Handout 9

Examples on Mean Squared Error

Let X_1 and X_2 be independent, each with unknown mean μ and known variance $\sigma^2 = 1$.

a. Let $\widehat{\mu}_1 = \dfrac{X_1 + X_2}{2}$. Find the bias, variance, and mean squared error of $\widehat{\mu}_1$.

We denote the mean of $\widehat{\mu}_1$ by $E(\widehat{\mu}_1)$ and the variance of $\widehat{\mu}_1$ by $V(\widehat{\mu}_1)$.

$$E(\widehat{\mu}_1) = \frac{\mu_{X_1} + \mu_{X_2}}{2} = \frac{\mu + \mu}{2} = \mu.$$

The bias of $\widehat{\mu}_1$ is $E(\widehat{\mu}_1) - \mu = \mu - \mu = 0$.

The variance of $\widehat{\mu}_1$ is $V(\widehat{\mu}_1) = \dfrac{\sigma^2 + \sigma^2}{4} = \dfrac{\sigma^2}{2} = \dfrac{1}{2}$.

The mean squared error of $\widehat{\mu}_1$ is the sum of the variance and the square of the bias, so

$$MSE(\widehat{\mu}_1) = \frac{1}{2} + 0^2 = \frac{1}{2}.$$

Let $\widehat{\mu}_3 = \dfrac{X_1 + X_2}{4}$. Find the bias, variance, and mean squared error of $\widehat{\mu}_3$.

We denote the mean of $\widehat{\mu}_3$ by $E(\widehat{\mu}_3)$ and the variance of $\widehat{\mu}_3$ by $V(\widehat{\mu}_3)$.

$$E(\widehat{\mu}_3) = \frac{\mu_{X_1} + \mu_{X_2}}{4} = \frac{\mu + \mu}{4} = \frac{\mu}{2}.$$

The bias of $\widehat{\mu}_3$ is $E(\widehat{\mu}_3) - \mu = \dfrac{\mu}{2} - \mu = -\dfrac{\mu}{2}$.

The variance of $\widehat{\mu}_3$ is $V(\widehat{\mu}_3) = \dfrac{\sigma^2 + \sigma^2}{16} = \dfrac{\sigma^2}{8}$.

The mean squared error of $\widehat{\mu}_3$ is the sum of the variance and the square of the bias, so

$$MSE(\widehat{\mu}_3) = \frac{\sigma^2}{8} + \left(-\frac{\mu}{2}\right)^2 = \frac{2\mu^2 + 1}{8}.$$

For what values of μ does $\widehat{\mu}_3$ have smaller mean squared error than $\widehat{\mu}_1$?

$\widehat{\mu}_3$ has smaller mean squared error than $\widehat{\mu}_1$ whenever $\dfrac{2\mu^2+1}{8} < \dfrac{1}{2}$.
Solving for μ yields $-1.2247 < \mu < 1.2247$.