Handout 7
Basics of probability

An experiment is a process that results in an outcome that cannot be predicted in advance with certainty.

Example: Tossing a coin, rolling a die, measuring the diameter of a bolt, weighing the contents of a box of cereal, and measuring the breaking strength of a length of fishing line are all examples of experiments.

The set of all possible outcomes of an experiment is called the sample space for the experiment.

For tossing a coin, we can use the set {Heads, Tails} as the sample space.
For rolling a six-sided die, we can use the set {1, 2, 3, 4, 5, 6}. These sample spaces are finite.

Some experiments have sample spaces with an infinite number of outcomes.
For example, imagine that a punch with diameter 10 mm punches holes in sheet metal.

When discussing experiments, we are often interested in a particular subset of outcomes.
For example, we might be interested in the probability that a die comes up an even number. The sample space for the experiment is {1, 2, 3, 4, 5, 6}, and coming up even corresponds to the subset {2, 4, 6}.

In the hole punch example, we might be interested in the probability that a hole has a diameter less than 10.1 mm. This corresponds to the subset $\{x \mid 10.0 < x < 10.1\}$. There is a special name for a subset of a sample space:Event

A subset of a sample space is called an event.
Note that for any sample space, the empty set $\emptyset$ is an event, as is the entire sample space.

A given event is said to have occurred if the outcome of the experiment is one of the outcomes in the event. For example, if a die comes up 2, the events {2, 4, 6} and {1, 2, 3} have both occurred, along with every other event that contains the outcome “2.”

Combining Events

We often construct events by combining simpler events. Because events are subsets of sample spaces, it is traditional to use the notation of sets to describe events constructed in this way.

We review the necessary notation here.

The union of two events A and B, denoted $A \cup B$, is the set of outcomes that belong either to A, to B, or to both. In words, $A \cup B$ means "A or B." Thus the event $A \cup B$ occurs whenever either A or B (or both) occurs.

The intersection of two events A and B, denoted $A \cap B$, is the set of outcomes that belong both to A and to B. In words, $A \cap B$ means "A and B." Thus the event $A \cap B$ occurs whenever both A and B occur.

The complement of an event A, denoted A^c , is the set of outcomes that do not belong to A. In words, A^c means "not A." Thus the event A^c occurs whenever A does not occur.

Mutually Exclusive Events

There are some events that can never occur together. For example, it is impossible that a coin can come up both heads and tails, and it is impossible that a steel pin can be both too long and too short. Events like this are said to be mutually exclusive.

Definition

The events A and B are said to be mutually exclusive if they have no outcomes in common.
More generally, a collection of events $A_1, A_2, \ldots, A_n$ is said to be mutually exclusive if no two of them have any outcomes in common.

Probabilities

Each event in a sample space has a probability of occurring. Intuitively, the probability is a quantitative measure of how likely the event is to occur.

Given any experiment and any event A:

The expression P(A) denotes the probability that the event A occurs.
P(A) is the proportion of times that event A would occur in the long run,sif the experiment were to be repeated over and over again.

Axioms of Probability

The subject of probability is based on three commonsense rules, known as axioms.

They are:

1. Let S be a sample space. Then $P(S) = 1$.

2. For any event A, $0 \leq P(A) \leq 1$.

3. If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.
   More generally, if $A_1, A_2, \ldots$ are mutually exclusive events, then
   $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

For any event A,

$P(A^c) = 1 - P(A)$

Let ∅ denote the empty set. Then $P(\emptyset) = 0$

If A is an event containing outcomes $O_1, \ldots, O_n$, that is, if $A = \{O_1, \ldots, O_n\}$,

Then $P(A) = P(O_1) + P(O_2) + \cdots + P(O_n)$

Sample Spaces with Equally Likely Outcomes

For some experiments, a sample space can be constructed in which all the outcomes are equally likely.

A simple example is the roll of a fair die, in which the sample space is {1, 2, 3, 4, 5, 6} and each of these outcomes has probability 1/6. Another type of experiment that results in equally likely outcomes is the random selection of an item from a population of items. The items in the population can be thought of as the outcomes in a sample space, and each item is equally likely to be selected.

A population from which an item is sampled at random can be thought of as a sample space with equally likely outcomes.

If a sample space contains N equally likely outcomes, the probability of each outcome is 1/N . This is so, because the probability of the whole sample space must be 1, and this probability is equally divided among the N outcomes.

If A is an event that contains k outcomes, then P(A) can be found by summing the probabilities of the k outcomes, so $P(A) = k/N$.

If S is a sample space containing N equally likely outcomes, and if A is an event containing k outcomes, then

$P(A) = k/N$

The Addition Rule

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

This rule can be generalized to cover the case where A and B are not mutually exclusive.

Let A and B be any events. Then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note that if A and B are mutually exclusive, then $P(A \cap B) = 0$,

The Fundamental Principle of Counting

Assume that k operations are to be performed. If there are n_1 ways to perform the first operation, and if for each of these ways there are n_2 ways to perform the second operation, and if for each choice of ways to perform the first two operations there are n_3 ways to perform the third operation, and so on, then the total number of ways to perform the sequence of k operations is $n_1 n_2 \cdots n_k$.

Permutations

A permutation is an ordering of a collection of objects. For example, there are six permutations of the letters A, B, C: ABC, ACB, BAC, BCA, CAB, and CBA.

For any positive integer n, $n! = n(n-1)(n-2) \cdots (3)(2)(1)$.

Also, we define $0! = 1$.

The number of permutations of n objects is $n!$.

The number of permutations of k objects chosen from a group of n objects is

$n!/(n-k)!$

Combinations

In some cases, when choosing a set of objects from a larger set, we don’t care about the ordering of the chosen objects; we care only which objects are chosen. For example,we may not care which lifeguard occupies which station; we might care only which three lifeguards are chosen. Each distinct group of objects that can be selected, without regard to order, is called a combination.

The number of combinations of k objects chosen from n is often denoted by the symbol $\binom{n}{k}$. The reasoning used to derive the number of combinations of three objects chosen from five can be generalized to derive an expression for $\binom{n}{k}$

The number of combinations of k objects chosen from a group of n objects is

$\binom{n}{k}$=n!/k!(n − k)!

The number of ways of dividing a group of n objects into groups of k 1 , . . . , k r objects, where k 1 + · · · + k r = n, is $\frac{n!}{k_1!\cdots k_r!}$

Conditional Probability and Independence

A sample space contains all the possible outcomes of an experiment. Sometimes we obtain some additional information about an experiment that tells us that the outcome comes from a certain part of the sample space. In this case, the probability of an event is based on the outcomes in that part of the sample space. A probability that is based on a part of a sample space is called a **conditional probability.**

Let A and B be events with $P(B) \neq 0$. The conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (2.14)$$

The probability that is based on the entire sample space is called as the **unconditional probability**.

Independent Events

Sometimes the knowledge that one event has occurred does not change the probability that another event occurs. In this case the conditional and unconditional probabilities are sthe same, and the events are said to be **independent**

Definition

Two events A and B are **independent** if the probability of each event remains the same whether or not the other occurs.

In symbols: If $P(A) \neq 0$ and $P(B) \neq 0$, then A and B are independent if

$$P(B|A) = P(B) \quad \text{or, equivalently,} \quad P(A|B) = P(A) \tag{2.15}$$

If either $P(A) = 0$ or $P(B) = 0$, then A and B are independent.

Definition

Events $A_1, A_2, \ldots, A_n$ are independent if the probability of each remains the same no matter which of the others occur.

In symbols: Events $A_1, A_2, \ldots, A_n$ are independent if for each A_i, and each collection $A_{j1}, \ldots, A_{jm}$ of events with $P(A_{j1} \cap \cdots \cap A_{jm}) \neq 0$,

$$P(A_i|A_{j1} \cap \cdots \cap A_{jm}) = P(A_i) \tag{2.16}$$

If A and B are two events with $P(B) \neq 0$, then

$$P(A \cap B) = P(B)P(A|B) \tag{2.17}$$

If A and B are two events with $P(A) \neq 0$, then

$$P(A \cap B) = P(A)P(B|A) \tag{2.18}$$

If $P(A) \neq 0$ and $P(B) \neq 0$, then Equations (2.17) and (2.18) both hold.

If A and B are independent events, then

$$P(A \cap B) = P(A)P(B) \tag{2.19}$$

This result can be extended to any number of events. If $A_1, A_2, \ldots, A_n$ are independent events, then for each collection $A_{j1}, \ldots, A_{jm}$ of events

$$P(A_{j1} \cap A_{j2} \cap \cdots \cap A_{jm}) = P(A_{j1})P(A_{j2}) \cdots P(A_{jm}) \tag{2.20}$$

In particular,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n) \tag{2.21}$$

Law of Total Probability

If $A_1, \ldots, A_n$ are mutually exclusive and exhaustive events, and B is any event, then

$$P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B) \tag{2.23}$$

Equivalently, if $P(A_i) \neq 0$ for each A_i,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n) \tag{2.24}$$