Unit 2 – Vector Spaces

Vector Spaces and Subspaces (definitions only) , Linear Independence, Basis and Dimensions, The Four Fundamental Subspaces. Self Learning Component : Examples of vector spaces and subspaces

15-17	Vector Spaces and Subspaces (Definition only), Column Space and Null Space, Examples
18-20	Echelon Form, Row Reduced Form, Pivot Variables , Free variables
21- 24	Linear Independence, Basis and Dimensions
25	**Scilab Class Number 4 – Span of Column Space of A**
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29	**Scilab Class Number 5 –Four Fundamental Subspaces of A**

Class work Problems:

1	Describe (geometrically) the column space and null space of the following matrices: (do not do a detailed procedure) (i) $\begin{bmatrix} 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ Sample Answer : (i) C(A) = Z , N(A) = R^1
2	Find the column space and null space of $A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix}$. Give an example of a matrix whose column space is the same as that of A but null space is different. Answer : C(A) is a 2d plane in R^3 and N(A) is the origin in R^2. The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{bmatrix}$ has same C(A) but its N(A) is a line in R^3 passing through (1, 1, -1)
3	For which vectors b = (b_1 , b_2 , b_3) do the following systems Ax = b have a solution ? (this is same as finding C(A)) (i) $A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Answer : (i) all b (ii) $b_3 = 0$ (iii) $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}$ (iv) $A=\begin{bmatrix} 1 & 2 \\ 2 & 5 \\ -1 & -2 \end{bmatrix}$ Answer : (iii) $b_2 = 2b_1$, $b_3 = -b_1$, (iv) $b_3 = - b_1$

4	Reduce the matrix $A=\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$ to its echelon form U and hence find its rank. Identify the pivot and free variables. Find the special solutions to Ax = 0 by reducing A to its row reduced form R. Answer : $U=\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$, $R=\begin{bmatrix} 1 & 2 & 0 & 7 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, rank r = 3 Special solutions are (-2, 1, 0, 0, 0) and (-7 , 0, -2 , 1, 0)
5	Reduce these matrices to their echelon form to find their rank. Also find a special solution to each of the free variables. (i) $A=\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$ Answer : (i) (-2, 1, 0, 0, 0) , (0, 0, -2, 1, 0) , (0, 0, -3, 0, 1) (ii) (1, -1 , 1)
6	For every c , find R and the special solutions to Ax = 0 : (i) $A=\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$ Answer : Pl refer Gilbert Strang book.
7	Choose the number q so that (if possible) the ranks are (i) 1 (ii) 2 (iii) 3 $A=\begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$, $B=\begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$ Answer : For A, q = 3 gives rank 1 and every other q gives rank 2. For B , q = 6 gives rank 1 and every other q gives rank 2.
8	Which vectors (b_1 , b_2 , b_3) are in the column space of A ? Which combination of the rows of A give 0 ? (i) $A=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$ (ii) $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ Answer : (i) all b and no combination of the rows (ii) $b_3 = 2b_2$ and $R_3 - 2R_2 = 0$
9	Examine if the following sets of vectors are linearly independent. When the set is dependent find a relation between the vectors 1. { (1, 2, 0) , (0, 3, 1) , (-1, 0, 1) }. Answer : independent 2. { (-1 , 2, 1) , (3 , 0, -1) , (-5, 4, 3) } Answer : dependent, v3 = 2v1 – v2
10	If v_1, v_2, v_3 are vectors and a and b are scalars show that the set { v_1, v_2 , v_3 } is linearly dependent whenever the set { $v_1 + av_2 + bv_3$, v_2 , v_3 } is linearly dependent.
11	Find a maximal linearly independent set of vectors from the set v_1 = (2 , -2 , -4) , v_2 = (1, 9, 3) , v_3 = (-2 , -4 , 1) , v_4 = (3, 7 , -1) Answer : Any two of the given vectors

12	Find the condition on a, b, c so that the vector (a, b, c) belongs to the space spanned by u = (2, 1, 0) , v = (1, -1, 2) and w = (0, 3, -4). Do the vectors u, v, w span R^3 ? Answer: 2a -4b -3c = 0. The vectors u, v, w do not span R^3
13	For what value of λ will the vectors (1, 3, -5) , (0, 5, λ) and (-2, -1, 0) span a two dimensional subspace ? For this value of λ, (i) express (-2, -1, 0) as a linear combination of the other two vectors and (ii) find a vector in R^3 that is not in the span of these vectors. Answer : For λ = -10, the vectors span a 2-d subspace. (i) (-2, -1, 0) = -2 (1, 3, -5) + 1 (0, 5, -10) (ii) Any vector (a, b, c) that satisfies -a+2b+c ≠ 0 will not be in the span of the given vectors.
14	Let v_1 = (1, 2, 1) , v_2 = (3, 1, 5) and v_3 = (3, -4 , 7). Show that the subspaces spanned by { v_1 , v_2 } and { v_1, v_2 , v_3 } are the same.
15	If V_1 = { (a, b, 0) / a, b are real } and V_2 is spanned by (1, 2, 3) and (1, -1 , 1) find a nonzero vector in R^3 that is in both V_1 and V_2. Answer : any multiple of (-2 , 5 , 0)
16	If the set vectors { (1, x , 1) , (x, 1 , 0) ,(0, 1, x) } is linearly dependent find x. Answer : x = 0 , $\pm\sqrt{2}$.
17	Show that the vectors $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$ generate the vector $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of all 2 x 2 matrices.
18	Expand the set { (1, 2, 0) , (1, -2 , 4) } to a basis for R^3 by choosing an appropriate vector from the set { (-2 , 4 , -8) , (2, -1, -5) , (3 , -6 , 0) }. Justify your choice. Answer : (-2 , 4 , -8). The right combination is -2 a + b + c = 0.
19	Find a basis and the dimension of the subspaces V = { (a, b, 0) / a and b are real numbers } , W = { (0, b , c) / b and c are real } and V ∩ W. Answer : Basis for V is { (1, 0, 0) , (0, 1, 0) } , dim V = 2 Basis for W is { (0, 1, 0) , (0, 0, 1) } , dim W = 2 Basis for V ∩ W is { (0, 1 , 0) } , dim = 1
20	Let V = { (a, b, c, d) / b + c + d = 0 } and W = { (a, b, c, d) / a + b = 0 and c = 2d } be subspaces of R^4. Find the dimension of V ∩ W. Answer : (3, -3 , 2 , 1) is a vector in the intersection. Dim = 1.
21	If the column space of A is spanned by the vectors (1, 4, 2) , (2, 5, 1) and (3, 6, 0) find all those vectors that span the left null space of A. Determine whether or not the vector b = (4, -2, 2) is in that subspace. What are the dimensions of C (A^T) and N (A^T) ? Answer: Solutions of $A^T x = 0$ are (2, -1, 1). The vector b is in N(A^T). Dim C(A^T) = 2 and dim N(A^T) = 1
22	Find the four fundamental subspaces , their dimensions and a basis given A = $\begin{bmatrix} 1 & -1 & 2 & -2 & 3 \\ -2 & 2 & 0 & 4 & -2 \\ 0 & 3 & 1 & -1 & 6 \\ -1 & -2 & -3 & 3 & -9 \end{bmatrix}$ Answer : Basis for C(A) is columns 1, 2, 3 ; Basis for C(A^T) is rows 1, 2, 3 Basis for N(A) is { (7, 1, 0, 3, 0) , (-8, -5, -3, 0, 3) } ; N(A^T) = { c (1, 0, 1, 1) }

23	Obtain the four fundamental subspaces of A = $\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ Answer : Basis for C(A) is columns 1, 2, 3, 4 ; Basis for C(A^T) is all four rows ; Basis for N(A) is (0,-1, -1 , 0,1) and N(A^T) = Z.
24	Find a left inverse / right inverse for (i) A = $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (ii) A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ Answer : (i) $\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$