Unit 4 –Orthogonalization , Eigenvalues and Eigenvectors

Orthogonal Bases, The Gram- Schmidt Orthogonalization, Introduction to Eigenvalues and Eigenvectors, Properties of Eigenvalues and Eigenvectors, Power Method to find the Largest Eigenvalue, Diagonalization of a Matrix.

44- 46	Orthogonal Bases- Orthogonal Matrices, Properties, Rectangular Matrices with orthonormal columns
47-50	The Gram- Schmidt Orthogonalization, A = QR Factorization
51	**Scilab Class Number 7- The Gram- Schmidt process**
52-54	Introduction to Eigenvalues and Eigenvectors, Properties of eigenvalues and eigenvectors, Power Method to compute the largest eigenvalue
55-56	Diagonalization of a Matrix, Powers and Products of Matrices
57-58	**Scilab Class Number 8&9- Eigen Values and Eigen Vectors, The Power Method**

Class work Problems:

1	The vectors $q_1 = (1, 0, 0)$, $q_2 = (0, 3/5, 4/5)$ and $q_3 = (0, 4/5, -3/5)$ form an orthonormal basis for R^3. Express the vector v = (7, -5 , 10) as a linear combination of the q's. Answer : $v = 7\, q_1 + 5\, q_2 - 10\, q_3$
2	Let W = { (a, b, b) / a , b are real } and let v = (3, 2, 6). (i) Find an orthonormal basis for W (i) Find the projection of v onto W , say v_1 (ii) Decompose v into a sum of two vectors $v_1 + v_2$ where v_2 is projection of v onto $W^{\perp}$ Answer : v = (3, 4, 4) + (0, -2 , 2)
3	Find a third column so that the matrix Q = $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -- \\ 1/\sqrt{3} & 2/\sqrt{14} & -- \\ 1/\sqrt{3} & -3/\sqrt{14} & -- \end{bmatrix}$ is orthogonal. Answer : $\pm\ (1/\sqrt{6}\,, -2/\sqrt{6}\,, 1/\sqrt{6}\ )$ (two vectors)
4	Let Q = $\begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$, x = $\begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ and y = $\begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$. Verify that (i) $Q^TQ = I$ (ii) $\Vert Qx \Vert = \Vert x \Vert$, $\Vert Qy \Vert = \Vert y \Vert$ (iii) $(Qx)^T(Qy) = x^Ty$
5	If W is a subspace spanned by the orthogonal vectors (2, 5, -1) and (-2 , 1, 1) find the point in W that is closest to (1, 2, 3) Answer : (-2/5, 2, 1/5)

6	What multiple of $a_1 = (1, 1)$ should be subtracted from $a_2 = (4, 0)$ to make the result orthogonal to a_1 ? Factorize $A = [\, a_1 \quad a_2 \,]$ into QR. Answer : 2.
7	Find an orthonormal set q_1, q_2, q_3 for which q_1 and q_2 span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. Which fundamental subspace contains q_3 ? What is the least squares solution of $Ax = b$ if $b = (1, 2, 7)$? Answer : $q_3 = (-2/3, 2/3, 1/3)$. LS solution is (1 , 2)
8	Use the Gram – Schmidt process to find a set of orthonormal vectors from the independent vectors $a_1 = (1, 1, 1)$, $a_2 = (0, 1, 1)$ and $a_3 = (0, 0, 1)$. Also find the A = QR factorization where $A = [\, a_1 \; a_2 \; a_3 \,]$.
9	Find the matrices Q and R such that QR = A where A has columns (1, 1, 1) , (1 , 1, 0) and (2 , 0, 0).
10	Find orthogonal vectors A , B, C by Gram- Schmidt method from a = (1, -1 , 0, 0) , b = (0, 1, -1 , 0) and c = (0, 0, 1, -1)
11	Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of eigenvalues and the determinant equals their product. If we shift A to A – 7 I what are the eigenvalues and eigenvectors and how are they related to those of A ?
12	Find the eigenvalues and eigenvectors of A , A^2 , A^{-1} and A + 4I if $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Answer : e. values of A are 1, 3 with e. vectors (1, 1) , (1 , -1)
13	Write three different 2 x 2 matrices for which the eigenvalues are 4 , 5 and determinant is 20.
14	Find the eigenvalues and the corresponding eigenvectors of (i) $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ Answer : (i) 1, -2, -2 k1(1, -1, 1) , k2(-1, 1, 0) , k3 (-1, 0, 1) (ii) 1, -2, -2 k1(1 , -1 , 1) , k2 (-1 , 1 , 0) (iii) 0,3, 15 k1(1, 2, 2) , k2 (2, 1, -2) , k3 (2, -2 , 1)
15	Use the Cayley – Hamilton's theorem to find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$
16	Use the power method to find the numerically largest eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ given $x_o = (1,0,0)$. Compute 6 iterations and correct the values to 3 decimal places.

17	Factor A = $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and hence compute A^{100}. Answer : eigenvalues are 1 , 5 with eigenvectors (1, -1) , (3, 1)
18	Diadonalize the matrix A = $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots, a matrix R such that R^2 = A. How many such square root matrices are there ? Answer : eigenvalues are 9 and 1 with eigenvectors (1, 1) , (1 , -1). One square root is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. There are 4 of them.
19	Find all eigenvalues and eigenvectors of A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write two different diadonalizing matrices S. Answer : eigenvalues are 0, 0, 3 with eigenvectors (-1 , 1, 0) , (-1, 0, 1), (1, 1, 1)
20	Find the matrices Λ and S to diagonalize A = $\begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$.What are limits of Λ^k *and* $S\Lambda^k S^{-1}$ as k → ∞ ? Answer : eigenvalues of A are 1 and 0.2 with eigenvectors (1 , 1) , (1 , -1). $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $S\Lambda^k S^{-1} \to \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$