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Q1. $L: R^3 \to R^2$ by $L(x_1, x_2, x_3) = (x_3 - x_1, x_1 + x_2)$

(i) Representing in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_1 \\ x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore L(e_1) = (-1, 1) \qquad L(e_2) = (0, 1) \qquad L(e_3) = (1, 0)$

(ii) Take any (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in R^3$

$$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(\underbrace{x_1 + x_2}_{x_1}, \underbrace{y_1 + y_2}_{x_2}, \underbrace{z_1 + z_2}_{x_3})$$

$$= T((z_1 + z_2) - (x_1 + x_2), (x_1 + x_2) + (y_1 + y_2))$$

$$= T(z_1 + z_2 - x_1 - x_2, x_1 + x_2 + y_1 + y_2)$$

$$= T((z_1 - x_1) + (z_2 - x_2), (x_1 + y_1) + (x_2 + y_2))$$

$$= (z_1 - x_1, x_1 + y_1) + (z_2 - x_2, x_2 + y_2)$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

Also $T(0, 0, 0) = \cancel{T}(0, 0) = 0$

(iii) $L(e_1) = L(1, 0, 0) = (-1, 1)$

$L(e_2) = L(0, 1, 0) = (0, 1)$

$L(e_3) = L(0, 0, 1) = (1, 0)$

Q2 $L(1, 0, 1) = (-1, 1, 0, 2) \qquad L(0, 1, 1) = (0, 6, -2, 0) \qquad L(-1, 1, 1) = (4, -2, 1, 0)$

Using the property of $L(x_1, x_2, x_3) = x_1 L(e_1) + x_2 L(e_2) + x_3 L(e_3)$

$L(1, 0, 1) = L(e_1) + L(e_3) = (-1, 1, 0, 2)$

$L(0, 1, 1) = L(e_2) + L(e_3) = (0, 6, -2, 0)$

$L(-1, 1, 1) = -L(e_1) + L(e_2) + L(e_3) = (4, -2, 1, 0)$

Representing in matrix form

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} L(e_1) \\ L(e_2) \\ L(e_3) \end{bmatrix} = \begin{bmatrix} 4 & -2 & 1 & 0 \\ 0 & 6 & -2 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}$$

Gaussian elimination:

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & -2 & 1 & 0 \\ 0 & 6 & -2 & 0 \\ 3 & -1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & -2 & 1 & 0 \\ 0 & 6 & -2 & 0 \\ 3 & -7 & 3 & 2 \end{bmatrix}$$

$\therefore \; L(e_3) = (3, -7, 3, 2)$

$L(e_2) + L(e_3) = (3, -7, 3, 2) + L(e_2) = (0, 6, -2, 0)$

$\therefore \; L(e_2) = (-3, 13, -5, -2)$

$-L(e_1) + L(e_2) + L(e_3) = -L(e_1) + (0, 6, -2, 0) = (4, -2, 1, 0)$

$\therefore \; L(e_1) = -(+4, -8, 3, 0)$

$\therefore \; L(e_1) = (-4, 8, -3, 0)$

$L(1, 2, -1) = L(e_1) + 2L(e_2) - L(e_3)$

$= (-4, 8, -3, 0) + (-6, 26, -10, -4) + (-3, 7, -3, -2)$

$= (-13, 41, -16, -6)$

Q3. We multiply B with each vector in the standard basis of F
The four corresponding matrices are columns in the matrix L

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}$$

$$\therefore \quad L = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

Q4. Since they are orthogonal, it means they are independent.

$v = (7, 1, 9)$

$v_1 = (1, 2, 1)$ $\qquad v_2 = (2, 1, -4)$ $\qquad v_3 = (3, -2, 1)$

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

We get the following equations

$$c_1 + 2c_2 + 3c_3 = 7$$

$$2c_1 + c_2 - 2c_3 = 1$$

$$c_1 - 4c_2 + c_3 = 9$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 9 \end{bmatrix}$$

Performing gaussian elimination

$R_2 \to R_2 - 2R_1$ $\qquad\qquad$ $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & -3 & -8 & -13 \\ 0 & -6 & -2 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 14 & 28 \end{bmatrix}$$

$\therefore c_3 = 2$ $\qquad$ $\therefore c_2 = -1$ $\qquad$ $\therefore c_1 = 3$

$\therefore v = (7, 1, 9) = 3v_1 - v_2 + 2v_3$

Q5. We know, $P = A(A^TA)^{-1}A^T$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$ $\qquad$ $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad$ $(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 3 & -3 & 0 \end{bmatrix}$

$$A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/6 & 1/3 \\ -1/6 & 5/6 & 1/3 \\ 1/3 & +1/3 & 1/3 \end{bmatrix}$$

Q6. We need to find $\hat{x}$.

$$\hat{x} = (A^TA)^{-1}A^Ty$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \frac{1}{10}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 3 & -3 & -1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

$$(A^TA)^{-1}A^Ty = \frac{1}{10}\begin{bmatrix} 1 & 3 & -3 & -1 \\ 2 & 1 & 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 7/10 \\ 7/5 \end{bmatrix}$$

Q7. Converting the equations in matrix form

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 5 \\ 2.09 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

To find x, we find $\hat{x} = (A^TA)^{-1}A^Tb$

$$A^TA = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ 9 & 9 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{18} \begin{bmatrix} 9 & -9 \\ -9 & 11 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \frac{1}{18} \begin{bmatrix} 9 & -9 \\ -9 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -9 & 9 & 0 \\ 13 & -5 & 2 \end{bmatrix}$$

$$(A^TA^{-1})A^Tb = \frac{1}{18} \begin{bmatrix} -9 & 9 & 0 \\ 13 & -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2.09 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 18 \\ 18.18 \end{bmatrix}$$

$$\therefore \hat{x} = \begin{bmatrix} 1 \\ 1.01 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore x = 1 \qquad y = 1.01$$

Q8. $T(x, y, z) = (5x - 3y + z, \; 2z + 4y, \; 5x + 3y)$

To find the standard matrix, we use vectors from the identity matrix

$T(1, 0, 0) = (5, 0, 5)$

$T(0, 1, 0) = (-3, 4, 3)$

$T(0, 0, 1) = (1, 2, 0)$

These are now columns in the final matrix.

$$\therefore A = \begin{bmatrix} 5 & -3 & 1 \\ 0 & 4 & 2 \\ 5 & 3 & 0 \end{bmatrix}$$

99. $T(x, y, z) = (2x + 3y,\ 3y - z)$

$T(1,0,0) = (2,0)$

$T(0,1,0) = (3,3)$

$T(0,0,1) = (0,-1)$

$$\therefore A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

We need to find $T(0,1,-1)$

So $Ax = b$ where $x = (0,1,-1)$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

~~verify~~ Verifying with $T(0,1,-1) = (3(1),\ 3(1) - (-1))$

$= (3,4)$

$\therefore T(0,1,-1) = (3,4)$

10. Given,

T is a reflection of the line $y=x$ in $\mathbb{R}^2$.

so, $T(x,y) = (y,x)$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

Standard matrix of $T = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$$T\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$

$$T\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$

11. Given,

$$\vec{x} = \begin{bmatrix}6\\5\\4\end{bmatrix}, \quad \vec{a} = \begin{bmatrix}1\\-1\\3\end{bmatrix}$$

Let, projection be $\hat{x}.a$

$$p = \hat{x}.a = \left[\frac{a^T x}{a^T a}\right].a$$

$$= \left(\frac{\begin{bmatrix}1 & -1 & 3\end{bmatrix}\begin{bmatrix}6\\5\\4\end{bmatrix}}{11}\right)\begin{bmatrix}1\\-1\\3\end{bmatrix}$$

$$p = \hat{x}.a = \begin{bmatrix}\frac{13}{11}\\ -13/11\\ +39/11\end{bmatrix}$$

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13. Given, $(1,0,1,0,2)$, $(0,1,1,1,0)$ & $(1,1,1,1,1)$

Let, $S = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - (1) R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix} = U = R$

The basis for $S^{\perp} = (-1,-1,1,0,0), (0,-1,0,1,0), (-2,0,0,0,1)$

14. P_1 = Projection of b onto A

$$= \frac{a^T b}{a^T . a} . a$$

$$= \frac{19}{38} (3,2,5)$$

$$= (3/2, 1, 5/2)$$

P_2 = Projection of b onto c

$$= \frac{c^T b}{c^T c} . c$$

$$= (0,0,0)$$

15. Reflection matrix $H = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$

1st reflection about y-axis $\theta = 90^\circ$, Let T be the matrix

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2nd reflection about ~~y~~ x-axis $\theta = 0^\circ$, Let s be the matrix

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$ST = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

16. Given,

$(1,-1), (4,11), (-1,-9)$ & $(-2,-13)$

Let, $b = C + Dt$ be the best fitting line,

Now, $C + D(1) = -1$

$C + D(4) = 11$

$C + D(-1) = -9$

$C + D(-2) = -13$

i.e, $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ -9 \\ -13 \end{bmatrix}$$

consider,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 11 \\ -9 \\ -13 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 22 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -12 \\ 78 \end{bmatrix}$$

$R_2 \to R_2 - (1/2) R_1$

$$\begin{bmatrix} 4 & 2 \\ 0 & 21 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -12 \\ 84 \end{bmatrix}$$

$C = -5, D = 4$ the best fitting line is $b = 4t - 5$

17. Given,

x	0	1	2	3
$f(x)$	1	0	1	2

Let, $f(x) = C + Dx$ be the best fitting line,

$C + D(0) = 1$, $C + D(1) = 0$, $C + D(2) = 1$, $C + D(3) = 2$

$C = 1$, $C + D = 0$, $C + 2D = 1$, $C + 3D = 2$

i.e; $Ax = b$

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$$\begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\0\\1\\2\end{bmatrix}$$

Consider, $A^T A \hat{x} = A^T b$

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\0\\1\\2\end{bmatrix}$$

$$\begin{bmatrix}4 & 6\\6 & 14\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}4\\8\end{bmatrix}$$

$$R_2 \rightarrow R_2 - (3/2) R_1$$

$$\begin{bmatrix}4 & 6\\0 & 5\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}4\\8\end{bmatrix}$$

$$4C + 6D = 4, \quad 5D = 8$$

$$4C + 6(8/5) = 4 \qquad D = 8/5.$$

$$C + 6(2/5) = 1$$

$$C = 1 - \frac{12}{5}$$

$$C = -7/5$$

The best fitting line is $-\frac{7}{5} + x\left(\frac{8}{5}\right) = b$

$$-7 + 8x = 5b$$

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Q12 $W^{\perp}$ is a line through the point $(1,1,1)$
Find the matrix P that projects onto this line

$P = A(A^TA)^{-1}A^T$ where $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$

$(A^TA)^{-1} = \frac{1}{3}$

$A(A^TA)^{-1}A^T = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The matrix that projects onto W is $(I-P)$.

$$proj_W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$\therefore proj_W(y) = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Q18. If $v \in V$ then $v = c(1,1)$

If $w \in W$ then $w = c(1,2)$

$\therefore \; c_1(1,1) + c_2(1,2) = (2,-1)$

we get the equations:

$$c_1 + c_2 = 2$$

$$c_1 + 2c_2 = -1$$

We get, $c_1 = 5$ and $c_2 = -3$.

$\therefore \; v = 5(1,1) = (5,5)$

$w = -3(1,2) = (-3,-6)$