SELF-LEARNING COMPONENT

UNIT - 1

n^{th} derivatives

Let $y = f(x)$ be a real-valued function possessing derivatives of all orders.

The first derivative of y wrt x is written as

$$y' = y_1 = f'(x) = \frac{dy}{dx} = Dy$$

Similarly, the higher order derivatives of y wrt x can be written as

$$y'' = y_2 = f''(x) = \frac{d^2y}{dx^2} = D^2 y$$

In general, the n^{th} derivative of y wrt x is written as

$$y^{(n)} = y_n = f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n y$$

Note : $f^n(x)$
$= f(f(f(\cdots)))$

The process of finding the derivatives of y wrt x is called successive differentiation.

n^{th} Derivatives of Standard Functions $(n = 1, 2, 3 \dots)$

(1) $D^n e^{ax+b} = a^n e^{ax+b}$

(2) $D^n a^{mx} = (m \ln a)^n a^{mx}$

(3) $D^n (ax+b)^m = m(m-1)(m-2)\cdots(m-n+1)(ax+b)^{m-n} a^m$

if m is +ve int, ${}^nP_r$

(4) $D^n x^n = n!$

(5) $D^n \left(\frac{1}{ax+b}\right) = \frac{(-1)^n n! \, a^n}{(ax+b)^{n+1}}$

(6) $D^n \ln(ax+b) = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

$y = \ln(ax+b)$

$y_1 = \left(\dfrac{1}{ax+b}\right) a$

$y_n = D^{n-1}\left(\dfrac{a}{ax+b}\right)$

(7) $D^n \sin(ax+b) = a^n \sin\left(ax+b+\dfrac{n\pi}{2}\right)$

$y = \sin(ax+b)$

$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$y_2 = -a^2\sin(ax+b) = a^2\sin\left(ax+b+2\cdot\frac{\pi}{2}\right)$

$y_3 = -a^3\cos(ax+b) = a^3\sin\left(ax+b+3\cdot\frac{\pi}{2}\right)$

(8) $D^n \cos(ax+b) = a^n\cos(ax+b+n\pi/2)$

(9) $D^n\left(e^{ax}\sin(bx+c)\right) = (a^2+b^2)^{n/2} e^{ax} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

(page 89, unit 1)

(10) $D^n\left(e^{ax}\cos(bx+c)\right) = (a^2+b^2)^{n/2} e^{ax} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

PROBLEMS

1. Obtain the n^{th} derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ as

$$2(-1)^{n-1}(n-1)!\,\sin(n\theta)\sin^n\theta \quad , \quad \theta = \tan^{-1} 1/x$$

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x$$

$$y = 2\tan^{-1}x$$

Let $s = \tan^{-1}x$

$$s_1 = \frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)} = \frac{A}{x+i} + \frac{B}{x-i}$$

$$1 = A(x-i) + B(x+i)$$

At $x = i$,

$$1 = B(2i) \Rightarrow B = \frac{1}{2i}$$

At $x = -i$,

$$1 = A(-2i) \Rightarrow A = \frac{-1}{2i}$$

$$\frac{1}{(x+i)(x-i)} = \frac{1}{2i(x+i)} \quad \frac{-1}{2i(x-i)}$$

$$\boxed{s_1 = \frac{-1}{2ix-2} + \frac{1}{2ix+2}}$$

Differentiating n-1 times

$$S_n = D^n\left(\frac{1}{2ix+2}\right) - D^n\left(\frac{1}{2ix-2}\right)$$

$$= \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

let $x = r\cos\theta$, $1 = r\sin\theta$

$\frac{1}{r} = \sin\theta \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$$y_n = \frac{2(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{i}\left[\frac{1}{r^n e^{-in\theta}} - \frac{1}{r^n e^{in\theta}}\right]$$

$$y_n = \frac{2(-1)^{n-1}(n-1)!}{2ir^n}\left(e^{in\theta} - e^{-in\theta}\right)$$

$$= \frac{2(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$\frac{1}{r} = \sin\theta$

$$\boxed{y_n = 2(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin(n\theta)}$$

UNIT-2

Taylor's Expansion for a Function of 2 Variables

If $f(x,y)$ is a function of 2 independent variables with continuous partial derivatives of all orders, then

continuous functions: $f_{xy} = f_{yx}$

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$

$$+\frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big)$$

$$+\frac{1}{3!}\Big((x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\Big)$$

$$+ \dots \infty$$

In particular, if $a=b=0$, then the series is called Maclaurin's Series.

UNIT 3

Curve Tracing (186, Grewal)

1. Trace the curve $y^2(2a-x) = x^3$, $a>0$

From the given equation, we make the following observations.

(a) Symmetry

The equation of the curve remains the same when y is changed to $-y$.
$\therefore$ curve is symmetric about x-axis.

If $x \to -x$, equation lost
Not symmetric about $y$axis.

(b) Region of existence:

$$y = \pm\sqrt{\frac{x^3}{2a-x}} \rightarrow +ve$$

if x -ve, then does not exist (y not real)

$\therefore$ no part of curve $x<0$

$2a-x > 0$

$\boxed{x < 2a}$ and $\boxed{x \geq 0}$

(c) Passing through (0,0) & tangents.

$$y^2(2a-x) = x^3$$

(i) if (0,0) satisfies equation, passes through origin.

(ii) Tangents at origin

Equate the <u>lowest-degree terms</u> to 0 to find tangents

2nd degree

$$2ay^2 - y^2x = x^3$$

3rd degree

$$2ay^2 = 0$$

(a is a const.)

2 tangents: double point

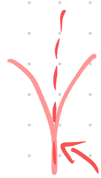
$$y^2 = 0 \Rightarrow y = 0, 0$$

DOUBLE POINT

↳ If 2 tangents diff & real: node

node

↳ If 2 tangents same & real: cusp



cusp

Here: $y=0$ and $y=0$ are tangents (cusp)

(d) Asymptotes

tangents at ∞

(i) Parallel to axes

equating to 0 the coefficients of highest power of y, we get

$$2a - x = 0$$

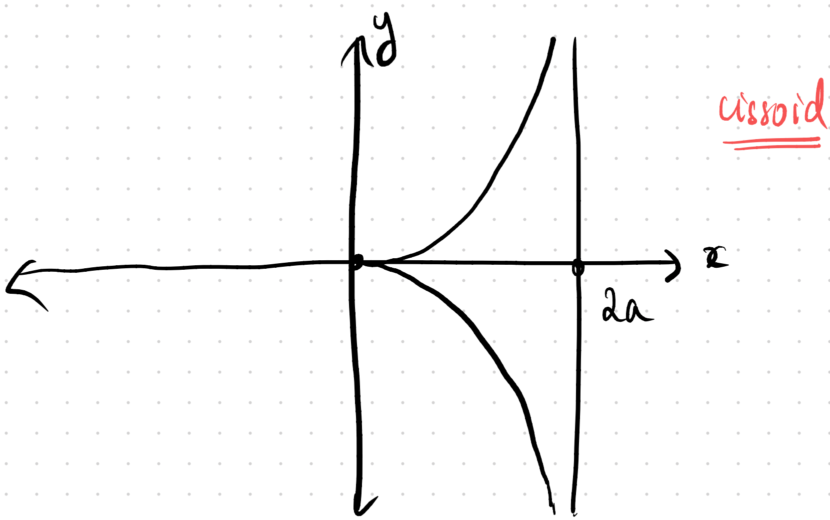
$x = 2a$ is an asymptote.

coeff of $x^3 = 1 \Rightarrow$ no asymptote $\parallel$ to x-axis.

(ii) Oblique asymptotes

next example

Based on these observations, the curve is shown as



2. Trace the curve $y^2(a^2+x^2) = x^2(a^2-x^2)$

(a) Symmetry

- $\left.\begin{array}{l} x \rightarrow -x \\ y \rightarrow -y \end{array}\right]$ symmetric about both axes.

(b) Origin

$x=0$ & $y=0$ satisfies the eqn.

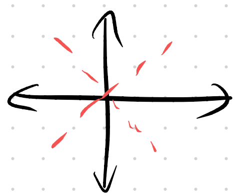
$$y^2a^2 + y^2x^2 = x^2a^2 - x^4$$

$$(y^2-x^2)a^2 = 0 \quad \text{2nd degree}$$

$$y^2 = x^2$$

$$y = \pm x$$

two tangents: $y=x$ and $y=-x$ (node)



(c) Asymptotes

(i) Parallel to axes

$-x^4 = 0$: no asymptote $\parallel$ to x axis.

$y^2(x^2+a^2) = 0$

$x = \pm ai$: imaginary tangents

no asymptote $\parallel$ to x axis

(ii) Oblique asymptotes.

(d) Region of existence

solving for y, we get

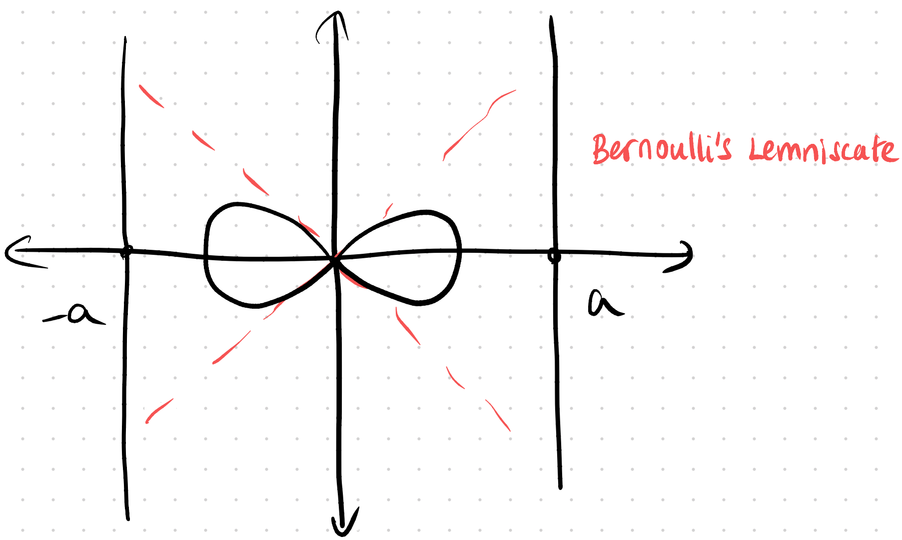
$$y = \pm x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

$$a^2 - x^2 \geq 0$$

$$x^2 \leq a^2$$

$x \leq a$ and $x \geq -a$

$x = a$ and $x = -a$ bound curve



UNIT 3

Curve tracing

Cartesian

1. Symmetry
2. Origin and tangents at (0,0)
3. Region of Existence & points
4. Asymptotes

1. $y^2(2a-x) = x^3$

<u>Symm</u>: x-axis

<u>Origin</u>: Passing

<u>Tangents at (0,0)</u>

$2ay^2 - xy^2 = x^3$

$\downarrow$

$y^2 = 0$

$y = 0$

$y = 0$

$\rightarrow$ <u>Double points</u>

1. Cusp
2. Node
3. Conjugate

<u>Cusp at (0,0)</u>

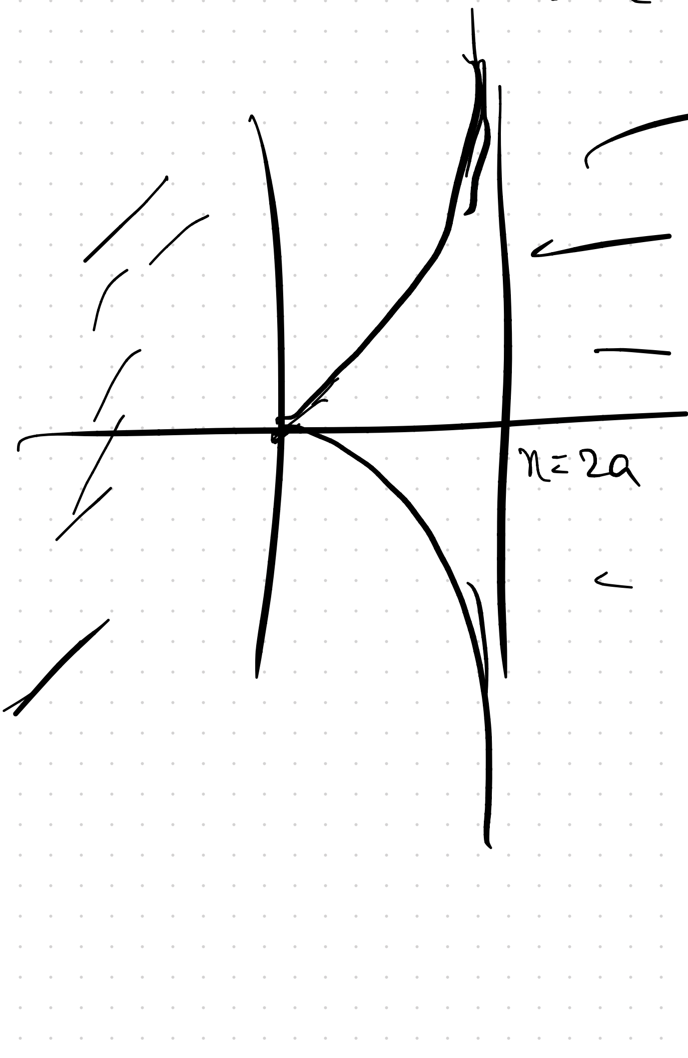
<u>Asymptotes</u>

i. $2a - x = 0$

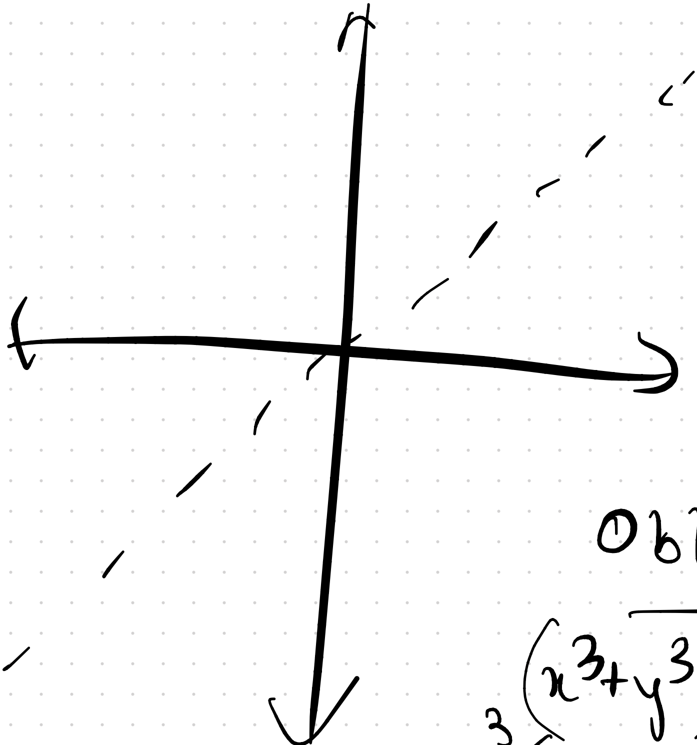
$x = 2a$ $\rightarrow$ Asymptote $\parallel^{le}$ to y

Region of existence

$$y^2 = \frac{x^3}{2a - x}$$



2: $x^3+y^3 = 3axy$ ← Folium of Descartes



Oblique/Inclin

$$(x^3+y^3) - 3axy = 0$$

$$x^3\left(1+\frac{y^3}{x^3}\right) - x^2\left(3a\frac{y}{x}\right) = 0$$

$$x^3\phi_3\left(\frac{y}{x}\right) - x^2\phi_2\left(\frac{y}{x}\right) = 0$$

$$x^m\phi_m\left(\frac{y}{x}\right) - - - - - \Rightarrow 0$$

$$x_m \phi_m\left(\frac{y}{x}\right) + x_{m-1}\phi_{m-1}\left(\frac{y}{x}\right) - \cdots = 0$$

$$y = mx + c$$

$$\frac{y}{x} = m + \frac{c}{x}$$

$$x_m \phi_m\left(m + \frac{c}{x}\right) + x_{m-1}\phi_{m-1}\left(m + \frac{c}{x}\right) - -$$

$$f(a+h) = f(a) + f'(a)h + \frac{h^2}{2!}f''(a)$$

$$\phi_m(m + \quad \phi_n(m) + \phi_n'(m)\frac{c}{x} + \frac{c^2}{x^2 2!}\phi$$

$$\boxed{c = -\frac{\phi_2(m)}{\phi_3'(m)}}$$

$$\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

Subs $\boxed{x = 1, \; y = m}$

$$\phi_3(m) = 1 + m^3$$

$$\phi_2(m) = 34m$$