Problems on Inverse L.T

I Shifting property of Inverse L.T

If $L^{-1}\{F(s)\} = f(t)$, $L^{-1}\{F(s-a)\} = e^{at}f(t)$

Problems: [Using table of Inverse L.T]

1. $L^{-1}\left\{\frac{2}{s^3} + \frac{4}{s}\right\} = 2L^{-1}\left\{\frac{1}{s^3}\right\} + 4L^{-1}\left\{\frac{1}{s}\right\}$

$$= 2\left[\frac{t^2}{2!}\right] + 4$$

$$= \underline{\underline{t^2+4}}$$

2. $L^{-1}\left\{\frac{1}{s+2} - \frac{2}{s-3}\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} - 2L^{-1}\left\{\frac{1}{s-3}\right\}$

$$= \underline{\underline{e^{-2t} - 2e^{3t}}}$$

3. $L^{-1}\left\{\frac{2s+5}{s^2+16}\right\} = 2L^{-1}\left\{\frac{s}{s^2+16}\right\} + 5L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$$= 2L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + \frac{5}{4}L^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$

$$= \underline{\underline{2\cos 4t + \frac{5}{4}\sin 4t}}$$

4. $L^{-1}\left\{\frac{1}{s^2+6s+18}\right\} = L^{-1}\left\{\frac{1}{s^2+6s+9+9}\right\}$

$$= \frac{1}{3}L^{-1}\left\{\frac{3}{(s+3)^2+3^2}\right\} = \underline{\frac{1}{3}e^{-3t}\sin 3t}$$

5. $\mathcal{L}^{-1}\left\{\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{2s+5}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{3s-2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+5/2}\right\}-\frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2/3}\right\}$

$e^{-2t}+\frac{3}{2}e^{-\frac{5}{2}t}-\frac{4}{3}e^{\frac{2}{3}t}$.

NOTE: Make the Coefficient of 's' always 1

6. $\mathcal{L}^{-1}\left\{\frac{5s-2}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{5s-2}{s^2+4s+4+4}\right\}$

$\mathcal{L}^{-1}\left\{\frac{5s-2}{(s+2)^2+2^2}\right\}=\mathcal{L}^{-1}\left\{\frac{5(s+2)-10-2}{(s+2)^2+2^2}\right\}$

$=5\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2^2}\right\}-12\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2^2}\right\}$

$=5e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}-6e^{-2t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

$=5e^{-2t}\cos 2t-6e^{-2t}\sin 2t$

$=e^{-2t}[5\cos 2t-6\sin 2t]$

7. $\mathcal{L}^{-1}\left\{\frac{s}{3s^2-2s-5}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2-\frac{2}{3}s-\frac{5}{3}}\right\}$

$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2-\frac{2}{3}s+\frac{1}{9}-\frac{1}{9}-\frac{5}{3}}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s-1/3+1/3}{(s-\frac{1}{3})^2-\frac{16}{9}}\right\}$

$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{(s-1/3)}{(s-1/3)^2-(\frac{4}{3})^2}+\frac{1/3}{(s-1/3)^2-(\frac{4}{3})^2}\right\}=\frac{1}{3}e^{1/3\,t}\left(\cosh\frac{4}{3}t+\frac{1}{4}\sinh\frac{4}{3}t\right)$

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II Method of Partial Fractions:

NOTE: Split the function using the technique of Partial fractions and then, find its inverse Laplace transforms.

1) $\mathcal{L}^{-1}\left\{\frac{s+1}{(s-1)^2(s+2)}\right\}$

Consider, $\frac{s+1}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$$s+1 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

Put $s=1$, $2 = B(3)$ $\boxed{B = \frac{2}{3}}$

Put $s=-2$, $-1 = 9C$ $\boxed{C = \frac{-1}{9}}$

Put $s=0$, $1 = -2A + 2B + C$

$$1 - \frac{4}{3} + \frac{1}{9} = -2A$$

$$\frac{9-12+1}{9} = -2A; \quad -2A = -\frac{2}{9}; \quad \boxed{A = \frac{1}{9}}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s-1)^2(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/9}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2/3}{(s-1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/9}{s+2}\right\}$$

$$= \frac{1}{9}e^{t} + \frac{2}{3}e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}e^{-2t}$$

$$= \frac{1}{9}e^{t} + \frac{2}{3}e^{t}\cdot t - \frac{1}{9}e^{-2t}$$

$$= \frac{1}{9}\left[e^{t} + 6te^{t} - e^{-2t}\right]$$

2. $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$

Consider, $\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$

$s+1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

Comparing the coefficients of s^3: $0 = A + C \rightarrow (1)$

" " s^2: $0 = B + D \rightarrow (2)$

" " s : $1 = 4A + C \rightarrow (3)$

" " Constant term : $1 = 4B + D \rightarrow (4)$

Solving eqns (1) & (3)

$$\begin{array}{r} A + \cancel{C} = 0 \\ -4A \pm \cancel{C} = 1 \\ \hline -3A = -1 \end{array} \quad \boxed{A = 1/3} \quad \boxed{C = -1/3}$$

Solving eqns (2) & (4)

$$\begin{array}{r} B + \cancel{D} = 0 \\ -4B \pm \cancel{D} = 1 \\ \hline -3B = -1 \end{array} \quad \boxed{B = 1/3} \quad \boxed{D = -1/3}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/3\, s + 1/3}{s^2+1} + \frac{-1/3\, s - 1/3}{s^2+4}\right\}$$

$$= \frac{1}{3}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right]$$

$$= \frac{1}{3}\left[\cos t + \sin t - \cos 2t - \frac{1}{2}\sin 2t\right]$$

Method of partial Fractions contd...

3. Find inverse Laplace Transform of $\frac{s}{s^4+s^2+1}$

$$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = L^{-1}\left\{\frac{s}{s^4+2s^2+1-s^2}\right\}$$

$$L^{-1}\left\{\frac{s}{(s^2+1)^2-s^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+1-s)(s^2+1+s)}\right\}$$

After this step, we can use either partial fractions or rearrange the numerator like,

$$\frac{1}{2}L^{-1}\left\{\frac{2s}{(s^2-s+1)(s^2+s+1)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{s^2+s+1-s^2+s-1}{(s^2-s+1)\,(s^2+s+1)}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{\cancel{(s^2+s+1)}}{(s^2-s+1)\cancel{(s^2+s+1)}} - \frac{\cancel{(s^2-s+1)}}{\cancel{(s^2-s+1)}(s^2+s+1)}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+\frac{1}{4}+\frac{3}{4}} - \frac{1}{s^2+s+\frac{1}{4}+\frac{3}{4}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{\frac{\sqrt{3}}{2}}e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t - \frac{1}{\frac{\sqrt{3}}{2}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{2}{2}\cdot\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left[\frac{e^{t/2}-e^{-t/2}}{2}\right] = \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,\sinh\frac{t}{2}$$

4. Find the $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Consider $s^4+4a^4 = s^4+4a^2s^2+4a^4-4a^2s^2$

$= (s^2+2a^2)^2-(2as)^2$

$= (s^2+2a^2+2as)(s^2+2a^2-2as)$

Numerator $s = \frac{1}{4a}(4as)$

$= \frac{1}{4a}(s^2+4as+2a^2-s^2-2a^2)$

$= \frac{1}{4a}\left[(s^2+2as+2a^2)-(s^2-2as+2a^2)\right]$

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = L^{-1}\left\{\frac{\frac{1}{4a}\left[(s^2+2as+2a^2)-(s^2-2as+2a^2)\right]}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{\cancel{s^2+2as+2a^2}}{(\cancel{s^2+2a^2+2as})(s^2+2a^2-2as)} - \frac{\cancel{s^2-2as+2a^2}}{(s^2+2a^2+2as)(\cancel{s^2-2as+2a^2})}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2-2as+a^2+a^2} - \frac{1}{s^2+2as+a^2+a^2}\right\}$$

$$= \frac{1}{4a^2}L^{-1}\left\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a^2}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

$$= \frac{1}{2a^2}\sin at\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2a^2}\sin at \sinh at$$

5. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+4)(s^2+9)}\right\}$

Let $s^2 = v$

$\mathcal{L}^{-1}\left\{\dfrac{v}{(v+4)(v+9)}\right\}$

Using Partial Fractions

$$\frac{v}{(v+4)(v+9)} = \frac{A}{v+4} + \frac{B}{v+9}$$

$$v = A(v+9) + B(v+4)$$

Put $v = -4$, $-4 = 5A \Rightarrow A = -4/5$

$v = -9$, $-9 = -5B \Rightarrow B = 9/5$

$$\mathcal{L}^{-1}\left\{\frac{-4/5}{v+4} + \frac{9/5}{v+9}\right\}$$

Substituting for v.

$$\mathcal{L}^{-1}\left\{\frac{-4/5}{s^2+4} + \frac{9/5}{s^2+9}\right\}$$

$$= -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{-2}{5}\sin 2t + \frac{3}{5}\sin 3t$$

III Multiplication by 's' and Division by 's'

i> $L^{-1}\{sF(s)\} = f'(t)$ if $f(0)=0$, where $f(t) = L^{-1}\{F(s)\}$.

In General if $f(0) = f'(0) = \dots = f^{n-1}(0) = 0$ then

$$L^{-1}\{s^n F(s)\} = \frac{d^n}{dt^n} f(t).$$

(ii) If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t\int_0^t f(t)\,dt\,dt.$$

Problems: 1) Find $L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$

$$L^{-1}\left\{\frac{1/(s+1)}{s^2}\right\} = \int_0^t\int_0^t L^{-1}\left\{\frac{1}{s+1}\right\} dt\,dt$$

$$= \int_0^t\int_0^t e^{-t}\,dt\,dt = \int_0^t \frac{e^{-t}}{-1}\Big|_0^t dt$$

$$= \int_0^t (-e^{-t}+1)\,dt = e^{-t}\Big|_0^t + t\Big|_0^t = \underline{\underline{e^{-t}-1+t}}$$

2. $$L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\} = L^{-1}\left\{\frac{s}{s^2(s+3)}\right\} + 2L^{-1}\left\{\frac{1}{s^2(s+3)}\right\}$$

$$L^{-1}\left\{\frac{1}{s(s+3)}\right\} + 2L^{-1}\left\{\frac{1}{s^2(s+3)}\right\}$$

$$\int_0^t e^{-3t}\,dt + 2\int_0^t\int_0^t e^{-3t}\,dt\,dt$$

$$-\frac{e^{-3t}}{3}\Big|_0^t + 2\int_0^t -\frac{e^{-3t}}{3}\Big|_0^t dt = -\frac{e^{-3t}}{3} + \frac{1}{3} + \frac{2}{3}\left(\frac{e^{-3t}}{3}\Big|_0^t\right)$$

$$= -\frac{e^{-3t}}{3} + \frac{1}{3} + \underline{\underline{\frac{2}{9}\left(e^{-3t}-1\right)}}$$

IV Inverse L.T of logarithmic function and Inverse trigonometric fns.

or

Inverse Laplace Transform of derivatives.

We have $L\{t\, f(t)\} = -\frac{d}{ds}F(s)$

$$\Rightarrow L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\, f(t).$$

|||ly $$L^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\} = (-1)^n t^n f(t).$$ OR

$$-\frac{1}{t} L^{-1}\left\{\frac{d}{ds}F(s)\right\} = L^{-1}\{F(s)\}$$

Problems:

1. Find $L^{-1}\left\{\ln\left(1+\frac{a^2}{s^2}\right)\right\}$

Let $L^{-1}\left\{\ln\left(1+\frac{a^2}{s^2}\right)\right\} = L^{-1}\{\ln(s^2+a^2) - \ln s^2\} = f(t)$

$$\Rightarrow L\{t\, f(t)\} = -\frac{d}{ds}\left[\ln(s^2+a^2) - \ln s^2\right]$$

$$= -\left[\frac{1\cdot 2s}{s^2+a^2} - \frac{1\cdot 2\cancel{s}}{s^{\cancel{2}}}\right]$$

$$L\{t\, f(t)\} = \frac{2}{s} - \frac{2s}{s^2+a^2}$$

$$t\cdot f(t) = L^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2+a^2}\right\}$$

$$t\, f(t) = 2 - 2\cos at$$

$$f(t) = \frac{2}{t}[1-\cos at]$$

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2. Find $\mathcal{L}^{-1}\left\{ \ln\left(\frac{s^2+1}{s(s+1)}\right)\right\}$

Let $\mathcal{L}^{-1}\left\{ \ln(s^2+1) - \ln(s) - \ln(s+1)\right\} = f(t)$

then, $\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}\left[\ln(s^2+1) - \ln s - \ln(s+1)\right]$

$$= -\frac{1\cdot 2s}{s^2+1} + \frac{1}{s} + \frac{1}{s+1}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$t\, f(t) = -2\cos t + 1 + e^{-t}$$

$$f(t) = \frac{-2\cos t + 1 + e^{-t}}{t}$$

3. Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$

Let $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \cancel{t f(t)} - \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\}$

$\mathcal{L}\{t\cdot f(t)\} = \frac{s+2}{(s^2+4s+5)^2} = \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)$

$-2t\, f(t) =$

3. $\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+4+1)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{((s+2)^2+1)^2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$= \frac{-1}{2}e^{-2t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$= \frac{-1}{2}e^{-2t}\cdot(-t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}t\,e^{-2t}\sin t$$

<u>NOTE</u>: $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2}t\sin at$

4. $\mathcal{L}^{-1}\left\{s\ln\left(\frac{s-1}{s+1}\right)\right\} = \mathcal{L}^{-1}\{sF(s)\} = f'(t)$

Where $F(s) = \ln\left(\frac{s-1}{s+1}\right)$ & $f'(t) = \frac{d}{dt}\mathcal{L}^{-1}\{F(s)\}$

with $f(0)=0$.

To check if $f(0)=0$, Let's find $f(t)$.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\ln(s-1) - \ln(s+1)\}$$

$$-t\,f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\}$$

$$f(t) = \frac{-(e^{t}-e^{-t})}{t} = \frac{e^{-t}-e^{t}}{t} \neq 0$$

at $t=0$.

Hence the above mentioned property fails.

$$\mathcal{L}^{-1}\left\{s \ln\left(\frac{s-1}{s+1}\right)\right\} = \mathcal{L}^{-1}\left\{s \ln(s-1) - s\ln(s+1)\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s-1}} + \ln(s-1) - \underbrace{\frac{s}{s+1}} - \ln(s+1)\right\}$$

↳ By differentiating using Product rule

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\right.$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{\cancel{s^2} + s - \cancel{s^2} + s}{s^2-1} + \ln(s-1) - \ln(s+1)\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2-1} + \ln(s-1) - \ln(s+1)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2-1}\right\} + \frac{1}{t^2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{t^2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{-1}{t}2\cosh t + \frac{1}{t^2}e^{t} - \frac{1}{t^2}e^{-t}$$

$$= \frac{-1}{t}2\cosh t + \frac{1}{t^2}(e^{t} - e^{-t})$$

$$= \frac{-1}{t}2\cosh t + \frac{1}{t^2}2\sinh t$$

$$= \frac{2}{t^2}\left[\sinh t - \cosh t \cdot t\right]$$

5) Find $\mathcal{L}^{-1}\{\tan^{-1}(\frac{s}{2})\}$

$$\mathcal{L}^{-1}\{\tan^{-1}(\tfrac{s}{2})\} = \frac{-1}{t}\mathcal{L}^{-1}\{\frac{d}{ds}(\tan^{-1}(\tfrac{s}{2}))\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\{\frac{1}{1+(s/2)^2}\cdot\frac{1}{2}\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\}$$

$$= \frac{-1}{t}\sin 2t = \frac{-\sin 2t}{t}$$

6) $\mathcal{L}^{-1}\{\cot^{-1}(\frac{a}{s+b})\}$

$$\mathcal{L}^{-1}\{\cot^{-1}(\tfrac{a}{s+b})\} = \frac{-1}{t}\mathcal{L}^{-1}\{\frac{-1}{1+(\frac{a}{s+b})^2}\left(\frac{-a}{(s+b)^2}\right)\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\{\frac{+\cancel{(s+b)^2}a}{(s+b)^2+a^2}\cdot\frac{1}{\cancel{(s+b)^2}}\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\{\frac{a}{(s+b)^2+a^2}\} = \frac{-1}{t}e^{-bt}\sin at$$

7) $\mathcal{L}^{-1}\{\tanh^{-1}(\frac{2}{s})\}$

$$\mathcal{L}^{-1}\{\tanh^{-1}(\tfrac{2}{s})\} = \frac{-1}{t}\mathcal{L}^{-1}\{\frac{1}{1-(2/s)^2}\cdot\left(\frac{-2}{s^2}\right)\}$$

$$\left[\because \frac{d}{dx}[\tanh^{-1}x = \frac{1}{1-x^2}\right]$$

$$\frac{-1}{t}\mathcal{L}^{-1}\{\frac{s^2}{s^2-2^2}\left(\frac{-2}{s^2}\right)\} = \frac{2}{t}\mathcal{L}^{-1}\{\frac{2}{s^2-2^2}\}$$

$$= \frac{1}{t}\sinh 2t$$

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8) Find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s(s^2+a^2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s/(s^2+a^2)^2}{s}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} dt$$

$$\frac{-1}{2}\int_0^t -t\, \mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\} dt = \frac{-1}{2}\int_0^t -t\, \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$\frac{1}{2a}\int_0^t t\, \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} dt$$

$$\frac{1}{2a}\int_0^t t \sin at\, dt = \frac{1}{2a}\left[t\left(-\frac{\cos at}{a}\right)\Big|_0^t - \left(-\frac{\sin at}{a^2}\right)\Big|_0^t\right]$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

V ~~Mult~~ Division by 't' Property:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds \Rightarrow \mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$$

Problems:

1. $\mathcal{L}^{-1}\left\{\int_s^\infty \frac{a}{s^2+a^2}\,ds\right\} = \frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \frac{\sin at}{t}$

2) $\mathcal{L}^{-1}\left\{\int_s^{\infty}\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds\right\}$

$$= \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right\} = \frac{1}{t}(\cos at - \cos bt)$$

3) $\mathcal{L}^{-1}\left\{\int_s^{\infty}\ln\left(\frac{s+2}{s+1}\right)ds\right\}$

$$= \frac{1}{t}\mathcal{L}^{-1}\left\{\ln(s+2)-\ln(s+1)\right\}$$

$$= \frac{1}{t}\left[-\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right]$$

$$= -\frac{1}{t^2}e^{-2t}+\frac{1}{t^2}e^{-t} = \underline{\underline{\frac{e^{-t}-e^{-2t}}{t^2}}}$$

VI Second shifting property:

$\mathcal{L}^{-1}\{f(t-a).u(t-a)\} = e^{-as}F(s)$, then

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a).u(t-a)$, , where

$$u(t-a) = \begin{cases} 0 & 0<t<a \\ 1 & t>a. \end{cases}$$

Problems: Find $\mathcal{L}^{-1}\left\{\frac{2}{s}-\frac{2e^{-\pi s}}{s}+\frac{e^{-2\pi s}}{s^2+1}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s}\right\}-2\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}$$

$$2-2u(t-\pi)+\sin(t-2\pi)\,u(t-2\pi)$$

The solution can be expressed as a discontinuous function.

$$f(t) = \begin{cases} 2, & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$$

Verification: Expressing $f(t)$ using unit-step fn

$$f(t) = 2 + (0-2)\, u(t-\pi) + (\sin t - 0)\, u(t-2\pi)$$

$$= 2 - 2u(t-\pi) + \sin t\, u(t-2\pi)$$

Extra problems : CLASS WORK PROBLEMS

1. $$\mathcal{L}^{-1}\left\{\frac{3s-12}{s^2+8}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+8} - \frac{12}{s^2+8}\right\}$$

$$3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+8}\right\} - 2\sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{3\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$3\cos 2\sqrt{2}t - 3\sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$3\cos 2\sqrt{2}t - 3\sqrt{2}\sin 2\sqrt{2}t$$

2. $$\mathcal{L}^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\} = \mathcal{L}^{-1}\left\{\frac{s-2\sqrt{s}+1}{s^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s^{3/2}} + \frac{1}{s^2}\right\} = \cancel{1+t}$$

$$1 - \frac{2\cdot 2\sqrt{t}}{\sqrt{\pi}} + t = 1 - \frac{4\sqrt{t}}{\sqrt{\pi}} + t$$

3. $L^{-1}\left\{ \frac{1}{s} \sin\left(\frac{1}{s}\right) \right\}$

$$L^{-1}\left\{ \frac{1}{s}\left[\frac{1}{s} - \frac{\left(\frac{1}{s}\right)^3}{3!} + \frac{\left(\frac{1}{s}\right)^5}{5!} - \cdots \right] \right\}$$

$$L^{-1}\left\{ \frac{1}{s^2} - \frac{1}{3!\,s^4} + \frac{1}{5!\,s^6} - \cdots \right\} \qquad L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

$$\frac{t}{1!} - \frac{t^3}{3!\cdot 3!} + \frac{t^5}{5!\cdot 5!} - \cdots$$

$$\underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{[(2n-1)!]^2}\, t^{2n-1}}}$$

4. $$L^{-1}\left\{\frac{s}{(s+1)^5}\right\} = L^{-1}\left\{\frac{s+1-1}{(s+1)^5}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)^4} - \frac{1}{(s+1)^5}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^4} - \frac{1}{s^5}\right\}$$

$$= e^{-t}\left[\frac{t^3}{3!} - \frac{t^4}{4!}\right] = \frac{e^{-t}}{24}\left[4t^3 - t^4\right]$$

5. $$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = \frac{-1}{t} L^{-1}\left\{\frac{d}{ds} \tan^{-1}\left(\frac{2}{s^2}\right)\right\}$$

$$= \frac{-1}{t} L^{-1}\left\{\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{-4}{s^3}\right)\right\}$$

$$= +\frac{1}{t} L^{-1}\left\{\frac{4s^4}{s^3(s^4+4)}\right\} = \frac{+1}{t} L^{-1}\left\{\frac{4s}{(s^2)^2 + 4s^2 + 4 - 4s^2}\right\}$$

$$= \frac{+4}{t} L^{-1}\left\{\frac{s}{(s^2+2)^2 - (2s)^2}\right\} = \frac{+4}{t} L^{-1}\left\{\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

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$$+\frac{4}{t}\, L^{-1}\left\{\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

$$+\frac{1}{t}\, L^{-1}\left\{\frac{s^2+2s+2-s^2+2s-2}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

$$+\frac{1}{t}\, L^{-1}\left\{\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

$$+\frac{1}{t}\, L^{-1}\left\{\frac{\cancel{(s^2+2s+2)}}{\cancel{(s^2+2s+2)}(s^2-2s+2)} - \frac{\cancel{(s^2-2s+2)}}{(s^2+2s+2)\cancel{(s^2-2s+2)}}\right\}$$

$$+\frac{1}{t}\, L^{-1}\left\{\frac{1}{s^2-2s+1+1} - \frac{1}{s^2+2s+1+1}\right\}$$

$$+\frac{1}{t}\, L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$+\frac{1}{t} e^{t}\sin t + \frac{1}{t} e^{-t}\sin t$$

$$+\frac{\sin t}{t}\left(e^{t}-e^{-t}\right) = +\frac{2\sin t \sinh t}{t}.$$

6. $$L^{-1}\left\{\ln\left(1-\frac{a^2}{s^2}\right)\right\}$$

$$L^{-1}\left\{\ln\left(\frac{s^2-a^2}{s^2}\right)\right\} = L^{-1}\left\{\ln(s^2-a^2)-\ln(s^2)\right\}$$

$$-\frac{1}{t}\, L^{-1}\left\{\frac{1\cdot 2s}{s^2-a^2} - \frac{1}{s^2}\cdot 2s\right\}$$

$$-\frac{1}{t}\, L^{-1}\left\{\frac{2s}{s^2-a^2} - \frac{2}{s}\right\} = -\frac{2}{t}\, L^{-1}\left\{\frac{s}{s^2-a^2} - \frac{1}{s}\right\}$$

$$= -\frac{2}{t}\left[\cosh at - 1\right] = \left[1-\cosh at\right]\frac{2}{t}$$

7. $\mathcal{L}^{-1}\left\{ \cot^{-1}\left(\frac{s+3}{2}\right) \right\}$

$= \frac{-1}{t} \mathcal{L}^{-1}\left\{ \frac{-1}{1+\left(\frac{s+3}{2}\right)^2} \cdot \frac{1}{2} \right\}$

$= \frac{1}{t} \mathcal{L}^{-1}\left\{ \frac{4}{4+(s+3)^2} \cdot \frac{1}{2} \right\}$

$= \frac{1}{t} \mathcal{L}^{-1}\left\{ \frac{2}{(s+3)^2+2^2} \right\} = \frac{1}{t} e^{-3t} \sin 2t$

8. $\mathcal{L}^{-1}\left\{ \frac{s}{(s^2+1)^2} \right\}$

$\mathcal{L}^{-1}\left\{ \frac{s}{(s^2+1)^2} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{2} \frac{d}{ds}\left(\frac{-1}{s^2+1}\right) \right\}$

$= \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{d}{ds}\left(\frac{-1}{s^2+1}\right) \right\}$

$= \frac{1}{2} (-t) \mathcal{L}^{-1}\left\{ \frac{-1}{s^2+1} \right\} = +\frac{t}{2} \sin t$

9. $\mathcal{L}^{-1}\left\{ \frac{1}{s^2}\left[\frac{s+1}{s^2+1}\right] \right\} = \int_0^t \int_0^t \mathcal{L}^{-1}\left\{ \frac{s+1}{s^2+1} \right\} dt\, dt$

$= \int_0^t \int_0^t \left(\mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} \right) dt\, dt$

$= \int_0^t \int_0^t (\cos t + \sin t)\, dt\, dt$

$$f(t) = \begin{cases} 2, & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$$

Verification: Expressing $f(t)$ using unit-step fn

$$f(t) = 2 + (0-2)\, u(t-\pi) + (\sin t - 0)\, u(t-2\pi)$$

$$= 2 - 2u(t-\pi) + \sin t\, u(t-2\pi)$$

Extra problems : CLASS WORK PROBLEMS

1. $L^{-1}\left\{\frac{3s-12}{s^2+8}\right\} = L^{-1}\left\{\frac{3s}{s^2+8} - \frac{12}{s^2+8}\right\}$

$$3\, L^{-1}\left\{\frac{s}{s^2+8}\right\} - 2\sqrt{2}\, L^{-1}\left\{\frac{3\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$3\cos 2\sqrt{2}t - 3\sqrt{2}\, L^{-1}\left\{\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$\underline{\underline{3\cos 2\sqrt{2}t - 3\sqrt{2}\sin 2\sqrt{2}t}}$$

2. $L^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\} = L^{-1}\left\{\frac{s-2\sqrt{s}+1}{s^2}\right\}$

$$L^{-1}\left\{\frac{1}{s} - \frac{2}{s^{3/2}} + \frac{1}{s^2}\right\} =$$

$$1 - \frac{2\cdot 2\sqrt{t}}{\sqrt{\pi}} + t = \underline{\underline{1 - \frac{4\sqrt{t}}{\sqrt{\pi}} + t}}$$

3. $L^{-1}\left\{ \frac{1}{s} \sin\left(\frac{1}{s}\right) \right\}$

$$L^{-1}\left\{ \frac{1}{s}\left[\frac{1}{s} - \frac{\left(\frac{1}{s}\right)^3}{3!} + \frac{\left(\frac{1}{s}\right)^5}{5!} - \cdots \right] \right\}$$

$$L^{-1}\left\{ \frac{1}{s^2} - \frac{1}{3!\,s^4} + \frac{1}{5!\,s^6} - \cdots \right\} \qquad L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

$$\frac{t}{1!} - \frac{t^3}{3!\,3!} + \frac{t^5}{5!\cdot 5!} - \cdots$$

$$\underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{[(2n-1)!]^2} t^{2n-1}}}$$

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$$= L^{-1}\left\{\frac{1}{(s+1)^4} - \frac{1}{(s+1)^5}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^4} - \frac{1}{s^5}\right\}$$

$$= e^{-t}\left[\frac{t^3}{3!} - \frac{t^4}{4!}\right] = \frac{e^{-t}}{24}\left[4t^3 - t^4\right]$$

5. $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

$$= \frac{-1}{t} L^{-1}\left\{\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(-\frac{4}{s^3}\right)\right\}$$

$$= +\frac{1}{t} L^{-1}\left\{\frac{4s^4}{s^3(s^4+4)}\right\} = \frac{+1}{t} L^{-1}\left\{\frac{4s}{(s^2)^2+4s^2+4-4s^2}\right\}$$

$$= \frac{+4}{t} L^{-1}\left\{\frac{s}{(s^2+2)^2-(2s)^2}\right\} = +\frac{4}{t} L^{-1}\left\{\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

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$$+\frac{4}{t}\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

$$+\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{s^2+2s+2-s^2+2s-2}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

$$+\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right\}$$

$$+\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{\cancel{(s^2+2s+2)}}{\cancel{(s^2+2s+2)}(s^2-2s+2)}-\frac{\cancel{(s^2-2s+2)}}{(s^2+2s+2)\cancel{(s^2-2s+2)}}\right\}$$

$$+\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+1+1}-\frac{1}{s^2+2s+1+1}\right\}$$

$$+\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$+\frac{1}{t}\,e^{t}\sin t+\frac{1}{t}\,e^{-t}\sin t$$

$$+\frac{\sin t}{t}\,(e^{t}-e^{-t}) = +\frac{2\sin t\,\sinh t}{t}.$$

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$$-\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{1\cdot 2s}{s^2-a^2}-\frac{1\cdot 2s}{s^2}\right\}$$

$$-\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{2s}{s^2-a^2}-\frac{2}{s}\right\} = -\frac{2}{t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}-\frac{1}{s}\right\}$$

$$= -\frac{2}{t}\left[\cosh at-1\right] = \left[1-\cosh at\right]\frac{2}{t}$$

10

7. $L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{-1}{1+\left(\frac{s+3}{2}\right)^2}\cdot\frac{1}{2}\right\}$

$= \frac{1}{t} L^{-1}\left\{\frac{4}{4+(s+3)^2}\cdot\frac{1}{2}\right\}$

$= \frac{1}{t} L^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\} = \frac{1}{t} e^{-3t}\sin 2t$

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$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{2}\frac{d}{ds}\left(\frac{-1}{s^2+1}\right)\right\}$

$= \frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{-1}{s^2+1}\right)\right\}$

$= \frac{1}{2}(-t) L^{-1}\left\{\frac{-1}{s^2+1}\right\} = +\frac{t}{2}\sin t$

9. $L^{-1}\left\{\frac{1}{s^2}\left[\frac{s+1}{s^2+1}\right]\right\} = \int_0^t\int_0^t L^{-1}\left\{\frac{s+1}{s^2+1}\right\} dt\, dt$

$= \int_0^t\int_0^t \left(L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}\right) dt\, dt$

$= \int_0^t\int_0^t (\cos t + \sin t)\, dt\, dt$

$$f(t) = \begin{cases} 2, & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$$

<u>Verification:</u> Expressing $f(t)$ using unit-step $\underline{fn}$

$$f(t) = 2 + (0-2)\,u(t-\pi) + (\sin t - 0)\,u(t-2\pi)$$

$$= 2 - 2u(t-\pi) + \sin t \; u(t-2\pi)$$

<u>Extra problems</u> : <u>CLASS WORK PROBLEMS</u>

1. $$\mathcal{L}^{-1}\left\{\frac{3s-12}{s^2+8}\right\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+8} - \frac{12}{s^2+8}\right\}$$

$$3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+8}\right\} - 2\sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{3\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$3\cos 2\sqrt{2}t - 3\sqrt{2}\,\mathcal{L}^{-1}\left\{\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$\underline{\underline{3\cos 2\sqrt{2}t - 3\sqrt{2}\sin 2\sqrt{2}t}}$$

2. $$\mathcal{L}^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\} = \mathcal{L}^{-1}\left\{\frac{s-2\sqrt{s}+1}{s^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s^{3/2}} + \frac{1}{s^2}\right\} = \cancel{1+t}$$

$$1 - \frac{2\cdot 2\sqrt{t}}{\sqrt{\pi}} + t = \underline{\underline{1 - \frac{4\sqrt{t}}{\sqrt{\pi}} + t}}$$

3. $L^{-1}\left\{\frac{1}{s}\sin\left(\frac{1}{s}\right)\right\}$

$L^{-1}\left\{\frac{1}{s}\left[\frac{1}{s}-\frac{\left(\frac{1}{s}\right)^3}{3!}+\frac{\left(\frac{1}{s}\right)^5}{5!}-\cdots\right]\right\}$

$L^{-1}\left\{\frac{1}{s^2}-\frac{1}{3!\,s^4}+\frac{1}{5!\,s^6}-\cdots\right\}$ $\qquad L^{-1}\left\{\frac{1}{s^n}\right\}=\frac{t^{n-1}}{(n-1)!}$

$\frac{t}{1!}-\frac{t^3}{3!\,3!}+\frac{t^5}{5!\cdot 5!}-\cdots$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{[(2n-1)!]^2}t^{2n-1}$$

4. $L^{-1}\left\{\frac{s}{(s+1)^5}\right\}=L^{-1}\left\{\frac{s+1-1}{(s+1)^5}\right\}$

$=L^{-1}\left\{\frac{1}{(s+1)^4}-\frac{1}{(s+1)^5}\right\}=e^{-t}L^{-1}\left\{\frac{1}{s^4}-\frac{1}{s^5}\right\}$

$=e^{-t}\left[\frac{t^3}{3!}-\frac{t^4}{4!}\right]=\frac{e^{-t}}{24}\left[4t^3-t^4\right]$

5. $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

$=\frac{-1}{t}L^{-1}\left\{\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{-4}{s^3}\right)\right\}$

$=\frac{+1}{t}L^{-1}\left\{\frac{4s^4}{s^3(s^4+4)}\right\}=\frac{+1}{t}L^{-1}\left\{\frac{4s}{(s^2)^2+4s^2+4-4s^2}\right\}$

$=\frac{+4}{t}L^{-1}\left\{\frac{s}{(s^2+2)^2-(2s)^2}\right\}=\frac{+4}{t}L^{-1}\left\{\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right\}$

$$= \int_0^t \sin t - \cos t \Big|_0^t dt = \int_0^t (\sin t - \cos t + 1)\, dt$$

$$= -\cos t - \sin t + t \Big|_0^t$$

$$= -\cos t - \sin t + t + 1$$

10. $$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} dt$$

$$\frac{1}{a}\int_0^t \sin at \, dt = \frac{1}{a}\, \frac{(-\cos at)}{a}\Big|_0^t$$

$$= \frac{1}{a^2}(-\cos at + 1) = \frac{1-\cos at}{a^2}$$

11. $$\mathcal{L}^{-1}\left\{\frac{e^{-4s} - e^{-7s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-7s}}{s^2}\right\}$$

$$= (t-4)\, u(t-4) - (t-7)\, u(t-7)$$

Expressing it in terms of discontinuous fn

$$f(t) = \begin{cases} 0 & 0<t<4 \\ t-4 & 4<t<7 \\ 3 & t>7 \end{cases}$$

12. $$\mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4e^{-3s}}{s^2}\right\} = 3 - 4(t-1)u(t-1) + 4(t-3)u(t-3)$$

$$f(t) = \begin{cases} 3 & 0<t<1 \\ -4t+7 & 1<t<3 \\ -5 & t>3 \end{cases}$$

13. $\mathcal{L}^{-1}\left\{\frac{s^2+9s-9}{s(s^2-9)}\right\} = \mathcal{L}^{-1}\left\{\frac{\cancel{s^2+9s+9-18}}{\cancel{s(s^2-9)}}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s-9}{s(s^2-9)}\right\} = \mathcal{L}^{-1}\left\{\frac{(s^2-9)}{s(s^2-9)} + \frac{9\cancel{s}}{\cancel{s}(s^2-9)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{9}{s^2-9}\right\} = \underline{\underline{1+3\sinh 3t}}$$

14. $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$

$$\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$s+1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

Comparing Coefficients of s^3 : $0 = A+C \rightarrow$ (1)

" " s^2 : $0 = B+D \rightarrow$ (2)

" " s : $1 = 4A+C \rightarrow$ (3)

" " Constant : $1 = 4B+D \rightarrow$ (4)

Solving equations (1) & (3)

$$A+\cancel{C}=0$$
$$\underline{-4A+\cancel{C}=1}$$
$$-3A=-1$$

$\boxed{A=1/3}$ & $\boxed{C=-1/3}$

Solving equations (2) & (4)

$$B+\cancel{D}=0$$
$$\underline{-4B+\cancel{D}=1}$$
$$-3B=-1$$

$\boxed{B=1/3}$ & $\boxed{D=-1/3}$

$$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

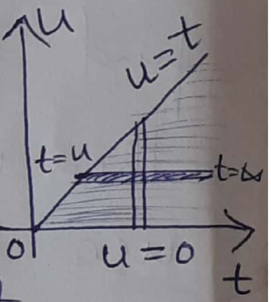
$$\frac{1}{3}\left[\cos t + \sin t - \cos 2t - \frac{1}{2}\sin 2t\right]$$

CONVOLUTION THEOREM PROOF [Self learning component]

Definition of Convolution: The convolution of two functions $f(t)$ and $g(t)$ denoted by $f(t)*g(t)$ or $(f*g)t$ is defined by $f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$.

Convolution Theorem: If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du = f(t)*g(t)$$



Proof:- By the definition of LT,

$$L\{f*g\} = \int_0^\infty e^{-st}(f*g)\,dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)\,g(t-u)\,du\right]dt$$

$$= \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st} f(u)\,g(t-u)\,du\,dt$$

changing the order of integration

$$L\{f*g\} = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)\,g(t-u)\,dt\,du$$

Put $t-u = v$

$dt = dv$ ($\because$ u is constant)

$t = u,\ t = \infty$

$t = \infty,\ v = \infty$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u) g(v)\, dv\, du$$

$$= \int_0^\infty e^{-su} f(u)\, du \cdot \int_0^\infty e^{-sv} g(v)\, dv$$

$$= F(s)\, G(s)$$

$$\Rightarrow L^{-1}\{F(s)\, G(s)\} = \int_0^t f(u)\, g(t-u)\, du = f(t) * g(t)$$

Note: ① In the problems if it is mentioned to find inverse LT using convolution theorem, then use convolution theorem only, otherwise use any other method to find inverse LT i.e partial fractions or properties of inverse LT or convolution theorem which ever is easy to find the solution.

② Choose $G(s)$ as simple function, s.t integration will be easier.

I) Use convolution theorem to find inverse Laplace transform of the following.

1) $\frac{1}{(s+1)(s+9)^2}$

Solution: Let $F(s)\,G(s) = \frac{1}{(s+1)(s+9)^2}$ such that

$$F(s) = \frac{1}{(s+9)^2} \quad \text{and} \quad G(s) = \frac{1}{s+1}$$

$$f(t) = t e^{-9t} \quad , \quad g(t) = e^{-t}$$

$$\therefore L^{-1}\{F(s)\,G(s)\} = \int_0^t u e^{-9u} e^{-(t-u)} du$$

$$= \int_0^t u e^{-9u} e^{-t} e^{+u} du$$

$$= \int_0^t e^{-t} u e^{-8u} du$$

$$= e^{-t}\left[u\left(\frac{e^{-8u}}{-8}\right) - (1)\left(\frac{e^{-8u}}{(-8)^2}\right)\right]_0^t$$

$$= -\frac{e^{-t}}{64}\left[8ue^{-8u} + e^{-8u}\right]_0^t$$

$$= -\frac{e^{-t}}{64}\left\{\left[8te^{-8t} + e^{-8t}\right] - \left[0+1\right]\right\} = \frac{e^{-t}}{64}\left[8te^{-8t} + e^{-8t} - 1\right]$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)(s+9)^2}\right\} = \frac{1}{64}\left[e^{-t} - 8te^{-9t} - e^{-9t}\right]$$

2) $\dfrac{s^2}{(s^2+a^2)^2}$

Sol: Let $L^{-1}\{F(s)*G(s)\} = L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$ s.t

$$F(s) = \frac{s}{s^2+a^2}, \quad G(s) = \frac{s}{(s^2+a^2)}$$

$\Rightarrow f(t) = \cos at, \quad g(t) = \cos at$

$$\therefore L^{-1}\{F(s)*G(s)\} = \int_0^t \cos au \cos a(t-u)\,du$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$= \frac{1}{2}\int_0^t \cos au \cdot \cos(at-au)\,du$$

$$= \frac{1}{2}\int_0^t \cos at + \cos(2au-at)]\,du$$

$$= \frac{1}{2}\left[u\cos at + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} - \frac{\sin(-at)}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\left[\sin at + at\cos at\right].$$

3) $\dfrac{1}{(s^2+4s+13)^2}$

Sol: Let $F(s)\,G(s) = \dfrac{1}{(s^2+4s+13)^2} = \dfrac{1}{[(s+2)^2+9]^2}$ s.t

$F(s) = \dfrac{1}{(s+2)^2+9}$ & $G(s) = \dfrac{1}{(s+2)^2+9}$

$\Rightarrow f(t) = e^{-2t}\dfrac{\sin 3t}{3}$ & $g(t) = e^{-2t}\dfrac{\sin 3t}{3}$

$\therefore L^{-1}\left\{\dfrac{1}{(s^2+4s+13)^2}\right\} = L^{-1}\left\{F(s)G(s)\right\} =$

$$= \int_0^t \left[e^{-2u}\frac{\sin 3u}{3} \times e^{-2(t-u)}\frac{\sin 3(t-u)}{3}\right]du$$

$$= \frac{1}{9}\int_0^t e^{-2t}\sin 3u \,\sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{9}\times\frac{1}{2}\int_0^t \left\{\cos[3u-(3t-3u)] - \cos[3u+(3t-3u)]\right\}du$$

using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \frac{e^{-2t}}{18}\int_0^t \left[\cos(6u-3t) - \cos(3t)\right]du$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin(6u-3t)}{6} - u\cos 3t\right]_0^t$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$= \frac{e^{-2t}}{54}\left[\sin 3t - 3t\cos 3t\right]$$

4) Verify Convolution theorem for the functions $f(t) = t$ and $g(t) = \cos t$.

Sol W.K.T $L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$$L^{-1}\{F(s)G(s)\} = \int_0^t u\cos(t-u)\, du$$

$$= \left\{u\left[\frac{\sin(t-u)}{-1}\right] - (1)\left[-\frac{\cos(t-u)}{(-1)^2}\right]\right\}_0^t$$

$$= [0+1] - [0+\cos t] = 1 - \cos t = RHS$$

Here $f(t) = t \Rightarrow L\{f(t)\} = \frac{1}{s^2}$ & $g(t) = \cos t$

$$F(s) = \frac{1}{s^2} \quad \& \quad L\{g(t)\} = \frac{s}{s^2+1}$$

$$G(s) = \frac{s}{s^2+1}$$

$$LHS = L^{-1}\left\{\frac{1}{s^2} \times \frac{s}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{1/s^2+1}{s}\right\} = \int_0^t f(t)\, dt$$

using $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\, dt$

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$$LHS = \int_0^t \sin t \, dt = -[\cos t]_0^t = 1 - \cos t$$

$\therefore$ LHS = RHS. Hence Convolution theorem is verified.

5) $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} \text{ where } a \neq b.$$

Let $F(S)\,G(S) = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ s.t

$F(S) = \dfrac{s}{s^2+a^2}$ & $G(S) = \dfrac{s}{s^2+b^2}$

$f(t) = \cos at$ & $g(t) = \cos bt$

$$L^{-1}\{F(S)\,G(S)\} = \int_0^t f(u)\, g(t-u)\, du \text{ by Convolution thm}$$

$$\therefore L^{-1}\left\{\frac{s}{s^2+a^2} \times \frac{s}{s^2+b^2}\right\} = \int_0^t \cos au \, \cos b(t-u)\, du$$

$$= \int_0^t \cos au \, \cos(bt - bu)\, du$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{2}\int_0^t \left\{\cos\left[au + bt - bu\right] + \cos\left[au - bt + bu\right]\right\} du$$

$$= \frac{1}{2}\int_0^t \left\{\cos\left[(a-b)u + bt\right] + \cos\left[(a+b)u - bt\right]\right\} du$$

$$= \frac{1}{2}\left\{ \frac{\sin[(a-b)u+bt]}{(a-b)} + \frac{\sin[(a+b)u-bt]}{(a+b)} \right\}_0^t$$

$$= \frac{1}{2}\left\{ \frac{\sin at}{(a-b)} + \frac{\sin(at)}{(a+b)} - \frac{\sin bt}{(a-b)} - \frac{\sin bt}{a+b} \right\}$$

$$= \frac{1}{2}\left\{ \frac{(a+b)\sin at + (a-b)\sin at}{(a^2-b^2)} \right\} - \frac{1}{2}\left\{ \frac{(a+b)\sin bt - (a-b)\sin bt}{a^2-b^2} \right\}$$

$$= \frac{1}{2}\left\{ \frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2} \right\}$$

$$L^{-1}\left\{ \frac{s^2}{(s^2+a^2)(s^2+b^2)} \right\} = \left[\frac{a\sin at - b\sin bt}{a^2-b^2} \right] \text{ where } a \neq b.$$

i.e $$L^{-1}\left\{ \frac{s^2}{(s^2+a^2)(s^2+b^2)} \right\} = \frac{a\sin at - b\sin bt}{a^2-b^2} \quad \text{given } a \neq b.$$

6) $\frac{4s+5}{(s-1)^2(s+2)}$

Sol: Let $F(s)G(s) = \frac{4s+5}{(s-1)^2(s+2)}$ s.t

Let $F(s) = \frac{1}{s+2}$, $G(s) = \frac{4s+5}{(s-1)^2}$ (or interchange F(s) & G(s) & try)

$f(t) = e^{-2t}$, $g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{4s+5}{(s-1)^2}\right\}$

$g(t) = L^{-1}\left\{\frac{4(s-1)+9}{(s-1)^2}\right\} = e^{t} L^{-1}\left\{\frac{4s+9}{s^2}\right\}$

$\therefore g(t) = e^{t}(4+9t)$

W.K.T $L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$L^{-1}\left\{\frac{1}{(s+2)} \frac{4s+5}{(s-1)^2}\right\} = \int_{u=0}^{t} e^{-2u} e^{(t-u)} [4+9(t-u)] du$

$= e^{t} \int_0^t e^{-3u}(4+9t-9u) du$

$= e^{t} \int_0^t \underset{v}{e^{-3u}} \underset{u}{(4+9t-9u)} du$

$= e^{t} \left[(4+9t-9u) \frac{e^{-3u}}{-3} - (-9) \frac{e^{-3u}}{9} \right]_{u=0}^{t}$

$L^{-1}\left\{\frac{4s+5}{(s+2)(s-1)^2}\right\} = e^{t}\left[\frac{1}{3} - \frac{1}{3}e^{-3t} + 3t\right]$

Class work problems on Convolution theorem

1) $\frac{1}{(s+1)(s^2+1)}$ Use convolution theorem find inverse LT of $\frac{1}{(s+1)(s^2+1)}$.

Sol: Let $F(s)\,G(s) = \frac{1}{(s+1)(s^2+1)}$

Let $F(s) = \frac{1}{s^2+1}$, $G(s) = \frac{1}{s+1}$

$f(t) = \sin t$, $g(t) = e^{-t}$

$\therefore L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$L^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \int_0^t \sin u \; e^{-(t-u)} du$

$= e^{-t} \int_0^t e^{u} \sin u \, du$

$= e^{-t} \left\{ \frac{e^{u}}{1+1} [1 \cdot \sin u - \cos u] \right\}_0^t$

using $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

$\therefore L^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) - \frac{1}{2}(-1)\right]$

$$\boxed{L^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \frac{1}{2}\left[e^{-t} + \sin t - \cos t\right]}$$

2. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$

Let $F(s) = \frac{1}{s^2+4}$, $f(t) = \frac{1}{2}\sin 2t$

$G(s) = \frac{1}{(s+1)^2}$, $g(t) = e^{-t}t$.

Using convolution theorem.

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \frac{1}{2}\sin 2u\, (t-u)\, e^{-(t-u)}\, du$$

$$= \frac{e^{-t}}{2}\left[\int_0^t e^{u}\sin 2u \cdot t - \int_0^t e^{u}\sin 2u \cdot u\, du\right]$$

$$= \frac{e^{-t}}{2}\left[\operatorname{Im}\int_0^t e^{u}e^{2iu}\cdot t\, du - \operatorname{Im}\int_0^t e^{u}\cdot e^{2iu}\cdot u\, du\right]$$

$$= \frac{e^{-t}}{2}\left[t\operatorname{Im}\int_0^t e^{u(1+2i)}\, du - \operatorname{Im}\int_0^t u\, e^{(1+2i)u}\, du\right]$$

$$= \frac{e^{-t}}{2}\left[t\cdot\operatorname{Im}\left(\frac{e^{u(1+2i)}}{1+2i}\right)\Bigg|_0^t - \operatorname{Im}\left[u\cdot\frac{e^{(1+2i)u}}{1+2i} - 1\cdot\frac{e^{(1+2i)u}}{(1+2i)^2}\right]_0^t\right.$$

$$= \frac{e^{-t}}{2}\left[t\operatorname{Im}\left(\frac{e^{(1+2i)t}-1}{1+2i}\right) - \operatorname{Im}\left[\frac{t\,e^{(1+2i)t}}{1+2i} - \frac{e^{(1+2i)t}}{(1+2i)^2} + \frac{1}{(1+2i)^2}\right]\right.$$

$$= \frac{e^{-t}}{2}\left[\cancel{t\operatorname{Im}\left(\frac{e^{t}(\cos 2t + i\sin 2t)-1}{1+2i}\right)}\right. -$$

(18)

$$\frac{e^t}{2}\left[t\,\mathrm{Im}\left(\frac{e^t(\cos 2t+i\sin 2t)-1}{1+2i}\right)-\right.$$

$$\left.\mathrm{Im}\left(\frac{te^t(\cos 2t+i\sin 2t)}{1+2i}-\frac{e^t(\cos 2t+i\sin 2t)}{(1+2i)^2}+\frac{1}{(1+2i)^2}\right)\right]$$

$$\frac{e^t}{2}\left[t\,\mathrm{Im}\left(\frac{\{e^t(\cos 2t+i\sin 2t)-1\}(1-2i)}{5}\right)-\right.$$

$$\mathrm{Im}\left(\frac{te^t(\cos 2t+i\sin 2t)(1-2i)}{5}-\frac{e^t(\cos 2t+i\sin 2t)(4i+3)}{(4i-3)(4i+3)}\right.$$

$$\left.\left.+\frac{(4i+3)}{(4i+3)(4i-3)}\right)\right]$$

$$\frac{e^t}{2}\left[\cancel{t}\left(-\frac{2e^t\cos 2t\,t}{\cancel{5}}+\frac{te^t\cancel{\sin 2t}}{5}+\frac{2t}{5}\right)+\frac{2e^t t\cancel{\cos 2t}}{5}\right.$$

$$\left.-\frac{te^t\cancel{\sin}2t}{5}+\frac{4e^t\cos 2t}{25}+\frac{3\sin 2t\,e^t}{25}-\frac{4}{25}\right]$$

$$\frac{e^t}{2}\left[\frac{2t}{5}+\frac{4e^t\cos 2t}{25}+\frac{3\sin 2t\,e^t}{25}-\frac{4}{25}\right]$$

$$\frac{e^t}{50}\left[10t+4e^t\cos 2t+3e^t\sin 2t-4\right]$$

3. $L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$ using Convolution theorem.

$$L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$F(s) = \frac{1}{(s+2)^2} = e^{-2t}t = f(t)$$

$$G(s) = \frac{1}{s-2} = e^{2t} = g(t)$$

$$L^{-1}\{F(s)\, G(s)\} = \int_0^t e^{-2u}\cdot u \cdot e^{2(t-u)} du$$

$$= \int_0^t u \cdot e^{-2u+2t-2u} du$$

$$= e^{2t} \int_0^t u\, e^{-4u} du$$

$$= e^{2t}\left[u\cdot \frac{e^{-4u}}{-4} - (1)\frac{e^{-4u}}{16}\Big|_0^t \right]$$

$$= e^{2t}\left[\frac{t\, e^{-4t}}{-4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right]$$

$$= \frac{(-4t\, e^{-2t} - e^{-2t} + e^{2t})}{16}$$

$$= \frac{1}{16}\left[e^{2t} - (4t+1)\, e^{-2t}\right]$$

Solution of Differential Equations by Laplace Transforms

Procedure :

I step: Take LT on both sides

II step: Convert LT eqn to an algebraic eqn using LT of derivatives and boundary conditions.

III step: By grouping find $y(t)$.

$$L\{y'(t)\} = sL\{y(t)\} - y(0) \text{ OR } sF(s) - f(0)$$

$$L\{y''(t)\} = s^2 L\{y(t)\} - sy(0) - y'(0)$$

$$L\{y'''(t)\} = s^3 L\{y(t)\} - s^2 y(0) - sy'(0) - y''(0)$$

$$L\{y^{IV}(t)\} = s^4 L\{y(t)\} - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)$$

1) Solve the differential equation using LT.

$$y'' - 3y' + 2y = 12e^{-2t}, \quad y(0) = 2, \quad y'(0) = 6$$

Sol: $L\{y''(t)\} - 3L\{y'(t)\} + 2L\{y(t)\} = 12\, L\{e^{-2t}\}$

$$s^2 L\{y(t)\} - s\,y(0) - y'(0) - 3\left[sL\{y(t)\} - y(0)\right] + 2L\{y(t)\} = 12 \times \frac{1}{s+2}$$

$$s^2 L\{y(t)\} - 2s - \cancel{6} - 3sL\{y(t)\} + \cancel{6} + 2L\{y(t)\} = \frac{12}{s+2}$$

$$L\{y(t)\}\left[s^2 - 3s + 2\right] = \frac{12}{s+2} + 2s$$

$$L\{y(t)\} = \frac{12}{(s+2)(s^2-3s+2)} + 2\frac{s}{(s^2-3s+2)} \quad \text{--- (1)}$$

consider $\dfrac{12}{(s+2)(s-2)(s-1)} = \dfrac{A}{s+2} + \dfrac{B}{s-2} + \dfrac{C}{s-1}$

$A = 1, B = 3, C = -4$

$\dfrac{s}{s^2-3s+2} = \dfrac{A}{s-2} + \dfrac{B}{s-1} \Rightarrow A = 4, B = -2$

$\therefore$ From (1)

$$y(t) = L^{-1}\left\{\frac{1}{s+2} + \frac{3}{s-2} - \frac{4}{s-1}\right\} + 2L^{-1}\left\{\frac{4}{s-2} - \frac{2}{s-1}\right\}$$

$$\Rightarrow y(t) = e^{-2t} + 3e^{2t} - 4e^{t} + 8e^{2t} - 4e^{t}$$

$$\therefore \boxed{y(t) = e^{-2t} + 11e^{2t} - 8e^{t}}$$

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2) solve the following differential eqn by the method of LT.

$$y'' + y = f(t) \; ; \; y(0)=1, \; y'(0)=0 \text{ and}$$

$$f(t) = \begin{cases} 3 & 0 \le t \le 4 \\ 2t-5 & t > 4 \end{cases}$$

Sol: $L\{y''(t)\} + L\{y(t)\} = L\{f(t)\} \rightarrow (1)$

where $L\{f(t)\} = L\{3 + ((2t-5)-3)u(t-4)\}$

$$= \frac{3}{s} + L\{(2t-8)u(t-4)\} \rightarrow (2)$$

Consider $L\{(2t-8)u(t-4)\} = e^{-4s} F(s)$

Here $f(t-4) = 2t-8 \Rightarrow f(t) = 2(t+4) - 8$

$$= 2t + 8 - 8$$

$$F(s) = \frac{2}{s^2} \quad \text{Subst}$$

$\therefore L\{(2t-8)u(t-4)\} = e^{-4s} \frac{2}{s^2}$ Subst in (2)

$\therefore L\{f(t)\} = \frac{3}{s} + e^{-4s} \frac{2}{s^2}$ $\quad \therefore$ eqn (1) becomes

$$s^2 L\{y(t)\} - s y(0) - y'(0) + L\{y(t)\} = \frac{3}{s} + \frac{2}{s^2} e^{-4s}$$

$$L\{y(t)\}(s^2+1) - s - 0 = \frac{3}{s} + \frac{2}{s^2} e^{-4s}$$

$$L\{y(t)\} = \frac{3}{s(s^2+1)} + \frac{2}{s^2(s^2+1)} e^{-4s} + \frac{s}{s^2+1}$$

$$y(t) = L^{-1}\left\{\frac{3+s^2}{s(s^2+1)} + \frac{2}{s^2(s^2+1)}e^{-4s}\right\} \rightarrow (3)$$

consider $\frac{3+s^2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow A = 3, B = -2, C = 0$

$$\frac{e^{-4s}}{s^2(s^2+1)} = \frac{e^{-4s}\left[(s^2+1)-s^2\right]}{s^2(s^2+1)}$$

$$\therefore L^{-1}\left\{\frac{e^{-4s}}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{e^{-4s}}{s^2} - \frac{e^{-4s}}{s^2+1}\right\}$$

$$= (t-4)u(t-4) - \sin(t-4)u(t-4)$$

From (3)

$$\therefore y(t) = L^{-1}\left\{\frac{3}{s} - \frac{2s}{s^2+1} + 2\frac{e^{-4s}}{s^2} - 2\frac{e^{-4s}}{s^2+1}\right\}$$

$$\boxed{y(t) = 3 - 2\cos t + 2(t-4)u(t-4) - 2\sin(t-4)u(t-4)}$$

3) solve $\frac{dy}{dt} + 3y + 2\int_0^t y\,dt = t$ for $y(0)=0$

Sol: $L\{y'(t)\} + 3L\{y(t)\} + 2L\{\int_0^t y\,dt\} = L\{t\}$

$sL\{y(t)\} - \cancel{y(0)}^{\nearrow 0} + 3L\{y(t)\} + 2\frac{F(s)}{s} = \frac{1}{s^2}$

$\because L\{\int_0^t y\,dt\} = \frac{F(s)}{s}$

$L\{y(t)\}\left(s+3+\frac{2}{s}\right) = \frac{1}{s^2}$

$L\{y(t)\} = \frac{s}{s^2(s^2+3s+2)}$

$y(t) = L^{-1}\left\{\frac{1}{s(s^2+3s+2)}\right\} = L^{-1}\left\{\frac{1/(s^2+3s+2)}{s}\right\} = \int_0^t f(t)\,dt$ — ①

where $f(t) = L^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = L^{-1}\left\{\frac{-1}{s+2} + \frac{1}{s+1}\right\}$ by partial fractions

$f(t) = -e^{-2t} + e^{-t}$ subst in ①

$\therefore y(t) = \int_0^t \left[-e^{-2t} + e^{-t}\right]dt$

$= \left[-\frac{e^{-2t}}{-2} - e^{-t}\right]_0^t = \left[\frac{e^{-2t}}{2} - e^{-t}\right] - \left[\frac{1}{2} - 1\right]$

$$\boxed{y(t) = \frac{e^{-2t}}{2} - e^{-t} + \frac{1}{2}}$$

4) solve, $ty'' - (2+t)y' + 3y = t-1$, $y(0)=0$

Sol: $L\{ty''\} - 2L\{y'(t)\} - 2L\{ty'\} + 3L\{y(t)\} = L\{t-1\}$

$$-\frac{d}{ds}\left[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}\right] - 2\left[sY(s) - \cancel{y(0)}^{0}\right]$$

$$+\frac{d}{ds}\left[sY(s) - \cancel{y(0)}^{0}\right] + 3Y(s) = \frac{1}{s^2} - \frac{1}{s}.$$

Using the given boundary conditions & simplifying we get,

$$-\left[s^2 Y'(s) + 2sY(s)\right] - 2sY(s) + \left[sY'(s) + Y(s)\right]$$

$$+ 3Y(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$-s^2 Y'(s) - 4sY(s) + sY'(s) + 4Y(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$-Y'(s)\, s\cancel{(s-1)} - 4Y(s)\cancel{(s-1)} = \frac{(1-s)}{s^2} \text{ or } \frac{-\cancel{(s-1)}}{s^2}$$

$$sY'(s) + 4Y(s) = \frac{1}{s^2} \quad \div \text{ by } s$$

$$Y'(s) + \frac{4}{s}Y(s) = \frac{1}{s^3}$$ is a l.d.e^n of the form $\frac{dy}{dx} + Py = Q$

where $IF = e^{\int P dx}$

sol^n is $y(IF) = \int Q(IF)\,dx$

$$IF = e^{\int \frac{4}{s} ds}$$

$$= e^{4\log s} = s^4$$

$$\therefore Y(s) \cdot s^4 = \int \frac{1}{s^3} s^4 ds + c$$

$Y(s)\, s^4 = \frac{s^2}{2} + c$

$Y(s) = \frac{1}{2s^2} + \frac{c}{s^4}$

$\therefore y(t) = L^{-1}\left\{ \frac{1}{2s^2} + \frac{c}{s^4} \right\}$

$$\boxed{\therefore y = \frac{1}{2}t + c\frac{t^3}{3!}}$$

5) Solve $\frac{d^2y}{dt^2} + 9y = \cos 2t$ given $y(0) = 1$, $y(\frac{\pi}{2}) = -1$ using LT.

Sol: Since $y'(0)$ is not given, we assume $y'(0) = a$

$\therefore L\{y''(t)\} + 9L\{y(t)\} = L\{\cos 2t\}$

$s^2 L\{y(t)\} - s\,y(0) - y'(0) + 9L\{y(t)\} = \frac{s}{s^2+4}$

$(s^2+9)L\{y(t)\} = s + a + \frac{s}{s^2+4}$

$L\{y(t)\} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$

$L\{y(t)\} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}$

*(explanation is given below).

$\Rightarrow y(t) = \cancel{a\sin} \; L^{-1}\left\{ \frac{a}{s^2+9} + \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9} \right\}$

$$\Rightarrow y(t) = a\frac{\sin 3t}{3} + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$$

at $t = \pi/2$, $-1 = \frac{a}{3}(-1) + \frac{1}{5}(-1) + \frac{4}{5}(0)$

$$\Rightarrow \frac{a}{3} = \frac{4}{5} \Rightarrow a = \frac{12}{5}$$

solution is $\boxed{y = \frac{1}{5}\left(\cos 2t + 4\sin 3t + 4\cos 3t\right)}$

* To find $L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$

consider $\frac{1}{(s^2+4)(s^2+9)} = \frac{1}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9}$

$$\Rightarrow A = 1/5 \ \& \ B = -1/5$$

$$\therefore \frac{s}{(s^2+4)(s^2+9)} = \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9}.$$

Class Work Problems:

Use Laplace Transforms to solve the following Differential Equations:

1. $y' - y = e^{3t}$, $y(0) = 2$

Taking Laplace Transform on both sides

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{e^{3t}\}$$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$$

$$S F(s) - y(0) - F(s) = \frac{1}{S-3}$$

$$F(s)(S-1) - 2 = \frac{1}{S-3}$$

$$F(s)(S-1) = \frac{1}{S-3} + 2$$

$$F(s) = \frac{1}{(S-3)(S-1)} + \frac{2}{S-1}$$

Finding Inverse $\mathcal{L}.T$

$$\mathcal{L}^{-1}\{F(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(S-3)(S-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{S-1}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(S-3)(S-1)}\right\} + 2e^{t}$$

using partial fractions $\frac{1}{(S-3)(S-1)} = \frac{A}{S-3} + \frac{B}{S-1}$

$$1 = A(S-1) + B(S-3)$$

$$A = \frac{1}{2}, \quad B = \frac{-1}{2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1/2}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/2}{s-1}\right\} + 2e^{t}$$

$$= \frac{1}{2}e^{3t} - \frac{1}{2}e^{t} + 2e^{t}$$

$$= \frac{1}{2}e^{3t} + \frac{3}{2}e^{t}$$

2. $y'' - 6y' + 9y = 0, \; y(0) = 2, \; y'(0) = 9.$

Taking L.T on both sides,

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = 0$$

$$Y(s)(s^2 - 6s + 9) - 2s - 9 + 12 = 0$$

$$Y(s)(s^2 - 6s + 9) = 2s - 3$$

$$Y(s) = \frac{2s-3}{s^2-6s+9} = \frac{2s-3}{(s-3)^2} = \frac{2(s-3)+3}{(s-3)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s-3} + \frac{3}{(s-3)^2}\right\}$$

$$= 2e^{-3t} + 3e^{-3t}\cdot t$$

$$y(t) = (3t+2)e^{-3t}$$

3. $y'' + y = e^{-2t}\sin t \quad , \; y(0)=0, \; y'(0)=0$

Taking L.T on both sides

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{-2t}\sin t\}$$

$$s^2 Y(s) - \cancel{s y(0)}^{0} - \cancel{y'(0)}^{0} + Y(s) = \frac{1}{(s+2)^2+1}$$

$$Y(s)\,(s^2+1) = \frac{1}{(s+2)^2+1}$$

$$Y(s) = \frac{1}{[(s+2)^2+1](s^2+1)} = \frac{1}{(s^2+4s+5)(s^2+1)}$$

Taking inverse Laplace transform

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4s+5)(s^2+1)}\right\}$$

Splitting the RHS using partial fractions

$$\frac{1}{(s^2+4s+5)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5}$$

$$1 = (As+B)(s^2+4s+5) + (Cs+D)(s^2+1)$$

Comparing coeff of s^3: $0 = A+C \rightarrow (1)$

" " s^2: $0 = 4A+B+D \rightarrow (2)$

" " s: $0 = 5A+4B+C \rightarrow (3)$

" constant term: $1 = 5B+D \rightarrow (4)$

Solving equations 1 2, 3 & 4

$$4A+4B=0 \Rightarrow A + B = 0$$
$$\underline{-A \pm C = 0}$$
$$B - C = 0 \Rightarrow B = C$$

$$B + \cancel{D} = -4A$$
$$\underline{5B + D = 1}$$
$$-4B = -4A - 1$$

$$4A - 4B = -1$$
$$4A - \cancel{4}C = -1$$
$$\underline{4A + 4\cancel{4} = 0}$$
$$8A = -1$$

$$\boxed{A = -1/8} \quad \boxed{C = 1/8} \quad \boxed{B = 1/8}$$

$$D = \frac{1}{2} - \frac{1}{8} = 3/8$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{8}s + \frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s + \frac{3}{8}}{(s+2)^2+1} \right\}$$

$$= \frac{-1}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$+ \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \frac{-1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t + \frac{1}{8}e^{-2t}\sin t$$

$$= \frac{1}{8}\left[\sin t - \cos t + e^{-2t}\cos t + e^{-2t}\sin t\right]$$

4. $y'''' - 16y = 30\sin t$, $y''(0)=0$, $y''(\pi)=0$, $y'''(0) = -18$

$y'''(\pi) = -18$

$\mathcal{L}\{y''''\} - 16\mathcal{L}\{y\} = 30\mathcal{L}\{\sin t\}$

$$\left[s^4 Y(s) - s^3 y(0) - s^2 y'(0) - \cancel{s y''(0)} - y'''(0)\right] - 16Y(s) = \mathcal{L}\{30\sin t\}$$

$$(s^4-16)Y(s) - s^3 k - s^2 k_1 - 18 = \frac{30}{s^2+1}$$

$$(s^4-16)Y(s) = \frac{30}{s^2+1} + 18 + s^3 k + s^2 k_1$$

$$Y(s) = \frac{30}{(s^2+1)(s^2-4)(s^2+4)} + \frac{18}{(s^2+4)(s^2-4)} + \frac{s^3 k}{(s^2+4)(s^2-4)} + \frac{k_1(s^2)}{(s^2+4)(s^2-4)}$$

$$Y(s) = \frac{30}{(s^2-4)(s^2+4)(s^2+1)} + \frac{18}{(s^2+4)(s^2-4)} + \frac{ks(s^2+4-4)}{(s^2+4)(s^2-4)} + \frac{k_1(s^2+4-4)}{(s^2+4)(s^2-4)}$$

$$= \frac{30}{(s^2-4)(s^2+4)(s^2+1)} + \frac{18}{(s^2+4)(s^2-4)} + \frac{ks\cancel{(s^2+4)}}{\cancel{(s^2+4)}(s^2-4)} - \frac{4ks}{(s^2+4)(s^2-4)}$$

$$+ \frac{k_1\cancel{(s^2+4)}}{\cancel{(s^2+4)}(s^2-4)} - \frac{4k_1}{(s^2+4)(s^2-4)}$$

5. An impulse voltage $E\delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. Find the limit of I as $t \to 0$ where I is the current at any subsequent time t. Equation of the circuit is

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I\,dt = E\,\delta(t) \quad \text{where } I(0) = 0.$$

Soln:

$$\mathcal{L}\left\{L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I\,dt = E\,\delta(t)\right\}$$

$$L\left[sI(s) - \cancel{I(0)}^{\,0}\right] + R\,I(s) + \frac{1}{C}\frac{I(s)}{s} = E$$

$$\left[Ls + R + \frac{1}{Cs}\right] I(s) = E$$

$$I(s) = \frac{E}{Ls^2 + Rs + \frac{1}{C}} = \frac{E}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$\downarrow$ Completing the square

$$I(s) = \frac{E}{L\left[\left(s+\frac{R}{2L}\right)^2 + \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)^2\right]}$$

$$-\frac{R^2}{4L^2} + \frac{1}{LC}.$$

$$= \frac{E}{L}\,\frac{e^{-R/2L\,t}\,.\sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t}{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$