LINEAR ALGEBRA AND ITS APPLICATIONS

Renna Sultana
Department of Science and Humanities

LINEAR ALGEBRA AND ITS APPLICATIONS

MATRICES AND GAUSSIAN ELIMINATION

Renna Sultana
Department of Science and Humanities

BREAKDOWN OF GAUSSIAN ELIMINATION:

Course Content: Breakdown of Gaussian Elimination

- If a zero appears in a pivot position, elimination has to stop –either temporarily or permanently. In this case the system may or may not be singular.

  In many cases this problem can be cured and elimination can proceed. Such a system is non-singular and has a full set of pivots.

  In other cases, when the breakdown is unavoidable (permanent). These systems are singular and have no solution or have infinitely many solutions. For such systems a full set of pivots cannot be found.

- Non-Singular & Curable $\left(|A| \neq 0\right)$
- Singular & InCurable $\left(|A| = 0\right)$
- Singular $\left(|A| = 0\right)$

BREAKDOWN OF GAUSSIAN ELIMINATION:

❖(i) Non-Singular & Curable $\left(|A| \neq 0\right)$:

❖ Consider the system

$$\begin{aligned} x+y+z&=6 \\ x+y+3z&=10 \\ x+2y+4z&=12 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1: & 6 \\ 1 & 1 & 3: & 10 \\ 1 & 2 & 4: & 12 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1: & 6 \\ 0 & 0 & 2: & 4 \\ 0 & 1 & 3: & 6 \end{pmatrix}$$

Here there is a zero in the second pivot position which can be avoided by a row exchange. Thus breakdown is **temporary** and **curable**.

Hence this system reduces to an upper triangular system which can be solved by Back Substitution and so system will become **consistent** and will have **a unique solution**.

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1: & 6 \\ 0 & 1 & 3: & 6 \\ 0 & 0 & 2: & 4 \end{pmatrix} \Longrightarrow \left.\begin{aligned} x+y+z&=6 \\ y+3z&=6 \\ 2z&=4 \end{aligned}\right\} \Rightarrow (x,y,z)=(4,0,2)$$

BREAKDOWN OF GAUSSIAN ELIMINATION:

- (ii) Singular & InCurable $\left(|A| = 0\right)$:
- Consider the system

$$
\begin{aligned}
x + y + z &= 6 \\
x + y + 3z &= 10 \\
x + y + 4z &= 13
\end{aligned}
$$

$$
\begin{pmatrix} 1 & 1 & 1 : 6 \\ 1 & 1 & 3 : 10 \\ 1 & 1 & 4 : 13 \end{pmatrix}
\xrightarrow[R_3 - R_1]{R_2 - R_1}
\begin{pmatrix} 1 & 1 & 1 : 6 \\ 0 & 0 & 2 : 4 \\ 0 & 0 & 3 : 7 \end{pmatrix}
$$

Here there is a zero in the second pivot position which cannot be avoided by any row exchange.

Hence breakdown cannot be avoided which is **incurable**.

The system is **singular** and has **no solution.**

$$
\Rightarrow \left.\begin{aligned}
x + y + z &= 6 \\
2z &= 4 \\
3z &= 7
\end{aligned}\right\}
$$

Here we get z=2 and z=7/3 which is not possible.

BREAKDOWN OF GAUSSIAN ELIMINATION:

- (iii) Singular $\left(|A| = 0\right)$:
- Consider the system

$$\begin{aligned} x + y + z &= 6 \\ x + y + 3z &= 10 \\ x + y + 4z &= 12 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1: & 6 \\ 1 & 1 & 3: & 10 \\ 1 & 1 & 4: & 12 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1: & 6 \\ 0 & 0 & 2: & 4 \\ 0 & 0 & 3: & 6 \end{pmatrix}$$

Here there is a zero in the second pivot position which cannot be avoided. But since last two equations are **consistent** and we get **infinite number of solutions**.

$$\Rightarrow \left.\begin{aligned} x + y + z &= 6 \\ 2z &= 4 \\ 3z &= 6 \end{aligned}\right\} \Rightarrow z = 2 \ and \ x + y = 4$$

$$y = k \Rightarrow x = 4 - k$$

$$\Rightarrow (x, y, z) = (4 - k, k, 2)$$

PES UNIVERSITY ONLINE

BREAKDOWN OF GAUSSIAN ELIMINATION:

(1) Apply Gaussian elimination to the system of equations

$$\begin{aligned} u+v+w&=-2\\ 3u+3v-w&=6\\ u-v+w&=-1 \end{aligned}$$

When does elimination fail at which pivot position.
Is the breakdown temporary or permanent, discuss.
What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed-and force elimination to break down?

$$\begin{pmatrix} 1 & 1 & 1:-2 \\ 3 & 3 & -1:\ 6 \\ 1 & -1 & 1:\ -1 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-3R_1} \begin{pmatrix} 1 & 1 & 1:\ -2 \\ 0 & 0 & -4:\ 12 \\ 0 & -2 & 0:\ 1 \end{pmatrix}$$

Elimination fails at second pivot position. This breakdown is **temporary** which can be cured by exchanging 2nd and 3rd row. Then system becomes consistent and we get a unique solution.

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1:-2 \\ 0 & -2 & 0:\ 1 \\ 0 & 0 & -4:12 \end{pmatrix} \Rightarrow \left.\begin{aligned} u+v+w&=-2\\ -2v&=1\\ -4w&=12 \end{aligned}\right\}$$

$$\Rightarrow (u,v,w)=\left(\frac{3}{2},-\frac{1}{2},-3\right)$$

If the coefficient of v in the third equation is 1 instead of -1, then elimination breaks down **permanently** and it is impossible to proceed.

BREAKDOWN OF GAUSSIAN ELIMINATION:

(2) For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$
$$ax + ay + 4z = b_2$$
$$ax + ay + az = b_3$$

$$\begin{pmatrix} a & 2 & 3 : b_1 \\ a & a & 4 : b_2 \\ a & a & \mathrm{a} : b_3 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} a & 2 & 3 : \; b_1 \\ 0 & a-2 & 1 : \; b_2 - b_1 \\ 0 & a-2 & a-3 : \; b_3 - b_1 \end{pmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} a & 2 & 3 : \; b_1 \\ 0 & a-2 & 1 : \; b_2 - b_1 \\ 0 & 0 & a-4 : \; b_3 - b_2 \end{pmatrix}$$

If **a=0, a=2, a=4** will fail to give 3 pivots.

BREAKDOWN OF GAUSSIAN ELIMINATION:

3. For what values of a and b does the following system have (i)a unique solution (ii) Infinitely many solutions (iii) No solution.

$$x+2y+3z=2$$
$$-x-2y+az=-2$$
$$2x+by+6z=5$$

$$\begin{pmatrix} 1 & 2 & 3:2 \\ -1 & -2 & a:-2 \\ 2 & b & 6:5 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{pmatrix} 1 & 2 & 3:2 \\ 0 & 0 & a+3:0 \\ 0 & b-4 & 0:1 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3:2 \\ 0 & b-4 & 0:1 \\ 0 & 0 & a+3:0 \end{pmatrix}$$

(i) If $a \neq -3$ and $b \neq 4$ then r(A)=r(A:b)=3=n hence system will be **consistent** and will have **a unique solution**.

(ii) If $a=-3$ then r(A)=r(A:b)=2<n(=3) hence system will be **consistent** and will have infinitely **many solutions**.

(iii) If $a=-3,\ b=4\ (\text{or } a \neq -3,\ b=4)$, then r(A)=1(or 2) and r(A:b)=2(or 3) hence system will be **inconsistent** and will have **no solution**.

THANK YOU

Renna Sultana

Department of Science and Humanities

rennasultana@pes.edu