Unit – 5

Singular Value Decomposition

1. Consider an n*n symmetric matrix, A then any two eigen vectors from different eigen spaces are orthogonal

2. A symmetric matrix, A is orthogonally diagonalizable, (i.e) the eigen vector matrix, S is a orthogonal matrix then $S^{-1} = S^{T}$.

Then Diagonal matrix, $\Lambda = S^{-1} A S$

and Factorization of matrix , $A = S \Lambda S^{-1}$.

Then it is possible to find powers of A as $A^{n} = S \Lambda^{n} S^{-1}$

3. The set of all eigen values of A is called Spectrum of A

4 .Quadratic Forms, Q(x): Q(x) has only quadratic terms in the form purely x^2, or Cross products, xy, xz or zx

A Q(x) is represented as $Q(x) = x^{T} A x$, where A is called matrix of the quadratic form and A is a symmetric matrix

$Q(x) = x^2 + y^2 + z^2$, then A is exactly diagonal matrix

5. Classification of QF :

a) Positive definite : if $Q(x) > 0$, Eigen values of $A > 0$

b) Negative definite : if $Q(x) < 0$, Eigen values of $A < 0$

c) Indefinite : If $Q(x)$ assumes both positive & negative & Eigen values of A are both positive & negative

d) Positive Semi definite : if $Q(x) \geq 0$

e) Negative Semi definite : if $Q(x) \leq 0$

Tests for positive definiteness :

a) $x^T A x > 0$

b) All the eigen values of $A > 0$

c) All the upper sub matrices of A must have positive determinant

d) All the pivots > 0

Tests for positive semi definite:

a) $x^T A x \geq 0$

b) All the eigen values of A ≥ 0

c) All the upper sub matrices of A must have non negative determinant

d) All the pivots ≥ 0

Singular value decomposition, SVD:

SVD is used to factorize m*n matrix, A by using the fact that $A^T A$ is symmetric matrix

Factorization of $A = U \Sigma V^T$ where,

U = m*m orthogonal matrix

V = n*n orthogonal matrix

Σ = diagonal matrix

$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$

Singular values of A , σ = square root of λ = length of the vector AX

1. Compute quadratic form for A=$\begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}$

   Sol: x= (x,y)

   QF = $x^T Ax$ = 3 x*x -4xy+7 y*y

2. Check the positive definiteness of A = $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

   Sol: Eigen values = 1,1,7 > 0

   L = $\begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 2/3 & 2/5 & 1 \end{pmatrix}$ D = $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5/3 & 0 \\ 0 & 0 & 7/5 \end{pmatrix}$

   Pivots = 3, 5/3, 7/5 > 0

   |A| = 7

   A is positive definite

3. Find SVD of A = $\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$

   Sol: $A^T A$ = $\begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$

   Eigen values of A = 0, 18

   Factorization of A = U ΣV^T where,

   U = m*m orthogonal matrix

   V = n*n orthogonal matrix

   Σ = diagonal matrix

   Σ = $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$

Singular values = $\sqrt{0}$, $\sqrt{18}$ = 0, $3\sqrt{2}$

v^T=(1/$\sqrt{2}$) $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\varepsilon = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

U=$\begin{pmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 1/\sqrt{45} \end{pmatrix}$