LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-5

INTRODUCTION TO EIGEN VALUES AND EIGEN VECTORS

Definition :

Let A be a square matrix of order n. If there exists a real or complex number λ and a non zero vector x such that $Ax = \lambda x$ then x is called the ***Eigen vector of A*** and λ is its corresponding ***Eigen value***.

Note :

- The vector x is in the null space of A- λ I.
- The number λ is chosen so that A- λ I has a null space.
- A- λ I must be singular.
- Det(A- λ I)=0 is called the *characteristic equation of A* and roots of this equation are called *characteristic roots or Eigen values or Latent roots*.

Corresponding to ‘n’ distinct Eigen values we get ‘n’ independent Eigen vectors. But when 2 or more eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to repeated roots.

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Procedure to find eigenvalues and eigenvectors

- Compute the determinant of $A - \lambda I$. With a λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.
- Find the roots of this polynomial. The n roots are the eigenvalues of A.

For each eigenvalue λ, solve the equation $(A - \lambda I)x = 0$. Since the determinant of $A - \lambda I$ is zero, there are solutions other than $x = 0$. Those are the eigenvectors.

Example: Find Eigen values and Eigen vectors

If

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

then the characteristic equation is

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} \right| = \lambda^2 + 3\lambda + 2 = 0$$

and the two eigenvalues are

λ_1=-1, λ_2=-2

Let's find the eigenvector, $\mathbf{v}_1$, associated with the eigenvalue, λ_1=-1, first.

$$\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$(\mathbf{A} - \lambda_1) \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{bmatrix} \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

so clearly from the top row of the equations we get

$$v_{1,1} + v_{1,2} = 0, \quad \text{so}$$

$$v_{1,1} = -v_{1,2}$$

Note that if we took the second row we would get

$$-2 \cdot v_{1,1} + -2 \cdot v_{1,2} = 0, \quad \text{so again}$$

$$v_{1,1} = -v_{1,2}$$

In either case we find that the first eigenvector is any 2 element column vector in which the two elements have equal magnitude and opposite sign.

$$\mathbf{v}_1 = k_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

where k_1 is an arbitrary constant. Note that we didn't have to use +1 and -1, we could have used any two quantities of equal magnitude and opposite sign.

Going through the same procedure for the second eigenvalue:

$$\mathbf{A} \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2$$

$$(\mathbf{A} - \lambda_2) \cdot \mathbf{v}_2 = \begin{bmatrix} -\lambda_2 & 1 \\ -2 & -3-\lambda_2 \end{bmatrix} \cdot \mathbf{v}_2 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0 \quad \text{so}$$

$$2 \cdot v_{2,1} + 1 \cdot v_{2,2} = 0 \quad \left(\text{or from bottom line: } -2 \cdot v_{2,1} - 1 \cdot v_{2,2} = 0\right)$$

$$2 \cdot v_{2,1} = -v_{2,2}$$

$$\mathbf{v}_2 = k_2 \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

Again, the choice of +1 and -2 for the eigenvector was arbitrary; only their ratio is important.

THANK YOU

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