Uncertainties in Least Squares Coefficients

Consider Bivariate data $(x_{i,} y_i)$ for i=1,2,3,…………….n

$$y = \beta_0 + \beta_1 x$$

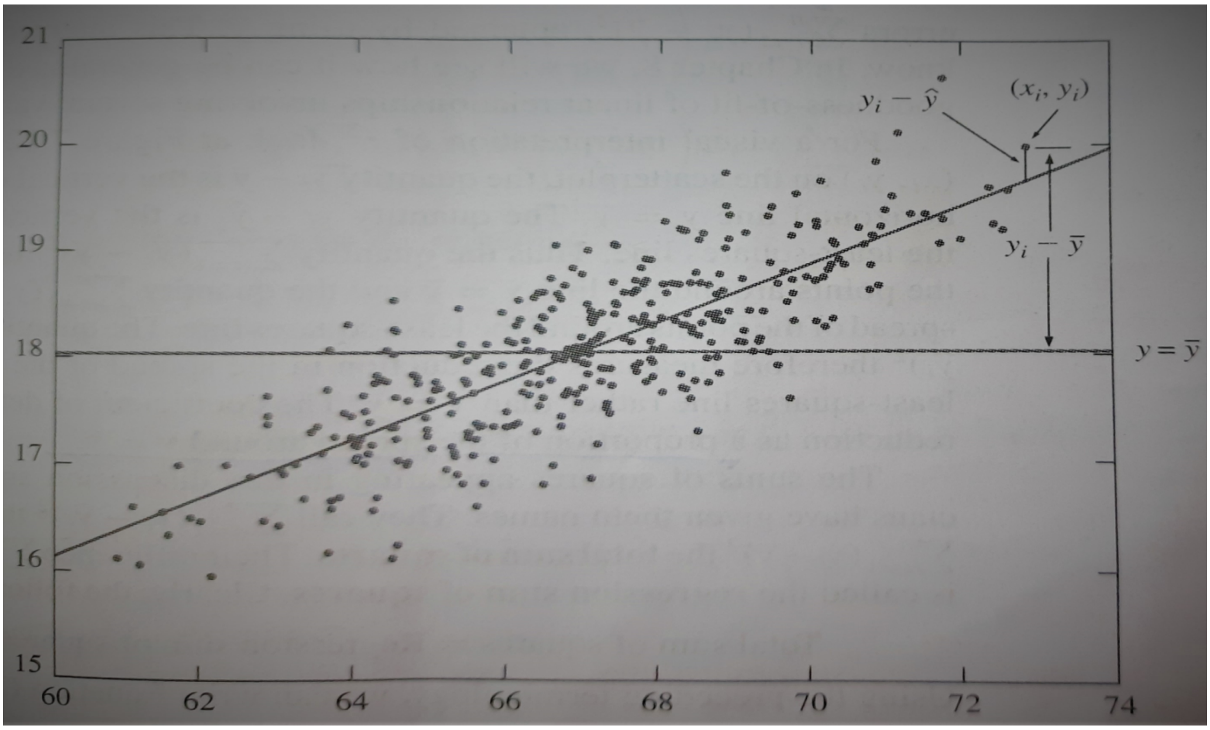
The line $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, ε_i is the error, that best fits the data in the sense of minimizing the sum of the squared errors. It is called the least squares regression line

$\widehat{\beta_0}$, $\widehat{\beta_1}$ are estimates of β_0 , β_1 .

$\widehat{y_i} = \widehat{\beta_0} + \widehat{\beta_1} x_i$ = Fitted line

If ε_i tend to be large then $(x_{i,} y_i)$ are widely scattered around the line.

If ε_i tend to be small then $(x_{i,} y_i)$ are tightly clustered around the line.



the quantities $\widehat{\beta_0}$, $\widehat{\beta_1}$ are obtained from

$$S = \sum_{i=1}^{n} e_i^{\,2} = \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = \text{Minimum}$$

Where $\frac{\partial S}{\partial \widehat{\beta}_0} = 0$ and $\frac{\partial S}{\partial \widehat{\beta}_1} = 0$

$\widehat{\beta_0}$, $\widehat{\beta_1}$ are called Least Squares Coefficients and defined as

$\widehat{\beta_1} = \sum_{i=1}^{n} \left[\frac{x_i - \bar{x}}{\sum_{i-1}^{n}(x_i - \bar{x})^2} \right] y_i$

$\widehat{\beta_0} = \sum_{i=1}^{n} \left[\frac{1}{n} - \frac{\bar{x}\,(x_i - \bar{x})}{\sum_{i-1}^{n}(x_i - \bar{x})^2} \right] y_i$

Linear combination of the points x, y, z is

$P = c_1 x + c_2 y + c_3 z$, where $c_{1,}\ c_{2,}\ c_3$ are constants

This indicates that $\widehat{\beta_0}$, $\widehat{\beta_1}$ are linear combination of ${}_i$.

Since each time the experiment is repeated, the values ε_i ,

$\widehat{\beta_0}$, $\widehat{\beta_1}$ will also be different.

Hence, the quantities ε_i , $\widehat{\beta_0}$, $\widehat{\beta_1}$ are random in nature.

The error ε_i creates Uncertainty in the estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$.

Uncertainty in the estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ is the standard deviation.

The spread of the points can be measured by the sum of the squared residuals as

The estimate of the error variance, $s^2 = \frac{\sum_{i=1}^{n} e_i^{\,2}}{n-2} = \frac{\sum (y_i - \widehat{y}\,)^2}{n-2}$

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum_{i-1}^{n}(y_i - \bar{y})^2 * \sum_{i-1}^{n}(x_i - \bar{x}\,)^2}$$

$$s^2 = \frac{(1 - r^2)\ \sum (y_i - \bar{y})^2}{n - 2}$$

The line $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ has Normal distribution with

$$\mu_{y_i} = \beta_0 + \beta_1 x_i$$

$$\sigma_{yi}^{\ 2} = \sigma^2$$

$\widehat{\beta_1} = \sum_{i=1}^{n} \left[\frac{x_i - \bar{x}}{\sum_{i-1}^{n}(x_i - \bar{x})^2} \right] yi$

$\widehat{\beta_0} = \sum_{i=1}^{n} \left[\frac{1}{n} - \frac{\bar{x}\,(x_i - \bar{x})}{\sum_{i-1}^{n}(x_i - \bar{x})^2} \right] yi$

Mean of the estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ are

$\mu_{\widehat{\beta_0}} = \beta_0 \qquad \mu_{\widehat{\beta_1}} = \beta_1$

Uncertainty in the estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ are

$$\sigma_{\widehat{\beta_0}} = \sigma\sqrt{\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

$$\sigma_{\widehat{\beta_1}} = \sigma\sqrt{\left[\frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

Since the value of σ is unknown it is approximated with s

$$s_{\widehat{\beta_0}} = s\sqrt{\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

$$s_{\widehat{\beta_1}} = s\sqrt{\left[\frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

Where s is the estimate of the error standard deviation σ and

$$s = \sqrt{\frac{(1-r^2)\sum(y_i-\bar{y})^2}{n-2}}$$

Problem: A chemical reaction is ran 12 times. The temperature and yield is recorded each time.

$\bar{x} = 65$ $\quad \bar{y} = 29.05$ $\quad \sum_{i-1}^{n}(x_i - \bar{x})^2 = 6032$

$\sum_{i-1}^{n}(y_i - \bar{y})^2 = 835.42$ $\quad \sum_{i-1}^{n}(x_i - \bar{x})(y_i - \bar{y}) =$

1988.4 Compute the least squares estimates, error variance estimate.

Sol: $\widehat{\beta_0} = 7.6234$ $\quad \widehat{\beta_1} = 0.32964$

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum_{i-1}^{n}(y_i - \bar{y})^2 * \sum_{i-1}^{n}(x_i - \bar{x})^2} = 0.8858$$

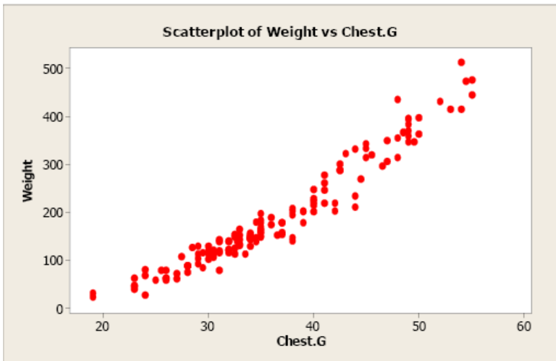
$$s = \sqrt{\frac{(1-r^2)\sum(y_i-\bar{y})^2}{n-2}} \quad \text{then} \quad s^2 = 17.99$$

$$s_{\widehat{\beta_0}} = s\sqrt{\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]} \qquad s_{\widehat{\beta_1}} = s\sqrt{\left[\frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

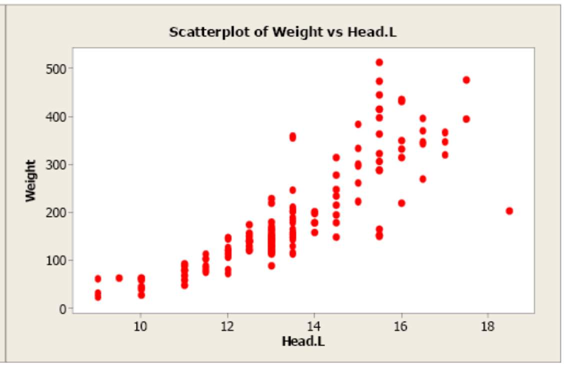
$$s_{\widehat{\beta_1}} \ \alpha \ \frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}$$

If x – values are more spread then the uncertainty of estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ are Smaller.

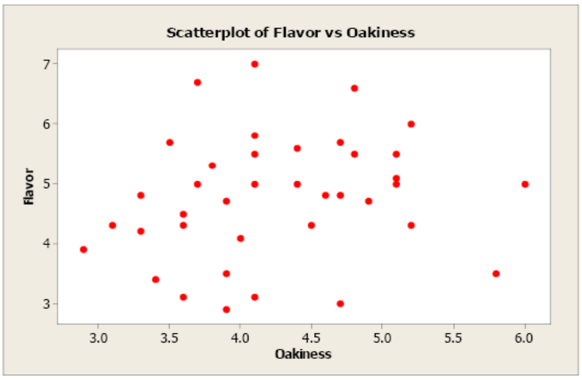
The standard deviation of x is more.



Strong positive relationship
r = 0.96



Moderate positive relationship
r = 0.67



Very weak positive relationship
r = 0.07

Problem: Two engineers are conducting independent experiments to estimate spring constant for a particular spring. The first engineer suggests measuring the length of the spring with no load, then applying loads of 0,1,2,3,& 4 lb. The second engineer suggests using loads of 0, 2, 4, 6 & 8 lb. Which will be more precise?

Sol: X ----- 0, 1, 2, 3, 4 Y-------- 0, 2, 4, 6, 8

σ_y is twice as great as σ_x .

Uncertainty of X is twice as large as the uncertainty of Y. Hence, the Engineer, Y 's estimate is twice as precise.