

ENGINEERING PHYSICS

UNIT -1

INTRODUCTION TO QUANTUM MECHANICS

Feedback/corrections: vibha@pesu.pes.edu

References: VR Sunitha's Lectures
Vishruth V's notes

Vibha Masti

Electric & Magnetic fields

Vector Fields

- Wind, fluids
- Gravitational field
- Electric & Magnetic field
- Represented by vectors in space

Vector Operator (del)

- del operator - ∇

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

- Vector operator with no magnitude
- partial differential operator
- operator acts on vector field as either cross or dot product
- $\vec{F}(x, y) = i x^2 + j y^2$
- When ∇ operates on scalar field: gradient
- $\nabla \phi$ where $\phi(x, y, z)$ is a scalar field is the gradient
- Gradient gives direction along which steepest change of field occurs
- Gradient of a scalar is a vector

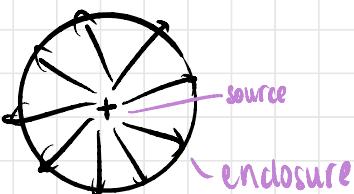
— Divergence ($\nabla \cdot \vec{F}$)

- Divergence means flow
- If $\nabla \cdot \vec{F} = +ve$: flow is outward
- If $\nabla \cdot \vec{F} = -ve$: flow is inward
- If $\nabla \cdot \vec{F} = 0$: inward flow = outward flow

at a
given
point

Electric Fields

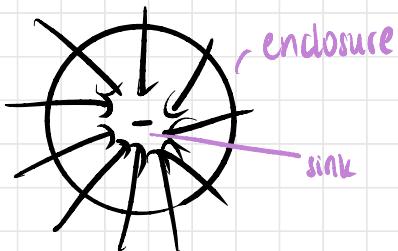
- i) isolated +ve charge



Divergence - flow

$\nabla \cdot \vec{F}$: +ve

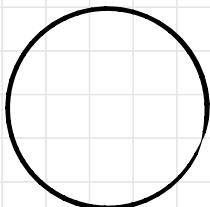
2) isolated -ve charge



Divergence - flow

$$\vec{D} \cdot \vec{n} = -ve$$

3) Free space

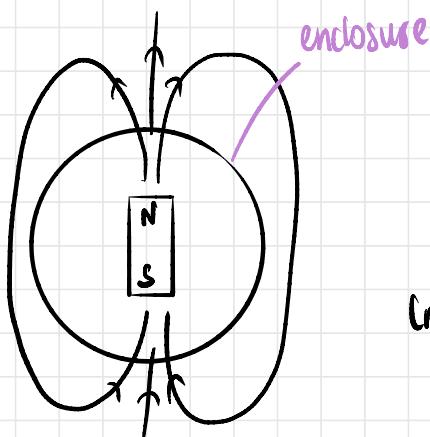


no flow

$$\vec{D} \cdot \vec{n} = 0$$

magnetic fields

i) Magnetic dipole



no flow

$$\vec{D} \cdot \vec{n} = 0$$

(no magnetic monopole)

$$\vec{F}(x, y) = i x^2 + j y^2$$

$$\vec{D} \cdot \vec{F} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (i x^2 + j y^2)$$

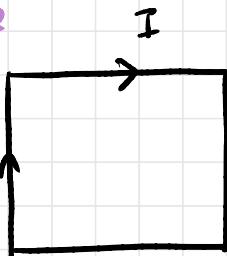
$$\vec{D} \cdot \vec{F} = 2x + 2y$$

Divergence of a vector field gives a scalar function.

— curl ($\nabla \times \vec{F}$)

- rotation of vector fields
- whirlpools, tornado, ocean current, centrifuges
- can be clockwise or anticlockwise

I) Purely resistive circuit



J = current density

$$\vec{\nabla} \cdot \vec{J} = 0 ; \text{ no flow}$$

There is curl

$$\vec{\nabla} \times \vec{J} \neq 0$$

— Laplacian Operator ($\nabla^2 = \nabla \cdot \nabla$)

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

MAXWELL'S EQUATIONS

INTEGRAL FORM

$$1) \int_S \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$$

] Gauss' Law

$$2) \int_S \vec{B} \cdot d\vec{s} = 0$$

] (larger regions of space)

$$3) \int_L \vec{E} \cdot d\vec{l} = -\frac{d\phi}{dt}$$

] Faraday's Law

$$4) \int_L \vec{B} \cdot d\vec{l} = \mu_0 (I_0 + I)$$

] modified Ampère's Law

DIFFERENTIAL FORM

$$1) \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

] Gauss' Law

$$2) \nabla \cdot \vec{B} = 0$$

] (smaller regions of space)

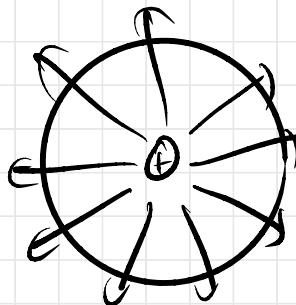
$$3) \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

] Faraday's Law

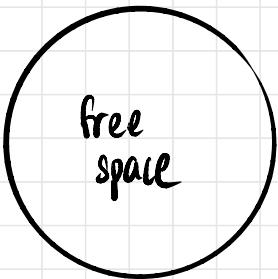
$$4) \nabla \times \vec{B} = \mu_0 \left(J + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

] modified Ampère's Law

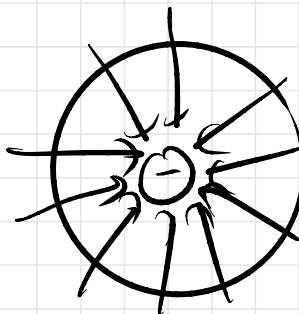
$$1) \nabla \cdot E = \frac{\rho}{\epsilon_0}$$



$$\nabla \cdot E = +ve$$

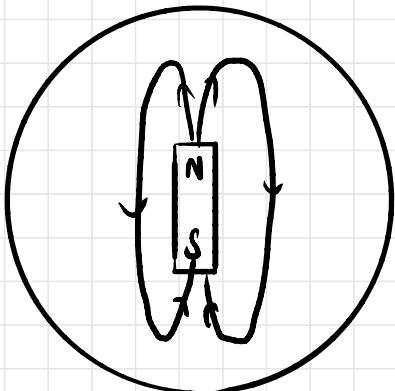


$$\nabla \cdot E = 0$$



$$\nabla \cdot E = -ve$$

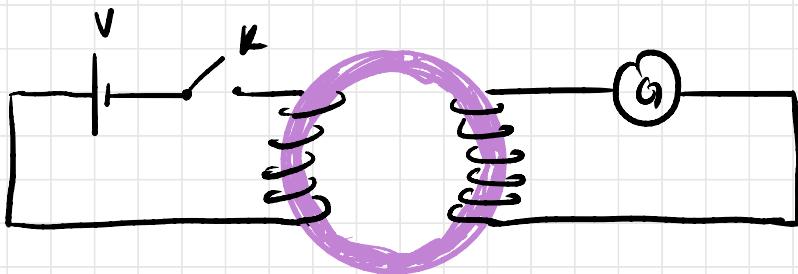
$$2) \nabla \cdot B = 0$$



$$\nabla \cdot B = 0$$

magnetic monopoles do
not exist

$$3) \nabla \times E = -\frac{\partial B}{\partial t}$$



- circulating \vec{E} can give rise to time-varying \vec{B}
- time-varying \vec{B} gives rise to circulating \vec{E}

$$4) \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Ampere's Law

$$\nabla \times H = J$$

IDENTITIES

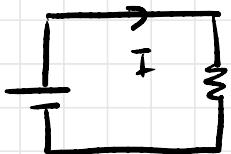
- Scalar triple product
1. $\nabla \cdot (\nabla \times A) = 0$
 2. $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$
- Vector triple product
(Lagrange's formula)

$$\vec{A} \times (\vec{B} \times \vec{C}) = (A \cdot C)B - (A \cdot B)C$$

$$\nabla \times H = J$$

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J$$

When only resistive element

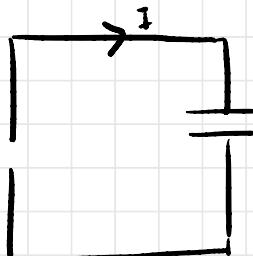


$$\mathbf{J} = \mathbf{I}/A$$

$$\nabla \cdot \mathbf{J} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0$$

With capacitor



$$\nabla \cdot \mathbf{J} \neq 0$$

\vec{E} is varying

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0$$

Maxwell's Correction

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_D$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t} \Rightarrow \mathbf{J}_D = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \Rightarrow \mathbf{H} = \frac{\mathbf{B}}{\mu_0}$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

MAXWELL'S EQUATIONS IN FREE SPACE

no charge, no current ($\rho=0, J=0$)

$$1) \vec{\nabla} \cdot \vec{E} = 0$$

$$2) \vec{\nabla} \cdot \vec{B} = 0$$

$$3) \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$4) \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Using Maxwell's Equations in Free Space, Derive Wave Equation in terms of Electric Field and Magnetic Field.

ELECTRIC FIELD

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$\vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = 0 - \nabla^2 \vec{E}$$

$$\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t} \right) = \nabla^2 \vec{E}$$

$$\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = \nabla^2 \vec{E}$$

$$\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = \nabla^2 \vec{E}$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E} \quad \text{--- (1)}$$

General wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{--- (2)}$$

where v is the velocity of propagation of the wave

Expanding (1), we get

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{--- (3)}$$

Comparing ② and ③, we get

wave equation in terms of \vec{E}

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{where } c^2 = \frac{1}{\mu_0 \epsilon_0}$$

\vec{E} propagate through free space at the speed of light.

MAGNETIC FIELD

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}$$

$$\vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0 - \nabla^2 \vec{B}$$

$$\mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) = - \nabla^2 \vec{B}$$

$$-\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\nabla^2 \vec{B}$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \nabla^2 \vec{B}$$

wave equation in terms of \vec{B}

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\text{where } c^2 = \frac{1}{\mu_0 \epsilon_0}$$

Magnetic fields propagate through free space at the speed of light.

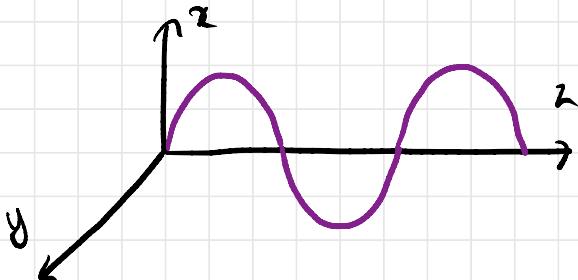
Varying \vec{E} and \vec{B} represent light waves.

Light waves are electromagnetic waves.

Show that \vec{E} and \vec{B} are perpendicular to each other and to the direction of propagation.

ELECTRIC WAVE

$$E_x = E_{0x} \cos(\omega t - kz)$$



$$\vec{E} = \hat{i} E_x + \hat{j} E_y + \hat{k} E_z$$

Let us consider only the x-component of \vec{E}
 $\therefore E_y = 0, E_z = 0$

(Polarised wave)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$\nabla \times E = \hat{j} \left(\frac{\partial E_x}{\partial z} \right) - \hat{i} \left(\frac{\partial E_x}{\partial y} \right)$$

E_x is independent
of y

$$\nabla \times E = \hat{j} \frac{\partial E_x}{\partial z}$$

$$\nabla \times E = \hat{j} \frac{\partial}{\partial z} \left(E_{0x} \cos(\omega t - kz) \right)$$

$$= \hat{j} (-k) \hat{t} E_{0x} \sin(\omega t - kz)$$

$$\nabla \times E = \hat{j} E_{0x} k \sin(\omega t - kz)$$

$$-\frac{\partial B}{\partial t} = \hat{j} E_{0x} k \sin(\omega t - kz)$$

Integrate wrt t

$$-B = j \epsilon_{0x} k \left(\frac{-\cos(\omega t - kz)}{\omega} \right)$$

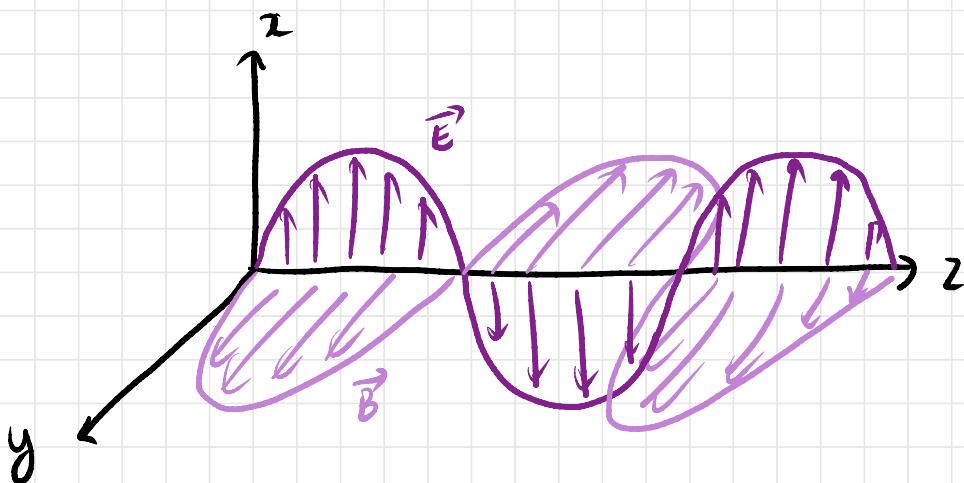
$$B = j \frac{k}{\omega} \epsilon_{0x} \cos(\omega t - kz)$$

$$\omega = 2\pi f = \frac{2\pi c}{\lambda} = kc \Rightarrow \frac{k}{\omega} = \frac{1}{c}$$

$$B = j \frac{\epsilon_0}{c}$$

\therefore Magnetic field is along y-direction

$$By = \frac{E_x}{c}$$



\vec{E} and \vec{B} are always coupled to each other and cannot be isolated.

Properties of EM Waves

1. $E \perp B$
2. $E \& B \perp$ direction of propagation
3. $E \& D \rightarrow$ speed of light
4. EM waves carry energy

ENERGY DENSITY

Energy carried by electric field

$$\text{energy density of } \vec{E} = \frac{1}{2} \epsilon_0 E^2$$

Energy carried by magnetic field

$$\text{energy density of } \vec{B} = \frac{1}{2\mu_0} B^2$$

Total energy density

$$U_T = \frac{1}{2} \epsilon_0 E_x^2 + \frac{1}{2} \frac{B_z^2}{\mu_0}$$

$$B_y = \frac{E_x}{c}$$

$$V_T = \frac{1}{2} \epsilon_0 E_x^2 + \frac{1}{2} \frac{E_x^2}{C^2 \mu_0}$$

$$\frac{1}{C^2} = \mu_0 \epsilon_0$$

$$V_T = \frac{1}{2} \epsilon_0 E_x^2 + \frac{1}{2} \epsilon_0 E_x^2$$

$$V_T = \epsilon_0 E_x^2 \quad \text{or} \quad V_T = \frac{B_y^2}{\mu_0}$$

POYNTING VECTOR

Amount of energy flowing through EM waves per unit area per unit time

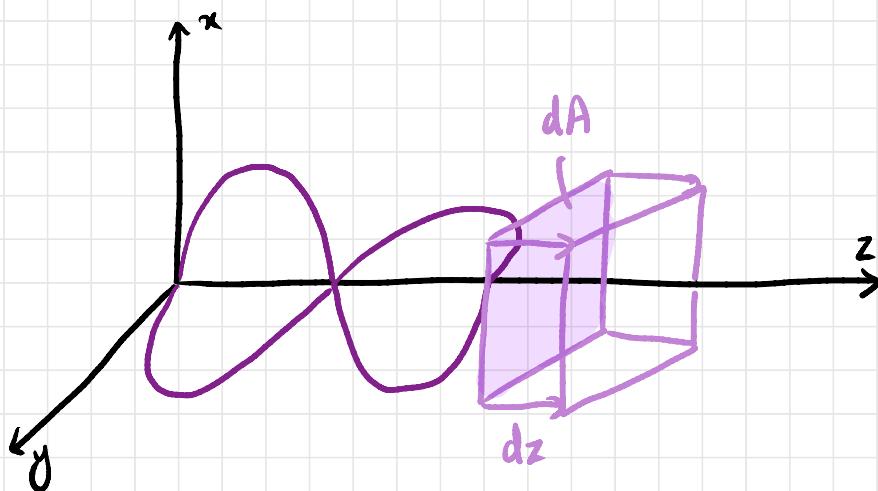
$$S = \frac{dU}{dA dt}$$

power / area = intensity

$$\text{units} = \frac{W}{m^2 s} = W m^{-2}$$

Consider a polarised EM wave propagating in space

Over a time dt , the wave moves from z to $z+dz$



$$\begin{aligned} dV &= dA \cdot dz \\ &= dA \cdot c \cdot dt \end{aligned}$$

$$dU = U_T dV$$

$$dU = \epsilon_0 E_x^2 dA c dt \quad E_y = c B_y$$

$$dU = \epsilon_0 E_x c B_y c dA dt$$

$$= \epsilon_0 E_x B_y c^2 dA dt$$

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$dU = \frac{E_x B_y}{\mu_0} dA dt$$

$$\frac{dU}{dA dt} = S = \frac{E_x B_y}{\mu_0}$$

We take the direction of \vec{S} in the direction of propagation of the wave

\therefore We take $\vec{S} \parallel \vec{E} \times \vec{B}$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Average Value of \vec{S} ($\langle S \rangle$)

$$\begin{aligned} S &= \frac{E_x B_y}{\mu_0} \\ &= \frac{E_x E_x}{\mu_0 C} \\ &= E_x^2 \epsilon_0 C \end{aligned}$$

$$\mu_0 C = \frac{1}{\epsilon_0 C}$$

$$S = C \epsilon_0 E_x^2$$

$$E_x = E_{0x} \cos(\omega t - kz)$$

$$\langle S \rangle = C \epsilon_0 \langle E_x^2 \rangle$$

$$= C \epsilon_0 E_{0x}^2 \langle \cos^2(\omega t - kz) \rangle$$

$$\langle \cos^2(\omega t - kz) \rangle = \frac{1}{T} \int_0^T \cos^2(\omega t - kz) dt = \frac{1}{2}$$

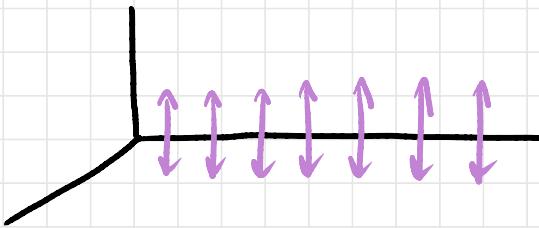
$$\boxed{\langle S \rangle = \frac{1}{2} C \epsilon_0 E_{0x}^2}$$

- $S \propto E_{0x}^2$, where E_{0x} is the amplitude
- ∴ S talks about the intensity of radiation (I)
- EM waves only talk about intensity, not frequency

POLARISATION

- Note: \vec{E} of matter interacts only with \vec{E} of EM wave, not \vec{B}
- Only in some cases (MRI scans), it interacts with \vec{B}

LINEAR POLARISATION / PLANE POLARISATION



\vec{E} only along x -direction

$$E_x = E_{0x} \sin(\omega t - kz)$$

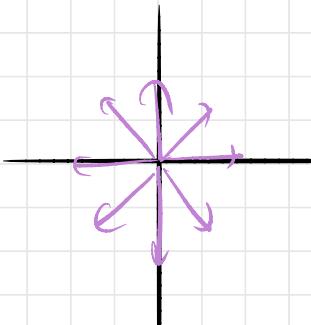
$$E_y = 0$$

CIRCULAR POLARISATION

- two mutually perpendicular waves of equal amplitude with a phase difference of $\pi/2$

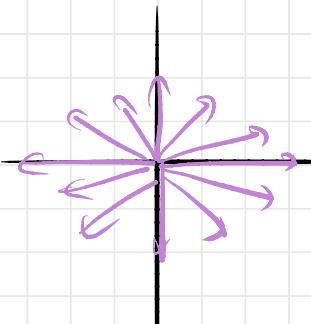
$$E_x = E_0 \sin(\omega t - kz) \hat{i}$$

$$E_y = E_0 \sin(\omega t - kz + \frac{\pi}{2}) \hat{j}$$



ELLIPTICAL POLARISATION

- two mutually perpendicular waves of different amplitude with a phase difference of $\pi/2$ (right elliptical)



DUAL NATURE OF RADIATION

Radiation as a Wave

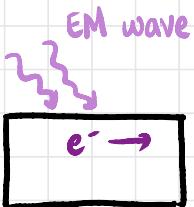
1. interference
2. diffraction
3. polarisation
4. reflection/refraction

Radiation as Particles

1. photoelectric effect
2. blackbody radiation
3. atomic spectra
4. compton effect

Photoelectric Effect

- Observation, experiment - Hertz



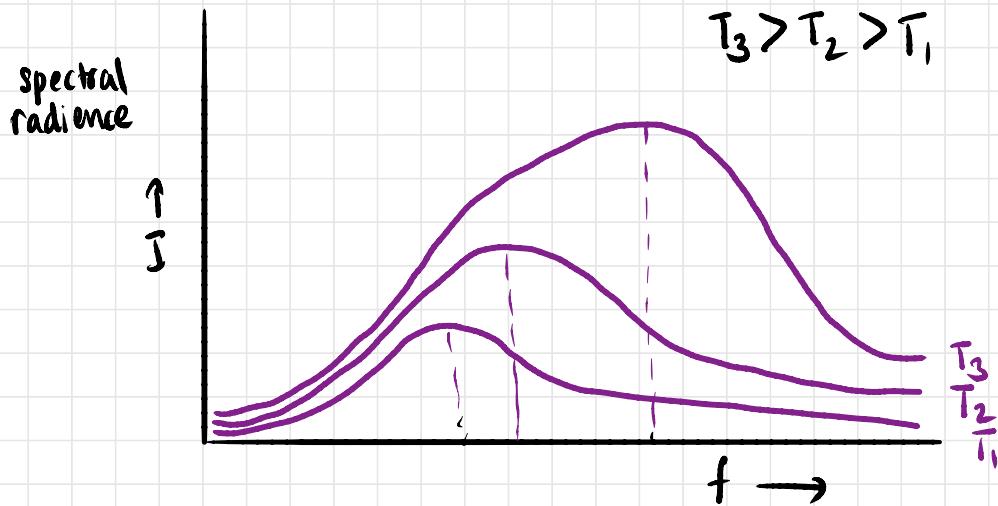
- Light incident on metals creates photo electrons
- Instantaneous emission of photo electrons
- Wave theory could not explain this phenomenon
- Discrete bundles of energy from EM wave
- Photon completely transfers energy to e^-
- Particle-particle interaction

$$h\nu = h\nu_0 + \frac{1}{2}mv^2$$

- Frequency \propto energy; intensity \propto photocurrent

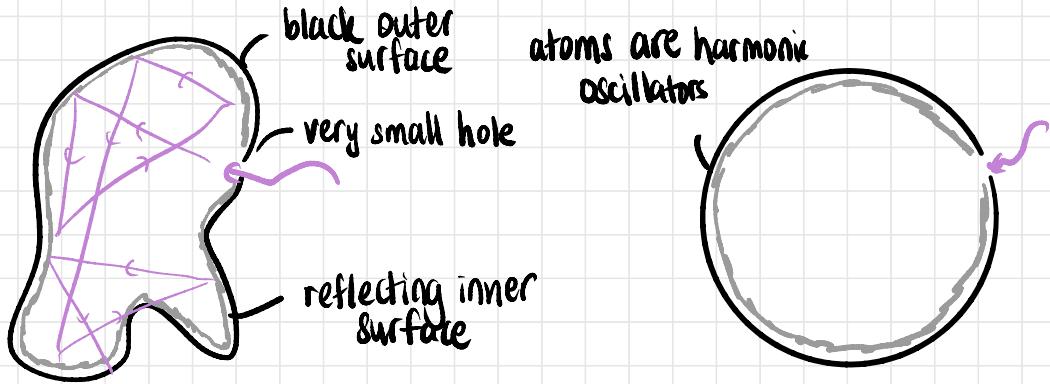
Blackbody Radiation

- blackbodies can be used for solar cells
- spectrum: variation of intensity as a function of λ or f
- experimental graph



Blackbody

- body that completely absorbs all incident radiation
- completely emits all absorbed energy
- zero reflection
- carbon black (soot), sun



Observations

- As $T \uparrow$, max intensity shifts towards higher frequency

$T \propto \frac{1}{\lambda_{\text{max}}}$, $T \propto f_{\text{max}}$ Wein's displacement Law

$$E(\lambda) d\lambda = c_1 \lambda^{-5} e^{-c_2/\lambda T} d\lambda$$

- Energy radiated is proportional to T^4

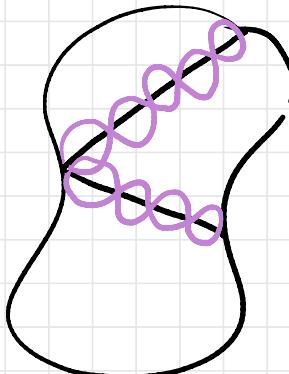
$E \propto T^4$ Stefan's Law

Spectral Density / Spectral Radiance

the amount of energy contained in the cavity per unit volume in the interval $\nu + d\nu$ or $\lambda + d\lambda$ at a constant temperature

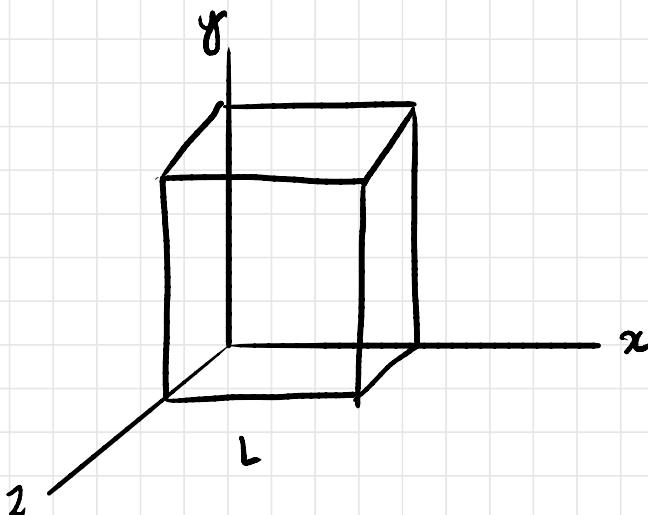
$$U_v d\nu = \frac{\text{no. of oscillators}}{\text{volume}} \times \text{average energy}$$

$$= \frac{\text{no. of standing waves}}{\text{volume}} \times \text{average energy}$$

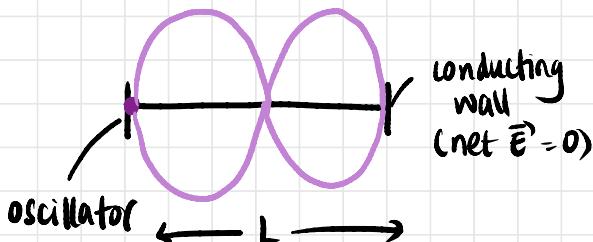


Derivation of Rayleigh-Jeans Expression for Energy Density

They imagined EM waves in a cubic volume due to oscillating dipoles that make up the walls of the cavity



For one-dimensional cavity, only certain frequencies can stabilise to form standing waves



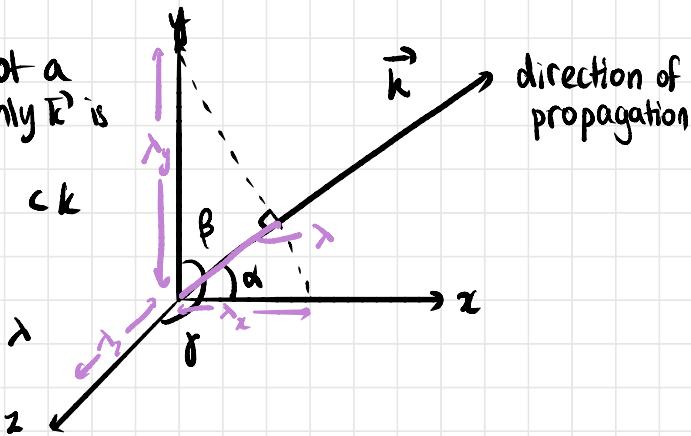
$$L = \frac{n\lambda}{2} \Rightarrow \lambda = \frac{2L}{n} \quad \text{or} \quad v = \frac{cn}{2L}$$

In a 3-D cubic cavity, wave can form standing wave in a direction only if each of its components independently forms standing waves in x-y-z directions

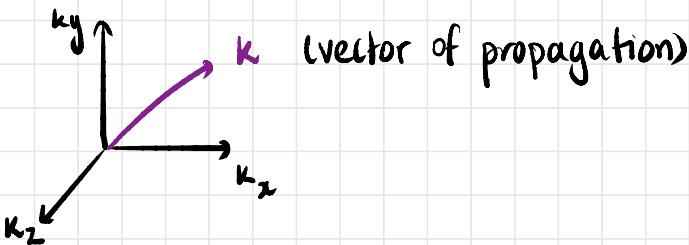
Note: Δ is not a vector, only \vec{k} is

$$\therefore k_x, k_y, k_z < k$$

$$\text{but} \\ \Delta_1, \Delta_2, \Delta_3 > \lambda$$



We get components of k in all three directions



$$\text{Now, } k = \frac{2\pi}{\lambda} \Rightarrow \text{We get } \lambda \text{ (scalar) along 3 axes}$$

For standing waves along 3 axes,

$$\lambda_x = \frac{2L}{n_x} \quad \lambda_y = \frac{2L}{n_y} \quad \lambda_z = \frac{2L}{n_z}$$

Taking α , β and γ as directions,

$$\cos \alpha = \frac{1}{\lambda_x}, \quad \cos \beta = \frac{1}{\lambda_y}, \quad \cos \gamma = \frac{1}{\lambda_z}$$

We know (dir. ratios)

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\alpha^2 \left(\frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} + \frac{1}{\lambda_z^2} \right) = 1$$

$$\frac{c^2}{v^2} \left(\frac{v_x^2}{c^2} + \frac{v_y^2}{c^2} + \frac{v_z^2}{c^2} \right) = 1$$

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$\text{We know } v = \frac{nL}{2L}$$

$$v^2 = \frac{c^2}{4L^2} (n_x^2 + n_y^2 + n_z^2)$$

$$v = \frac{c}{2L} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

Lowest possible mode of frequency,

$$n_x = 1, n_y = 1, n_z = 1$$

$$\nu_{111} = \frac{c}{2L} \sqrt{3}$$

Second frequency:

Three possible modes (112, 121, 211)

$$\nu_{112} = \nu_{121} = \nu_{211} = \frac{c}{2L} \sqrt{6}$$

Modes are all the possible standing waves.

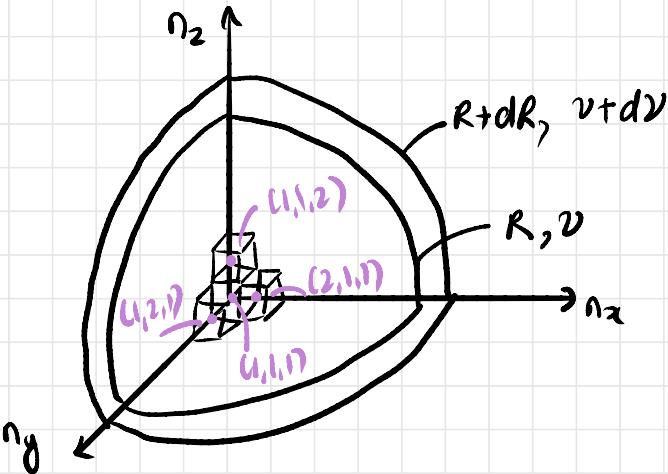
The frequencies $\nu_{112}, \nu_{121}, \nu_{211}$ are equal but are different modes, as they are physically different (different directions of propagation)

Phase space

We plot the modes on a phase space.

We imagine an octet of a sphere to get the number of possible modes

Each possible mode can be represented as a unit cube



Volume of Octet = no. of unit cubes (approx as octet is huge)

For each cube there is only 1 point which represents 1 possible mode

\therefore volume = no. of unit cubes = no. of points = no. of modes

$$\text{volume} = \text{no. of modes}$$

According to the equation of a sphere $x^2 + y^2 + z^2 = R^2$

$$\text{Here } n_x^2 + n_y^2 + n_z^2 = R^2$$

$$\therefore V = \frac{C}{2L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{CR}{2L}$$

Let sphere be of radius R; all points on the surface have frequency v

No. of modes within frequency ν = volume of octet

$$= \frac{1}{8} \times \frac{4}{3} \pi R^3$$

Slightly bigger octet of radius $R+dr$ with modes within frequency $\nu+dv$

No. of modes within frequency $\nu+dv$

$$\frac{1}{8} \times \frac{4}{3} \pi (R+dr)^3$$

No. of modes with frequency lying between ν and $\nu+dv$

$$\begin{aligned} &= \frac{1}{8} \times \frac{4}{3} \pi \left[(R+dr)^3 - R^3 \right] \\ &= \frac{\pi}{6} \left[R^5 + 3R^2 dr + 3R(dr)^2 + (dr)^3 - R^5 \right] \end{aligned}$$

dr is very small (neglecting higher order terms)

$$= \frac{\pi}{6} (3R^2 dr) = \frac{\pi}{2} R^2 dr$$

$$\text{Now, } \nu = \frac{CR}{2L} \Rightarrow R = \frac{2L\nu}{C} \Rightarrow dr = \frac{2L}{C} d\nu$$

No. of modes

$$\text{from } \nu \text{ to } \nu+dv = \frac{\pi}{6} \left(\frac{4L^2\nu^2}{C^2} \right) \left(\frac{2L}{C} d\nu \right)$$

$$N(\nu) d\nu = \frac{4\pi L^3 \nu^2}{C^3} d\nu$$

Now, Rayleigh - Jean assumed average energy per mode is kT at temperature T (equipartition theorem)

Energy of nodes lying between ν and $\nu + d\nu$

$$E(\nu) d\nu = \frac{4\pi\nu^2}{c^3} kT d\nu$$

$$c = \nu \lambda \Rightarrow c^2 = \nu^2 \lambda^2$$

$$\nu = \frac{c}{\lambda} \Rightarrow d\nu = -\frac{c}{\lambda^2} d\lambda$$

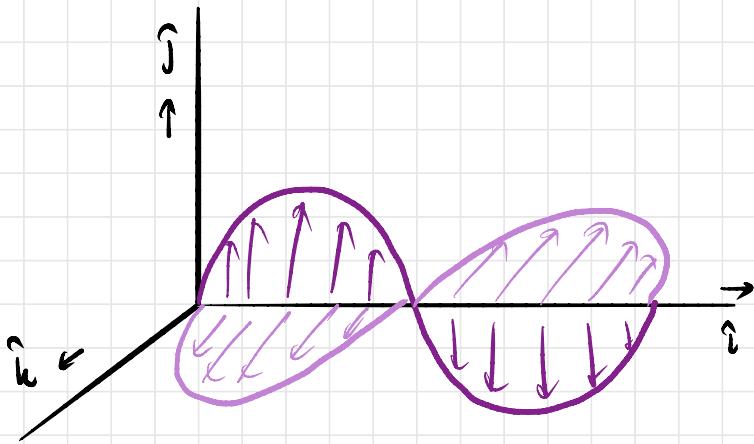
$$E(\lambda) d\lambda = \frac{4\pi c^2}{\lambda^2 c^3} \left(-\frac{c}{\lambda^2} \right) kT d\lambda$$

$$E(\lambda) d\lambda = -\frac{4\pi}{\lambda^4} kT d\lambda$$

*energy possessed
by oscillators*

This energy is only considering 1 direction of oscillations of waves for every direction of propagation

For direction of propagation \hat{i} , there are 2 orthogonal modes (along \hat{j} and \hat{k})



Now, no. of modes between ν and $\nu + d\nu$

$$N(\nu) d\nu = 2 \times \frac{4\pi\nu^2}{c^3} d\nu$$

$$= \frac{8\pi\nu^2}{c^3} d\nu$$

Energy per mode is kT

$$E(\nu) d\nu = \frac{8\pi\nu^2 kT}{c^3} d\nu$$

$$E(\lambda) d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

Max Planck's Theory

Due to failure of R-J, he assumed that oscillators can only oscillate at certain frequencies

Therefore, energy is quantised (multiples of $h\nu$)

Number of oscillators with energy $n h\nu$

$$N_n \propto e^{-\frac{n h\nu}{kT}} \quad (\text{Boltzmann equation})$$

$$N_n = A e^{-\frac{n h\nu}{kT}}$$

Energy of N_n oscillators

$$E_n = N_n n h\nu$$

Total energy

$$\sum_{n=0}^{\infty} E_n = \sum_{n=0}^{\infty} N_n n h\nu$$

Average energy

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} A n h\nu e^{-\frac{E_n}{kT}}}{\sum_{n=0}^{\infty} A e^{-\frac{E_n}{kT}}} = \frac{\sum_{n=0}^{\infty} n h\nu e^{-\frac{n h\nu}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{n h\nu}{kT}}} = \frac{h\nu \sum_{n=0}^{\infty} n e^{-\frac{n h\nu}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{n h\nu}{kT}}}$$

$$kT \propto = \frac{h\nu}{4}$$

$$\langle E \rangle = \frac{kT \sum_{n=0}^{\infty} \alpha n e^{-\alpha n}}{\sum_{n=0}^{\infty} e^{-\alpha n}}$$

Writing in differential form

$$\langle E \rangle = -kT \alpha \frac{d}{d\alpha} \left(\ln \left(\sum_{n=0}^{\infty} e^{-\alpha n} \right) \right)$$

$$\sum_{n=0}^{\infty} e^{-\alpha n} = 1 + e^{-\alpha} + e^{-2\alpha} + \dots$$

LHP with $C_R = e^{-\alpha}$

$$\text{sum} = \frac{1}{1-e^{-\alpha}} = (1-e^{-\alpha})^{-1}$$

$$\sum_{n=0}^{\infty} e^{-\alpha n} = (1-e^{-\alpha})^{-1}$$

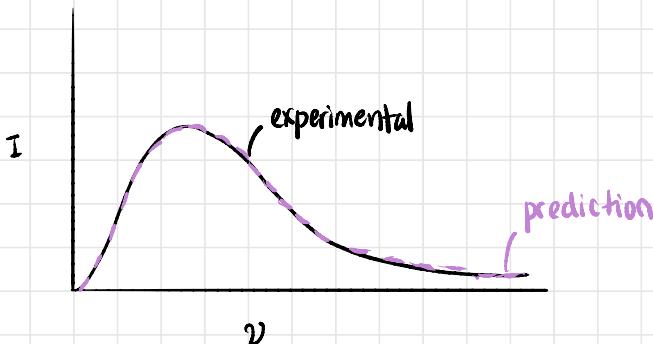
$$\langle E \rangle = -\alpha kT \frac{d}{d\alpha} \ln (1-e^{-\alpha})^{-1}$$

$$= \alpha kT \frac{d}{d\alpha} \ln (1-e^{-\alpha})^{-1} - \alpha kT \left(\frac{e^{-\alpha}}{1-e^{-\alpha}} \right)$$

$$= h\nu \left(\frac{1}{e^{\alpha}-1} \right) = \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\langle E \rangle = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} \quad - \text{average energy}$$

$$\begin{aligned} U(\nu) d\nu &= \frac{8\pi\nu^2}{c^3} d\nu \frac{h\nu d\nu}{e^{\frac{h\nu}{kT}} - 1} \quad - \text{Max} \\ &= \frac{8\pi h\nu^3}{c^3} \left(\frac{1}{e^{\frac{h\nu}{kT}} - 1} \right) d\nu \quad \text{Planck's Law} \end{aligned}$$



Case I:

- high frequencies
- as $h\nu \gg kT$, $I \rightarrow 0$
- no ultraviolet catastrophe

Case II

- low frequencies
- $h\nu \ll kT$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \dots \\ e^{\frac{h\nu}{kT}} &= 1 + \frac{h\nu}{kT} + \frac{1}{2} \left(\frac{h\nu}{kT} \right)^2 + \dots \end{aligned}$$

$$e^{\frac{h\nu}{kT}} = 1 + \frac{h\nu}{kT}$$

Substituting in Max Planck's Law,

$$U(\nu) d\nu = \frac{8\pi\nu^2}{c^3} d\nu \frac{h\nu}{h\nu/kT}$$

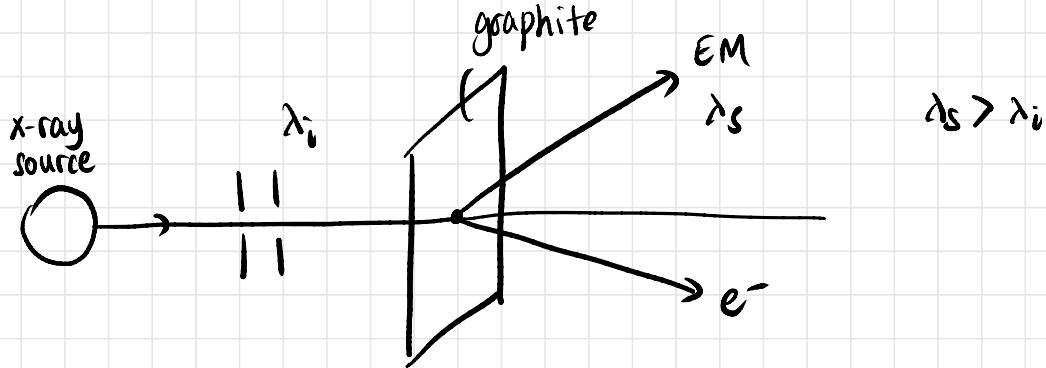
$$= \frac{8\pi\nu^3}{c^3} d\nu kT \quad \text{— Rayleigh-Jeans Law}$$

- at low frequencies, reduces to Rayleigh-Jeans Law

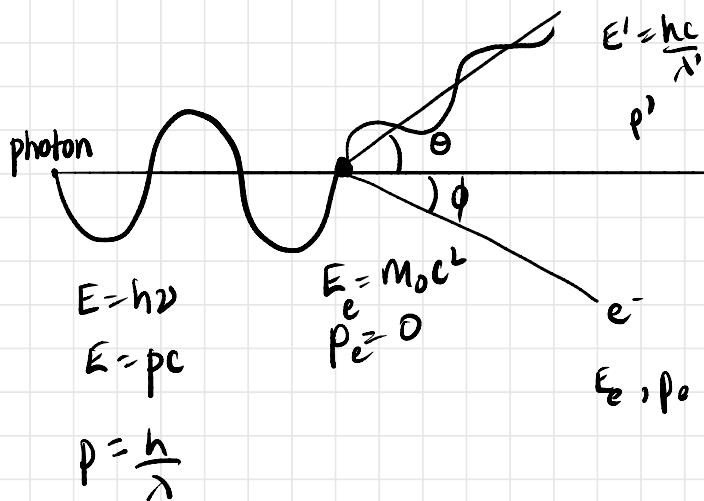
Planck's Law of blackbody radiation proves that radiation is a particle (discrete energy)

Compton Effect

- Compton Scattering
- Experiment that supported particle behaviour of EM radiation



Using particle picture of light (photons)



Law of conservation of energy

35

Total energy before collision = total energy after collision

$$E + m_0 c^2 = E' + E_e \quad \text{--- (1)}$$

$$E_e = \sqrt{m_0^2 c^4 + p_e^2 c^2} \quad \begin{aligned} &\text{--- Einstein's Theory} \\ &\text{(relativistic energy)} \\ &\text{of moving particle} \\ &m_0 = \text{rest mass} \end{aligned}$$

Law of conservation of momentum

x-component of momentum

$$p + 0 = p' \cos \theta + p_e \cos \phi$$

$$p - p' \cos \theta = p_e \cos \phi \quad \text{--- (2)}$$

y-component of momentum

$$0 = p' \sin \theta - p_e \sin \phi$$

$$p' \sin \theta = p_e \sin \phi \quad \text{--- (3)}$$

Squaring and adding (2) and (3)

$$(p - p' \cos \theta)^2 + (p' \sin \theta)^2 = (p_e \cos \phi)^2 + (p_e \sin \phi)^2$$

$$p^2 - 2pp' \cos 2\theta + p'^2 = p_e^2$$

$$P_e^2 = p^2 + p'^2 - 2pp' \cos \theta \quad \text{--- (4)}$$

Using equation (1)

$$(E - E') + m_0 c^2 = E_e$$

$$E - E' + m_0 c^2 = \sqrt{m_0^2 c^4 + P_e^2 c^2}$$

$$(pc - p'c) + m_0 c^2 = m_0^2 c^4 + P_e^2 c^2$$

$$(pc - p'c)^2 + m_0^2 c^4 + 2(p(c - p')m_0 c^2) = m_0^2 c^4 + P_e^2 c^2$$

$$p^2 c^2 + p'^2 c^2 - 2pp' c^2 + 2c^2(p - p')m_0 = P_e^2 c^2$$

Substituting $p^2 + p'^2$ from (4)

$$p^2 + p'^2 - 2pp' + 2(p - p')m_0 c = P_e^2$$

$$P_e^2 + 2pp' \cos \theta - 2pp' + 2(p - p')m_0 c = P_e^2$$

$$2pp'(\cos \theta - 1) = 2(p' - p)m_0 c$$

$$\frac{2h^2}{\lambda_i \lambda_s} (\cos \theta - 1) = 2 \left(\frac{h}{\lambda_s} - \frac{h}{\lambda_i} \right) m_0 c$$

$$2h (\cos \theta - 1) = 2(\lambda_i - \lambda_s) m_0 c$$

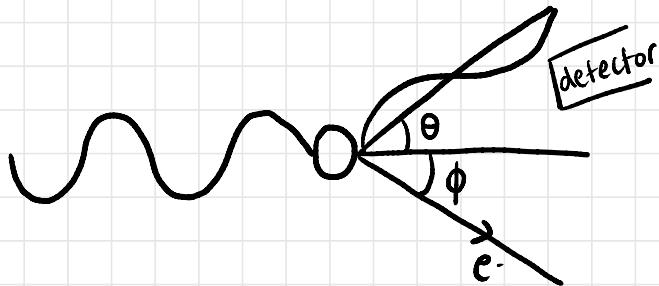
$$\Delta\lambda = \lambda_s - \lambda_i$$

$$\lambda_s - \lambda_i = \frac{h}{m_0 c} (1 - \cos\theta)$$

↗ Compton shift

$\Delta\lambda$: Compton Shift

$\frac{h}{m_0 c}$: Compton wavelength $\approx 2.427 \times 10^{-12} \text{ m} = \lambda_c$



X-rays from Mo target ($\lambda = 0.074 \text{ nm}$)

(i) $\theta = 0^\circ$

$$\Delta\lambda = 0$$

photon not interacting with e^-

(ii) $\theta = 45^\circ$

$$\Delta\lambda = 0.71 \text{ pm}$$

(iii) $\theta = 90^\circ$

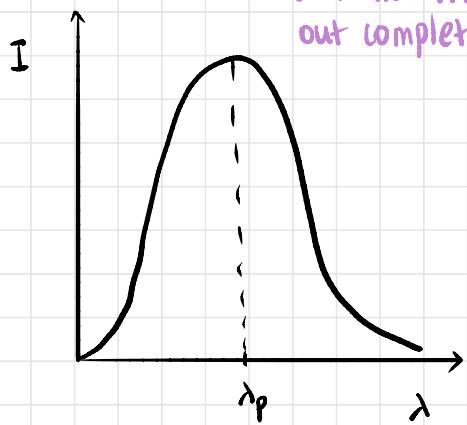
$$\Delta\lambda = \lambda_c = 2.427 \text{ pm}$$

(iv) $\theta = 180^\circ$

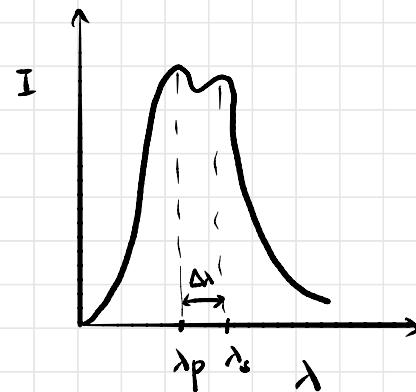
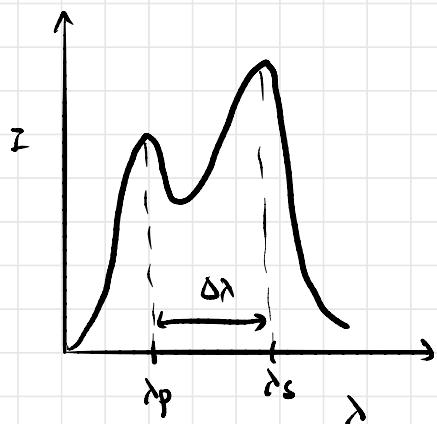
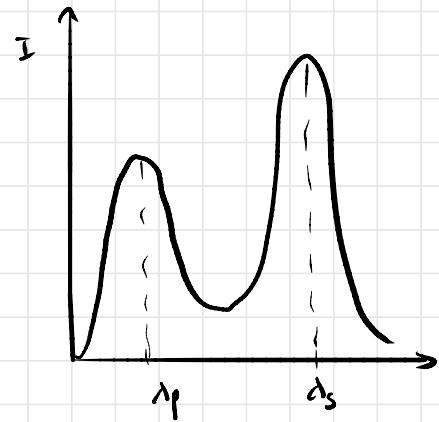
$$\Delta\lambda = 2\lambda_c = 4.854 \text{ pm}$$

photons undergo backscattering
 e^- gains maximum energy

I vs λ graph

(i) $\theta = 0^\circ$ 

could not filter out completely

(ii) $\theta = 45^\circ$ (iii) $\theta = 90^\circ$ (iv) $\theta = 180^\circ$ 

If a photon interacts with a tighter bound electron, the e^- would not move/gain energy as it is in a stable state and photon would not lose energy and the wavelength stays λ .

This is because $\Delta\lambda = \frac{h}{M_{\text{nucleus}} c} (1 - \cos\theta)$ and M_{nucleus} is very high

photon with
tighter e^-

$$\Delta\lambda \approx 0$$

Conclusion

- Compton Shift does not depend on the incident wavelength
- Depends only on the scattering angle θ

de-Broglie Hypothesis

- Dual nature of matter
- Argued that if radiation shows dual nature, matter should too
- Every object in motion is associated with a wave, called matter waves

$$\lambda = \frac{h}{p}$$

- Cannot be observed for macroscopic objects as the momentum is large and the associated λ is extremely small
- Only in atomic/subatomic scale
- Won Nobel prize in 1924
- Proven first by Davisson-Germer experiment
- Used Ni crystal, e^- was accelerated at different potentials
- λ for a free particle

$$E_k = \frac{p^2}{2m} \Rightarrow p = \sqrt{2m E_k}$$

$$\lambda = \frac{h}{\sqrt{2m E_k}}$$

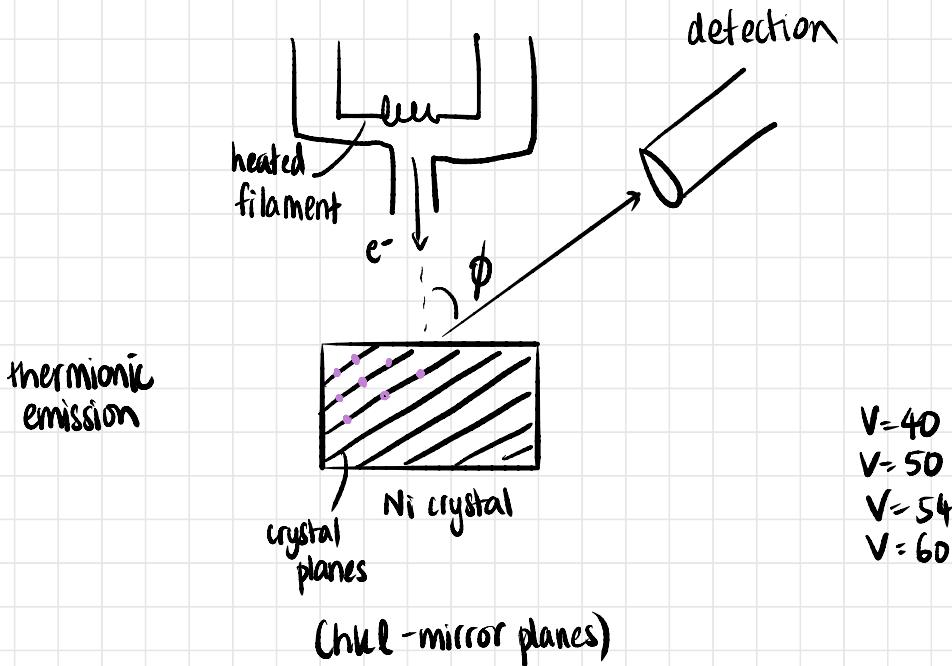
- For an accelerated charged particle

$$E_k = eV$$

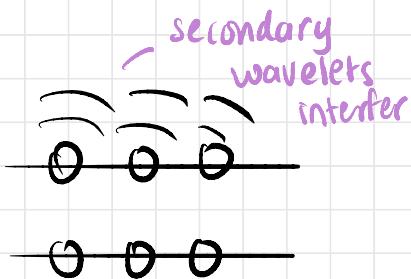
$$\frac{1}{2}mv^2 = eV \Rightarrow \frac{p^2}{2m} = eV \Rightarrow p = \sqrt{2meV}$$

$$\lambda = \frac{h}{\sqrt{2meV}}$$

Davission-Germer Experiment



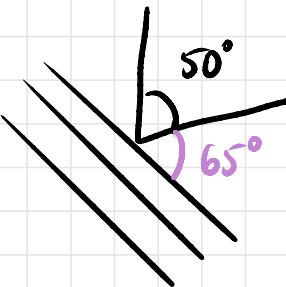
It was noticed that at $V=54V$ and $\phi=50^\circ$, intensity was maximum



$$2d \sin\theta = n\lambda$$

Bragg's Law

- Interplanar distance known $d = \frac{a}{\sqrt{h^2+k^2+l^2}}$
- Using X-rays, they found lattice planes, Miller indices found and interplanar space found to be $d = 0.91\text{\AA}$
- For e; $2d \sin\theta = n\lambda$ (always take first order)



- Diffraction angle = 65° (Bragg's diffraction)
- Use these values to calculate $\lambda \Rightarrow \lambda = 1.65\text{\AA}$
- de Broglie λ for accelerated charged particle

$$\lambda = \frac{h}{\sqrt{2eVm}} = 1.67\text{\AA}$$

Q: Find the KE and V of proton of mass 1.67×10^{-27} kg associated with deBroglie wavelength of 0.2865 \AA

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

$$v = \frac{h}{m\lambda} = 13848 \text{ m s}^{-1}$$

$$KE = \frac{1}{2}mv^2 = 1 \text{ eV}$$

Q: The shift in the λ of x-rays scattered in a Compton experiment is 0.2 pm . $\lambda_s = 1.002 \text{ nm}$. Find θ at which x-ray photon is scattered and what is the momentum gained by the e^- ?

$$\Delta\lambda = 0.2 = \lambda_s - \lambda_i$$

$$\lambda_i = 1.0018$$

$$\Delta\lambda = \frac{h}{mc} (1 - \cos\theta)$$

$$\cos\theta = 0.917$$

$$\theta = 23.42^\circ$$

$$\text{energy transferred} = \frac{hc}{\lambda_i} - \frac{hc}{\lambda_s}$$

$$\frac{1}{2} \frac{p^2}{m} = hc \left(\frac{\Delta\lambda}{\lambda_i \Delta s} \right)$$

homework

$$p^2 = \frac{2mh c \Delta\lambda}{\lambda_i \Delta s} = 2.685 \times 10^{-25} \text{ kg m s}^{-1}$$

$$\text{Ans: } 6.6 \times 10^{-25} \text{ kg m s}^{-1}$$

- Q: Compare the momentum and energy of e^- and photon whose $\lambda_b = 650 \text{ nm}$

electron: $p = \frac{h}{\lambda}$

photon: $p = \frac{E}{c} = \frac{h}{\lambda}$

$$E = \frac{p^2}{2m}$$

$$E = \frac{hc}{\lambda}$$

$$p_e = 1.019 \times 10^{-27} \text{ kg m s}^{-1}$$

$$E_e = 3.565 \times 10^{-6} \text{ eV}$$

$$p_p = 1.019 \times 10^{-27} \text{ kg m s}^{-1}$$

$$E_p = 1.91 \text{ eV}$$

$$\frac{E_e}{E_p} = 1.867 \times 10^{-6}$$

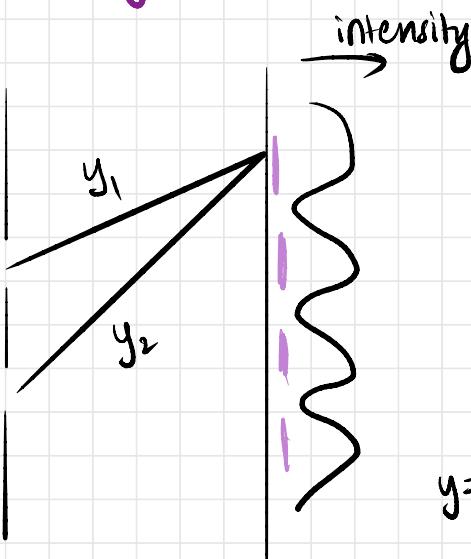
Q: What is λ of H atom moving with the mean v corresponding to the avg. KE of H atoms under thermal eq at 293 K (mass of H = 1.008 amu)

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$

$$\frac{p^2}{2m} = \frac{3}{2}kT \Rightarrow p = \sqrt{3kTm}$$

$$\lambda = \frac{h}{p} = 1.466 \text{ \AA}$$

Interference (light-revision)



$$y_1 = a \sin wt$$

$$y_2 = a \sin (wt + \phi)$$

$$y = y_1 + y_2$$

$$y = a (\sin wt + \sin(wt + \phi))$$

$$y = 2a \sin\left(\omega t + \frac{\phi}{2}\right) \cos\left(\frac{\phi}{2}\right)$$

$$I = y^2 = 4a^2 \sin^2\left(\omega t + \frac{\phi}{2}\right) \cos^2\left(\frac{\phi}{2}\right)$$

$$\langle I \rangle = 4a^2 \left(\frac{1}{2}\right) \cos^2\left(\frac{\phi}{2}\right)$$

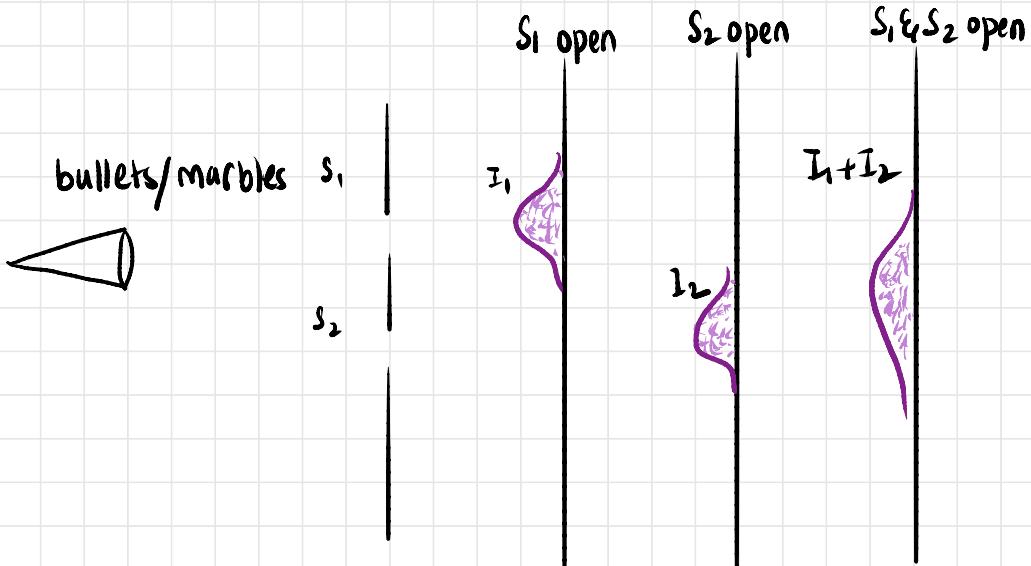
$$I \propto \cos^2 \frac{\phi}{2}$$

Single Particle Double Slit Experiment

A single particle (photon, electron) can only go to one spot at a time.

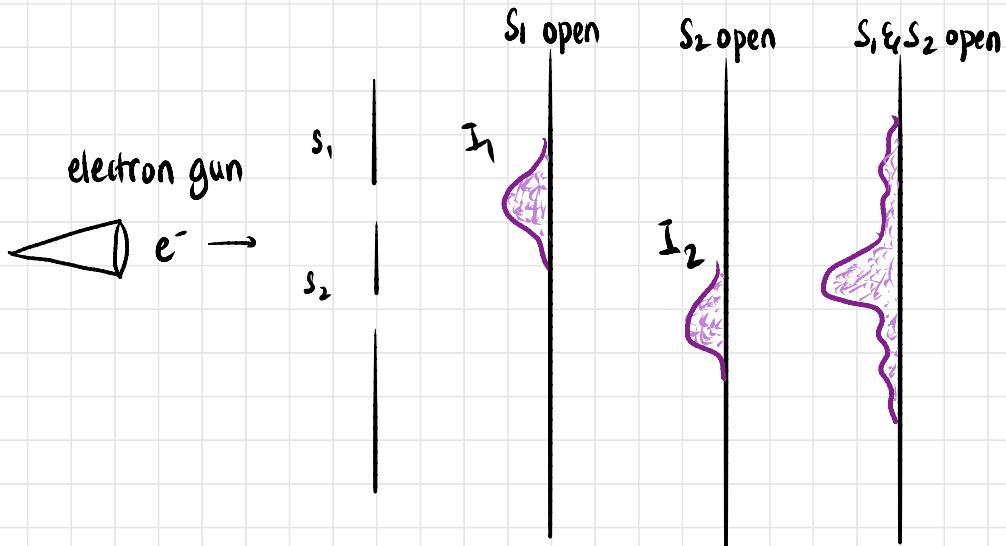
When one slit is open, we observe a normal distribution I_1 & I_2 .

When both are opened, we expect to observe $I_{\text{res}} = I_1 + I_2$ as in the case of bullets. However, we notice an interference pattern. But this does not make sense as individual particles were sent one at a time (not light waves)



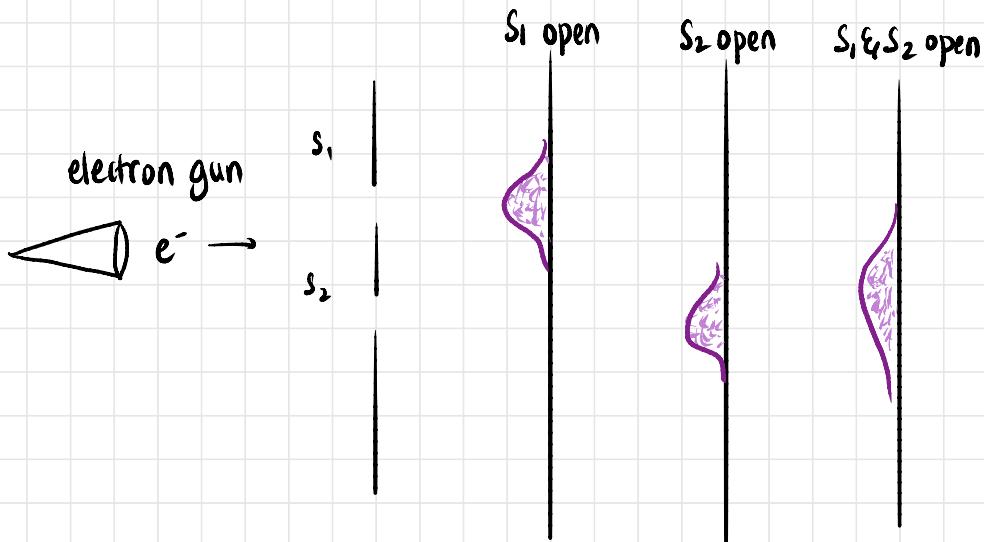
Experimental setup

46



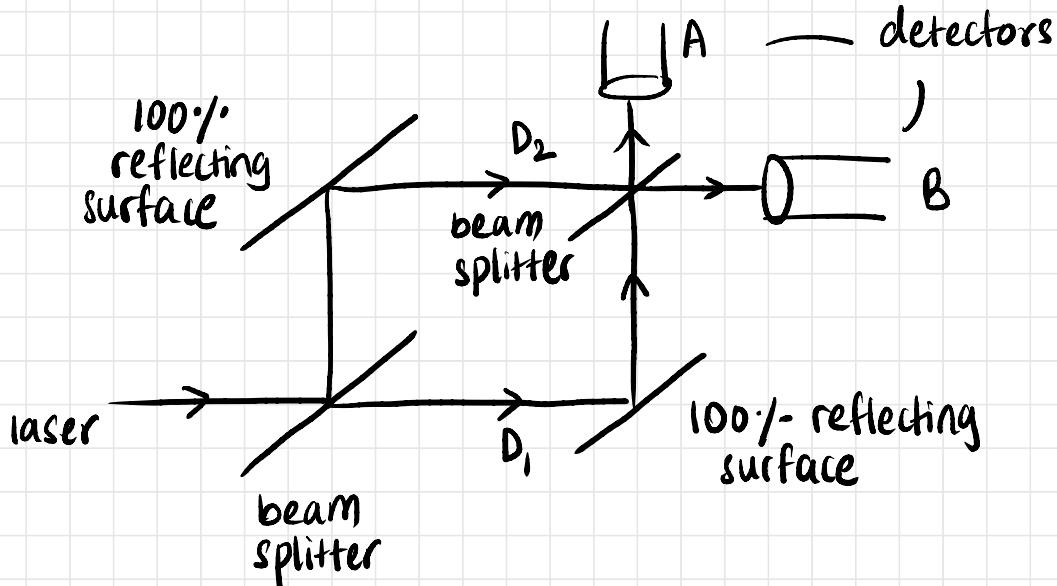
interference fringes
— wave nature

If detectors are placed, \vec{E} disturbs wave nature of e^- and particle behaviour observed.



Mach Zehnder Experiment (Interferometer)

47



Similar to single photon interference, a laser is shone as shown above.

Beam splitter

splits beam into 50% intensity reflection, 50% intensity transmission.

When light is sent, all light reaches detector B and no light reaches detector A.

This is because the two paths to B result in constructive interference (in phase), while the two paths to A result in destructive interference (phase diff = π)

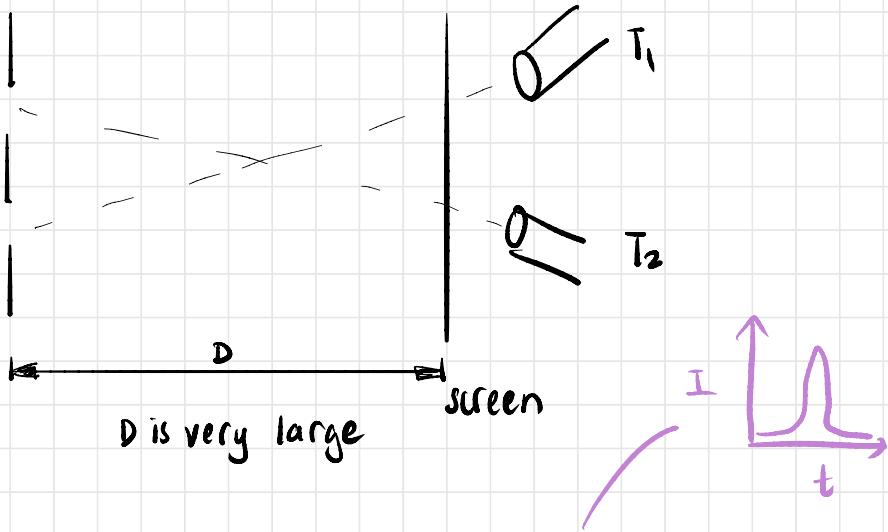
Paths to B: trans → refl → refl and refl - refl - trans
(0) (π) (π) (π) (π) (0)

Paths to A: trans → refl → trans and refl - refl - refl
(π) (0) (0) (π) (π) (π)

Even when performed with single particles, 100% of the particles go to B and 0% to A.

However, when detectors D_1 and D_2 are placed, 50% of the intensity is at A and 50% at B.

John Wheeler's Delayed Choice Experiment



Send a short pulse of light (femtosecond) with many photons travelling towards the screen.

Two detectors T_1 and T_2 are placed behind the screen and are focused on each slit.

The screen can be made translucent in a fraction of a second by applying E.

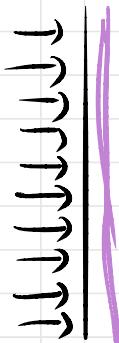
When experiment performed, we observe interference as usual when the screen is present.

This must mean that the photons were directed in such a way as to form the interference pattern.



photon momentum
must have
been as shown

If the screen is removed and the detectors are exposed, the interference pattern does not form. Instead, a continuous distribution of light is observed, which means photons were travelling like this.



momentum of
photons

How did the photons change their momentum to travel in all places instead of those certain areas?

Wavefunction (Ψ -psi)

de Broglie assumed that all matter has associated with it a wave, known as a matter wave (hypothetical)

This model accurately predicted experimental observations like interference, diffraction etc.

Waves signify variation of a certain parameter (E, pressure, water height)

$$\begin{array}{c} \text{EM waves} \xrightarrow{\quad} \vec{E} \& \vec{B} \xrightarrow{\quad} E(x,t) \\ \text{strings} \xrightarrow{\quad} \text{displacement} \xrightarrow{\quad} y(x,t) \\ \text{sound wave} \xrightarrow{\quad} \text{pressure} \xrightarrow{\quad} P(x,t) \end{array}$$

In matter waves, what is varying?

Max Born assumed that all particles have associated with them a wavefunction with an associated wavelength.

$$\Psi = A \sin\left(\frac{2\pi}{\lambda} z\right)$$

The wavefunction Ψ has no physical meaning; it is only a mathematical representation.

The wavefunction is defined as

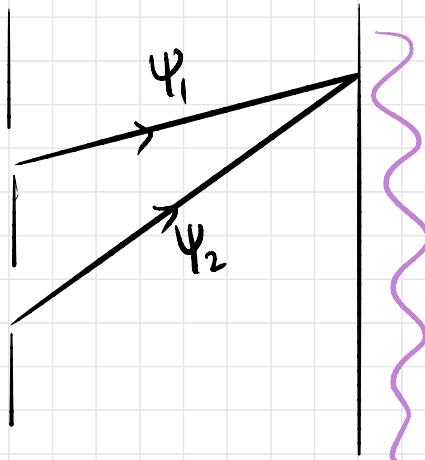
$$\Psi(z,t) = A \sin(wt - kz)$$

We assume a complex wavefunction for simplicity

$$\Psi(x,t) = A e^{i(wt - kz)} \quad \text{— purely mathematical}$$

For interference,

51



$$I = |\Psi_1 + \Psi_2|^2$$

- We associate Ψ_1, Ψ_2 as wavefunctions for each slit (fictitious wave)
- Now, we notice that the intensity observed on the screen perfectly matches $|\Psi_1 + \Psi_2|^2$, and not $\Psi_1^2 + \Psi_2^2$.
- We imagine each particle sends 2 mathematical curves through both slits (the particle kind of "splits")
- Fringe width $\beta = \frac{\lambda D}{d} = \frac{\hbar D}{pd}$; depends on λ
- Ψ is purely mathematical (complex)
 $|\Psi|^2$ is real (probability density)
- e^- knows its surroundings and only goes to areas where the Ψ constructively interferes.

- $|\Psi|^2 = \Psi^* \Psi$, where Ψ^* is complex conjugate
- eg: $\Psi = A e^{ikx}$, $\Psi^* = A e^{-ikx}$
- $|\Psi|^2 = A^2$ ————— observable

Probability density $|\Psi|^2$

- Probability of finding the particle at a particular place when the space $\rightarrow 0$
- To find the probability of finding the particle in a finite place:
 $P = |\Psi|^2 \Delta x$ or $|\Psi|^2 \Delta A$ or $|\Psi|^2 \Delta V$
- $|\Psi|^2$ becomes probability only when multiplied by some dimension.

$$dP = |\Psi|^2 dx$$

- The probability that the particle lies in the region $a \leq x \leq b$ at any given time is given by

$$P_{a \leq x \leq b} = \int_a^b |\Psi|^2 dx$$

- Total sum of probability = 1

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{z=0}^{\infty} |\Psi|^2 dz dy dx = 1$$

This is called **normalisation**

- $|\Psi|^2$ is the probability per unit area/volume/distance
- $|\Psi|^2$ can be greater than 1 as it is density
- But $P_i = |\Psi|^2 dx < 1$ as $|\Psi|^2 dx$ gives probability itself.

Conditions on Ψ

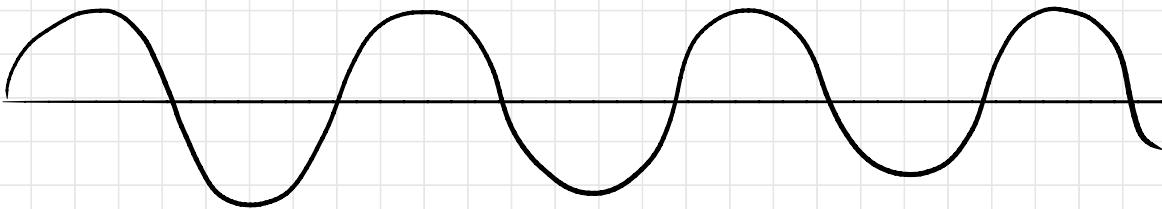
- Ψ must be continuous everywhere (probability must be defined everywhere)
- Ψ must be single-valued (single probability per point)
- Ψ must be finite and as $x \rightarrow \infty$, $\Psi \rightarrow 0$ (due to normalisation)

$$\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} |\Psi|^2 dx dy dz = 1$$
 and Ψ cannot be infinite
- $\frac{\partial \Psi}{\partial x}$, $\frac{\partial \Psi}{\partial y}$, $\frac{\partial \Psi}{\partial z}$ and $\frac{\partial \Psi}{\partial t}$ must be continuous everywhere.
- Ψ must be a solution to Schrödinger's Equation
- Ψ must be **normalisable**

- In EM waves, $A^2 = I$
- In matter waves, A^2 gives probability density (probability of finding the particle)

Consider a sine wave

Amplitude same from $-\infty$ to ∞



To represent matter wave, we look for wave with varying amplitude

We superimpose many sine waves of slightly different frequencies and get wave packets.

Only a mathematical representation; not real

When many waves of slightly different frequencies are superimposed, the resultant is a wave packet / envelop

Construction of Wave Packet

Superimpose waves with slightly different wavelengths.

For simplicity, we consider 2 waves and add / superimpose them

Phase & Group velocities

For any wave $y(x,t) = A \sin(\omega t - kx)$

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{2\pi f \lambda}{2\pi} = f\lambda$$

phase velocity of the wave (how fast phase varies)

Consider two waves of slightly different frequencies

$$y_1 = A \sin(\omega t - kx) \quad (1)$$

$$y_2 = A \sin((\omega + \Delta\omega)t - (k + \Delta k)x) \quad (2)$$

Superimposing (1) and (2)

$$y = A \sin(\omega t - kx) + A \sin((\omega + \Delta\omega)t - (k + \Delta k)x)$$

$$y = 2A \sin\left(\omega t - kx + \frac{\Delta\omega t - \Delta k x}{2}\right) \cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$$

$$y = 2A \sin\left(\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right) \cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$$

$$y = \underbrace{2A \cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)}_{\text{amplitude}} \underbrace{\sin\left(\omega' t - k' x\right)}_{\text{phase}}$$

$$\text{where } \omega' = \omega + \frac{\Delta\omega}{2}$$

$$k' = k + \frac{\Delta k}{2}$$

Amplitude varies with time (amplitude modulation)

There are two velocities in the wave.

Phase velocity

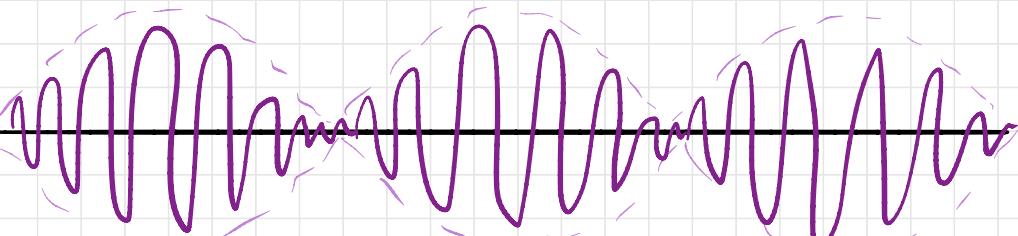
- actual velocity of the wave (how fast one phase moves)
- $\frac{\omega}{k} \approx \frac{\omega}{K} \rightarrow$ high frequency component
- gives us momentum

Group velocity

- velocity of the wave packet / envelope/group
- $\frac{\Delta\omega}{\Delta k} \rightarrow$ low frequency component
- gives us position

If $v_{\text{phase}} = v_{\text{group}}$, wave looks stationary; only horizontal movement is seen.

group/
wave packet



$$v_{\text{group}} = \lim_{\Delta k \rightarrow 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Show that particle velocity = group velocity

$$v_{\text{group}} = \frac{dw}{dk}$$

$$\omega = 2\pi f = \frac{2\pi E}{h}$$

$$\therefore v_{\text{group}} = \frac{dE}{dp}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

$$E = \frac{p^2}{2m} \Rightarrow \frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m} = v_{\text{particle}}$$

$$\therefore v_{\text{particle}} = v_{\text{group}}$$

In a dispersive medium, $v_{\text{phase}} = v_{\text{group}}$

In a non-dispersive medium, $v_{\text{phase}} \neq v_{\text{group}}$

Relationship Between v_{phase} and v_{group}

$$v_{\text{phase}} = \frac{\omega}{k}$$

$$v_{\text{group}} = \frac{dw}{dk}$$

$$\omega = k v_{\text{phase}}$$

$$\frac{dw}{dk} = v_{\text{phase}} + k \frac{dv_{\text{phase}}}{dk}$$

$$V_{\text{group}} = V_{\text{phase}} + \frac{2\pi}{\lambda} \frac{dV_{\text{phase}}}{d\lambda} \frac{d\lambda}{d\mu}$$

$$k = \frac{2\pi}{\lambda} \Rightarrow dk = -\frac{2\pi}{\lambda^2} d\lambda$$

$$V_{\text{group}} = V_{\text{phase}} + \frac{2\pi}{\lambda} \left(\frac{-\lambda^2}{2\pi} \right) \frac{dV_{\text{phase}}}{d\lambda}$$

$$V_{\text{group}} = V_{\text{phase}} - \lambda \frac{dV_{\text{phase}}}{d\lambda}$$

Q: Evaluate the condition under which

$$1) V_{\text{group}} = \frac{1}{2} V_{\text{phase}}$$

$$2) V_{\text{group}} = 2 V_{\text{phase}}$$

$$V_g - V_p = -\lambda \frac{dV_{\text{phase}}}{d\lambda}$$

$$\frac{1}{2} V_p = \lambda \frac{dV_p}{d\lambda}$$

$$\int \frac{1}{2} \frac{d\lambda}{\lambda} = \int \frac{dV_p}{V_p}$$

$$\frac{1}{2} \ln \lambda = \ln V_p + C$$

$$V_p \propto \lambda^{1/2}$$

$$V_g - V_p = -\lambda \frac{dV_p}{d\lambda}$$

$$V_p = -\lambda \frac{dV_p}{d\lambda}$$

$$\int -\frac{d\lambda}{\lambda} = \int \frac{dV_p}{V_p}$$

$$-\ln \lambda = \ln V_p + C$$

$$V_p \propto \frac{1}{\lambda}$$

Q: Phase velocity of ripples on a liquid surface is $\sqrt{\frac{2\pi s}{\lambda \rho}}$ where s is the surface tension, ρ is density.

Find v_g in terms of v_p .

$$v_g = v_p - \lambda \frac{d v_p}{d \lambda}$$

$$\begin{aligned} v_g &= \sqrt{\frac{2\pi s}{\lambda \rho}} - \lambda \sqrt{\frac{2\pi s}{\rho}} \left(-\frac{1}{2} \lambda^{-3/2} \right) \\ &= v_p - \lambda^{3/2} v_p \left(-\frac{1}{2} \lambda^{-3/2} \right) \end{aligned}$$

$$v_g = \frac{3}{2} v_p$$

Q: v_p of ocean waves is $\sqrt{\frac{g \lambda}{2\pi}}$ where g = acc. due to gravity
Find v_g in terms of v_p .

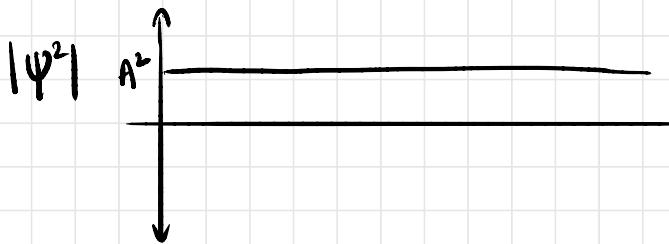
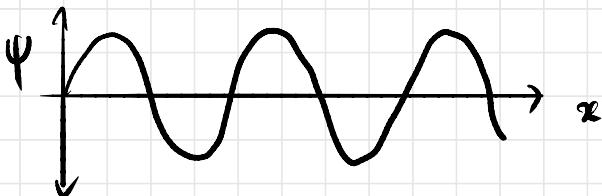
$$v_g = v_p - \lambda \sqrt{\frac{g}{2\pi}} \left(\frac{1}{2} \lambda^{-1} \right) = v_p - \frac{1}{2} v_p$$

$$v_g = \frac{1}{2} v_p$$

HEISENBERG'S UNCERTAINTY PRINCIPLE

According to deBroglie, $\lambda = \frac{h}{mv}$ where λ represents a wave

Let $\Psi = Ae^{ikx}$ which is a wavefunction of a particle and we get $|\Psi|^2 = A^2$



- We know λ exactly $\Rightarrow p$ is exactly known
- $|\Psi|^2 \rightarrow$ probability density is constant everywhere, which means the probability of finding the particle is constant everywhere. Therefore, the position of particle is unknown.
- To find position, we apply fourier transforms.
- We saw by adding two waves, we got packets, but those packets were everywhere.

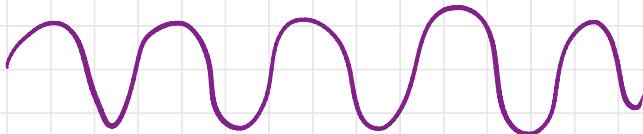
- If we add multiple waves,

$$\Psi_1 - \omega_1, k_1$$



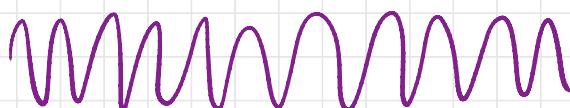
+

$$\Psi_2 - \omega_2, k_2$$



+

$$\Psi_3 - \omega_3, k_3$$

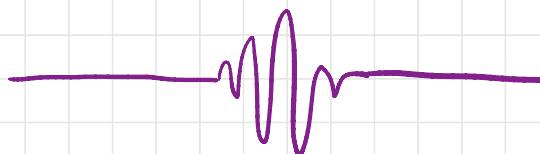


+

:

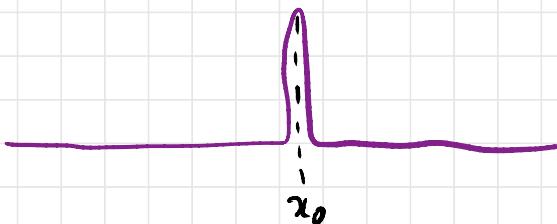
we get **localised packets** (packets only in one position)

$$\Psi =$$



- If we make $\Delta\omega \rightarrow$ very, very small
 $n \rightarrow$ very, very large
 (add many waves), we get

$$\Psi$$



- We know the position of the particle fairly accurately, but since we added so many waves of different λ , the momentum of the particle is unknown.
- Fourier transform gives localised peak called as Dirac-Delta function
- If $\Delta x = 0$, $\Delta p = \infty$
 $\Delta p = 0$, $\Delta x = \infty$

FOURIER INTEGRAL

- more on it later

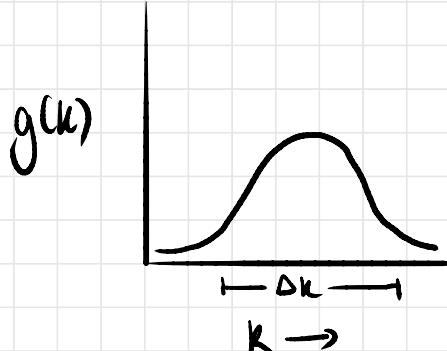
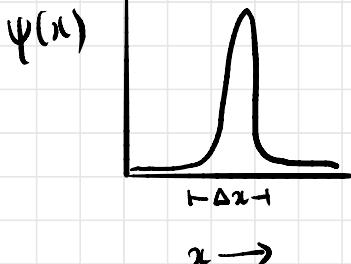
$$\Psi(x) = \int_0^{\infty} g(k) \cos(kx) dk \rightarrow \text{fourier integral}$$

↑
amplitude

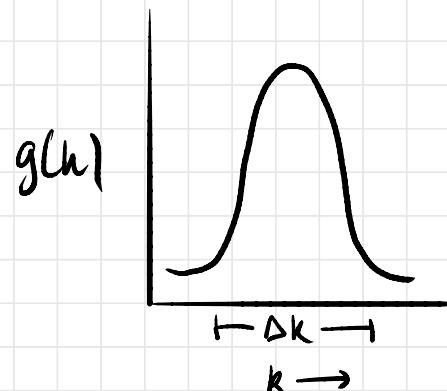
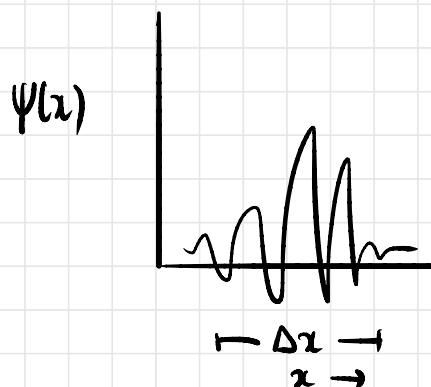
gives us wavefunction

If we take various fourier integral waveforms

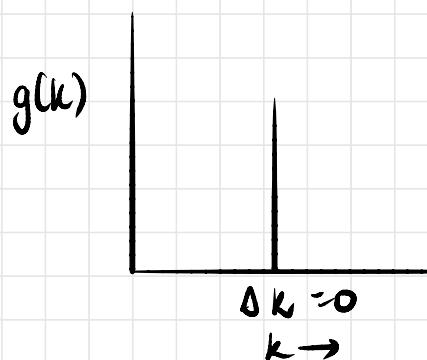
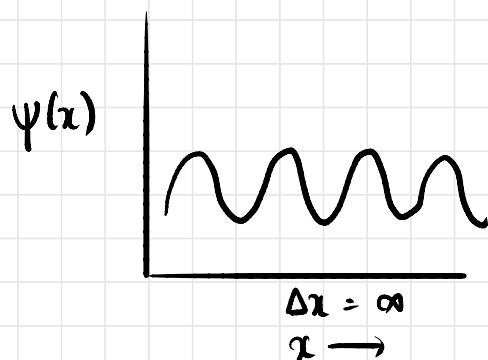
i) pulse



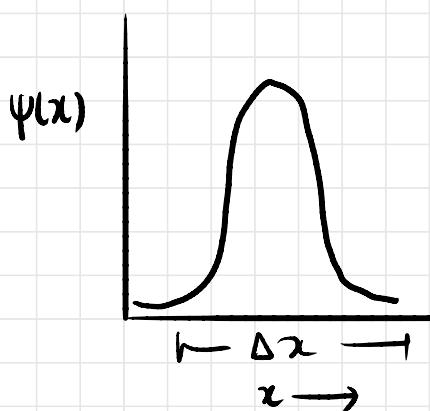
2) Wave group



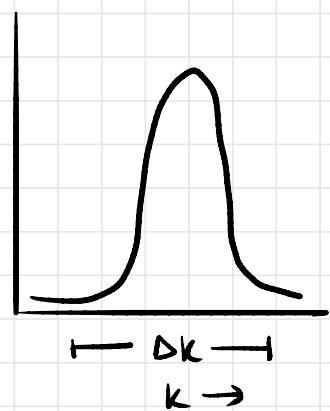
3) Wave train



4) Gaussian



fourier
integral



The product of Δx and Δk is minimum for Gaussian wavepackets.

standard deviation of Δx and Δk , as functions of $\psi(x)$ and $g(k)$, we get $\Delta x \Delta k = \frac{1}{2}$

Generally, wavepackets are not of Gaussian type

$$\Delta x \Delta k \geq \frac{1}{2}$$

$$k = \frac{2\pi}{\hbar} p \Rightarrow \Delta k = \frac{2\pi}{\hbar} \Delta p$$

$$\Delta x \Delta p \geq \frac{\hbar}{4\pi} = \frac{\hbar}{2}$$

Other Uncertainty relations

$$\Delta L \Delta \theta \geq \frac{\hbar}{4\pi} \quad (\text{Angular})$$

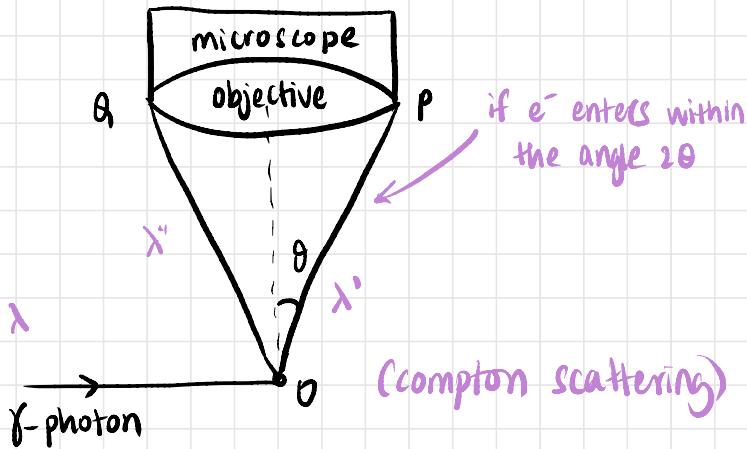
$$\Delta E \Delta t \geq \frac{\hbar}{4\pi} \quad (\text{Energy})$$

Statement: It is impossible to measure momentum and position simultaneous with unlimited precision.

Illustration of Uncertainty Principle

Gamma Ray Microscope

- A thought experiment



- Limit to which position of e^- can be measured is resolving power

$$\Delta x = \frac{\lambda}{2 \sin \theta} \quad \text{--- (1)}$$

- Using Compton scattering, find Δp

Extreme cases

- If photon enters eyepiece through DP (p_{\min})

- Momentum in x-direction

$$\frac{h}{\lambda} + 0 = \frac{h}{\lambda'} \sin \theta + p_{\min}$$

2) If scattered photon enters through OQ (p_{\max})

$$\frac{h}{\lambda} + 0 = \frac{h}{\lambda''} \cos(\theta) + p_{\max}$$

Uncertainty in momentum

Momentum can actually lie between p_{\min} and p_{\max}

$$p_{\max} - p_{\min} = \frac{h}{\lambda'} \sin \theta + \frac{h}{\lambda''} \sin \theta = h \sin \theta \left(\frac{\lambda' + \lambda''}{\lambda' \lambda''} \right)$$

$$\Delta p = \frac{2h \sin \theta}{\lambda}$$

$$\lambda \approx \lambda' \approx \lambda''$$

$$\Delta p = \frac{2h}{\lambda} \sin \theta \quad \text{--- (2)}$$

from (1) and (2)

$$\Delta x \Delta p = \frac{\lambda}{2 \sin \theta} \cdot \frac{2h \sin \theta}{\lambda} = h$$

$$\boxed{\Delta x \Delta p \approx h}$$

Note: $h/4\pi$ comes from a different derivation involving standard deviation and fourier transforms.

Important: Here, we see Δx & Δp are limitations due to our measurement, but in reality these uncertainties are inherent to the particle itself.

Nonexistence of e^- Inside of Nucleus

- Let us assume e^- exists inside nucleus
- If the e^- is part of the nucleus, then the position of the e^- is uncertain to the extent of the nuclear diameter.

$$\Delta x = D = 10^{-14} \text{ m}$$

- According to HUP, $\Delta x \Delta p \geq \frac{\hbar}{4\pi}$

$$\therefore \Delta p \approx 5.27 \times 10^{-21} \text{ kg m s}^{-1}$$

- We know from β -decay studies that the energy of the e^- is about 3-4 MeV.
- We make an assumption that the momentum is of the order of the error
- The minimum momentum of the e^- has to be the uncertainty Δp
- Therefore, $p = \Delta p$
- $E = \frac{p^2}{2m} \approx \frac{(\Delta p)^2}{2m} = 95.48 \text{ MeV}$
- The order of the energy of the e^- we get is out of range of the energy of e^-
- Therefore, the e^- cannot exist inside the nucleus.