Harmonic Analysis

If the function is not defined explicitly as a function of an independent variable but defined (in terms of a graph) or by a table of corresponding values then the process of finding Fourier series for such a data is known as Harmonic analysis. Here we can't use Euler's formulas to find a_0, a_n, b_n & hence we use the following method.

$a_1 \cos x + b_1 \sin x$ is I harmonic, $a_2 \cos 2x + b_2 \sin 2x$ - II harmonic.

* The mean value of a function $y = f(x)$ over the range (a, b) is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$

If a set of "n" values for a function $y = f(x)$ having 2π as period at equidistant points of x is given in the interval $(c, c+2\pi)$ [$c \le x < c+2\pi$ or $c < x \le c+2\pi$] then the Fourier co-effs a_0, a_n, b_n using above property are as follows:

W.K.T $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\,dx\right]$

$= 2\left[\text{Mean value of } y = f(x) \text{ in } (c, c+2\pi)\right]$

$= 2\left[\frac{\sum y}{n}\right] \Rightarrow \boxed{\frac{2}{n}\sum y = a_0}$

$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\cos nx\,dx\right]$

$= 2\left[\text{Mean value of } y\cos nx \text{ in } (c, c+2\pi)\right]$

$$\boxed{a_n = \frac{2}{n}\sum y\cos nx}$$

$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx = 2\left[\text{Mean value of } y\sin nx \text{ in } (c, c+2\pi)\right]$

$$\boxed{b_n = \frac{2}{n}\sum y\sin nx}$$

HA is the process of finding the harmonics in the F S of a fn $f(x)$ which is given by a table of its values.

note: If the period is not 2π i.e if it is $2l$ then

$$a_n = \frac{2}{n}\sum y\cos\left(\frac{n\pi x}{l}\right) \; ; \; b_n = \frac{2}{n}\sum y\sin\left(\frac{n\pi x}{l}\right)$$

Let $\theta = \frac{\pi x}{l}$ s.t

$$a_n = \frac{2}{n}\sum y\cos n\theta \quad \& \quad b_n = \frac{2}{n}\sum y\sin\theta$$

① Obtain con term & co-eff of I sine & cosine terms in the F. series of $f(x)$, given for the following data.

i.e. find fourier sine & cosine if asked.

x:	0	1	2	3	4	5
$f(x)$:	9	18	24	28	26	20.

Sol:

Here the interval of x is $0 \le x < 6$ i.e $[0,6)$ & $n = 6$

$\therefore$ Length of interval is $6 - 0 = 6$ $\therefore 2l = 6 \Rightarrow l = 3$

$0 \le x < 5$ not correct $\because 5 - 0 = 5$

$\therefore$ Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}.$$

$\therefore$ Required I harmonics are

$$y = f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3}.$$

Let $\theta = \frac{\pi x}{3}$ $\therefore$ $\boxed{y = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta}$

The values of 0, 1, 2, 3, 4, 5. are given $\boxed{n=6}$ $\therefore$ interval of x should be $0 \le x < 6$.

note: If the values of y at $x = c$ & $x = c + 2\pi$ are given we must omit one of them,

(3)

x	$\theta = \frac{\pi x}{3}$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1.0	9	0	0
1	$\pi/3$	18	0.5	9	0.87	15.66
2	$2\pi/3$	24	-0.5	-12	0.87	20.88
3	$3\pi/3$	28	-1.0	-28	0	0
4	$4\pi/3$	26	-0.5	-13	-0.87	-22.62
5	$5\pi/3$	20	0.5	10	-0.87	-17.4
		$\Sigma y = 125$		$\Sigma y\cos\theta = -25$		-3.468

$$\therefore a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{n}\Sigma y\cos\theta = \frac{2}{6}(-25) = -8.33$$

$$b_1 = \frac{2}{n}\Sigma y\sin\theta = \frac{2}{6}(-3.468) = -1.16$$

$\therefore$ Reqd F S is $y = f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$

$$= \frac{41.67}{2} + (-8.33)\cos\frac{\pi x}{3} + (-1.16)\sin\frac{\pi x}{3}$$

(2)

917795188961
26669039

(2)

3

Here $A = A(t)$ is defined over time interval $[0, T]$

$\therefore \boxed{2l = T} \Rightarrow l = T/2$. $n = 6$

Here $\theta = \frac{\pi x}{l} = \frac{2\pi x}{T}$ or $\frac{2\pi t}{T}$

$\therefore a_0 = \frac{2}{n} \Sigma A$, $a_1 = \frac{2}{n} \Sigma A \cos\left(\frac{2\pi}{T}\right)t$

2

$b_1 = \frac{2}{n} \Sigma A \sin\left(\frac{2\pi}{T}\right)t$.

A	t	$\left(\frac{2\pi}{T}\right)t = \theta$	$\cos\theta$ $\cos\left(\frac{2\pi}{T}\right)t$	$\sin\theta$ $\sin\left(\frac{2\pi}{T}\right)t$	$A\cos\left(\frac{2\pi}{T}\right)t$	$A\sin\left(\frac{2\pi}{T}\right)t$
1.98	0	0	1	0	1.98	0
1.30	$T/6$	$\pi/3$	$1/2$	$\sqrt{3}/2$	0.65	1.1258
1.05	$T/3$	$2\pi/3$	$-1/2$	$\sqrt{3}/2$	−0.525	0.9093
1.30	$T/2$	π	−1	0	−1.30	0
−0.88	$2T/3$	$4\pi/3$	$-1/2$	$-\sqrt{3}/2$	0.44	0.7621
−0.25	$5T/6$	$5\pi/3$	$1/2$	$-\sqrt{3}/2$	−0.125	0.2165
$\Sigma A = 4.5$					1.12	3.0137.

$a_0 = \frac{2}{n}\Sigma A = a_0 = \left[\frac{2(4.5)}{6}\right] = 1.5$

$a_1 = \frac{2}{n}\Sigma A \cos\left(\frac{2\pi}{T}\right)t = \frac{2}{6}(1.12) = 0.3733$

$b_1 = \frac{2}{6}(3.0137) = 1.00457 = 1.0046$

$\therefore$ Reqd Fourier expansion is

$y = 0.75 + 0.3733 \cos\left(\frac{2\pi}{T}\right)t + 1.0046 \sin\left(\frac{2\pi}{T}\right)t$

$\therefore$ Direct current part $= \frac{a_0}{2} = \frac{1.5}{2} = 0.75$, amplitude of I harmonic $= \sqrt{a_1^2 + b_1^2} = 1.072$

) Find I two harmonics for the function $f(\theta)$ given by the following table.

θ°	0	60	120	180	240	300	360
$f(\theta)$	0.8	0.6	0.4	0.7	0.9	1.1	0.8

$a_0 = 1.5 \qquad a_1 = 0.1, \; a_2 = 0$

$b_1 = -0.29, \quad b_2 = 0$

$\therefore \; f(\theta) = 0.75 + (0.1)\cos\theta - 0.29\sin\theta$

5) Find the constant term and the first three co-efficients in the Fourier Cosine series for the function $f(x)$ given by following table.

x	0	1	2	3	4	5
$f(x)$	4	8	15	7	6	2

$a_0 = 14, \quad a_1 = -2.8, \; a_2 = 1.5, \; a_3 = 2.7$

6) The following table gives displacement u (in mms) of a sliding piece from a fixed reference point for every 30° of rotation of the crank.

θ°	0	30	60	90	120	150	180	210	240	270	300	33
u	298	356	373	337	254	155	80	51	60	93	147	22

$u = 202 + (107\cos\theta + 121\sin\theta) + (-13\cos 2\theta + 9\sin 2\theta).$

7) The fo table gives turning moment T for va values of crank angle θ. Find half range Fourier cosine series for T upto II har

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

$T = 4882.6 + 1186.4\cos\theta - 3656.7\cos 2\theta.$

Here the interval is $0 \le \theta < 2\pi$. $\therefore$ period of T is 2π

$\therefore a_0 = \frac{2}{n}\Sigma T,\ a_1 = \frac{2}{N}\Sigma T\cos\theta,\ b_1 = \frac{2}{N}\Sigma T\sin\theta,\ n = 12$

θ°	T	$\cos\theta$	$T\cos\theta$	$\sin\theta$	$T\sin\theta$
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	-0.5	-4.05	0.866	7.0146
150	8.3	-.866	-7.1878	0.5	4.15
180	7.9	-1	-7.9	0	0
210	6.8	-0.866	-5.8888	-0.5	-3.4
240	5.5	-0.5	-2.75	-.866	-4.763
270	4.1	0	0	-1	-4.1
300	2.6	0.5	1.3	-.866	-2.2516
330	1.2	0.866	1.0392	-0.5	-0.6
	59.4		~~-20.46~~ -20.4992		8.9032

$a_0 = \frac{1}{6}\Sigma T = 9.9$

$a_1 = \frac{1}{6}\Sigma T\cos\theta = -3.4165$

$b_1 = \frac{1}{6}\Sigma T\sin\theta = 1.4839$

$\therefore T = \frac{9.9}{2} - 3.4165\cos\theta$

$+ 1.4839\sin\theta$

1) x° 0 45 90 135 180 225 270 315
y 3 3/2 2 5/2 0 3/2 4 3/2

Here interval is 0° to 360° ∴ $\boxed{0 \le x < 360}$ or $0 \le x < 2\pi$

Here $n = 8$ $a_0 = \frac{2}{n}\Sigma y$ $a_1 = \frac{2}{n}\Sigma y\cos x$, $b_1 = \frac{2}{n}\Sigma y\sin x$

2) x° 45 90 135 180 225 270 315 360
y 2.4 4.0 3.8 -1.5 2.0 2.8 0 4.3

Here interval is $\boxed{0 < x \le 2\pi}$ & period of $y = f(x)$ ie 2π

& $n = 8$ $\Sigma y = 17.8$ $\Sigma y\cos\theta = 3.396$ $\Sigma y\sin\theta = 4.156$
$a_0 = 4.45$, $a_1 = 0.85$, $b_1 = 1.03$

3) x : 0 60° 120° 180° 240° 300° 360°
y : 1.98 1.30 1.05 1.30 -0.88 -.25 1.98

Here interval is $\boxed{0 \le x \le 2\pi}$. the values of y at $x = 0$ & $x = 2\pi$ ∴ are same by property of periodic fn. i.e $f(x) = f(x+2\pi)$.

Here we have to consider only one of these two values ~~I~~ omit I or last value ∴ $n = 6$

4) obtain constant term & I harmonic

x : 0 1 2 3 4 5 , $0 \le x < 6$ ∴ $n = 6$

y : 8 10 11 7 9 12 ∴ $2l = 6 \Rightarrow \boxed{l = 3}$

Both cosine & sine term is reqd, so full range.

5) Obtain const term & first two co-eff in Fourier Cosine series for
No half range problems in Harmonic Analysis.

x : 0 1 2 3 4 5
y : 8 4 3 1 5 2
$0 \le x < 6$ & $n = 6$

Comparing with $(0, l)$, $\boxed{l = 6}$

Here it is half range series ∴ only Cosine series is reqd.

Find the direct current part & amplitude of the first harmonic from the following table consisting of the variations of periodic current.

t sec:	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp):	1.98	1.30	1.05	1.30	-.88	-0.25	1.98

Sol: Length of the interval is $[0, T)$. $b-a=T$

$\therefore 2l = T$ & $l = T/2$.

L: Since last value of y is repeated, only the first six values will be used.

$\therefore f(x) = A = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} + \cdots$

Reqd table:

A	t	$\theta = \frac{2\pi t}{T}$	$\cos\frac{2\pi t}{T}$	$A\cos\frac{2\pi t}{T}$ or $y\cos\frac{2\pi t}{T}$	$\sin\frac{2\pi t}{T}$	$A\sin\frac{2\pi t}{T}$ or $y\sin\frac{2\pi t}{T}$
1.98	0	0	1.0	1.980	0.000	0.000
1.30	$\frac{T}{6}$	$\frac{\pi}{3}$	0.5	0.650	0.866	1.126
1.05	$\frac{T}{3}$	$\frac{2\pi}{3}$	-0.5	-0.525	0.866	0.909
1.30	$\frac{T}{2}$	π	-1.0	-1.300	0.000	0.000
-.88	$\frac{2T}{3}$	$\frac{4\pi}{3}$	-0.5	0.440	-.866	0.762
-.25	$\frac{5T}{6}$	$\frac{5\pi}{3}$	0.5	-.125	-0.866	0.217
1.98	T					
4.5				1.12		3.014

$a_0 = \frac{2}{n}\sum A = \frac{2}{6}(4.5) = 1.5$

$a_1 = \frac{2}{n}\sum A\cos\frac{2\pi t}{T} = \frac{2}{6}\sum(1.12) = 0.373.$

$b_1 = \frac{2}{n}\sum A\sin\frac{2\pi t}{T}$

$b_1 = \frac{2}{6}(3.014)$
$= 1.005$

The direct current part in the variable current:-

$= \frac{a_0}{2} = 0.75$

Amplitude of I harmonic $= \sqrt{a_1^2 + b_1^2} =$

$= \sqrt{(0.373)^2 + (1.005)^2}$

$= 1.072$

2) compute a_0, a_1, a_2, a_3 in the Fourier ~~cosine~~ series for y which is tabulated below.

x : 0 1 2 3 4 5

y : 4 8 15 7 6 2

Sol: Here $N = 6$ (no of observations) + Interval of x is $0 \le x < 6$ i.e $(0,6)$. $\therefore$ comparing with $2l$, we get $2l = 6 \Rightarrow \boxed{l = 3}$

~~Fourier series in the interval $(0,6)$ is~~

$y = \frac{a_0}{2} + a_1 \cos\frac{2\pi x}{3} + a_2 \cos 2\left(\frac{2\pi x}{3}\right) + a_3 \cos 3\left(\frac{2\pi x}{3}\right)$

Let $\theta = \frac{2\pi x}{3}$ $\quad a_n = \frac{2}{N}\Sigma y \cos n\theta$ where $\theta = \frac{\pi x}{l} = \frac{\pi x}{3}$

x	y	$\theta = \frac{\pi x}{3}$	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	4	0	1	1	1	4	4	4
1	8	60°	1/2	-1/2	-1	4	-4	-8
2	15	120°	-1/2	-1/2	1	-7.5	-7.5	15
3	7	180°	-1	1	-1	-7	7	-7
4	6	240°	-1/2	-1/2	1	-3	-3	6
5	2	300°	1/2	-1/2	-1	1	-1	-2
Total:	42					-8.5	-4.5	8

$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(42) = 14$

$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(8.5) = -2.8$

$a_2 = \frac{2}{N}\Sigma y\cos 2\theta = -1.5$

$a_3 = \frac{2}{N}\Sigma y\cos 3\theta = 2.7$.