UE19CS251
Design and Analysis of Algorithms
Unit 5: Limitations of Algorithmic Power and Coping with the Limitations

Transitive Closure (Warshall's Algorithm)

PES University

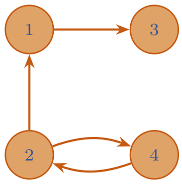
Outline

Concepts covered

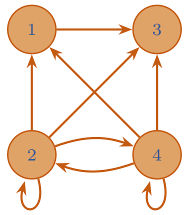
- Transitive Closure (Warshall's Algorithm)
  - Motivation
  - Algorithm
  - Example

1 Transitive Closure

- Computes the transitive closure of a relation
- Alternatively: existence of all nontrivial paths in a digraph (directed graph)
- Example of transitive closure:



$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$



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\end{array}
$$

2 Warshall's Algorithm

- Constructs transitive closure T as the last matrix in the sequence of $n \times n$ matrices $R^{(0)}, \ldots, R^{(k)}, \ldots, R^{(n)}$ where $R^{(k)}[i,j] = 1$ iff there is nontrivial path from i to j with only first k vertices allowed as intermediate vertices
  - $R^{(0)} = A$ (adjacency matrix), $R^{(n)} = T$ (transitive closure)
- On the k^{th} iteration, the algorithm computes $R^{(k)}$

$$R^{(k)}[i,j] = \begin{cases} 1 & \text{if path from } i \text{ to } k \text{ and } k \text{ to } j, \text{i.e., } R^{(k-1)}[i,k] = R^{(k-1)}[k,j] = 1 \\ R^{(k-1)}[i,j] & \text{otherwise} \end{cases}$$

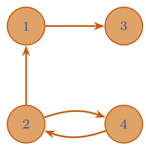
$$R^{(k)}[i,j] = R^{(k-1)}[i,j] \ \textbf{ or } \ (R^{(k-1)}[i,k] \textbf{ and } R^{(k-1)}[k,j])$$

3 Algorithm

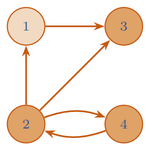
Transitive Closure (Warshall's Algorithm)

procedure WARSHALL(()$A[1 \ldots n, 1 \ldots n]$)
    ▷ **Input:** The adjacency matrix A of a digraph with n vertices
    ▷ **Output:** The transitive closure of the digraph
    $R^{(0)} \leftarrow A$
    for $k \leftarrow 1$ **to** n **do**
        for $i \leftarrow 1$ **to** n **do**
            for $j \leftarrow 1$ **to** n **do**
                $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j]);$
    return $R^{(n)}$

4 Warshall's Algorithm



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



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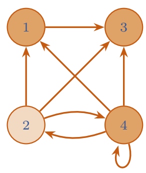

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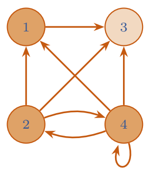

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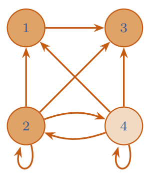

$$
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 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & 0 & 1 & 0 \\
2 & 1 & \mathbf{1} & 1 & 1 \\
3 & 0 & 0 & 0 & 0 \\
4 & 1 & 1 & 1 & 1
\end{array}
$$

5 Think About It

- Is Warshall's algorithm efficient for sparse graphs? Why / why not?
- Can Warshall's algorithm be used to determine if a graph is a DAG (Directed Acyclic Graph)? If so, how?