Pumping lemma for Context free languages

- Pumping lemma for CFLs is not linear like it was for Regular lang's.
- In FSM, if we have a long string

$\underset{\text{Before loop}}{\underline{11}}\ \ \underset{\text{loop}}{\overset{010}{\downarrow}}\ \ \underset{\text{After loop}}{\underline{01}}$

pumping the loop any no. of times gives us the strings $\in$ Lang.

- In PDA, decision not only depends on state no. but also on the stack contents.
- Hence execution of second 010 might change the stack contents.
- Hence we need a better way of using Pumping lemma in CFLs.
- Unlike FSMs, we define pumping lemma by looking at the grammar (in CNF) instead of the machine

Pumping lemma for Regular languages

- we look at FSM
- Pumping constant - #states in FSM
- String z is broken in 3 parts

$z = x y^i z$ $\quad |xy| \le n$ $\quad |y| \ge 1$

x → Before loop, y → loop, z → After loop

Pumping lemma for Context free languages

- we look at grammar in CNF
- Pumping constant - # Non terminals in grammar (in CNF)
- String z is broken into 5 parts

$z = u v^i w x^i y$ $\quad |vxy| \le n$ $\quad |vx| \ge 1$

v → loop 1, w → fixed center, x → loop 2

Both the loops pump simultaneously.

Where & how could we figure out the loop is?

CFG ⇒ via the Parse tree where the loop

CNF #NT: 3+1

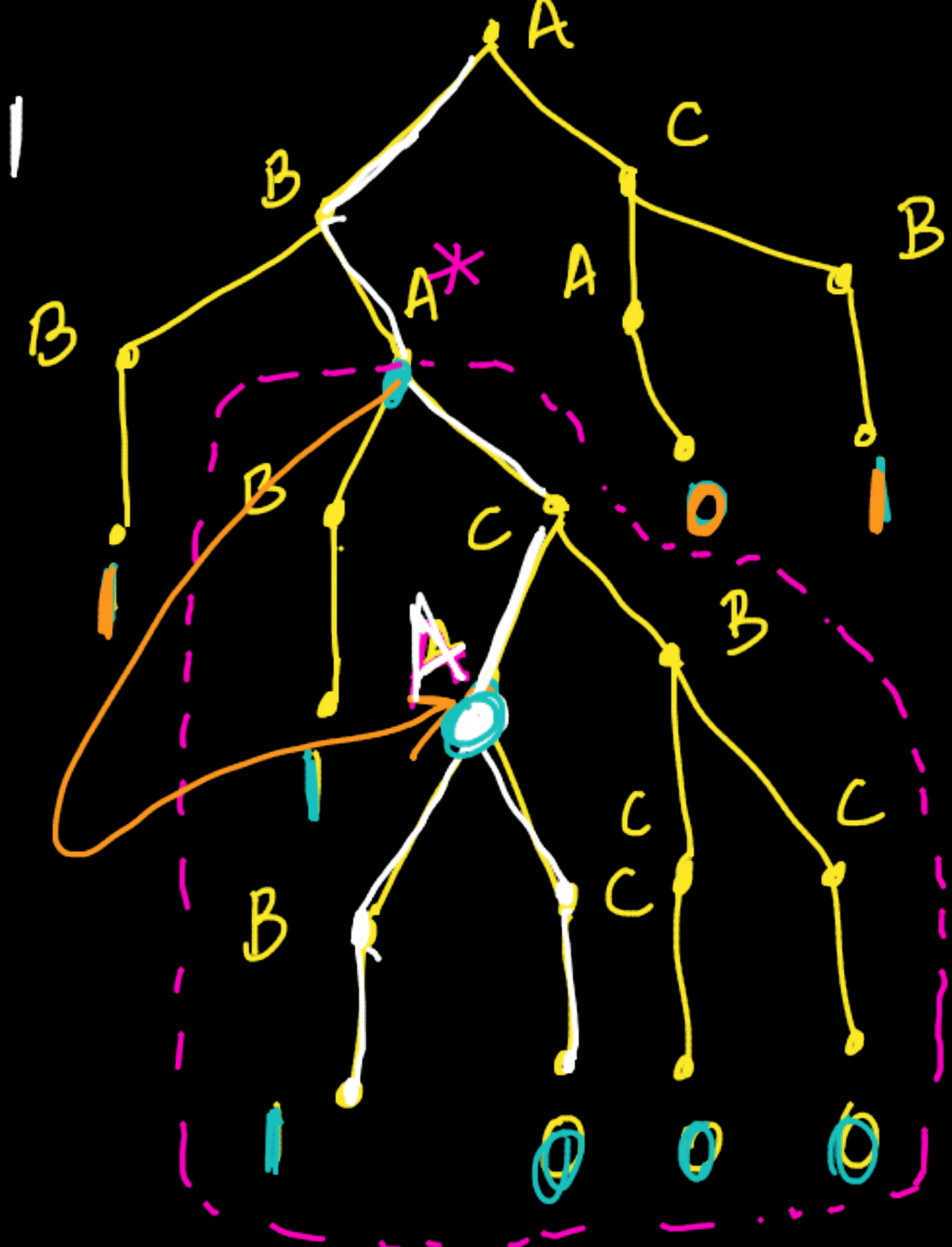
$A \to BC \mid 0$

$B \to BA \mid CC \mid 1$

$C \to AB \mid 0$

z = 111 0000 1

String z should be long enough



1 1 10 00 01

B 1 111 10 00 00 00 00 01

1 11 10 00 00 01

u $\boxed{v}$ $\boxed{w}$ $\boxed{x}$ y

loops pump simultaneously

$|z| \geq \boxed{2^{N+1}}$

Pumping Property for CFLs

if $\mathcal{L}$ is a CFL,

$\exists N$ is the # NT in CNF form of grammar for $\mathcal{L}$.

Pumping constant (min string length)

$$n = 2^{N+1}$$

$\forall z \in L$ such that $|z| \geq n$

$\exists\, uvwxy = z$ such that $\boxed{|vwx| \leq n}$

$|vx| \geq 1$

$\forall i \geq 0 \quad uv^iwx^iy \in \mathcal{L}$

Prove $\mathcal{L} = \{ n_a(w) = n_b(w) \}$ is a CFL using Pumping property.

$$S \to aSb \mid bSa \mid SS \mid \lambda$$

CNF:-

$S \to AC \mid BD \mid SS \mid \lambda$

$A \to a$

$C \to SB$

$B \to b$

$D \to SA$

$N = 5$ (# NT)

Pumping const, $n \geq 2^{5+1}$

$n \geq 64$

$|vwx| \leq 64$

Choose a string z, $|z| \geq n$

$$a^{64}\, b^{64} \Rightarrow a^{63}\, a^{1}\, \lambda\, b^{1}\, b^{63}$$

$|v\ w\ x| \leq 64$

Pumping property for CFLs

Given a CFL L,

$\exists N$ where N is the # NT in CNF grammar for L

We choose the pumping constant n

$n = 2^{N+1}$

$\forall z \in L, \ |z| \geq n$

$\exists uvwxy = z$ such that $|vwx| \leq n$

$|vx| \geq 1$

$\forall i \geq 0 \quad uv^iwx^iy \in L$

~ Pumping property

Adversary claims that L is context free.

$\forall N$, where N is the # NT in the hypothetical CNF grammar for L.

pumping constant $n = 2^{N+1}$

$\exists z \in L, \ |z| \geq n$

$\forall uvwxy = z$ such that

$|vwx| \leq n$

$|vx| \geq 1$

$\exists i \geq 0$

$uv^iwx^iy \notin L$

Hence the claim is wrong! L is not context free.

Prove using Pumping lemma that the Lang $L = \{0^n 1^n 0^n$ is not context free $\}$

Soln

(∀N) * Adversary claims $L = \{0^n 1^n 0^n, n \geq 0\}$ is context free.

* Let's say there are $N = 2$ NT's in the hypothetical grammar for L (CNF)

* We calculate the pumping constant 'n' $= 2^{N+1} = 2^{2+1} = 2^3$

$\Rightarrow n = 8$

* We choose $z = 0^8 1^8 0^8$

$z \in L$ & $|z| \geq n$

Case ① ② ③ ④ ⑤

$\forall uvwxy = z$, $|vwx| \leq n$ & $|vx| \geq 1$

$\exists i$, $uv^iwx^iy \notin L$

Hence L is not context free.

Case	loc^n of vwx	i	n_0	n_1	n_0	Comment
1	only 0's	≥2	↑	–	–	Too many 0's at start
2	some 0's & some 1's	≥2	↑	↑	–	few 0's at end
3	Only 1's	≥2	–	↑	–	Too many 1's
4	some 1's & some 0's	≥2	–	↑	↑	few 0's at start
5	Only 0's	≥2	–	–	↑	Too many 0's at end

Prove using pumping lemma that ww is not context free.
$w \in \{a, b\}^*$

Soln

* Adversary claims that Lang $L = \{ww, w \in \{a, b\}^*\}$ is context free.
* Let's say there are N non-terminals in the hypothetical grammar (CNF form) for L
* we calculate the pumping constant $n = 2^{N+1}$
* we choose a string $\boxed{z = a^n b a^n b}$ such that $|z| \geq n$
* we prove $\forall uvwxy = z$ where $|vwx| \leq n$, $|vx| \geq 1$
  $\exists i \geq 0$, $uv^iwx^iy \notin L$.

$\forall i \geq 0$ $a^{n-1} a^i b a^i a^{n-1} b \in L$

$\underbrace{a^i}_{v}\ \underbrace{b}_{w}\ \underbrace{a^i}_{x}$

$w \downarrow$ fixed center

You Lost !!

Prove ww is not context free.

Soln

We choose $z = \underbrace{a^n b^n}_{w} \underbrace{a^n b^n}_{w}$ (positions 1–7 marked)

$|z| \geq n$

$|vx| \geq 1$

$\boxed{|vwx| \leq n}$

ww

$\forall\, uvwxy = z \quad \exists i, \quad uv^iwx^iy \notin L$

Case	loc^n of vwx	i	n_a	n_b	n_a	n_b	comment ($z \notin L$)
1	Only a's (start)	≥ 2	$\uparrow$	–	–	–	Too many a's in start
2	somes a's & some b's	≥ 2	$\uparrow$	$\uparrow$	–	–	$w \neq w$
3	only b's	≥ 2	–	$\uparrow$	–	–	Too many b's at start
4	Some b's & some a's	≥ 2	–	$\uparrow$	$\uparrow$	–	$w \neq w$
5	some a's (second w)	≥ 2	–	–	$\uparrow$	–	Too many a's later
6	Some a's & some b's	≥ 2	–	–	$\uparrow$	$\uparrow$	$w \neq w$
7	Some b's	≥ 2	–	–	–	$\uparrow$	Too many b's at last

Hence L is not context free.

Prove using Pumping lemma that $L = \{ a^n b^m c^k ; k = n*m , n,m \geq 1 \}$ is not context free.

Soln:- * Adversary claims that the lang L is context free

* Let's say there are N non-terminals in the hypothetical grammar in CNF form.

* We calculate pumping constant $n = 2^{N+1}$

* We choose the string $z = a^n b^n c^{n*n}$, $|z| \geq n$ $|vwx| \leq n$

~~$\forall$~~ $uvwxy = z$,

$\exists i$,

$uv^iwx^iy \notin L$

case no.	for $\lvert vwx \rvert$	i	n_a	n_b	n_c	Comment $\lvert vx \rvert \geq 1$
1	only a's	≥ 2	$\uparrow$	-	-	# c's will not be equal to the product of #a's & #b's ⇒ more a's
2.	some a's & some b's	≥ 2	$\uparrow$	$\uparrow$	-	
3.	only b's	≥ 2	-	$\uparrow$	-	$\notin L$
4.	Some b's & some c's	≥ 2	-	$\uparrow$	$\uparrow$	
5.	only c's	≥ 2	-	-	$\uparrow$	

Hence L is not context free!

Prove using Pumping lemma that Lang $L = \{ a^{n!}, n \geq 0 \}$ is not context free.

Soln:
* Adversary claims that L is context free
* Let's say N is the # NT in Adversary's hypothetical grammar in CNF for L.
* we calculate pumping constant $n = 2^{N+1}$
* we choose the string $z = a^{n!}$, $|z| \geq n$

$\boxed{|vx| \geq 1}$

$\boxed{|vwx| \leq n}$

p a's

range of p :- $(1 \leq p \leq n)$

$\boxed{a^{n!}}$

$a \ldots a \;(a^p)^i\; a \ldots a$

default $i = 2$

$\boxed{a^{n!+p}}$

$n! < n! + p < (n+1)!$

$n! + n < (n+1)!$

$n! + n < (n+1) \cdot n! \Rightarrow L$ is not context free!

Prove using Pumping lemma that Lang $L = \{ a^{n^2}, n \geq 0 \}$ is not context free.

Soln:-

* Adversary claims that L is context free.
* Let's say N is the # NT's in Adversary's hypothetical grammar in CNF for L.
* We calculate pumping constant $n = 2^{N+1}$
* We choose a string $z = a^{n^2}$, $|z| \geq n$

$\forall \, uvwxy = z$, $|vwx| \leq n$ and $|vx| \geq 1$

$\exists \, i \geq 0$

$uv^iwx^iy \notin L$

loop is made up of p symbols

range of p :- $1 \leq p \leq n$

a^{n^2}

$a \; a \; (a^p)^i \; a \; a$

for $i = 2$ $\quad a^{n^2+p}$

we will prove,

$n^2 < n^2 + p < (n+1)^2$

Assume $p = n$ (max value)

$n^2 + n < (n+1)^2 \Rightarrow n^2 + n < n^2 + 2n + 1 \Rightarrow L$ is not context free!

Prove using pumping lemma that Lang $L = \{0^k$ where k is composite$\}$ is not context free!

Soln:-

Skipping first 3 points $\Rightarrow$ Pumping constant $= n$

we choose the string $z = 0^{2n}$ $> n$

$|vwx| \leq n$

$|vx| \geq 1$

w can be λ

fixed center

$|w| \geq 0$

0^{3n}

$= u\,v\,w\,x\,y$

$0^i \; (0 \quad 0)^i$

$i = 2$

$0\,0 \quad 0\,0\,0\,0$

$0^{2n} = u\,v\,w\,x\,y$, $w \to \lambda$, $v \to 0^i$, $x \to 0^i$

multiple of 2 $\forall i \geq 0$

$z' \in L$

We can never prove L is not context free using pumping lemma.

Satisfies pumping prop for CFL's But is still not context free!

Note:

- Unlike Regular languages we cannot use its complement $L^c = \{0^p$, where p is prime$\}$ to prove that L is not context free.
- This is because, CFLs are not closed under complement.

$\therefore$ We can never prove using pumping lemma that the language $L = \{0^k$ where k is composite$\}$ is not context free.

Lang L although not context free, satisfies the pumping property.

Summary :-

We have proved using pumping lemma that following lang's are not context-free :-

1) $0^n 1^n 0^n : n \geq 0$
2) $ww : w \in \{a, b\}^*$
3) $a^n b^m c^k : k = n * m, n, m \geq 1$
4) $a^{n!} : n \geq 0$
5) $a^{n^2} : n \geq 0$

We saw that $L = \{0^k, k \text{ is composite}\}$ although not context free, satisfies pumping property.

Homework :-

Prove using pumping lemma that following languages are not context free :-

1) $a^n b^m c^n d^m$, $n, m \geq 0$

2) 0^p, where p is prime

Acknowlegement :- Notes are prepared by Prof. Preet Kanwal.