* Fourier Series are used in the Analysis of Periodic Function.

Ex: The current & Voltage.

* These periodic funtions can be analysed into their constituent components (fundamentals & harmonics) by a process called F.S.

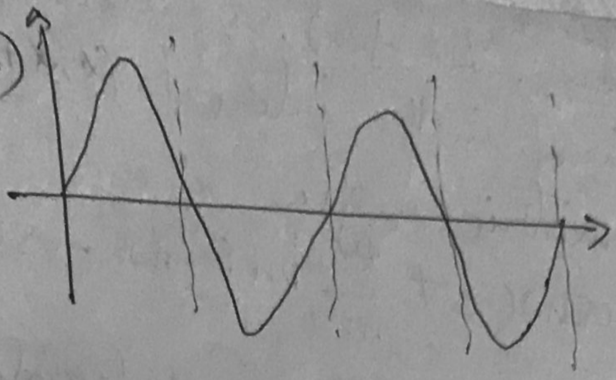
<u>Comparision</u>

Taylor	Fourier
1. Series are computed using derivatives	Series are determined using integrals
2. Series ~~are~~ is interms of a polynomial	Series is interms of Sine & Cosine.
3. Function must be cont.	Discontinuous fun can also be expressed as F.S.
4. Function can be more than one variable	Fourier series is for one variable only

∴ Fourier series is expressing periodic function interms of Sine ~~or~~ & Cosine.

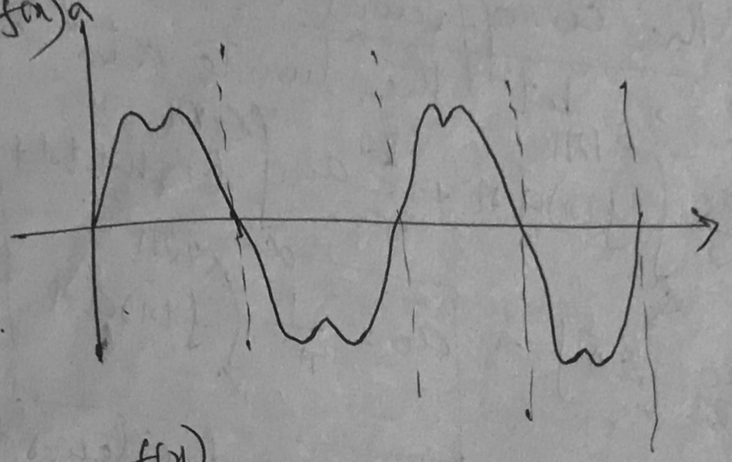
aaa

1)



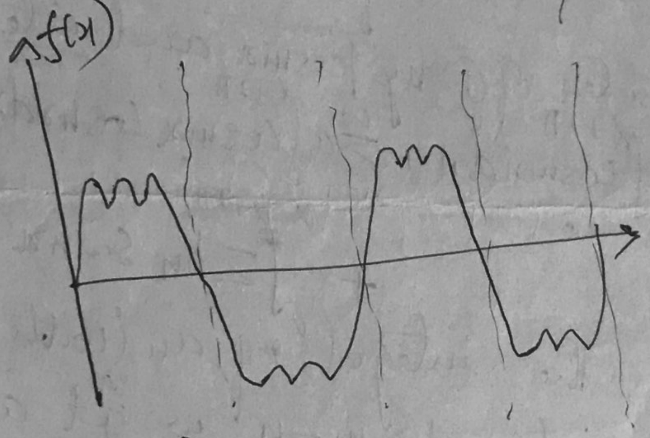
one sine component

aaa

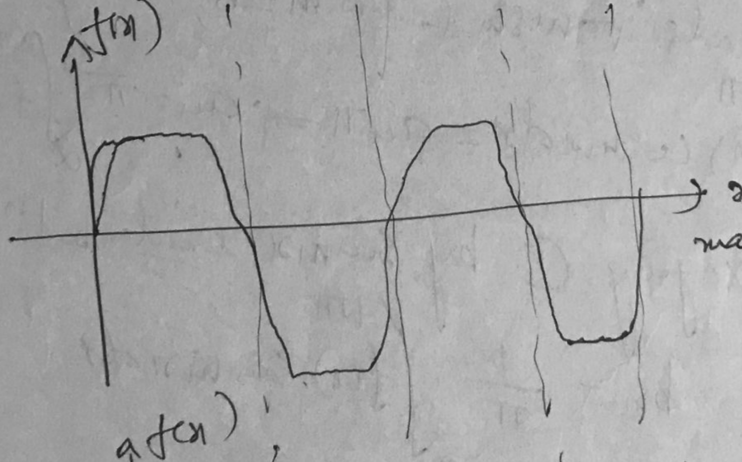


Addition of two sine components

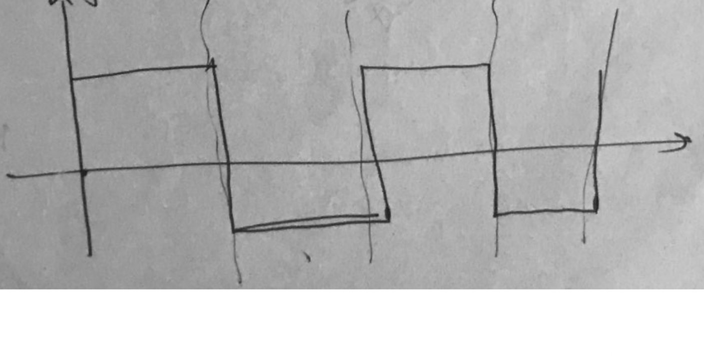
diff amplitude?



Addition of 3 sine components



Addition of many sine components



square function.

b **Euler's Formulae**: The Fourier series for the function $f(x)$, in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

desmos

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \longrightarrow (1)$$

$n = 1, 2, 3 \ldots$

$a_0, a_1, a_2 \ldots a_n, b_1, b_2 \ldots b_n$ are F. co-effts.

n a +ve integer.

where $a_1 \cos x + b_1 \sin x$ is called I harmonic, $a_2 \cos x + b_2 \sin x$ — II harmonic

a_0, a_n, b_n are known as Eulers formulae or Eulers co-effts

$a_0/2$ is called mean value, or direct current part in electrical signal

Case i: If $\alpha = 0$, the interval becomes $0 < x < 2\pi$, & formula (1) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Case ii: If $\alpha = -\pi$, the interval becomes $-\pi < x < \pi$ & formula (1) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

~~Case iii~~: If $f(x)$ is defined in the interval $(a, a+2l)$ (i.e. s.t $f(x)$ is a periodic fn of period $2l$)

3 $$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x$$

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx,\quad b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$n = 1, 2 \ldots$

$\left(\frac{n\pi}{l}\right)x$ — angular ... (1)

~~If~~ Case iii :- If $a=0$, $f(x)$ is defined over the interval $(0, 2l)$ & above formula reduces to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx. \qquad n=1,2\ldots$$

Case IV : If $a=-l$, then $f(x)$ is defined over the interval $(-l, l)$, s.t

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

Fourier Series of even function ...

Proof of Euler's formulae: Let $f(x)$, a periodic function with period 2π defined in the interval $(\alpha, \alpha+2\pi)$, be the sum of trigonometric series i.e

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (1)$$

To determine the co-efficient a_0: Integrate both sides of (1) w.r.t "x" betn the limits α to $\alpha+2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx \rightarrow 0 + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \rightarrow 0$$

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{2}[2\pi] \Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$$

$\sin(2n\pi + n\alpha) = \sin n\alpha$

" " " w.r.t x in $(\alpha, \alpha+2\pi)$

a_n: xlying both sides of (1) by $\boxed{\cos mx}$ and integ w.r.t x in $(\alpha, \alpha+2\pi)$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} \cos mx\,dx \rightarrow 0 + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \sum b_n \sin nx \cos mx\,dx \rightarrow 0$$

In RHS, except the integral of a_n (with $m=n$) all other integrals vanish & for $m=n$, we get $a_n\pi$.

$$\therefore \int_{\alpha}^{\alpha+2\pi} f(x)\cos mx\,dx = a_n\pi \Rightarrow a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos mx\,dx.$$

b_n: |||ly xlying (1) by $\sin mx$ & inte betn α to $\alpha+2\pi$

we get $$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin mx\,dx$$

$\cos(2n\pi + n\alpha) = \cos n\alpha.$

for $m=n$ we get

$\cos^2 mx \rightarrow \frac{1+\cos 2mx}{2}$ ($\cos 2mx \rightarrow 0$)

only $\frac{1}{2} \times 2\pi$ remains so we get $a_n\pi$.

6b Applications: 1) FS is used in Electromagnetic spectrum, radio waves, TV waves & cell phone waves.

2) FS is used to modulate or demodulate voice signals in audio system.

3) MP3, JP3

Periodic function: If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself i.e $f(x) = f(x+T)$ $\forall x$, then T is called the period of the function $f(x)$.

Ex: 1) $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic fns with period 2π.

2) $\sin nx = \sin(nx + 2\pi) = \sin n\left(x + \frac{2\pi}{n}\right)$ $\therefore$ $\sin nx$ is a periodic fn with period $\frac{2\pi}{n}$.

3) e^{x}, $\sinh x$ are non periodic functions.

Periodic functions (in daily life):

i) Rotating parts of machines

ii) Current & Voltage in alternating current.

iii) Motion of planets.

iv) Heart beat under normal conds is periodic in nature.

Conditions for a Fourier expansion: (Dirichlet's condition)

Any function $f(x)$ can be developed as a Fourier series provided,

i) $f(x)$ is periodic, single valued & finite (~~or bounded~~)

ii) $f(x)$ has a finite number of discontinuities in any one period.

iii) $f(x)$ has at the most a finite number of maxima and minima.

Even and odd functions:

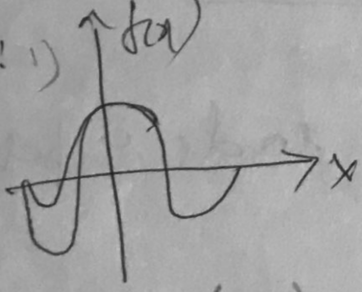
(i) A function defined in the interval $(-\pi,\pi)$ or $(-l,l)$ is said to be an even function if $f(-x) = f(x)$ & odd function if $f(-x) = -f(x)$.

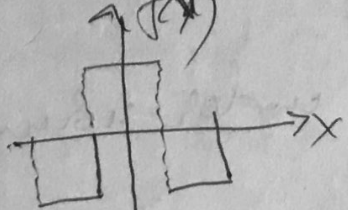
(ii) A function defined in the interval $(0,2\pi)$ is said to be an even function if $f(2\pi-x) = f(x)$ & odd function if $f(2\pi-x) = -f(x)\ \forall x$.

In $(0, 2l)$: even $\Rightarrow f(2l-x) = f(x)$

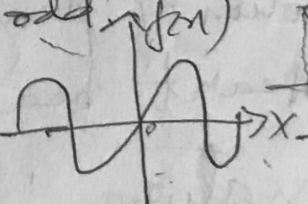
odd $\Rightarrow f(2l-x) = -f(x)$

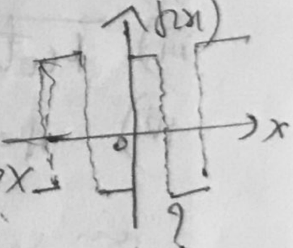
even! 1)





odd:

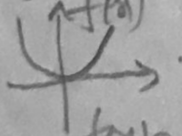




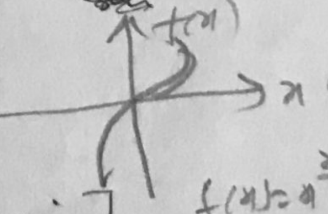
$f(x) = \sin x$ + odd f

$\cos x = f(x)$

2)



$f(x) = x^2$



odd function

$f(x) = x^3$ –

[time domain to frequency domain].

Fourier Series of even & odd Functions

1 i) If $f(x)$ is <u>even function</u> in $(-\pi,\pi)$, $b_n = 0$ (since integrand is odd fn)

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ & $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

& <u>in $(-l, l)$</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$ & $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$.

ii) If $f(x)$ is odd function in $(-\pi,\pi)$, $a_0 = 0$ & $a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, \sin nx\,dx$

& <u>in $(-l, l)$</u>: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x$

where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$.

<u>Note</u>:- i) If the interval is $(-2, 2)$ then $2l = 2-(-2) = 4 \Rightarrow l = 2$

ii) in $(0, 3)$ $2l = 3-0 \Rightarrow l = 3/2$

product of
even × even fn = even
odd × odd = even
even × odd = odd.

iii) $f(x)$ is said to be even function in $(0, 2\pi)$ iff

$f(2\pi - x) = f(x)$ & odd fun iff $f(2\pi - x) = -f(x)$

OR $f(2l - x) = f(x)$ & ,, ,, $f(2l - x) = -f(x)$

Ex ① $\sin x$ in $(0, 2\pi)$

$b_n = 0$ & $a_0 = a_n = 0$ $\sin(2\pi - x) = -\sin x$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, $a_0 = \frac{2}{\pi}\int^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int^{\pi} f(x)\cos nx\,dx$ $= -f(x)$ odd

② $\cos x$ in $(0, 2\pi)$ $\cos(2\pi - x) = \cos x$ even

(iv) For discontinuous fn defined on $(-\pi, \pi)$ or $(-l, l)$.

If $f(x) = \begin{cases} \phi(x) & -l < x < 0 \text{ or } -\pi < x < 0 \\ \psi(x) & 0 < x < l \quad 0 < x < \pi. \end{cases}$

Then, $f(x)$ is even if $\phi(-x) = \psi(x)$

$f(x)$ is odd if $\phi(-x) = -\psi(x)$.

(v) For discontinuous fn defined on $(0, 2\pi)$ or $(0, 2l)$. If $f(x) = \begin{cases} \phi(x) & 0 < x < \pi \text{ or } 0 < x < l \\ \psi(x) & \pi < x < 2\pi, \quad l < x < 2l. \end{cases}$

If $\phi(2\pi - x) = \psi(x)$ then $f(x)$ is even

& $\phi(2\pi - x) = -\psi(x)$ then $f(x)$ is odd.

$$* \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$$

IV & V examples

i) $f(x) = \begin{cases} x - \frac{\pi}{2} & \text{in } -\pi < x < 0 \\ x + \frac{\pi}{2} & \text{in } 0 < x < \pi \end{cases}$

is an odd fun $\because$ $\phi(x) = x - \frac{\pi}{2}$

$\phi(-x) = -x - \frac{\pi}{2}$

$= -(x + \frac{\pi}{2}) = -\psi(x)$

ii) $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$

is an even fun. $\phi(-x) = 1 - \frac{2x}{\pi} = \psi(x)$

iii) $|x|$ is an even fun

$\because |-x| = |x|$