UNIT 3 - INTEGRAL CALCULUS

JACOBIAN

1. If $u = x\sin y$, $v = y\sin x$ find $\frac{\partial(u,v)}{\partial(x,y)}$

↳ Given $u = x\sin y$ & $v = y\sin x$

$\frac{\partial u}{\partial x} = \sin y$ $\qquad$ $\frac{\partial v}{\partial x} = y\cos x$

$\frac{\partial u}{\partial y} = x\cos y$ $\qquad$ $\frac{\partial v}{\partial y} = \sin x$

$$\because J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = J = \begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x \end{vmatrix}$$

$$\Rightarrow \boxed{J = \sin x \sin y - xy\cos x \cos y}$$

2. $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

↳ Please refer class notes

3. Find Jacobian of u, v, w, wrt x, y, z when $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$.

↳ given, $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$u_x = -\frac{yz}{x^2}$ $\qquad$ $u_y = \frac{z}{x}$ $\qquad$ $u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$ $\qquad$ $v_y = -\frac{zx}{y^2}$ $\qquad$ $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$ $\qquad$ $w_y = \frac{x}{z}$ $\qquad$ $w_z = -\frac{xy}{z^2}$

$$\therefore J = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{xz}{y} & x \\ y & x & -\frac{xy}{z} \end{vmatrix}$$

$$= \frac{1}{xyz}\left[-\frac{yz}{x}\left(x^2 - x^2\right) - z\left(-xy - xy\right) + y\left(xz + xz\right)\right]$$

$$= \frac{1}{xyz}\left(2xyz + 2xyz\right) = \frac{4xyz}{xyz} \Rightarrow \boxed{J = 4}$$

4. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

↳ Here, $u = xyz$

$v = x^2 + y^2 + z^2$

$w = x + y + z$

$u_x = yz \qquad u_y = xz \qquad u_z = xy$

$v_x = 2x \qquad v_y = 2y \qquad v_z = 2z$

$w_x = 1 \qquad w_y = 1 \qquad w_z = 1$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} yz & xz & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} z(y-x) & x(z-y) & xy \\ x-y & y-z & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad J = 2\begin{vmatrix} z(y-x) & x(z-y) \\ x-y & y-z \end{vmatrix} = 2\begin{vmatrix} z(y-x) & x(z-y) \\ -1(y-x) & -1(z-y) \end{vmatrix}$$

$$= 2(y-x)(z-y)\begin{vmatrix} z & x \\ -1 & -1 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

$$\cancel{= 2(x-y)(y-z)(z-z)}$$

since $J \cdot J' = 1$ where, $J' = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$$\Rightarrow \boxed{J' = \frac{-1}{2(x-y)(y-z)(z-x)}}$$

5. If $u = x+y+z$, $uv = y+z$, $uvw = z$, find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

↳ Please refer notes

6. Calculate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = \dfrac{2yz}{x}$, $v = \dfrac{3zx}{y}$, $w = \dfrac{4xy}{z}$.

↳ Please refer notes

in notes, we have found $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 96$

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{96}}$$ (This is the correct answer.

PROPERTIES OF JACOBIAN

1. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$.

↳ given $x = u(1-v)$

& $y = uv$

let $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

$$\Rightarrow \quad J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$J = u(1-v) + uv$$

$$\Rightarrow \quad J = x+y \text{ ------ } (1)$$

now, $\dfrac{x}{y} = \dfrac{u(1-v)}{uv} = \dfrac{1-v}{v}$

$$\Rightarrow \quad xv = y - vy$$

$$(x+y)v = y$$

$$\boxed{v = \frac{y}{x+y}}$$

$$u = \frac{y}{v} = \frac{y}{\frac{y}{x+y}} \Rightarrow \boxed{u = x+y}$$

Let $J' = \dfrac{\partial(u,v)}{\partial(x,y)}$

$$\Rightarrow J' = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \dfrac{-y}{(x+y)^2} & \dfrac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

$$\Rightarrow J' = \frac{1}{x+y} \quad \text{---} \quad (2)$$

From (1) & (2)

$$J \cdot J' = (x+y) \cdot \frac{1}{(x+y)} = 1$$

$$\therefore \boxed{JJ' = 1}$$

Hence Proved.

2. If $u = x + \dfrac{y^2}{x}$, $v = \dfrac{y^2}{x}$, prove that $JJ' = 1$.

$\hookrightarrow$ $u = x + \dfrac{y^2}{x}$, $v = \dfrac{y^2}{x}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 - \dfrac{y^2}{x^2} & \dfrac{2y}{x} \\ -\dfrac{y^2}{x^2} & \dfrac{2y}{x} \end{vmatrix}$$

$$= \frac{2y}{x}\begin{vmatrix} 1-\frac{y^2}{x^2} & 1 \\ -\frac{y^2}{x^2} & 1 \end{vmatrix}$$

$$= \frac{2y}{x}\left[1-\frac{y^2}{x^2}+\frac{y^2}{x^2}\right] = \frac{2y}{x}$$

$$\Rightarrow J = \frac{2y}{x} \longrightarrow (1)$$

now $u = x+\frac{y^2}{x}$. $v = \frac{y^2}{x^2}$

$u - v = x \Rightarrow \boxed{x = u-v}$

$\frac{y^2}{x} = u - x = u-u+v$

$\frac{y^2}{x} = v \qquad y^2 = v(x) = v(u-v)$

$\boxed{y = \sqrt{v(u-v)}} \Rightarrow \boxed{y = \sqrt{uv-v^2}}$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ \frac{v}{2\sqrt{uv-v^2}} & \frac{(u-2v)}{2\sqrt{uv-v^2}} \end{vmatrix}$$

$$\Rightarrow J' = \frac{v+u-2v}{2\sqrt{uv-v^2}} = \frac{u-v}{2\sqrt{v}.\sqrt{u-v}} \cancel{= \frac{1}{2}\frac{\sqrt{u-v}}{\sqrt{v}}}$$

now, $x = u-v$. $\Rightarrow J' = \frac{x}{2y} \longrightarrow (2)$

$y = \sqrt{uv-v^2}$

From (1) & (2), $JJ' = \frac{2y}{x}.\frac{x}{2y} = 1 \Rightarrow \boxed{JJ' = 1}$ Hence Proved.

3. If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}(x) + \tan^{-1}(y)$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related? If so, find this relationship.

↳ Please refer class notes.

4. If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ and $v = \sin^{-1}(x) + \sin^{-1}(y)$, show that u and v are functionally related, and find this relationship.

↳ $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$v = \sin^{-1}(x) + \sin^{-1}(y)$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{1-y^2} - \frac{yx}{\sqrt{1-x^2}} & \frac{-yx}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$\Rightarrow J = 1 - \frac{yx}{\sqrt{1-y^2}\sqrt{1-x^2}} + \frac{yx}{\sqrt{1-y^2}\sqrt{1-x^2}} - 1 = 0$$

$$\Rightarrow \boxed{J = 0}$$

Hence, u & v are functionally dependend.

now, $v = \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \sin^{-1}(u)$

$\Rightarrow \boxed{u = \sin v}$ → This is the relationship.

5. If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

(done in notes)

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (4x^2 + 4y^2) \times \left[r(\cos^2\theta + \sin^2\theta) \right]$$

$$= 4(r^2) \cdot r$$

$$\Rightarrow \boxed{J = 4r^3}$$

6. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)}$$

$$= \begin{vmatrix} 0 & \frac{w}{2\sqrt{vw}} & \frac{v}{2\sqrt{vw}} \\ \frac{w}{2\sqrt{wu}} & 0 & \frac{u}{2\sqrt{wu}} \\ \frac{v}{2\sqrt{vu}} & \frac{u}{2\sqrt{vu}} & 0 \end{vmatrix} \times \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{2uvw}\begin{vmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{vmatrix} \times r^2 \times \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{8uvw}\left[+uvw + uvw\right] \times r^2 \times \left[\cos\theta\left[\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta . \sin^2\phi\right] + \sin\theta\left[\sin^2\theta . \cos^2\phi + \sin^2\theta . \sin^2\phi\right]\right]$$

$$= \frac{1}{4} \times r^2 \times \left[\sin\theta . \cos^2\theta + \sin\theta . \sin^2\theta\right]$$

$$\Rightarrow \boxed{J = \frac{1}{4} r^2 \sin\theta}$$

DOUBLE INTEGRALS

1. Evaluate $\int_3^4 \int_1^2 \frac{dy\,dx}{(x+y)^2}$

↳ Please refer illustration in notes

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

↳ Please refer illustration in notes

3. Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ & $y=x$.

↳ Please refer illustration in notes

4. Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the first quadrant by the lines $x=y$, $y=0$, $x=8$ & the curve $xy=16$.

↳ Please refer problem in notes

5. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} (1+x^2+y^2)^{-1}\,dx\,dy$

↳ Please refer problem in notes.

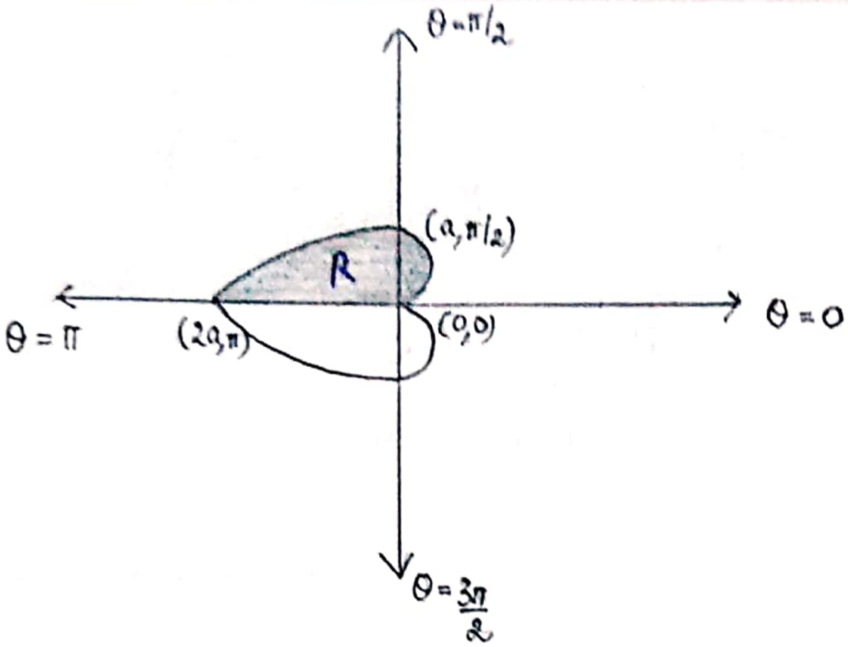
6. Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ & $r=4\sin\theta$.

↳ Please refer problem in notes

7. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.

↳ Given curve: $r=a(1-\cos\theta)$

Sketch is as shown:



here, $r: 0 \rightarrow a(1-\cos\theta)$

$\theta: 0 \rightarrow \pi$

$$\therefore \quad I = \int_0^{\pi} \int_0^{a(1-\cos\theta)} r \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \cdot \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta \quad = \frac{a^2}{2} \int_0^{\pi} (1-\cos\theta)^2 \sin\theta \, d\theta$$

let $1-\cos\theta = t$

$\Rightarrow \sin\theta \, d\theta = dt$

also, lower limit, ~~$1-\cos\theta = 0$~~

$\Rightarrow 1-1=0$

$t=0$

upper limit $t = 1-(-1) = 2$

$$\therefore \quad I = \frac{a^2}{2} \int_0^2 t^2 \, dt \quad = \frac{a^2}{2} \cdot \left[\frac{t^3}{3}\right]_0^2$$

$$= \frac{a^2}{2} \cdot \frac{8}{3}$$

$$\Rightarrow \boxed{I = \frac{4a^2}{3}}$$

CHANGE OF VARIABLES IN DOUBLE INTEGRATION (POLAR)

Evaluate the following integrals, by changing into polar co-ordinates.

1. $$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

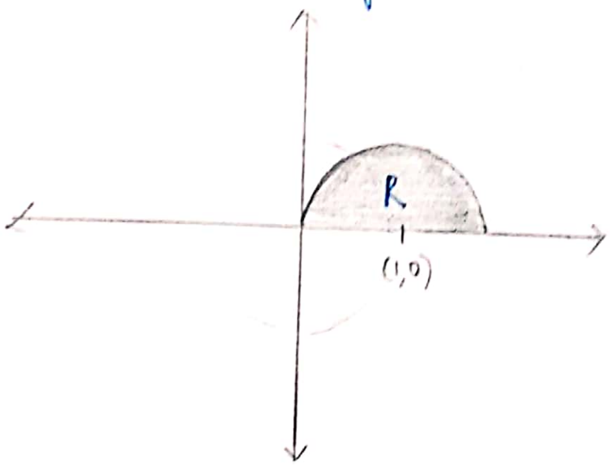
↳ given, $$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

$y = \sqrt{2x-x^2}$

$y^2 + x^2 - 2x = 0$

$c = (1,0)$

$r = 1$



here $\theta : 0 \to \frac{\pi}{2}$

$r : 0 \to 2\cos\theta.$

$x^2 + y^2 = 2x$

$r^2 = r\,2\cos\theta$

$\Rightarrow r = 2\cos\theta$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{r} . r . dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r . \cos\theta . dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta . \left.\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta \quad = 2\int_0^{\pi/2} \cos^3\theta . d\theta$$

Using reduction formula:

$$I = 2 \times \frac{2}{3} \cancel{x} \Rightarrow \boxed{I = \frac{4}{3}}$$

2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$

$\hookrightarrow I = \int_0^\infty \int_0^\infty e^{-x^2+y^2}\, dy\, dx$

The given region of integration is the entire XOY plane.

here:- $r : 0 \to \infty$

$\theta : 0 \to \frac{\cancel{2}\pi}{2}$

$$\therefore I = \int_{\theta=0}^{\frac{\cancel{2}\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r.\, dr\, d\theta$$

Let $r^2 = t$, $\quad 2r.\, dr = dt$

$r.\, dr = \frac{dt}{2}$

$r^2 = 0,\ t = 0$

$r = \infty,\ t = \infty$

$$\Rightarrow I = \int_0^{\cancel{2}\pi/2} d\theta . \int_0^\infty \frac{e^{-t}}{2}\, dt$$

$$= \frac{\cancel{2}\pi}{2} . \left(-\frac{e^{-t}}{2}\right)\Bigg]_0^\infty = \frac{\cancel{2}\pi}{4}\left[-0+1\right]$$

$$\Rightarrow \boxed{I = \pi/4}$$

also, $I = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2} \cdot dy\, dx = \frac{\pi}{4}$

$$\Rightarrow \left(\int_0^\infty e^{-x^2} dx\right) \cdot \left(\int_0^\infty e^{-y^2} \cdot dy\right) = \frac{\pi}{4}$$

$$\Rightarrow \left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4}$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^2} \cdot dx = \frac{\sqrt{\pi}}{2}}$$

Hence Proved.

3. $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx\, dy$

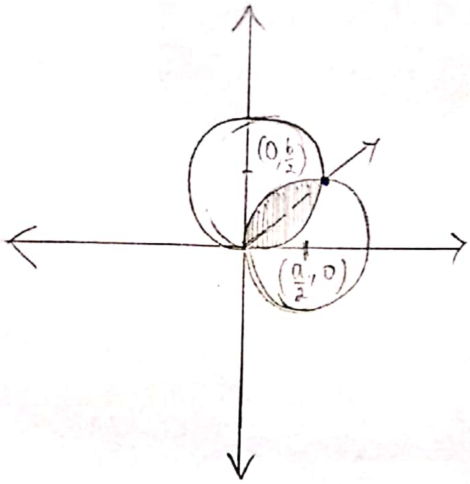
↳ Please refer class notes.

4. $\iint_R \sqrt{x^2+y^2}\, dx\, dy$, where R is the region bounded by the circles $x^2+y^2=4$.

↳ QUESTION IS INCOMPLETE.

5. $\iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$ over the area common to $x^2+y^2=ax$ and $x^2+y^2=by$, $a,b>0$.

↳



$ax=by$

$\Rightarrow y=\frac{a}{b}x$

and $\tan\theta=\frac{a}{b}$

$\Rightarrow \theta=\tan^{-1}\frac{a}{b}$

here, the region needs to be split into 2 parts.

$$\therefore\ I=\int_{\theta=0}^{\tan^{-1}\frac{a}{b}}\int_{r=0}^{b\sin\theta}\frac{r^4}{r^4\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta+\int_{\theta=\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}\int_{r=0}^{a\cos\theta}\frac{r^4}{r^4\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta$$

$$\Rightarrow I=\int_0^{\tan^{-1}\frac{a}{b}}\int_0^{b\sin\theta}\frac{1}{\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta+\int_{\tan^{-1}\frac{a}{b}}^{\cancel{\tan^{-1}\frac{a}{b}}\ \frac{\pi}{2}}\int_0^{a\cos\theta}\frac{1}{\sin^2\theta\cos^2\theta}\,r\,dr\,d\theta$$

$$=\int_0^{\tan^{-1}\frac{a}{b}}\frac{1}{\sin^2\theta\cos^2\theta}\left.\frac{r^2}{2}\right]_0^{b\sin\theta}d\theta+\int_{\tan^{-1}\frac{a}{b}}^{\pi/2}\frac{1}{\sin^2\theta\cos^2\theta}\left.\frac{r^2}{2}\right]_0^{a\cos\theta}d\theta$$

$$=\frac{b^2}{2}\int_0^{\tan^{-1}\frac{a}{b}}\sec^2\theta\,d\theta+\frac{a^2}{2}\int_{\tan^{-1}\frac{a}{b}}^{\pi/2}\operatorname{cosec}^2\theta\,d\theta$$

$$\cancel{= \int_0^{2\pi} (1-\sin 2\theta)\, d\theta \times r\Big]_0^1}$$

$$\cancel{= \left. 0 + \frac{\cos 2\theta}{2} \right]_0^{2\pi} \times 1}$$

$$\cancel{\Rightarrow \boxed{I = 2\pi + \frac{1}{2}}}$$

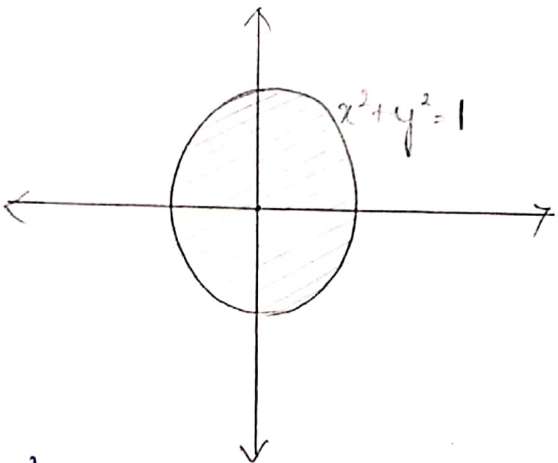
$$\left. \frac{b^2}{2} \tan\theta \right]_0^{\tan^{-1}\frac{a}{b}} - \frac{a^2}{2} \cot\theta \Big]_{\tan^{-1}\frac{a}{b}}^{\pi/2}$$

$$= \frac{b^2}{2} \cdot \frac{a}{b} + \frac{a^2}{2} \cdot \frac{b}{a}$$

$$= \frac{ab}{2} + \frac{ab}{2}$$

$$\boxed{I = ab}$$

6. $\iint \frac{(x-y)^2}{x^2+y^2}\, dx\, dy$ over the circle $x^2+y^2 \leq 1$.



$r^2 = 1$

$\Rightarrow r = 1$

here $\frac{(x-y)^2}{x^2+y^2} = \frac{x^2+y^2-2xy}{x^2+y^2} = 1 - \frac{2xy}{x^2+y^2}$

$$= 1 - \frac{2r^2 \sin\theta \cos\theta}{r^2}$$

$$= 1 - \sin 2\theta$$

here, $r: 0 \to 1$, $\theta: 0 \to 2\pi$

$$\therefore I = \int_0^{2\pi} \int_0^1 (1-\sin 2\theta)\, r \cdot dr\, d\theta$$

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$$\Rightarrow I = \int_0^{2\pi} (1 - \sin 2\theta)\, d\theta \cdot \int_0^1 r\, dr$$

$$= \left. \theta + \frac{\cos 2\theta}{2} \right]_0^{2\pi} \cdot \frac{1}{2}$$

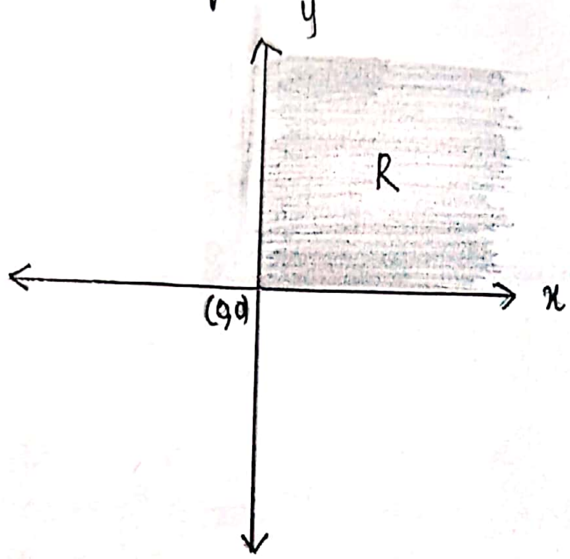
$$= \frac{2\pi}{2} \quad \Rightarrow \boxed{I = \pi}$$

CHANGE OF ORDER OF INTEGRATION

Evaluate by changing the order of integration:

1. $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy$ and show that $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$.

$\hookrightarrow$



By changing the order of integration:

$$I = \int_0^\infty \int_0^\infty e^{-xy} \sin px \,.\, dy \, dx$$

$$= -\int_0^\infty \frac{\sin px}{x} .\, e^{-xy}\Big]_0^\infty dx = -\int_0^\infty \frac{\sin px}{x} .\left[e^{-\infty} - e^{0}\right] dx$$

$$= \int_0^\infty \frac{\sin px}{x} dx \longrightarrow (1)$$

$\cancel{\int uv\,dx = u\int v\,dx - \int \frac{du}{dx}\left(\int v\,dx\right)dx}$

Keeping the given order: $I = \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy$

$$= \int_0^\infty \frac{1}{p^2+y^2} e^{-xy}\left(-y \sin px + b\cos px\right)\Big]_0^\infty = \int_0^\infty \frac{p}{p^2+y^2} dy$$

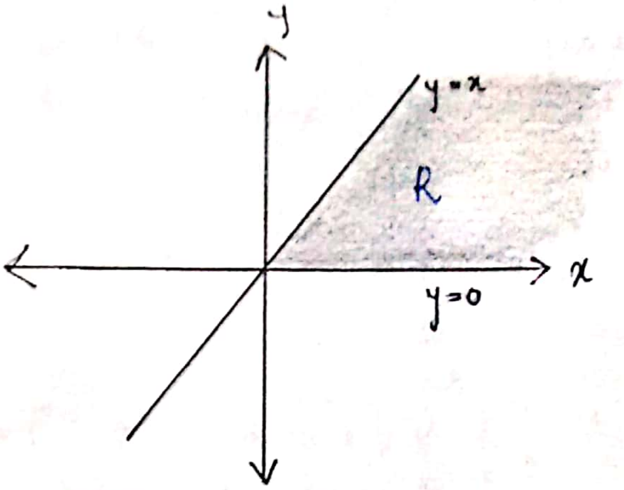
$\left[\int e^{ax} \sin bx = \frac{1}{a^2+b^2} e^{ax}\left[a \sin bx - b\cos bx\right]\right]$

$$= \int_0^\infty \frac{p}{p} \tan^{-1}\left(\frac{y}{p}\right)\Big]_0^\infty = \frac{\pi}{2} \longrightarrow (2)$$

$$\therefore \boxed{I = \int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}}$$

Hence proved.

3. $\int_0^\infty \int_0^x e^{-xy}\, y\, dy\, dx$

↳ 

Changing the order of integration:

$$\int_0^\infty \int_{x=y}^\infty e^{-xy}\, y\, dx\, dy = -1\int_0^\infty \frac{y}{y} e^{-xy}\Big]_y^\infty dy$$

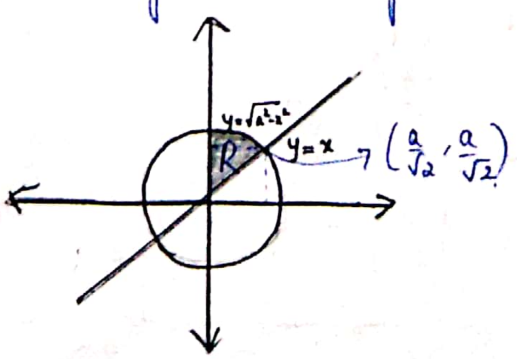
$$= -1\int_0^\infty 0 - e^{-y^2} dy = \int_0^\infty e^{-y^2} dy = \boxed{\frac{\sqrt{\pi}}{2}}$$

[For proof of $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, please refer notes]

---

4. $\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} y^2\, dy\, dx$

↳ inner limits : $y = x$ to $y^2 = \sqrt{a^2-x^2} \Rightarrow y^2 = a^2 - x^2$ & $x^2+y^2=a^2$



$y = x$

$x^2 + y^2 = a^2$

$2y^2 = a^2$

$y = \frac{a}{\sqrt{2}}, \frac{-a}{\sqrt{2}}$

new limits :

$x : 0 \rightarrow y \quad [y : 0 \rightarrow \frac{a}{\sqrt{2}}]$

$x : 0 \rightarrow \sqrt{a^2-y^2} \quad [y : \frac{a}{\sqrt{2}} \rightarrow a]$

$$\therefore \; I = \int_{y=0}^{\frac{a}{\sqrt{2}}} \int_{x=0}^{y} y^2 .\, dx\, dy + \int_{\frac{a}{\sqrt{2}}}^{a} \int_{0}^{\sqrt{a^2-y^2}} y^2 .\, dx\, dy$$

$$= \int_{0}^{\frac{a}{\sqrt{2}}} y^2 \, x\Big]_0^{y} dy + \int_{\frac{a}{\sqrt{2}}}^{a} y^2 \, x\Big]_0^{\sqrt{a^2-y^2}} dy$$

$$I = \underbrace{\int_{0}^{\frac{a}{\sqrt{2}}} y^3 \, dy}_{I_1} + \underbrace{\int_{\frac{a}{\sqrt{2}}}^{a} y^2 . \sqrt{a^2-y^2}\, dy}_{I_2}$$

$$\Rightarrow \; I = I_1 + I_2$$

$$I_1 = \int_0^{a/\sqrt{2}} y^3 \, dy = \frac{y^4}{4}\Big]_0^{a/\sqrt{2}}$$

$$= \frac{a^4}{4 \times 4} \Rightarrow \boxed{I_1 = \frac{a^4}{16}}$$

$$I_2 = \int_{\frac{a}{\sqrt{2}}}^{a} \underset{\mathrm{I}}{y^2} \; \underset{\mathrm{II}}{\sqrt{a^2-y^2}} \, dy$$

Let $y = a\sin\theta \Rightarrow dy = a\cos\theta . d\theta$

$\Rightarrow \sqrt{a^2-y^2} = a\cos\theta$

~~using integration by parts:~~

$$\cancel{I_2 = \left[\sqrt{a^2-y^2}\cdot\frac{y^3}{3}\right]_{\frac{a}{\sqrt{2}}}^{a} + \int_{\frac{a}{\sqrt{2}}}^{a}\frac{y}{\sqrt{a^2-y^2}}\cdot\frac{y^3}{3}\cdot dy}$$

$$\cancel{I_2 = y^2\Big]_{a/\sqrt{2}}^{a}\int\sqrt{a^2-y^2}\cdot dy - \int_{\frac{a}{\sqrt{2}}}^{a} 2y\cdot\int\sqrt{a^2-y^2}\cdot dy}$$

$\Rightarrow$ Also, $y = \frac{a}{\sqrt{2}} \Rightarrow a\sin\theta = \frac{a}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

$y = a \Rightarrow a\sin\theta = a$ and $\theta = \frac{\pi}{2}$

$$\therefore I_2 = \int_{\frac{\pi}{4}}^{\pi/2} a^2\sin^2\theta\cdot a\cdot\cos\theta\cdot a\cdot\cos\theta\, d\theta$$

$$= a^4\int_{\frac{\pi}{4}}^{\pi/2}\sin^2\theta\cdot\cos^2\theta\cdot d\theta = a^4\int_{\frac{\pi}{4}}^{\pi/2}(1-\cos^2\theta)\cos^2\theta\cdot d\theta$$

$$= a^4\int_{\frac{\pi}{4}}^{\pi/2}(\cos^2\theta - \cos^4\theta)\,d\theta$$

$$\Rightarrow I_2 = \frac{a^4}{2}\int_0^{\pi/2}\cos^2\theta\cdot d\theta - \frac{a^4}{2}\int_0^{\pi/2}\cos^4\theta\cdot d\theta$$

using reduction formula:

$$I_2 = \frac{a^4}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{a^4}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{a^4\pi}{8} - \frac{3a^4\pi}{32}$$

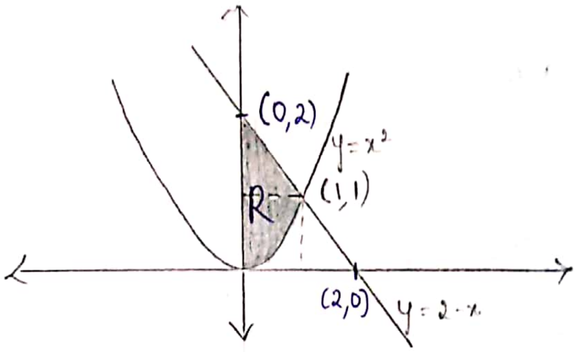
$$\Rightarrow \boxed{I_2 = \frac{a^4\pi}{32}}$$

$$\therefore I = I_1 + I_2 = \frac{a^4}{16} + \frac{a^4\pi}{32}$$

$$\Rightarrow \boxed{I = \frac{a^4(\pi+2)}{32}}$$

2. $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy.$

↳ The given integral is, $I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy \, dy \, dx.$



$y = x^2$, $y = 2-x$

$x^2 = 2-x$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x = -2, 1$

$y = 4, 1$

$y = x^2$
$\Rightarrow x = \sqrt{y}$

By changing the order of integration,

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_0^1 y \cdot \frac{x^2}{2}\Big]_0^{\sqrt{y}} dy + \int_1^2 y \cdot \frac{x^2}{2}\Big]_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 y \cdot \frac{(2-y)^2}{2} dy$$

$$= \frac{1}{2}\left[\int_0^1 y^2 \, dy + \int_1^2 (4y + y^3 - 4y^2) \, dy \right.$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\Big]_0^1 + 4 \cdot \frac{y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\Big]_1^2 \right]$$

$$= \frac{1}{2}\left[\frac{1}{3} + 8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3} \right]$$

$$= \frac{1}{2}\left[10 - \frac{1}{4} - 9\right] = \frac{1}{2}\left[1 - \frac{1}{4}\right] = \frac{1}{2} \cdot \frac{3}{4}$$

$$\Rightarrow \boxed{I = \frac{3}{8}}$$

6. $\displaystyle\int_0^a \int_0^{bx/a} x \, dy \, dx$

↳ given, $I = \displaystyle\int_0^a \int_0^{bx/a} x \cdot dy \, dx$

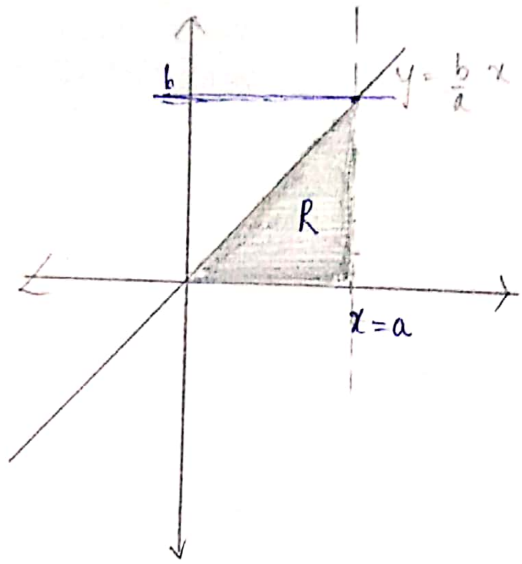
$y = 0 \to y = \dfrac{bx}{a}$

$\Rightarrow ay = bx$

$bx - ay = 0$

$y = 0,$

$\underline{y = \dfrac{b}{a} \cdot a = b}$



Changing the order of integration:

$$I = \int_{y=0}^{b} \int_{x=\frac{a}{b}y}^{x=a} x \, dx \, dy$$

$$= \int_0^b \left.\frac{x^2}{2}\right]_{\frac{ay}{b}}^{a} dy = \int_0^b \left(\frac{a^2}{2} - \frac{a^2}{b^2} \cdot \frac{y^2}{2} \right) dy$$

$$= \left.\frac{a^2}{2} \cdot y\right]_0^b - \left.\frac{a^2}{b^2} \cdot \frac{y^3}{6}\right]_0^b$$

$$= \frac{a^2 b}{2} - \frac{a^2}{b^2} \cdot \frac{b^3}{6} = \frac{a^2 b}{2} - \frac{a^2 b}{6}$$

$$\Rightarrow \boxed{I = \frac{2}{6} a^2 b} \Rightarrow \boxed{I = \frac{1}{3} a^2 b}$$

Scanned by CamScanner

Evaluate

1. $\int_0^2 \int_1^z \int_0^{yz} xyz \, dx \, dy \, dz$

↳ $I = \int_0^2 \int_1^z \int_0^{yz} xyz \, dx \, dy \, dz$

$= \int_0^2 \int_1^z yz \cdot \frac{x^2}{2} \Big]_0^{yz} dy \, dz$

$= \int_0^2 \int_1^z \frac{y^3 z^3}{2} \, dy \, dz$

$= \int_0^2 \frac{z^3}{2} \cdot \frac{y^4}{4} \Big]_1^z = \int_0^2 \frac{z^3}{2} \cdot \frac{z^4}{4} \, dz - \int_0^2 \frac{z^3}{8} \, dz$

$= \frac{1}{8} \cdot \frac{z^8}{8} \Big]_0^2 - \frac{z^4}{32} \Big]_0^2$

$= \frac{256}{64} - \frac{16}{32} = \frac{4-1}{2} = \frac{8-1}{2}$

$\Rightarrow \boxed{I = \frac{7}{2}}$

3. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} \, dx\, dy\, dz$

↳ Given, $I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} \, dz\, dy\, dx$

$\Rightarrow I = \int_0^1 \int_0^{1-x} \left. \frac{-2}{(x+y+z+1)^2} \right]_0^{1-x-y} dy\, dx$

$= \int_0^1 \left[\int_0^{1-x} \frac{-2}{4} \, dy - \int_0^{1-x} \frac{-2}{(1+x+y)^2} \, dy \right] dx$

$= \int_0^1 \left. -\frac{1}{2} y \right]_0^{1-x} + 2 \cdot \left. \left(\frac{-1}{1+x+y} \right) \right]_0^{1-x} dx$

$= \int_0^1 \left(\frac{x-1}{2} - 1 + \frac{2}{1+x} \right) dx$

$= \left. \frac{(x-1)^2}{4} \right]_0^1 - x \Big]_0^1 + 2 \log (1+x) \Big|_0^1$

$= -\frac{1}{4} - 1 + 2 \log 2$

$= -\frac{5}{4} + 2 \log 2 = \boxed{-2\left(\frac{5}{8} - \log 2 \right)}$

4. $\displaystyle\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

$$\hookrightarrow \quad I = \int_{z=0}^{4} \int_{x=0}^{2\sqrt{z}} \int_{y=0}^{\sqrt{4z-x^2}} dy\, dx\, dz$$

$$= \int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z-x^2}\; dx\, dz$$

$$\sqrt{4z-x^2} = \sqrt{(2\sqrt{z})^2 - x^2}$$

$$= \int_0^4 \left[\frac{x}{2}\sqrt{4z-x^2} + \frac{\overset{4z}{\cancel{2\sqrt{z}}}}{2} \sin^{-1}\left(\frac{x}{2\sqrt{z}}\right)\right]_0^{2\sqrt{z}} dz$$

$$= \int_0^4 \left[\frac{\cancel{2}\sqrt{z}}{\cancel{2}} \cdot \frac{\pi}{2} - \frac{1}{2}\sqrt{4z-1} - \frac{\cancel{2}\sqrt{z}}{\cancel{2}} \sin^{-1}\left(\frac{1}{2\sqrt{z}}\right)\right] dz$$

$$= \frac{\pi}{2}\int_0^4 \sqrt{z}\cdot dz - \frac{1}{2}\int_0^4 \sqrt{4z-1}\cdot dz - \int_0^4 \sin^{-1}\left(\frac{1}{2\sqrt{z}}\right)\cdot\sqrt{z}\cdot dz -$$

$$= \frac{\pi}{2}\,\frac{2}{3} z^{3/2}\Big]_0^4 - \frac{1}{2}\cdot\frac{2}{3}(4z-1)^{3/2}\Big]_0^4 -$$

$$= \int_0^4 2z\,\frac{\pi}{2}\, dz$$

$$= \pi\int_0^4 z\, dz = \pi\cdot\frac{z^2}{2}\Big]_0^4 = \pi\times\frac{16}{2}$$

$$\Rightarrow \boxed{I = 8\pi}$$

5. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$

↳ given $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$

$$\Rightarrow I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left.\frac{xy}{2} . z^2\right]_0^{\sqrt{a^2-x^2-y^2}} dy \, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{x}{2}\left[(a^2-x^2)y - y^3\right] dy \, dx$$

$$= \int_0^a \frac{x}{2}\left[(a^2-x^2) . \frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\frac{x}{2} \frac{(a^2-x^2)(a^2-x^2)}{2} - \frac{x}{2} \frac{(a^2-x^2)^4}{4}\right) dx$$

$$= \int_0^a \left(\frac{x(a^2-x^2)^2}{4} - \frac{x(a^2-x^2)^4}{8}\right) dx$$

$$= \int_0^a \frac{x(a^2-x^2)^2}{8} dx$$

$$= \frac{1}{8} \int_0^a (a^4 x + x^5 - 2a^2 x^3)\, dx$$

$$= \frac{1}{8} \left(a^4 \frac{x^2}{2} \Big]_0^a + \frac{x^6}{6} \Big]_0^a - 2a^2 \frac{x^4}{4} \Big]_0^a \right)$$

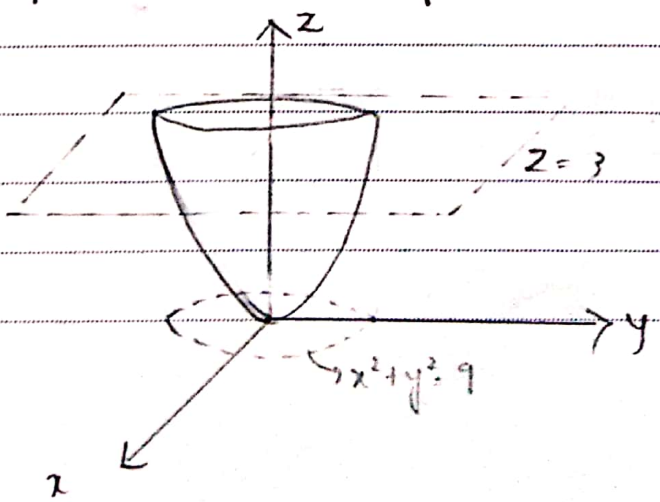
$$= \frac{1}{8} \left(\frac{a^6}{2} + \frac{a^6}{6} - \frac{a^6}{2} \right)$$

$$= \frac{1}{8} \cdot a^6$$

$$\Rightarrow \boxed{I = \frac{a^6}{48}}$$

6. $\iiint (x^2+y^2)\, dx\, dy\, dz$ over the region bounded by the paraboloid $x^2+y^2 = 3z$, and the plane $z = 3$.

↳



2. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \cdot dz\, dx\, dy$

↳ Given $I = \int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$

$$\Rightarrow I = \int_1^e \int_1^{\log y} \Big[z(\log z - 1) \Big]_1^{e^x} dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left[e^x(x-1) + 1 \right] dx\, dy$$

$$= \int_1^e \int_1^{\log y} (xe^x - e^x + 1)\, dx\, dy$$

$$= \int_1^e \Big[e^x(x-2) + x \Big]_1^{\log y} dy$$

$$= \int_1^e \left(y(\log y - 2) + \log y + e - 1 \right) dy$$

$$= \int_1^e \left(y \log y - 2y + \log y + e - 1 \right) dy$$

$$= \frac{y^2 \log y}{2} - \frac{y^2}{4} - y^2 + y(\log y - 1) + (e-1)y \Big]_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - e^2 + e(0) + (e-1)e - \left[-\frac{1}{4} - 1 - 1 + (e-1) \right]$$

$$= \frac{e^2}{4} + e^2 - e^2 - e + \frac{1}{4} + 2 - e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

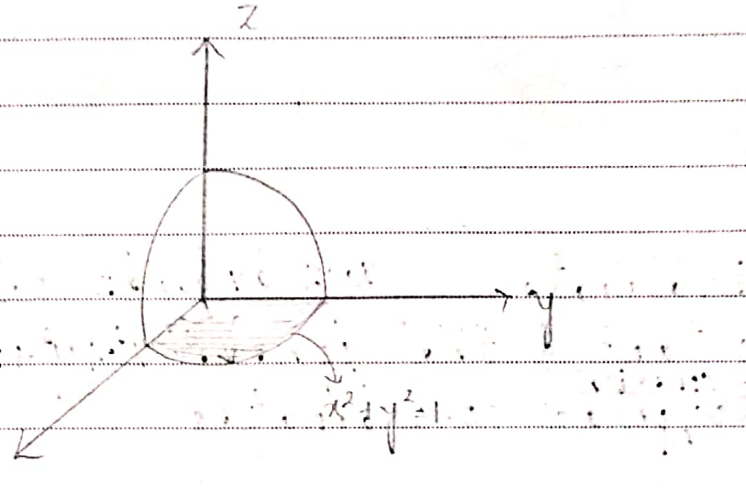
$$\Rightarrow \boxed{I = \frac{1}{4}\left(e^2 - 8e + 13\right)}$$

CHANGE OF VARIABLES IN TRIPLE INTEGRATION [SPHERICAL & CYLINDRICAL]

1. Use spherical co-ordinates to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dx\, dy\, dz$$

↳



here, $r : 0 \to 1$ $\qquad x^2+y^2+z^2=1$

$\theta : 0 \to \frac{\pi}{2}$ $\qquad \Rightarrow r^2 = 1$

$\phi : 0 \to \frac{\pi}{2}$ $\qquad \Rightarrow r = 1$

$$\therefore \; I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{1}{\sqrt{1-r^2}} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} d\phi \, . \int_0^{\pi/2} \sin\theta \, . \, d\theta \, . \int_0^1 \frac{r^2}{\sqrt{1-r^2}} \, dr$$

↓

Let $r = \sin\alpha$ $\qquad \sqrt{1-r^2} = \cos\alpha$

$\Rightarrow dr = \cos\alpha \, d\theta$

& $r^2 = \sin^2\alpha$

also, $r = 0, \Rightarrow \alpha = 0$

$r = 1 \Rightarrow \alpha = \pi/2$

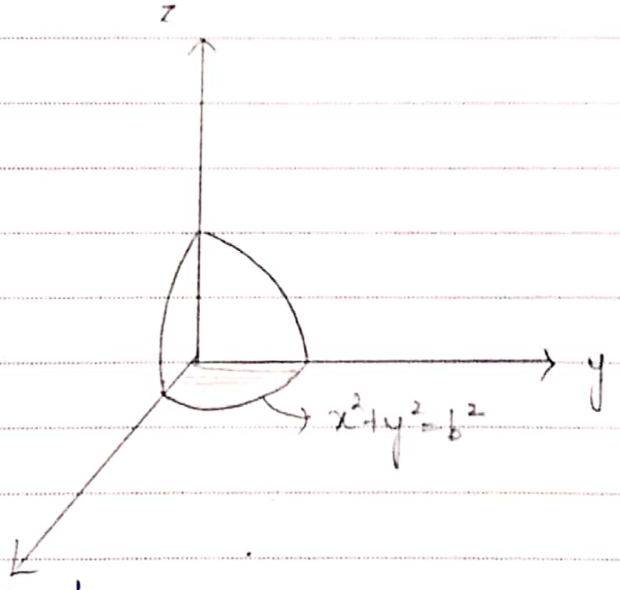
$$\therefore\ I = \frac{\pi}{2}\cdot\left(-\cos\theta\right)_0^{\pi/2}\cdot\int_0^{\pi/2}\frac{\sin^2\alpha\,\cos\alpha}{\cos\alpha}\,.d\alpha$$

$$= \frac{\pi}{2}\cdot 1\cdot\int_0^{\pi/2}\sin^2\alpha\,.d\alpha$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \Rightarrow \boxed{I = \frac{\pi^2}{8}}$$

2. Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=b^2$ by transforming to spherical polar co-ordinates

↳



$$I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{b} r\sin\theta\cos\phi\,.\,r\sin\theta\,.\sin\phi\,.\,r\,.\cos\theta\ r^2\sin\theta\ dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\,.\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\,.\int_0^{b} r^5\,dr$$

$\sin\theta = t$

$\cos\theta\,d\theta = dt$

$t : 0 \to 1$

$$= \frac{1}{2}\int_0^{\pi/2} \sin 2\phi \, d\phi \cdot \int_0^1 t^3 \, dt \cdot \int_0^b r^5 \, dr$$

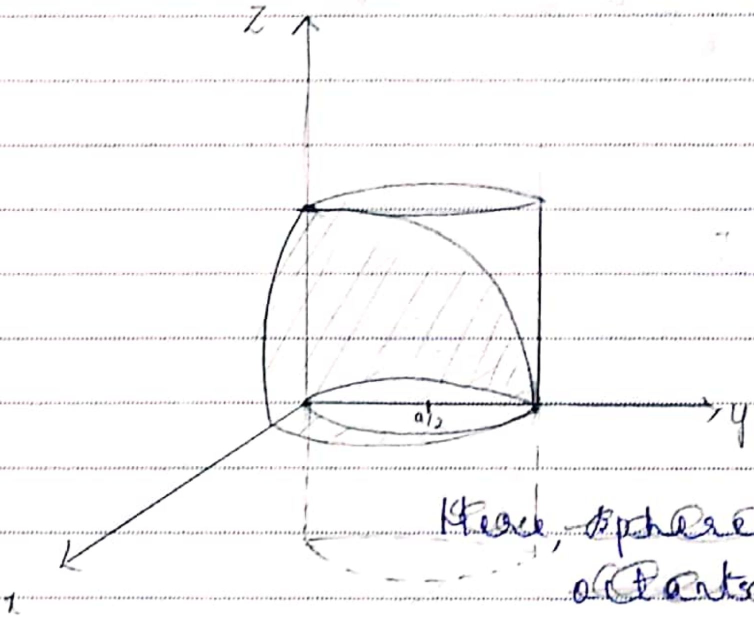
$$= \frac{1}{4}\left(-\cos 2\phi\right)_0^{\pi/2} \cdot \left.\frac{t^4}{4}\right]_0^1 \cdot \left.\frac{r^6}{6}\right]_0^b = \frac{1}{4}(2) \cdot \frac{1}{4} \cdot \frac{b^6}{6}$$

$$= \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{b^6}{6} = \frac{b^6}{48}$$

$$\Rightarrow \boxed{I = \frac{b^6}{48}}$$

3. Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.

↳



$x^2+y^2-ay=0$

$C=(0,\frac{a}{2})$

Hence, sphere lies in 2 octants

Switching to cylindrical co-ordinates:

~~$z: 0 \to a$~~ $\rho^2+z^2=a^2 \Rightarrow z=\sqrt{a^2-\rho^2}$ [sphere]

~~ρ:~~ also $\rho = a\sin\phi$ [paraboloid]

& $\phi: 0 \to \pi$

The required ~~area~~ volume is 2 times the shaded ~~area~~ part (i.e, required portion lies in 4 octants).

$$\therefore V = 2\int_{\phi=0}^{\pi}\int_{\rho=0}^{a\sin\phi}\int_{z=0}^{\sqrt{a^2-\rho^2}} \rho \, dz \, d\rho \, d\phi$$

$$= 2\int_0^{\pi}\int_0^{a\sin\phi} \rho\sqrt{a^2-\rho^2} \, d\rho \, d\phi$$

Let $a^2-\rho^2 = t$

$\Rightarrow -2\rho \, d\rho = dt$

$-\rho \, d\rho = \frac{dt}{2}$

also ~~$\rho = 0 \Rightarrow t =$~~

$$= 2\int_0^{\pi} \frac{1}{2}\left[-\frac{2}{3}(a^2-\rho^2)^{3/2}\right]_0^{a\sin\phi} d\phi$$

$$= -\frac{2}{3}\int_0^{\pi} (a^3\cos^3\phi - a^3) \, d\phi$$

$$= \cancel{\frac{2a^3}{3}\int_0^{\pi}(\cos^3\phi - 1)\,d\phi} = \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\phi)\,d\phi$$

$$= \cancel{\frac{2a^3}{3}\left[\right.} = \frac{2a^3 \cdot \pi}{3} - \frac{2a^3}{3}\left(\frac{2}{3}\cdot 1\right)$$

(using reduction formulae)

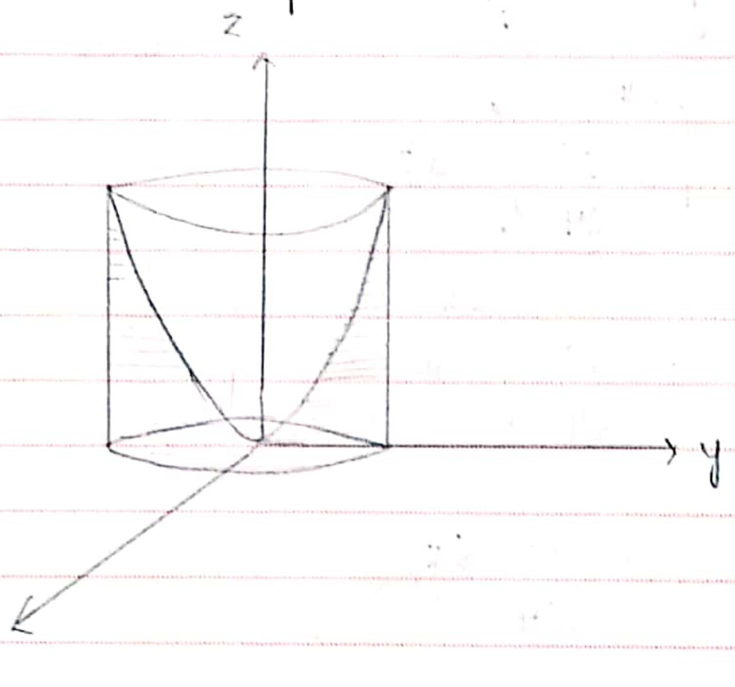
$$\Rightarrow \boxed{V = \frac{2a^3}{9}(3\pi - 4)}$$

4. Evaluate $\iiint \sqrt{x^2+y^2}\, dx\, dy\, dz$ over the volume bounded by the right circular cone $x^2+y^2=z^2$ $z>0$ and the planes $z=0$ & $z=1$.

↳ Please refer class notes.

5. Evaluate $\iiint z^2\, dx\, dy\, dz$ over the volume bounded by the cylinder $x^2+y^2=a^2$ & the paraboloid $x^2+y^2=z$ and the plane $z=0$.

↳



Switching to cylindrical co-ordinates,

$x^2+y^2=a^2 \Rightarrow \rho^2=a^2 \Rightarrow \rho: 0 \rightarrow a$

$x^2+y^2=z \Rightarrow z=\rho^2 \Rightarrow z: 0 \rightarrow \rho^2$

$\phi: 0 \rightarrow 2\pi.$

$\therefore$ The given integral becomes:

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\int_{z=0}^{\rho^2} z^2 \cdot \rho \, dz \, d\rho \, d\phi$$

$$= \int_0^{2\pi}\int_0^{a} \frac{\rho}{3} \cdot z^3 \Big]_0^{\rho^2} d\rho \, d\phi$$

$$= \int_0^{2\pi}\int_0^{a} \frac{\rho^7}{3} d\rho \, d\phi$$

$$= \int_0^{2\pi} \frac{\rho^8}{24} \Big]_0^{a} d\phi$$

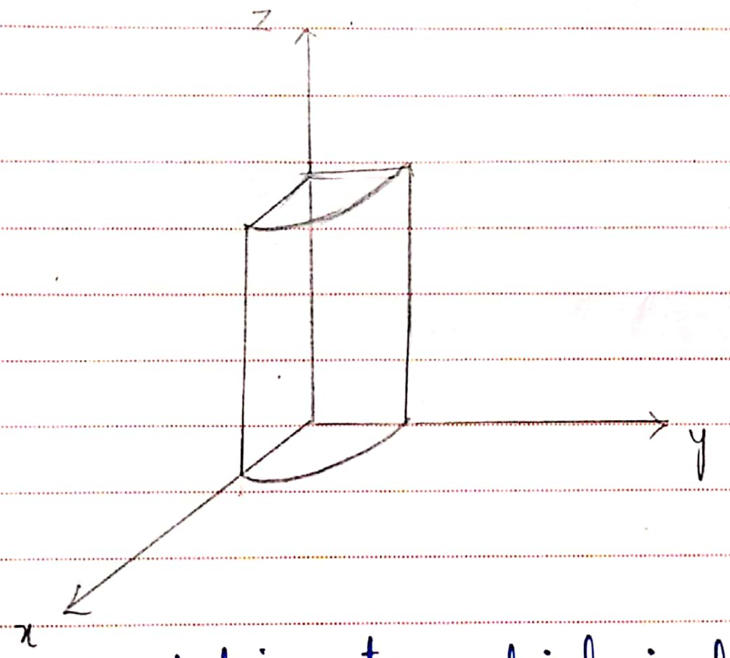
$$= \frac{a^8}{24}\int_0^{2\pi} d\phi$$

$$= \frac{a^8}{24}(2\pi)$$

$$\Rightarrow \boxed{I = \frac{\pi a^8}{12}}$$

6. Evaluate $\iiint xyz\, dx\, dy\, dz$ over the region bounded by the planes $x=0$, $y=0$, $z=0$ and $z=1$ and the cylinder $z=1$.

↳



Switching to cylindrical co-ordinates:

$z : 0 \to 1$

$\rho : 0 \to 1$

$\phi : 0 \to \frac{\pi}{2}$

$$\left[xyz = \rho\cos\phi \cdot \rho\sin\phi \cdot z = \rho^2 \frac{\sin 2\phi}{2} \cdot z \ \ \& \ \ dx\,dy\,dz = \rho\, d\rho\, d\phi\, dz \right]$$

$$\Rightarrow I = \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \int_{z=0}^{1} \rho^3 \frac{\sin 2\phi}{2}\, z\, dz\, d\rho\, d\phi$$

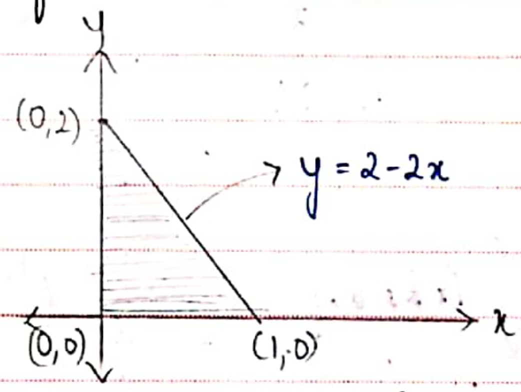
$$= \int_0^{\pi/2} \frac{\sin 2\phi}{2} \cdot \int_0^1 \rho^3 d\rho \cdot \int_0^1 z\, dz$$

$$= -\frac{\cos 2\phi}{2}\Bigg]_0^{\pi/2} \cdot \frac{\rho^4}{4} \cdot \frac{z^2}{2} = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \Rightarrow \boxed{I = \frac{1}{16}}$$

APPLICATION OF MULTIPLE INTEGRALS – CENTRE OF MASS, MOMENT OF INERTIA, AREA AND VOLUME

1. Find the mass and center of mass of a triangular lamina with vertices (0,0), (1,0), (0,2) if the density function is $1+3x+y$.

↳ Mass (M) $= \iint \rho \, dx \, dy$



given $\rho = 1 + 3x + y$.

also, here $y: 0 \to 2-2x$

& $x: 0 \to 1$

$$\therefore M = \int_{x=0}^{1} \int_{y=0}^{2-2x} (1+3x+y) \, dy \, dx$$

$$= \int_0^1 \left(y + 3xy + \frac{y^2}{2} \right)_0^{(2-2x)} dx$$

$$= \int_0^1 \left(2 - 2x + 6x - 6x^2 + \frac{4 + 4x^2 - 8x}{2} \right) dx$$

$$= \int_0^1 \left(2 - 2x + 6x - 6x^2 + 2 + 2x^2 - 4x \right) dx$$

$$= \int_0^1 (4-4x^2)\,dx = 4x - \frac{4x^3}{3}\Big]_0^1$$

$$= 4 - \frac{4}{3}$$

$$\Rightarrow \boxed{M = \frac{8}{3}}$$

$$\bar{x} = \frac{\iint \rho x \, dx\, dy}{\text{Mass}} = \frac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + yx)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left(xy + 3x^2y + \frac{xy^2}{2}\right)\Big]_0^{2-2x} dx$$

$$= \frac{3}{8}\int_0^1 (4-4x)x\,dx = \frac{3}{8}\left[\frac{4x^2}{2} - \frac{4x^4}{4}\right]_0^1$$

$$= \frac{3}{8}(2-1) \Rightarrow \boxed{\bar{x} = 3/8}$$

$$\bar{y} = \frac{\iint \rho y \, dx\, dy}{\text{Mass}} = \frac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left(2 + 2x^2 - 4x + 6x + 6x^3 - 12x^2 + \frac{8}{3} - \frac{8}{3}x^3 - 8x + 8x^2\right) dx$$

$$= \frac{3}{8}\left(2x + \frac{2x^3}{3} - \frac{4x^2}{2} + \frac{6x^2}{2} + \frac{6x^4}{4} - \frac{12x^3}{3} + \frac{8x}{3} - \frac{8x^4}{12} - \frac{8x^2}{2} + \frac{8x^3}{3}\right)_0^1$$

$$= \frac{3}{8}\left(\frac{11}{6}\right) \Rightarrow \boxed{\bar{y} = \frac{11}{16}}$$

$\therefore$ Center of Mass $= \boxed{\left(\frac{3}{8}, \frac{11}{16}\right)}$

6. Find the volume of the solid surrounded by the surface

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

↳ Let $\left(\frac{x}{a}\right)^{1/3} = X \Rightarrow x = X^3 a^{\cancel{3}}$

$\left(\frac{y}{b}\right)^{1/3} = Y \Rightarrow y = b^{\cancel{3}} Y^3$

& $\left(\frac{z}{c}\right)^{1/3} = Z \Rightarrow z = cZ^3$

$$J = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \begin{vmatrix} 3X^2 a & 0 & 0 \\ 0 & 3Y^2 b & 0 \\ 0 & 0 & 3Z^2 c \end{vmatrix}$$

$$= 27abc \cdot X^2 Y^2 Z^2$$

$$\therefore \iiint dx\,dy\,dz = 27abc \iiint X^2 Y^2 Z^2 \, dX\,dY\,dZ$$

and the given solid transforms to the sphere

$$X^2 + Y^2 + Z^2 = 1$$

Converting into spherical polar coordinates, $r^2 = 1 \Rightarrow r =$

Considering the positive octant,

$r : 0 \to 1$

$\theta : 0 \to \pi/2 \qquad \phi : 0 \to \pi/2$

$\therefore$ required volume is:

$$27abc \times 8 \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin^2\theta \cos^2\phi \cdot r^2 \sin^2\theta \cdot \sin^2\phi \cdot r^2 \cos^2\theta \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 216abc \int_{\phi=0}^{\pi/2} \sin^2\phi \cos^2\phi \, d\phi \cdot \int_{\theta=0}^{\pi/2} \sin^5\theta \cdot \cos^2\theta \, d\theta \cdot \int_{r=0}^{1} r^8 \, dr$$

$$= \frac{216abc}{\cancel{4}} \int_0^{\pi/2} \sin^2\phi(1-\sin^2\phi)\, d\phi \cdot \int_0^{\pi/2} (\sin^5\theta - \sin^7\theta)\, d\theta \cdot \int_0^1 r^8\, dr$$

$$= \frac{216abc}{\cancel{4}} \int_0^{\pi/2} (\sin^2\phi - \sin^4\phi)\, d\phi \cdot \int_0^{\pi/2} (\sin^5\theta - \sin^7\theta)\, d\theta \cdot \int_0^1 r^8\, dr$$

$3 \cdot 5 \cdot 7 = 105$

$$= 216abc \left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) \cdot \left(\frac{4}{5}\cdot\frac{2}{3}\cdot 1 - \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right) \cdot \left[\frac{r^9}{9}\right]_0^1$$

$\frac{\pi}{4} - \frac{3\pi}{16} = \frac{\pi}{16}$

$$= 216abc \cdot \frac{\pi}{16} \cdot \frac{8}{105} \cdot \frac{1}{9}$$

$\frac{8}{15} - \frac{6}{7}\cdot\frac{8}{15}$

$\frac{8}{15}\cdot\frac{1}{7} = \frac{8}{105}$

$$= \frac{12\pi abc}{105} = \frac{4\pi abc}{35}$$

$$\therefore \boxed{V = \frac{4\pi abc}{35}}$$

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JACOBIAN

1. If $u = e^x \sin y$, $v = x \log(\sin y)$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

↳ Here, $\frac{\partial u}{\partial x} = e^x \sin y$ $\qquad \frac{\partial u}{\partial y} = e^x \cos y$

$\frac{\partial v}{\partial x} = \log(\sin y)$ $\qquad \frac{\partial v}{\partial y} = x \cot y$

$$\therefore J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ \log(\sin y) & x \cot y \end{vmatrix}$$

$$= e^x \left[x \cos y - \cos y \log(\sin y) \right]$$

$$\Rightarrow \boxed{J = e^x \cos y \left[x - \log(\sin y) \right]}$$

PROPERTIES OF JACOBIAN

1. If $x = u$, $y = u \tan v$, $z = w$, prove that $JJ' = 1$.

↳ $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v$

$$\Rightarrow \boxed{J = u \sec^2 v} \text{ --- (1)}$$

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now, $u = x \qquad w = z$

$\Rightarrow y = x \tan v$

$v = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow J' = \frac{x}{x^2+y^2} = \frac{u}{u^2+u^2\tan^2 v}$$

$$= \frac{u}{u^2(1+\tan^2 v)} \Rightarrow \boxed{J' = \frac{1}{u\sec^2 v}} \longrightarrow ②$$

From ① & ② :

$$JJ' = u\sec^2 v \cdot \frac{1}{u\sec^2 v} = 1$$

$\therefore \boxed{JJ' = 1}$ Hence Proved.

Q. If $u = x+y-z$, $v = x-y+z$ and $w = x^2+y^2+z^2-2yz$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Are u, v, w functionally related? If so, find this relationship.

↳ $u = x+y-z \qquad w = x^2+y^2+z^2$

$v = x-y+z$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ x & y-z & z-y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ x & y-z & x \end{vmatrix} = \underline{\underline{0}}$$

i.e, $\boxed{J=0}$

Hence, u, v, w are functionally dependent.

now, $u^2 = (x+y-z)^2 = x^2+y^2+z^2+2xy-2yz-2zx$

+

& $v^2 = (x-y+z)^2 = x^2+y^2+z^2-2xy-2yz+2zx$

$u^2+v^2 \quad = 2(x^2+y^2+z^2)$

$\therefore \boxed{u^2+v^2 = 2w}$ → This is the required relation.

P.T.O→

3. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$.

$\hookrightarrow$ $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$

$$= \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= 4xy \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \times r$$

$$= 4xy \times 3 \times r$$

$$= 4r\cos\theta \cdot r\sin\theta \cdot 3 \times r$$

$$= 12r^3 \sin\theta \cos\theta$$

$$\Rightarrow \boxed{J = 6r^3 \sin 2\theta}$$ Hence proved.

4. If $u^3 + v + w = x + y^2 + z^2$, $u + v^3 + w = x^2 + y + z^2$, $u + v + w^3 = x^2 + y^2 + z$, then calculate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

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DOUBLE INTEGRALS

1. Evaluate $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx$.

↳ $I = \int_1^2 \left[\frac{xy^2}{2}+e^y\right]_3^4 dx = \int_1^2\left(8x-\frac{9}{2}x+e^4-e^3\right)dx$

$= \left(8-\frac{9}{2}\right)\cdot\left.\frac{x^2}{2}\right]_1^2 + \left(e^4-e^3\right)x\Big]_1^2$

$= \left(8-\frac{9}{2}\right)\left(\frac{3}{2}\right)+e^4-e^3$

$\Rightarrow \boxed{I=\frac{21}{4}+e^4-e^3}$

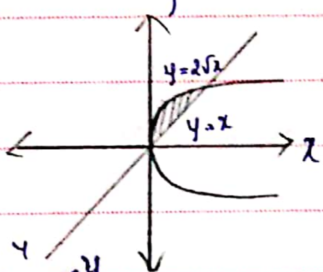
2. Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is bounded by $y=x$ and $y^2=4x$.

$2\sqrt{x}=x$

$\Rightarrow \sqrt{x}=2$

$x=4$

↳



here, $x: \frac{y^2}{4}\to y$

$y: 0\to 4$

$\therefore\ I = \int_{y=0}^{4}\int_{x=\frac{y^2}{4}}^{y}(x^2+y^2)\,dx\,dy = \int_0^4 \left[\frac{x^3}{3}+y^2x\right]_{y^2/4}^{y} dy$

$= \int_0^4\left(\frac{y^3}{3}+y^3-\frac{y^6}{64\times3}-\frac{y^4}{4}\right)dy$

$= \frac{4}{3}\frac{y^4}{4}\Big]_0^4 - \frac{y^7}{7\times64\times3}\Big]_0^4 - \frac{y^5}{4\times5}\Big]_0^4 = \frac{4^4}{3}-\frac{4^4}{7\times3}-\frac{4^4}{5}$

$=4^4\left(\frac{1}{3}-\frac{1}{21}-\frac{1}{5}\right) = \underline{64\times4}$

$=4^4\left(\frac{35-5-21}{105}\right)=\frac{4^4\times3}{\cancel{105}\,35} \Rightarrow \boxed{I=\frac{768}{35}}$