DISTRIBUTED FORCES (1/2)

At the end of this session student will be able to:

- Differentiate between centroid, center of mass and center of gravity.
- Understand body diagonal and lamina
- Identify the difference between axis of summetry, centroidal axis and reference axis.
- Derive centroid of triangle, quarter circle and segment of a circle

DISTRIBUTED FORCES (2/2)

- When forces are continuously distributed over a region of a structure, the cumulative effect of this distribution must be determined.
- In case of Hoover Dam, water pressure is distributed over the area of the Dam and varies greatly with the depth of the water.

Unnumbered 5 p232
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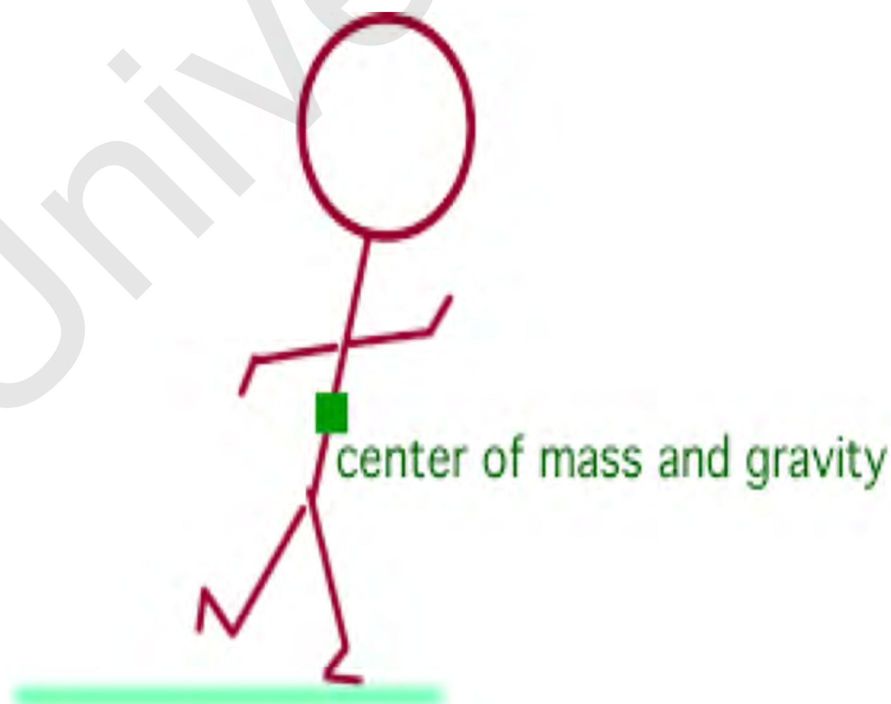
Center of Mass (1/3)

- A point representing mean position of the matter in a body or system.



CENTER OF MASS (2/3)

- Consider a three-dimensional body of any size and shape, having a mass m.
- If we suspend the body from any point such as A, the body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational forces acting on all particles of the body.
- This resultant is clearly collinear with the cord.

CENTER OF MASS (3/3)

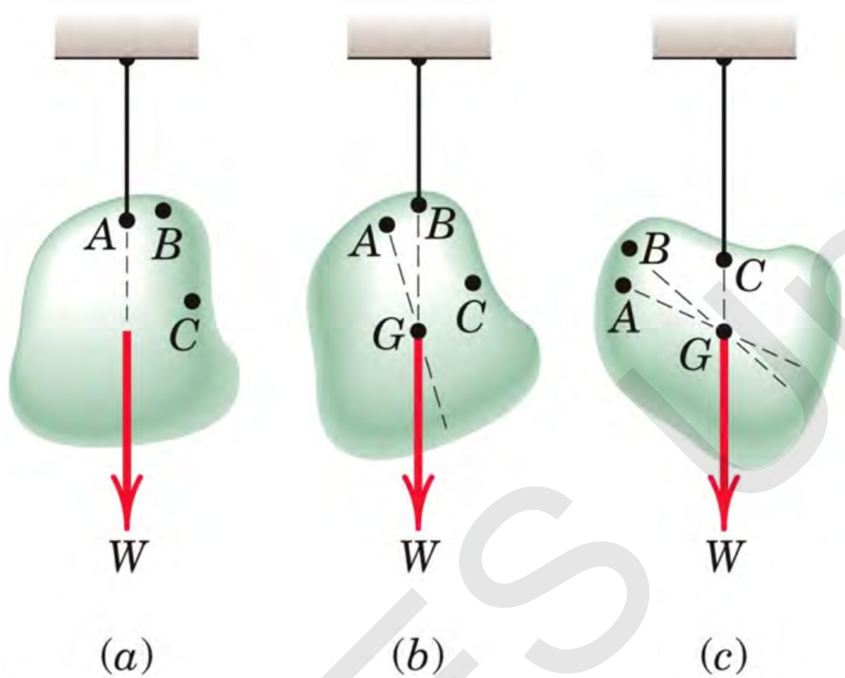
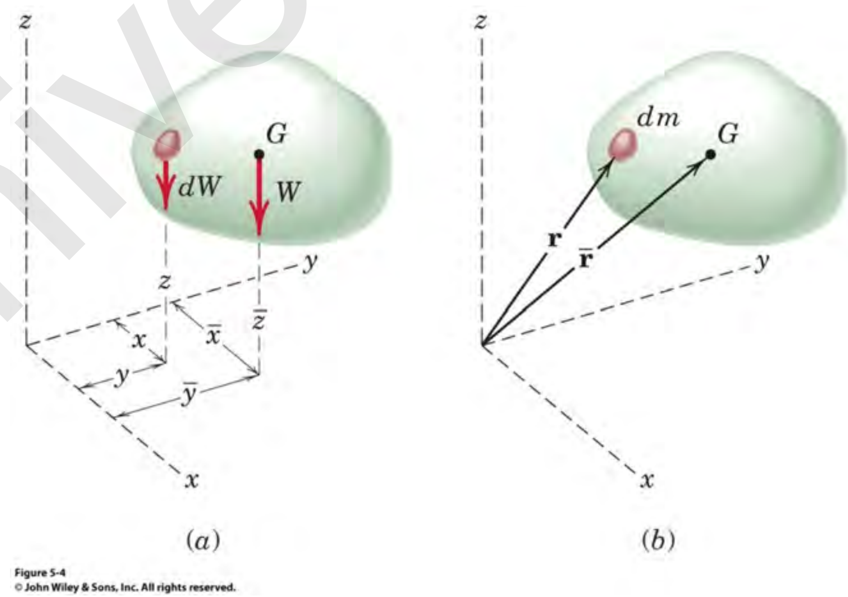


Figure 5-3

- **Definition** of **center of gravity(1/4)**. The point at which the entire weight of a body may be considered as concentrated so that if supported at this point the body would remain in equilibrium in any position.
- To determine mathematically the location of the center of gravity of any body we apply the principle of moments to the parallel system of gravitational forces

CENTER OF GRAVITY(2/4)

- The moment of the resultant gravitational force W about any axis equals the sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements of the body.



CENTER OF GRAVITY (3/4)

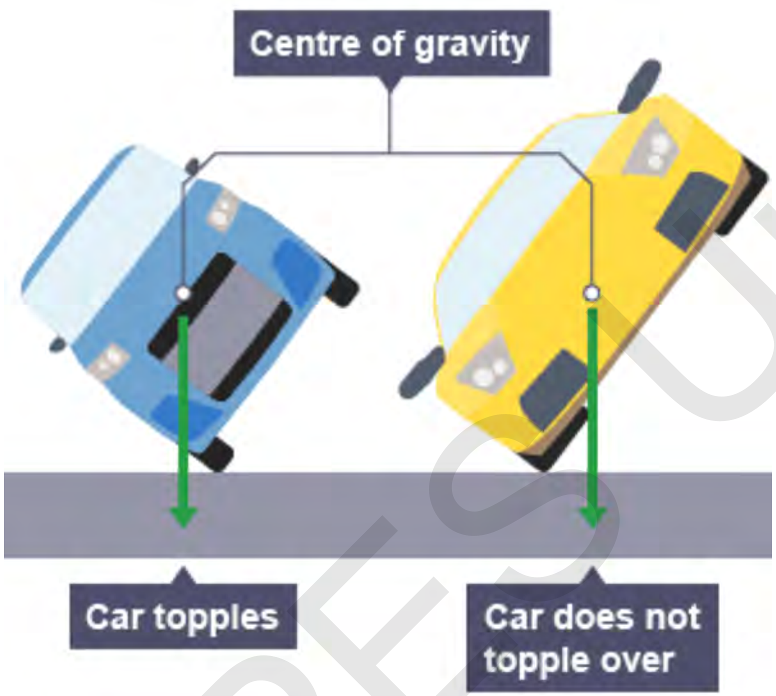
- The resultant of the gravitational forces acting on all elements is the weight of the body and is given by the sum $W=\int dW$
- This sum of moments must equal $W\bar{x}$, the moment of the sum.

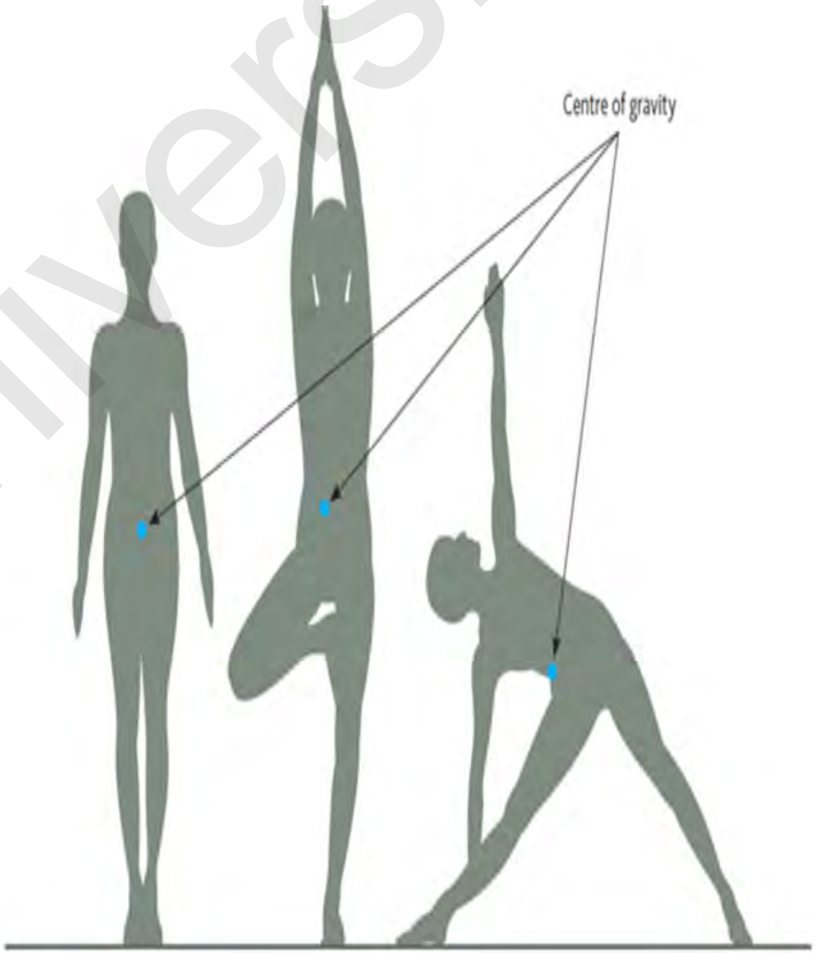
With similar expressions for the other two components, we may express the coordinates of the center of gravity *G* as

$$\bar{x} = \frac{\int x\, dW}{W} \qquad \bar{y} = \frac{\int y\, dW}{W} \qquad \bar{z} = \frac{\int z\, dW}{W}$$

Thus, $\bar{x}W = \int x\, dW.$

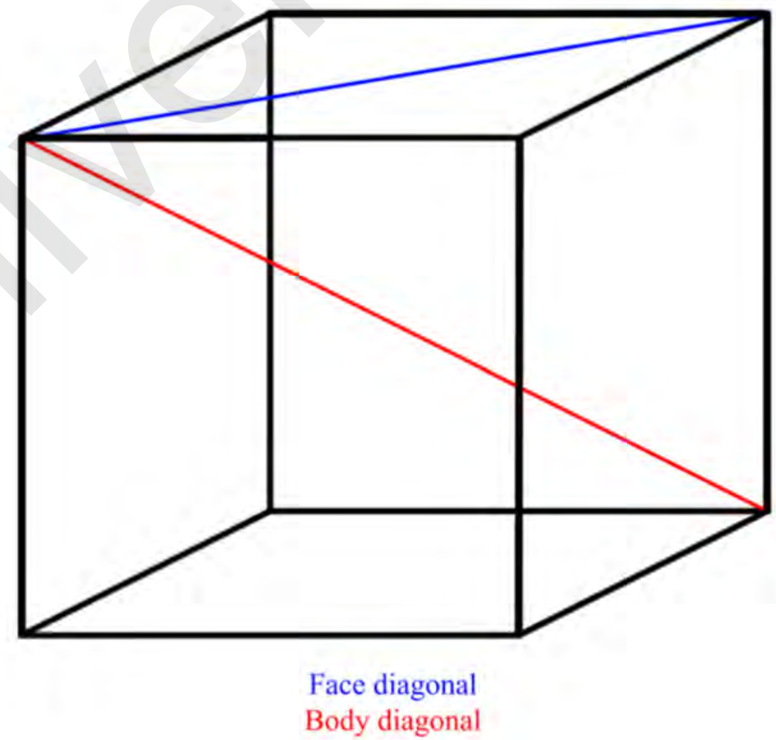
CENTER OF GRAVITY (4/4)





BODY DIAGONAL

- In geometry a space **diagonal** (also interior **diagonal** or **body diagonal**) of a polyhedron is a line connecting two vertices that are not on the same face.

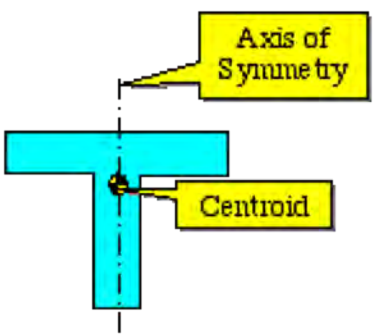


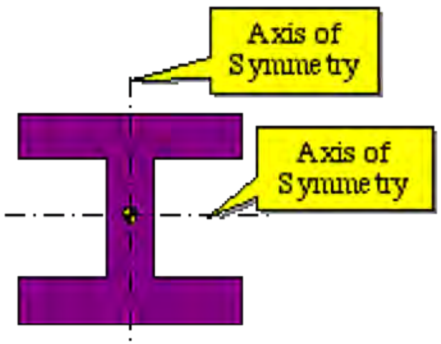
LAMINA

- A **lamina** is a 2-dimensional object. In other words, it is a flat object whose thickness is we can ignore.
- If a body has a line of symmetry, the centre of mass will lie on this line.

AXIS OF SYMMETRY

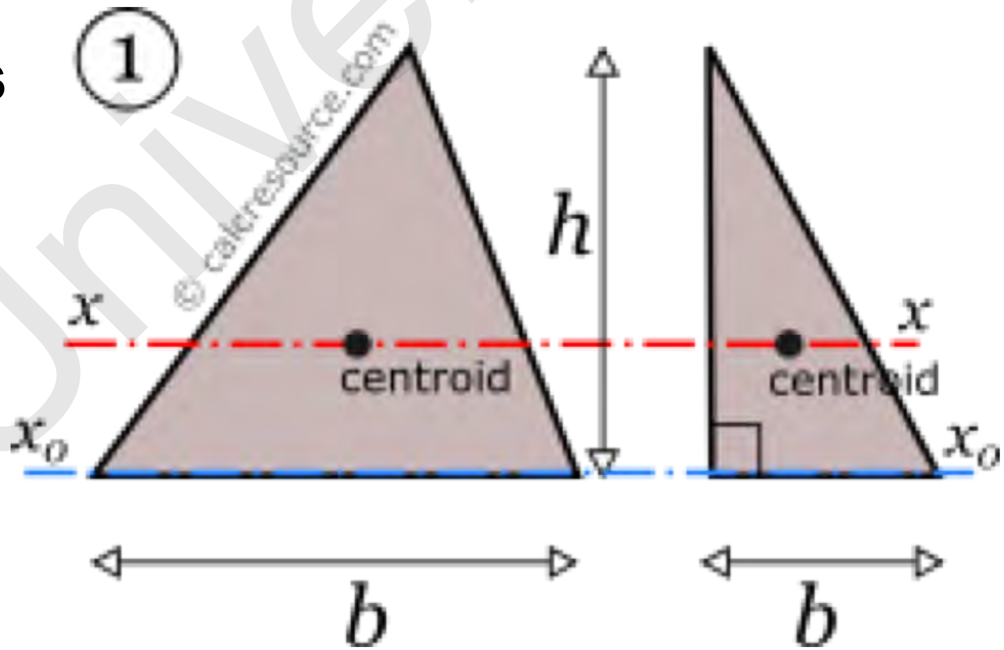
- The line that divides an object into two equal halves, thereby creating a mirror like reflection of either side of the object



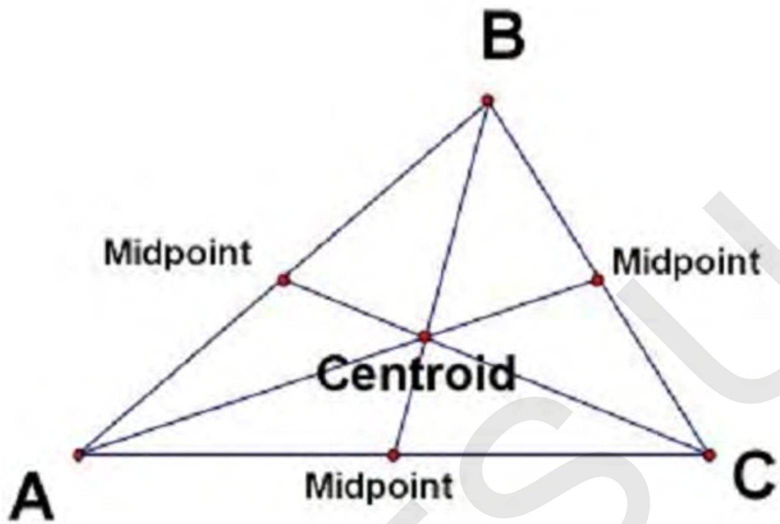


CENTROIDAL AXIS

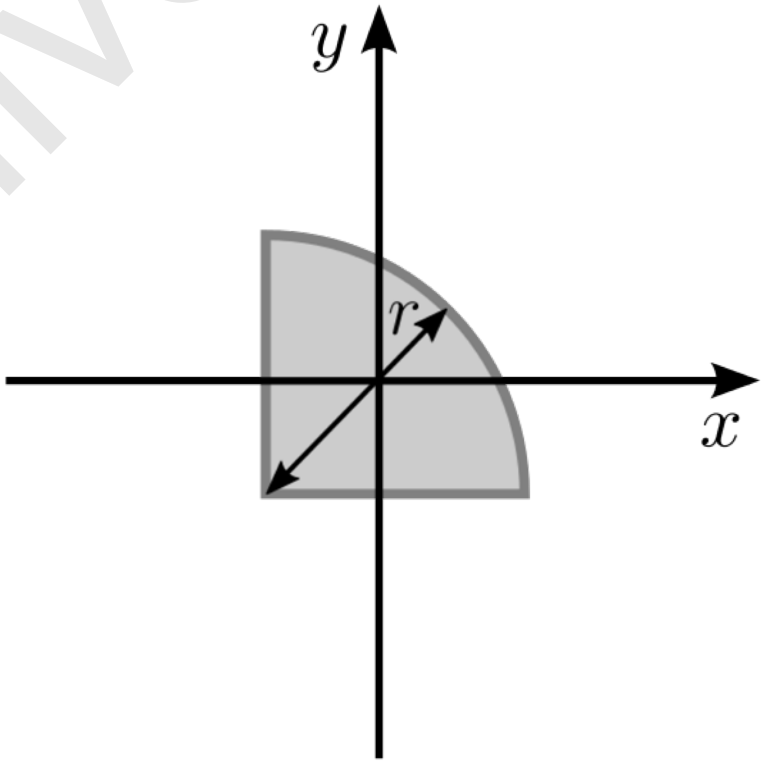
- **Centroidal axis** is any **axis** that passes through the **centroid** of the cross section.
- There can be an infinite number of**centroidal axes**.



- CENTROID OF TRIANGLE

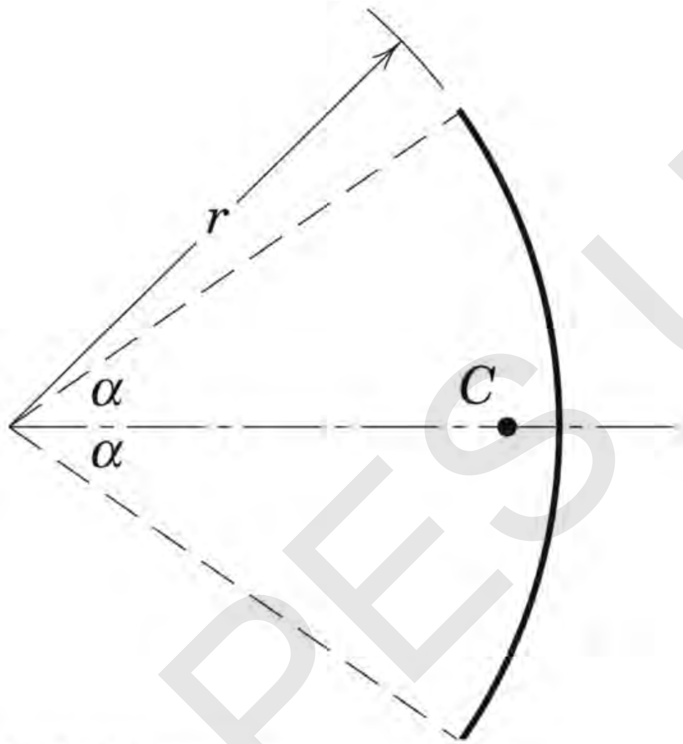


- CENTROID OF QUARTER CIRCLE

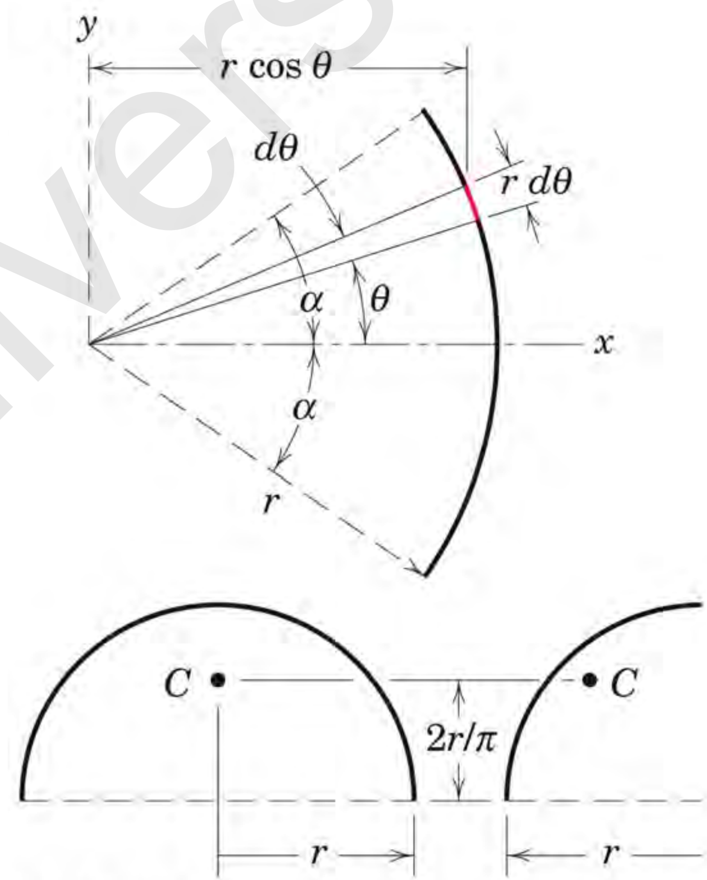


CENTROID OF A CIRCULAR ARC(1/2)

Centroid of a circular arc: Locate the centroid of a circular arc as shown in the figure.



Sample Problem 5-1a



Sample Problem 5-1b

CENTROID OF A CIRCULAR ARC SOLUTION (2/2)

Solution. Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length $dL = r\,d\theta$ expressed in polar coordinates, and the x-coordinate of the element is $r \cos\theta$.

Applying the first of Eqs. 5/4 and substituting $L = 2\alpha r$ give

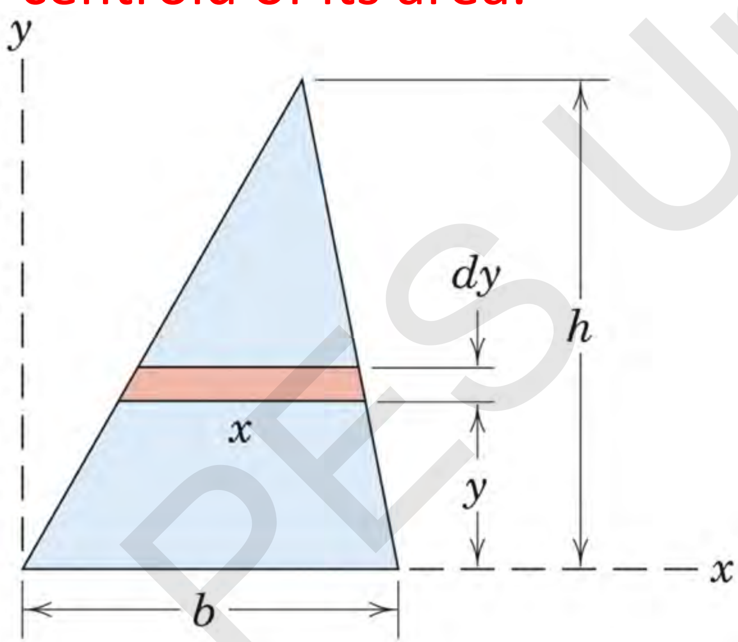
$$\left[L\bar{x} = \int x\,dL\right] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\, r\,d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r \sin\alpha}{\alpha} \qquad \textit{Ans.}$$

CENTROID OF A TRIANGULAR AREA

- Centroid of a triangular area. Determine the distance from the base of a triangle of altitude h to the centroid of its area.

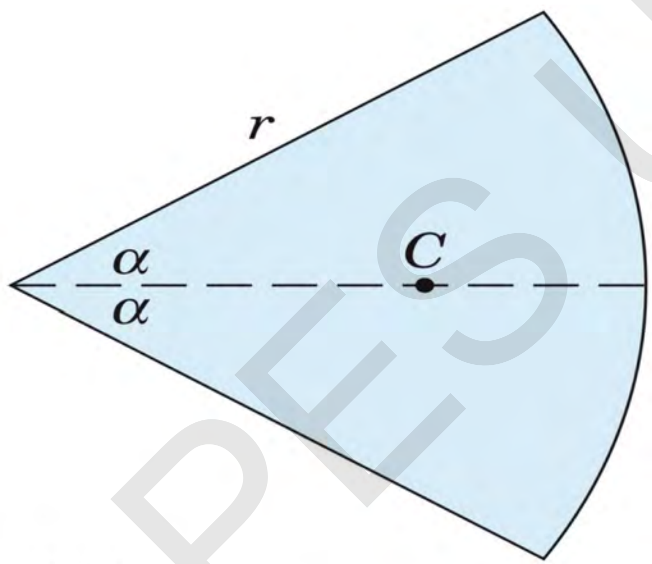


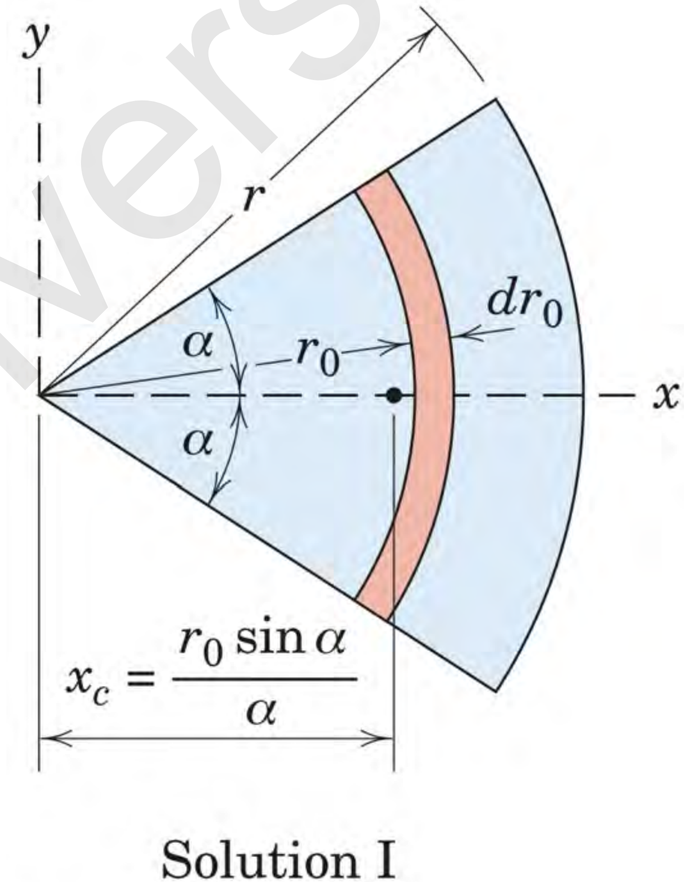
$$[A\bar{y} = \int y_c \, dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{h} dy = \frac{bh^2}{6}$$

and $$\bar{y} = \frac{h}{3}$$

CENTROID OF THE AREA OF A CIRCULAR SECTOR(1/2)

- Centroid of the area of a circular sector. Locate the centroid of the area of a circular sector with respect to its vertex.





CENTROID OF THE AREA OF A CIRCULAR SECTOR: SOLUTION (2/2)

Solution I. The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is $dA = 2r_0\alpha\, dr_0$.

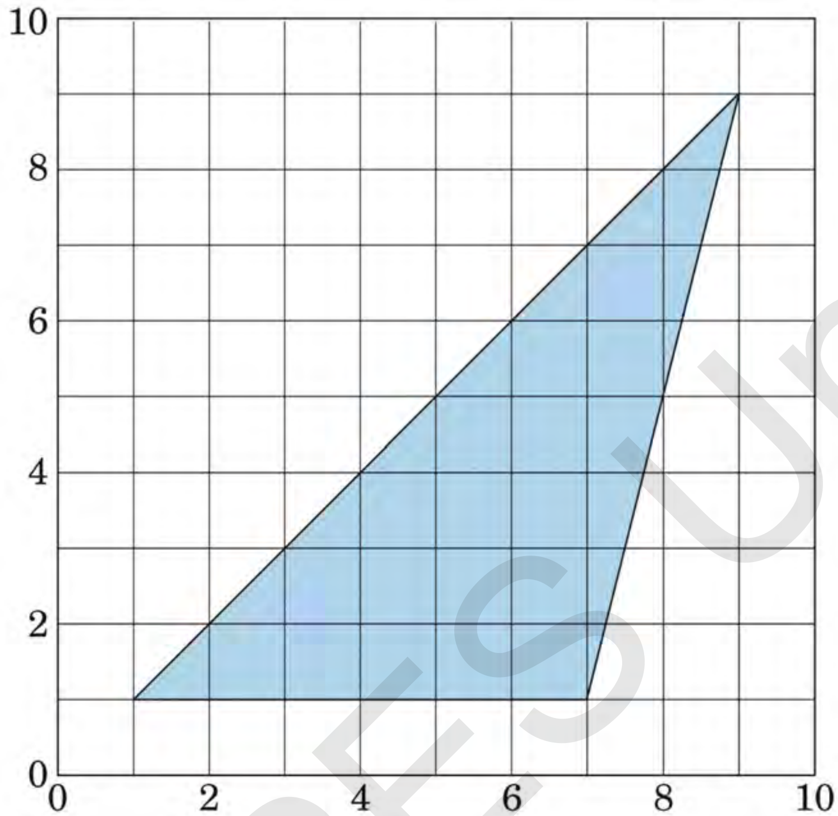
The x-coordinate to the centroid of the element from Sample Problem 5/1 is $x_c = r_0 \sin\alpha/\alpha$, where r_0 replaces r in the formula. Thus, the first of Eqs. 5/5a gives

$$\left[A\bar{x} = \int x_c\, dA\right] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\bar{x} = \tfrac{2}{3}r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

With your pencil, make a dot on the position of your best visual estimate of the centroid of the triangular area.



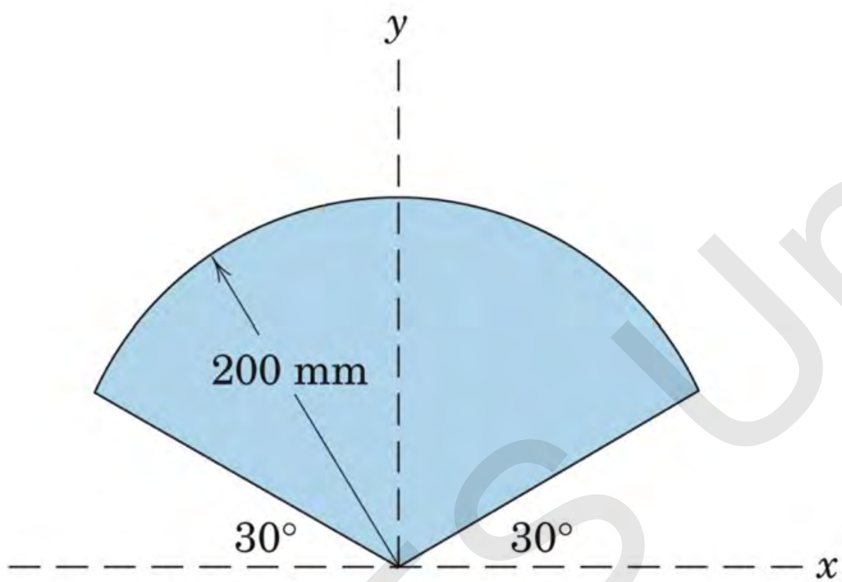
Problem 5-1

$$\bar{x} = 1 + \frac{8+6}{3} = 5.64$$

$$\bar{y} = 1 + \frac{8}{3} = 3.67$$

$$(\bar{x}, \bar{y}) = (5.67, 3.67)$$

With your pencil, make a dot on the position of your best visual estimate of the centroid of the area of the circular sector.



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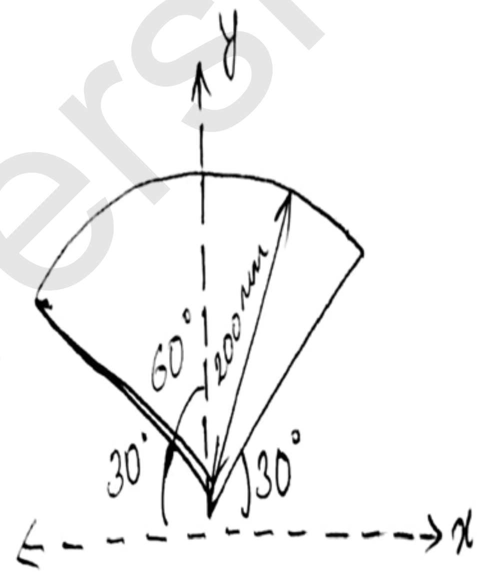
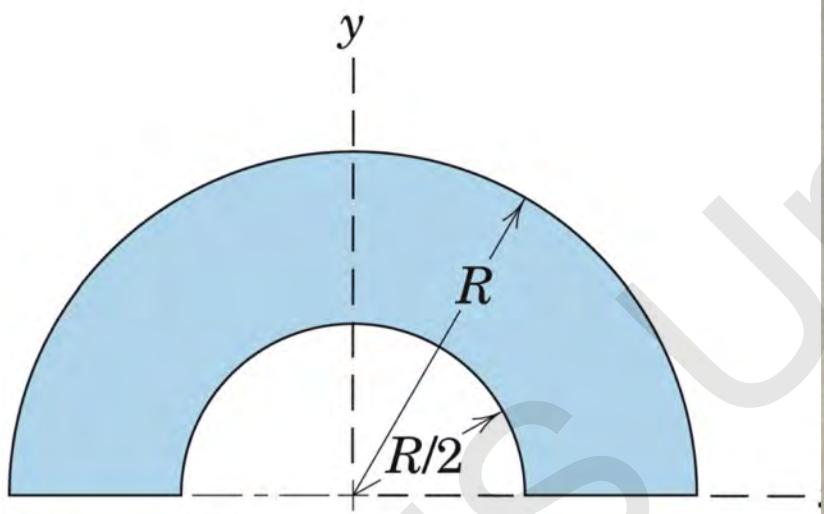


$\alpha = 60°$

$$\bar{X} = 0;\quad \bar{Y} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} = \frac{\frac{2}{3}\times 200\times\frac{\sqrt{3}}{2}}{60\times\frac{\pi}{180}} = \underline{110.26\text{ mm}}$$

Determine the y-coordinate of the centroid of the area by direct integration.



Problem 5-6

$$dA = r \cdot dr \cdot d\theta$$

$$A\bar{y} = \int dA \cdot y$$

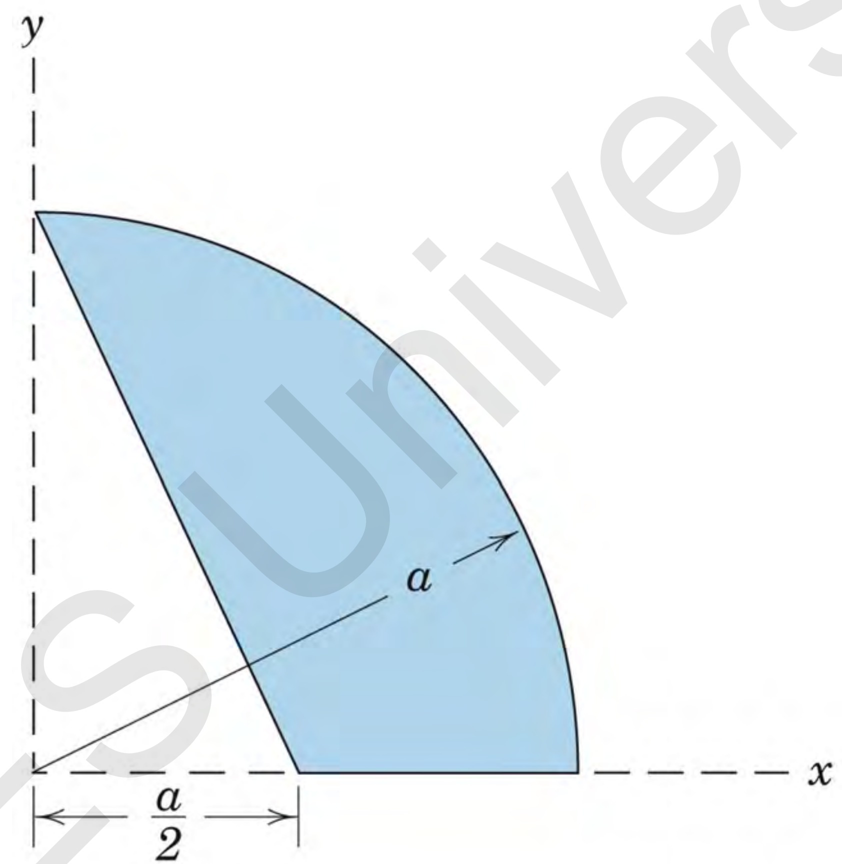
$$A\bar{y} = \frac{7R^3}{12}$$

$$\bar{y} = \frac{14R}{9\pi}$$

$$\int dA = \int r \cdot dr \cdot d\theta$$

$$= \frac{3\pi R^2}{8}$$

Determine the x- and y-coordinates of the centroid of the shaded area.



Problem 5-12

$$\bar{X} = \frac{a_1x_1 - a_2x_2}{a_1 - a_2} \quad ; \quad \bar{Y} = \frac{a_1y_1 - a_2y_2}{a_1 - a_2}$$

$$a_1 = \frac{\pi a^2}{4} \; ; \quad a_2 = \frac{1}{2} \times a \times \frac{a}{2} = \frac{a^2}{4}$$

$$x_1 = \frac{4a}{3\pi} \; ; \quad y_1 = \frac{4a}{3\pi} \; ; \quad x_2 = \frac{a}{6} \; ; \quad y_2 = \frac{a}{3}.$$

$$\bar{X} = \frac{\frac{\cancel{\pi} a^2}{\cancel{4}} \cdot \frac{\cancel{4} a}{3\cancel{\pi}} - \frac{a^2}{4} \cdot \frac{a}{6}}{\frac{\pi a^2}{4} - \frac{a^2}{4}}$$

$$= \frac{\frac{a^3}{3} - \frac{a^3}{24}}{\frac{a^2(\pi - 1)}{4}} = \frac{8a^3 - a^3}{\cancel{24}\,6} \times \frac{\cancel{4}}{a^2(\pi - 1)}$$

$$= \underline{\frac{7a}{6(\pi - 1)}}$$

$$\bar{Y} = \frac{\frac{\cancel{\pi} a^2}{\cancel{4}} \cdot \frac{\cancel{4} a}{3\cancel{\pi}} - \frac{a^2}{4} \times \frac{a}{3}}{\frac{a^2(\pi - 1)}{4}}$$

$$= \left(\frac{a^3}{3} - \frac{a^3}{12}\right) \times \frac{4}{a^2(\pi - 1)} = \frac{\cancel{3} a^3}{\cancel{12}} \times \frac{\cancel{4}}{a^2(\pi - 1)} = \underline{\frac{a}{\pi - 1}}.$$

Composite Bodies & Figures ; Approximations

- When a body or figure can be conveniently divided into several parts whose mass centers are easily determined, we use the principle of moments and treat each part as a finite element of the whole.
- Its parts have masses *m*1, *m*2, *m*3 with the respective mass-center coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$
- We generalize, then, for a body of any number of parts and express the sums in condensed form to obtain the mass-center coordinates

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$

An Approximation Method

- In practice the boundaries of an area or volume might not be expressible in terms of simple geometrical shapes or as shapes which can be represented mathematically. For such cases we must resort to a method of approximation.

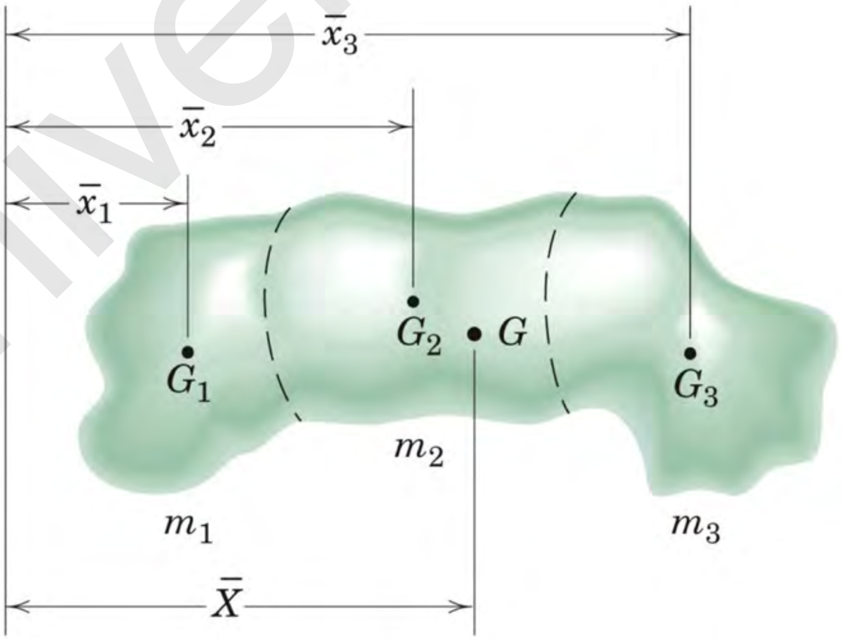


Figure 5-13

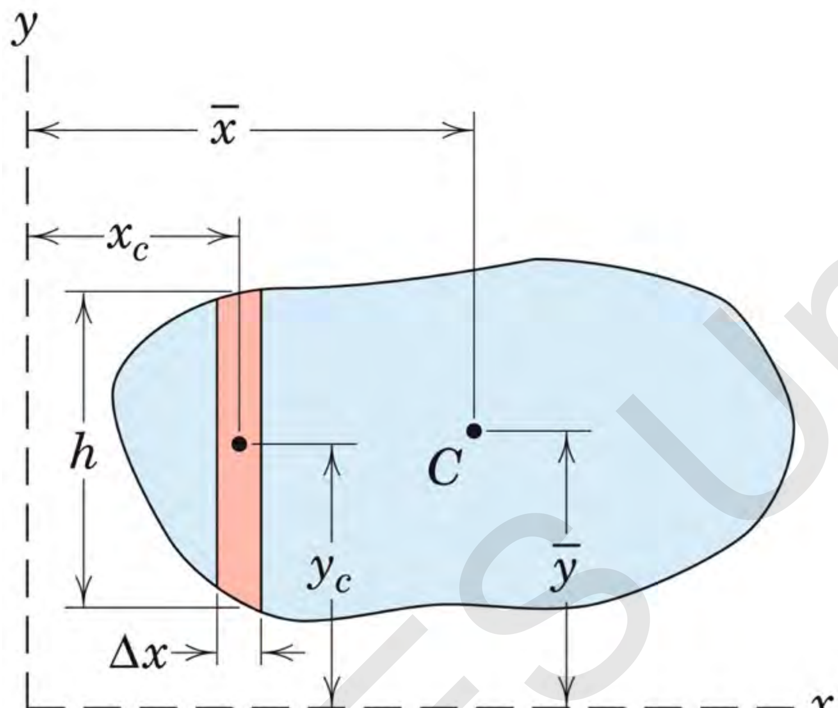
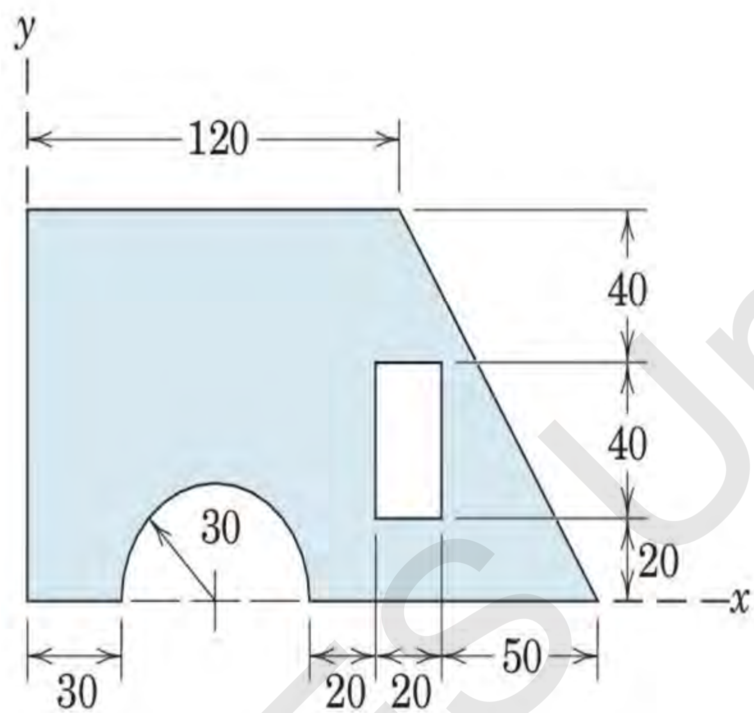


Figure 5-14

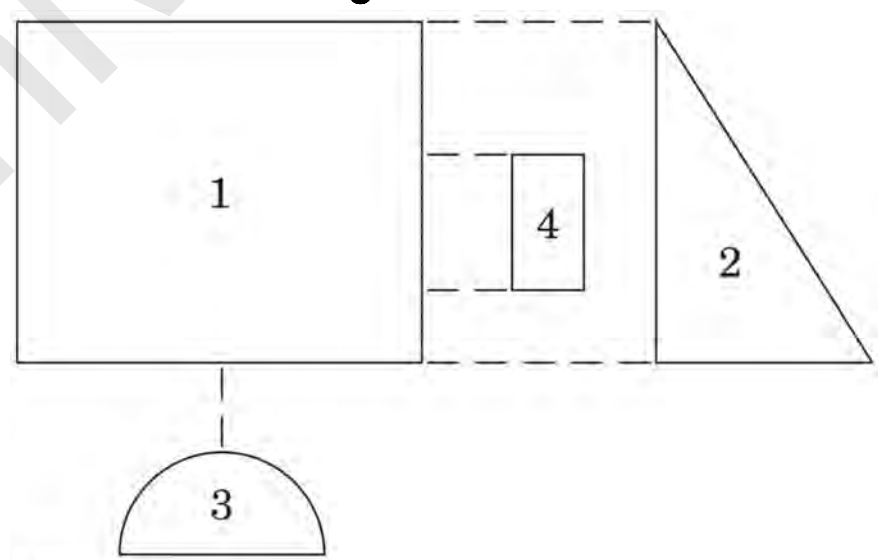
- The area is divided into strips of width *x* and variable height *h*. The area *A* of each strip, such as the one shown in red, is *h x* and is multiplied by the coordinates *xc* and *yc* of its *centroid* to obtain the moments of the element of area

Locate the centroid of the shaded area.(1/2)



Dimensions in millimeters

- **Solution.** The composite area is divided into the four elementary shapes shown in the lower figure.



Locate the centroid of the shaded area.(2/2)

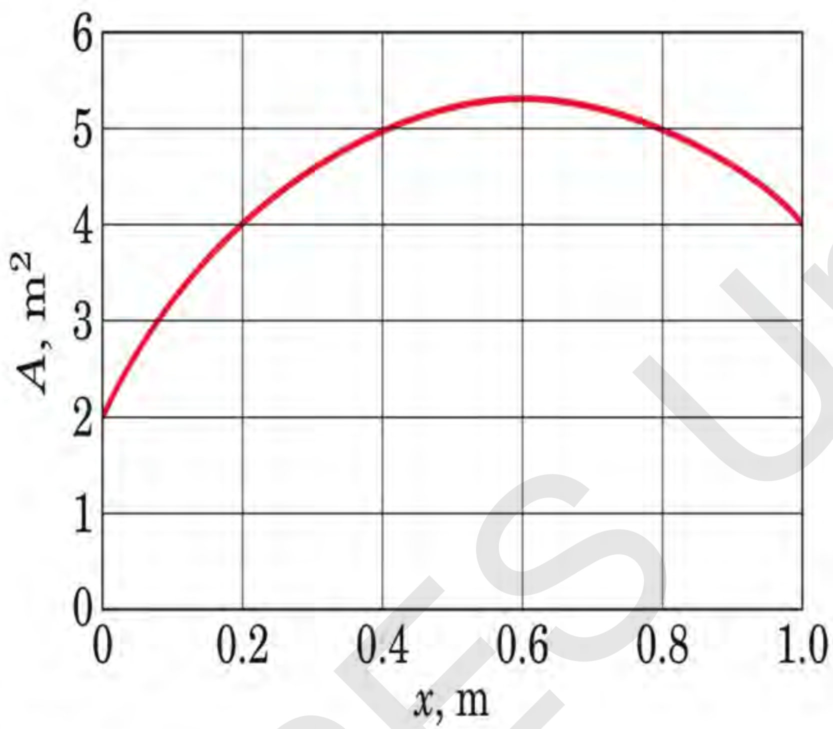
PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	−14.14	6	1.273	−84.8	−18
4	−8	12	4	−96	−32
TOTALS	127.9			959	650

The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.} \qquad \textit{Ans.}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.} \qquad \textit{Ans.}$$

Approximate the *x*-coordinate of the volume centroid of a body whose length is 1 m and whose cross-sectional area varies with *x* as shown in the figure.

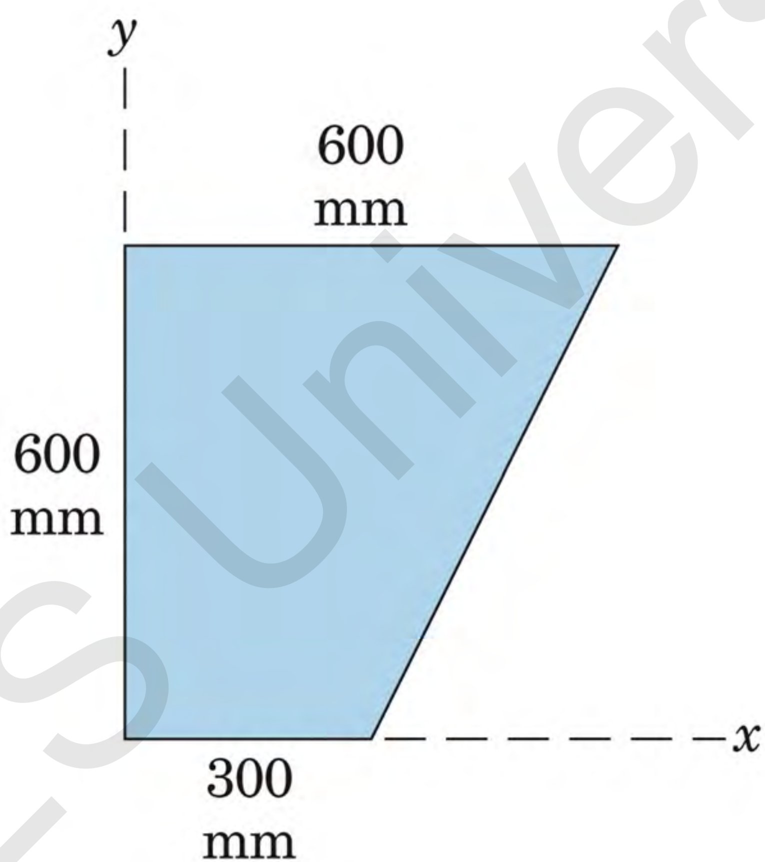


Sample Problem 5-7

- **Solution.** The body is divided into five sections.

INTERVAL	A_{av} m^2	Volume V m^3	$\bar{x}$ m	$V\bar{x}$ m^4
0–0.2	3	0.6	0.1	0.060
0.2–0.4	4.5	0.90	0.3	0.270
0.4–0.6	5.2	1.04	0.5	0.520
0.6–0.8	5.2	1.04	0.7	0.728
0.8–1.0	4.5	0.90	0.9	0.810
TOTALS		4.48		2.388

5/49 Determine the coordinates of the centroid of the trapezoidal area shown.(1/2)



Problem 5-49

Solution (2/2)

Section	Area	x-dist	y-dist	Ax	Ay
① Rectange	$600 \times 300 = 180000$	$300/2 = 150$	$600/2 = 300$	27000000 (180000×150)	$180000 \times 300 = 54000000$
② triangle	$\frac{600 \times 300}{2} = 90000$	$300 + \frac{300}{3} = 400$	$\frac{600}{3} = 200$	36000000	18000000

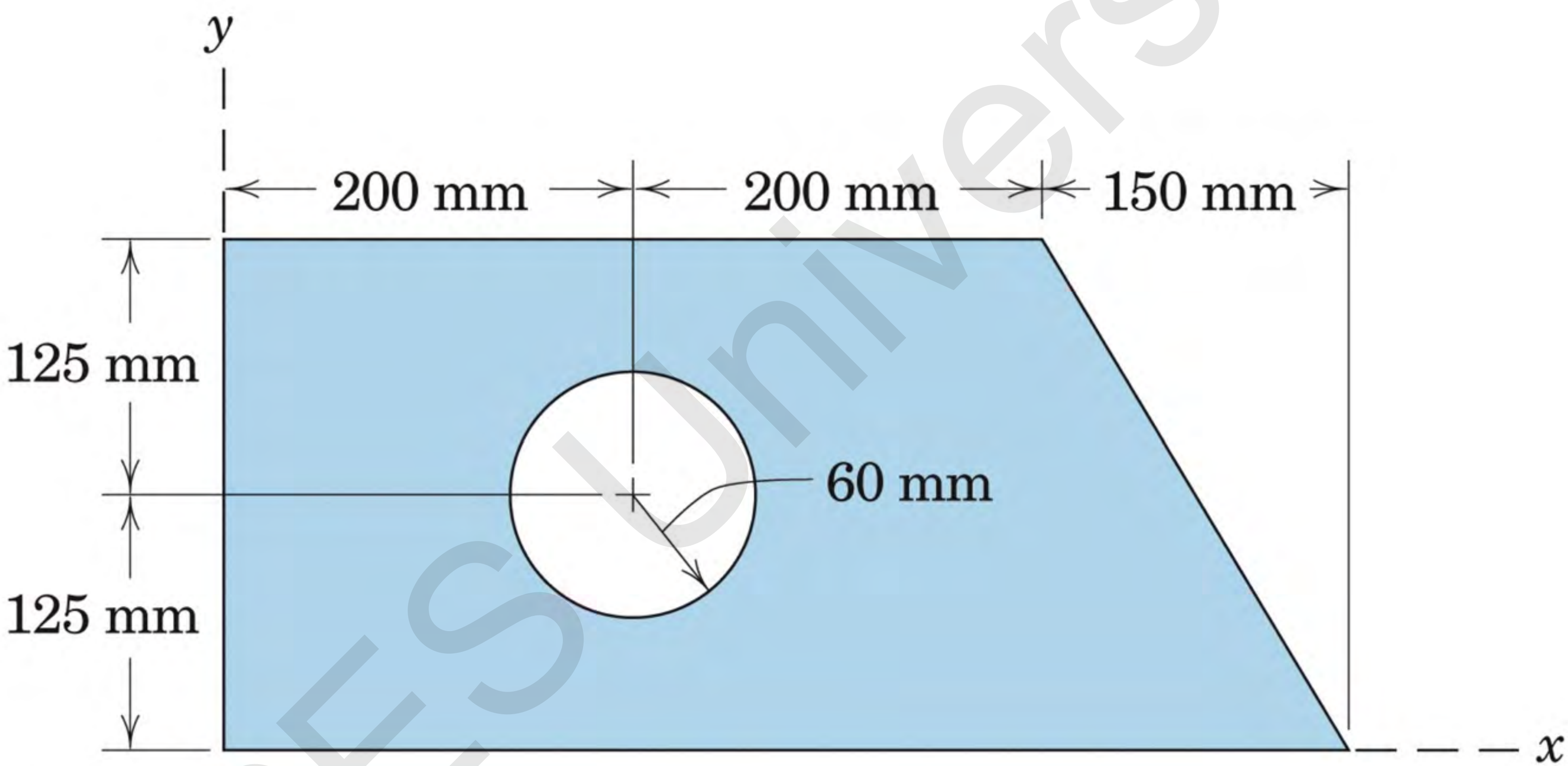
$$\Sigma A = 270000 \text{ mm}^2$$

$$\Sigma Ax = 63000000 \qquad \Sigma Ay = 72000000$$

$$\therefore \bar{X} = \frac{63000000}{270000} = \underline{233.33 \text{ mm}}$$

$$\bar{Y} = \frac{72000000}{270000} =$$

Determine the coordinates of the centroid of the shaded area.(1/2)



Problem 5-51

Solution (2/2)

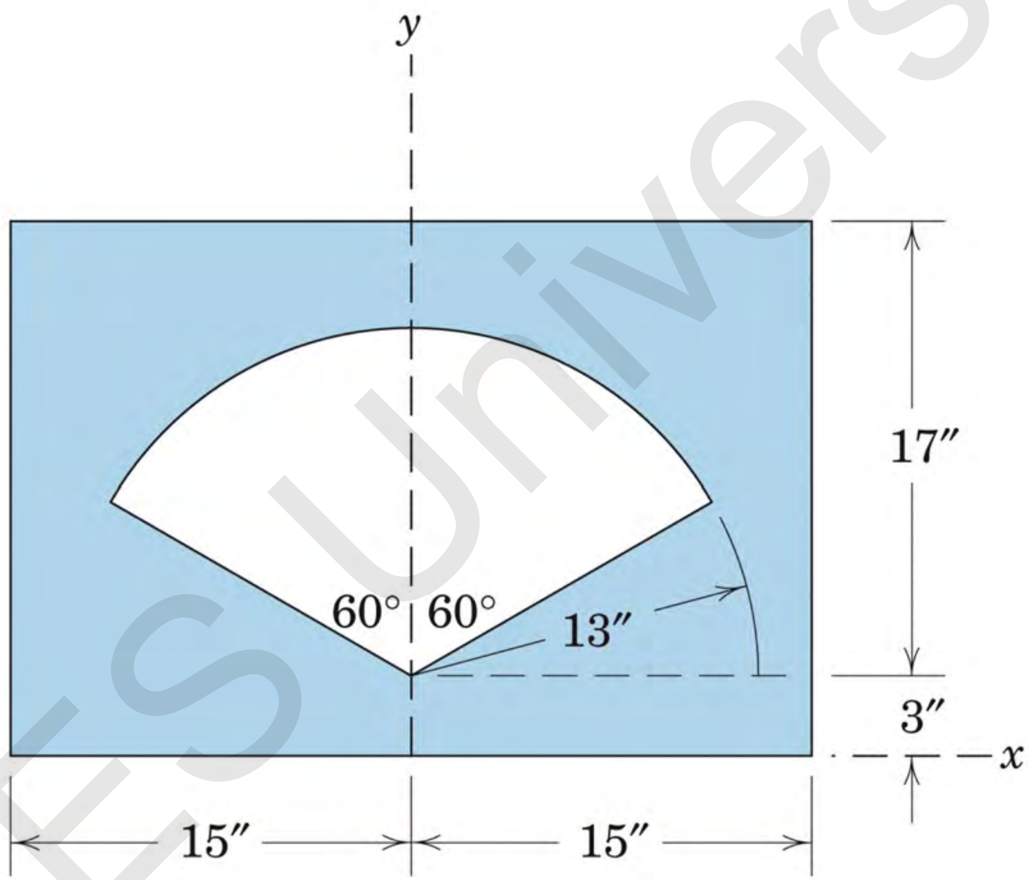
Section	Area (mm^2)	x-dist (mm)	y-dist (mm)	Ax $(mm)^3$	Ay (mm^3)
① Rectangle	400 × 250 = 100000	400/2 = 200	125	20000000	12500000
② Triangle	18750	150/3 = 50 + 400 = 450	250/3 = 83.33	8437500	1562437.5
③ Circle	πr^2 11309.73	200	125	2261946	1413716.25

$$\Sigma A = 107440.27$$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 - a_3 x_3}{a_1 + a_2 - a_3} = 244 \text{ mm}$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 - a_3 y_3}{a_1 + a_2 - a_3} = 117.7 \text{ mm}$$

Calculate the y-coordinate of the centroid of the shaded area.(1/2)



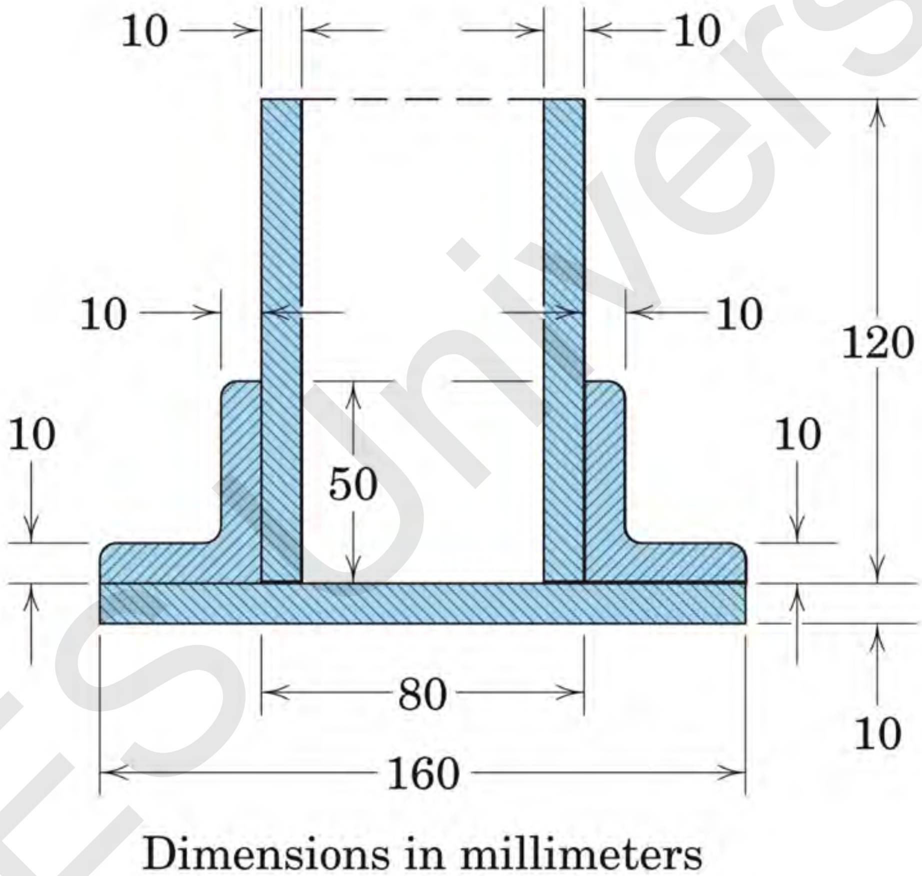
Problem 5-52

Solution (2/2)

Section	Area (mm^2)	Y-distance (mm)	Ay (mm^3)
① Rectangle	$200 \times 300 = 60000$	$\frac{200}{2} = 100$	6000000
② Sector	$\frac{1}{2} \times (130)^2 \times 120 \times \frac{\pi}{180}$ 17697.63	$\frac{\frac{2}{3} \times 130 \times \frac{\sqrt{3}}{2}}{\frac{\pi}{3}} + 30$ 71.67	1268389.14

$$\bar{Y} = \frac{4731610.86}{42302.37} = 99.85\,mm$$

Determine the distance from the bottom of the base plate to the centroid of the built-up structural section shown. (1/2)



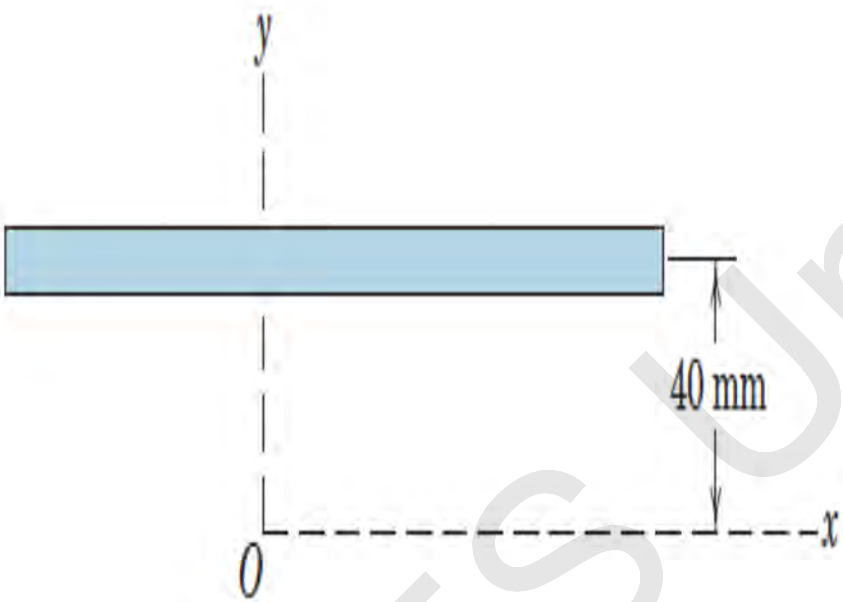
Problem 5-57

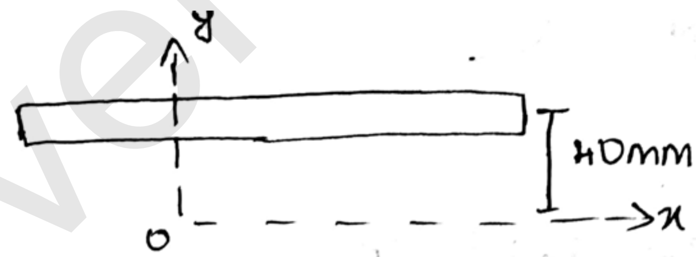
Solution (2/2)

Section	Area	y-distance	Ay
(1)	160 X 10 = 1600	10/2 = 5	8000
(2)	(40 X 10)2 = 800	10 + 10/2 = 15	12000
(3)	(40 X 10)2 = 800	$10 + 10 + \frac{40}{2} = 40$	32000
(4)	(120 X 10)2 = 2400	$10 + \frac{120}{2} = 70$	168000

$$\bar{y} = \frac{220000}{5600} = \underline{39.3\,mm}$$

If the moment of inertia of the thin strip of area about the *x*-axis is 2.56(106),determine the area *A* of the strip to within a close approximation.





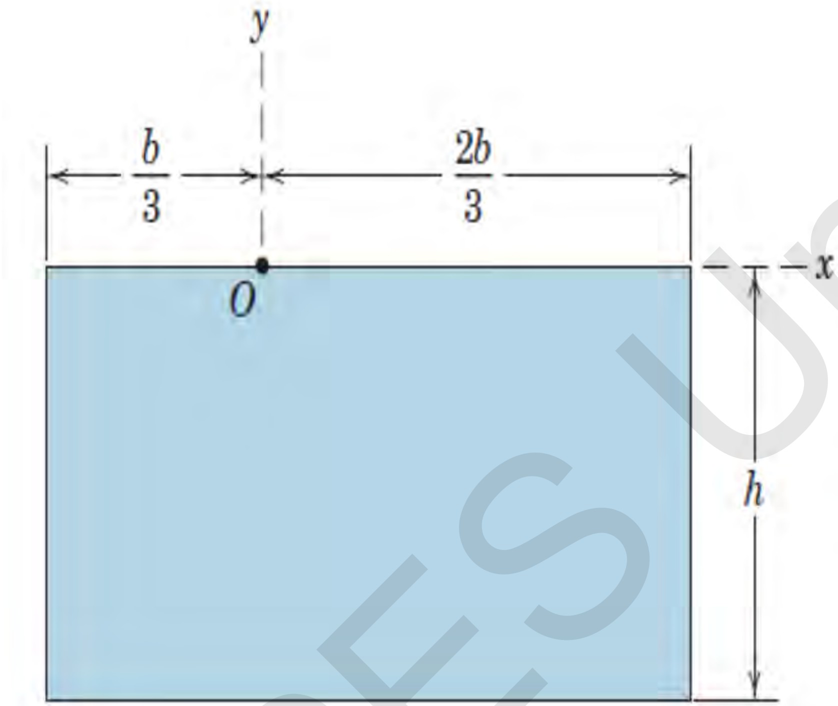
$$I_x = \cancel{\bar{I}_x}^{\,0} + Ad^2$$

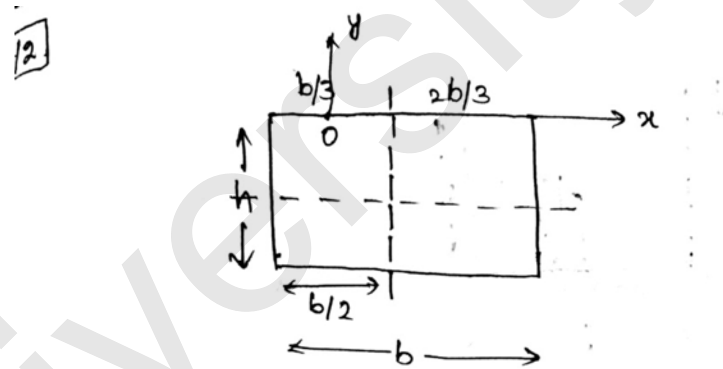
$$2.56 \times 10^6 = A(40)^2$$

$$A = \frac{2.56 \times 10^6}{40 \times 40}$$

$$= \underline{1.600 \times 10^3 \ mm^2}$$

Determine the moments of inertia of the rectangular area about the *x*- and *y*-axes and find the polar moment of inertia about point *O*.





$$I_x = \bar{I}_x + Ad^2$$

$$= \frac{bh^3}{12} + bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{12} + \frac{bh^3}{4} = \underline{\frac{bh^3}{3}}$$

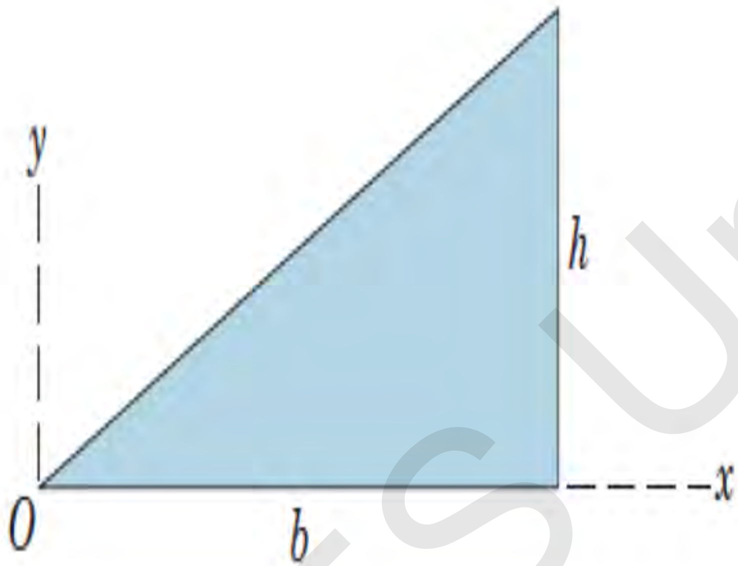
$$I_y = \bar{I}_y + Ad^2$$

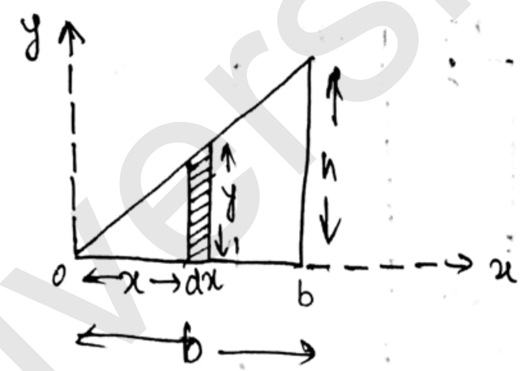
$$= \frac{hb^3}{12} + bh\left(\frac{b}{2} - \frac{b}{3}\right)^2 = \frac{hb^3}{12} + bh \times \frac{b^2}{36} = \underline{\frac{b^3h}{9}}$$

$$I_z = I_x + I_y$$

$$= \frac{bh^3}{3} + \frac{hb^3}{9} = \underline{\frac{3bh^3 + hb^3}{9}}$$

Determine by direct integration the moment of inertia of the triangular area about the *y*-axis.



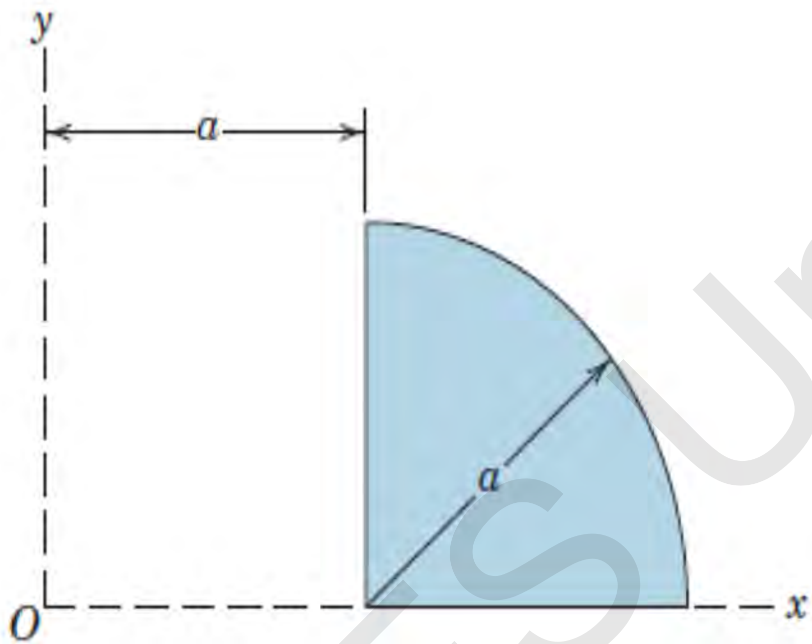


$$I_y = \int x^2 \cdot dA = \int_0^b x^2 \cdot y \, dx$$

From the Δ, $\frac{h}{b} = \frac{y}{x}$ ∴ $y = \frac{hx}{b}$.

$$I_y = \frac{h}{b}\int_0^b x^3 dx \cdot = \frac{h\,b^4}{b\;4} = \frac{hb^3}{4\cancel{b^3}}$$

Determine the moments of inertia of the quartercircular area about the *x*- and *y*-axes, and find the polar radius of gyration about point *O*.



$$I_x = \frac{\pi a^4}{16}$$

$$= 0.196a^4$$

$$I_y = 0.055a^4 + \frac{\pi a^2}{4}\left[a + \frac{4a}{3\pi}\right]^2$$

$$= 1.648$$

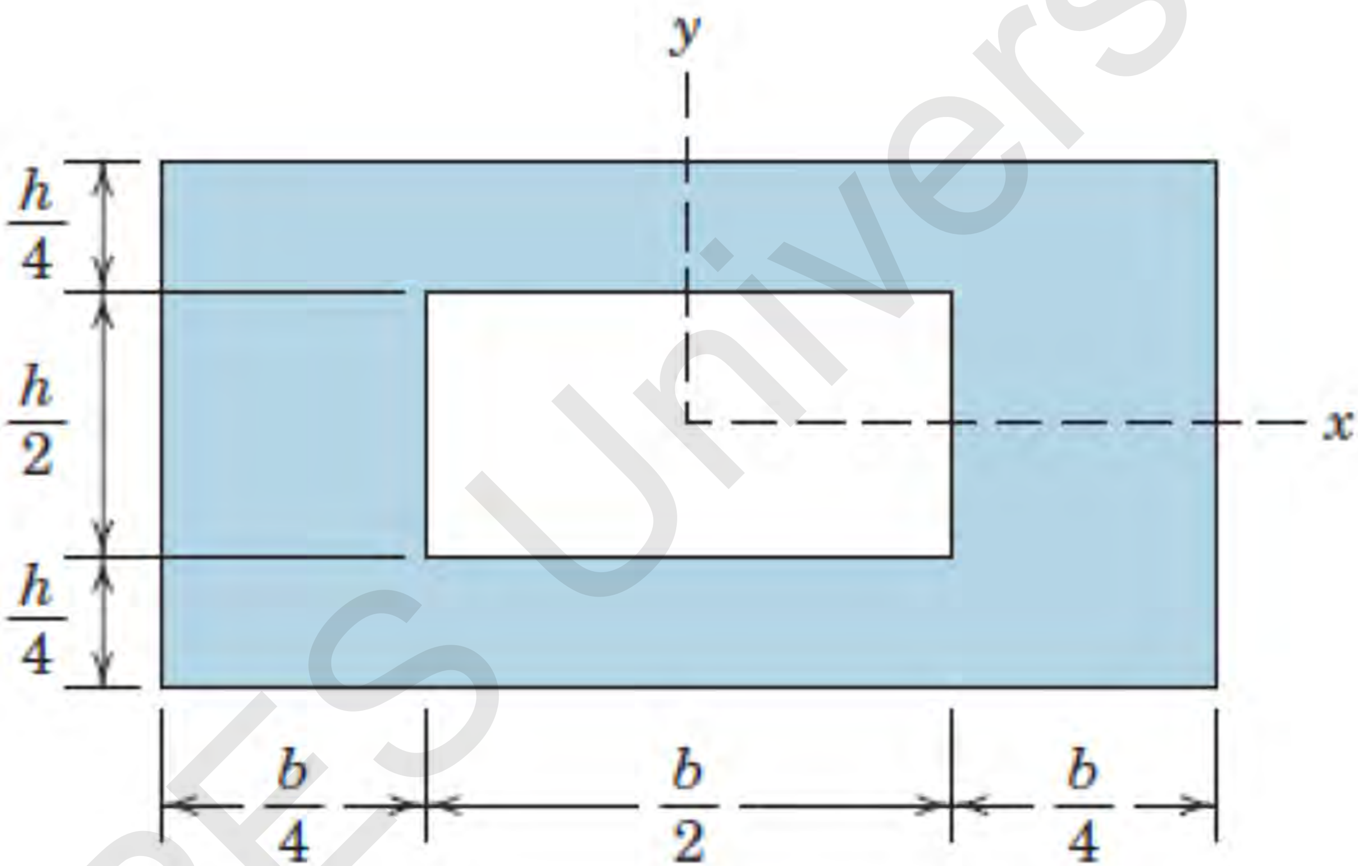
$$I_z = I_x + I_y$$

$$= 1.844\,a^4$$

$$K_o = \sqrt{\frac{1.844a^4 \times 4}{\pi a^2}}$$

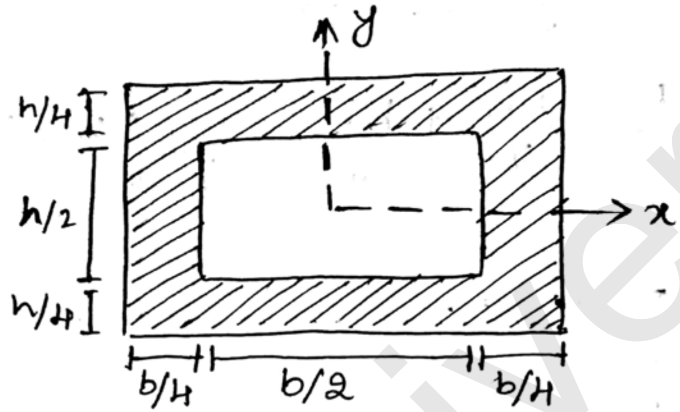
$$= 1.53\,a$$

Determine the moment of inertia about the *x*-axis of the rectangular area without and with the central rectangular hole. (1/2)



Solution (2/2)

A/35



Ans: without the cut piece.

$$I_x = \frac{bh^3}{12} = 0.083bh^3$$

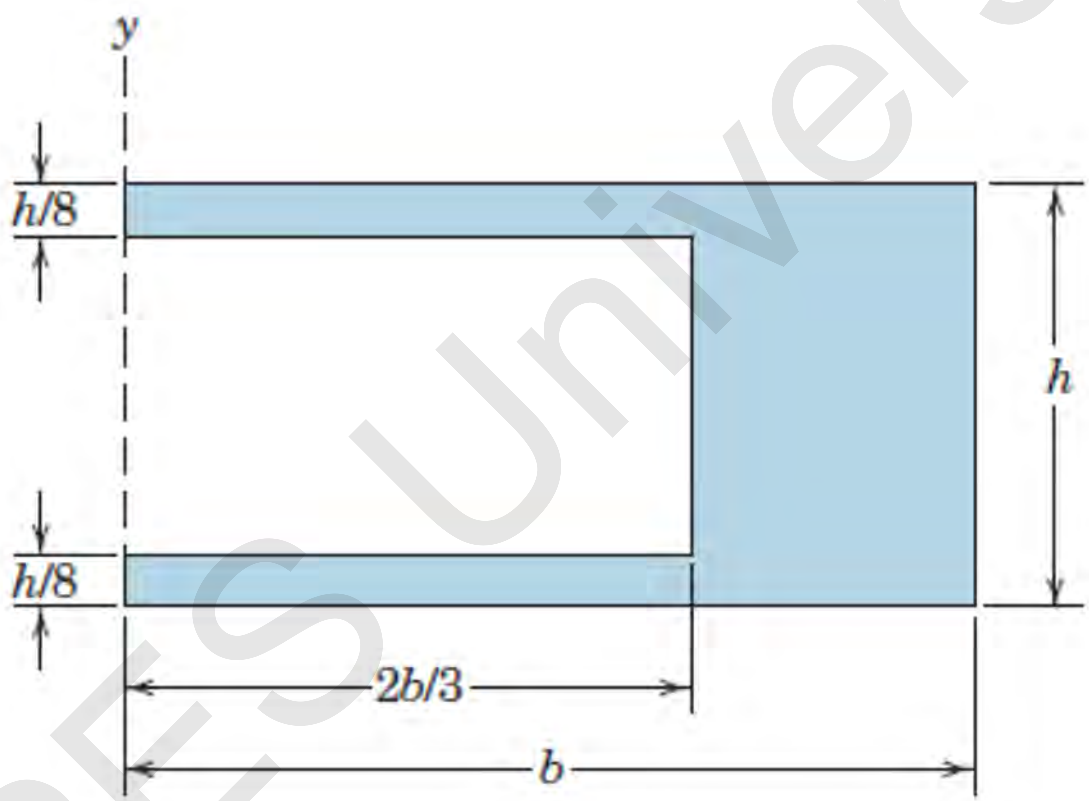
cut piece.

$$I_x = \frac{b}{2} \times \frac{h^3}{8 \times 12} = \frac{bh^3}{192} = 0.0052bh^3$$

with the cut piece

$$I_x = (0.083 - 0.0052)bh^3 = 0.078bh^3$$

Determine the percent reductions in both area and area moment of inertia about the *y*-axis caused by removal of the rectangular cutout from the rectangular plate of base *b* and height *h*. (1/2)



Solution (2/2)

Percentage reduction in area :

$$= \frac{\text{Area of cut portion}}{\text{Area of full rectangle}} \times 100$$

$$= \frac{\frac{2b}{3} \times \frac{3h}{4}}{h \times b} \times 100$$

$$= 50\ \%$$

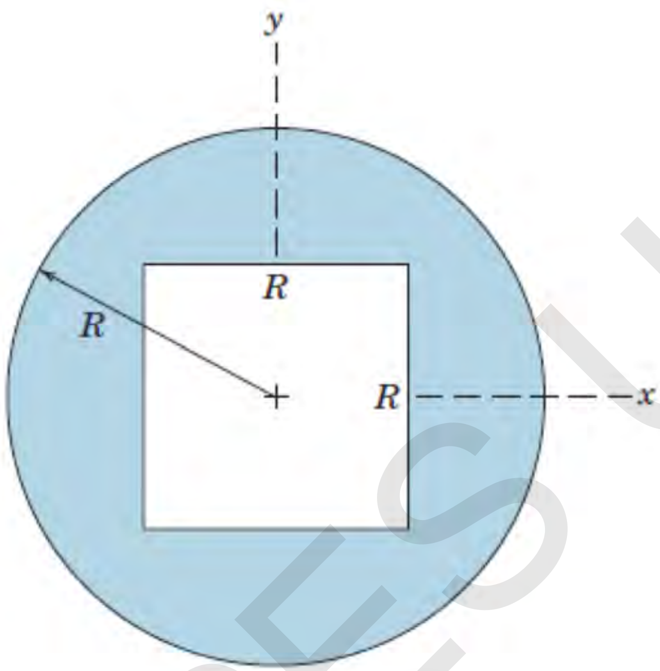
Percentage reduction in I_y

$$= \frac{I_y \text{ of cut portion}}{I_y \text{ of full rectangle}} \times 100$$

$$= \frac{3h}{4} \times \frac{8b^3}{27} \times \frac{12}{hb^3} \times \frac{100}{12}$$

$$= 22.22\ \%$$

Determine the moment of inertia about the y-axis of the circular area without and with the central square hole.



Percentage reduction in area :

$$= \frac{\text{Area of cut portion}}{\text{Area of full rectangle}} \times 100$$

$$= \frac{\frac{2b}{3} \times \frac{3h}{4}}{h \times b} \times 100$$

$$= 50\ \%$$

Percentage reduction in I_y

$$= \frac{I_y \text{ of cut portion}}{I_y \text{ of full rectangle}} \times 100$$

$$= \frac{3h}{4} \times \frac{8b^3}{27} \times \frac{12}{hb^3} \times \frac{100}{12}$$

$$= 22.22\ \%$$