STATISTICS FOR DATA SCIENCE

Generation of Random Variates

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Computer Science and Engineering

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Topics to be covered...

- Random Numbers

- Random Variate Generator

- Random Variates

- Techniques for Generating Random Variates

Random Numbers

A random sequence of numbers obtained from a stochastic process.

In real world, random numbers may be generated using a dice or a roulette wheel.

Random Numbers

- Random numbers are very important for a simulation.

- Since all the randomness required by the model is simulated by a random number generator, – Whose output is assumed to be a sequence of independent and identically (uniformly) distributed random numbers between 0 and 1.

- These random numbers are transformed into required probability distributions.

Source: http://www.cs.bilkent.edu.tr/~cagatay/cs503/_M&S_04_Random_Variate_Generation.pdf,Sigma, medium.com

Random Number Generators

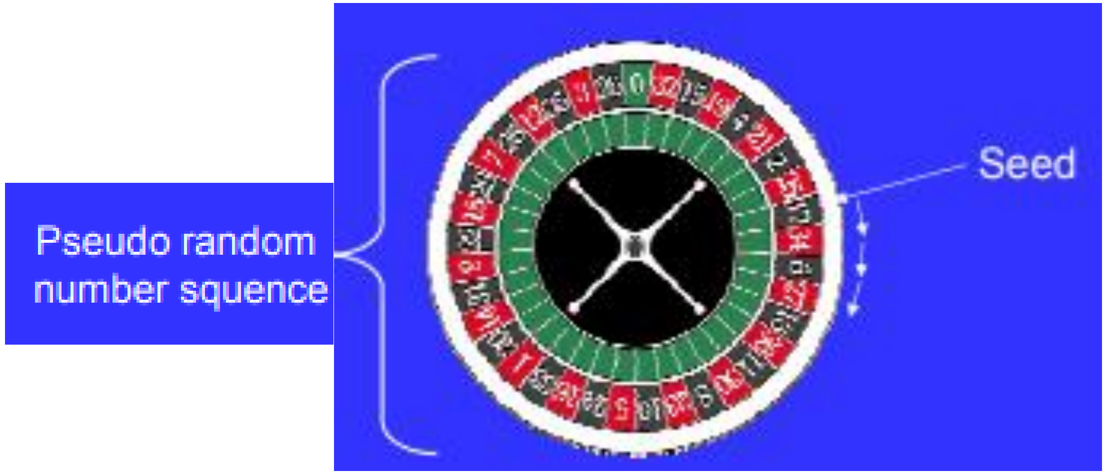
- A computational or physical device designed to generate a sequence of numbers that lack any pattern (i.e. appear random).
- Computer-based generators are simple deterministic programs trying to fool the user by producing a deterministic sequence that looks random (pseudo random numbers).
- They should meet some statistical tests for randomness intended to ensure that they do not have any easily discernible patterns.

Source: http://www.cs.bilkent.edu.tr/~cagatay/cs503/_M&S_04_Random_Variate_Generation.pdf

Random Number Seed

- Computer-based generators use random number seeds for setting the starting point of the random number sequence.

- These seeds are often initialized using a computer's real time clock in order to have some external noise.



Source: http://www.cs.bilkent.edu.tr/~cagatay/cs503/_M&S_04_Random_Variate_Generation.pdf

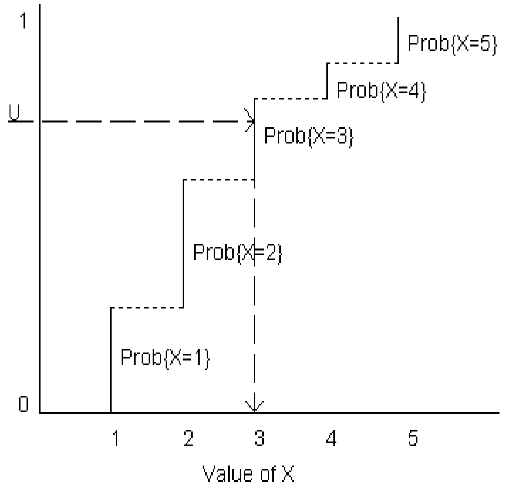
Random Variate Generation

- Process of producing observations that have the distribution of the given random variables.

- This is to develop simulation models for the purpose of analysis and decision making.

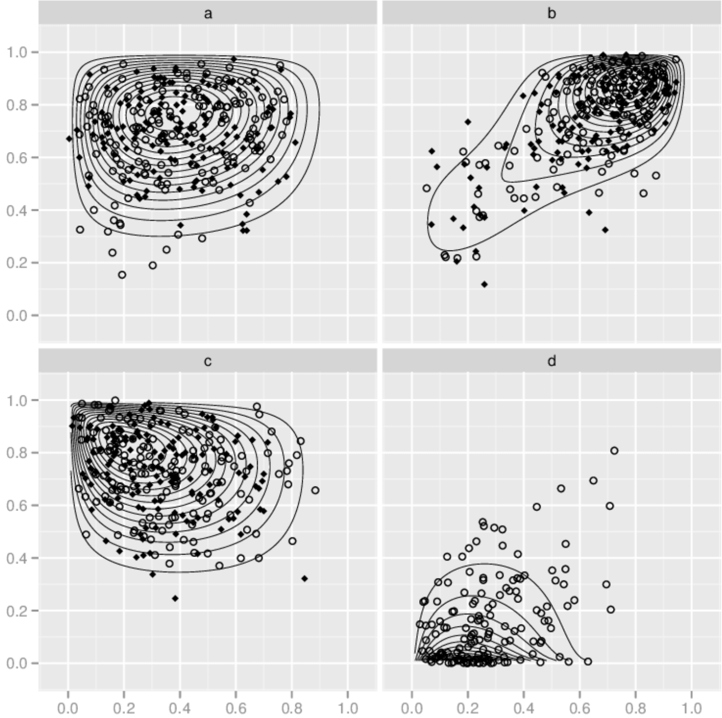
- It rely on generating uniformly distributed random number on the interval (0,1).

- Random variate generators use as starting point random numbers distributed U[0,1].



Random Variate Generation

It is assumed that a distribution is completely specified and we wish to generate samples from this distribution as input to a simulation model.



Source: mathworld.wolfram.com , www.slideshare.net,ResearchGate, Lambert

Random Variate

A **random variate** is a variable generated from uniformly distributed pseudorandom numbers.

Depending on how they are generated, a **random variate** can be uniformly or non-uniformly distributed.

Random variates are frequently used as the input to simulation models.

Examples: Inter-arrival time and service time.

Source: mathworld.wolfram.com , www.slideshare.net

Factors to Be Considered

• Exactness: – Exact if distribution of variates generated has the exact form desired.

• Speed: – Related with computing time required to generate variate.
Contributions to time are: • Setup time, • Variable generation time.

• Space: – Computer memory required for the generator.

• Simplicity: – Refers to both algorithmic simplicity and implementation simplicity.

Ref :Wikipedia

- Devroye defines a random variate generation algorithm (for real numbers) as follows:

- Assume that
- Computers can manipulate real numbers.
- Computers have access to a source of random variates that are uniformly distributed on the closed interval [0,1].
- Then a random variate generation algorithm is any program that halts almost surely and exits with a real number x. This x is called a random variate.

Random Variate

RV Generators – Techniques used to generate random variates.

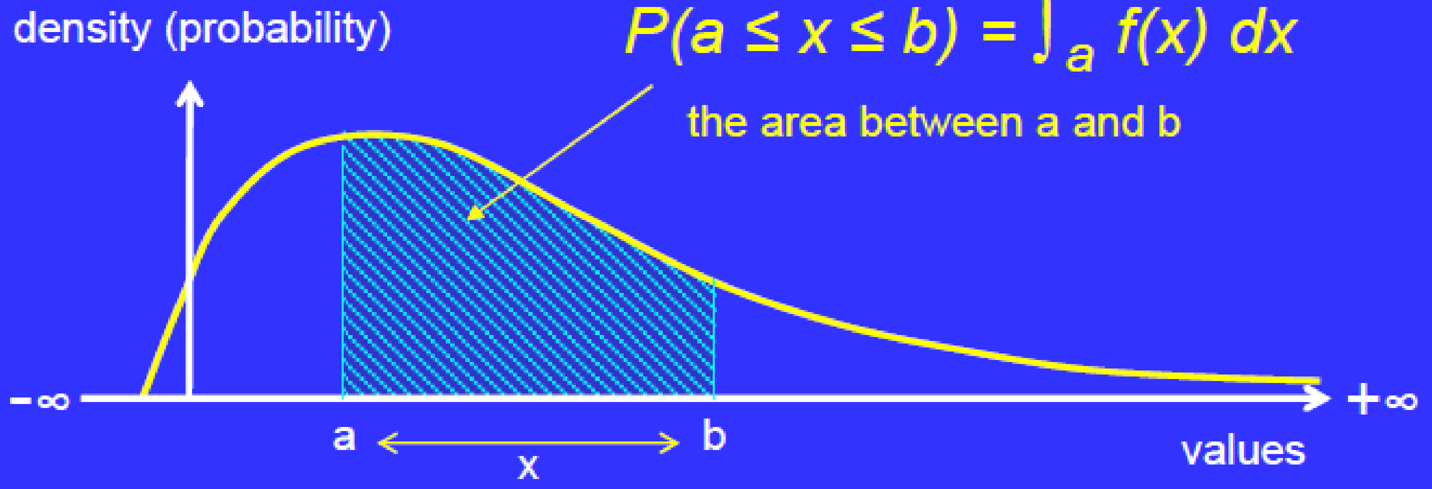
- Inverse transform technique
- Direct transformation for the Normal Distribution
- Convolution Method
- Acceptance and Rejection Technique

Source: mathworld.wolfram.com , www.slideshare.net

Inverse Transform Method

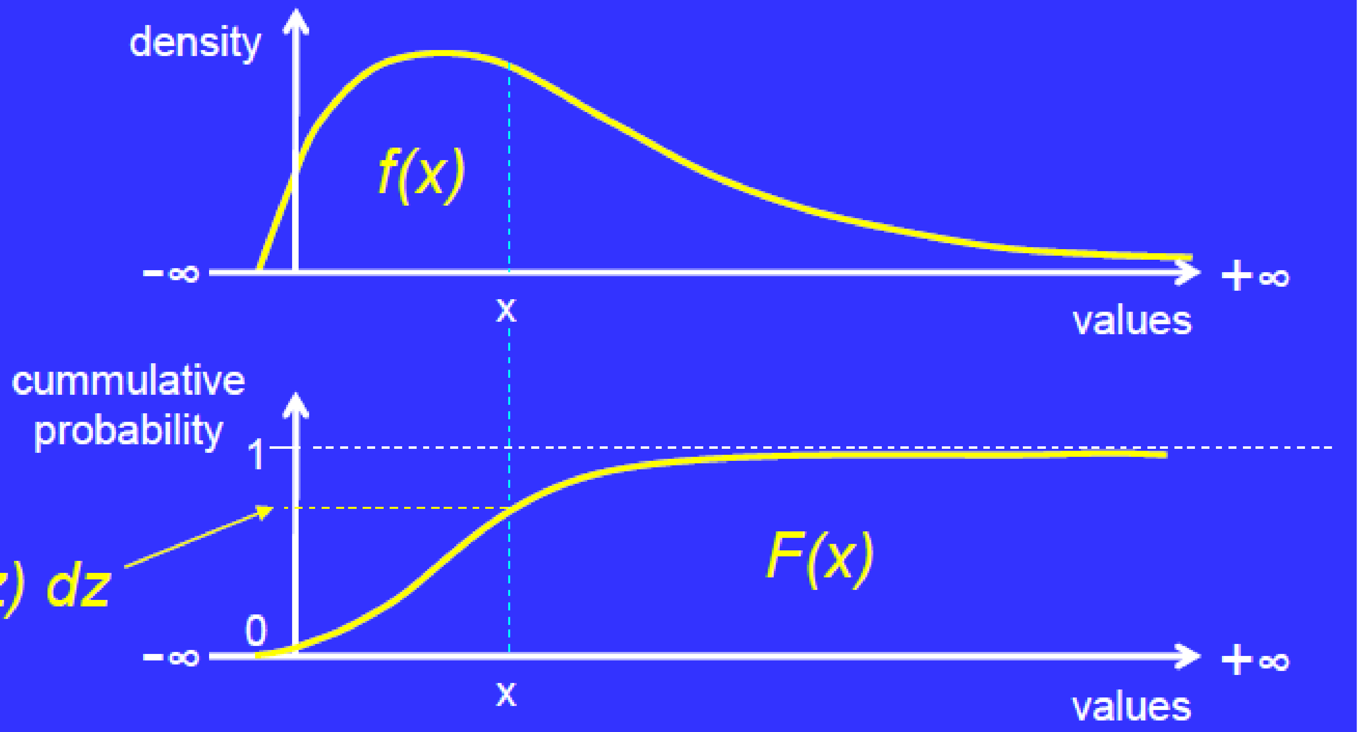
- A continuous distribution can be defined by its density function *f(x)*.
- The probability that a value x lies between a and b, where a>b, is the integral of function between a and b.

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$



Inverse Transform Method

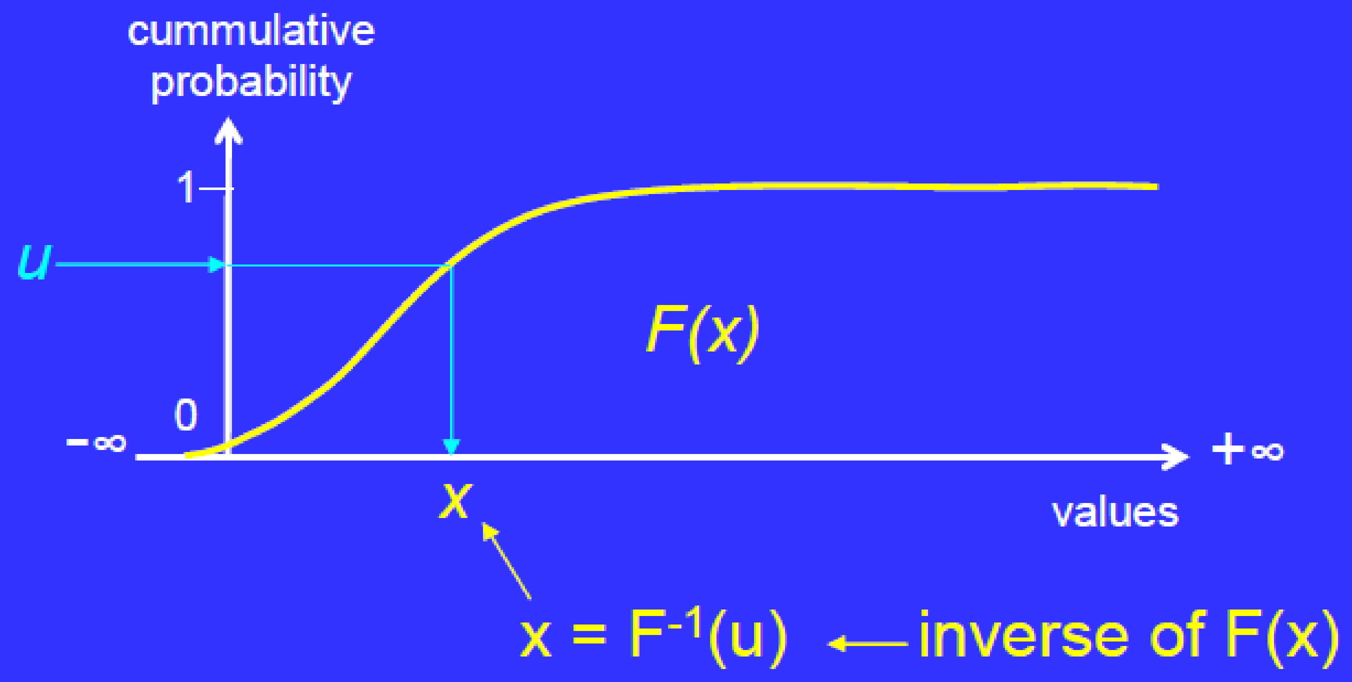
- The cummulative distribution function can be generated for all x by computing the integral of function between -infinite and x.



$$F(x) = \int_{-\infty}^{x} f(z)\, dz$$

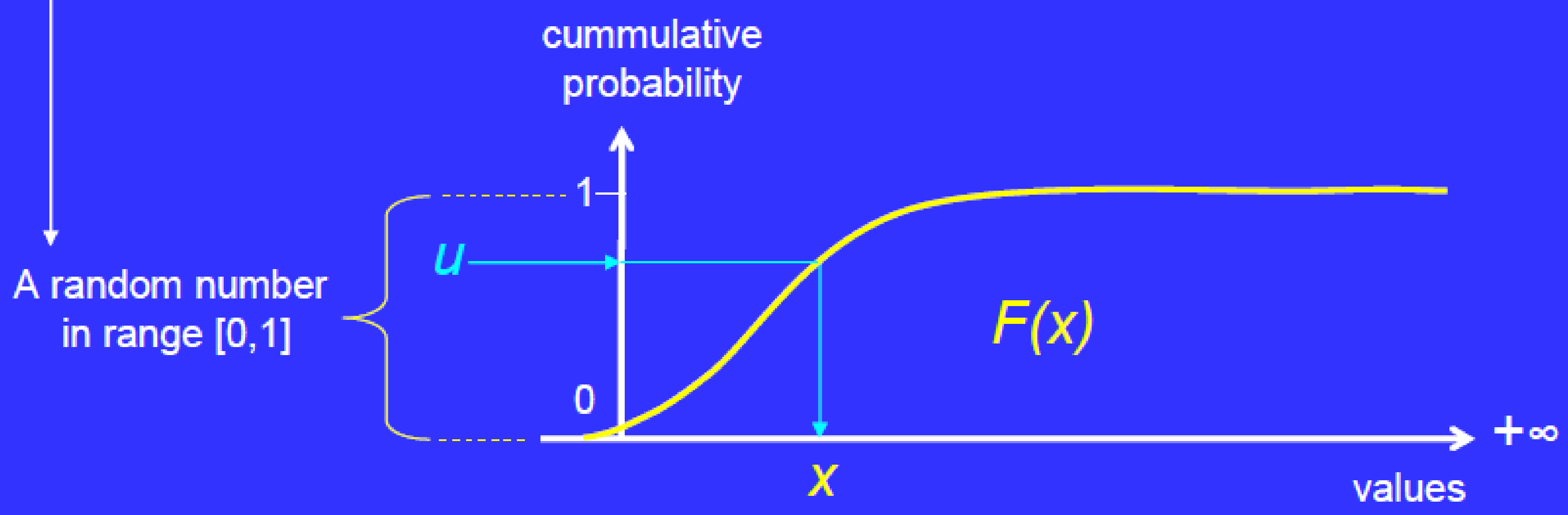
Inverse Transform Method

- Function is strictly increasing and contiuous.
- $F(x) = u$ gives a unique x.



Inverse Transform Method

- For generating random variates with f(x):
  - *Let u = RN(0,1)*
  - *Return* $x = F^{-1}(u)$



Steps in Inverse Transform technique

1. Compute CDF of the desired random variable X.
2. Set F(X)=R on the range of X.
3. Solve the equation F(X)=R for X in terms of R.
4. Generate uniform random numbers R1,R2,R3… and compute the desired random variate by

$$X_i = F^{-1}(R_i)$$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Steps in Inverse Transform Technique

Inverse transform method – Uniform Distibution Example:

Step 1 – compute *cdf* of the desired random variable X

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

Step 2 – Set $F(X) = R$ where R is a random number ~U[0,1)

$$F(x) = R = \frac{x-a}{b-a}$$

Step 3 – Solve $F(X) = R$ for X in terms of R. $X = F^{-1}(R)$.

$$R(b-a) = X - a, \quad X = R(b-a) + a$$

Step 4 – Generate random numbers R_i and compute desired random variates:

$$X_i = R_i(b-a) + a$$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Inverse Transform Method (Sample)

- For exponential distribution:

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0, \\ 0 & , x < 0. \end{cases}$$

Probability density function

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & , x \ge 0, \\ 0 & , x < 0. \end{cases}$$

Cummulative distribution function

$x = F^{-1}(x,\lambda) = -ln(1-u)/\lambda$

Bernoulli Distributions

- Density:
  - Value 1 with success probability p, and
  - Value 0 with failure probability $q = 1 - p$.
- Random variate production:

Let U = $RN(0,1)$

If ($U \leq p$)

Return $X = 1$

Else

Return $X = 0$

Binomial

A random variable X has a binomial distribution with parameters n and p if

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, \ldots, n;$$

X is the number of successes in n independent Bernoulli trials, each with success probability p.

Algorithm:

- Generate n Bernoulli(p) random variables $Y_1, \ldots, Y_n$;
- Set $X = Y_1 + Y_2 + \cdots + Y_n$.

Alternative algorithms can be derived by using the following results.

Poisson

A random variable X has a Poisson distribution with parameter λ if

$$P(X = i) = \frac{\lambda^i}{i!}e^{-\lambda}, \qquad i = 0, 1, 2, \ldots;$$

X is the number of events in a time interval of length 1 if the inter-event times are independent and exponentially distributed with parameter λ.

Algorithm:

- Generate exponential inter-event times $Y_1, Y_2, \ldots$ with mean 1; let I be the smallest index such that

$$\sum_{i=1}^{I+1} Y_i > \lambda;$$

- Set $X = I$.

Poisson (alternative)

- Generate U(0,1) random variables $U_1, U_2, \ldots;$ let I be the smallest index such that

$$\prod_{i=1}^{I+1} U_i < e^{-\lambda};$$

- Set $X = I.$

Inverse-transform Technique: Other Distributions

Examples of other distributions for which inverse CDF works are:

- Uniform distribution
- Weibull distribution
- Triangular distribution

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Inverse-transform Technique: Discrete Distribution

All discrete distributions can be generated via inverse transform technique.

Method:

Numerically, table-lookup procedure, algebraically, or a formula

Examples of application:

- Empirical
- Discrete uniform
- Geometric

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Inverse-transform Technique: Continuous Distributions

A number of continuous distributions do not have a closed form expression for their CDF, e.g. Normal, Gamma and Beta.

The presented method does not work for these distributions.

Solution

- Approximate the CDF or numerically integrate the CDF

Problem

- Computationally slow

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Acceptance and Rejection Technique

- Useful particularly when inverse CDF does not exist in closed form
- **Illustration:** To generate random variates, X ~ U(1/4,1)

Procedure:

Step 1. Generate $R \sim U(0,1)$

Step 2. If $R \geq ¼$, accept $X=R$.

Step 3. If $R < ¼$, reject R, return to Step 1

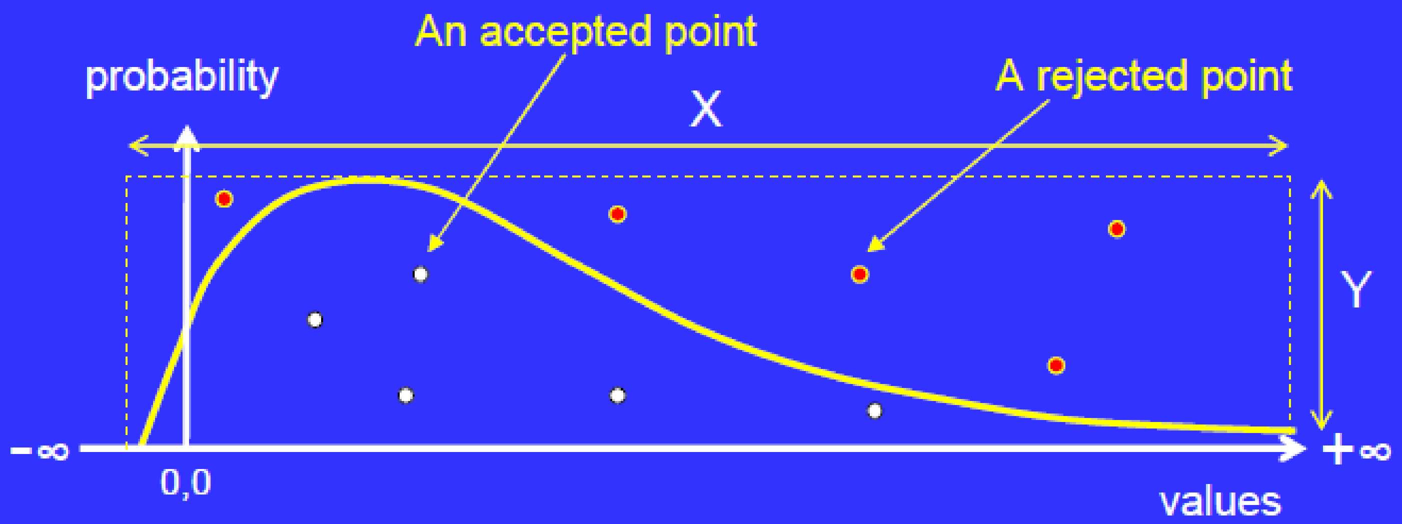
Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Acceptance-Rejection Method

- Generate a random point (X,Y) on the graph.
- If (X,Y) lies under the graph of f(X) then

  Accept X
- Otherwise

  Reject X



Acceptance-Rejection Method (Drawback)

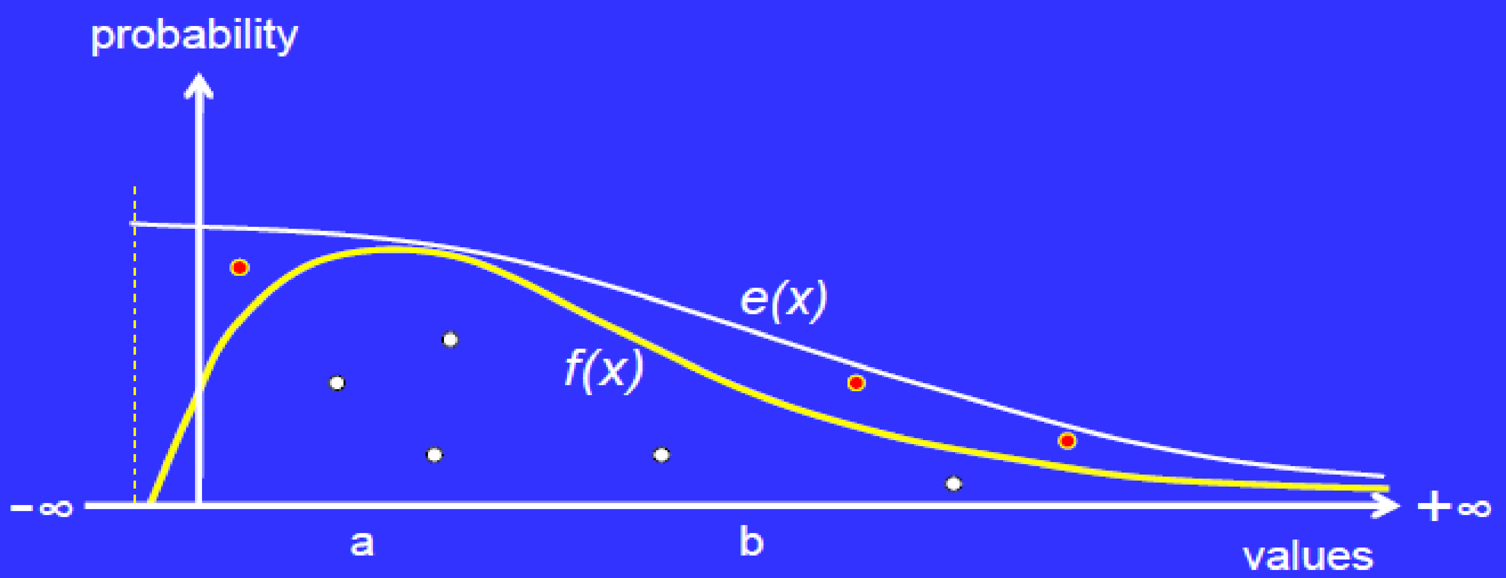
Trials ratio: Average number of points (X,Y) needed to produce one accepted X.

- We need to make trial ratio close to 1.

- Else generator may not be efficient enough because of wasted computation effort.

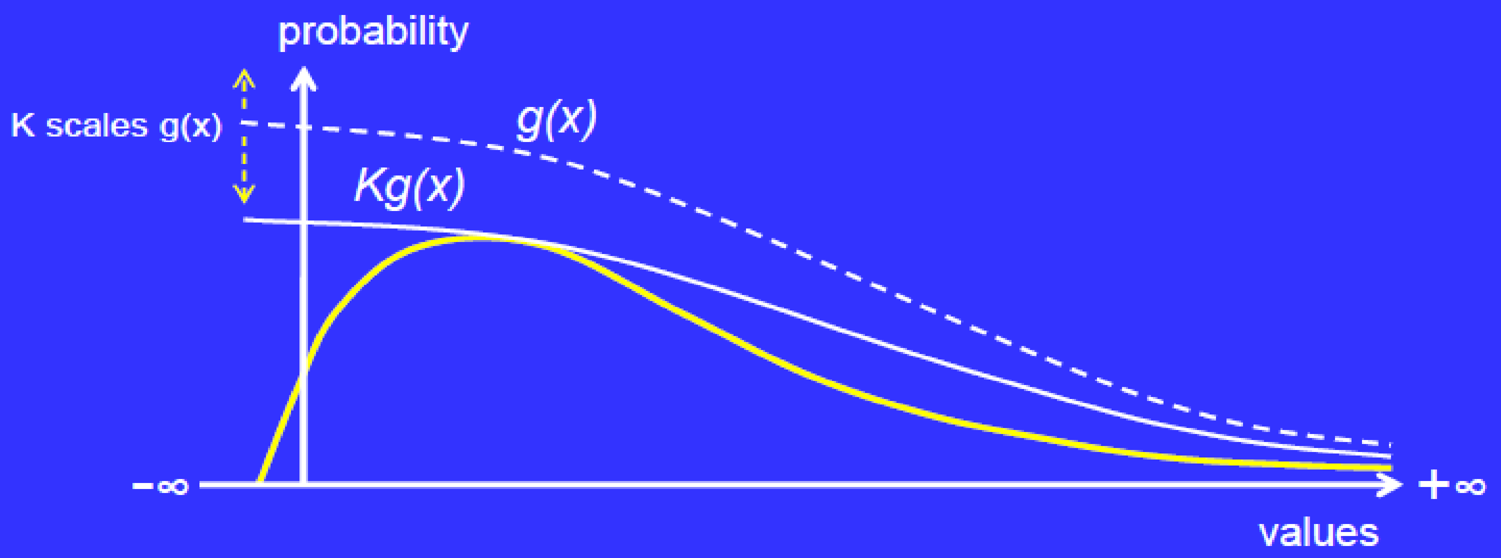
Acceptance-Rejection Method (Making More Efficient)

- One way to make generator efficient is;
  - To generate points uniformly scattered under a function *e(x)*, where area between the graph of *f* and *e* be small.



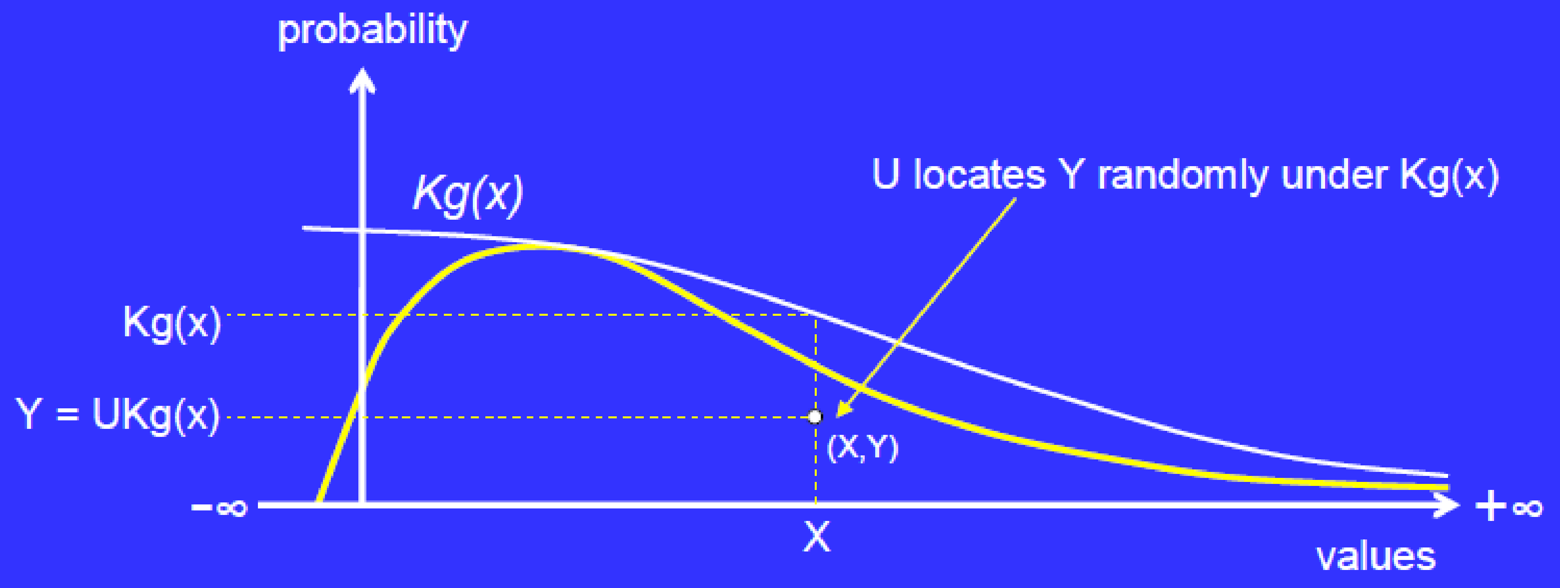
Acceptance-Rejection Method (Constructing e(x))

- Take $e(x) = Kg(x)$
- *$g(x)$ = density function of a distribution for which an easy way of generating variates already exists.*
- *K = scale factor*

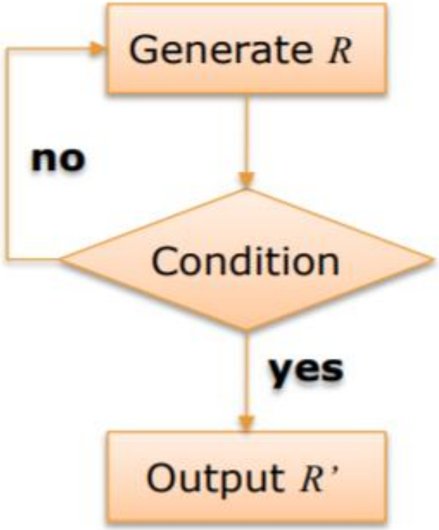


Acceptance-Rejection Method (Producing (X,Y))

- Let X = a variate produced from Kg(x)
- Let U = *RN(0,1)*
- (X,Y) = (X, UKg(X))



Acceptance and Rejection Technique



R does not have the desired distribution, but R conditioned (R') on the event {R ≥ ¼} does.

- Efficiency: Depends heavily on the ability to minimize the number of rejections.

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Acceptance and Rejection Technique – Poisson Distribution

Procedure of generating a Poisson random variate N is as follows

1. Set $n=0$, $P=1$
2. Generate a random number R_{n+1}, and replace P by $P \times R_{n+1}$
3. If $P < \exp(-\alpha)$, then accept $N=n$
   - Otherwise, reject the current n, increase n by one, and return to step 2.

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Acceptance and Rejection Technique – Poisson Distribution

- Example: Generate three Poisson variates with mean α=0.2
  - $\exp(-0.2) = 0.8187$
- Variate 1
  - Step 1: Set $n = 0, P = 1$
  - Step 2: $R1 = 0.4357, P = 1 \times 0.4357$
  - Step 3: Since $P = 0.4357 < \exp(-0.2)$, accept $N = 0$
- Variate 2
  - Step 1: Set $n = 0, P = 1$
  - Step 2: $R1 = 0.4146, P = 1 \times 0.4146$
  - Step 3: Since $P = 0.4146 < \exp(-0.2)$, accept $N = 0$
- Variate 3
  - Step 1: Set $n = 0, P = 1$
  - Step 2: $R1 = 0.8353, P = 1 \times 0.8353$
  - Step 3: Since $P = 0.8353 > \exp(-0.2)$, reject $n = 0$ and return to Step 2 with $n = 1$
  - Step 2: $R2 = 0.9952, P = 0.8353 \times 0.9952 = 0.8313$
  - Step 3: Since $P = 0.8313 > \exp(-0.2)$, reject $n = 1$ and return to Step 2 with $n = 2$
  - Step 2: $R3 = 0.8004, P = 0.8313 \times 0.8004 = 0.6654$
  - Step 3: Since $P = 0.6654 < \exp(-0.2)$, accept $N = 2$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Acceptance and Rejection Technique – Poisson Distribution

- It took five random numbers to generate three Poisson variates
- In long run, the generation of Poisson variates requires some overhead!

N	R_{n+1}	P	Accept/Reject		Result
0	0.4357	0.4357	$P < \exp(-\alpha)$	Accept	$N=0$
0	0.4146	0.4146	$P < \exp(-\alpha)$	Accept	$N=0$
0	0.8353	0.8353	$P \geq \exp(-\alpha)$	Reject	
1	0.9952	0.8313	$P \geq \exp(-\alpha)$	Reject	
2	0.8004	0.6654	$P < \exp(-\alpha)$	Accept	$N=2$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Normal distribution

Methods:

- Acceptance-Rejection method
- Box-Muller method

Acceptance-Rejection method

If X is $N(0, 1)$, then the density of $|X|$ is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \qquad x > 0.$$

Now the function

$$g(x) = \sqrt{2e/\pi}\, e^{-x}$$

majorizes f. This leads to the following algorithm:

1. Generate an exponential Y with mean 1;
2. Generate U from $U(0, 1)$, independent of Y;
3. If $U \leq e^{-(Y-1)^2/2}$, then accept Y; else reject Y and return to step 1.
4. Return $X = Y$ or $X = -Y$, both with probability $1/2$.

Box-Muller method

If U_1 and U_2 are independent $U(0, 1)$ random variables, then

$$\begin{aligned} X_1 &= \sqrt{-2 \ln U_1} \cos(2\pi U_2) \\ X_2 &= \sqrt{-2 \ln U_1} \sin(2\pi U_2) \end{aligned}$$

are independent standard normal random variables.

Direct Transformation

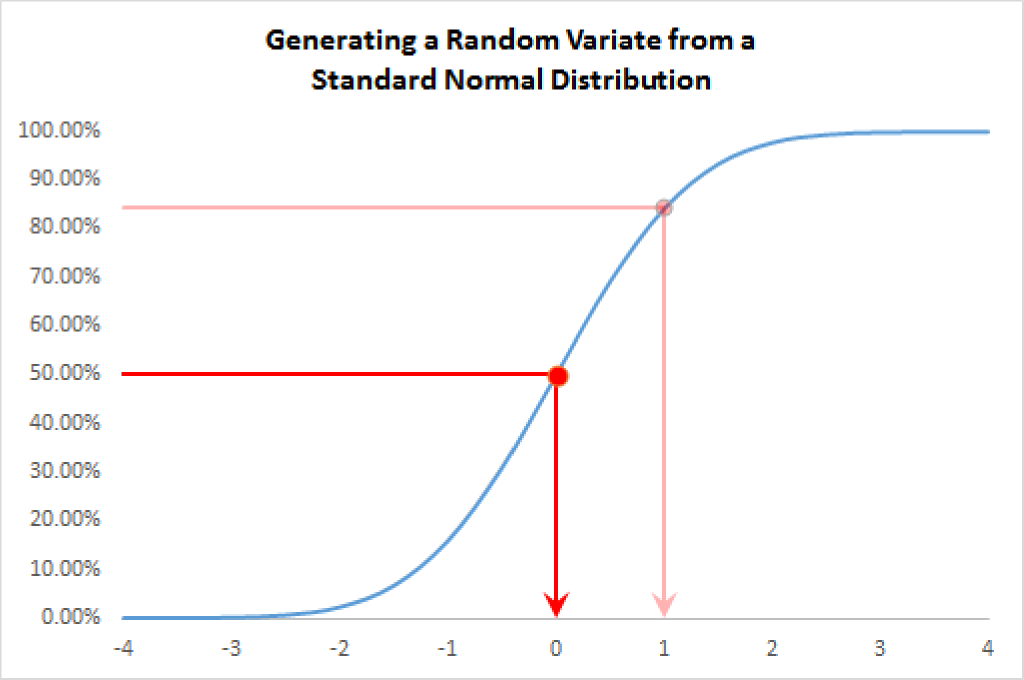
Approach for $N(0,1)$

- PDF

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

- CDF, No closed form available

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$



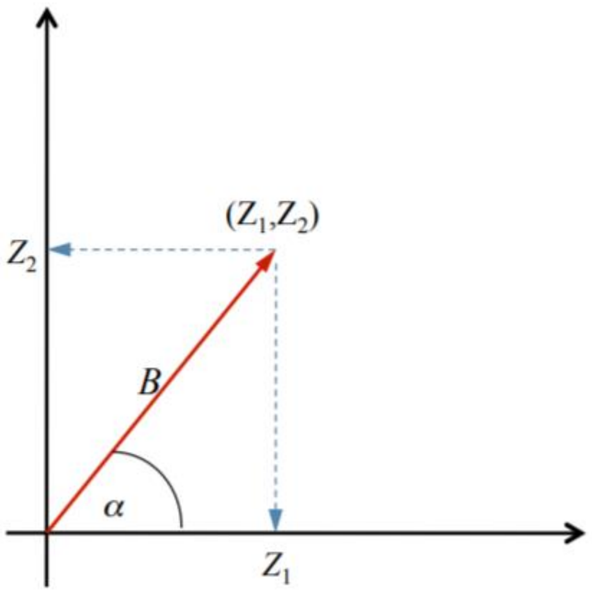
Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf, tvmcalcs.com

Direct Transformation

Approach for N(0,1)

- Consider two standard normal random variables, Z1 and Z2, plotted as a point in the plane:

- In polar coordinates:
  - Z1 = B cos(α)
  - Z2 = B sin(α)



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Direct Transformation

- Approach for $N(\mu, \sigma^2)$:
  - Generate $Z_i \sim N(0,1)$

$$\boxed{X_i = \mu + \sigma Z_i}$$

- Approach for Lognormal(μ,σ^2):
  - Generate $X \sim N(\mu,\sigma^2)$

$$\boxed{Y_i = e^{X_i}}$$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Direct Transformation-Example

Let R1 = 0.1758 and R2=0.1489

- Two standard normal random variates are generated as follows:

$$Z_1 = \sqrt{-2\ln(0.1758)}\cos(2\pi 0.1489) = 1.11$$

$$Z_2 = \sqrt{-2\ln(0.1758)}\sin(2\pi 0.1489) = 1.50$$

- To obtain normal variates Xi with mean μ=10 and variance $\sigma^2 = 4$

$$X_1 = 10 + 2 \cdot 1.11 = 12.22$$

$$X_2 = 10 + 2 \cdot 1.50 = 13.00$$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Do It Yourself !!!!

Implement Random Variate Generation for **Poisson Distribution.**

Implement Random Variate Generation for **Normal Distribution**.

THANK YOU

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