ASSIGNMENT (UNIT-5)

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Q1) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$

All sub-matrices should be positive

i. $|1| > 0$

ii. $\begin{vmatrix} 1 & 2 \\ 2 & b \end{vmatrix} > 0$

$b - 4 > 0$

$\therefore b \in (4, \infty)$

iii. $\begin{vmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{vmatrix} > 0$

$1(7b-64) - 2(14-32) + 4(16-4b) > 0$

$-9b + 36 > 0$

$4 > b$

$\therefore b \in (-\infty, 4)$

Intersection of both sets, reveals that A can never be positive definite.

Q3) $f: 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

$$\therefore x^TAx = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 5 & -0.5 & 0 \\ -0.5 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

as f is of the form:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3.$$

Q4) $f_A: 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$

$f_A: (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2$

$f_B: x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$

$f_B: (x_1 + x_2 + x_3)^2$

Q5) $g(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

$$g = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Applying Gaussian elimination:

$$U = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2/3 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

Pivots are: $3, 2/3, -5$

$\therefore g$ is not positive definite.

Q6) $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$$

Finding Eigen vectors:

$(5-\lambda)(80-\lambda) - 400 = 0$

$\lambda^2 - 85\lambda = 0$

$\lambda(\lambda - 85) = 0$

$\therefore \lambda_1 = 85$ $\lambda_2 = 0$

Eigen vectors:

(i) $\lambda_1 = 85$ $A - 85I = \begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix}$

$\therefore v_1 = (0.25, 1)$ $\|v_1\| = 1.03$

(ii) $\lambda_2 = 0$ $A = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$

$\therefore v_2 = (-4, 1)$ $\|v_2\| = 4.123$

$\therefore$ Matrix $V = \begin{bmatrix} 0.24 & -0.97 \\ 0.97 & 0.24 \end{bmatrix}$

(Entered values after normalizing the vectors)

$\Sigma = \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9.22 & 0 \\ 0 & 0 \end{bmatrix}$

Finding U:

$u_1 = \dfrac{\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.24 \\ 0.97 \end{bmatrix}}{9.22} = \dfrac{Av_1}{\sigma_1} = \begin{bmatrix} 0.45 \\ 0.89 \end{bmatrix}$

For u_2, we can't use this formula as $\sigma_2 = 0$

We use, $(AA^T)x = 0$

$\begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

This reduces to:

$\begin{bmatrix} 17 & 34 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

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$17x + 34y = 0$ or $x = -2y$

$\therefore x_2 = (-2, 1)$

To find U_2, we use Gram-Schmidt method

$$U_2 = \frac{x_2}{||x_2||} \qquad ||x_2|| = \sqrt{5} = 2.24$$

$\therefore U_2 = (-0.89, 0.45)$

$$\therefore U = \begin{bmatrix} 0.45 & -0.89 \\ 0.89 & 0.45 \end{bmatrix}$$

And $A = U \Sigma V^T$ is verified.

Q7) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Finding Eigen vectors:

$(2-\lambda)(1-\lambda) - 1 = 0$

$\lambda^2 - 3\lambda + 1 = 0$

$\therefore \lambda_1 = 2.618 \qquad \lambda_2 = 0.382$

Finding Eigen vectors:

1) $\lambda_1 = 2.618 \qquad A - 2.618 I = \begin{bmatrix} -0.618 & 1 \\ 1 & -1.618 \end{bmatrix}$

$\therefore v_1 \ (1.618, 1)$

$||v_1|| = 1.9$

ii) $\lambda_2 = 0.382$ $\quad A - 0.382I = \begin{bmatrix} 1.62 & 1 \\ 1 & 0.62 \end{bmatrix}$

$\therefore v_2 (-0.62, 1)$ $\quad \|v_2\| = 1.18$

$\therefore$ Matrix $V = \begin{bmatrix} 0.85 & 0.53 \\ 0.53 & -0.85 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sqrt{2.62} & 0 \\ 0 & \sqrt{0.38} \end{bmatrix} = \begin{bmatrix} 1.62 & 0 \\ 0 & 0.62 \end{bmatrix}$$

finding U:

$$U_1 = \frac{AV_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.85 \\ 0.53 \end{bmatrix}}{1.62} = \begin{bmatrix} 0.85 \\ 0.52 \end{bmatrix}$$

$$U_2 = \frac{-AV_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.53 \\ -0.85 \end{bmatrix}}{0.62} = -\begin{bmatrix} -0.52 \\ +0.85 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 0.85 & 0.52 \\ 0.52 & -0.85 \end{bmatrix}$$

And $A = U \Sigma V^T$ is verified.

98) (i) $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

finding Eigen values:

$|A - \lambda I| = 0 = \cancel{\text{added}}\ (80-\lambda)\left((170-\lambda)(200-\lambda)\right) - 140^2) - 100\left(100(200-\lambda) - 140\times 40\right)$

$+ 40\left(14000 - 40(170-\lambda)\right) = 0$

$\therefore \lambda_1 = 360 \qquad \lambda_2 = 90, \qquad \lambda_3 = 0$

Finding Eigen values:

i) $\lambda_1 = 360$

$$A - 360I = \begin{bmatrix} -240 & 100 & 40 \\ 100 & -190 & 140 \\ 40 & 140 & -160 \end{bmatrix}$$

$\therefore v_1 \, (0.5, 1, 1) \qquad \|v_1\| = 1.5$

ii) $\lambda_2 = 90$

$$A - 90I = \begin{bmatrix} -10 & 100 & 40 \\ 100 & 80 & 140 \\ 40 & 140 & 110 \end{bmatrix}$$

$\therefore v_2 = (-1, -0.5, 1) \qquad \|v_2\| = 1.5$

iii) $\lambda_3 = 0$

$$A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$\therefore v_3 \, (2, -2, 1) \qquad \|v_3\| = 3.$

$\cancel{\therefore \text{Matrix } V = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{bmatrix}}$

$$\therefore \text{Matrix } V = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}$$

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{\frac{1}{3}\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\sqrt{360}} = \begin{bmatrix} 0.95 \\ 0.32 \end{bmatrix}$$

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$$U_2 = \frac{-AV_2}{\sigma_2} = \frac{\frac{1}{3}\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}}{\sqrt{90}} = -\begin{bmatrix} 0.32 \\ -0.95 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 0.95 & -0.32 \\ 0.32 & 0.95 \end{bmatrix}$$

and $A = U \Sigma V^T$ is verified

ii) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

finding Eigen values:

$(9-\lambda)(9-\lambda) - 81 = 0$

$\lambda^2 - 18\lambda = 0$

$\therefore \lambda_1 = 18$ $\qquad \lambda_2 = 0$

finding Eigen vectors

(i) $\lambda_1 = 18$ $\qquad A - 18I = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}$

$\therefore v_1 (-1, 1)$ $\qquad \|v_1\| = \sqrt{2} = 1.41$

(ii) $\lambda_2 = 0$ $\qquad A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

$\therefore v_2 (1, 1)$ $\qquad \|v_2\| = \sqrt{2} = 1.414$

$\therefore$ Matrix $V = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}$

(Normalising and putting)

$$\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4.24 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For U:

$$U_1 = \frac{AV_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 0.7 \\ -0.7 \end{bmatrix}}{\sqrt{18}} = \frac{1}{3}\begin{bmatrix} +1 \\ -2 \\ +2 \end{bmatrix}$$

To find U_2 and U_3, we will have to use Gram Shimdt method.

$$AA^T = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

$(AA^T)x = 0$

$$\begin{bmatrix} 2 & -4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x = 2y - 2z$ (Equation obtained from above matrix)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y-2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore x_2 (2, 1, 0)$ and $x_3 (-2, 0, 1)$

$$U_2 = \frac{x_2}{\|x_2\|} = (0.89, 0.45, 0)$$

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$$U_3 = \frac{X_3}{\|X_3\|} \qquad X_3 = x_3 - (u_2^T x_3)\, u_2$$

$$u_2^T x_3 = \begin{bmatrix} 0.89 & 0.45 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = 1.78$$

$$X_3 = (-2, 0, 1) - 1.78(0.89, 0.45, 0)$$
$$= (-2, 0, 1) - (1.59, 0.8, 0) = (-3.59, -0.8, 1)$$

$\|X_3\| = 3.81$

$$U_3 = (-0.94, -0.2, 0.26)$$

$$\therefore U = \begin{bmatrix} 0.33 & 0.89 & -0.94 \\ -0.67 & 0.45 & 0.2 \\ 0.67 & 0 & 0.26 \end{bmatrix}$$

and $A = U \Sigma V^T$ is verified.

Q2) (i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$

Finding Eigen values :

$$(1-\lambda)((5-\lambda)(9-\lambda) - 16) - 2(2(9-\lambda) - 12) + 3(8 - 3(5-\lambda)) = 0$$

$$\lambda^3 - 15\lambda^2 + 30\lambda + 4 = 0$$

$\lambda_1 = 12.6 \qquad \lambda_2 = 2.5 \qquad \lambda_3 = -0.12$

Since λ are both positive and negative. This is indefinite.

(2) $B = A^{-1} = \frac{1}{3}\begin{bmatrix} -29 & 12 & 7 \\ 12 & 0 & -2 \\ 7 & -2 & -1 \end{bmatrix}$

Since we are only observant on the sign of λ, we can exclude $\left(\frac{1}{3}\right)$

$$(-29-\lambda)\left(((-\lambda)(-1-\lambda))-4\right) - 12\left(+12(1-\lambda)+14\right) + 7\left(-24-7(-\lambda)\right)$$

$$\lambda^3 + 30\lambda^2 - 168\lambda + 76 = 0$$

$\therefore \lambda_1 = 4.4 \quad \lambda_2 = 0.5$ and $\lambda_3 = -34.8$

$\therefore$ B is also indefinite.

OR

We can use the concept:

If A has eigen values as λ

A^{-1} has eigen values $\frac{1}{\lambda}$

$\left(\frac{1}{\lambda}\right)$ does not lead to any change in the sign of λ.

Therefore A and A^{-1} (B) would have the same definiteness.

$\therefore$ B is also indefinite.