VECTOR SPACES

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CLASS 11 : CONTENT

- Uniqueness, Existence of right and left inverse
- Matrix of rank 1

EXISTENCE OF INVERSE FOR A RECTANGULAR MATRIX

Existence of Inverses

Definition:

Let Amxn ($m \leq n$) be a matrix such that rank of A = m. Then Ax = b has at least one solution x for every b if and only if the columns span R^m.

In this case, A has a right inverse C such that AC= I (m*m).

Let Amxn ($m \geq n$) be a matrix such that rank of A = n. Then Ax = b has at most one solution x for every b if and only if the columns are linearly independent. In this case, A has a left inverse B such that BA= I (n*n).

EXISTENCE AND UNIQUENESS

A has a left inverse if BA = I

A has a right inverse if AC= I

Rank always satisfies $r \leq m$ and $r \leq n$ .An m by n matrix cannot have More than 'm' independent rows or 'n' independent columns .there is not a space for more than m pivots or more than n .

When r= m there is right inverse and AX=b always has a solution

When r=n there is a left inverse and the solution (if it exists) is unique .

Only a square matrix has both r=m=n hence a square matrix has both existence and uniqueness achieved ,so only square matrix has two sided inverse.

EXISTENCES OF INVERSES

Case (i) If $\rho(A) = m$, (m is the number of rows) then A will has right inverse of order m x n such that $A_{m\times n}\, C_{n\times m} = I_{m\times m}$

Case (ii) If $\rho(A) = n$, (n is the number of columns) then A will has left inverse of order n x m such that $B_{n\times m}\, A_{m\times n} = I_{n\times n}$

[Best right inverse, $C = A^T\,(A\,A^T)^{-1}$]

[Best left inverse, $B = (A^T A)^{-1} A^T$]

LEFT AND RIGHT INVERSE

E.g. : Obtain left inverse or a right inverse if it exists for the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

Solution: Here $\rho(A) = 2 = n$

Therefore A has left inverse, say B

$$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$$ then we have

EXISTENCES OF INVERSES

$$BA = I$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a = 1, b = 0, d = 0, e = 1$$

$c = 1, f = 1\}$ free variables

$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is the required left inverse.

EXISTENCES OF INVERSES

Example:

Let $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

Then, a right inverse of A is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}$

Since the third row is arbitrary, there are infinitely many right inverses for A

MATRICES OF RANK 1

Every matrix of rank 1 has the simple form $A = uv^T =$ column times row

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \rho(A) = 1$$

Every row is a multiple of first row, so row space is one dimensional. In fact , we can write the whole matrix as the product of a column vector and row vector

MATRIX OF RANK 1

i.e, $A = (column)(row)$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Where the rows are all multiples of the vectors v^T

columns are all multiples of the vector u

MATRIX OF RANK 1

Matrices Of Rank One:

When the rank of a matrix is as small as possible,

a complicated system of equations can be broken into simple pieces.

Those simple pieces are matrices of rank one.

The matrix has rank r = 1.

We can write such matrices as a column times row.

That is $$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

THANK YOU

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