Unit - 5

Singular Value Decomposition

(Chapter 6 – 6.2 , 6.3)

Tests for Positive Definiteness

Each of the following tests is a necessary and sufficient condition for the real symmetric matrix A to be *positive definite*:

- $x^T Ax > 0$ for all nonzero real vectors x.
- All the eigenvalues of A satisfy $\lambda_i > 0$.
- All the upper left submatrices A_k have positive determinants.
- All the pivots (without row exchanges) satisfy $d_k > 0$.

Positive Definite Matrices and Least Squares

Note:

1. A symmetric matrix A is positive definite if and only if there is a matrix R with independent columns such that $A = R^TR$.

2. Using LDU decomposition $A=R^TR$ where $R=\sqrt{D}L^T$.

   This *Cholesky decomposition* has the pivots split evenly between L and L^T.

3. Usingdiagonalization

,

$$A = Q\Lambda Q^T, therefore\ R = \sqrt{\Lambda}\, Q^T.$$

where Q is an orthogonal matrix with orthonormal eigenvectors of A.

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4. A third possibility is $R = Q\sqrt{\Lambda}\,Q^T$.

the symmetric positive definite square root of

A. If we multiply any R by a matrix Q with orthonormal columns, then $(QR)^T(QR) = R^TQ^TQR = R^TIR = A$. Therefore QR is another choice.

Semi Definite matrices

Each of the following tests is a necessary and sufficient condition for a symmetric matrix A to be positive semidefinite:

- $x^TAx \geq 0$ for all vectors x (this defines positive semidefinite).
- All the eigenvalues of A satisfy $\lambda_i \geq 0$
- No principal submatrix has negative determinants.
- None of the pivots is negative.
- There is a matrix R, possibly with dependent columns, such that $A = R^TR$.

Singular Value Decomposition

Any m by n matrix A can be factored into

$$A = U \Sigma V^T$$

(orthogonal)(diagonal)(orthogonal)

The columns of U (m by m) are eigenvectors of AA^T, and the columns of V (n by n) are eigenvectors of A^TA. The r singular values on the diagonal of Σ(m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Note:

- For positive definite matrices, Σ is Λ.
- *U and V give orthonormal bases for all four fundamental subspaces:*

  first *r columns of U: column space of A*

  last *m-r columns of U: left nullspace of A*

  first *r columns of V: row space of A*

  last *n-r columns of V: nullspace of A*