Handout 10

Examples on Maximum Likelihood

Let $X_1, \ldots, X_n$ be a random sample from a population with the Poisson(λ) distribution. Find the MLE of λ.

The joint probability mass function of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n; \lambda) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda_i^x}{x_i!} = e^{-n\lambda} \frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}.$$

The MLE is the value of λ that maximizes $f(x_1, \ldots, x_n; \lambda)$, or equivalently, $\ln f(x_1, \ldots, x_n; \lambda)$.

$$\frac{d}{d\lambda} \ln f(x_1, \ldots, x_n; \lambda) = \frac{d}{d\lambda}(-n\lambda + \sum_{i=1}^{n} x_i \ln \lambda - \sum_{i=1}^{n} \ln x_i!) = -n + \frac{\sum_{i=1}^n x_i}{\lambda} = 0.$$

Now we solve for λ:

$$-n + \frac{\sum_{i=1}^n x_i}{\lambda} = 0.$$

$$\lambda = \frac{\sum_{i=1}^n x_i}{n} = \overline{x}.$$

The MLE is $\widehat{\lambda} = \overline{X}$.

Maximum likelihood estimates possess the property of *functional invariance*, which means that if $\widehat{\theta}$ is the MLE of θ, and $h(\theta)$ is any function of θ, then $h(\widehat{\theta})$ is the MLE of $h(\theta)$.

a. Let $X \sim \text{Bin}(n, p)$ where n is known and p is unknown. Find the MLE of the odds ratio $p/(1-p)$.

(a) The probability mass function of X is $f(x;\, p) = \dfrac{n!}{x!(n-x)!}(p)^x(1-p)^{n-x}$.

The MLE of p is the value of p that maximizes $f(x;\, p)$, or equivalently, $\ln f(x;\, p)$.

$$\frac{d}{dp}\ln f(x;\, p) = \frac{d}{dp}[\ln n! - \ln x! - \ln(n-x)! + x\ln p + (n-x)\ln(1-p)] = \frac{x}{p} - \frac{n-x}{1-p} = 0.$$

Solving for p yields $p = \dfrac{x}{n}$. The MLE of p is $\widehat{p} = \dfrac{X}{n}$.

The MLE of $\dfrac{p}{1-p}$ is therefore $\dfrac{\widehat{p}}{1-\widehat{p}} = \dfrac{X}{n-X}$.

Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu,\, 1)$ population. Find the MLE of μ.

The joint probability density function of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n;\, \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2} = (2\pi)^{-n/2} e^{-\sum_{i=1}^{n}(x_i-\mu)^2/2}.$$

The MLE is the value of μ that maximizes $f(x_1, \ldots, x_n;\, \mu)$, or equivalently, $\ln f(x_1, \ldots, x_n;\, \mu)$.

$$\frac{d}{d\mu}\ln f(x_1, \ldots, x_n;\, \mu) = \frac{d}{d\mu}\left[-(n/2)\ln 2\pi - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2}\right] = \sum_{i=1}^{n}(x_i-\mu) = 0.$$

Now we solve for μ:

$$\textstyle\sum_{i=1}^{n}(x_i-\mu) = \sum_{i=1}^{n} x_i - n\mu = 0, \text{ so } \mu = \dfrac{\sum_{i=1}^{n} x_i}{n} = \overline{x}.$$

The MLE of μ is $\widehat{\mu} = \overline{X}$.

Let $X_1, \ldots, X_n$ be a random sample from a $N(0,\, \sigma^2)$ population. Find the MLE of σ.

The joint probability density function of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n; \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-x_i^2/2\sigma^2} = (2\pi)^{-n/2}\sigma^{-n}e^{-\sum_{i=1}^{n} x_i^2/2\sigma^2}.$$

The MLE is the value of σ that maximizes $f(x_1, \ldots, x_n; \sigma)$, or equivalently, $\ln f(x_1, \ldots, x_n; \sigma)$.

$$\frac{d}{d\sigma}\ln f(x_1, \ldots, x_n; \sigma) = \frac{d}{d\sigma}\left[-(n/2)\ln 2\pi - n\ln\sigma - \sum_{i=1}^{n}\frac{x_i^2}{2\sigma^2}\right] = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} x_i^2}{\sigma^3} = 0.$$

Now we solve for σ:

$$-\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} x_i^2}{\sigma^3} = 0$$

$$-n\sigma^2 + \sum_{i=1}^{n} x_i^2 = 0$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$$

The MLE of σ is $\widehat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} X_i^2}{n}}$.