Predicates:

Statements involving variables, such as
"**x > 3**" and "Computer x is functioning properly"
can be true or false depending on the value of x.

Statement "**x** is greater than **3**" has two parts:

1. Subject: variable **x**
2. Predicate: "**is greater than 3**"

We can denote the statement "x is greater than 3" by
P(x), where x is a variable.
P denotes the predicate "is greater than 3".
P is also called as "Propositional Function".

Predicate Logic (Predicate Calculus):
The area of logic that deals with predicates and quantifiers.

Essentially, it's dealing with the declarative statements which has **propositional variables** and convert them into **propositions**.
How?
1. By assigning a value to the variable.
2. By quantifying the value of the variable.

P(x) is the value of the **propositional function** P at x.

P(x) becomes a proposition when a value is assigned to x (and P is defined).

Eg: Let P(x) denote the statement “x > 3”.

P(4) denotes “4 > 3”, which is true.

∴ P(4) is true.

P(2) denotes “2 > 3”, which is false.

∴ P(2) is false.

Eg: Let $P(x,y)$ denote the statement "$x = y + 3$".

$P(1, 2)$ denotes "$1 = 2 + 3$", which is false.

$\therefore P(1, 2)$ is false.

$P(4, 1)$ denotes "$4 = 1 + 3$", which is true.

$\therefore P(4, 1)$ is true.

$P(x, y)$ is a binary predicate.

$P(x)$ is an unary predicate.

n-ary predicate:

A statement having n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$.

P is the called as n-place predicate or an n-ary predicate.

Quantifiers:
Another way to convert a propositional function to a proposition.

Universal Quantification:
$\forall x P(x)$: “P(x) for all values of x in the domain”
Eg: “All politicians are dishonest”

Existential Quantification:
$\exists x P(x)$: “There exists an element x in the domain such that P(x)”
Eg: “There is a politician who is honest”
Let dishonest ≡ ¬honest.

Universal Quantification:

$\forall x P(x)$: "P(x) for all values of x in the domain"

or "for all x P(x)"

or "for every x P(x)"

or *for each*, *all of, given any, for arbitrary,* or *for any*

$\forall x P(x)$ is true when P(x) is true for all the values of x in the domain.

Just one counterexample is good enough to prove $\forall x P(x)$ is false.

Existential Quantification:

$\exists xP(x)$: “There exists an element x in the domain such that P(x)”

or “there exists an x such that P(x)”

or “there is at least one value of x such that x P(x)”

or *for some*, there is a

$\exists xP(x)$ is true when P(x) is true for at least one value of x in the domain.

$\exists xP(x)$ is false only when P(x) is false for all x.

Q: Let P(x) be the statement "$x^2 > 0$". What is the truth value of the quantification **∀xP(x)**, where the domain consists of the set of **integers**.

Soln: P(x) is not true for x=0. P(0) is false.

It is a counterexample.

∴ ∀xP(x) is a false proposition.

Q: … ∀xP(x) … positive integers.

Soln: P(x) is true for all the values of x in the domain.

∴ ∀xP(x) is a true proposition.

Q: … ∃xP(x) … integers.

Soln: P(x) is true for x=1. P(1) is true.

∴ ∃xP(x) is a true proposition.

When all the elements of the domain can be listed like x_1, x_2, …, x_n (domain is a finite set),

$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

$\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Eg: P(x): "$x^2 < 9$" and x is an int in the range [1, 3].

$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$

P(3), which is "$3^2 < 9$" is false.

∴ **∀xP(x) is a false** proposition

$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$

P(1), which is "$1^2 < 9$" is true.

∴ **∃xP(x) is a true** proposition

Binding Variables:

$\exists x(x+y=1)$, the variable x is bound, but y is free. Hence, it is not a proposition. If any variable in the expression is not bound, the expression is not yet a proposition.

$\boldsymbol{\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)}$ and $\boldsymbol{\forall x \exists y(x+y = 1)}$ are propositions because all the involved variables are bound.

Precedence of quantifiers and logical operators:

$\forall \; \exists$

$\neg$

$\wedge \; \vee$

$\rightarrow \; \leftrightarrow$

Eg: $p \vee \neg q \wedge r \rightarrow s$ is same as $(p \vee ((\neg q) \wedge r)) \rightarrow s$

Eg: $\forall x P(x) \vee Q(x)$ is same as $(\forall x P(x)) \vee Q(x)$

$$\sum_{1}^{n}(\mathrm{a}+\mathrm{b}) = \sum_{1}^{n}\mathrm{a} + \sum_{1}^{n}\mathrm{b}$$

$$\sum_{1}^{4}(3+5) = \sum_{1}^{4}3 + \sum_{1}^{4}5$$

$$(3+5)+(3+5)+(3+5)+(3+5)$$

$$= (3+3+3+3)+(5+5+5+5)$$

$$8+8+8+8 = 12 + 20$$

$$32 = 32$$

Logical Equivalences involving quantifiers

Which of the following logical equivalences are valid?

$\forall x(\ \mathbf{P(x)} \land \mathbf{Q(x)}\ ) \equiv \forall x P(x) \land \forall x Q(x)$

$\forall x(\ \mathbf{P(x)} \lor \mathbf{Q(x)}\ ) \equiv \forall x P(x) \lor \forall x Q(x)$

$\exists x(\ \mathbf{P(x)} \lor \mathbf{Q(x)}\ ) \equiv \exists x P(x) \lor \exists x Q(x)$

$\exists x(\ \mathbf{P(x)} \land \mathbf{Q(x)}\ ) \equiv \exists x P(x) \land \exists x Q(x)$

Logical Equivalences involving quantifiers

Universal quantification of conjunction of two predicates is equivalent to conjunction of universal quantification of the two predicates, where same domain is used throughout.

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

$$\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$$

But, $\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$

$$\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$$

Logical Equivalences involving quantifiers

Q: Show that $\mathbf{\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)}$ where the same domain is used throughout.
(That is, a universal quantifier can be distributed over a conjunction)

Proof: $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$ means
$\forall x(P(x) \land Q(x)) \leftrightarrow \forall x P(x) \land \forall x Q(x)$ is a tautology.

That is, to prove
$\forall x(P(x) \land Q(x)) \rightarrow \forall x P(x) \land \forall x Q(x)$ is a tautology **and**
$\forall x P(x) \land \forall x Q(x) \rightarrow \forall x(P(x) \land Q(x))$ is a tautology.

To prove: $\forall x(P(x) \land Q(x)) \rightarrow \forall x P(x) \land \forall x Q(x)$ is a T.

Suppose, $\forall x(P(x) \land Q(x))$ is true.

$P(a) \land Q(a)$ is true for any ‘a’ in the domain.

$P(a)$ is true and $Q(a)$ is true.

Because ‘a’ is an arbitrary element in the domain and $P(a)$ is true, we can conclude that $\forall x P(x)$ is true.

Similarly, $\forall x Q(x)$ is true.

That means $\forall x P(x) \land \forall x Q(x)$ is true.

$\therefore \forall x(P(x) \land Q(x)) \rightarrow \forall x P(x) \land \forall x Q(x)$ is a tautology ..①

To prove: $\forall x P(x) \land \forall x Q(x) \rightarrow \forall x (P(x) \land Q(x))$ is a T.

Suppose, $\forall x P(x) \land \forall x Q(x)$ is true.

That is, $\forall x P(x)$ is true and $\forall x Q(x)$ is true.

$P(a)$ is true and $Q(a)$ is true for any ‘a’ in the domain.

Because $P(a)$ and $Q(a)$ true for all values in the domain,
for a given ‘a’, $P(a) \land Q(a)$.

Because ‘a’ is an arbitrary element in the domain,
$\forall x (P(x) \land Q(x))$ is true

$\therefore \forall x P(x) \land \forall x Q(x) \rightarrow \forall x (P(x) \land Q(x))$ is a tautology. ---②

By ① and ②,

$\forall x (P(x) \land Q(x)) \leftrightarrow \forall x P(x) \land \forall x Q(x)$ is a tautology.

$\therefore \forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

Eg: What is the **negation** of “All politicians are dishonest”?

Let $P(x)$ be “x is a dishonest politician”.
$\forall x P(x)$ is “All politicians are dishonest”.

$\neg\forall x P(x)$ would be “It is not the case that all politicians are dishonest”.
In other words, “Not all politicians are dishonest”.
That is, “There is a politician who is not dishonest”.
$\neg\forall x P(x) \equiv \exists x \neg P(x)$ ***not dishonest***
≡ *honest*

$\exists x \neg P(x)$ would be “There exists a politician who is **not** dishonest”.

Q: Prove that $\neg\forall x P(x) \equiv \exists x \neg P(x)$

Proof: $\neg\forall x P(x)$ is true iff $\forall x P(x)$ is false.

$\forall x P(x)$ is false iff there is an element x in the domain for which P(x) is false.

There is an element x in the domain for which $\neg P(x)$ is true.

That is, $\exists x \neg P(x)$ is true.

$\therefore \neg\forall x P(x) \equiv \exists x \neg P(x)$

Logical Equivalences involving quantifiers

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

$$\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$$

But, $$\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$$

$$\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$$

De Morgan's laws for quantifiers

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

$\neg \exists x P(x) \equiv \forall x \neg P(x)$

Eg: $\forall x\ (x^2>0)$ **where** $x \in R^-$

$\forall_{x<0}\ (x^2>0)$ **where** $x \in R$

For every real number x which is lesser than zero, $x^2>0$.

In other words, for every real number x,

if $x < 0$, then $x^2 > 0$.

i.e. $\forall x\ (x < 0 \rightarrow x^2 > 0)$ **where** $x \in R$

Eg: $\forall_{x<0}\ (x^2 > 0)$ **where** $x \in R$

For every real number x which is lesser than zero,

$x^2 > 0$.

In other words, for every real number x,

if $x < 0$, then $x^2 > 0$.

i.e. $\forall x\ (x < 0 \rightarrow x^2 > 0)$ **where** $x \in R$

Eg: $\exists_{x<0}\ (x^2 = 2)$ **where** $x \in R$

There exists a real number $x < 0$, such that $x^2 = 2$.

In other words, there exists a real number x,

such that $x < 0$ and $x^2 = 2$.

i.e. $\exists x\ ((x < 0) \land (x^2 = 2))$ **where** $x \in R$

Eg: $\forall_{x<0}\ (x^2 > 0)$ **where** $x \in R$

Equivalent to: $\forall x\ (x < 0 \rightarrow x^2 > 0)$ **where** $x \in R$

Not equivalent to: $\forall x\ (x < 0 \land x^2 > 0)$ **where** $x \in R$

Eg: $\exists_{x<0}\ (x^2 = 2)$ **where** $x \in R$

Equivalent to: $\exists x\ ((x < 0) \land (x^2 = 2))$ **where** $x \in R$

Not equivalent to: $\exists x\ ((x < 0) \rightarrow (x^2 = 2))$ **where** $x \in R$

Let the domain be all the animals.

D(x): x is a dog

M(x): x is a mammal

C(x): x is cute

"All dogs are mammals" is equivalent to

$\forall x\, (D(x) \rightarrow M(x))$ or $\forall x\, (D(x) \wedge M(x))$?

"Some dogs are cute" is equivalent to

$\exists x\, (D(x) \rightarrow C(x))$ or $\exists x\, (D(x) \wedge C(x))$?

Explanation for: “All dogs are mammals”

What does $\forall x\ (D(x) \rightarrow M(x))$ mean?

$\forall x\ (D(x) \rightarrow M(x)) \equiv \forall x\ (\neg D(x) \lor M(x))$

Each animal is a non-dog or a mammal.

It means all dogs are mammals.

What does $\forall x\ (D(x) \land M(x))$ mean?

$\forall x\ (D(x) \land M(x)) \equiv (\forall x\ D(x)) \land (\forall x\ M(x))$

All animals are dogs and all animals are mammals.

It is **not** same as “All dogs are mammals”.

Explanation for: “Some dogs are cute”

What does **$\exists x\ (D(x) \land C(x))$** mean?

There exists an animal, which is both a dog and a cute animal.

It means there exists a cute dog.

What does **$\exists x\ (D(x) \to C(x))$** mean?

$\exists x\ (D(x) \to C(x)) \equiv \exists x\ (\neg D(x) \lor C(x))$

$\equiv (\exists x\ \neg D(x)) \lor (\exists x\ C(x))$

There is a non-dog animal or there is a cute animal.

It is **not** same as “Some dogs are cute”

Nested Quantifiers

Eg: $\forall x \exists y\,(x+y=0)$ where $x,y \in \mathbf{R}$

In words, “For each real number x,
there exists a real number y such that x+y=0”.

$\forall x \exists y\,(x+y=0)$ is a proposition

$\forall x Q(x)$ where $Q(x)$ is $\exists y\,(x+y=0)$.

$\forall x Q(x)$ where $Q(x)$ is $\exists y\, P(x,y)$ and $P(x,y)$ is $x+y=0$

Other Quantifiers:

We can define quantifier other than the standard universal and existential quantifiers.

$\forall x \exists ! y\ (x+y=0)$ where $x,y \in \mathbf{R}$ means

"For each real number x, there exists **exactly one** real number y such that x+y=0".

$\exists!$ means "there exists exactly one"

$\exists_{10}$ means "there exists exactly 10"

Eg: $\forall x \forall y\ ((x<0) \land (y<0) \rightarrow (xy > 0))$ where $x,y \in \mathbf{R}$

"For each pair of real numbers (x,y), if both are greater than 0, then their product is greater than 0"

$\boldsymbol{\forall y \forall x}\ ((x>0) \land (y>0) \rightarrow (xy > 0))$ where $x,y \in \mathbf{R}$?

Eg: $\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{Z}^+$

"There is a 3-tuple (x,y,z) of non-trivial positive integers for which $(x^2 + y^2 = z^2)$ is true"

$\boldsymbol{\exists y \exists z \exists x}\ (x^2 + y^2 = z^2)$ where x,y,z are non-trivial positive integers ?

$\forall x \exists y\, P(x,y) \equiv \exists y \forall x\, P(x,y)$ **?**

Let $P(x, y)$: $x+y=0$

$\forall x \exists y\, (x+y=0)$

"For each real number x, there exists a real number y such that x+y=0"

$\exists y \forall x\, (x+y=0)$

"There is a real number y which matches with every real number x such that x+y=0"

$\therefore\ \forall x \exists y\, P(x,y) \not\equiv \exists y \forall x\, P(x,y)$

Nested Quantification of two variables

$\forall x \forall y\ P(x,y)$ $\forall y \forall x\ P(x,y)$	P(x,y) is true for every pair (x,y)
$\exists x \exists y\ P(x,y)$ $\exists y \exists x\ P(x,y)$	P(x,y) is true for at least one pair of (x,y)
$\forall x \exists y\ P(x,y)$	For each x, there is a y such that P(x,y) is true
$\exists x \forall y\ P(x,y)$	There is a x such that P(x,y) is true for every y

Are these statements true?

$\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$

$\forall \mathbf{x} \exists y \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$

$\forall \mathbf{x} \forall \mathbf{y} \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$

$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z}\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$

Are these statements true?

$\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$
"There is a 3-tuple (x,y,z) for which ($x^2 + y^2 = z^2$) is true"

$\forall \mathbf{x} \exists y \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$
"For all x, there is a pair (y,z) for which ($x^2+y^2=z^2$) is true"

$\forall \mathbf{x} \forall \mathbf{y} \exists z\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$
"For every pair (x,y) there is a z for which ($x^2+y^2=z^2$) is true"

$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z}\ (x^2 + y^2 = z^2)$ where $x,y,z \in \mathbf{R}^+$
"For every 3-tuple (x,y,z), ($x^2 + y^2 = z^2$) is **false**"

Eg: Negation of $\forall x \exists y\ (xy = 1)$.

Soln: Negation of $\forall x \exists y\ (xy = 1)$

$\equiv \neg \forall x \exists y\ (xy = 1)$

Eg: Negation of $\forall x \exists y\,(xy = 1)$.

Soln: Negation of $\forall x \exists y\,(xy = 1)$

$\equiv \neg \forall x \exists y\,(xy = 1)$

$\equiv \exists x \neg \exists y\,(xy = 1)$ $\quad \because \neg \forall x P(x) \equiv \exists x \neg P(x)$

$\equiv \exists x \forall y\, \neg(xy = 1)$ $\quad \because \neg \exists x P(x) \equiv \forall x \neg P(x)$

$\equiv \exists x \forall y\,(xy \neq 1)$