System of linear equations

A system of equations in which all the unknowns appear in 1st degree only is called ***system of linear equations.***

Consider 'n' equations with 'n' unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

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$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Matrix form of system of linear equations

The above system can be written as $Ax = b$

Here, $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ called coefficient matrix

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ called unknown matrix

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ called constant matrix

Geometry of linear equations

We have 2 geometric interpretations of equations and their solution. We call the interpretation the

(i) row picture

(ii) column picture

(i) Row picture

In the row picture, we draw a line for each equation . The solution of the system of linear equation is the unique intersection of all the lines.

(ii) Column picture

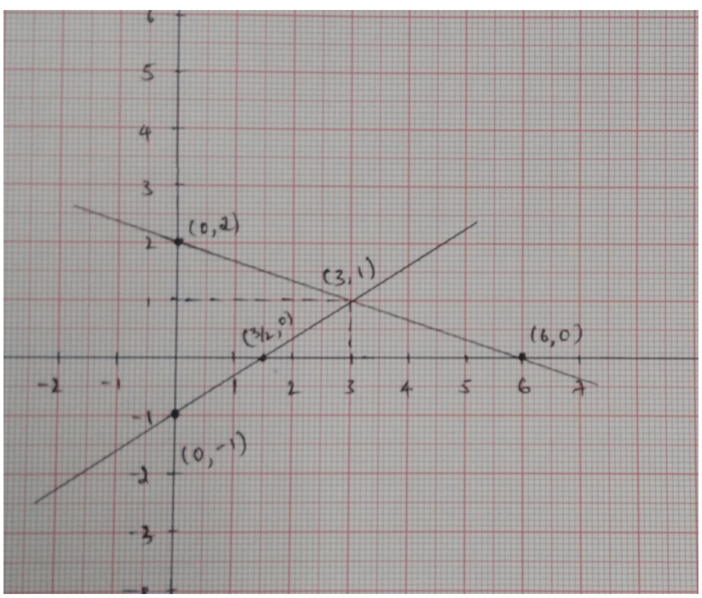
In column picture, we draw a vector for each column of the matrix A and draw the vector $\vec{b}$. Column picture is the linear combination of the columns of the matrix A which equal to vector $\vec{b}$.

Problems

1. Explain the row approach to solve the system $2x - 3y = 3, x + 3y = 6$ with neat diagram.

$$2x - 3y = 3 \quad \Rightarrow \quad \frac{x}{3/2} + \frac{y}{-1} = 1 \quad \Rightarrow \left(\frac{3}{2}, 0\right) \ and \ (0, -1)$$

$$x + 3y = 6 \ \Rightarrow \ \frac{x}{6} + \frac{y}{2} = 1 \ \Rightarrow \ (6\,,0) \ \ and \ \ (0\,,2)$$



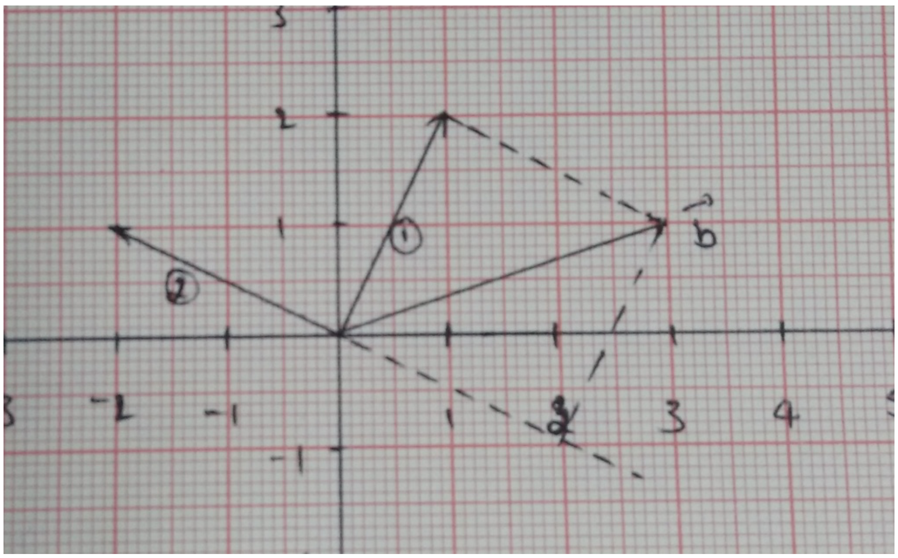
The two lines intersect at (3,1). $\quad x = 3, y = 1$. This is non-singular case.

2. Explain the column approach to solve the system $x - 2y = 3, 2x + y = 1$ with a neat diagram.

$Ax = b$

$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, x = \begin{bmatrix} x \\ y \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



From the graph

$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\Rightarrow x = 1, y = -1$ is the solution.

System is non-singular.

The method of Gaussian elimination

In this method unknowns are eliminated successively and the system is reduced to an upper triangular form using which unknowns are found by back substitution.

Consider

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The augmented matrix for the above given system is given by

$$[A\colon b] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1 ; R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & a_{32}' & a_{33}' & b_3' \end{array}\right]$$

$$R_3 \to R_3 - \frac{a_{32}'}{a_{22}'} R_2$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & 0 & a_{33}'' & b_3'' \end{array}\right]$$

by back substitution, we get the solution of given system of linear equation.

Problems

1.Solve the following system by the method of Gaussian elimination

$$x + 2y - z = 6$$

$$2x + y + z = 3$$

$$x - y + z = -2$$

Solution:

The augmented matrix is given by

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 1 & -2 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & -3 & 3 & -9 \\ 0 & -3 & 2 & -8 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & -3 & 3 & -9 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

The pivot elements are $a_{11} = 1, a_{22} = -3, a_{33} = -1$

From the above matrix: $x + 2y - z = 6$

$$-3y + 3z = -9$$

$$-z = 1$$

From the above equations one can get$x = 1,\ y = 2,\ z = -1.$

2. Solve the following system by the method of Gaussian elimination

$$u + v + w = 2$$

$$2u + 2v + 3w = 7$$

$$4u + 6v + 8w = 16$$

Solution:

The augmented matrix is given by

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 5 & 7 \\ 4 & 6 & 8 & 16 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right]$$

This is a case of temporary break down

Exchange $R_3 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

The pivot elements are $a_{11} = 1, a_{22} = 2, a_{33} = 3$

From the above matrix

$$u + v + w = 2$$

$$2v + 4w = 8$$

$$3w = 3$$

From the above equations one can get $u = -1,\ v = 2,\ w = 1.$

3. Solve the following system by the method of Gaussian elimination

$$3x - y + 2z = 1$$

$$x + 2y - 3z = 5$$

$$4x + y - z = 7$$

Solution:

The augmented matrix is given by

$$[A: b] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -1 & 7 \end{array}\right]$$

$$R_2 \to R_2 - \frac{1}{3}R_1;\ R_3 \to R_3 - \frac{4}{3}R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 7/3 & -11/3 & 14/3 \\ 0 & 7/3 & -11/3 & 17/3 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 7/3 & -11/3 & 14/3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is a case of permanent break down.

4. Investigate the values of λ and μ such that

$$x + 3y + 5z = 9$$

$$x - y + 2z = 1$$

$$2x + 2y + \lambda z = \mu$$

has (i) unique solution (ii) infinitely many solution (iii) no solution

Solution

The augmented matrix is given by

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & \lambda & \mu \end{array}\right]$$

$$R_2 \to R_2 - R_1; R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -4 & -3 & -8 \\ 0 & -4 & \lambda - 10 & \mu - 18 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -4 & -3 & -8 \\ 0 & 0 & \lambda - 7 & \mu - 10 \end{array}\right]$$

Unique solution:

From the Echelon form it is seeing that when$\lambda \neq 7, r(A) = 3$ & $r(A:b) = 3 = n$

hence we get a unique solution when$\lambda \neq 7$ [μ can be any real number].

No Solution

For the system to have no solution $r(A) \neq r(A:b)$. Thus $r(A)$ must be 2 and $r(A:b)$ must be 3.

For this to happen λ should be equal to 7 $(r(A) = 2)$ and $\mu - 10 \neq 0$ i.e.,

$\mu \neq 10$ $(r(A:b) = 3)$.

Many solutions

For the system to have infinitely many solution we should have

$r(A) = r(A:b) \neq n$. Thus $r(A) = 2,\ r(A:b) = 2$.

For this to happen $\lambda = 7,\ \mu = 10$

5. Use Gaussian elimination to test for consistency of the system of linear equations $x - 2y - 3z = 0,\ y + z = -8,\ -x + y + 2z = 3$. Find the solution of the system is consistent. In the case of inconsistency change the coefficient of z suitably in the third equation (viz 2) so that the system yields a unique solution with $z = 1$ (Do not substitute for z in the given set of equations). Solve also for x & y.

Solution

The augmented matrix is given by

$$[A:b] = \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

$$R_3 \to R_3 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right]$$

$$R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$$r(A) \neq r(A:b)$$

Therefore the system is inconsistent.

Given system yields unique solution $z = 1$

i.e.,

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & -5 & -5 \end{array}\right]$$

(Do the operation in reverse way)

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & -1 & -6 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ -1 & 1 & -3 & 3 \end{array}\right]$$

6. Do the three planes $x + 2y + z = 4, y - z = 1$ and $x + 3y = 0$ have atleast one common point of the intersection? Explain; is the system consistent if the last equation is changed to $x + 3y = 5$? If so solve the system completely.

Solution: The augmented matrix is given by

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$$r(A) \neq r(A:b)$$

Hence the system is inconsistent.

Now, replace last equation by $x + 3y = 5$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 5 \end{array}\right]$$

$$R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r(A) = r(A:b) = 2 < 3$$

Therefore$(3-2)=1$, unknowns can be chosen arbitrarily.

$$y - z = 1 \Longrightarrow y = 1 + z \Longrightarrow y = 1 + k \text{ (if } z = k)$$

$$x + 2y + z = 4 \Longrightarrow x + (2 + 2k) + k = 4 \Longrightarrow x = 2 - 3k$$

Elementary Matrices

An elementary matrix is a matrix obtained from the identity matrix by performing one single elementary row operation.

$$(E_{32}E_{31}E_{21})A = U \quad \text{(From } A \text{ to } U)$$

$$\Rightarrow A = {E_{21}}^{-1}{E_{31}}^{-1}{E_{32}}^{-1}\ U$$

$$\Rightarrow A = LU \qquad \text{(From } U \text{ to } A)$$

It is a square matrix with 1's on the main diagonal and almost one non-zero entry off the main diagonal.

1. Which E_{ij} put A into triangular form$U$? Multiply those$E$'s to get Msuch that $MA = U$. Also express Aas LU, where A is the coefficient matrix of the following system: $x + y + z = 6$

$$x + 2y - z = 2$$

$$2x + y - z = 1$$

Solution: The augmented matrix is given by

$$[A: b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & -1 & 2 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

$$R_2 \rightarrow R_2 - R_1;\ R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & -1 & -3 & -11 \end{array}\right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$(E_{32}E_{31}E_{21})A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix}$$

undoing the transformation to get back to $A$to see that

$\Rightarrow A = E_{21}{}^{-1}E_{31}{}^{-1}E_{32}{}^{-1}\ U \Rightarrow A = LU$ where$L = E_{21}{}^{-1}E_{31}{}^{-1}E_{32}{}^{-1}$is a lower triangular matrix. It can be easily written as

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, (E_{21})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, (E_{31})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, (E_{32})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = (E_{21})^{-1}(E_{31})^{-1}(E_{32})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

Triangular Factorization

(i) $A = LU$ (with no exchange of rows)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{21} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Here L is the lower triangular matrix with 1’s on the diagonal. U is the upper triangular matrix which appear after forward elimination. Diagonal entries of U are the pivots.

(ii) $A = LDU$ (with no exchange of rows)

where$L$ and U have 1’s on the diagonal &$D$is the diagonal matrix of pivots.

Permutation Matarix

If I_n is an identity matrix of order n, then permutation matrix is obtained by interchanging only two rows of I_n.

- When a matrix has to be factorized into LU at same point, row exachange might be needed, then A is multiplied by corresponding permutation matrix and then factorized into LU. Thus $PA = LU$.
- P^{-1} is always same as P^T.
- When using Gauss elimination method to find$U$, all the row operations involved. These row operations will help to find L using the identity matrix.
- To write L , start with the identity matrix & use the following rule
  → Any row that involves adding a multiple of one row to other
  Eg: $R_i \to R_i + kR_j$, put the value $-k$ in i^{th}row and j^{th} column of the identity matrix i.e., k_{ij} position.
- Order of L= no. of rows of given matrix.

1. Find LU factorization for $\begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 4 & -4 & -1 & 4 & 11 \\ 2 & -5 & -2 & 2 & -1 \\ 0 & 2 & 1 & 0 & 4 \end{bmatrix}$

Solution:

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & -2 & -1 & 0 & -4 \\ 0 & 2 & 1 & 0 & 4 \end{bmatrix}$$

$$R_3 \to R_3 + R_2; R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$R_4 \to R_4 + R_3$$

$$\sim \begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

2. $A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$

Solution:$R_2 \to R_2 + \frac{1}{2}R_1; R_3 \to R_3 + \frac{1}{2}R_1$

$$\sim \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$L \to$ Square matrix of order = no. of rows

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ -\dfrac{1}{2} & 3 & 1 \end{bmatrix}$$

3. Find LU and LDU factorization for $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

Solution: $R_2 \to R_2 - \frac{2}{3}R_1; R_3 \to R_3 - \frac{1}{3}R_1$

$$\sim \begin{bmatrix} 3 & 1 & 2 \\ 0 & -\dfrac{11}{3} & -\dfrac{7}{3} \\ 0 & \dfrac{5}{3} & \dfrac{1}{3} \end{bmatrix}$$

$$R_3 \to R_3 + \frac{5}{11}R_2$$

$$\sim \begin{bmatrix} 3 & 1 & 2 \\ 0 & -\frac{11}{3} & -\frac{7}{3} \\ 0 & 0 & -\frac{24}{33} \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{-5}{11} & 1 \end{bmatrix}$$

$A = LU$ & $A = LDU, D$is the diagonal matrix of pivots, Here L and U have 1's in the diagonal.

Therefore divide each row of U by its pivot.

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{-5}{11} & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{11}{3} & 0 \\ 0 & 0 & -\frac{24}{33} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

4. Find LDU factorization for $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

A cannot be factorized in the given form.

Introduce permutation matrix

$$P_{12}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$P_{12}A = LU \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P_{12}A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Factorize into LDU

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1; R_4 \to R_4 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$P_{23}A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$P_{23}A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$P_{34}P_{23}A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} = U$$

Dummy matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{34}P_{23}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Invertible Matrices

A matrix of order n x n is said to be is said to be invertible matrix if there (exist) is a matrix B of order n x n such that $AB = BA = I$ where $I$is the identity matrix of order n x n. A matrix B is called as inverse of A and it is denoted by A^{-1}.

- ❖ Inverse matrix is unique
- ❖ $AA^{-1} = A^{-1}A = I$
- ❖ If A is invertible then $A^{-1}$is also invertible
- ❖ $(AB)^{-1}$=$B^{-1}A^{-1}$
- ❖ Invertible matrix is also called as non-singular matrix
- ❖ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$& if $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- ❖ If A is invertible then only one solution to the system $Ax = b$ is $x = A^{-1}b$

<u>Inverse by row reduce to A to I</u>[Gauss-Jordan Method]

$$[A{:}\, I] \sim [LU{:}\, I] \sim [I{:} \sim (LU)^{-1}I] \sim [I{:} \sim U^{-1}L^{-1}] \sim [I{:} \sim A^{-1}]$$

$$\Rightarrow [A{:}\, I] \ \sim [I{:} \sim A^{-1}]$$

Let A be a matrix of order n x n and I be an identity matrix of some order. Then $[A{:}\, I]$ is called as augmented matrix. Transform the matrix A into identity matrix $I$by eliminating row operations. Then identity matrix Iin augmented matrix is transform into A^{-1}.

- ❖ The inverse of a matrix A exist if and only if Gauss elimination produces *'n'*
  [Permanent break down → inverse does not exists]
  or
  A square matrix A is invertible if and only if elimination yields the same pivots as rows

1. Obtain the inverse of A (or) use Gauss-Jordan method to solve

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution:

$$[A:I] = \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 2 & -1 & 1 & : & 0 & 1 & 0 \\ 1 & 3 & -1 & : & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & -3 & -1 & : & -2 & 1 & 0 \\ 0 & 2 & -2 & : & -1 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{2}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & -3 & -1 & : & -2 & 1 & 0 \\ 0 & 0 & -\frac{8}{3} & : & -\frac{7}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{3}{8}R_3; R_1 \to R_1 + \frac{3}{8}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & \frac{1}{8} & \frac{2}{8} & \frac{3}{8} \\ 0 & -3 & 0 & : & -\frac{9}{8} & \frac{6}{8} & -\frac{3}{8} \\ 0 & 0 & -\frac{8}{3} & : & \frac{7}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$R_1 \to R_1 + \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ 0 & -3 & 0 & : & -\frac{9}{8} & \frac{6}{8} & -\frac{3}{8} \\ 0 & 0 & -\frac{8}{3} & : & \frac{7}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$$R_2 \to -\frac{1}{3}R_2; R_3 \to -\frac{3}{8}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ 0 & 1 & 0 & : & \frac{3}{8} & -\frac{2}{8} & \frac{1}{8} \\ 0 & 0 & 1 & : & \frac{7}{8} & -\frac{2}{8} & -\frac{3}{8} \end{bmatrix}$$

$$I : A^{-1}$$