

Using Limits to Compare Order of Growth

$$\lim_{n \rightarrow \infty} \frac{t(n)}{g(n)} = \begin{cases} 0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n), \\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n), \\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n). \end{cases}$$

Case1: $t(n) \in O(g(n))$

Case2: $t(n) \in \Theta(g(n))$

Case3: $g(n) \in O(t(n))$

$t'(n)$ and $g'(n)$ are first-order derivatives of $t(n)$ and $g(n)$

L'Hopital's Rule $\lim_{n \rightarrow \infty} \frac{t(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{t'(n)}{g'(n)}$

Stirling's Formula $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ for large values of n

Using Limits to Compare Order of Growth: Example 1

Compare the order of growth of $f(n)$ and $g(n)$ using method of limits

$$t(n) = 5n^3 + 6n + 2, \quad g(n) = n^4$$

$$\lim_{n \rightarrow \infty} \frac{t(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{5n^3 + 6n + 2}{n^4} = \lim_{n \rightarrow \infty} \left(\frac{5}{n} + \frac{6}{n^3} + \frac{2}{n^4} \right) = 0$$

As per case1

$$t(n) = O(g(n))$$

$$5n^3 + 6n + 2 = O(n^4)$$

Using Limits to Compare Order of Growth: Example 2

$$t(n) = \sqrt{5n^2 + 4n + 2}$$

using the Limits approach determine $g(n)$ such that $f(n) = \Theta(g(n))$

Leading term in square root n^2

$$g(n) = \sqrt{n^2} = n$$

$$\lim_{n \rightarrow \infty} \frac{t(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\sqrt{5n^2 + 4n + 2}}{\sqrt{n^2}}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{5n^2 + 4n + 2}{n^2}} = \lim_{n \rightarrow \infty} \sqrt{5 + \frac{4}{n} + \frac{2}{n^2}} = \sqrt{5}$$

non-zero constant

Hence, $t(n) = \Theta(g(n)) = \Theta(n)$

Using Limits to Compare Order of Growth

$$\lim_{n \rightarrow \infty} t(n)/g(n) \neq 0, \infty \Rightarrow t(n) \in \Theta(g(n))$$

$$\lim_{n \rightarrow \infty} t(n)/g(n) \neq \infty \Rightarrow t(n) \in O(g(n))$$

$$\lim_{n \rightarrow \infty} t(n)/g(n) \neq 0 \Rightarrow t(n) \in \Omega(g(n))$$

$$\lim_{n \rightarrow \infty} t(n)/g(n) = 0 \Rightarrow t(n) \in o(g(n))$$

$$\lim_{n \rightarrow \infty} t(n)/g(n) = \infty \Rightarrow t(n) \in \omega(g(n))$$

Using Limits to Compare Order of Growth: Example 3

Compare the order of growth of $t(n)$ and $g(n)$ using method of limits

$$t(n) = \log_2 n, \quad g(n) = \sqrt{n}$$

$$\lim_{n \rightarrow \infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{(\log_2 n)'}{(\sqrt{n})'} = \lim_{n \rightarrow \infty} \frac{(\log_2 e) \frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 2 \log_2 e \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

$$\log_2 n \in o(\sqrt{n})$$