we non-periodic

2) Find the Fourier series of $f(x) = x\sin x$ in $(-\pi, \pi)$. or $-\pi \le x \le \pi$

Sol: $\because$ F.S. of $f(x)$ having the period 2π is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Here $f(-x) = -x\sin(-x) = -(x)[-\sin x] = x\sin x$ $\therefore f(x)$ is even, $\therefore b_n = 0$ & $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ & $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\,\underset{v}{\sin x}\,dx \qquad \therefore \boxed{a_0 = 2}$$

$$= \frac{2}{\pi}\Big[x(-\cos x) - (1)(-\cancel{\sin x})^{0}\Big]_0^{\pi} = \frac{-2}{\pi}\left[\pi(-1)^{1} - 0\right] = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx \qquad \sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$a_n = \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\left\{x\sin(1+n)x + x\sin(1-n)x\right\}dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \underset{I_1}{x\sin(n+1)x} - \underset{I_2}{x\sin(n-1)x}$$

$\sin(n+1)\pi = 0$
$\sin(n-1)\pi = 0$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1}\right] - 1\left[-\frac{\cancel{\sin(n+1)x}}{(n+1)^2}\right]\right\}_0^{\pi} - \frac{1}{\pi}\left\{+x\left[\frac{-\cos(n-1)x}{n-1}\right] - \left[\frac{-\cancel{\sin(n-1)x}}{(n-1)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi(-1)^{n+1}}{n+1} - 0\right\} + \frac{1}{\pi}\left\{\frac{\pi(-1)^{n-1}}{n-1}\right\} \quad \text{for } n \ne 1$$

$$= \frac{\pi}{\pi}(-1)^n\left\{+\frac{1}{n+1} - \frac{1}{n-1}\right\} = (-1)^n\left[\frac{n-1-(n+1)}{n^2-1}\right] = (-1)^n\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right]$$

$$= \frac{-2(-1)^n}{n^2-1} \quad \text{or} \quad \boxed{a_n = \frac{2(-1)^{n+1}}{n^2-1}} \quad \text{for } n \ne 1$$

B-2

To find a_n when $n=1$:

W.K.T $a_n = \frac{2}{\pi}\int_0^{\pi} \cancel{x}\sin x \cos nx\, dx$

$\therefore$ for $n=1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\, dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx.$

$a_1 = \frac{1}{\pi}\left\{x\left(\frac{-\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$ $\quad \cos 2\pi = 1$

$= \frac{-1}{2\pi}\left[x\cos 2x\right]_0^{\pi} = \frac{-1}{2\pi}\left[\pi - 0\right] \Rightarrow \boxed{\frac{-1}{2} = a_1}$

$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$

$f(x) = \frac{2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$

$$\boxed{x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx}$$

3) $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$. Hence evaluate

$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

$1-\cos x = 2\sin^2\frac{x}{2}$

$\sqrt{1-\cos x} = \sqrt{2}\times\sin\frac{x}{2}$

Sol. FS for $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1} b_n \sin nx.$

Here $f(x) = \sqrt{1-\cos x}$ $\therefore f(2\pi - x) = \sqrt{1-\cos(2\pi - x)}$

$= \sqrt{1-\cos x}$

$= f(x)$

$\boxed{b_n = 0}$ & $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx.$ $\Rightarrow f(x)$ is even $\therefore b_n = 0.$

6a) $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$ hence deduce that
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$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

Sol:- $f(x) = \sqrt{1-\cos x} \Rightarrow f(2\pi - x) = \sqrt{1-\cos(2\pi - x)} = \sqrt{1-\cos x} = f(x)$

$\therefore f(x)$ is is even & $b_n = 0$ $\qquad 1-\cos x = 2\sin^2\frac{x}{2}$

$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\, dx = \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \sin\frac{x}{2}\, dx = \frac{2\sqrt{2}}{\pi}\left[-\cos\frac{x}{2} \times 2\right]_0^{\pi}$

$a_0 = \frac{4\sqrt{2}}{\pi}\left[0 - (-1)\right] = \frac{4\sqrt{2}}{\pi}$

$a_n = \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\frac{x}{2} \times \cos nx\, dx = \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \cos nx \sin\frac{x}{2}\, dx$

$\boxed{\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]}$

$a_n = \frac{\cancel{2}\sqrt{2}}{\pi} \times \frac{1}{\cancel{2}}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right] dx$

$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{\pi}$

$\cos\left(n+\frac{1}{2}\right)\pi = \cos\left(\frac{\pi}{2}+n\pi\right) = -\sin n\pi = 0$

$\cancel{\cos\left(n-\frac{1}{2}\right)\pi = -\sin\left(\frac{\pi}{2}-n\pi\right) = -(\cos n\pi) = (-1)^n}$

$\cos\left(n-\frac{1}{2}\right)\pi = \cos\left(\frac{\pi}{2}-n\pi\right) = \sin n\pi = 0$

$= \frac{\sqrt{2}}{\pi}\left\{\left[0+0\right] - \left[\frac{-1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right]\right\} = \frac{\sqrt{2}}{\pi}\left[\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right]$

$= \frac{\sqrt{2}}{\pi}\left[\frac{\cancel{n}-\frac{1}{2}-\cancel{n}-\frac{1}{2}}{n^2-\frac{1}{4}}\right] = -\frac{\sqrt{2}}{\pi} \times \frac{4}{4n^2-1} \Rightarrow \boxed{\frac{-4\sqrt{2}}{\pi(4n^2-1)} = a_n}$

$\sqrt{1-\cos x} = f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n\geq 1}^{\infty} \cos nx \times \frac{1}{(4n^2-1)}$

put $x = 0$ $\quad \sqrt{1-1} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

5) Obtain the FS of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Sol: The FS of $f(x)$ having period as 2π is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where

$$f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{-\pi + x}{2} = \frac{x - \pi}{2} = -\frac{(\pi - x)}{2}$$

$\therefore f(2\pi - x) = -f(x) \Rightarrow f(x)$ is odd $\therefore a_0 = a_n = 0$

$$\& \ b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi - x}{2}\right) \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{0 - \pi\left(\frac{-1}{n}\right)\right\} = \frac{1}{n} \Rightarrow \boxed{b_n = \frac{1}{n}}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

$$\therefore \frac{\pi - x}{2} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots$$

Put $\boxed{x = \frac{\pi}{2}}$, $\frac{\pi}{4} = 1 + 0 + \frac{1}{3} \times (-1) + 0 + \frac{1}{5}(1) + \cdots$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots}$$

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