Question and answers

1. Solve the following system by the method of Gaussian elimination

$$x + 2y - z = 6$$

$$2x + y + z = 3$$

$$x - y + z = -2$$

Solution:

The augmented matrix is given by

$$[A: b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 1 & -2 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & -3 & 3 & -9 \\ 0 & -3 & 2 & -8 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & -3 & 3 & -9 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

The pivot elements are $a_{11} = 1, a_{22} = -3, a_{33} = -1$

From the above matrix: $$x + 2y - z = 6$$

$$-3y + 3z = -9$$

$$-z = 1$$

From the above equations one can get$x = 1,\ y = 2,\ z = -1.$

2. Find LU and LDU factorization for $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

Solution: $R_2 \to R_2 - \frac{2}{3}R_1; R_3 \to R_3 - \frac{1}{3}R_1$

$$\sim\begin{bmatrix}3 & 1 & 2\\ 0 & -\frac{11}{3} & -\frac{7}{3}\\ 0 & \frac{5}{3} & \frac{1}{3}\end{bmatrix}$$

$$R_3 \to R_3 + \frac{5}{11}R_2$$

$$\sim\begin{bmatrix}3 & 1 & 2\\ 0 & -\frac{11}{3} & -\frac{7}{3}\\ 0 & 0 & -\frac{24}{33}\end{bmatrix} = U$$

$$L = \begin{bmatrix}1 & 0 & 0\\ \frac{2}{3} & 1 & 0\\ \frac{1}{3} & \frac{-5}{11} & 1\end{bmatrix}$$

$A = LU$ & $A = LDU, D$is the diagonal matrix of pivots, Here L and U have 1's in the diagonal.

Therefore divide each row of U by its pivot.

$$A = LDU = \begin{bmatrix}1 & 0 & 0\\ \frac{2}{3} & 1 & 0\\ \frac{1}{3} & \frac{-5}{11} & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\ 0 & -\frac{11}{3} & 0\\ 0 & 0 & -\frac{24}{33}\end{bmatrix}\begin{bmatrix}1 & \frac{1}{3} & \frac{2}{3}\\ 0 & 1 & \frac{7}{11}\\ 0 & 0 & 1\end{bmatrix}$$

3. . Investigate the values of λ and μ such that

$$x + 3y + 5z = 9$$

$$x - y + 2z = 1$$

$$2x + 2y + \lambda z = \mu$$

has (i) unique solution (ii) infinitely many solution (iii) no solution

Solution

The augmented matrix is given by

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & \lambda & \mu \end{array}\right]$$

$$R_2 \to R_2 - R_1; R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -4 & -3 & -8 \\ 0 & -4 & \lambda - 10 & \mu - 18 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -4 & -3 & -8 \\ 0 & 0 & \lambda - 7 & \mu - 10 \end{array}\right]$$

Unique solution:

From the Echelon form it is seeing that when$\lambda \neq 7, r(A) = 3$ & $r(A:b) = 3 = n$

hence we get a unique solution when$\lambda \neq 7$ [μ can be any real number].

No Solution

For the system to have no solution $r(A) \neq r(A:b)$. Thus $r(A)$ must be 2 and $r(A:b)$ must be 3.

For this to happen λ should be equal to 7 ($r(A) = 2$) and $\mu - 10 \neq 0$ i.e.,

$\mu \neq 10$ ($r(A:b) = 3$).

Many solutions

For the system to have infinitely many solution we should have

$r(A) = r(A:b) \neq n$. Thus $r(A) = 2,\ r(A:b) = 2$.

For this to happen $\lambda = 7,\ \mu = 10$

4. Obtain the inverse of A (or) use Gauss-Jordan method to solve

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution:

$$[A:I] = \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 2 & -1 & 1 & : & 0 & 1 & 0 \\ 1 & 3 & -1 & : & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & -3 & -1 & : & -2 & 1 & 0 \\ 0 & 2 & -2 & : & -1 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{2}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & -3 & -1 & : & -2 & 1 & 0 \\ 0 & 0 & -\frac{8}{3} & : & -\frac{7}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{3}{8}R_3; R_1 \to R_1 + \frac{3}{8}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & \frac{1}{8} & \frac{2}{8} & \frac{3}{8} \\ 0 & -3 & 0 & : & -\frac{9}{8} & \frac{6}{8} & -\frac{3}{8} \\ 0 & 0 & -\frac{8}{3} & : & \frac{7}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$R_1 \to R_1 + \frac{1}{3}R_2$$

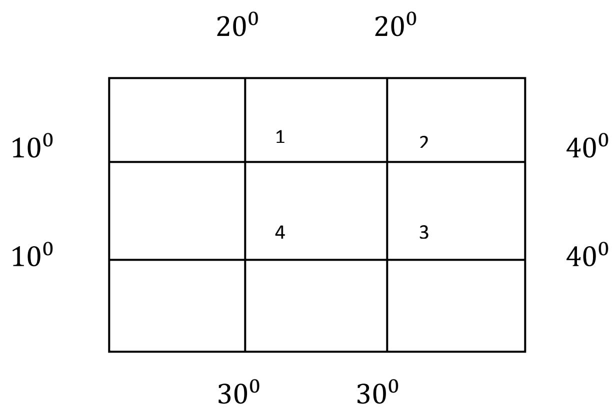
$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ 0 & -3 & 0 & : & -\frac{9}{8} & \frac{6}{8} & -\frac{3}{8} \\ 0 & 0 & -\frac{8}{3} & : & \frac{7}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$R_2 \to -\frac{1}{3}R_2; R_3 \to -\frac{3}{8}R_3$$

$$\sim\left[\begin{array}{ccccccc} 1 & 0 & 0 & : & -\frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ 0 & 1 & 0 & : & \frac{3}{8} & -\frac{2}{8} & \frac{1}{8} \\ 0 & 0 & 1 & : & \frac{7}{8} & -\frac{2}{8} & -\frac{3}{8} \end{array}\right]$$

$$I : A^{-1}$$

5. Assume that the plate shown in the fig. represents a cross section of a metal beam with negligible heat flow in the direction perpendicular to the plate. Let T_1, T_2, T_3 & T_4 denote the temperature at the 4 interior nodes of the mesh. The temperature at the node is approximately equal to the average of the 4 nearest nodes to the left, right, above and below. Write a system of 4 equation whose solution gives estimates for the temperature T_1, T_2, T_3 & T_4. Hence find its solution.



Solution:

$$T_1 = \frac{10+20+T_2+T_4}{4}$$

$$4T_1 = 30 + T_2 + T_4$$

$$4T_1 - T_2 - T_4 = 30 \ldots \ldots (1)$$

$$T_2 = \frac{20 + 40 + T_1 + T_3}{4}$$

$$4T_2 = 60 + T_1 + T_3$$

$$-T_1 + 4T_2 - T_3 = 60 \ldots \ldots (2)$$

$$T_3 = \frac{T_2 + T_4 + 40 + 30}{4}$$

$$4T_3 = T_2 + T_4 + 70$$

$$-T_{2+}4T_3 - T_4 = 70 \ldots . (3)$$

$$T_4 = \frac{\mathrm{T}_1 + \mathrm{T}_3 + 30 + 10}{4}$$

$$-T_1 - T_3 + 4T_4 = 40 \ldots \ldots . (4)$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} \mathrm{T}_1 \\ \mathrm{T}_2 \\ \mathrm{T}_3 \\ \mathrm{T}_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \\ 70 \\ 40 \end{bmatrix}$$

$$[A:d] = \begin{bmatrix} 4 & -1 & 0 & -1 & : & 30 \\ 0 & {}^{15}/_{4} & -1 & {}^{-1}/_{4} & : & {}^{135}/_{2} \\ 0 & 0 & {}^{56}/_{15} & {}^{-16}/_{15} & : & 88 \\ 0 & 0 & 0 & {}^{24}/_{7} & : & {}^{540}/_{7} \end{bmatrix}$$

$T_1 = 20,\ \ T_2 = 27.5,\ \ T_3 = 30,\ \ T_4 = 22.5$