Series Solution of Differential Equations.

Consider a DE of the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \text{—(1)}$$

If $P_0(a) \neq 0$ then $x=a$ is called an ordinary point of (1). Otherwise $x=a$ is a singular point.

A singular point $x=a$ of (1) is called regular if, when (1) is put in the form

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0$$

$Q_1(x)$ and $Q_2(x)$ possess derivatives of all orders in the neighbourhood of a.

A singular point which is not regular is called an irregular singular point.

<u>Theorem 1</u>: When $x=a$ is an ordinary point of (1) its every solution can be expressed in the form

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots \quad \text{—(2)}$$

<u>Theorem 2</u>: When $x=a$ is a regular of point of (1) atleast one of the solutions can be expressed as

$$y = (x-a)^m\left[a_0 + a_1(x-a) + a_2(x-a)^2 + \dots\right] \quad \text{—(3)}$$

Series Solution when $x=0$ is an ordinary point of the equation.

Consider $P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0$

where P's are polynomials in x and $P_0(0) \neq 0$.

To solve this equation

(i) assume its solution to be of the form

$y = a_0 + a_1 x + a_2 x^2 + \ldots$

(ii) find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and substitute in the given eqn

(iii) Equate to zero the coefficients of the various powers of x and determine $a_2, a_3, \ldots$ in terms of a_0 and a_1.

(iv) Substitute the values of $a_2, a_3, \ldots$ to get the desired series solution having a_0 and a_1 as its arbitrary constants.

(1) Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$.

<u>Solution</u>: The coeff of y'' is $1 \neq 0$ at $x = 0$.

$\therefore$ $x = 0$ is an ordinary point of the DE.

Assume $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

Then $y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$

$y'' = 2.1.a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \ldots + n(n-1) a_n x^{n-2} + \ldots$

Substituting for y' and y'' the given DE is

$2.1.a_2 + 3.2.a_3 x + \ldots + n(n-1) a_n x^{n-2} + \ldots +$

$x[a_0 + \cancel{2}a_1 x + \cancel{3}a_2 x^2 + \ldots + \cancel{n}a_n x^{n-1}] = 0.$

$\Rightarrow 2.1.a_2 + (3.2a_3 + a_0)x + (4.3.a_4 + a_1)x^2 +$

$(5.4.a_5 + a_2)x^3 + \ldots + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + \ldots = 0$

Equating to zero the coefficients of various powers of x we get

$a_2 = 0$, $a_3 = \frac{-a_0}{3.2} = \frac{-a_0}{3!}$

$a_4 = \frac{-a_1}{4.3.} = \frac{-2a_1}{4!}$

$a_5 = 0$ and so on.

In general,

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$\Rightarrow a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

This is called a recurrence relation.

Substituting $n = 4, 5, 6, \ldots$ successively,

$$a_6 = \frac{-a_3}{6.5} = \frac{a_0}{6.5.\cancel{3}.\cancel{2}} = \frac{4a_0}{6!}$$

$$a_7 = \frac{-a_4}{7.6} = \frac{2a_1}{7.6.4!} = \frac{10a_1}{7!}$$

$$a_8 = \frac{-a_5}{8.7} = 0, \quad a_9 = -\frac{a_6}{9.8} = -\frac{4a_0}{9.8.6!} = -\frac{28a_0}{9!}$$

$$\therefore y = \cancel{a_0 + a_1}$$

$$a_0 + a_1 x - \frac{a_0}{3!}x^3 - \frac{2a_1}{4!}x^4 + \frac{4}{6!}a_0 x^6 + \frac{10}{7!}a_1 x^7 + \ldots$$

$$= a_0\left(1 - \frac{x^3}{3!} + \frac{4}{6!}x^6 - \frac{4.7.x^9}{9!} + \ldots\right)$$

$$+ a_1\left(x - \frac{2}{4!}x^4 + \frac{\overset{2.5}{10}}{7!}x^7 - \ldots\right)$$

Frobenius Method : Series solution when $x=0$ is a regular singular point.

Consider $P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$

(i) Assume the solution to be

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) + \ldots)$$

(ii) find y', y'' and substitute

(iii) Equate to zero the coefficients of the lowest degree term in x. It gives a quadratic equation called the indicial equation

(iv) Equate to zero the coefficient of the other powers of x and find the values of $a_1, a_2, a_3, \ldots$ in terms of a_0.

(v) The complete solution depends on the nature of roots of the indicial equation.

(vi) When the roots of the indicial equation are distinct and do not differ by an integer the complete solution is

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

where m_1, m_2 are the roots.

(2) Solve in series the equation

$$9x(1-x)y'' - 12y' + 4y = 0.$$

Solution: $x = 0$ is a singular point since coefficient of y'' is zero at $x=0$.

$\therefore$ assume $y = x^m(a_0 + a_1 x + a_2 x^2 + \ldots)$

$$\text{or } y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$$

$$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \ldots$$

$$y'' = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ldots$$

Substituting in the given equation,

$$9x(1-x)\left[m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ldots\right]$$

$$-12\left[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \ldots\right]$$

$$+4\left[a_0 x^m + a_1 x^{m+1} + \ldots\right] = 0.$$

The lowest power of x is x^{m-1}. Equating to zero, its coefficient we get

$$9m(m-1)a_0 - 12 m a_0 = 0 \Rightarrow 9m^2 - 21m = 0$$

$$\text{or } m = 0, \; m = 7/3.$$

Roots are distinct and do not differ by an integer.

Equating to zero the coeff of x^m we get

$$-9m(m-1)a_0 + 9m(m+1)a_1 - 12(m+1)a_1 + 4a_0 = 0$$

$$\Rightarrow a_0[4 - 9m(m-1)] + a_1[9m(m+1) - 12(m+1)] = 0.$$

$$\Rightarrow a_0(-9m^2+9m+4) + a_1(9m^2-3m-12) = 0$$

$$\Rightarrow -a_0(9m^2-12m+3m-4) + 3a_1(3m^2-m-4) = 0$$

$$\Rightarrow -a_0[3m(3m-4)+1(3m-4)] + 3a_1[3m^2+3m-4m-4] = 0$$

$$\Rightarrow -a_0[(3m+1)(3m-4)] + 3a_1[(3m-4)(m+1)] = 0$$

$$\Rightarrow 3a_1(m+1) = a_0(3m+1)$$

Similarly, $3a_2(m+2) = a_1(3m+4)$

$3a_3(m+3) = a_2(3m+7)$ and so on.

Hence $a_1 = \frac{3m+1}{3(m+1)}a_0$, $a_2 = \frac{3m+4}{3(m+2)}a_1 = \frac{(3m+4)(3m+1)a_0}{3^2(m+1)(m+2)}$

$$a_3 = \frac{3m+7}{3(m+3)}a_2 = \frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)}a_0 \ldots$$

When $m=0$, $a_1 = \frac{1}{3}a_0$, $a_2 = \frac{1.4}{3^2.1.2}a_0 = \frac{1.4}{3.6}a_0$

$a_3 = \frac{1.4.7}{3.6.9}$ and so on.

When $m = 7/3$, $a_1 = \frac{8a_0}{10}$, $a_2 = \frac{8.11a_0}{10.13}$, $a_3 = \frac{8.11.14a_0}{10.13.16}$

The complete solution is

$$y = c_1\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\right]a_0$$

$$+ c_2\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \ldots\right]a_0 x^{7/3}$$