7. $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \ldots\ldots\right)$

8. $u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{\cos 2\omega t}{1.3} + \frac{\cos 4\omega t}{3.5} + \frac{\cos 6\omega t}{5.7} + \ldots\ldots\right).$

10.7. HALF RANGE SERIES

Sometimes it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or more generally in the range $(0, l)$ in a Fourier series of period $2l$.

If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

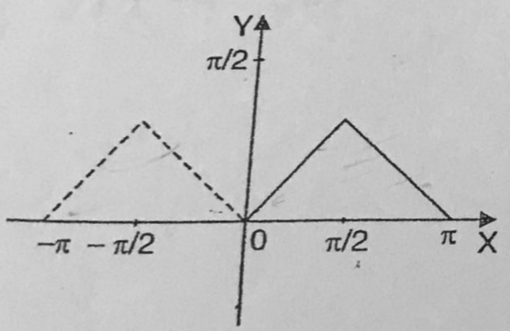
If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended functicn is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

For example, consider the function

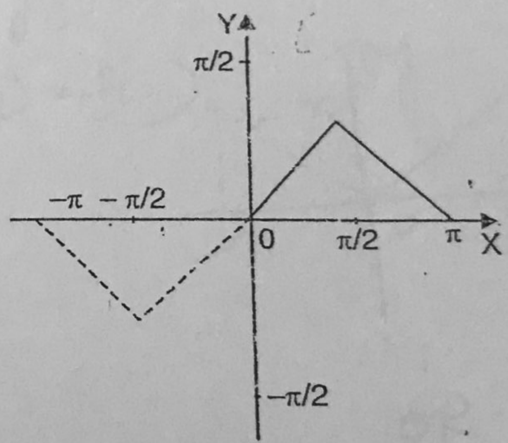
$$f(x) = x, \qquad 0 < x < \frac{\pi}{2}$$

$$= \pi - x, \qquad \frac{\pi}{2} < x < \pi$$



FS-7

(Reflection in the y-axis)



(Reflection in the origin)

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half-range series.

The half-range cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$; $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$

The half-range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$, where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

∴ It is our choice whether we extend to Y axis or origin.

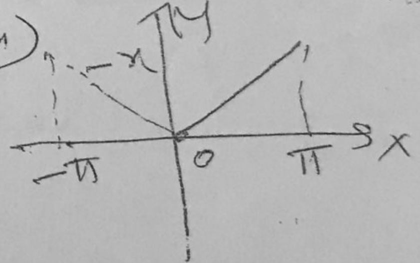
Question: Find FS of $f(x) = x$ in $(0,\pi)$?

Here it is not possible to expand $f(x)=x$ in FS $\because$ the interval is $(0,\pi)$ & it is not a periodic function of period 2π.

$\therefore$ we need to extend the fn w.r.t to Y axis or w.r.t origin.

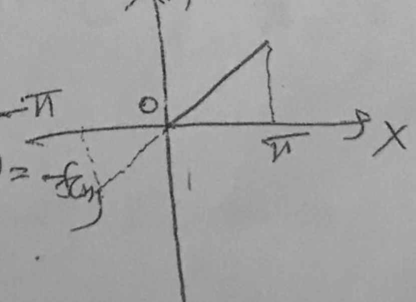
$\therefore$ if the interval is $(0,\pi)$ or $(0,l)$, we extend the fn w.r.t Y-axis or origin, so that we get $(-\pi,\pi)$ or $(-l,l)$.

i) $f(-x) = f(x)$,



OR

ii) $f(-x) = -f(x)$



HRE-3

Obtain the ~~half~~ half range sine series for the function $f(x)=x^2$ in the interval $0<x<3$.

Sol: Half range sine series is $\quad l=3$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n x\pi}{l}\right) \rightarrow ①$$

~~Here period of fun is 3−0=3~~ $\quad \therefore$ ~~$2l=3 \rightarrow l=3/2$~~

$$\therefore b_n = \cancel{\frac{2}{3/2}\int_0^{3/2} x^2 \sin\left(\frac{2n\pi x}{3}\right)dx} = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

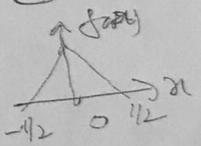
$$=\frac{2}{3}\int_0^3 x^2 \sin\left(\frac{n\pi x}{3}\right)dx$$

$$=\frac{2}{3}\left[x^2\left(\frac{-\cos\frac{n\pi x}{3}}{n\pi/3}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{3}}{n^2\pi^2/9}\right) + 2\left(\frac{\cos\frac{n\pi x}{3}}{n^3\pi^3/27}\right)\right]_0^3$$

$$=\frac{2}{3}\left\{\left[\frac{-3}{n\pi}\times 9(-1)^n + \frac{27}{n^3\pi^3}\times 2\times(-1)^n\right] - \left[0+\frac{27}{n^3\pi^3}\times 2\times 1\right]\right\}$$

$$b_n = \frac{2}{3}\left[-\frac{27(-1)^n}{n\pi}+\frac{54}{n^3\pi^3}(-1)^n-\frac{54}{n^3\pi^3}\right] \rightarrow \text{subst in eq}^n\ ①.$$

2) P.T $\frac{1}{2}-x = \frac{1}{\pi}\sum_{1}^{\infty} \frac{1}{n}\sin\frac{2n\pi x}{l}$, $0<x<1$.



Sol: Given $f(x)=\frac{1}{2}-x$

Not reqd: ~~$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2\int_0^1\left(\frac{1}{2}-x\right)dx = 2\left[\frac{1}{2}x-\frac{x^2}{2}\right]_0^1 = 0$~~

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = 2\int_0^1\left(\frac{1}{2}-x\right)\sin n\pi x\,dx$$

$$=2\left[\left(\frac{1}{2}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$=2\left[+\frac{1}{2}\frac{(-1)^n}{n\pi} - \left(\frac{1}{2}\right)\left(-\frac{1}{n\pi}\right)\right] = \frac{1}{n\pi}\left[(-1)^n+1\right].$$

$f(x) = a_0 + \sum^{\infty} a_n \cos$

$$= \frac{2}{l}\left[\left(\frac{l}{2}-x\right)\left(-\frac{\frac{\cos n\pi x}{l}}{n\pi/l}\right) - (-1)\left(-\cancel{\frac{\frac{\sin n\pi x}{l}}{n^2\pi^2/l^2}}\right)\right]_0^l$$

$$= \frac{-2}{l}\left\{\left[\left(\frac{l}{2}-l\right)(-1)^n \times \frac{l}{n\pi}\right] - \frac{l}{2} \times \frac{l}{n\pi}\right]$$

$$= \frac{-2}{n\pi}\left\{ \right.$$

in $(0,\pi)$.

Sol Reqd half range cosine series is

$\cos n\pi = (-1)^n$

$$f(x) = x\cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{---(1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\cos x\,dx = \frac{2}{\pi}\Big[x(\sin x)^{\,0} - (1)(-\cos x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\big[-1-1\big] \Rightarrow \boxed{\frac{-4}{\pi} = a_0} \quad \text{---(2)}$$

$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos x\cos nx\,dx = \frac{2}{\pi}\times\frac{1}{2}\int_0^{\pi} x\left[\cos(n+1)x + \cos(n-1)x\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cos(n+1)x\,dx + \int_0^{\pi} x\cos(n-1)x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\cancel{x\cos(n+1)x}\; x\left(\cancel{\frac{\sin(n+1)x}{n+1}}\right)^{0} - (1)\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} + \left[x\left(\cancel{\frac{\sin(n-1)x}{n-1}}\right)^{0} - (1)\left(\frac{-\cos}{(n-1)^2}\right.\right.\right.$$

$$= \frac{1}{\pi}\left\{\left[\frac{1}{(n+1)^2}(-1)^{n+1} - \frac{1}{(n+1)^2}\right] + \left[\frac{(-1)^{n-1}}{(n-1)^2} - \frac{1}{(n-1)^2}\right]\right\} \text{ for } n \neq 1 \quad \text{---(3)}$$

$= \cancel{\frac{1}{\pi}}\Big\{\Big[$

for $n=1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos^2 x\,dx = \frac{\cancel{2}}{\pi}\int_0^{\pi} x\left(\frac{1+\cos 2x}{\cancel{2}}\right.$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} x\,dx + \frac{\cancel{2}}{\pi}\int_0^{\pi} x\cos 2x\,dx$$

$(-1)^n$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] + \frac{1}{\pi}\left[x\left(\cancel{\frac{\sin 2x}{2}}\right)^{0} - (1)\left(\frac{-\cos 2x}{}\right)\right]^{\pi}$$

$$= \frac{\pi}{2} + \frac{1}{4\pi}\left[1-1\right] = \frac{\pi}{2} \rightarrow (4)$$

Subst ② ③ ④ in ① we get ~~reqd~~ result.

Extra

④ Expand $\pi x - x^2$ in a Half range sine series in the interval $(0, \pi)$ up to first 3 terms.

Ans: $$\pi x - x^2 = \frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right]$$

5) Obtain sine half range series of $f(x)$

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{in } \frac{1}{2} < x < 1 \end{cases}$$

Sol: Here $l = 1$ $\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

Ans: $$b_n = \frac{1}{2n\pi}\left[1 - (-1)^n\right] - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$b_n = 2\int_0^1 f(x)\sin\left(\frac{n\pi x}{1}\right)dx = 2\left\{\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\sin\right.$$

(i) The half-range cosine series is $f(x) = \dots$

$= \frac{2}{\pi}\int^{\pi} f(x) \cos nx \, dx.$

$\frac{2}{\pi}\int^{\pi} f(x) \sin nx$

(10) $f(x) = |x|, \quad -2 < x < 2$

$L = 2$

$$f(x) = \begin{cases} -x & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$$

$f(-x) = f(x)$ ∴ $f(x)$ is even

$\boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{\cancel{2}}{\cancel{2}}\int_0^2 x \, dx = \frac{x^2}{2}\Big|_0^2 = 2$$

$$a_n = \frac{\cancel{2}}{\cancel{2}}\int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx$$

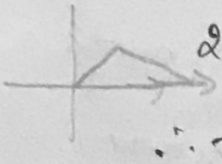
$$= \cancel{x}\left[\cancel{\frac{x \sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}}^{\,0} - \frac{(1)\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{\left(\frac{n\pi}{2}\right)^2}\Bigg|_0^2 \right]$$

$$a_n = \frac{\cos n\pi - 1}{\frac{n^2\pi^2}{4}} = \frac{\left((-1)^n - 1\right)4}{n^2\pi^2}.$$

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{2}\right)$$

7) Obtain FS for $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2. \end{cases}$

Sol: $\phi(x) = \pi x$ & $\psi(x) = \pi(2-x)$. period of $f(x) = 2-0 = 2$



$2l = 2 \Rightarrow l = 1$ $\quad \therefore \phi(2l-x) = \phi(2-x) = \pi(2-x)$

$= \psi(x)$

$\therefore f(x)$ is even. $\therefore b_n = 0$.

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2\int_0^1 \pi x\,dx = \cancel{2}\pi \times \left[\frac{x^2}{\cancel{2}}\right]_0^1 = \pi$

$a_n = 2\int_0^1 \pi x \cos\left(\frac{n\pi x}{1}\right) dx = 2\pi\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$

$= \frac{2\pi}{n^2\pi^2}\left[(-1)^n - 1\right]$

$$\boxed{\therefore f(x) = \frac{\pi}{2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \cos n\pi x}$$

(8) Find FS of $f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{"} \ -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{"} \ \frac{\pi}{2} < x < \pi. \end{cases}$

Sol: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow ①$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} -dx + 0 + \int_{\pi/2}^{\pi} dx\right\}$

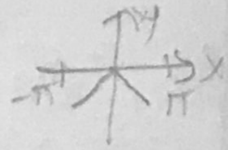
$= \frac{1}{\pi}\left\{-\left[-\frac{\pi}{2} - (-\pi)\right] + \left[\pi - \frac{\pi}{2}\right]\right\}$

$= \frac{1}{\pi}\left[+\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right] = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$

6) obtain FS of the function $f(x)=\begin{cases} x, & -\pi<x<0 \\ -x, & 0<x<\pi \end{cases}$

& hence S.T $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$



Sol: Here $\phi(x)=x$ & $\psi(x)=-x$

$\phi(-x)=-x=\psi(x) \Rightarrow f(x)$ is even $\therefore b_n=0$

& $a_0=\frac{2}{\pi}\int_0^\pi f(x)\,dx$ & $a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$

$a_0=\frac{2}{\pi}\int_0^\pi -x\,dx=\frac{-2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi=\frac{-1}{\pi}\left[\pi^2-0\right]=-\pi.$

$a_n=\frac{-2}{\pi}\int_0^\pi x\cos nx\,dx=-\frac{2}{\pi}\left[x\left(\cancelto{0}{\frac{\sin nx}{n}}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$

$=-\frac{2}{\pi n^2}\left[(-1)^n-1\right]=\frac{2(1-(-1)^n)}{n^2\pi}$

$f(x)=\frac{-\pi}{2}+\sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n^2\pi}\cos nx$

Put $x=0$.

10) obtain the Fourier half expansion of $x \sin x$ as a cosine series in $(0, \pi)$

$\cos n\pi = (-1)^n$

Sol: Required Half range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx} \rightarrow (1) \quad \text{where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \text{ \& } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx = \frac{2}{\pi}\left\{\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{[+\pi + 0] - [0+0]\right\} \quad \boxed{a_0 = 2} \rightarrow (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx = \frac{2}{\pi}\times\frac{1}{2}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx \sin x\,dx$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} x \sin(n+1)x\,dx - \int_0^{\pi} x \sin(n-1)x\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[x\left[-\frac{\cos(n+1)x}{n+1}\right] - (1)\left(-\frac{\cancel{\sin(n+1)x}}{(n+1)^2}\right)\right]_0^{\pi}\right.$$

$$\left. - \left[x\left(-\frac{\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\cancel{\sin(n-1)x}}{(n-1)^2}\right)\right]_0^{\pi}\right\}$$

$$a_n = \frac{1}{\cancel{\pi}}\left\{\left[-\frac{\cancel{\pi}}{(n+1)}(-1)^{n+1} - 0\right] - \left[-\frac{\cancel{\pi}}{n-1}(-1)^{n-1} + 0\right]\right\}$$

$$= \frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} = (-1)^n\left[\frac{-1}{n-1} + \frac{1}{n+1}\right] = (-1)^n\left[\frac{-\cancel{n}-1+\cancel{n}-1}{n^2-1}\right]$$

$$\boxed{a_n = \frac{2(-1)^{n+1}}{n^2-1}} \text{ where } n \neq 1 \rightarrow (3)$$

$$\therefore a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\,dx = \frac{1}{\pi}\left\{x\left(-\frac{\cos 2x}{2}\right) - (1)\left(\frac{\sin 2x}{2}\right)\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi + 0\right] \rightarrow \boxed{a_1 = -\frac{1}{2}} \rightarrow (4)$$

Sub (2), (3) & (4) in (1)

11) Find the Fourier half range (a) cosine series and (b) sine series of $f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$

Half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \text{ where } l=2$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 f(x)dx$$

$$a_0 = \int_0^1 x\,dx + \int_1^2 (2-x)dx = \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{(2-x)^2}{2(-1)}\right]_1^2$$

$$a_0 = \frac{1}{2} - \frac{1}{2}\left[0-1\right] = 1 \Rightarrow \boxed{a_0 = 1}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \int_0^2$$

$$= \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= x\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2}\right] - (1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{n^2\pi^2/4}\right)\Bigg]_0^1 + \left[(2-x)\left(\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{n^2\pi^2/4}\right)\right]_1^2$$

$$= \left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right] - \left[0 + \frac{4}{n^2\pi^2}\right]$$

$$+ \left[0 - \frac{4}{n^2\pi^2}\,\cancel{\cos\frac{4n\pi}{2}}\,(-1)^n\right] - \left[\frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$\boxed{a_n = \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left[1+(-1)^n\right]}$$

$\therefore$ Half range F cosine series is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$