Electric circuit theory of mechanical Vibrations. (RMS value)

Parseval's Formula.

* P.T $\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$,

provided the Fourier series for $f(x)$ converges uniformly in $(-l, l)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

* NOTE (No proof) Parsevals identities in different cases.

(i) If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$ in $(0, 2l)$

then $\int_0^{2l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$

(ii) If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$ in $(0, l)$, then

$\int_0^{l} [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + \cdots\right]$ or $\frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

(iii) If $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ in $(0, l)$, then

$\int_0^{l} [f(x)]^2 dx = \frac{l}{2}\left[b_1^2 + b_2^2 + b_3^2 + \cdots\right]$

iv) $\frac{1}{2l}\int_c^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$.

Prove that in $0 < x < 2$, $x = 1 - \frac{8}{\pi^2}\left\{\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right\}$,

deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^2}{96}$ &

hence determine the sum S of the series

$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

Note: $\int_c^{c+2l} [f(x)]^2 dx = 2l\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$.

* Obtain the FS for $y=x^2$ in $-\pi < x < \pi$ & hence show that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4}{n^2}[-1]^n \quad b_n = 0$$

$$\therefore x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \Big] \text{AN}$$

Using Parsevals identity

$$\int_{-\pi}^{\pi}[x^2]^2\,dx = \pi\left[\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2\right]$$

$$\frac{2}{5}[\pi^5] = \pi\left[\frac{4\pi^4}{9\times 2}+\sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

$$\frac{2\pi^4}{5} = \frac{4\pi^4}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{8\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}}$$

$\frac{x^5}{5}$

$=2\frac{1}{5}[$

$\frac{2}{5}=$

$\frac{18-}{45}$

(3) Find the Fourier sine series for unity in $0<x<\pi$ & hence S.T $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots=\frac{\pi^2}{8}$

We get Half range Fourier sine series for 1 in $(0,\pi)$,

Let $1=\sum_{n=1}^{\infty} b_n \sin nx$

$$b_n=\frac{2}{\pi}\int_0^{\pi} 1 \sin nx\,dx=\frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore b_n=\frac{4}{n\pi} \text{ when } n \text{ is odd}$$

Now from Parsevals thm on FS,

$$\int_0^{\pi}[1]^2 dx=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{16}{(2n-1)^2\pi^2}$$

$$x\Big]_0^{\pi}=\frac{8}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

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Parseval's formula: OR Parseval's theorem on Fourier constants

(9)

Sol P.T $\int_{-l}^{l} [f(x)]^2 dx = l\left\{ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$

provided the Fourier series for $f(x)$ converges uniformly in $(-l, l)$. or $\frac{1}{2l}\int_{c}^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

Proof:- W.K.T the FS for $f(x)$ in $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

xlying both sides of (1) by $f(x)$ & inte term by term from $-l$ to l, we get

$$\int_{-l}^{l} [f(x)]^2 = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left\{ a_n \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx + b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \right\} \quad \text{--- (2)}$$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \Rightarrow \boxed{\int_{-l}^{l} f(x)\,dx = l a_0}$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \Rightarrow \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = l a_n$

& $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \Rightarrow \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = l b_n$.

Subst in (2) $\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$ is Parseval's id

Fourier half expansion

P.T in $0 < x < 2$, $x = 1 - \frac{8}{\pi^2}\left\{\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right\}$. using Parsevals thm, & hence deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$,

hence determine the sum S of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

Proof:- Here $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ in $(0, 2)$

$\cancel{\therefore f(x) = -x \quad -2 < x < 2 \text{ i.e } (-2, 2)}$ $\quad \cancel{2l = 4} \quad \cancel{l = 2}$

[As there are only cosine terms in RHS it is Half range F. Series & that too cosine series] $\boxed{\therefore l = 2}$.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_0^l f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_0 = \frac{2}{2}\int_0^2 x\,dx = \frac{x^2}{2}\Big]_0^2 = 2 \Rightarrow \boxed{a_0 = 2}$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[(x)\left(\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{2}}{n^2\pi^2/4}\right)\right]_0^2 = \frac{+4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{+4}{n^2\pi^2}\left[(-1)^n - 1\right] = \frac{-4}{n^2\pi^2}\left[1 - (-1)^n\right] \Rightarrow a_n = \frac{-8}{n^2\pi^2} \text{ if } n \text{ is odd}$$
0 if n is even

$\frac{\pi^4}{90}$

$$\therefore f(x) = \frac{1}{\cancel{2}}(\cancel{2}) + \frac{8}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} \text{ OR}$$

$$= 1 + \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right]$$

using Parsevals identity : $\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

$$\int_0^2 x^2 dx = \frac{\cancel{2}}{\cancel{2}}\left[\frac{4}{2} + \sum_{n=1,3,5\ldots}^{\infty}\frac{64}{n^4\pi^4}\right]$$

$$\frac{8}{3} = 2\left[1 + \sum_{n=1,3,5\ldots}^{\infty}\frac{32}{n^4\pi^4}\right] \Rightarrow \frac{1}{3} = \sum_{n=1,3,5\ldots}^{\infty}\frac{32}{n^4\pi^4} \Rightarrow \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \cdots$$