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VECTOR SPACES

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CLASS 5 : CONTENT

- Problems on Linear independence and dependence of vectors
- Problems on Basis and Dimension

LINEAR DEPENDENCE AND INDEPENDENCE

Problem 1 : Decide the dependence or independence of the following vectors :

(1) vectors $(1, 3, 2)$, $(2, 1, 3)$, and $(3, 2, 1)$

Solution: Write the given set of vectors as columns of matrix A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{pmatrix} \xrightarrow[R_3 - \frac{1}{5}R_2]{R_1 - 3R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{pmatrix}$$

∴ All the columns of matrix A is with pivots
 $\text{S}(A) = n = 3$ Given vectors are linearly independent.

LINEAR DEPENDENCE AND INDEPENDENCE



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2) vectors $(1, -3, 2)$, $(2, 1, -3)$ and $(-3, 2, 1)$

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

\therefore There are columns without pivots also in A.

Given vectors are Linearly Dependent.

3) vectors $(1, 1)$, $(2, 3)$, $(1, 2)$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

vectors are linearly dependent

LINEAR DEPENDENCE AND INDEPENDENCE

Note: A set of 'n' vectors of \mathbb{R}^m must be linearly dependent if $n > m$.

4) vectors $(1 \ 1 \ 0 \ 0)^T, (1 \ 0 \ 1 \ 0)^T, (0 \ 0 \ 1 \ 1)^T, (0 \ 1 \ 0 \ 1)^T$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

\therefore All columns of A
are with pivots

Given set of vectors are
Linearly Independent.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xleftarrow{R_4 - R_3}$$

LINEAR DEPENDENCE AND INDEPENDENCE

Problem 2: If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3, v_2 = w_1 - w_3, v_3 = w_1 - w_2$ are dependent.

Solution: $\because w_1, w_2, w_3$ are independent
 $\alpha w_1 + \beta w_2 + \gamma w_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0$ only

To check if v_1, v_2 and v_3 are independent
check for the combination $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

LINEAR DEPENDENCE AND INDEPENDENCE

$$\begin{aligned}\therefore c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) &= 0 \\ \Rightarrow c_1w_2 - c_1w_3 + c_2w_1 - c_2w_3 + c_3w_1 - c_3w_2 &= 0 \\ \Rightarrow w_1(c_2 + c_3) + w_2(c_1 - c_3) + w_3(c_3 - c_1) &= 0 \\ \Rightarrow \alpha w_1 + \beta w_2 + \gamma w_3 &= 0 \\ \therefore w_1, w_2, w_3 \text{ are linearly independent} &\end{aligned}$$

$\alpha = \beta = \gamma = 0$ only Hence dependent.

$$\begin{aligned}c_2 + c_3 = 0 \Rightarrow c_2 = -c_3 \Rightarrow c_1 = c_3 = -c_2 \\ c_1 - c_3 = 0 \Rightarrow c_1 = c_3 \\ c_3 - c_1 = 0 \Rightarrow c_1 = c_3\end{aligned}$$

c_i 's can assume values other than zero also.

LINEAR DEPENDENCE AND INDEPENDENCE

Problem 3 : Find the Basis and hence find the dimension of subspaces of \mathbb{R}^4

i) All vectors whose components are equal.

" " vectors are in \mathbb{R}^4 it should contain 4 components

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \sim \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} = \left\{ x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; x \in \mathbb{R} \right\} \text{ Subspace}$$

Basis $(1 1 1 1)^T$; Dimension = 1

2) All vectors whose components add up to zero

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \text{ s.t. } x+y+z+t=0$$

$$\Rightarrow x = -y - z - t$$

$$\begin{pmatrix} -y-z-t \\ y \\ z \\ t \end{pmatrix} \sim \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; y, z, t \in \mathbb{R} \right\}$$

Subspace of \mathbb{R}^4

Basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ Dimension = 3



BASIS

Problem 4 :

Let 'V' be a subspace of four dimensional space \mathbb{R}^4

So $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4; x_1 - x_2 + x_3 - x_4 = 0 \right\}$

Find the Basis and Dimension of V.

Solution :- $x_1 - x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = x_2 - x_3 + x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \sim \begin{pmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \sim x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$

$v_1 \quad \quad \quad v_2 \quad \quad \quad v_3$

Any vector in subspace V of \mathbb{R}^4 can be obtained as a linear combination of vectors (v_1, v_2, v_3) .

BASIS

Problem 5: Find a Basis for each of the following subspaces of 2 by matrices.

- 1) All diagonal matrices
- 2) All symmetric matrices ($A^T = A$)
- 3) All skew symmetric matrices ($A^T = -A$)

Solution: Any 2 by 2 matrix is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a, b, c, d \in \mathbb{R}$

Basis of 2 by 2 matrix is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

\therefore There are 4 vectors in Basis set

Dimension of 2 by 2 matrices subspace is 4

BASIS

1) All diagonal matrices (2 by 2)

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; a, b \in \mathbb{R} \right\}$$

Basis is $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Dimension = 2

2) All symmetric matrices ($A^T = A$) (2 by 2)

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; a, b, c \in \mathbb{R} \right\}$$

Basis is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Dimension = 3

BASIS

3) All Skew symmetric matrices ($A^T = -A$) (2 by 2)

$$\left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \mid a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; a \in \mathbb{R} \right\}$$

Basis $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ Dimension = 1

Problem 6 : find a basis for subspace of polynomials $p(t)$ of degree ≥ 3 .

Solution : $\left\{ (1, t, t^2, t^3) \right\}$ Basis, Dimension = 4

SPAN OF VECTORS

Problem 7: Describe the subspace of \mathbb{R}^3 spanned by :

- 1) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$

Solution: Step 1: check the independence of given vectors before drawing conclusions w.r.t. subspace.

$$\therefore \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{\begin{array}{l} R_2 - R_1 \\ R_3 + R_1 \end{array}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

\because Only 1 column with pivot, these vectors are dependent
 Hence, the span of the given vectors is a Line.

- 2) $(0, 1, 1)$, $(1, 1, 0)$ and $(0, 0, 0)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

SPAN OF VECTORS

" Only 2 columns with pivots,
the given set of three vectors span a
2 dimensional plane in \mathbb{R}^3 .

3) The columns of a 3 by 5 Echelon form matrix
with two pivots
2 Dimensional plane in \mathbb{R}^3

4) all vectors with positive components

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \sim a, b, c \in \mathbb{R} \right\} \sim \left\{ a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; a, b, c \in \mathbb{R} \right\}$$

Whole of \mathbb{R}^3 i.e. 3 dimensional space.



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THANK YOU

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