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## VECTOR SPACES

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**Deepthi Rao**

Department of Science & Humanities

# CLASS 10 : CONTENT

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- Problems on Four Fundamental Subspaces

# FUNDAMENTAL SUBSPACES

Problem 1: Find the Basis and Dimension for the four fundamental subspaces.

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$$

Solution :- Step I :- Apply Gauss Elimination and reduce the matrix to Echelon form.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 - 3R_1}} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# FUNDAMENTAL SUBSPACES

Column Space :  $C(A)$

$$C(A) = \left\{ c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

$C(A)$  is a 2 dimensional space spanned by

$(1, 1, 3)$  and  $(2, 3, 7)$  in  $\mathbb{R}^3$ .

$$\text{Basis of } C(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}$$

Dim of  $C(A) = 2$ .

Note: Select columns of  $A$  and not from  $U$  to span  $C(A)$   
 $\because$  Columns are not Preserved in elementary Row operations.

## CLASS 10 : CONTENT

Row space :  $C(A^T)$

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

Or

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

$C(A^T)$  is a 2 dim plane spanned by the linear combination of vectors  $(1, 2, 1, 2)$  and  $(0, 0, 0, 1)$  in  $\mathbb{R}^4$ .

Basis  $C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \right\}$  Dimension of  $C(A^T) = 2$

Note: Either non zero rows of A or U can be selected to span  $C(A^T)$

# FUNDAMENTAL SUBSPACES

Null Space :  $N(A)$

Special solutions to  $Ax=0 \Rightarrow Ux=0$  forms the basis of  $N(A)$ .

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x + 2y + y + 2t = 0 \Rightarrow x = -2y - z, t = 0$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \sim \begin{pmatrix} -2y - z \\ y \\ 0 \\ 0 \end{pmatrix} \sim y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

# FUNDAMENTAL SUBSPACES

$N(A)$  is a 2 dim Plane spanned by linear combination of vectors  $(-2, 1, 0, 0)$  and  $(-1, 0, 1, 0)$  in  $\mathbb{R}^4$ .

Basis of  $N(A)$  :  $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$  Dimension = 2

$N(A) = \left\{ y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; y, z \in \mathbb{R} \text{ (as } y, z \text{ are free variables)} \right\}$ .

Left Null space :  $N(A^T)$  :-

To find  $N(A^T)$  Solve for  $A^T y = 0 \Rightarrow y^T A = 0$ .

# FUNDAMENTAL SUBSPACES

$$A = \left( \begin{array}{cccc} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{array} \right) \quad R_1 \\ R_2 \\ R_3$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{array} \right) \quad R_1 \\ R_2 \\ R_3$$

$$R_2 - R_1, \quad R_3 - 3R_1$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 1 & b_3 - 3b_1 \end{array} \right)$$

$$R_3 - R_2$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & (b_3 - 3b_1) - (b_2 - b_1) \end{array} \right) \sim \left( \begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array} \right)$$

## FUNDAMENTAL SUBSPACES

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$$N(A^T) = \left\{ c_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}; c_1 \in \mathbb{R} \right\}$$

$N(A^T)$  is a line spanned by  $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$  in  $\mathbb{R}^3$ .

$$\text{Basis for } N(A^T) = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Dimension for  $N(A^T) = 1$

Note:- To find the left Null Space, find the combination of the rows of  $A$  which produces zero rows. (instead of finding  $A^T$ )

# FUNDAMENTAL SUBSPACES

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Problem 2 : Describe the Column space and the Null space for the following matrices.

1.  $\begin{bmatrix} 0 \end{bmatrix}$

$$C(A) = \mathbb{Z} ; N(A) = \mathbb{R}$$

2.  $\begin{bmatrix} 0, -3 \end{bmatrix}$

$$C(A) = \mathbb{R} ; N(A) = \{x \text{ axis in } \mathbb{R}^2\}$$

3.  $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$C(A) = \{y \text{ axis in } \mathbb{R}^2\} ; N(A) = \mathbb{Z}$$

# FUNDAMENTAL SUBSPACES

4.  $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$C(A) = \{x \text{ axis in } \mathbb{R}^2\}; N(A) = \{ \text{line } x=y \text{ in } \mathbb{R}^2 \}$

5.  $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A) = \{x \text{ axis in } \mathbb{R}^2\}; N(A) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}; x \in \mathbb{R} \right\}$

$N(A)$  is line spanned by  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  in  $\mathbb{R}^3$

6.  $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$

$C(A) = \{ \text{whole of } \mathbb{R}^2 \}; N(A) = \{ \text{origin } (0,0) \text{ in } \mathbb{R}^2 \}$

## FUNDAMENTAL SUBSPACES

Problem 3: Find the Column Space and the Null space of  $A = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{pmatrix}$ . Give an example of a matrix whose  $C(A)$  is same as that of  $A$  but the Null space is different.

Solution:-  $C(A)$  is 2 dim plane in  $\mathbb{R}^3$

$N(A)$  is origin in  $\mathbb{R}^2$ .

$A' = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{pmatrix}$  has same  $C(A')$  as  $C(A)$

$N(A')$  is line spanned by  $(\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix})$  in  $\mathbb{R}^3$ .

## FUNDAMENTAL SUBSPACES

Problem 4 : Let  $V = \{(a, b, c, d) / b + c + d = 0\}$   
 and  $W = \{(a, b, c, d) / a + b = 0 \text{ and } c = 2d\}$  be  
 subspaces of  $\mathbb{R}^4$ . Find the dimension of  $V \wedge W$ .

Solution :

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; b + c + d = 0, \sim \begin{pmatrix} a \\ -c-d \\ c \\ d \end{pmatrix} \sim a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$V$  is a 3 Dim Plane in  $\mathbb{R}^4$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; a + b = 0 \text{ and } c = 2d \sim \begin{pmatrix} -b \\ b \\ 2d \\ d \end{pmatrix} \sim b \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

$W$  is a 2 Dim Plane in  $\mathbb{R}^4$ .

## FUNDAMENTAL SUBSPACES

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$V \cap W$

$$b+c+d=0 \quad \text{and} \quad a+b=0, \quad c=2d$$

$$b+3d=0 \Rightarrow b=-3d, \Rightarrow a=3d$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \sim \begin{pmatrix} 3d \\ -3d \\ 2d \\ d \end{pmatrix} \sim d \begin{pmatrix} 3 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

$V \cap W$  is  $\therefore$  a line spanned by  $(3, -3, 2, 1)$  in  $\mathbb{R}^4$ .



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**THANK YOU**

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**Deepthi Rao**

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