Important points :=

* Before finding a_0, a_n & b_n, first verify whether the given function $f(x)$ is even or odd

* If $f(x)$ is even, then $b_n = 0$.
  Find only a_0 & a_n
  (No need of finding b_n)

* If $f(x)$ is odd, then $a_0 = a_n = 0$.
  Find only b_n.
  No need to find b_n

* If $f(x)$ is neither even nor odd, then find a_0, a_n & b_n

* $\sin(0) = \sin(n\pi) = 0$

* $\cos(n\pi) = (-1)^n$

* $\cos(2n\pi) = 1$

* $\cos(2n+1)\pi = -1$

1) Obtain the Fourier Series for the function $f(x) = e^{-ax}$ in $-\pi \le x \le \pi$ & hence derive the Series for $\frac{\pi}{\sinh \pi}$

Ans: By data $f(x) = e^{-ax}$ in $-\pi \le x \le \pi$

This is neither even nor odd

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

{ To find l, Compare the width of the general interval with width of the given interval }

$$2l = \pi - [-\pi] \Rightarrow 2l = 2\pi \Rightarrow l = \pi \quad \therefore \frac{n\pi x}{l} = \frac{n\pi x}{\pi} = nx$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] \quad — ①$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax}\,dx = \left. \frac{e^{-ax}}{-\pi a} \right]_{-\pi}^{\pi}$$

$$= \frac{-1}{\pi a} \left[e^{-a\pi} - e^{a\pi} \right] = \frac{1}{\pi a} \left[e^{a\pi} - e^{-a\pi} \right]$$

$$a_0 = \frac{1}{\pi a} \, \frac{e^{a\pi} - e^{-a\pi}}{2} \times 2$$

$$a_0 = \frac{1}{\pi a} \, 2 \sinh a\pi$$

or $$\boxed{\frac{a_0}{2} = \frac{\sinh a\pi}{\pi a}}$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \qquad \boxed{\begin{matrix} l=\pi \\ c=-\pi \\ c+2l=\pi \end{matrix}}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos(nx) + n\sin(nx)\}\right]_{x=-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\{-a\cos(n\pi)+0\} - e^{a\pi}\{-a\cos(n\pi)+0\}\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[a\cos(n\pi)\left[e^{a\pi}-e^{-a\pi}\right]\right]$$

$$= \frac{a}{\pi(a^2+n^2)}\left[-1\right]^n\left[e^{a\pi}-e^{-a\pi}\right] = \underline{\underline{\frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi}}$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\,\cancel{\sin(nx)}^{\,0} - n\cos(nx)\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\{-n\cos(n\pi)\} - e^{a\pi}\{-n\cos(n\pi)\}\right]$$

$$= \frac{n(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right) = \frac{2n(-1)^n}{\pi(a^2+n^2)}\sinh(a\pi)$$

$\therefore$ Eqⁿ ① becomes

$$\boxed{e^{-ax} = \frac{\sinh a\pi}{\pi a} + \sum_{n=1}^{\infty}\left[\frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}\cos(nx) + \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}\sin(nx)\right]}$$

$$e^{-ax} = \frac{\sinh a\pi}{\pi a} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\left[a\cos(nx) + n\sin(nx)\right]$$

To derive the series for $\frac{\pi}{\sinh \pi}$

put $x=0$ & $a=1$

$$1 = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{\sinh\pi} = 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$= 1 + 2\left[-\frac{1}{1^2+1} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \cdots\right]$$

$$= 1 + 2\left[-\frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \cdots\right]$$

$$\frac{\pi}{\sinh\pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - + \cdots\right]$$

(2) Find the Fourier Series of $f(x) = x + x^2$ in $(-\pi, \pi)$
Hence deduce that (i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\frac{\pi^2}{12} = \sum \frac{(-1)^n}{n^2}$

Ans: By data $f(x) = x + x^2$ in $(-\pi, \pi)$

Replace x by $-x$

$f(-x) = -x + x^2 \neq f(x) \neq -f(x)$

This is neither even nor odd function

$2l = \pi - (-\pi)$ = Width of the given interval

$= 2\pi \Rightarrow l = \pi$ $\therefore$ $\frac{n\pi x}{l} = \frac{n\pi x}{l} = nx$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] \quad - (1)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3} \right) \right]$$

$$a_0 = \frac{1}{\pi} \left[\frac{2\pi^3}{3} \right] = \frac{2\pi^2}{3} \Rightarrow \boxed{\frac{a_0}{2} = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underset{\mathrm{I}}{(x + x^2)} \underset{\mathrm{II}}{\cos(nx)}\, dx$$

$$= \frac{1}{\pi} \left[(x + x^2) \left\{ \cancelto{0}{\frac{\sin(nx)}{n}} \right\} - \left\{ \frac{-\cos(nx)}{n^2} \right\} (1 + 2x) - \left\{ \cancelto{0}{-\frac{\sin(nx)}{n^3}} \right\} 2 \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2} \left[(1 + 2x) \cos(nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2} \left[\{ (1 + 2\pi) \cos(n\pi) \} - \{ (1 - 2\pi) \cos(n\pi) \} \right]$$

$$= \frac{1}{\pi n^2} \left[4\pi (-1)^n \right] = \frac{4(-1)^n}{n^2} \Rightarrow \boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left\{-\frac{\cos(nx)}{n}\right\} - \left\{-\cancel{\frac{\sin(nx)}{n^2}}\right\}^{0}(1+2x) + \left\{\frac{\cos(nx)}{n^3}\right\}(2)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left\{(x+x^2)\cos(nx)\right\} + \frac{2}{n^3}\cancel{\cos(nx)}^{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left\{(\pi+\pi^2)\cos(n\pi) - (-\pi+\pi^2)\cos(n\pi)\right\}\right]$$

$$= \frac{(-1)^n}{-\pi n}\left[\pi+\pi^2+\pi-\pi^2\right] = \frac{(-1)^n}{-\pi n}2\pi$$

$$\boxed{b_n = \frac{2(-1)^{n+1}}{n}}$$

$\therefore$ Eqn (1) becomes

$$x+x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos(nx) + \frac{2(-1)^{n+1}}{n}\sin(nx)\right] \quad \text{—— (2)}$$

In order to get the given series put $x=0$ in (2)

$$0+0^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2} + 0$$

$$\Rightarrow -\frac{\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}$$

$$= 4\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\right]$$

$$\frac{\pi^2}{12} = -\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\cdots$$

$$\boxed{\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}} \quad \text{—— (A)}$$

put $x=\pi$ in (1)

Here $x=\pi$ is one of the end point of range of x.

WKT at the end points $f(x) \to \frac{1}{2}[f(c)+f(c+2l)]$

ie $(x+x^2) \to \frac{1}{2}[f(-\pi)+f(\pi)]$

$$\to \frac{1}{2}[(-\pi+\pi^2)+(\pi+\pi^2)] = \pi^2$$

$$\therefore \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n + 0$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum\frac{1}{n^2}$$

$$\frac{3\pi^2-\pi^2}{3} = 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\therefore \boxed{\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots} \quad \text{— (B)}$$

NOTE: If we add (A) + (B), we get

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

3) Express $f(x) = (\pi - x)^2$ as a Fourier Series of period 2π in the interval $0 < x < 2\pi$ & hence deduce the sum of the series

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Ans : By data $f(x) = (\pi - x)^2$ in $(0, 2\pi)$

Replace x by $(2\pi - x)$,

$$f(2\pi - x) = (\pi - 2\pi + x)^2 = (-\pi + x)^2$$

$$= (x - \pi)^2 = f(x)$$

$$= (\pi - x)^2 = f(x)$$

$\Rightarrow f(x)$ is even $\Rightarrow \boxed{b_n = 0}$

<u>To find l</u> : $2l = 2\pi \Rightarrow l = \pi$ & $\frac{n\pi x}{l} = \frac{n\pi x}{\pi} = nx$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \left[\because f(x) \text{ is even}\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)^2\,dx = \frac{2}{\pi}\left.\frac{(\pi - x)^3}{-3}\right|_0^{\pi}$$

$$= -\frac{2}{3\pi}\left[(\pi - \pi)^3 - (\pi - 0)^3\right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3} \Rightarrow \boxed{\frac{a_0}{2} = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi-x)^2 \cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[(\pi-x)^2 \left\{\cancel{\frac{\sin(nx)}{n}}\right\} - \left\{\frac{-\cos(nx)}{n^2}\right\} 2(\pi-x)(-1) + \left\{\cancel{\frac{-\sin(nx)}{n^3}}\right\} 2 \right]_0^{\pi}$$

$$\left[\because \sin(0) = \sin(n\pi) = 0\right]$$

$$= -\frac{4}{\pi n^2}\left[(\pi-x)\cos(nx)\right]_0^{\pi}$$

$$a_n = \frac{-4}{\pi n^2}\left[0-\pi\right] = \frac{4}{n^2} \Rightarrow \boxed{a_n = \frac{4}{n^2}}$$

$\therefore$ Eqn ① becomes

$$(\pi-x)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) \quad \text{—— (2)}$$

To deduce the sum of the given series

Put $x=0$ in ②

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

4. Obtain the Fourier Series to represents the function $f(x) = |x|$ in $-\pi < x < \pi$

Ans: By data $f(x) = |x|$ in $(-\pi, \pi)$

Since $f(x) = |x|$ is even, $\boxed{b_n = 0}$ & $\lambda = \pi$

$\left\{ \because f(-x) = |-x| = f(x) \right\}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

Write directly

$$= \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left.\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2 - 0\right] = \pi \Rightarrow a_0 = \pi$$

$$\Rightarrow \boxed{\frac{a_0}{2} = \frac{\pi}{2}}$$

<u>NOTE</u>

$$|x| = \begin{cases} x & : x > 0 \\ -x & : x < 0 \end{cases}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left[\cancel{x\left\{\frac{\sin(nx)}{n}\right\}}^{\,0} - \left\{\frac{-\cos(nx)}{n^2}\right\}\right]_0^{\pi}$$

$$= +\frac{2}{\pi n^2}\Big[\cos(nx)\Big]_0^{\pi}$$

$$= \frac{+2}{\pi n^2}\left[\cos(n\pi) - 1\right] = \frac{-2}{\pi n^2}\left[1-(-1)^n\right]$$

$$\Rightarrow \boxed{a_n = \frac{-2}{\pi n^2}\left[1-(-1)^n\right]}$$

$\therefore$ Eqn (1) becomes

$$\boxed{|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2\left[1-(-1)^n\right]}{\pi n^2}\cos(nx)}$$

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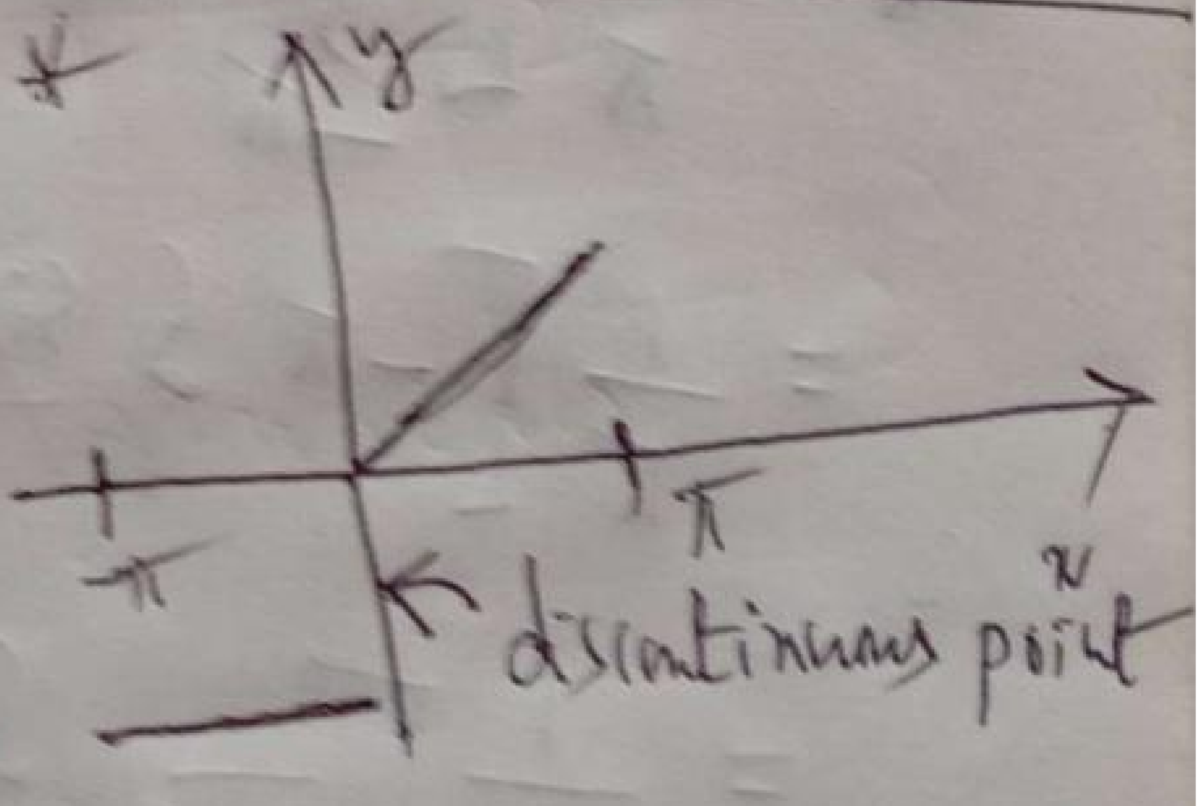
5. Find the Fourier Series of

$$f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi . \end{cases}$$



Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots\cdots = \dfrac{\pi^2}{8}$

Ans: $f(x)$ is neither even nor odd function & $l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right] \quad \text{—(1)}$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\pi x\right)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{x=0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-(0+\pi^2) + \frac{1}{2}(\pi^2 - 0)\right] = \frac{-\pi^2}{2\pi} = \frac{-\pi}{2}$$

$$\Rightarrow \boxed{\frac{a_0}{2} = \frac{-\pi}{4}}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left\{\cancel{\frac{\sin(nx)}{n}}^{\,0}\right\}_{-\pi}^{0} + \left\{x\left(\cancel{\frac{\sin nx}{n}}^{\,0}\right) - \left(-\frac{\cos(nx)}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\left\{\cos(nx)\right\}_{x=0}^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[\cos(n\pi) - 1\right]$$

$$\boxed{a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin(nx)\,dx + \int_{0}^{\pi} x\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left\{-\frac{\cos(nx)}{n}\right\}\Bigg|_{-\pi}^{0} + \left\{x\left\{-\frac{\cos(nx)}{n}\right\} - \cancel{\left\{-\frac{\sin(nx)}{n^2}\right\}}^{0}.1\right\}_{x=0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left\{1-\cos(n\pi)\right\} - \frac{1}{n}\left\{\pi\cos(n\pi) - 0\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{2\pi}{n}\cos(n\pi)\right] = \frac{1}{n}\left[1-2(-1)^n\right]$$

$$\boxed{b_n = \frac{1}{n}\left[1-2(-1)^n\right]}$$

$$\therefore f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}\cos(nx) + \frac{\left\{1-2(-1)^n\right\}\sin(nx)}{n}\right]$$

To deduce the given result, put $x=0$

$$f(0) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}$$

Since $f(x)$ is discontinuous at $x=0$,

$$f(0) \rightarrow \frac{1}{2}\left[f(0^-) + f(0^+)\right] = \frac{1}{2}\left[LHL + RHL\right] = \frac{1}{2}\left[-\pi + 0\right]$$

$$= -\frac{\pi}{2}$$

$$\therefore -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\left[\frac{-2}{1^2} - \frac{2}{3^2} - \frac{2}{5^2} - \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

6) Find the Fourier expansion of the function $f(x) = \cancel{x\sin x}\ x(1-x)(2-x)$ in $(0,2)$. Deduce the sum of the series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - + \cdots\cdots$$

Ans: By data $f(x) = x(1-x)(2-x)$ in $(0,2)$

Replace x by $(2-x)$ in $f(x)$

$$f(2-x) = (2-x)\,[1-(2-x)]\,[2-(2-x)]$$

$$= (2-x)(x-1)(x)$$

$$= -\,x(1-x)(2-x)$$

$$\boxed{f(2-x) = -f(x)} \Rightarrow \text{odd function}$$

$\therefore\ \boxed{a_0 = a_n = 0}$ & $2l = 2 \Rightarrow l=1$

$$\frac{n\pi x}{l} = \frac{n\pi x}{1} = n\pi x$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{—}\ ①$$

$$b_n = \frac{1}{1}\int_0^2 x(1-x)(2-x)\sin(n\pi x)\,dx = \int_0^2 (x^3-3x^2+2x)\sin(n\pi x)$$

$$= 2\int_0^1 [x^3-3x^2+2x]\sin(n\pi x)\,dx$$

$$= 2\Big[(x^3-3x^2+2x)\left\{\frac{-\cos(n\pi x)}{n\pi}\right\} - \left\{-\cancel{\frac{\sin(n\pi x)}{n^2\pi^2}}\right\}^0 (3x^2-6x+2)$$

$$+\left\{+\frac{\cos(n\pi x)}{n^3\pi^3}\right\}(6x-6) - \left\{-\cancel{\frac{\sin(n\pi x)}{n^4\pi^4}}\right\}^0 (6)\Big]_0^1$$

$$= 2\left[-\frac{1}{n\pi}\{0-0\} + \frac{6}{n^3\pi^3}\{0-(0-1)\}\right] = \frac{+12}{n^3\pi^3}$$

$$b_n = \frac{12}{n^3\pi^3}$$

$$\therefore \; x^3 - 3x^2 + 2x = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin(n\pi x)$$

put $x = \frac{\pi}{2}$

$$\frac{\pi^3}{8} - \frac{3\pi^2}{4} + \frac{2\pi}{2} = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi^3 - 6\pi^2 + 8\pi}{8} = \frac{12}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - + \cdots\right]$$

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - + \cdots = \frac{\pi^6 - 6\pi^5 + 8\pi^4}{96}$$