(3) **Jacobian of Implicit functions.** *If u_1, u_2, u_3 instead of being given explicitly in te... x_1, x_2, x_3, be connected with them by equations such as*

$$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0, \text{ then}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

Obs. This result can be easily generalised. It bears analogy to the result $\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$, where x, y ... connected by the relation $f(x, y) = 0$.

Example 5·29 *If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\partial(x, y, z)/\partial(u, v, w)$*

(U.P.T.U., 20..)

Sol. Let $f_1 = u - xyz$, $f_2 = v - x^2 - y^2 - z^2$, $f_3 = w - x - y - z$.

We have
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}$$

Now
$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$$= -2(x - y)(y - z)(z - x)$$

and
$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Substituting values from (*ii*) and (*iii*) in (*i*), we get

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1) \times 1/[-2(x - y)(y - z)(z - x)]$$

$$= 1/2(x - y)(y - z)(z - x).$$