Probability

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Course material created using various Internet resources and text book

Experiment, Event, Sample space, Probability, Counting rules, Conditional probability, Bayes's rule

5E Note 4

Descriptive and Inferential Statistics

Statistics can be broken into two basic types:

- Descriptive Statistics :

 We have already learnt this topic

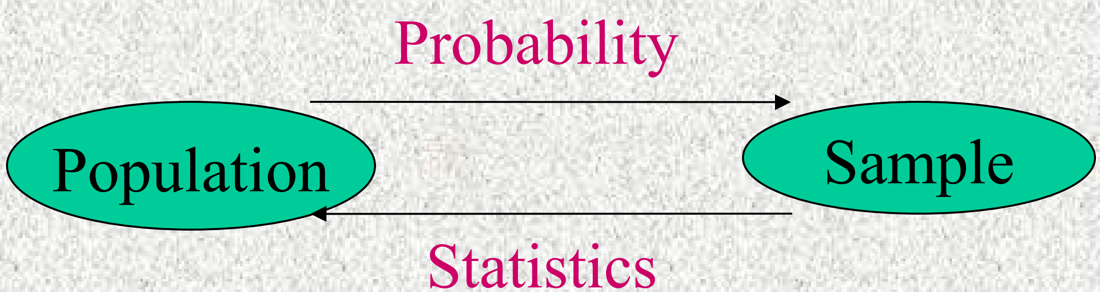
- Inferential Statistics

 Methods that making decisions or predictions about a population based on sampled data.

Probability

Why Learn Probability?

- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes, from business to medicine to the weather
- A probability provides a quantitative description of the chances or likelihoods associated with various outcomes
- It provides a bridge between descriptive and inferential statistics



Probabilistic vs Statistical Reasoning

- Suppose I know exactly the proportions of car makes in Bangalore. Then I can find the probability that the first car I see in the street is a Ford. This is probabilistic reasoning as I know the population and predict the sample
- Now suppose that I do not know the proportions of car makes in Bangalore, but would like to estimate them. I observe a random sample of cars in the street and then I have an estimate of the proportions of the population. This is statistical reasoning

What is Probability?

- we used graphs and numerical measures to describe data sets which were usually **samples.**
- We measured "how often" using

Relative frequency = f/n

- As n gets larger,

Sample → **Population**
And "How often"
= Relative frequency → **Probability**

Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.
- An **event** is an outcome of an experiment, usually denoted by a capital letter.
  - The basic element to which probability is applied
  - When an experiment is performed, a particular event either happens, or it doesn't!

Experiments and Events

- **Experiment: Record an age**
  - A: person is 30 years old
  - B: person is older than 65
- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2

Basic Concepts

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.
- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

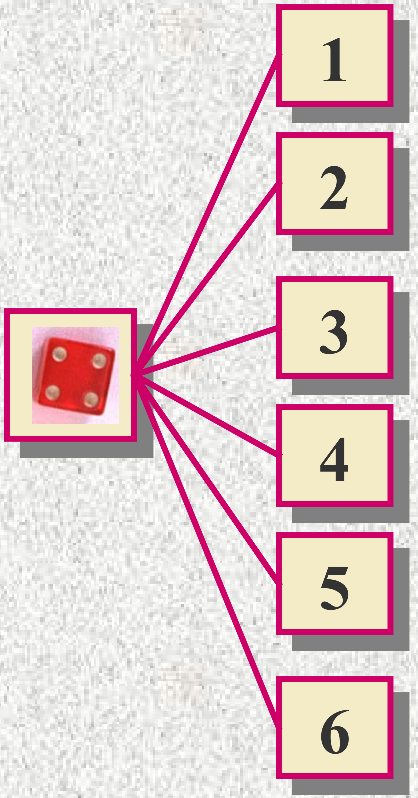
B and C?
B and D?

Basic Concepts

- An event that cannot be decomposed is called a **simple event**.
- Denoted by E with a subscript.
- Each simple event will be assigned a probability, measuring "how often" it occurs.
- The set of all simple events of an experiment is called the **sample space, S**.

Example

- **The die toss:**
- Simple events:



E_1

E_2

E_3

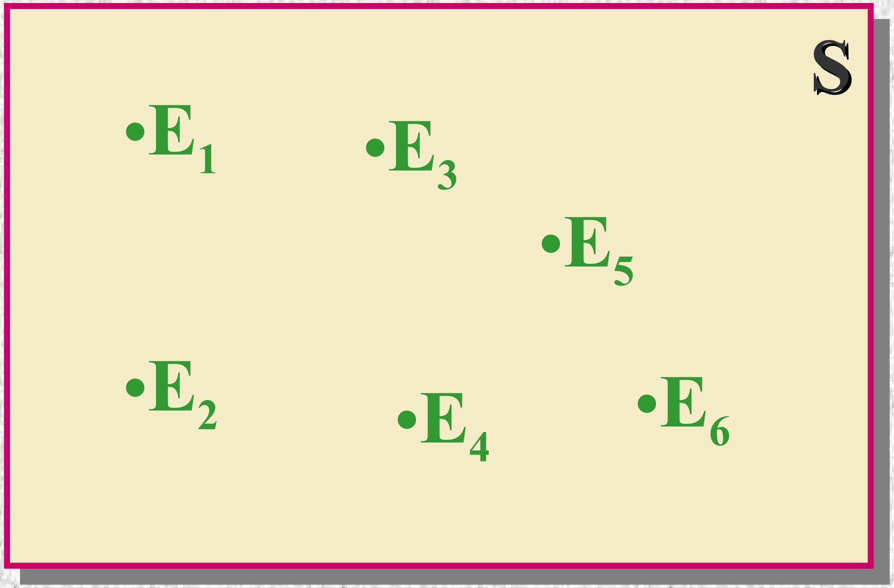
E_4

E_5

E_6

Sample space:

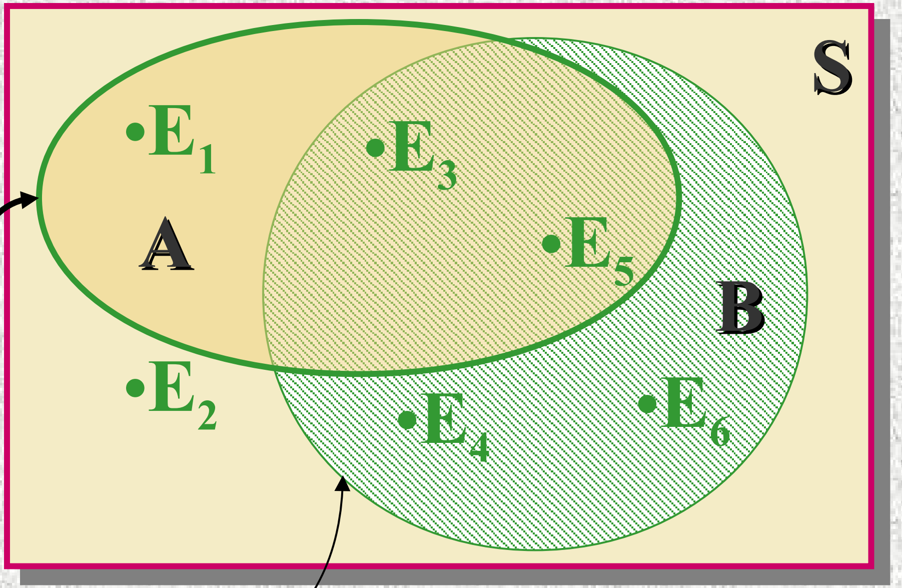
$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



Basic Concepts

- An **event** is a collection of one or more **simple events.**

- **The die toss:**
  - A: an odd number
  - B: a number > 2



A ={E_1, E_3, E_5}

B ={E_3, E_4, E_5, E_6}

The Probability of an Event

- The probability of an event A measures "how often" A will occur. We write **P(A).**
- Suppose that an experiment is performed n times. The relative frequency for an event A is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

- If we let n get infinitely large,

$$P(A) = \lim_{n \to \infty} \frac{f}{n}$$

The Probability of an Event

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

- The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

Finding Probabilities

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events.
- **Examples:**
  - Toss a fair coin. P(Head) = 1/2
  - Suppose that 10% of the U.S. population has red hair. Then for a person selected at random,

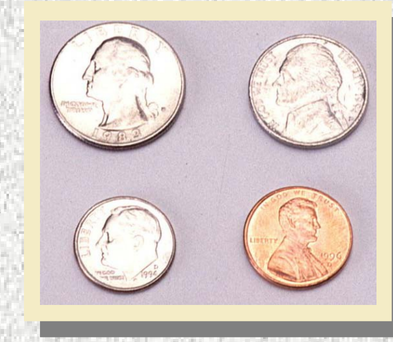
P(Red hair) = .10

Using Simple Events

- The probability of an event A is equal to the sum of the probabilities of the simple events contained in A
- If the simple events in an experiment are **equally likely**, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

Example 1



Toss a fair coin twice. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

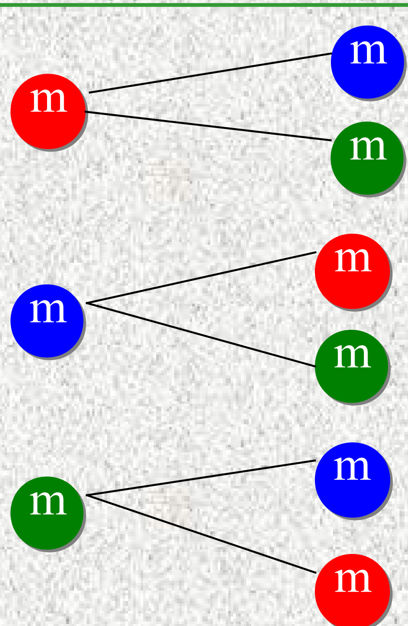
$= 1/4 + 1/4 + 1/4 = 3/4$

Note 5 of 5E

Example 2

A bowl contains three M&Ms®, one red, one blue and one green. A child selects two M&Ms at random. What is the probability that at least one is red?

1st M&M 2nd M&M



E_i	$P(E_i)$
RB	1/6
RG	1/6
BR	1/6
BG	1/6
GB	1/6
GR	1/6

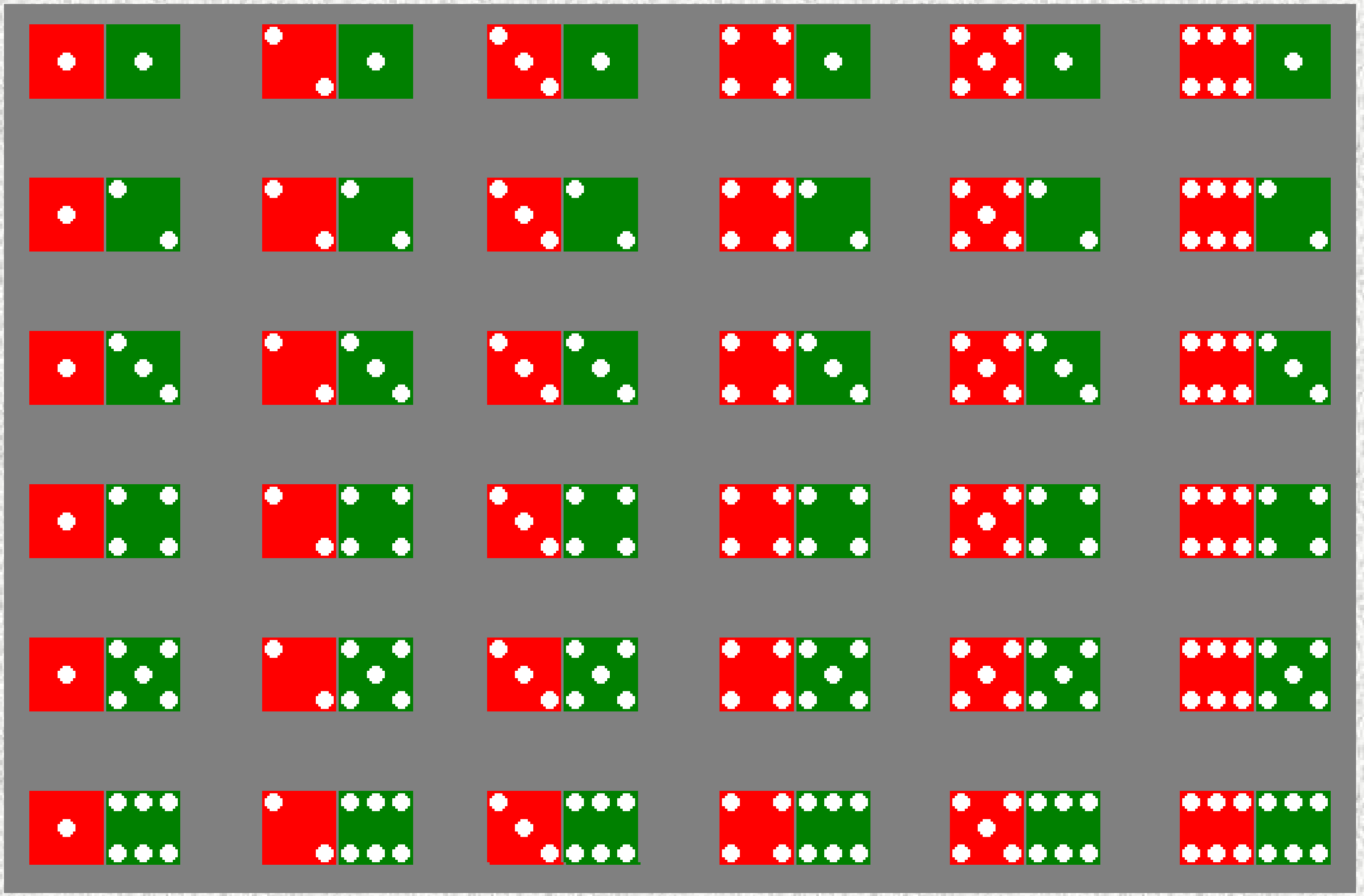
P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Example 3

The sample space of throwing a pair of dice is



Example 3

Event	Simple events	Probability
Dice add to 3	(1,2),(2,1)	2/36
Dice add to 6	(1,5),(2,4),(3,3),(4,2),(5,1)	5/36
Red die show 1	(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)	6/36
Green die show 1	(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)	6/36

Counting Rules

- Sample space of throwing 3 dice has 216 entries, sample space of throwing 4 dice has 1296 entries, …
- At some point, we have to stop listing and start thinking …
- We need some counting rules

The *mn* Rule

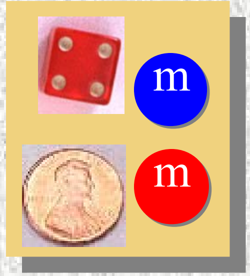
- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.
- This rule is easily extended to ***k*** stages, with the number of ways equal to

$$n_1 n_2 n_3 \dots n_k$$

Example: Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

Examples



Example: Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Example: Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

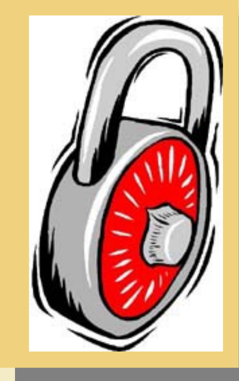
Example: Toss three dice. The total number of simple events is:

$$6 \times 6 \times 6 = 216$$

Example: Two M&Ms are drawn from a dish containing two red and two blue candies. The total number of simple events is:

$$4 \times 3 = 12$$

Permutations



- The number of ways you can arrange n distinct objects, taking them r at a time is $P_r^n = \dfrac{n!}{(n-r)!}$

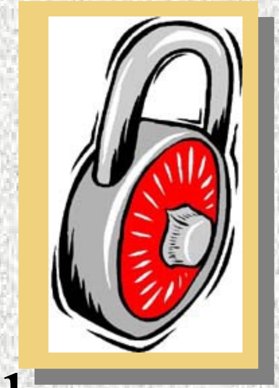
where $n! = n(n-1)(n-2)\ldots(2)(1)$ and $0! \equiv 1$.

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

Examples



Example: A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

$$P^5_5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

Combinations

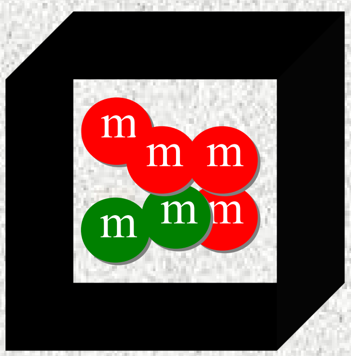
- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is $C_r^n = \dfrac{n!}{r!(n-r)!}$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example



- A box contains six M&Ms®, four red and two green. A child selects two M&Ms at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$C^6_2 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M&Ms.

$$C^2_1 = \frac{2!}{1!1!} = 2$$

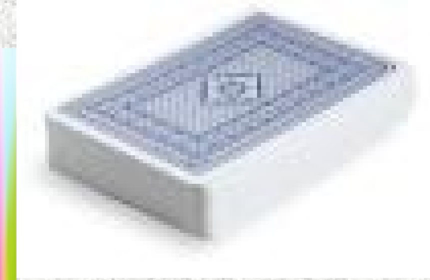
ways to choose 1 green M&M.

$$C^4_1 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M&M.

$4 \times 2 = 8$ ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

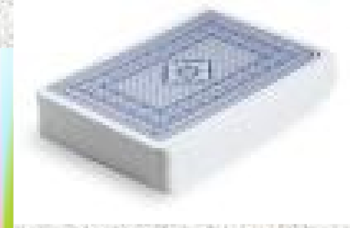


Example

A deck of cards consists of 52 cards, 13 "kinds" each of four suits (spades, hearts, diamonds, and clubs). The 13 kinds are Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), King (K). In many poker games, each player is dealt five cards from a well shuffled deck.

There are $C_5^{52} = \dfrac{52!}{5!\,(52-5)!} = \dfrac{52(51)(50)(49)48}{5(4)(3)(2)1} = 2,598,960$

possible hands



Example

Four of a kind: 4 of the 5 cards are the same "kind". What is the probability of getting four of a kind in a five card hand?

There are 13 possible choices for the kind of which to have four, and 52-4=48 choices for the fifth card. Once the kind has been specified, the four are completely determined: you need all four cards of that kind. Thus there are 13×48=624 ways to get four of a kind.

The probability=624/2598960=.000240096

Example

One pair: two of the cards are of one kind, the other three are of three different kinds. What is the probability of getting one pair in a five card hand?

There are 13 possible choices for the kind of which to have a pair; given the choice, there are $C_2^4 = 6$ possible choices of two of the four cards of that kind

Example

There are 12 kinds remaining from which to select the other three cards in the hand. We must insist that the kinds be different from each other and from the kind of which we have a pair, or we could end up with a second pair, three or four of a kind, or a full house.

Example

There are $C_3^{12} = 220$ ways to pick the kinds of the remaining three cards. There are 4 choices for the suit of each of those three cards, a total of $4^3 = 64$ choices for the suits of all three.

Therefore the number of *one pair* hands is $13 \times 6 \times 220 \times 64 = 1{,}098{,}240$.

The probability = 1098240/2598960=

¿.422569

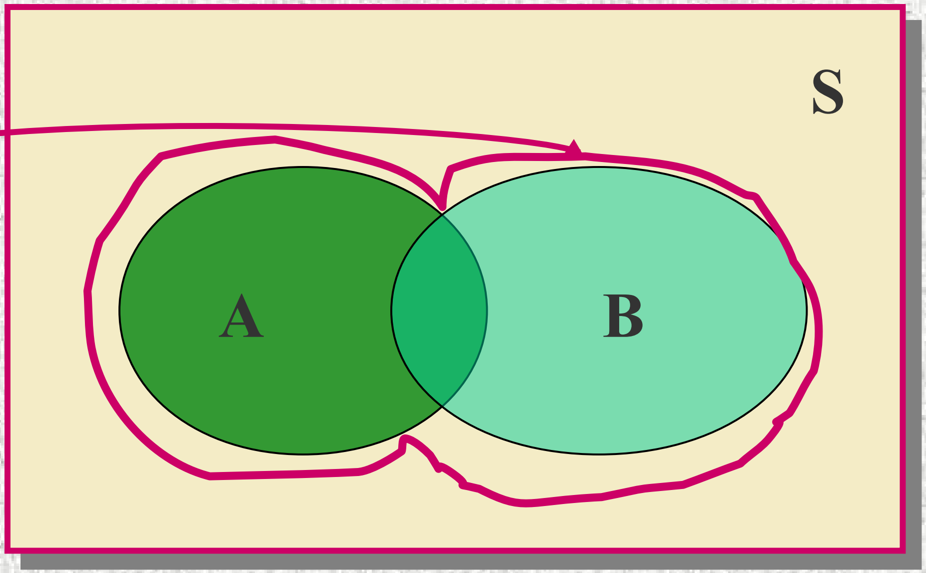
Event Relations

The beauty of using events, rather than simple events, is that we can **combine** events to make other events using logical operations: and, or and not.

The **union** of two events, **A** and **B**, is the event that either **A or B or both** occur when the experiment is performed. We write

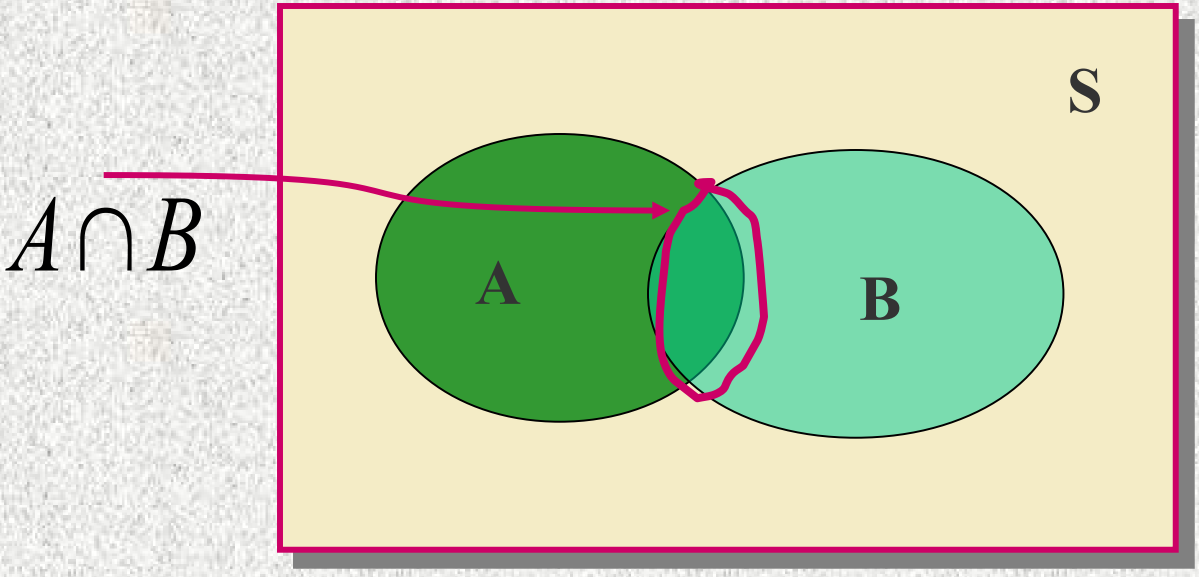
$$\mathbf{A \cup B}$$

$A \cup B$



Event Relations

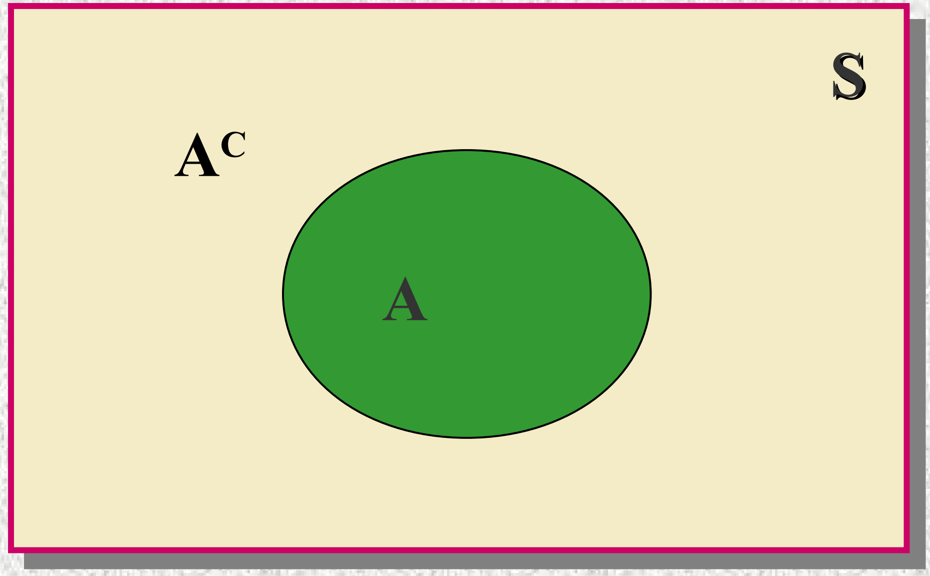
The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B**.



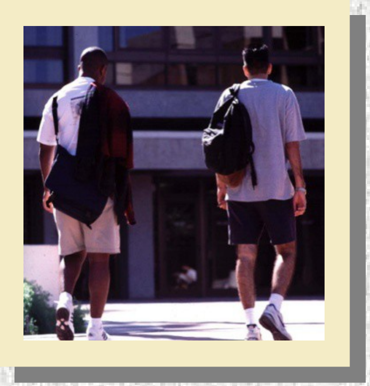
- If two events A and B are **mutually exclusive**, then $P(A \cap B) = 0$.

Event Relations

The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event **A**. We write A^C.



Example



Select a student from the classroom and record his/her **hair color** and **gender.**

– **A:** student has brown hair

– **B:** student is female

– **C:** student is male

Mutually exclusive; B = C^C

What is the relationship between events **B** and **C**?

- **A^C:** Student does not have brown hair
- **$B \cap C$:** Student is both male and female = $\varnothing$
- **$B \cup C$:** Student is either male and female = all students = S

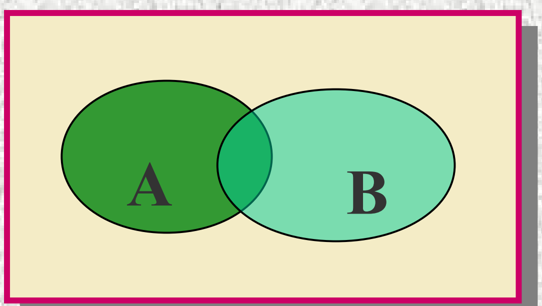
Calculating Probabilities for Unions and Complements

There are special rules that will allow you to calculate probabilities for composite events.

The Additive Rule for Unions:

For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example: Additive Rule

Example: Suppose that there were 120 students in the classroom, and that they could be classified as follows:

A: brown hair

P(A) = 50/120

B: female

P(B) = 60/120

	Brown	Not Brown
Male	20	40
Female	30	30

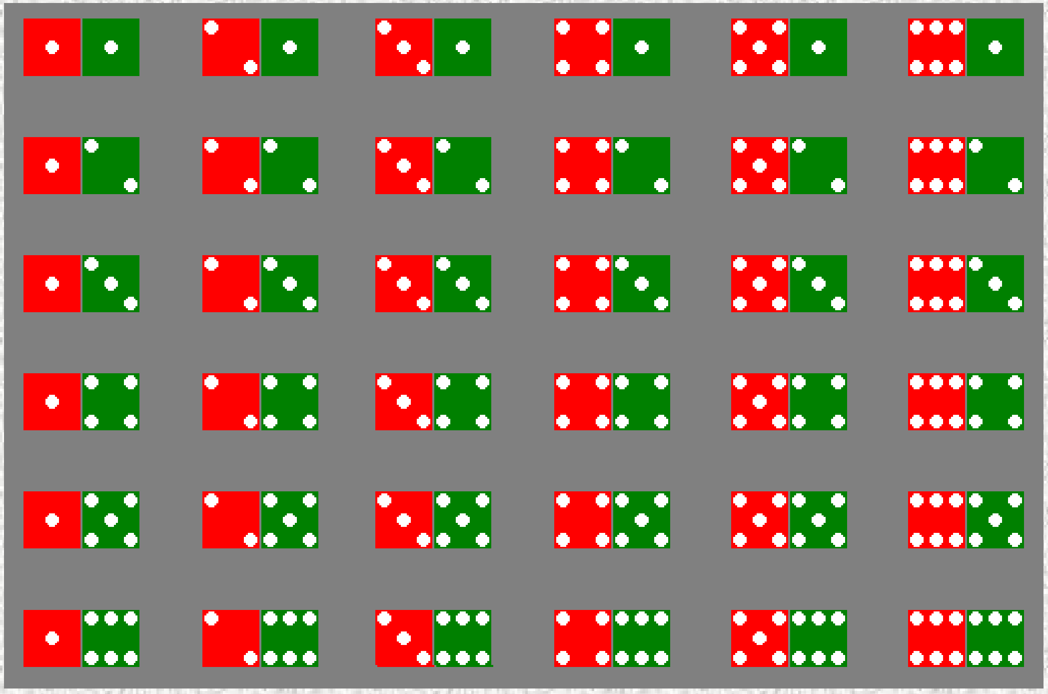
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 50/120 + 60/120 - 30/120$
$= 80/120 = 2/3$

Check: $P(A \cup B)$
$= (20 + 30 + 30)/120$

Example: Two Dice

A: red die show 1

B: green die show 1



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 6/36 + 6/36 - 1/36$$
$$= 11/36$$

A Special Case

When two events A and B are **mutually exclusive**, $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$.

A: male with brown hair

$P(A) = 20/120$

B: female with brown hair

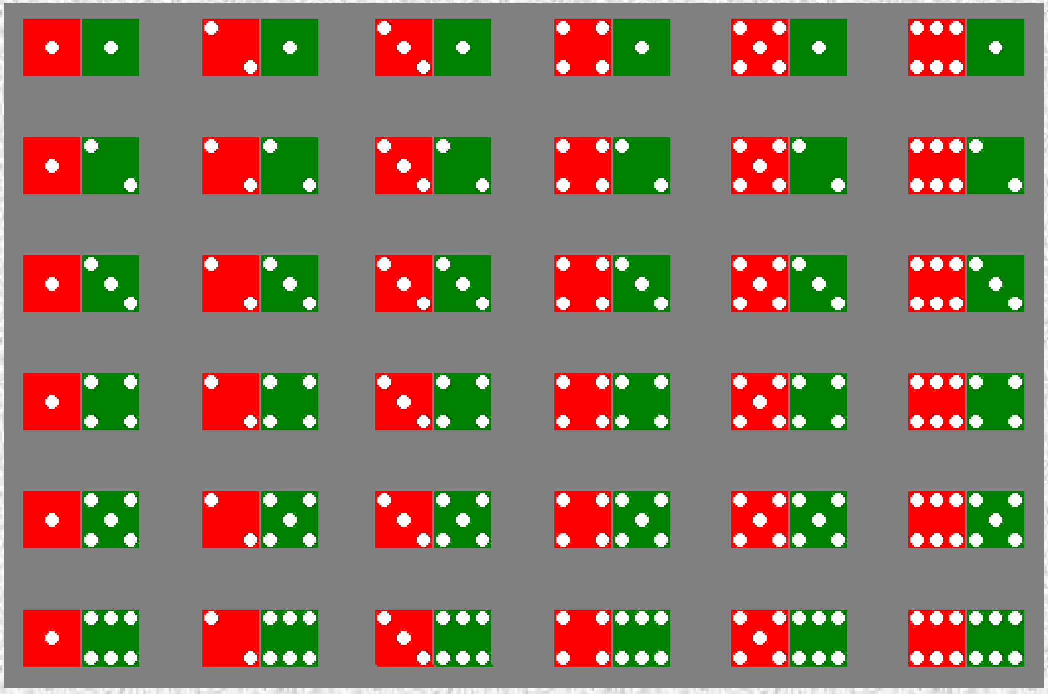
$P(B) = 30/120$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are mutually exclusive, so that

$P(A \cup B) = P(A) + P(B)$
$= 20/120 + 30/120$
$= 50/120$

Example: Two Dice



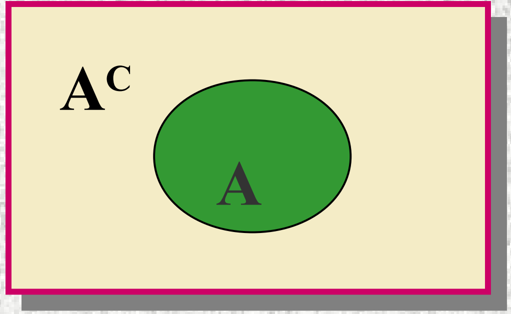
A: dice add to 3

B: dice add to 6

A and B are mutually exclusive, so that

$P(A \cup B) = P(A) + P(B)$
$= 2/36 + 5/36$
$= 7/36$

Calculating Probabilities for Complements



We know that for any event **A:**

– $P(A \cap A^C) = 0$

Since either **A** or A^C must occur,

$P(A \cup A^C) = 1$

so that $P(A \cup A^C) = P(A) + P(A^C) = 1$

$$\boxed{P(A^C) = 1 - P(A)}$$

Example

Select a student at random from the classroom. Define:

A: male

$P(A) = 60/120$

B: female

$P(B) = ?$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are complementary, so that

$P(B) = 1 - P(A)$

$= 1 - 60/120 = 60/120$

Calculating Probabilities for Intersections

In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events**.

Two events, **A** and **B**, are said to be **independent** if the occurrence or nonoccurrence of one of the events does not change the probability of the occurrence of the other event.

Conditional Probabilities

The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P\left(A \mid B\right) = \frac{P\left(A \cap B\right)}{P\left(B\right)} \text{ if } P\left(B\right) \neq 0$$

"given"

Example 1

Toss a fair coin twice. Define

- A: head on second toss
- B: head on first toss

P(A|B) = ½

P(A|not B) = ½

HH	1/4
HT	1/4
TH	1/4
TT	1/4

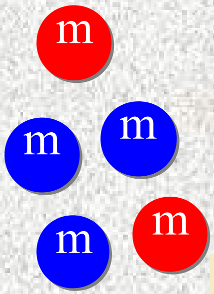
P(A) does not change, whether B happens or not…

A and B are independent!

Example 2

A bowl contains five M&Ms®, two red and three blue. Randomly select two candies, and define

- A: second candy is red.
- B: first candy is blue.



P(A|B) =P(2nd red|1st blue)= 2/4 = 1/2

P(A|not B) = P(2nd red|1st red) = 1/4

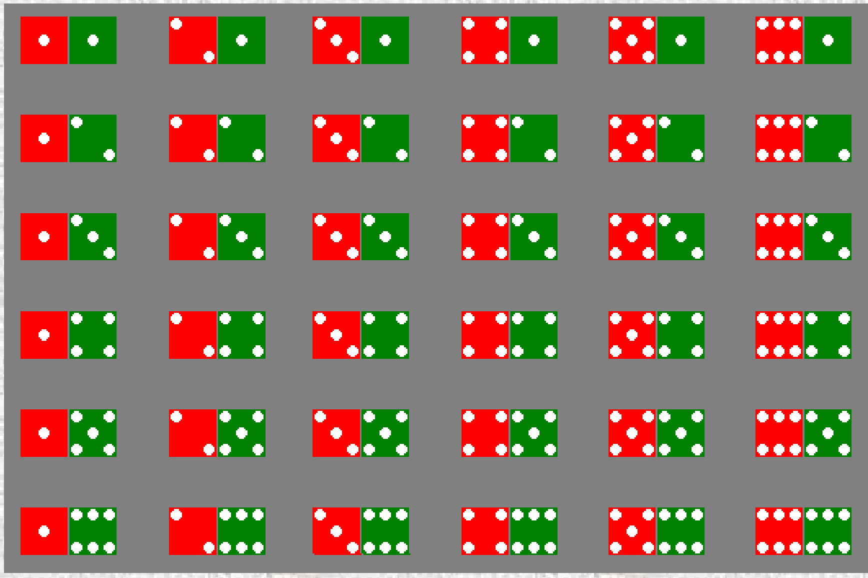
P(A) does change, depending on whether B happens or not…

A and B are dependent!

Example 3: Two Dice

Toss a pair of fair dice. Define

- A: red die show 1
- B: green die show 1



P(A|B) = P(A and B)/P(B)

=1/36/1/6=1/6=P(A)

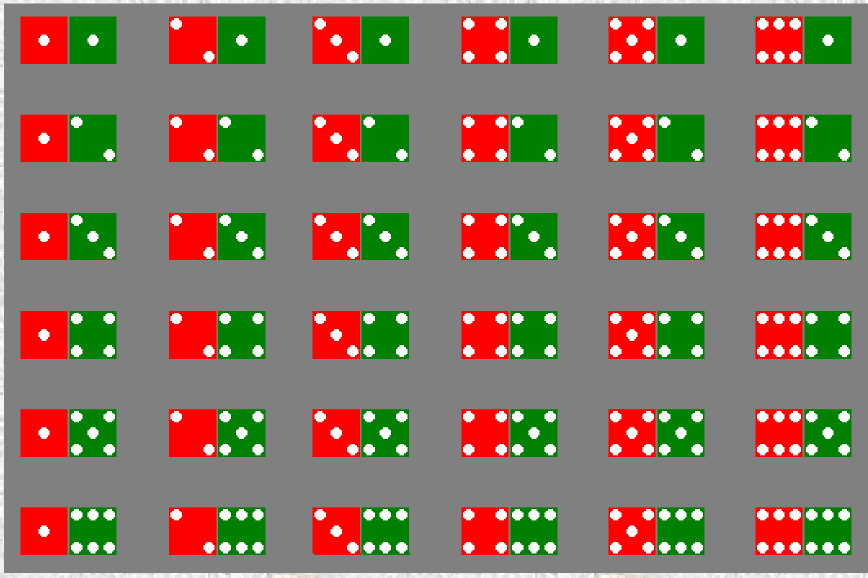
P(A) does not change, whether B happens or not…

A and B are independent!

Example 3: Two Dice

Toss a pair of fair dice. Define

- A: add to 3
- B: add to 6



P(A|B) = P(A and B)/P(B)

=0/36/5/6=0

P(A) does change when B happens

A and B are dependent! In fact, when B happens, A can't

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

- Once you've decided whether or not two events are independent, you can use the following rule to calculate their intersection.

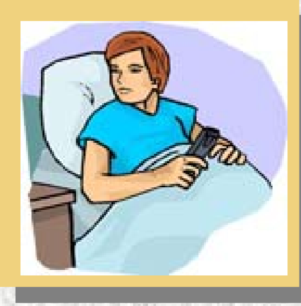
The Multiplicative Rule for Intersections

- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$\boxed{\begin{aligned} P(A \cap B) &= P(A)\, P(B \text{ given that } A \text{ occurred}) \\ &= P(A)P(B|A) \end{aligned}}$$

- If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

$$\boxed{P(A \cap B) = P(A)\, P(B)}$$



Example 1

In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk?

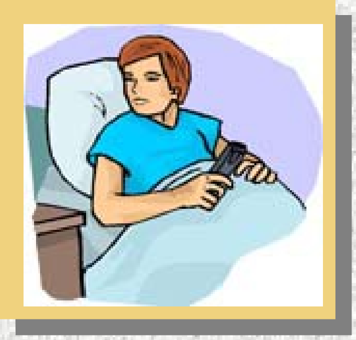
Define H: high risk N: not high risk

P(exactly one high risk) = P(HNN) + P(NHN) + P(NNH)

= P(H)P(N)P(N) + P(N)P(H)P(N) + P(N)P(N)P(H)

$= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$

Example 2



Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?

Define H: high risk F: female

From the example, $P(F) = .49$ and $P(H|F) = .08$.
Use the Multiplicative Rule:

$$P(\text{high risk female}) = P(H \cap F)$$

$$= P(F)P(H|F) = .49(.08) = .0392$$

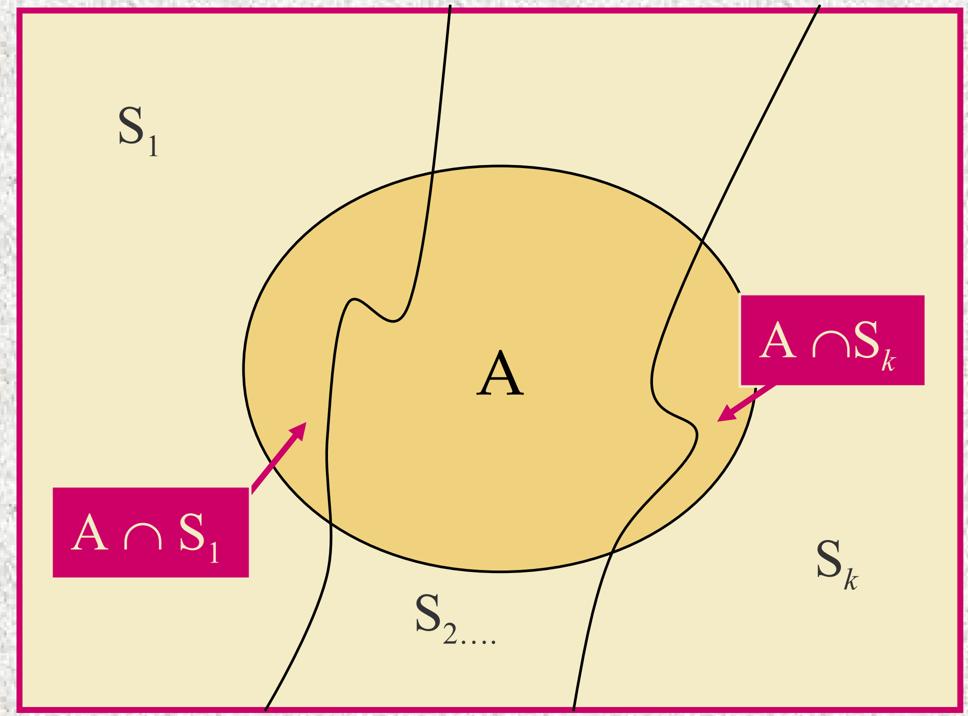
The Law of Total Probability

Let S_1, S_2, S_3 ,...., S_k be mutually exclusive and exhaustive events (that is, one and only one must happen). Then the probability of any event A can be written as

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)$$

$$= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)$$

The Law of Total Probability



$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)$$

$$= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)$$

Bayes' Rule

Let S_1 , S_2 , S_3 ,...., S_k be mutually exclusive and exhaustive events with prior probabilities $P(S_1)$, $P(S_2)$,…,$P(S_k)$. If an event A occurs, the posterior probability of S_i, given that A occurred is

$$P(S_i|A) = \frac{P(S_i)P(A|S_i)}{\sum P(S_i)P(A|S_i)} \quad \text{for } i = 1,\ 2, \ldots k$$

Proof

$$P(A|S_i) = \frac{P(AS_i)}{P(S_i)} \rightarrow P(AS_i) = P(S_i)P(A|S_i)$$

$$P(S_i|A) = \frac{P(AS_i)}{P(A)} = \frac{P(S_i)P(A|S_i)}{\sum P(S_i)P(A|S_i)}$$

Example

From a previous example, we know that 49% of the population are female. Of the female patients, 8% are high risk for heart attack, while 12% of the male patients are high risk. A single person is selected at random and found to be high risk. What is the probability that it is a male? Define H: high risk F: female M: male

We know:

P(F) = .49

P(M) = .51

P(H|F) = .08

P(H|M) = .12

$$P(M \mid H) = \frac{P(M)P(H \mid M)}{P(M)P(H \mid M) + P(F)P(H \mid F)}$$

$$\frac{.51(.12)}{.51(.12) + .49(.08)} = .61$$

Example

Suppose a rare disease infects one out of every 1000 people in a population. And suppose that there is a good, but not perfect, test for this disease: if a person has the disease, the test comes back positive 99% of the time. On the other hand, the test also produces some false positives: 2% of uninfected people are also test positive. And someone just tested positive. What are his chances of having this disease?

Example

Define A: has the disease B: test positive

We know:

$P(A) = .001$ $P(A^c) = .999$

$P(B|A) = .99$ $P(B|A^c) = .02$

We want to know $P(A|B) = ?$

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)}$$

$$= \frac{.001 \times .99}{.001 \times .99 + .999 \times .02} = .0472$$

Example

A survey of job satisfaction[2] of teachers was taken, giving the following results

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	74	43	117
	High School	224	171	395
	Elementary	126	140	266
	Total	424	354	778

[2] "Psychology of the Scientist: Work Related Attitudes of U.S. Scientists" (*Psychological Reports* (1991): 443 – 450).

Example

If all the cells are divided by the total number surveyed, 778, the resulting table is a table of empirically derived probabilities.

LEVEL		Job Satisfaction		
		Satisfied	Unsatisfied	Total
	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

For convenience, let C stand for the event that the teacher teaches college, S stand for the teacher being satisfied and so on. Let's look at some probabilities and what they mean.

$P(C) = 0.150$ is the proportion of teachers who are college teachers.

$P(S) = 0.545$ is the proportion of teachers who are satisfied with their job.

$P(C \cap S) = 0.095$ is the proportion of teachers who are college teachers and who are satisfied with their job.

Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

$$P(C\,|\,S) = \frac{P(C \cap S)}{P(S)} = \frac{0.095}{0.545} = 0.175$$

is the proportion of teachers who are college teachers given they are satisfied. Restated: This is the proportion of satisfied that are college teachers.

$$P(S\,|\,C) = \frac{P(S \cap C)}{P(C)} = \frac{P(C \cap S)}{P(C)} = \frac{0.095}{0.150} = 0.632$$

is the proportion of teachers who are satisfied given they are college teachers. Restated: This is the proportion of college teachers that are satisfied.

Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

Are C and S independent events?

$$P(C) = 0.150 \text{ and } P(C\,|\,S) = \frac{P(C \cap S)}{P(S)} = \frac{0.095}{0.545} = 0.175$$

$\boxed{P(C|S) \neq P(C) \text{ so C and S are dependent events.}}$

Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.658
	Total	0.545	0.455	1.000

$P(C \cap S)$?

$P(C) = 0.150$, $P(S) = 0.545$ and

$P(C \cup S) = 0.095$, so

$$P(C \cap S) = P(C)+P(S) - P(C \cup S)$$
$$= 0.150 + 0.545 - 0.095$$
$$= 0.600$$

Example

Tom and Dick are going to take a driver's test at the nearest DMV office. Tom estimates that his chances to pass the test are 70% and Dick estimates his as 80%. Tom and Dick take their tests independently.

Define D = {Dick passes the driving test}

T = {Tom passes the driving test}

T and D are independent.

P (T) = 0.7, P (D) = 0.8

Example

What is the probability that at most one of the two friends will pass the test?

P(At most one person pass)

$= P(D^c \cap T^c) + P(D^c \cap T) + P(D \cap T^c)$

$= (1 - 0.8)\,(1 - 0.7) + (0.7)\,(1 - 0.8) + (0.8)\,(1 - 0.7)$

$= .44$

P(At most one person pass)

$= 1 - P(\text{both pass}) = 1 - 0.8 \times 0.7 = .44$

Example

What is the probability that at least one of the two friends will pass the test?

P(At least one person pass)

$= P(D \cup T)$

= 0.8 + 0.7 - 0.8 x 0.7

= .94

P(At least one person pass)

= 1-P(neither passes) = 1- (1-0.8) x (1-0.7) = .94

Example

Suppose we know that only one of the two friends passed the test. What is the probability that it was Dick?

P(D | exactly one person passed)

= P(D $\cap$ exactly one person passed) / P(exactly one person passed)

= $P(D \cap T^c) / (P(D \cap T^c) + P(D^c \cap T))$

= 0.8 x (1-0.7)/(0.8 x (1-0.7)+(1-.8) x 0.7)

= .63

Key Concepts

I. Experiments and the Sample Space

1. Experiments, events, mutually exclusive events, simple events
2. The sample space

II. Probabilities

1. Relative frequency definition of probability
2. Properties of probabilities
    a. Each probability lies between 0 and 1.
    b. Sum of all simple-event probabilities equals 1.
3. *P(A)*, the sum of the probabilities for all simple events in *A*

Key Concepts

III. Counting Rules

1. *mn* Rule; extended *mn* Rule
2. Permutations: $P_r^n = \frac{n!}{(n-r)!}$
3. Combinations: $C_r^n = \frac{n!}{r!(n-r)!}$

IV. Event Relations

1. Unions and intersections
2. Events
   a. Disjoint or mutually exclusive: $P(A \cap B) = 0$
   b. Complementary: $P(A) = 1 - P(A^C)$

Key Concepts

3. Conditional probability: $P(A|B)=\frac{P(A\cap B)}{P(B)}$

4. Independent and dependent events

5. Additive Rule of Probability:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

6. Multiplicative Rule of Probability:

$$P(A\cap B)=P(A)P(B|A)$$

7. Law of Total Probability

8. Bayes' Rule