

UE19CS251

DESIGN AND ANALYSIS OF ALGORITHMS

UNIT 5: Limitations of Algorithmic Power and  
Coping with the Limitations

P, NP, and NP-Complete Problems

PES University

## Outline

### Concepts covered

- Class P
- Class NP
- NP-Complete
- NP-Hard

## 1 Classifying Problem Complexity

- Is the problem tractable, i.e., is there a polynomial-time ( $O(p(n))$ ) algorithm that solves it?
- Possible answers:
  - yes
  - no
    - \* because it's been proved that no algorithm exists at all (e.g., Turing's halting problem)
    - \* because it's been proved that any algorithm takes exponential time
  - unknown

## 2 Problem Types: Optimization and Decision

- Optimization problem: find a solution that maximizes or minimizes some objective function
- Decision problem: answer yes/no to a question

- Many problems have decision and optimization versions
- Example: traveling salesman problem
  - optimization: find Hamiltonian cycle of minimum length
  - decision: find Hamiltonian cycle of length  $\leq m$
- Decision problems are more convenient for formal investigation of their complexity

## 3 Class P

### Class P (Polynomial)

The class of decision problems that are solvable in  $O(p(n))$  time, where  $p(n)$  is a polynomial of problem's input size  $n$

- searching
- element uniqueness
- graph connectivity
- graph acyclicity
- primality testing

## 4 Class NP

### Class NP (Nondeterministic Polynomial)

class of decision problems whose proposed solutions can be verified in polynomial time = solvable by a nondeterministic polynomial algorithm

- A nondeterministic polynomial algorithm is an abstract two-stage procedure that:
  - generates a random string purported to solve the problem
  - checks whether this solution is correct in polynomial time
- By definition, it solves the problem if it's capable of generating and verifying a solution on one of its tries
- Why this definition?
  - led to development of the rich theory called “computational complexity”

## 5 Example: CNF satisfiability

### Boolean Satisfiability (CNF)

Is a Boolean expression in its conjunctive normal form (CNF) satisfiable, i.e., are there values of its variables that makes it true?

- This problem is in NP. Nondeterministic algorithm:
  - Guess truth assignment
  - Substitute the values into the CNF formula to see if it evaluates to true
- Example: Consider the Boolean expression in CNF form:

$$(a + \bar{b} + \bar{c})(\bar{a} + b)(\bar{a} + \bar{b} + \bar{c})$$

- Can values *false* and *true* (or 0 and 1) be assigned to  $a$ ,  $b$  and  $c$  such that above expression evaluates to 1?
- $a = 1, b = 1, c = 0$
- Checking phase:  $\Theta(n)$

## 6 What problems are in NP?

- Hamiltonian circuit existence
- Partition problem: Is it possible to partition a set of  $n$  integers into two disjoint subsets with the same sum?
- Decision versions of TSP, knapsack problem, graph coloring, and many other combinatorial optimization problems. (Few exceptions include: MST, shortest paths)

- All the problems in  $P$  can also be solved in this manner (but no guessing is necessary), so we have:

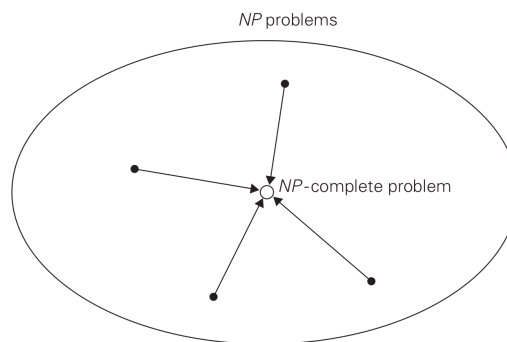
$$P \subseteq NP$$

- Big question:

$$P = NP ?$$

## 7 NP-Complete Problems

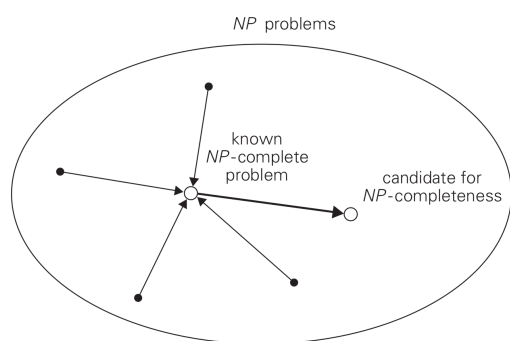
- A decision problem  $D$  is  $NP$ -complete if it's as hard as any problem in  $NP$ , i.e.,
  - $D$  is in  $NP$
  - every problem in  $NP$  is polynomial-time reducible to  $D$



- Cook's theorem (1971): CNF-sat is NP-complete

## 8 NP-Complete Problems

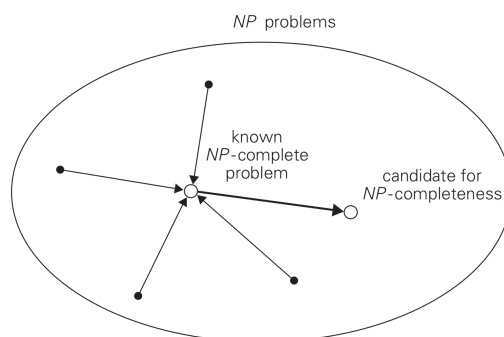
- Other NP-complete problems obtained through polynomial- time reductions from a known NP-complete problem



- Examples: TSP, knapsack, partition, graph-coloring and hundreds of other problems of combinatorial nature

## 9 $P = NP$ ? Dilemma Revisited

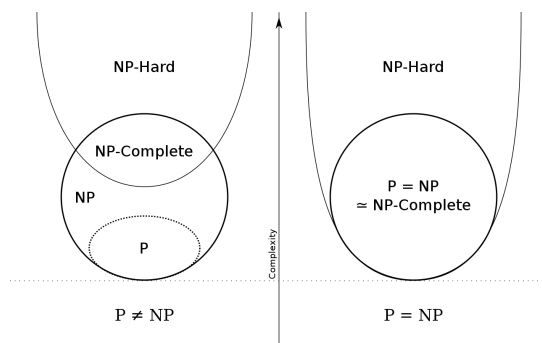
- $P = NP$  would imply that every problem in  $NP$ , including all  $NP$ -complete problems, could be solved in polynomial time
- If a polynomial-time algorithm for just one  $NP$ -complete problem is discovered, then every problem in  $NP$  can be solved in polynomial time, i.e.,  $P = NP$



- Most but not all researchers believe that  $P \neq NP$ 
  - Though others like Stephen Cook, Leonid Levin and Donald Knuth don't

## 10 NP-Hard Problems

- A decision problem  $D$  is  $NP$ -hard iff:
  - $D$  is in  $NP$
  - every problem in  $NP$  is polynomial-time reducible to  $D$



## 11 Complexity Hierarchy

