LINEAR ALGEBRA AND ITS APPLICATIONS

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MATRICES AND
GAUSSIAN ELIMINATION

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Course Content: Row Exchanges and Permutation Matrices

- Consider the system of equations Ax=b. While solving the system if a zero appears In the pivot position, it calls for a row exchange.

  This row exchange is taken care by **Permutation Matrices P .**

  Here **A ≠ LU** then **PA=LU** where **P is a Permutation Matrix** which is an Identity Matrix with rows in different order.

Ex:Consider the system $y=b_1;\ 2x-3y=b_2$

$$Ax=b \Rightarrow \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Here Gaussian elimination fails and so calls for a row exchange i.e., $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$

This is same as PAX=Pb

$$PA=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}=U$$ and $$Pb=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}=\begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$$

$$\therefore PAx=PB \Rightarrow \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$$

Permutation Matrices:

- **P is a Permutation Matrix** which is an Identity Matrix with rows in different order.
- **Product of two permutation matrices is also a Permutation Matrix.**
- **Inverse of a permutation matrices is also a Permutation Matrix.**
- **P^{-1} is always same as P^T .**
- **Permutation Matrices of order 2 are 2!=2 in number.**

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

❖**Permutation Matrices of order 3 are 3!=6 in number.**

$$I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad ; \qquad P_{21}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}=P_{21}^{-1} \quad ;$$

$$P_{31}=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}=P_{31}^{-1} \quad ; \qquad P_{32}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}=P_{32}^{-1} \quad ;$$

$$P_{21}P_{31}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}=\left(P_{21}P_{32}\right)^{-i} \quad ; \qquad P_{21}P_{32}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}=\left(P_{21}P_{32}\right)^{-1}$$

MATRICES AND GAUSSIAN ELIMINATION:

Ex: $A=\begin{pmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{pmatrix} \xrightarrow{R_2-3R_1} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 5 & 7 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = U$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} \neq A$$

$$P_{23}A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \xrightarrow{R_3-3R_1} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = U$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = P_{23}A$$

Problem: Explain why A is not factorizable into LU? How can A be modified so that the new matrix can be factored into LU? Also obtain the factors L,D,U for the new matrix. What is the relation between L and U thus obtained? Explain.

$$A=\begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{pmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A is not factorizable into LU as Gaussian elimination fails and Row exchange is required.

So A should be multiplied with permutation matrix P_{23} so that PA=LU.

$$P_{23}A=\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2+2R_1} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U \qquad L=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$L=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U=\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$L=U^T$

THANK YOU

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