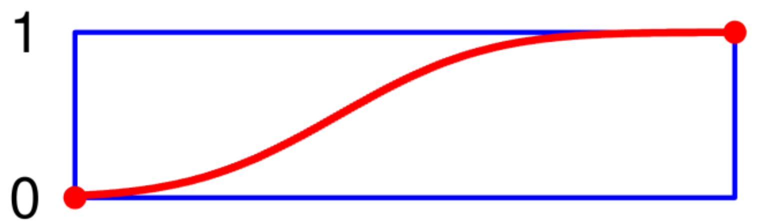
$$F_X(x) \text{ is continuous from the right}$$

$$\lim_{x \to -\infty} F(x) = 0, \qquad \lim_{x \to +\infty} F(x) = 1.$$

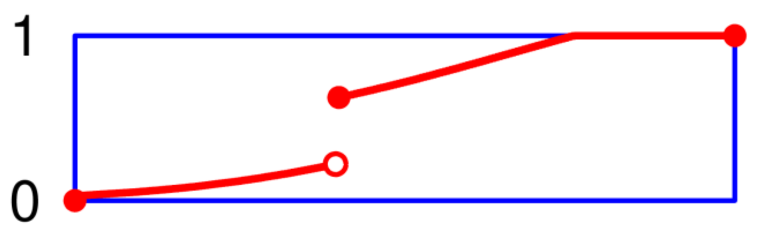
# Example CDFs




Of a discrete probability distribution




Of a continuous probability distribution




Of a distribution which has both a continuous part and a discrete part.

# Mean for Discrete Random Variables

- Let $X$ be a discrete random variable with probability mass function $p(x) = P(X = x)$.
- The **mean** of $X$ is given by

$$\mu_X = \sum_x xP(X = x),$$

where the sum is over all possible values of $X$.

- The mean of $X$ is sometimes called the expectation, or expected value, of $X$ and may also be denoted by $E(X)$ or by $\mu$.

# Variance for Discrete Random Variables

- Let *X* be a discrete random variable with probability mass function $p(x) = P(X = x)$.
- The **variance** of *X* is given by

$$\sigma_X^2 = \sum_x (x - \mu_X)^2 P(X = x)$$
$$= \sum_x x^2 P(X = x) - \mu_X^2.$$

- The variance of *X* may also be denoted by *V*(*X*) or by $\sigma$2.
- The standard deviation is the square root of the variance: $\sigma_X = \sqrt{\sigma_X^2}$

# The Probability Histogram

- When the possible values of a discrete random variable are evenly spaced, the probability mass function can be represented by a histogram, with rectangles centered at the possible values of the random variable.
- The area of the rectangle centered at a value $x$ is equal to $P(X = x)$.
- Such a histogram is called a **probability histogram**, because the areas represent probabilities.

# The Mean and Standard Deviation

Let *x* be a discrete random variable with probability distribution *p(x)*. Then the mean, variance and standard deviation of *x* are given as

$$\text{Mean}: \mu = \sum x p(x)$$

$$\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\text{Standard deviation}: \sigma = \sqrt{\sigma^2}$$

# Example

Toss a fair coin 3 times and record *x* the number of heads.

| $x$ | $p(x)$ | $xp(x)$ | $(x\text{-}\mu)2p(x)$ |
|---|---|---|---|
| 0 | 1/8 | 0 | (-1.5)2(1/8) |
| 1 | 3/8 | 3/8 | (-0.5)2(3/8) |
| 2 | 3/8 | 6/8 | (0.5)2(3/8) |
| 3 | 1/8 | 3/8 | (1.5)2(1/8) |

$$\mu = \sum xp(x) = \frac{12}{8} = 1.5$$

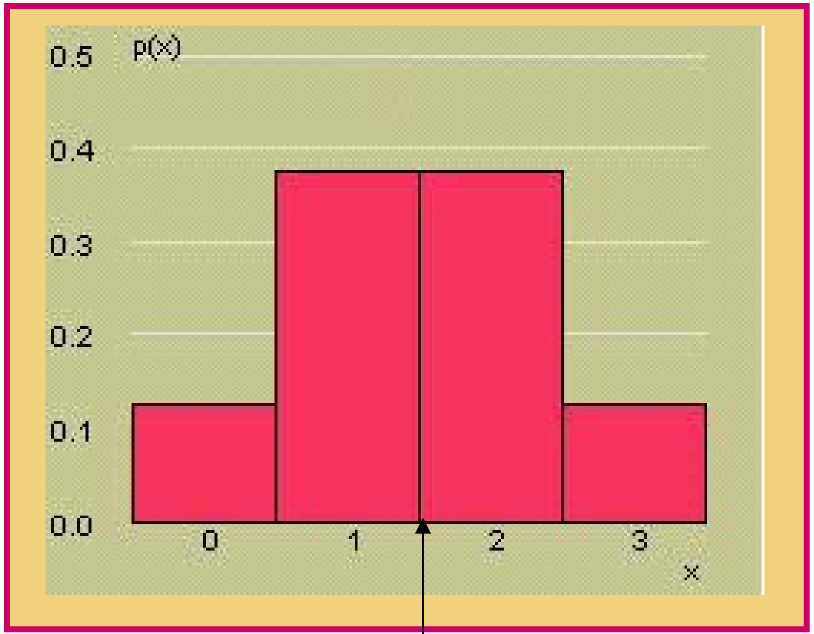
$$\sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\sigma^2 = .28125 + .09375 + .09375 + .28125 = .75$$

$$\sigma = \sqrt{.75} = .866$$

# Example

The probability distribution for *x* the number of heads in tossing 3 fair coins.




μ

- Shape?
- Outliers?
- Center?
- Spread?

Symmetric; mound-shaped

None

$\mu = 1.5$

$\sigma = .688$

# Key Concepts

**V. Discrete Random Variables and Probability Distributions**

1. Random variables, discrete and continuous
2. Properties of probability distributions

$$0 \le p(x) \le 1 \text{ and } \sum p(x) = 1$$

3. Mean or expected value of a discrete random variable: Mean: $\mu = \sum xp(x)$

4. Variance and standard deviation of a discrete random variable: Variance: $\sigma^2 = \sum (x-\mu)^2 p(x)$

Standard deviation: $\sigma = \sqrt{\sigma^2}$