Extra problems on Complex Form of Fourier series

7/4/2020.

1) Find the complex form of the Fourier series of $f(x) = e^{x}$ in $0<x<2$, $f(x+2)=f(x)$.

Solution: Given $f(x) = e^{x}$ in $0<x<2$

Here $2l = 2 \Rightarrow l=1$. $\therefore$ Complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad -(1)$$

where $C_n = \frac{1}{2l}\int_0^{2l} f(x) e^{\frac{-in\pi x}{l}} dx$

$$= \frac{1}{2}\int_0^2 f(x) e^{-in\pi x} dx$$

$$= \frac{1}{2}\int_0^2 e^{x} e^{-in\pi x} dx$$

$$= \frac{1}{2}\int_0^2 e^{(1-in\pi)x} dx$$

$$= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-in\pi}\right]_0^2$$

$$= \frac{1}{2(1-in\pi)}\left[e^{(1-in\pi)2} - e^{0}\right]$$

$$= \frac{1}{2(1-in\pi)}\left[e^{2} \cdot e^{-i2n\pi} - 1\right]$$

$\cos 2n\pi = 1$

$\sin 2n\pi = 0$

$$= \frac{1+in\pi}{2(1+n^2\pi^2)}\left[e^{2}(\cos 2n\pi - i\sin 2n\pi) - 1\right]$$

$\therefore C_n = \frac{1+in\pi}{2(1+n^2\pi^2)}(e^{2}-1)$

Subst in (1)

$$\boxed{\therefore e^{x} = \left(\frac{e^{2}-1}{2}\right)\sum_{n=-\infty}^{\infty} \frac{(1+in\pi)}{1+n^2\pi^2} e^{in\pi x}}$$

Scanned with CamScanner

2) Find the complex Fourier series of $f(x) = \cos ax$ in $-\pi < x < \pi$, where a is not an integer & $f(x+2\pi) = f(x)$.

Solution:

Here $f(x) = \cos ax$, $-\pi < x < \pi$.

complex form of Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ — ①

where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax\, e^{-inx}\, dx$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{i^2n^2+a^2}\left(-in\cos ax + a\sin ax\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[e^{-in\pi}\left(-in\cos a\pi + a\sin a\pi\right) - e^{in\pi}\left(-in\cos a\pi - a\sin a\pi\right)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi\left(e^{in\pi} - e^{-in\pi}\right) + a\sin a\pi\left(e^{in\pi} + e^{-in\pi}\right)\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[in\cos a\pi \cdot 2i\sin n\pi + a\sin a\pi \cdot 2\cos n\pi\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\, 2a\sin a\pi\cos n\pi \quad \text{as } \sin n\pi = 0$$

$$C_n = \frac{(-1)^n a\sin a\pi}{\pi(a^2-n^2)}$$ subst in ① required

complex form of Fourier series is

$$\boxed{\cos ax = \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty} (-1)^n \frac{e^{inx}}{a^2-n^2}}$$

Scanned with CamScanner

3) Find the complex form of Fourier series of $f(x) = e^{ax}$ in $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.

Deduce that $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{\pi}{a \sinh \pi a}$

Solution: Here $f(x) = e^{ax}$, $-\pi < x < \pi$

$\therefore$ complex form of Fourier series in $(-\pi, \pi)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad -①$$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} . e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)} \left[e^{(a-in)\pi} - e^{-(a-in)\pi} \right]$$

$$= \frac{(a+in)}{2\pi(a^2+n^2)} \left[e^{a\pi} e^{-in\pi} - e^{-a\pi} e^{in\pi} \right]$$

$$= \frac{a+in}{2\pi(a^2+n^2)} \left[e^{a\pi} (\cos n\pi - i \sin n\pi) - e^{-a\pi} (\cos n\pi - i \sin n\pi) \right]$$

$$= \frac{(a+in)}{2\pi(a^2+n^2)} \left[(e^{a\pi} - e^{-a\pi}) \cos n\pi \right] \quad \text{as } \sin n\pi = 0.$$

Scanned with CamScanner

$$C_n = \frac{(a+in)}{2\pi(a^2+n^2)} \cos n\pi \cdot 2 \sinh \pi a$$

$$C_n = \frac{(a+in)(-1)^n}{\pi(a^2+n^2)} \sinh \pi a$$

Reqd complex form of Fourier series from ① is

$$\boxed{e^{ax} = \frac{1}{\pi} \sinh \pi a \sum_{n=-\infty}^{\infty} \frac{(a+in)(-1)^n e^{inx}}{a^2+n^2}} \rightarrow ②$$

Put $x = 0$, $\therefore$ sum of FS at $x = 0$ is $f(0) = e^0 = 1$

$$\therefore 1 = \frac{1}{\pi} \sinh \pi a \sum_{n=-\infty}^{\infty} \frac{a+in}{a^2+n^2} (-1)^n$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{a^2+n^2} = \frac{\pi}{\sinh \pi a}$$

Equating real part we get

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{\pi}{a \sinh \pi a}.$$

Scanned with CamScanner

Complex Form of Fourier Series in various intervals

7/4/2020.

1) In the interval $(0, 2l)$, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}}$, where $C_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-i\frac{n\pi x}{l}} dx$

2) In the interval $(-l, l)$, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}}$, where $C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-i\frac{n\pi x}{l}} dx$

3) In the interval $(0, 2\pi)$, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$, where $C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} dx$

4) In the interval $(-\pi, \pi)$, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$, where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

Scanned with CamScanner