

## Pumping lemma for Context free languages

- Pumping lemma for CFLs is not linear like it was for Regular lang's.
- In PDA, if we have a long string
  - pumping the loop any no. of times gives w the strings  $\in$  Lang.
  - In PDA, decision not only depends on state no. but also on the stack contents.
  - Hence execution of second 010 might change the stack contents.
  - Hence we need a better way of using Pumping lemma in CFLs.
  - Unlike PSMs, we define pumping lemma by looking at the grammar (in CNF) instead of the machine

## Pumping lemma for Regular languages

- we look at FSM
  - Pumping constant - # states in FSM
  - String  $x$  is broken in 3 parts
- $z = \overset{o}{x} \underset{\substack{\downarrow \\ \text{loop}}}{y} \underset{\substack{\rightarrow \\ \text{After loop}}}{z}$
- $|zy| \leq n$   
 $|y| \geq 1$

## Pumping lemma for Context free languages

- we look at grammar in CNF
  - Pumping constant - # Non terminals in grammar(in CNF)
  - String  $z$  is broken into 5 parts
- $z = u \overset{i}{v} w \overset{i}{x} y \underset{\substack{\downarrow \\ \text{fixed} \\ \text{center}}}{z} \underset{\substack{\leftarrow \\ \text{loop 1}}}{\text{loop 1}} \underset{\substack{\rightarrow \\ \text{loop 2}}}{\text{loop 2}}$
- $|vxy| \leq n$   
 $|vx| \geq 1$
- Both the loops pump simultaneously.

where & how would we figure out the hoop is?

CFG  $\Rightarrow$  via the Parse tree where the loop  
A

CNF

#NT: 3

$$A \rightarrow BC|D$$

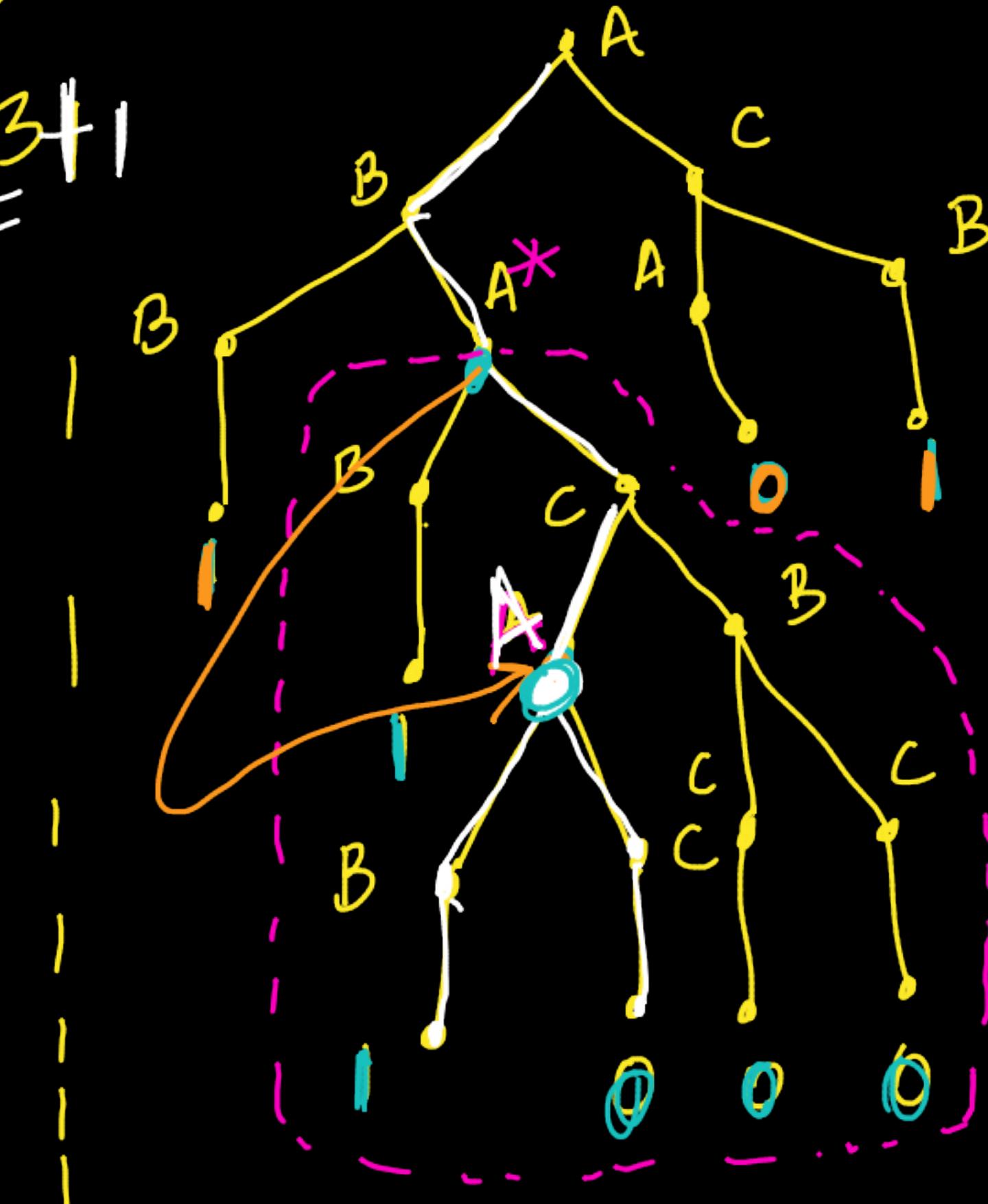
$B \rightarrow BA \mid CC$

$$C \rightarrow AB \mid 0$$

$$z = \underline{111} \quad \underline{0000} \quad |$$

Shift z should be

long enough



B | |||| 10 00 00 01  
| ||| 10 00 00 00 01  
| ||| 10 00 00 00 01  
=

$$= \overbrace{u \bar{v} w x}^{\sim} y$$

A hand-drawn diagram consisting of two lines. One line is wavy and labeled "loops" in yellow. The other line is straight and labeled "pump" in yellow.

Simultaneously

$$|z| \geq 2^{N+1}$$

Pumping property for CFLs

if  $L$  is a CFL,

$\exists N$  is the #NT in CNF form of grammar for  $L$ .

Pumping constant (min string length)

$$n = 2^{N+1}$$

$\forall z \in L$  such that  $|z| \geq n$

$\exists u v w x y : z$  such that  $|vwx| \leq n$

$$|vx| \geq 1$$

$\forall i \geq 0 \quad u^i v^i w^i x^i y^i \in L$

Prove  $L = \{n_a(w) = n_b(w)\}$  is a CFL using Pumping property.

$$S \rightarrow aSb | bSa | SS | \lambda$$

CNF:-

$$S \rightarrow AC |$$

$$BD | SS | \lambda$$

$$\begin{aligned} A &\rightarrow a \\ C &\rightarrow SB \\ B &\rightarrow b \\ D &\rightarrow SA \end{aligned}$$

$$N = 5 \quad (\#NT)$$

Pumping const,  $n \geq 2^{N+1}$   
 $n \geq 64$

$$|vwx| \leq 64$$

Choose a string  $z$ ,  $|z| \geq 64$

$$a^{64} b^{64} \Rightarrow a^{63} a^1 \lambda b^1 b^{63}$$

$|vwx| \leq 64$

## Pumping property for CFLs

Given a CFL  $L$ ,

$\exists N$  where  $N$  is the # NT in  
CNF grammar for  $L$

We choose the pumping constant  $n$

$$n = 2^{N+1}$$

$\forall z \in L, |z| \geq n$

$\exists u v w x y = z$  such that  $|vwx| \leq n$   
 $|vz| \geq 1$

$\forall i \geq 0 \quad uv^i w x^i y \in L$

## "Pumping property"

Adversary claims that  $L$  is context-free.

$\forall N$ , where  $N$  is the # NT  
in the hypothetical CNF  
grammar for  $L$ .

pumping constant  $n = 2^{N+1}$

$\exists z \in L, |z| \geq n$

$\forall u v w x y = z$  such that  
 $|vwx| \leq n$   
 $|vx| \geq 1$

$\exists i \geq 0$

$uv^i w x^i y \notin L$

Hence the claim is wrong!  $L$  is not context-free.

Prove using Pumping lemma that the Lang  $L = \{0^n 1^n 0^n \mid n \geq 0\}$  is not context free

Soln

\* Adversary claims  $L = \{0^n 1^n 0^n, n \geq 0\}$  is context free.

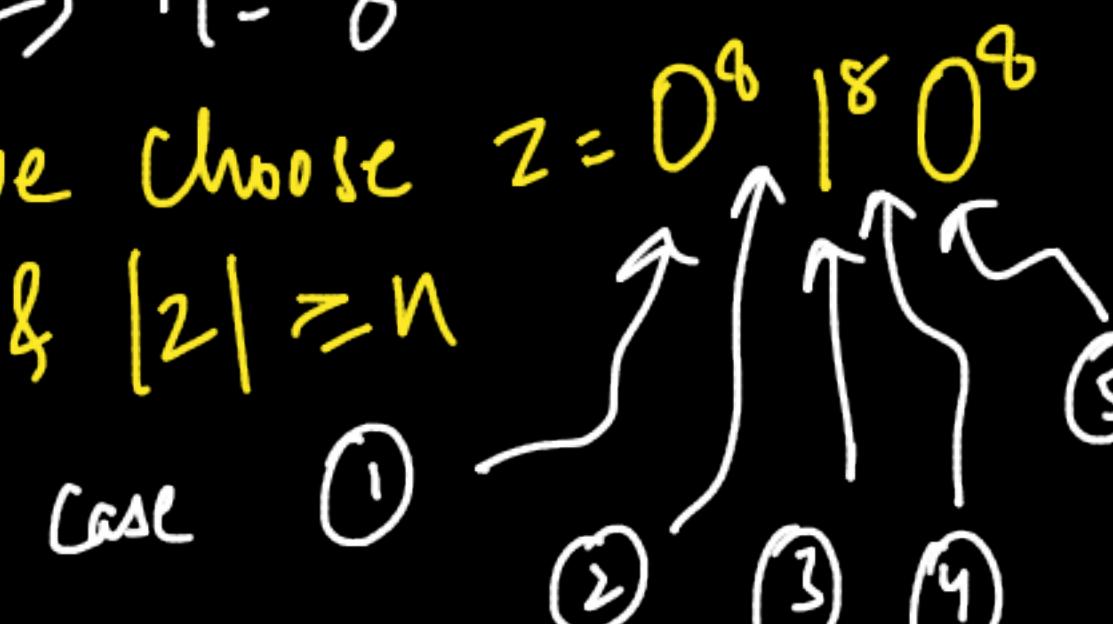
(\*) \* Let's say there are  $N=2$  NT's in the hypothetical grammar for  $L$  (CNF)

\* We calculate the pumping constant ' $n$ ' =  $2^{N+1} = 2^2 + 1 = 2^3 = 8$

$\Rightarrow n = 8$

\* We choose  $z = 0^8 1^8 0^8$

$z \in L$  &  $|z| \geq n$

Case ① 

$uvwxy = z$ ,  $|vwx| \leq n$  &  $|vz| \geq 1$

$\exists i, uv^iwy \notin L$

Hence  $L$  is not context free.

Case	locn of $vwx$	$i$	No. $n_1, n_0$	Comment
1	only 0's	$\geq 2$	$\uparrow - -$	Too many 0's at start
2	some 0's & some 1's	$\geq 2$	$\uparrow \uparrow -$	few 0's at end
3	only 1's	$\geq 2$	$- \uparrow -$	Too many 1's
4	some 1's & some 0's	$\geq 2$	$- \uparrow \uparrow$	few 0's at start
5	only 0's	$\geq 2$	$- - \uparrow$	Too many 0's at end

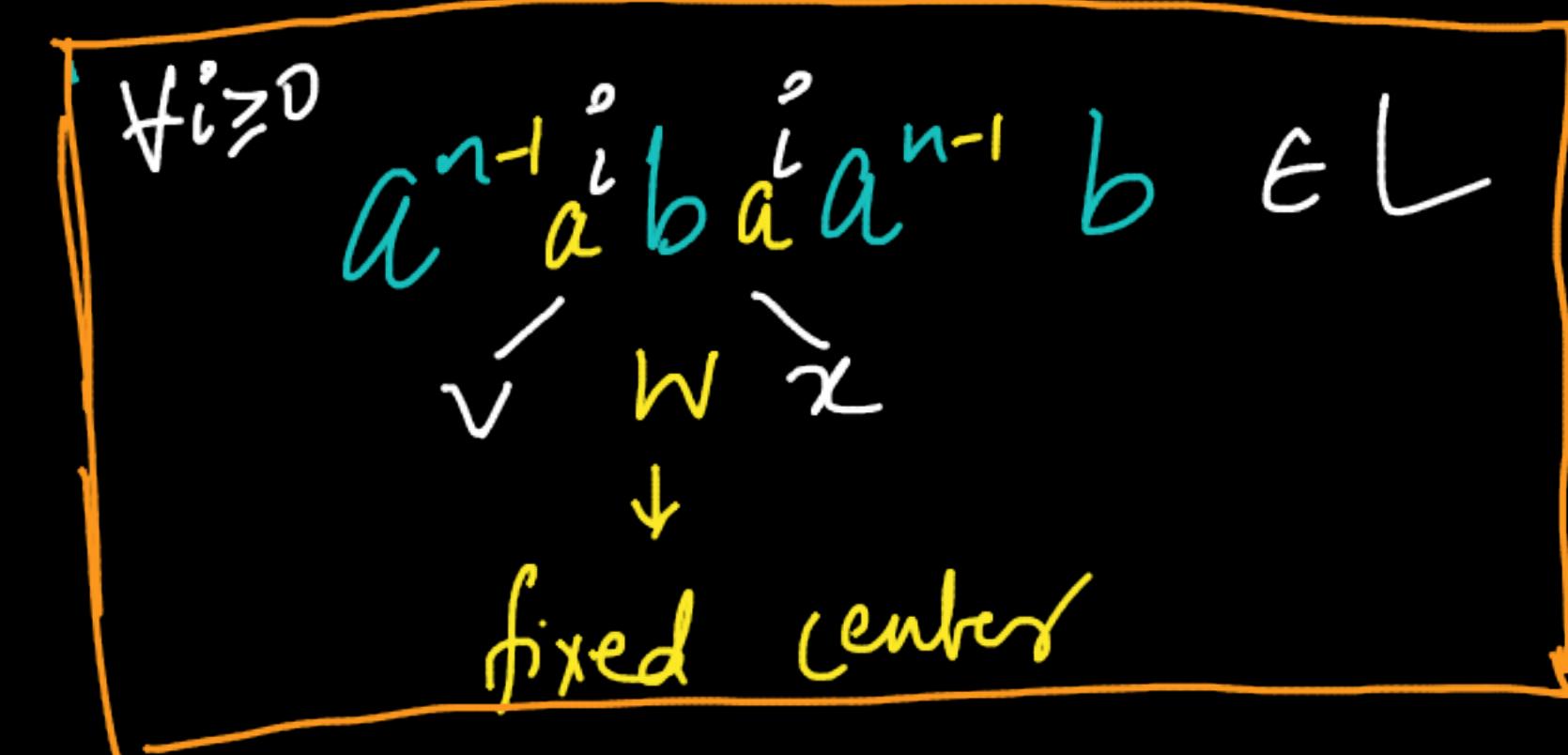
Prove using pumping lemma that  $ww$  is not context free.

$$w \in \{a, b\}^*$$

Soln

- \* Adversary claims that Lang  $L = \{ww, w \in \{a, b\}^*\}$  is context free.
- \* Let's say there are  $N$  non-terminals in the hypothetical grammar (CNF form) for  $L$ .
- \* we calculate the pumping constant  $n = 2^{N+1}$
- \* we choose a string  $z = a^n b a^n b$  such that  $|z| \geq n$
- \* we prove  $\nexists u v w x \text{ s.t. } z = uvwx$  where  $|vwx| \leq n, |vx| > 0$   
 $\exists i \geq 0, uv^iwx^i \notin L$ .

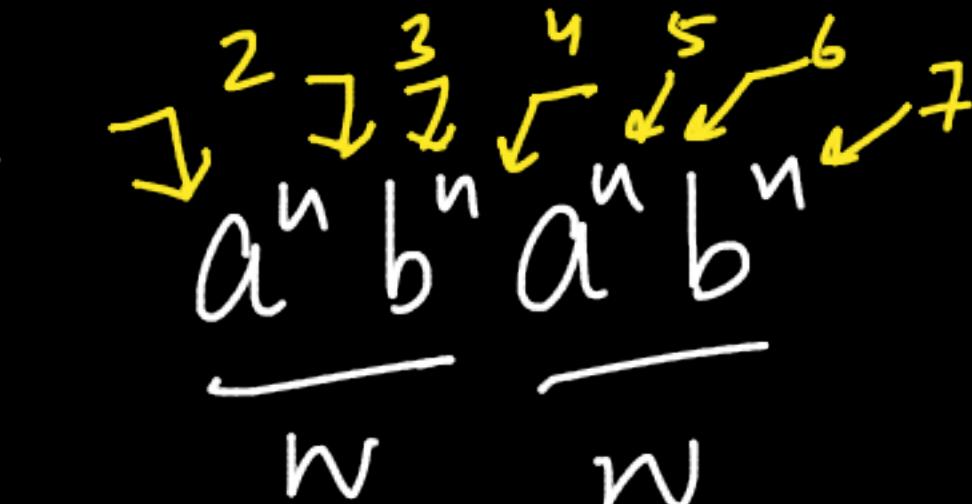
You lost !!



Prove  $ww$  is not context-free.

Soln

We choose  $z =$



$$|z| \geq n$$

$$|vx| \geq 1$$

$$|vwx| \leq n$$

$ww$

$\boxed{H_{uvwxy} = z}$  Fig,

$uv^i w x^i y \notin L$

Case	loc <sup>n</sup> of vwx	i	$n_a$	$n_b$	$n_a$	$n_b$	Comment ( $z \notin L$ )
1	Only a's (start)	$\geq 2$	..↑	-	-	-	Too many a's instant
2	some a's & some b's	$\geq 2$	↑	↑	-	-	$w \neq w$
3	only b's	$\geq 2$	-	↑	-	-	Too many b's at start
4	some b's & some a's	$\geq 2$	-	↑	↑	-	$w \neq w$
5	some a's (second w)	$\geq 2$	-	-	↑	-	Too many a's later
6	some a's & some b's	$\geq 2$	-	-	↑↑		$w \neq w$
7	some b's	$\geq 2$	-	-	-	↑	Too many b's at start

Hence  $L$  is not context free.

Prove using Pumping lemma that  $L = \{a^n b^m c^k ; k=n+m, n,m \geq 1\}$  is not context free.

- Soln :-
- \* Adversary claims that the lang  $L$  is context free
  - \* Let's say there are  $N$  non-terminals in the hypothetical grammar in CNF form.
  - \* we calculate pumping constant  $n = 2^{N+1}$
  - \* we choose the string  $z = a^n b^n c^{n+m}$ ,  $|z| \geq n$   $|vwx| \leq n$

~~$uv^i w^j x^k y = z$ ,~~

case no.	$ vwx $	$i$	$n_a$	$n_b$	$n_c$	Comment	$ vx  \geq 1$
1.	only a's	$\geq 2$	$\uparrow$	-	-	$\#Cs$ will not be equal to the product of $\#as$ & $\#bs$ $\Rightarrow$ more as	
2.	some a's, some b's	$\geq 2$	$\uparrow$	$\uparrow$	-		
3.	only b's	$\geq 2$	-	$\uparrow$	-		
4.	some b's, some c's	$\geq 2$	-	$\uparrow$	$\uparrow$		
5.	only c's	$\geq 2$	-	-	$\uparrow$		

$\#L$   
Hence  $L$  is not context free.

Prove using Pumping lemma that lang  $L = \{ a^{n!} \mid n \geq 0 \}$  is not context free.

Soln:

- \* Adversary claims that  $L$  is context free
- \* Let's say  $N$  is the #NT in Adversary's hypothetical grammar in CNF for  $L$ .

- \* We calculate pumping constraint  $n = 2^{N+1}$
- \* We choose the string  $z = a^{n!}$ ,  $|z| > n$

$$|vz| \geq 1$$

$$|vwr| \leq n$$

#p a's

range of p:-

$$(1 \leq p \leq n)$$

$$a \cdots a \underbrace{(a^p)^i a}_{\text{defaut } i=2}$$

$$n! < n! + p < (n+1)!$$

$$a^{n!+p}$$

$$\frac{n!+n}{n!+n} < \frac{(n+1)!}{(n+1) \cdot n!}$$

$\Rightarrow L$  is not context free!

Prove using Pumping lemma that lang  $L = \{ a^{n^2}, n \geq 0 \}$  is not context free.

Soln:- \* Adversary claims that  $L$  is context free.

\* Let's say  $N$  is the #NT's in Adversary's hypothetical grammar in CNF for  $L$ .

\* We calculate pumping constant  $n = 2^{N+1}$

\* We choose a string  $z = a^{n^2}$ ,  $|z| \geq n$

$vuwxyz = z$ ,  $|vwx| \leq n$  and  $|wx| \geq 1$

$\exists i \geq 0$

$uv^iw^jx^y \notin L$

$a \cdot \overbrace{a}^{\text{loop}} \cdot \overbrace{(a^b)^i}^{\text{a}} \cdot a$

for  $i=2$  range of  $b$  :-  $1 \leq b \leq n$

a we will prove,

Assume  $b = n$  (max value)

$$n^2 + n < (n+1)^2 \Rightarrow$$

$$n^2 + n < n^2 + 2n + 1 \Rightarrow L \text{ is not context free!}$$

$$n^2 < n^2 + b < (n+1)^2$$

Prove using pumping lemma that lang  $L = \{0^k \text{ where } k \text{ is composite}\}$  is not context free!

Soln:-

Skipping first 3 points  $\Rightarrow$  Pumping constant  $= n$

we choose the string  $z = 0^{2n} > n$

$$|vwx| \leq n$$

$$0^{3n}$$

$$= \underbrace{uvwx}_i \underbrace{y}_{(0^i)(0^i)} \quad i=2$$

$$00 \quad 0000$$

$$0^{2n} = u v \overset{\lambda}{\underset{\downarrow}{w}} x \underset{\downarrow}{y}$$

$$\text{multiple of 2} \quad \forall i \geq 0 \quad z' \in L$$

$$|vz| \geq 1$$

$w$  can be  $\lambda$

fixed center

$$|w| \geq 0$$

free!

We can never prove  $L$  is not context free using pumping lemma.

$L$  is not context free

satisfies  
pumping  
prop for  
CFL's

But it

will not

(context)

### Note:

- Unlike Regular languages we cannot use its complement  $L^c = \{0^p\}$ , where  $p$  is prime} to prove that  $L$  is not Context free.
- This is because, CFLs are not closed under complement.  
∴ We can never prove using pumping lemma that the language  $L = \{0^K\}$  where  $K$  is composite} is not context free.  
Lang  $L$  although not context free, satisfies the pumping property.

## Summary :-

We have proved using pumping lemma that following lang's are not context-free :-

- 1)  $0^n 1^n 0^n : n \geq 0$
- 2)  $ww : w \in \{a, b\}^*$
- 3)  $a^n b^m c^k : k = n+m, n, m \geq 1$
- 4)  $a^{n!} : n \geq 0$
- 5)  $a^{n^2} : n \geq 0$

We saw that  $\mathcal{L} = \{0^k, k \text{ is composite}\}$  although not context-free, satisfies pumping property.

Homework :-

Prove using pumping lemma that following languages are  
not context free :-

1)  $a^n b^m c^n d^m$ ,  $n, m \geq 0$

2)  $O^p$ , where  $p$  is prime

Acknowledgement :- Notes are prepared by Prof. Preet Kaurwal.