AFLL

UNIT-4

CLASS NOTES

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CLOSURE PROPERTIES OF CONTEXT-FREE LANGUAGES

context-free languages

Deterministic (DCFL)

Non-deterministic (NCFL)
also simply CFLs

Closure Property

A set is closed under an operation if the operation can always be completed with elements in the set

CLOSURE PROPERTIES OF CFLs

Closed under
- union
- concatenation
- Kleene closure
- Reversal

Not closed under
- intersection
- complement

CLOSURE PROPERTIES

UNION

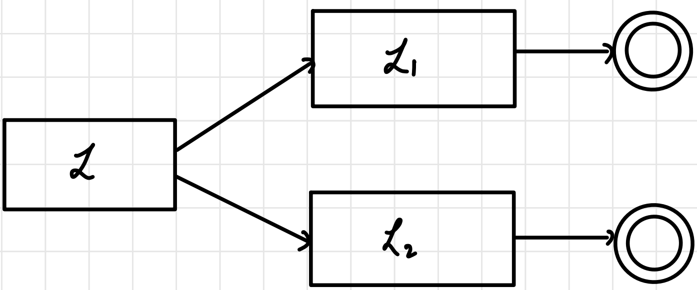
If L_1 is a CFL and L_2 is a CFL, then $L_1 \cup L_2$ is a CFL

- or operation ($\mathcal{L}_1$ or $\mathcal{L}_2$)

Question 1

$$\mathcal{L}_1 = \{ a^n b^n c^m \mid m \geq 0 \text{ and } n \geq 0 \}$$
$$\mathcal{L}_2 = \{ a^n b^m c^m \mid m \geq 0 \text{ and } n \geq 0 \}$$

Show that they are closed under union as CFLs



$$\left.\begin{array}{l} S \rightarrow S_1 \mid S_2 \\ S_1 \rightarrow (\mathcal{L}_1) \\ S_2 \rightarrow (\mathcal{L}_2) \end{array}\right] \text{context fre}$$

CONCATENATION

If L_1 is a CFL and L_2 is a CFL, then $L_1 L_2$ is a CFL

$$S \rightarrow S_1 S_2$$
$$S_1 \rightarrow (\mathcal{L}_1)$$
$$S_2 \rightarrow (\mathcal{L}_2)$$

Question 2

$$\mathcal{L}_1 = \{a^n b^n \mid n \geq 0\}$$
$$\mathcal{L}_2 = \{c^m d^m \mid m \geq 0\}$$

$$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 = \{a^n b^n c^m d^m \mid n \geq 0, m \geq 0\}$$

PDA:

- push a's to stack & pop b's once b reached
- once z_0 reached, push c's to stack until a d is seen
- resultant PDA $\rightarrow$ CFL

KLEENE CLOSURE

If $\mathcal{L}$ is a CFL, $\mathcal{L}^*$ is a CFL

Question 3

$$\mathcal{L} = \{a^n b^n \mid n \geq 0\}$$

$$\mathcal{L}^* = \{(a^n b^n)^* \mid n \geq 0\}$$

$$S \rightarrow AS \mid \lambda$$
$$A \rightarrow aAb \mid \lambda$$

REVERSAL

If $\mathcal{L}$ is a CFL, $\mathcal{L}^R$ is a CFL

NON-CLOSURE PROPERTIES

INTERSECTION

- Intersection of a CFL with an RG is a CFL
- Intersection of a CFL with a CFL is not a CFL

Question 4

$\mathcal{L}_1 = \{a^m b^n c^n \mid m,n \geq 0\}$
$\mathcal{L}_2 = \{a^n b^n c^m \mid m,n, \geq 0\}$

$\mathcal{L}_1 \cap \mathcal{L}_2 = \{a^n b^n c^n \mid n \geq 0\} \longrightarrow$ not context free
cannot construct PDA

RE and CFL

- cross product of PDA and DFA
- run paths parallelly from single start state
- if both reach final state, accepted

COMPLEMENT

$\mathcal{L}_1 \longrightarrow$ CFL , $\overline{\mathcal{L}_1} \longrightarrow$ not CFL

- According to DeMorgan's Law, $\mathcal{L}_1 \cap \mathcal{L}_2 = \overline{\overline{\mathcal{L}_1} \cup \overline{\mathcal{L}_2}}$
- Assume complement to be closed and let $\mathcal{L}_1$ & $\mathcal{L}_2$ be CFLs
- Then, $\overline{\overline{\mathcal{L}_1} \cup \overline{\mathcal{L}_2}}$ should also be a CFL. However, the LHS $\mathcal{L}_1 \cap \mathcal{L}_2$ is not a CFL.
- Therefore, complement is not closed under CFLs

CLOSURE PROPERTIES OF DCFLs

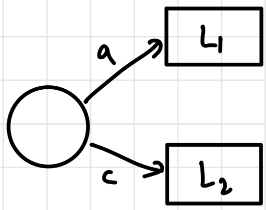
- only closed under complement
- sure about when to push/pop (single path)

NON-CLOSURE PROPERTIES

UNION

$\mathcal{L}_1 = \{a^n b^n \mid n \geq 0\}$
$\mathcal{L}_2' = \{c^m d^m \mid m \geq 0\}$

$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is DCFL

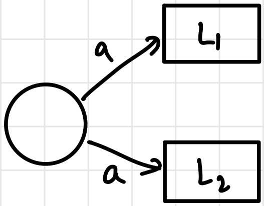


DCFL
CLOSED

Question 5

$\mathcal{L}_1 = \{a^n b^m c^m \mid m,n \geq 0\}$
$\mathcal{L}_2 = \{a^n b^n c^m \mid m,n \geq 0\}$

$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$



not DCFL   NOT CLOSED

- Not necessarily closed

INTERSECTION

- Not necessarily closed under intersection

$\mathcal{L}_1 = a^n b^n c^m$
$\mathcal{L}_2 = a^n b^m c^m$

$\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2 \longrightarrow$ not DCFL

CONCATENATION

- Not necessarily closed under intersection

$\mathcal{L}_1 = \{a^n b^n \mid n \geq 0\}$
$\mathcal{L}_2' = \{c^m d^m \mid m \geq 0\}$

$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2$ is DCFL   can make PDA

Question 6

$\mathcal{L}_1 = \{a^n b^m \mid m < n\}$
$\mathcal{L}_2 = \{w c w^R\}$

$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 = \{a^n b^m w c w^R\}$

for example $w =$ abb bbbacabbb

don't know to start w from here

non-determinism

$\mathcal{L}$ is not a DCFL

REVERSAL

If $\mathcal{L}$ is a DCFL, $\mathcal{L}^R$ is not necessarily a DCFL

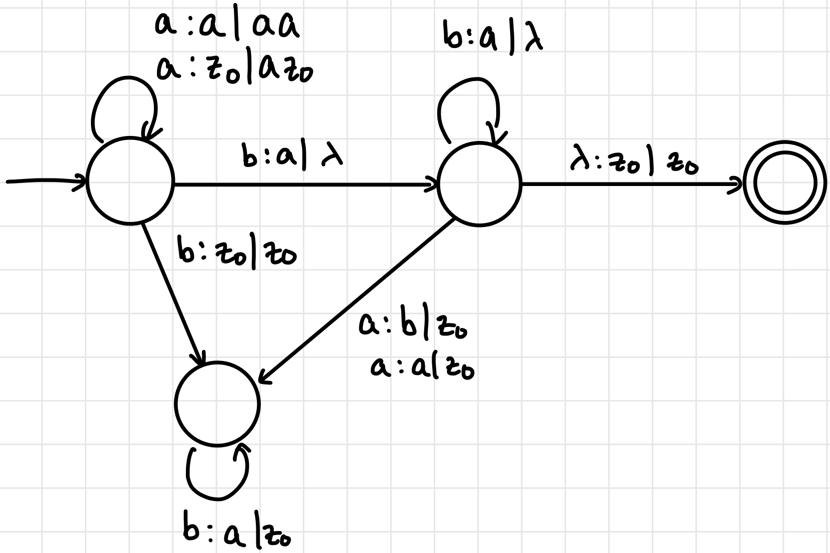
CLOSURE PROPERTIES

COMPLEMENT

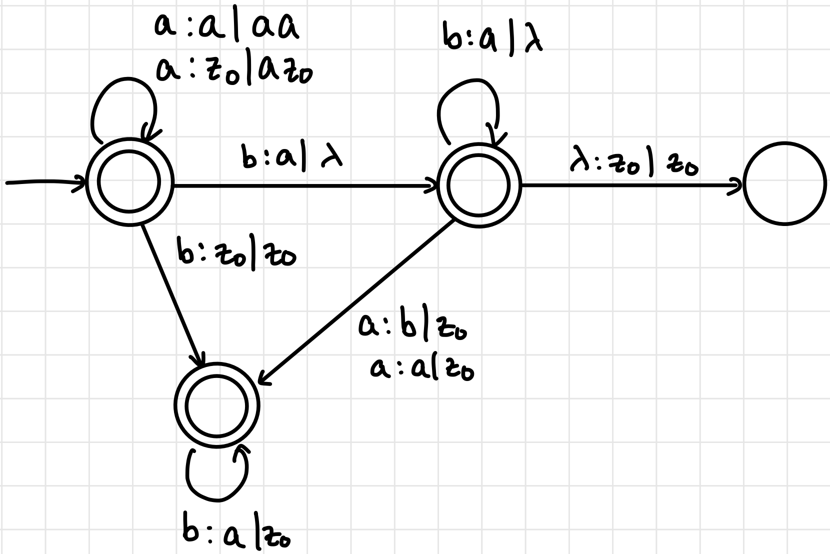
$\mathcal{L}_1 \longrightarrow$ DCFL , $\overline{\mathcal{L}_1} \longrightarrow$ DCFL (closed under complement)

Question 7

$\mathcal{L} = \{a^n b^n \mid n \geq 0\}$



- flipped states



ANSWERING QUESTIONS ABOUT CFLs

1. Is a CFL empty?

- if start state S is non-generating

  $S \longrightarrow$ non generating

- if start symbol is non-terminating

$$
\begin{aligned}
S &\rightarrow aA \mid bB \mid C \\
A &\rightarrow aA \mid Aa \mid bB \\
B &\rightarrow bB \mid Ba \mid aA \\
C &\rightarrow bA \mid aB \mid aSb \mid bSa
\end{aligned}
$$

non-terminating

- emptiness is a decidable property

2. Is a CFL finite?

- if there is a loop, the language is infinite

  (a) $S \rightarrow aSb \mid \lambda$ infinte

  (b)
$$
\begin{aligned}
S &\rightarrow aA \mid bB \mid C \\
A &\rightarrow Ab \mid bB \\
B &\rightarrow bB \mid aS
\end{aligned}
$$

  infinite

- decidable

3. Is a string w a member of the CFL?

- derivable strings ⟶ can use CYK algorithm
- decidable

4. Are two CFLs equal?

- no algorithm to prove equality of CFLs
- algorithm exists for RLs (if start states indistinguishable)

5. Is the CFL ambiguous?

- not decidable; no algorithm

PUMPING LEMMA

- Have to consider state as well as the stack contents
- Pumping lemma cannot be used in the same way as a regular language
- Work with CNF grammars and we work with long strings greater than number of non-terminals in the grammar
- Unlike RLs where we focus on number of states
- Divide into 5 parts
- Only can prove if $\mathcal{L}$ is not CFL (contradiction)

Question 8

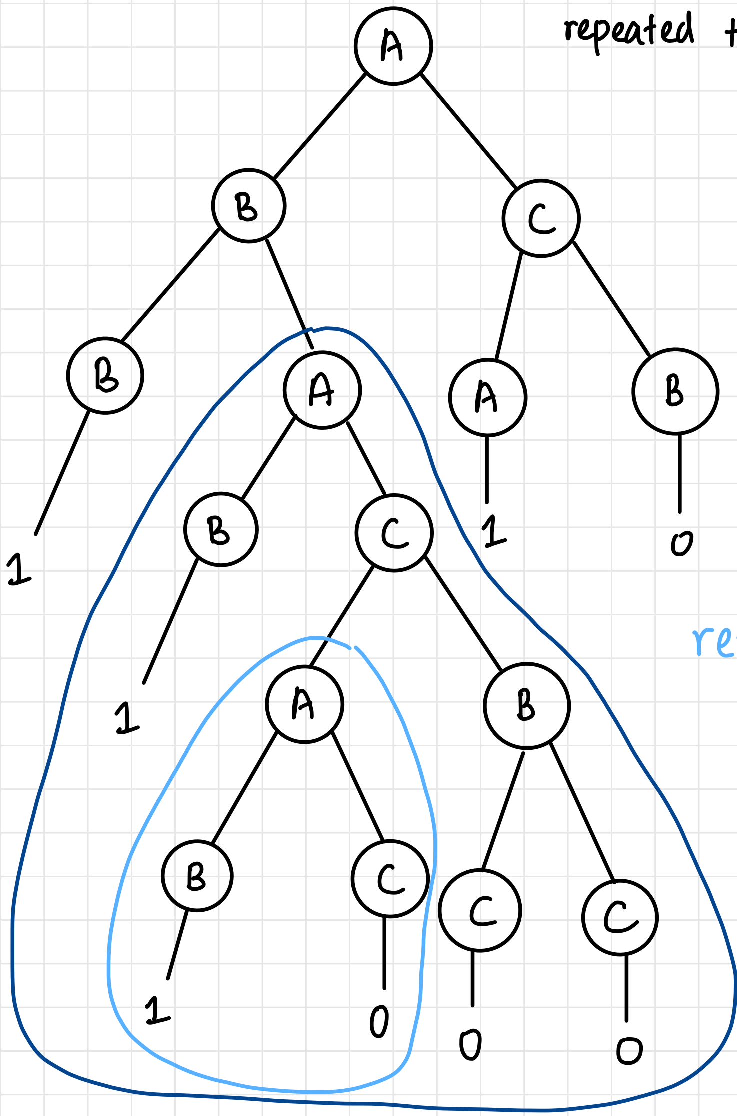
$A \to BC \mid 0 \mid 1$
$B \to BA \mid CC \mid 1 \mid 0$
$C \to AB \mid 0$

pumping using parse tree

non-terminals = 3
$|w| \geq 2^3 = 8$

$w = 11100010$ parse tree

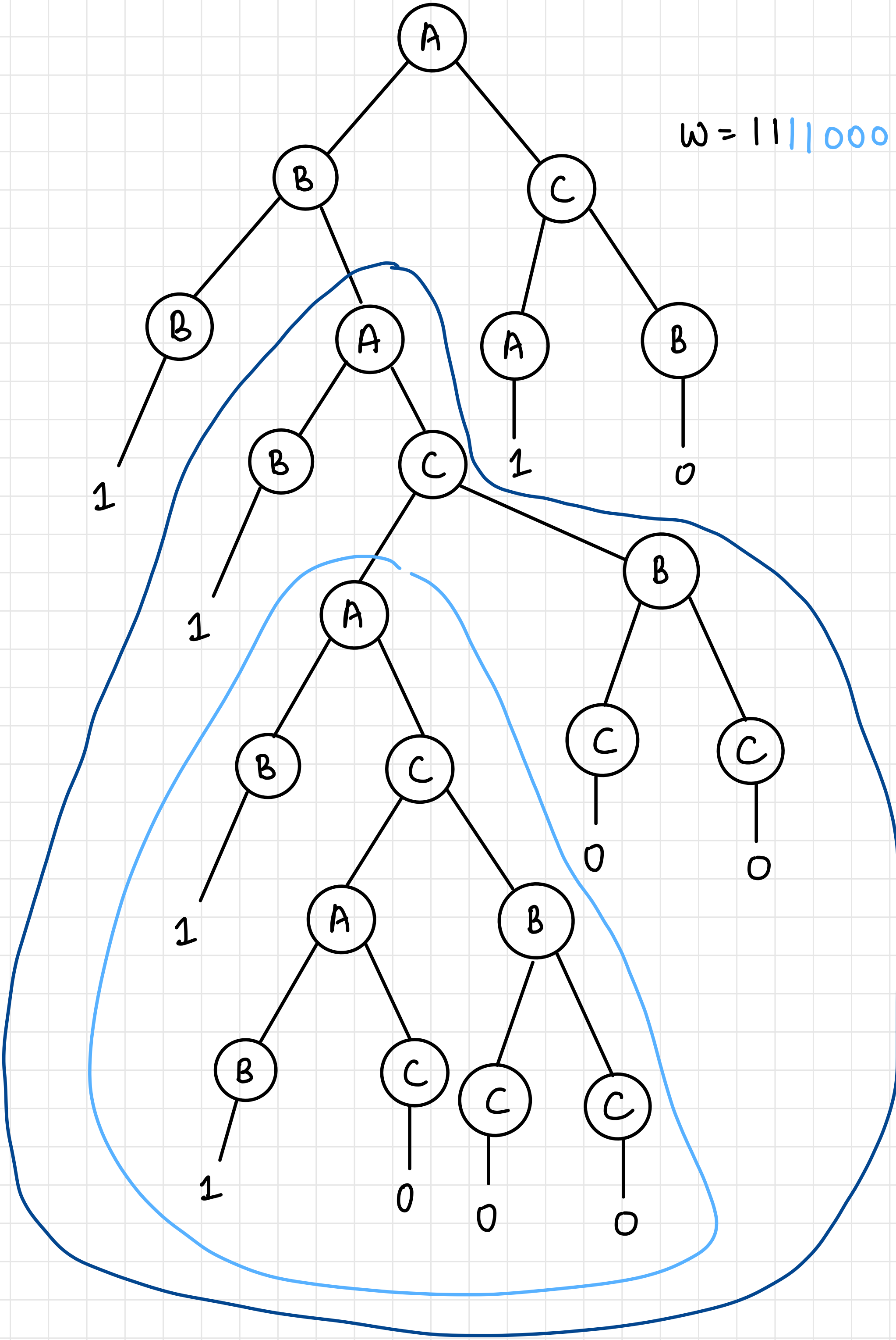
repeated terminal $\longrightarrow$ infinite



replace inner A with outer subtree

$w = 11100010$

$w = 11110000010$



$w = \underbrace{1}_{u}\,\underbrace{1}_{v}\,\underbrace{10}_{w}\;\underbrace{00}_{x}\;\underbrace{10}_{y}$ first iteration

$w = \underbrace{1}_{u}\,\underbrace{11}_{v}\,\underbrace{10}_{w}\;\underbrace{00\;00}_{x}\;\underbrace{10}_{y}$ second iteration

$w = \underbrace{1}_{u}\,\underbrace{111}_{v}\,\underbrace{10}_{w}\;\underbrace{000000}_{x}\;\underbrace{10}_{y}$ third iteration

- only v & x keep changing

$|word| \geq n$ n = no. of variables

$|vwx| \leq m$

$|vx| \geq 1$

- note: not used to prove $\mathcal{L}$ is CFL; only used when we cannot construct a PDA

Question 9

$\mathcal{L} = \{a^n b^n\}$

word

$W = a^{20} b^{20}$ $m = 20$

word, not segment ω

$|W| \geq m$

$\underbrace{a^{19}}_{u}\;\underbrace{a}_{v}\;\underbrace{\lambda}_{w}\;\underbrace{b}_{x}\;\underbrace{b^{19}}_{y}$

Question 10

$$\mathcal{L} = \{a^n b^n c^n \mid n \geq 0\}$$

- Assume $a^n b^n c^n$ is a CFL
- Let $s = a^p b^p c^p$ $p =$ pumping constant/ no. of variables

$$|s| \geq p$$
$$|vwx| \leq p$$
$$|vx| \geq 1$$

How to split

Case 1

- vwx is made only of a's

$$\underset{vwx}{a^p} \mid b^p \; c^p$$

- vx together must contain at least one a

$$\underset{u}{} \; \underset{v}{a} \; \underset{w}{} \; \underset{x}{a} \; \underbrace{bb \;\; cc}_{y} \longrightarrow \text{not in } \mathcal{L}$$

Case 2

- vwx is made only of b's

$$\underset{u}{a^p} \quad \underset{vwx}{b^p} \quad \underset{y}{c^p} \quad \notin \mathcal{L}$$

Case 3

- vwx is made only of c's

$$\underbrace{a^p \quad b^p}_{u} \quad \underset{vwx}{c^p} \quad \underset{y}{\lambda} \quad \notin \mathcal{L}$$

Case 4

- vwx is made of some a's and some b's

$$\underset{u}{a^{p-2}} \; \underset{v}{a} \; \underset{w}{ab} \; \underset{x}{b} \; \underset{y}{b^{p-2} \; c^p} \quad \notin \mathcal{L}$$

Case 5

- vwx is made of some b's and some c's

$$\underset{u}{a^p \, b^{p-2}} \; \underset{v}{b} \; \underset{w}{bc} \; \underset{x}{c} \; \underset{y}{c^{p-2}} \quad \notin \mathcal{L}$$

- cannot span over all (a,b,c) as $|vwx| \le p$

Question 11

$$L = \{ ss \mid s \in \{a,b\}^* \}$$

- cannot construct PDA (only for ss^R)
- Let $s = a^m b a^m b$, m = no. of variables
- $|s| \geq m$ ✓
- divide into u, v, w, x, y

Placement of u, v, w

Case 1

- vwx contains only a's

$$v^i w x^i \quad \text{for } i = 2$$

$$\underset{u}{\lambda} \quad \underset{v}{a}\ \underset{w}{a^{m-2}}\ \underset{x}{a} \quad \underset{y}{b\ a^m\ b} \qquad \notin L$$

- pump in, more a's in the first part

Case 2

- vwx contains both a's and b's

$$\underset{u}{a^{m-1}} \quad \underset{vwx}{ab} \quad \underset{y}{a^m b} \qquad \notin L$$

Case 3

- vwx contains only b's

$$\underbrace{a^m}_{u}\ \underbrace{b}_{vwx}\ \underbrace{a^m\ b}_{y} \notin L$$

Case 4

- vwx contains only a's

$$\underbrace{a^m b}_{u}\ \underbrace{a^m}_{vwx}\ \underbrace{b}_{y} \notin L$$

Case 5

- vwx contains b's and a's

$$\underbrace{a^m}_{u}\ \underbrace{b\,aa}_{vwx}\ \underbrace{a^{m-2}\ b}_{y} \notin L$$

Case 6

- vwx contains both a's and b's

$$\underbrace{a^m\ b\ a^{m-2}}_{u}\ \underbrace{aa\,b}_{vwx}\ \underbrace{\lambda}_{y} \notin L$$

Question 12

$L = \{a^n b^m a^n \mid n \geq m\}$

Assume L is CFL, p = no. of variables in CNF

w, $|w| \geq p$

$u\,v\,x\,y\,z = w$

loop

$|vxy| \leq p$ cannot span entire string

$|vy| \geq 1$

$w = a^p b^p c^p$

Case 1

$\underbrace{a^p}_{uvxy} \mid \underbrace{b^p a^p}_{z}$

$a^{p+2} b^p a^p \notin L$

Case 2

$\underset{u}{a^p} \quad \underset{vxy}{b^p} \quad \underset{z}{a^p}$

$a^p b^{p+2} a^p \notin L$

Case 3

$\underbrace{a^p b^p}_{u} \,|\, \underbrace{a^p}_{vxy} \,|\, \underbrace{\lambda}_{z} \quad \notin \mathcal{L}$

Case 4

$a^p b^p a^p$

$\underbrace{aaa\ldots a}_{u}\underbrace{\boxed{ab}}_{vxy}\underbrace{b\ldots bbaa\ldots aa}_{z}$

Case 5

$a^p b^p a^p$

$\underbrace{aa\ldots aabb\ldots}_{u}\underbrace{\boxed{bb\,aa}}_{vxy}\underbrace{\ldots aa}_{z}$

Question 13

$\mathcal{L} = \{a^n b^m c^n d^{n+m} \mid n, m \geq 0\}$

Assume $\mathcal{L}$ is CFL, p = no. of variables in CNF

w, $|w| \geq p$

$u\,v\,x\,y\,z = w$

loop

$|vxy| \leq p$

$|vy| \geq 1$

$w = a^p b^q c^p d^{p+q}$

Case 1

vxy in only a's $\longrightarrow$ $n_a(w) \neq n_c(w)$

Case 2

$$\underbrace{aa\ldots}_{u}\underbrace{aabb}_{vxy}\underbrace{\ldots bbcc\ldots ccdd\ldots dd}_{z}$$

- d will not match n+m
- $a \neq c$

Case 3

only b's $\longrightarrow$ d's will not match

Case 4

bc $\longrightarrow$ d's not matched

Case 5

only c's $\longrightarrow$ a & c not match

Case 6

vxy over (abc) $\longrightarrow$ ruled out

Case 7

vxy over cd $\longrightarrow$ $a \neq c$

Case 8

only d's $\longrightarrow$ won't be n+m

* Question 14

$L = \{a^{n^2} \mid n \geq 0\}$

perfect squares

$L = \{\lambda, a, a^4, a^9, a^{16}, \dots\}$

Assume L is CFL, k = no. of variables in CNF (pumping constant)

w, $|w| \geq k$

$uvxyz = w$

loop

$|vxy| \leq k$

$|vy| \geq 1$

Let $n = k^2$

Let $w = a^{k^4}$

$\forall\ i \geq 0$, $w \in L$, $w = uv^ixy^iz$

w contains only a's

$vy \rightarrow a^p$ (pump p times)

$w = a^{k^4+p} = s$ (new string)

- to get the next perfect square of $n^2 \rightarrow (n+1)^2$
- for a^{k^4}, next string is $a^{(k^2+1)^2}$

$$a^{(k^4+2k^2+1)}$$

if $p = 2k^2+1$, only then the language is accepted

$\therefore$ $|vy|$ would be $2k^2+1$

but $|vy| \leq k$

* Question 15

$L = \{a^{n!} \mid n \geq 0\}$

$w = a^{m!}$

$\overbrace{u\ v\ x\ y\ z}^{m!}$

$v = a^{k_1}$

$y = a^{k_2}$

$ak_1 + ak_2 = k$

$\underbrace{u\ v\ x\ y\ z}$

$m! + ak_1 + ak_2 = m! + k$

$m! + k \leq m! + m \leq (m+1)!$

$a^{n!}$ not CFL

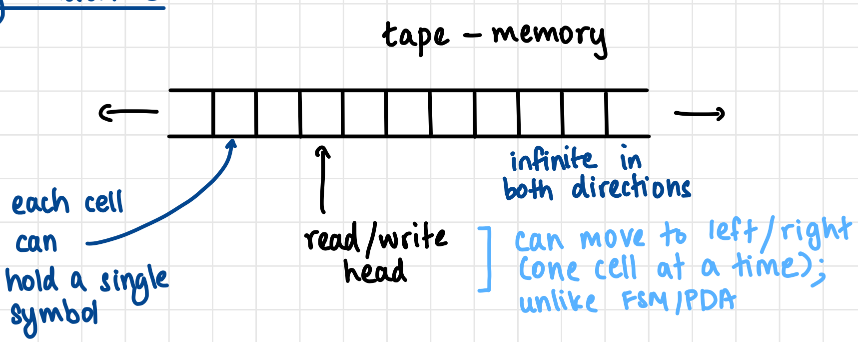
turing MACHINE

- More powerful than CFLs
- Languages — recursively enumerable languages
- Turing award – highest award given

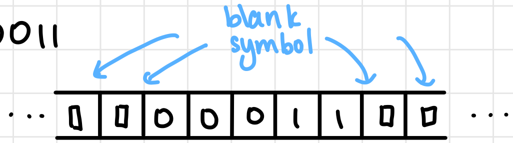
Alan Turing, 1936

- created an abstract machine (paper on Computable Numbers — 1936)
- Computing done on paper using symbols
- Imagine paper to be tape divided into square boxes and the tape is an abstract machine
- Behaviour of a computer (person) determined by the symbols they are using and their state of mind
- Transition depends on what state one is in and what one is looking at on the tape

Turing Machine



For w = 00011



Special symbols called blank symbols (not empty)

B or ␣ or ☐

- Any computation that a modern computer is capable of performing can be performed (computed) by a Turing Machine
- If a computation cannot be performed by a Turing Machine, then it cannot be performed by a modern computer
- Tape data structure acts as input, output and stores intermediate memory (infinite memory)

Formal Definition of Turing Machine

7-tuple, $M = (Q, \Gamma, \square, \Sigma, \delta, q_0, F)$

gamma (toe)

Q — finite set of states
Γ — tape alphabets
$\square$ — blank symbol
Σ — alphabet
δ — transition function
q_0 — start state
F — set of final states

state   input symbol   read/write

$$\delta = Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$$

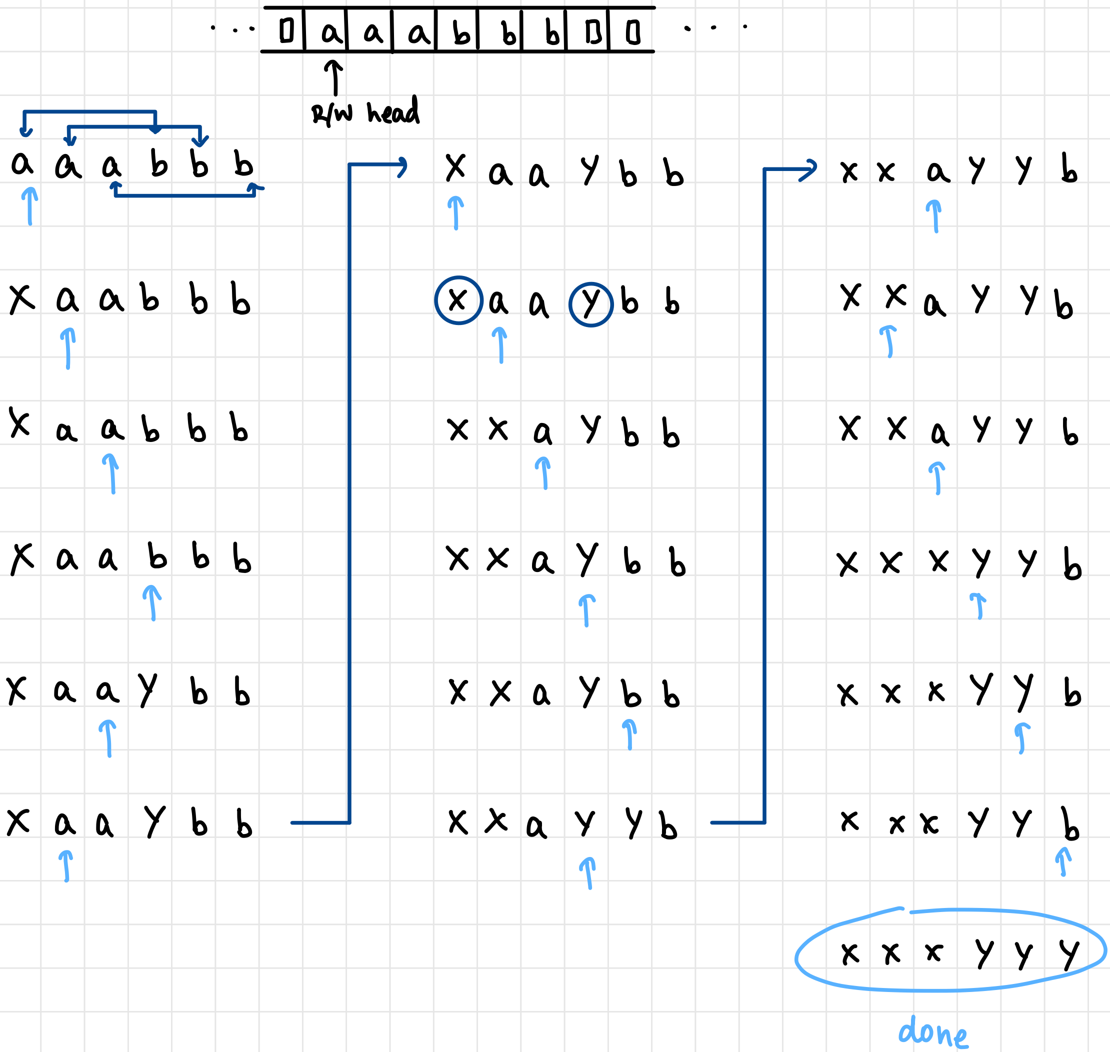
move to the left or right

- TM is more powerful than FA and PDA and any string accepted by them is accepted by TM

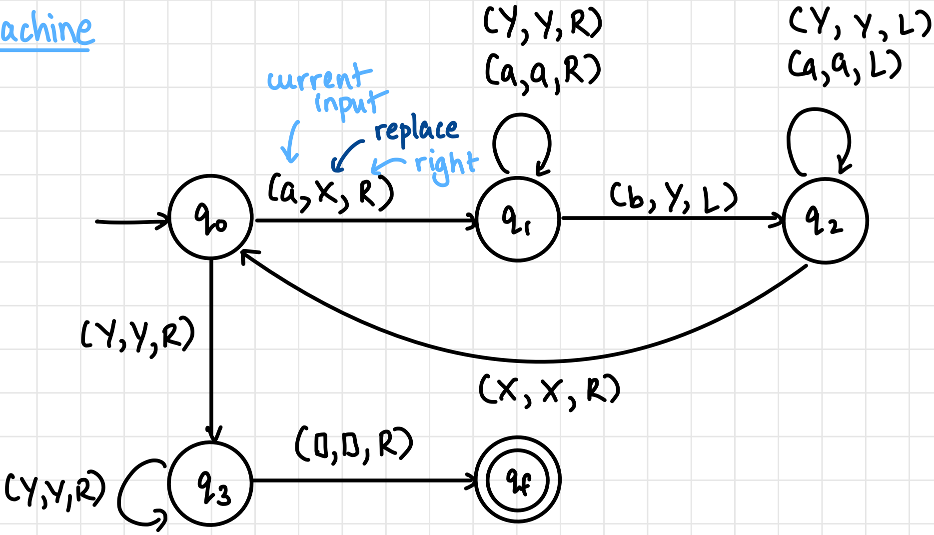
Question 16

$L = \{a^n b^n \mid n \geq 1\}$ show acceptance by TM

$w = aaabbb$



Machine

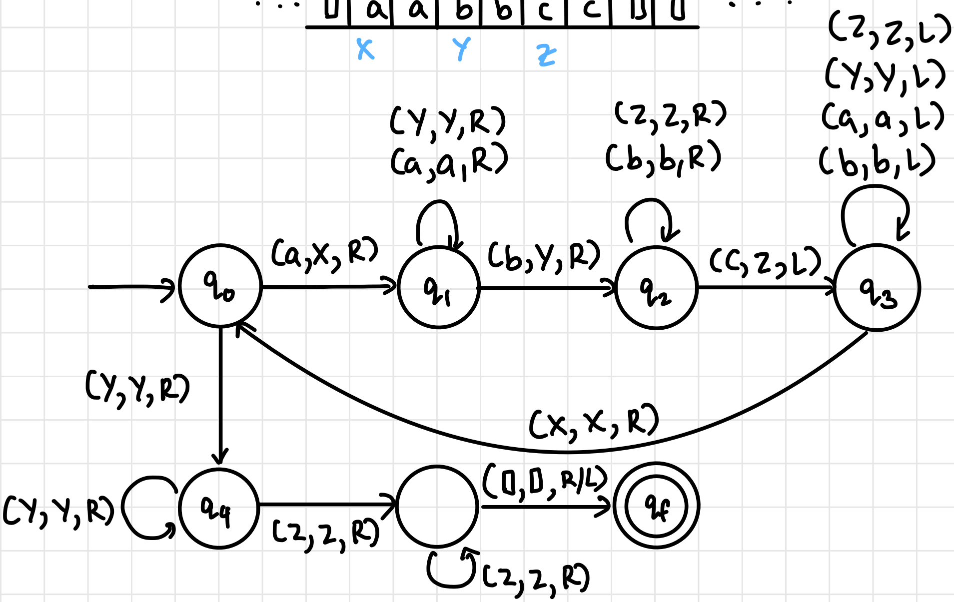


Question 17

$\mathcal{L} = \{a^n b^n c^n \mid n \geq 1\}$

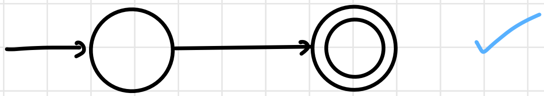
$w = aabbcc$

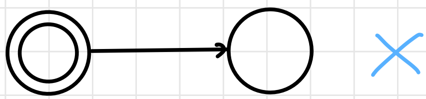
...	□	a	a	b	b	c	c	□	□	...
		X		Y		Z				



Acceptance

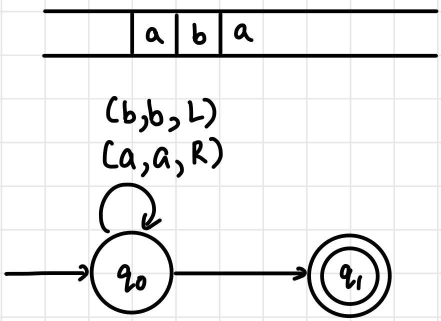
- No outgoing transition from final state





- If machine halts in a non-final state or enters an infinite loop, the string is rejected

Infinite loop



- If machine halts on final state, string is accepted

Turing Machine

Decider

decidable or recursive

algorithms

- $\forall w \in \mathcal{L}$
- all strings of a language
- either accepts all or rejects all

Halt and accept

Halt and reject

- Turing accepted machine
- Recursively enumerable language

Acceptor

- $w \in \mathcal{L}$
- particular string in a language
- only concerned about a single string

Halt and accept

Reject

halt

∞

unpredictable

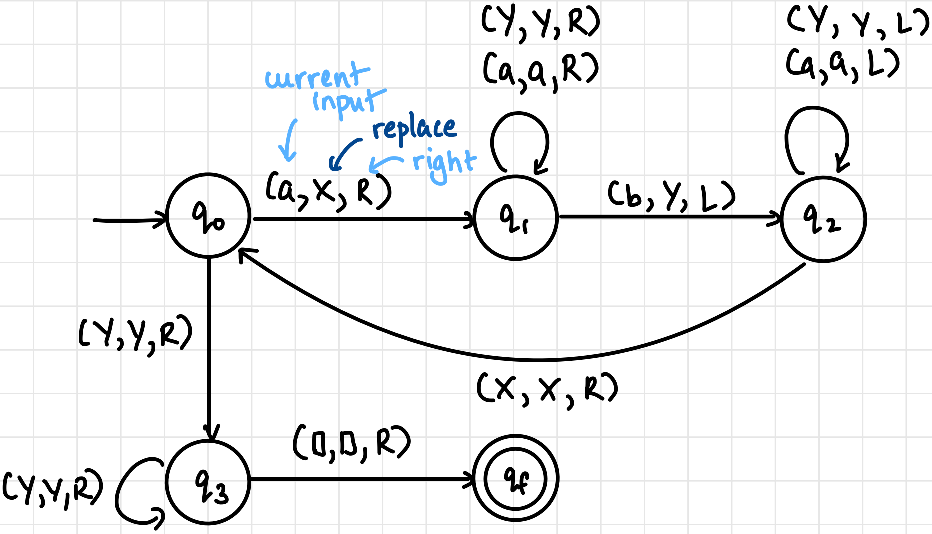
Continuation of Question 16

$$\mathcal{L} = \{a^n b^n \mid n \geq 0\}$$

Transition function

$$\delta(q_0, a) = (q_0, X, R)$$
$$\delta(q_0, b) = \text{halt}$$
$$\delta(q_1, a) = (q_1, a, R)$$
$$\delta(q_1, b) = (q_2, Y, L)$$

write for X & Y also (all tape symbols)



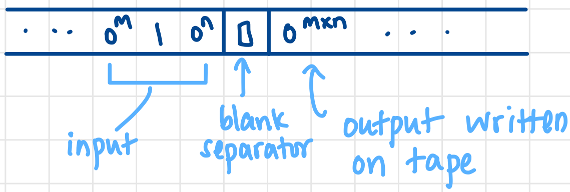
Transition Table

state	a	b	X	Y	□
$\rightarrow q_0$	q_1, X, R	halt	halt	q_3, Y, R	halt
q_1	q_1, a, R	q_2, Y, L	halt	q_1, Y, R	halt
q_2	q_2, a, L	halt	q_0, X, R	q_2, Y, L	halt
q_3	halt	halt	halt	q_3, Y, R	q_f, □, R
q_f	halt	halt	halt	halt	halt

Question 18

$\mathcal{L} = \{0^m 1 0^n 1 \;\; n, m \geq 1\}$ Language of Multiplication

i/p $\rightarrow$ o/p $= 0^{m \times n}$



- Perform repeated addition

- $w =$ 000 1 00 (the three 0s numbered 1 2 3)

(00) x 3

1st 0 $\longrightarrow$ print 00
2nd 0 $\longrightarrow$ print 00
3rd 0 $\longrightarrow$ print 00

] 000000 on tape

$w =$ 0001 00 ▯ ▯ ▯ ▯ ▯ ▯ ▯ ▯

A

X ▯0

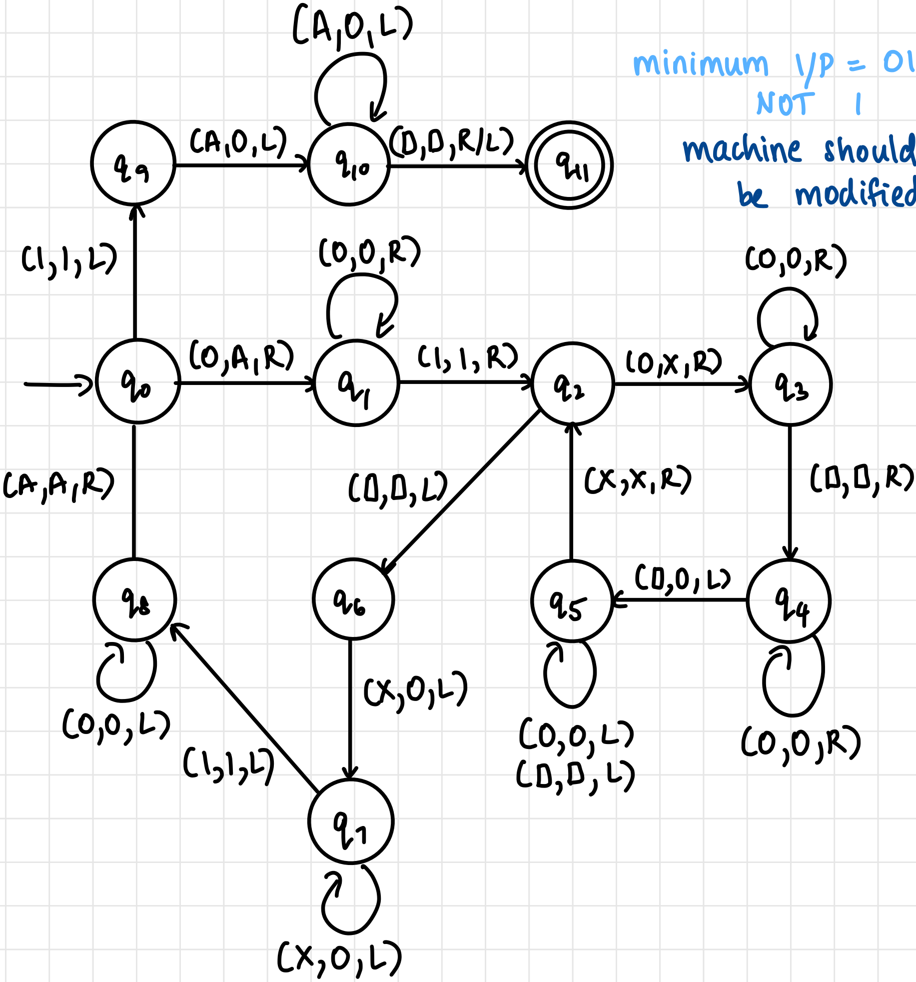
X ▯00

00

A continue

minimum I/P = 010
NOT 1

machine should be modified



w = 0 0 0 1 0 0 □ □ □ □ □ □ □
= A 0 0 1 X 0 □ 0 □ □ □ □ □
= A 0 0 1 X X □ 0 0 □ □ □ □
= A 0 0 1 0 0 □ 0 0 □ □ □ □
= A A 0 1 X 0 □ 0 0 0 □ □ □
= A A 0 1 X X □ 0 0 0 0 □ □
= A A 0 1 0 0 □ 0 0 0 0 □ □
= A A A 1 X 0 □ 0 0 0 0 0 □
= A A A 1 X X □ 0 0 0 0 0 0
= A A A 1 0 0 □ 0 0 0 0 0 0
= 0 0 0 1 0 0 □ 0 0 0 0 0 0

Question 19

$$\mathcal{L} = \{ww \mid w \in \{a,b\}^*\}$$

- write down logic & then construct – mandatory
- must find centre
- divide string of even length into 2 equal parts
- if length not even, half
- match first half with second half

$w =$ abaaba

□	□	a	b	a	a	b	a	□	□

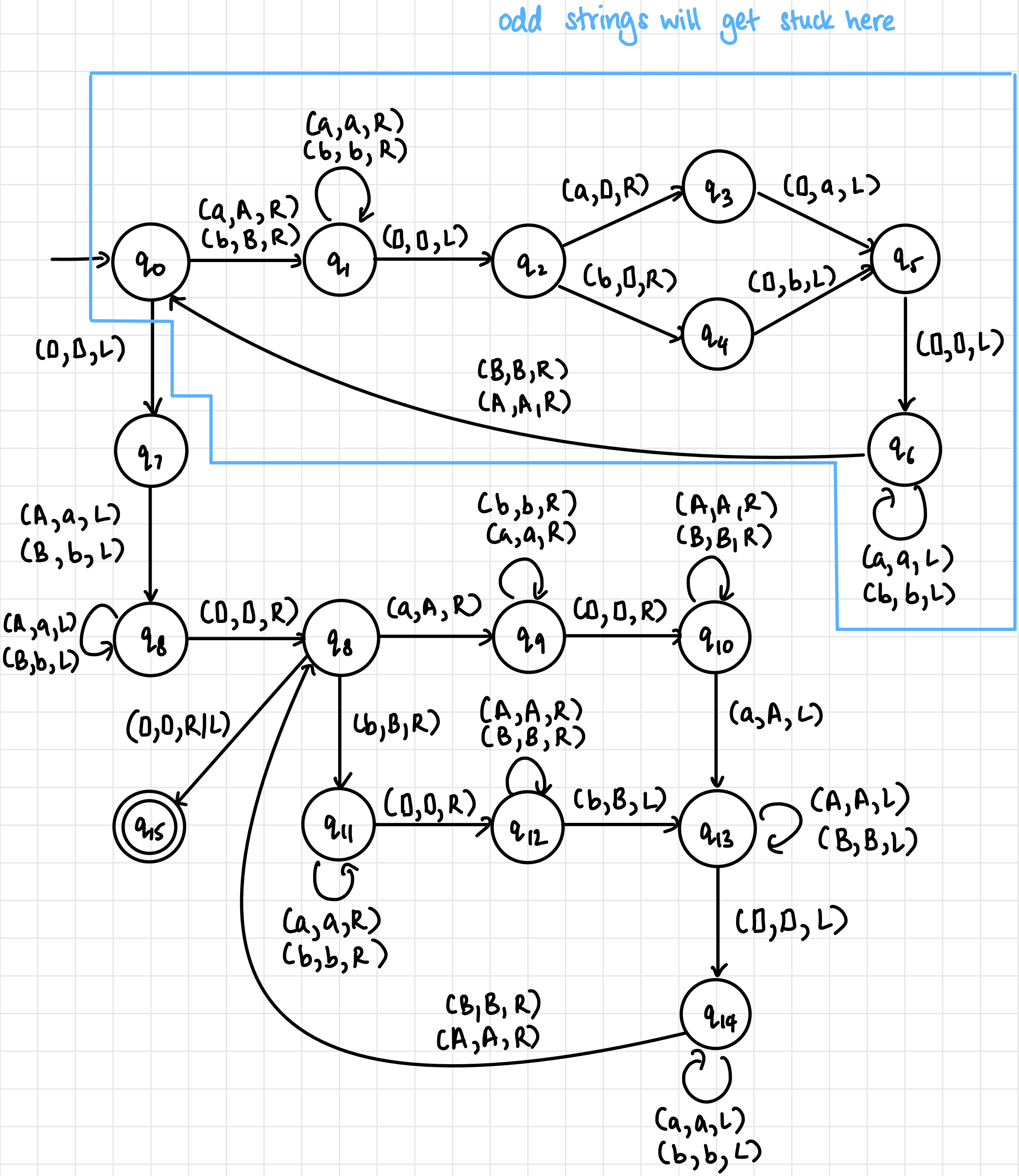
find centre

□	□	A	b	a	a	b	□	a	□
□	□	A	B	a	a	□	b	a	□
□	□	A	B	A	□	a	b	a	□

↑ centre

match

□	□	a	b	a	□	a	b	a	□
□	□	A	b	a	□	A	b	a	□
□	□	A	B	a	□	A	B	a	□
□	□	A	B	A	□	A	B	A	□



odd strings will get stuck here
(a,a,R)
(b,b,R)
(a,A,R)
(b,B,R)
(□,□,L)
(a,□,R)
(□,a,L)
(b,□,R)
(□,b,L)
(□,□,L)
(B,B,R)
(A,A,R)
(a,a,L)
(b,b,L)
(□,□,L)
(A,a,L)
(B,b,L)
(A,a,L)
(B,b,L)
(□,□,R)
(a,A,R)
(b,b,R)
(a,a,R)
(□,□,R)
(A,A,R)
(B,B,R)
(a,A,L)
(□,□,R/L)
(b,B,R)
(A,A,R)
(B,B,R)
(□,□,R)
(b,B,L)
(A,A,L)
(B,B,L)
(a,a,R)
(b,b,R)
(□,□,L)
(B,B,R)
(A,A,R)
(a,a,L)
(b,b,L)
q0
q1
q2
q3
q4
q5
q6
q7
q8
q8
q9
q10
q11
q12
q13
q14
q15

Question 20

Language of subtraction

write c's onto tape
- if a>b : c
- if b>a : -c

$\mathcal{L} = \{ a^m b^n c^k \mid k = m-n \mid n,m \geq 1 \}$

- #a > #b
- # a < #b
- #a = #b

#a's > # b's

copy # of c's [aabccc → should halt]

$\Rightarrow$ $k+n=m$ or

$a = \#b + \#c$

O/P

a	a	a	a	b	b	c	c	▯	▯	▯
x	a	a	a	y	b	c	c	▯	▯	▯
x	x	a	a	y	y	c	c	▯	▯	▯
x	x	x	a	y	y	z	c	▯	c	▯
x	x	x	x	y	y	z	z	▯	c	c

match a's & b's

aaaa bb cc

X a a a Y

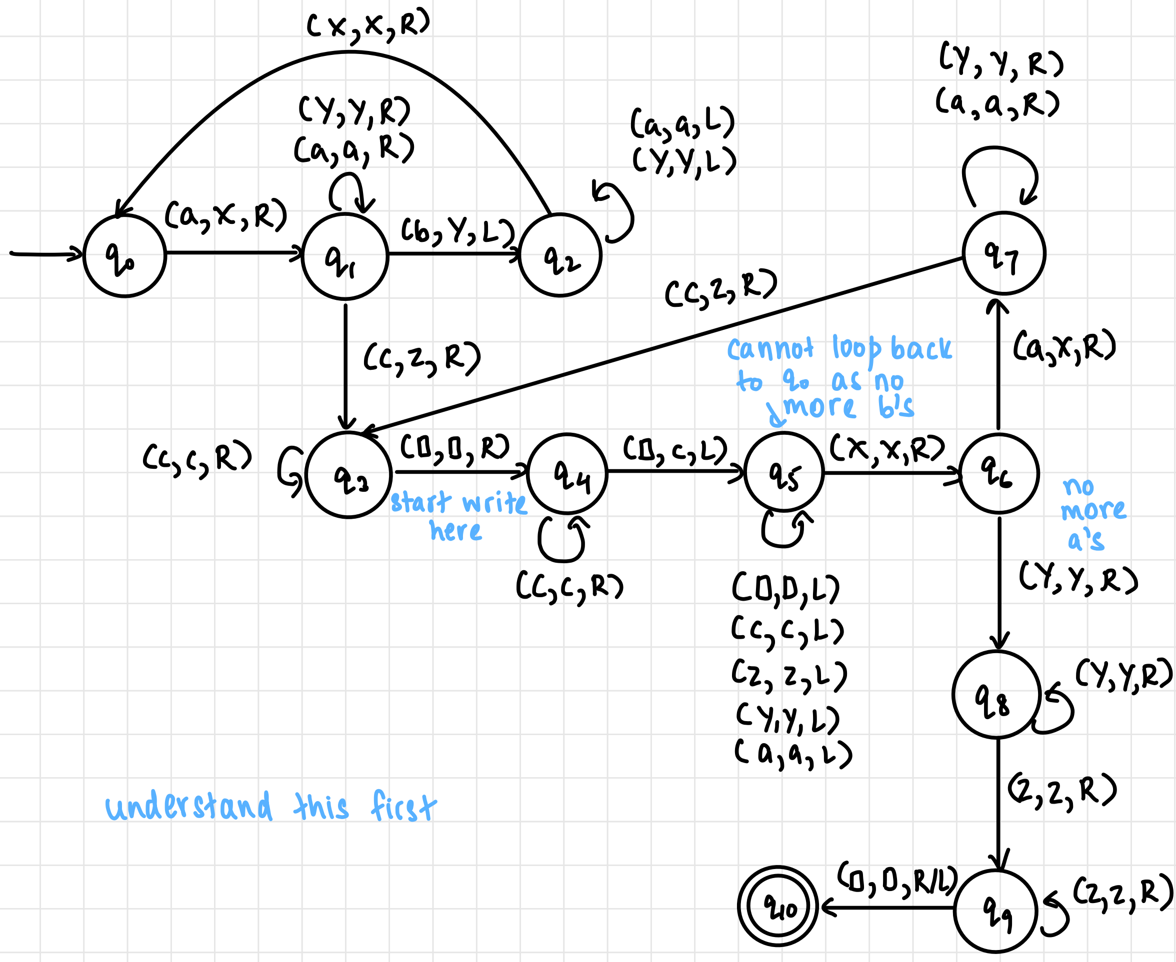
X X a a Y Y c

X X X a Y Y Z c □ c

X X X X Y Y Z Z □ c c

↑ $q_6 \to q_8$

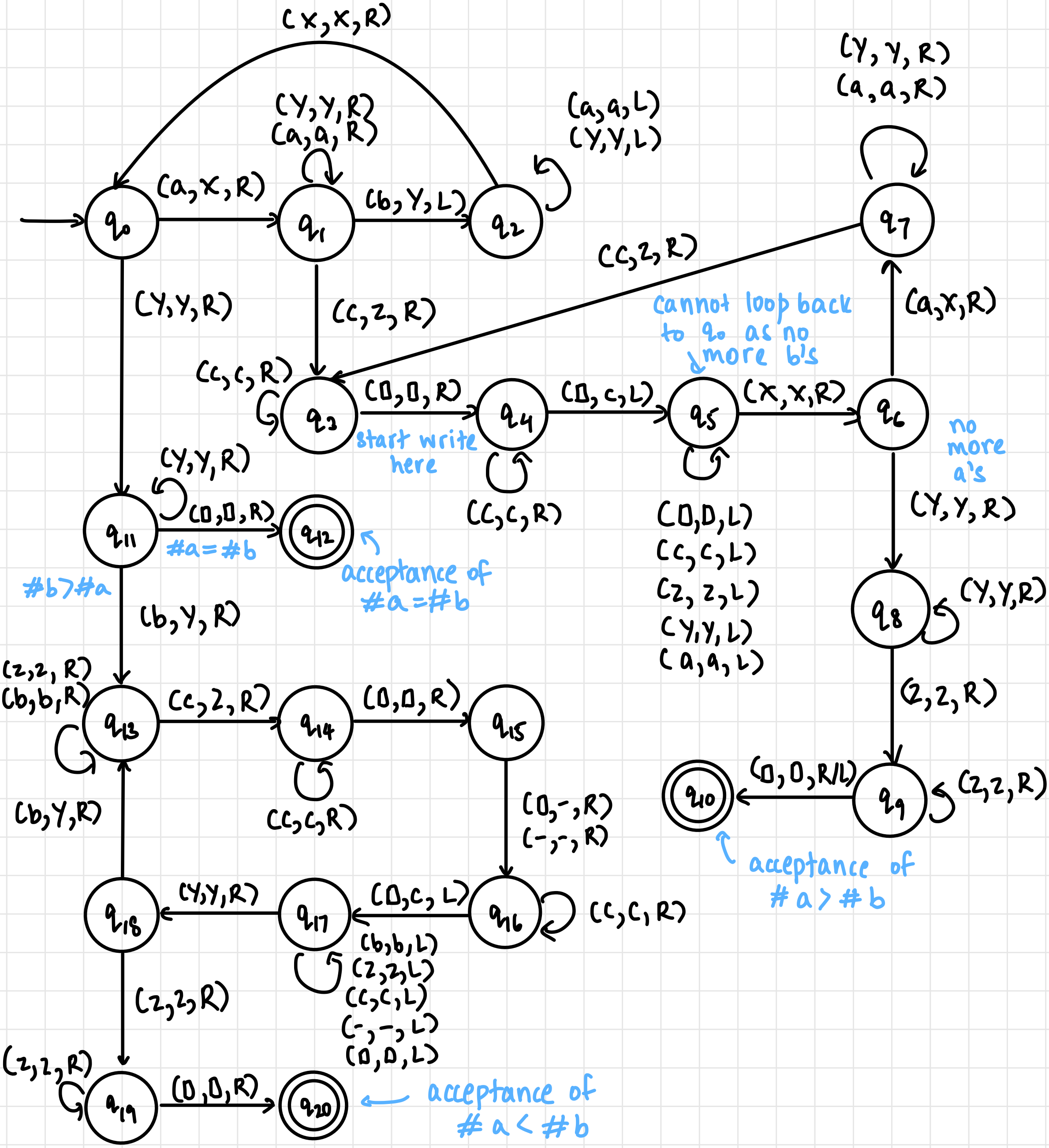
1) Only for #a's = #b's ; NOT COMPLETE YET



understand this first

2) #a's = #b's & #a's < #b's

aabbbbcc
XaYbbbcc
XXYYbbcc → XXYYYbZc□-c
XXYYYYZZ□-CC
↑ q18 – q19



Question 21

$$L = \{0^n 1^{n^2},\ n \geq 1\}$$

$n = 3$

$w = 000\,111111111$
$x\,00\,yyy\,111111$
$x\,x\,0\,yyy\,yyy\,111$
$xxx\,yyyyyyyyy$

roughly

Tape contents

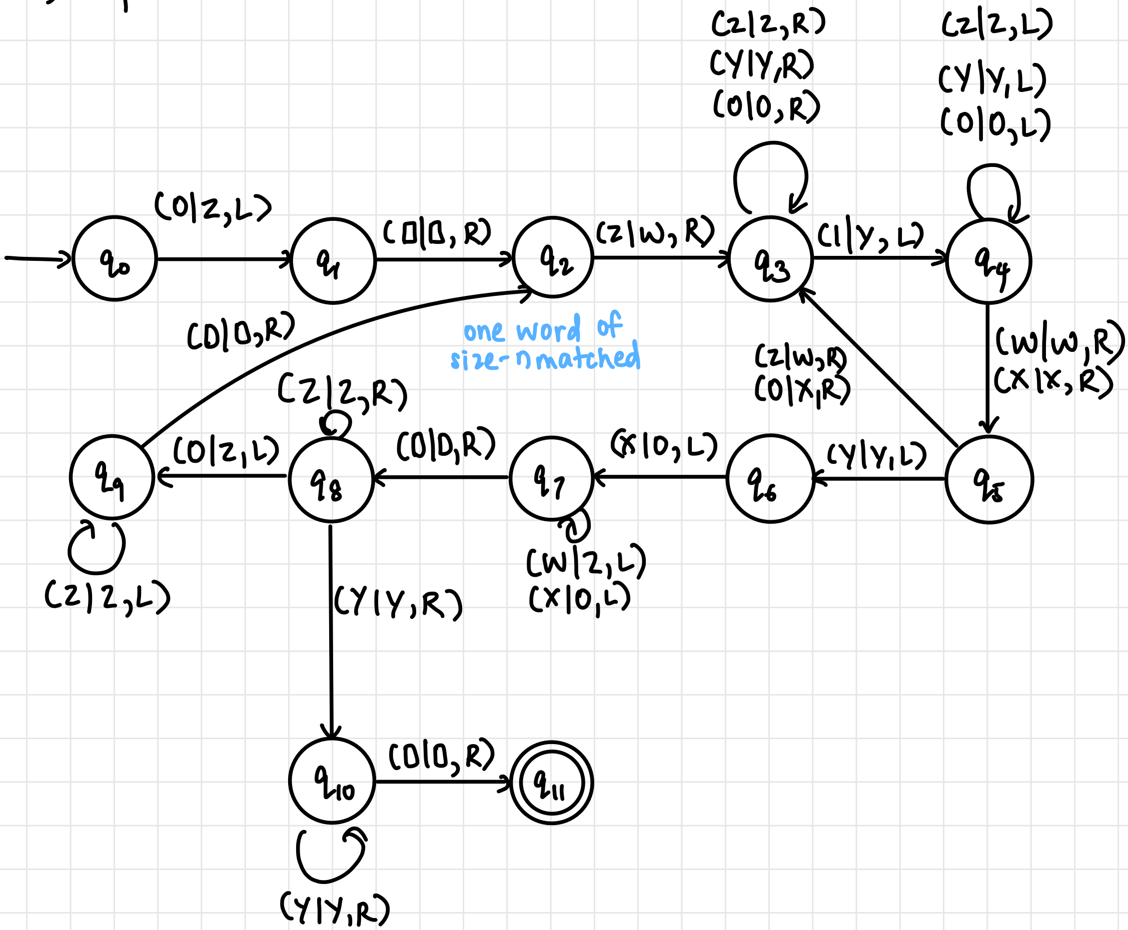
	☐	0	0	0	1	1	1	1	1	1	1	1	1	☐
1st →	☐	Z	0	0	1	1	1	1	1	1	1	1	1	☐
	☐	W	0	0	Y	1	1	1	1	1	1	1	1	☐
	☐	W	x	0	Y	Y	1	1	1	1	1	1	1	☐
	☐	W	x	x	Y	Y	Y	1	1	1	1	1	1	☐
	☐	Z	0	0	Y	Y	Y	1	1	1	1	1	1	☐
2nd →	☐	Z	Z	0	Y	Y	Y	1	1	1	1	1	1	☐
	☐	W	Z	0	Y	Y	Y	Y	1	1	1	1	1	☐
	☐	W	W	0	Y	Y	Y	Y	Y	1	1	1	1	☐
	☐	W	W	X	Y	Y	Y	Y	Y	Y	1	1	1	☐
	☐	Z	Z	0	Y	Y	Y	Y	Y	Y	1	1	1	☐
3rd →	☐	Z	Z	Z	Y	Y	Y	Y	Y	Y	1	1	1	☐
	☐	W	Z	Z	Y	Y	Y	Y	Y	Y	Y	1	1	☐
	☐	W	W	Z	Y	Y	Y	Y	Y	Y	Y	Y	1	☐
	☐	W	W	W	Y	Y	Y	Y	Y	Y	Y	Y	Y	☐
	☐	Z	Z	Z	Y	Y	Y	Y	Y	Y	Y	Y	Y	☐ ✓ accepted

Logic

1) Check if 0 exists

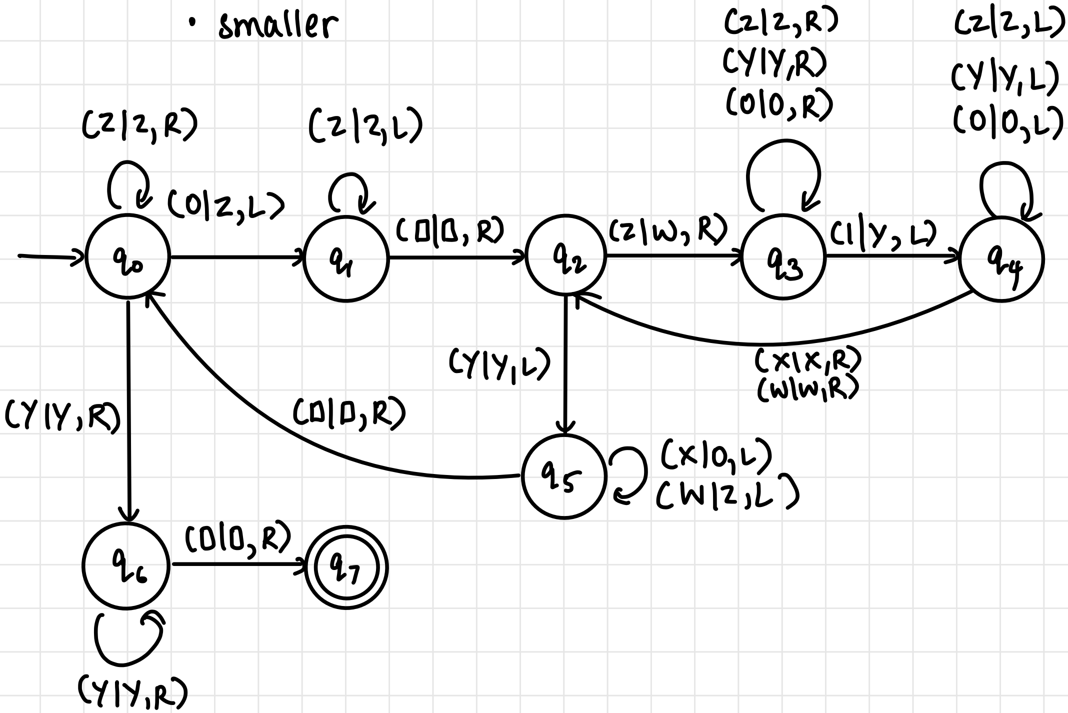
2) Check if n 1's corresponding to that 0 exist

3) Repeat



Alternate Machine

- smaller



Question 22

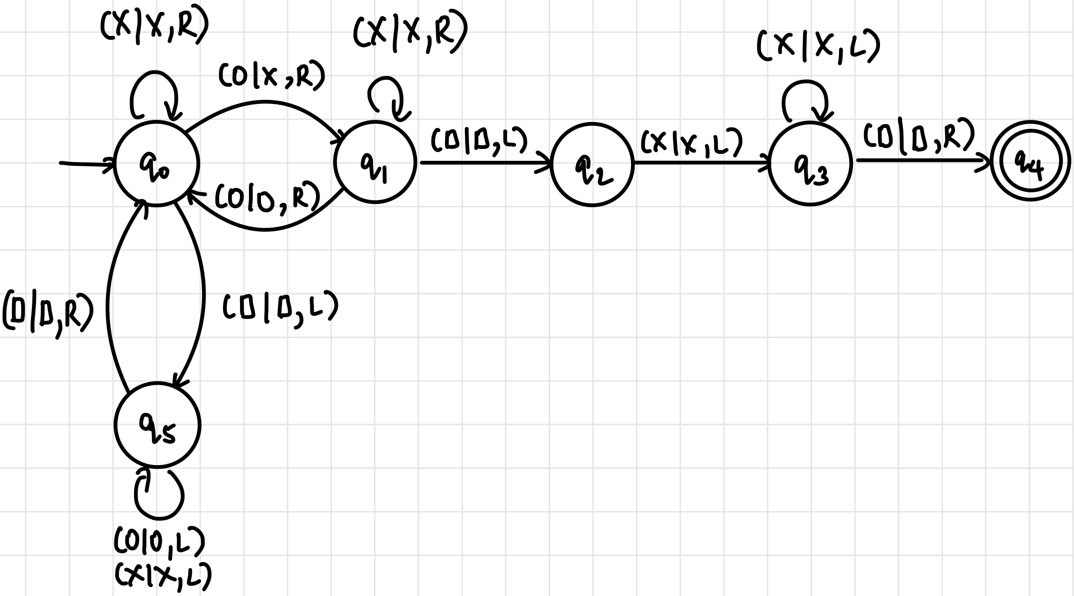
$$L = \{0^{2^n} \mid n \geq 0\}$$

$n = 3$

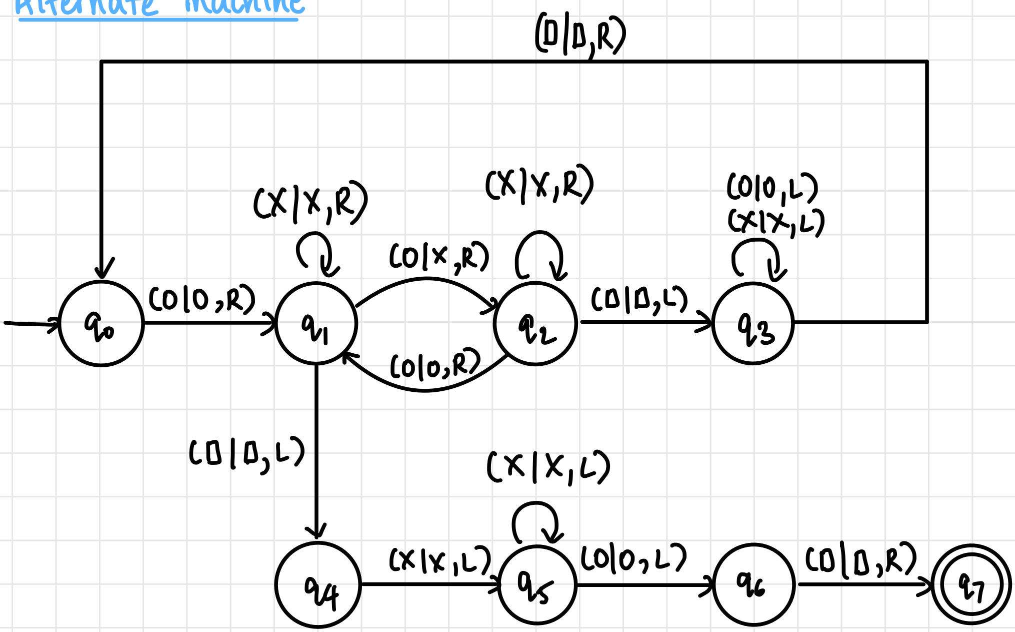
□	0	0	0	0	0	0	0	0	□
□	X	0	X	0	X	0	X	0	□
□	X	X	X	0	X	X	X	0	□
□	X	X	X	X	X	X	X	0	□
□	X	X	X	X	X	X	X	X	□

Logic

- every round, check for (1/2) of the remaining zeroes
- strike out alternate zeroes
- eg: for 8 0's, first round 4 will go, then 2, then 1 then 1



Alternate machine

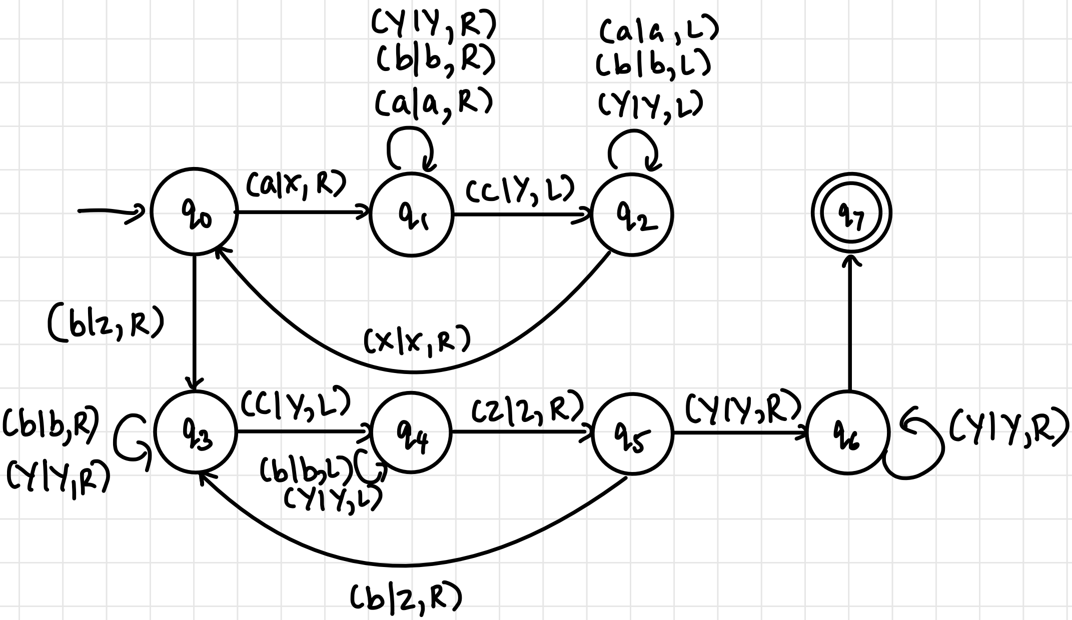


Question 23

$\mathcal{L} = \{a^n b^m c^{n+m}\}$ accept

- match #a's with #c's
- match #b's with #c's
- accept

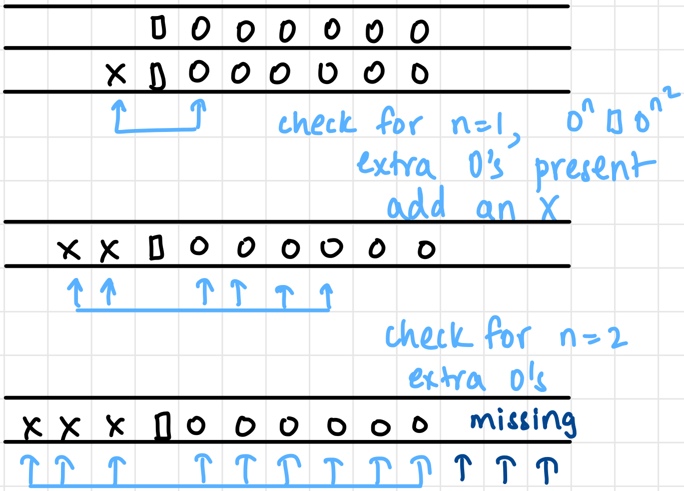
□	a	a	b	b	b	c	c	c	c	c	□	□
□	X	a	b	b	b	Y	c	c	c	c	□	□
□	X	X	b	b	b	Y	Y	c	c	c	□	□
□	X	X	Z	b	b	Y	Y	Y	c	c	□	□
□	X	X	Z	Z	b	Y	Y	Y	Y	c	□	□
□	X	X	Z	Z	Z	Y	Y	Y	Y	Y	□	□



Question 24

$\mathcal{L} = \{0^{n^2}\}$

- in principle, hit and trial



- possible to construct

Church-Turing thesis (not a theory)

- anything that can be computed can be computed by a turing machine (standard TM)
- variations of the machine add no power (only time/space)
- multi-tape can be converted to single tape by separating the tapes with a symbol (#)

Computable Function

- Function that TM can compute
- Alonzo Church and Alan Turing

  Lamba calculus   TM

- Only functions that are computable by TM's are computable
- Algorithm: TM decider
- Set of all TMs

  $\{ M_1, M_2, M_3, \dots \}$ set of all Turing Machines

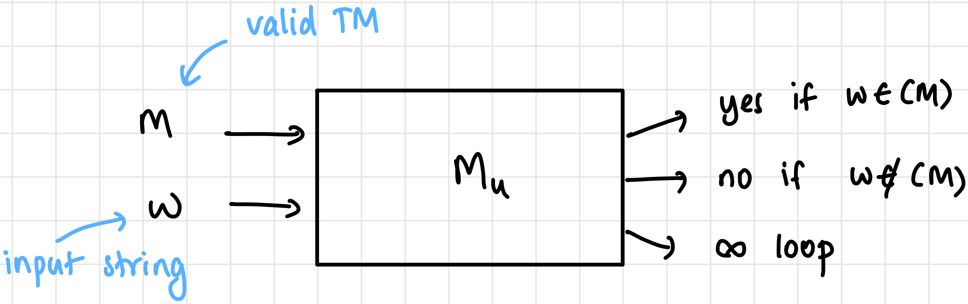
  $\{ \mathcal{L}(M_1), \mathcal{L}(M_2), \mathcal{L}(M_3), \dots \}$

  set of all possible computable functions

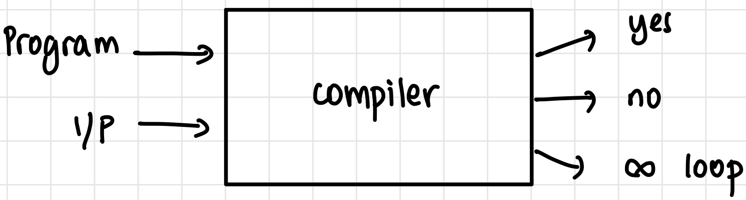
  $\{ P_1, P_2, P_3 \dots \}$ valid computer programs one could write

Universal Turing Machine

- Single TM that can compute any computable function
- Single TM that can simulate a TM (TM for a TM)
- Similar to compiler

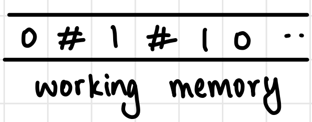


Similar to compiler

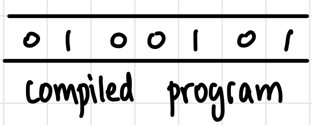


Structure of UTM

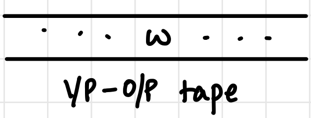
- 3-tape machine



current state no. in binary



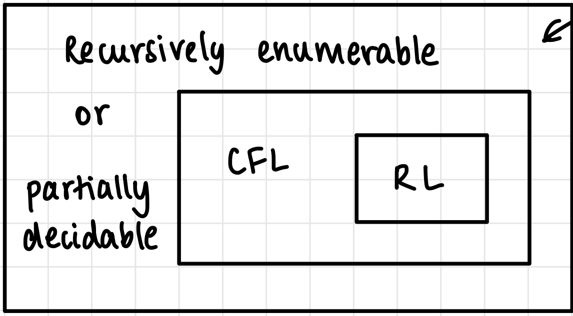
encoding of TM (binary string)
$\delta(q_2, b, L)$ left = 0 etc.



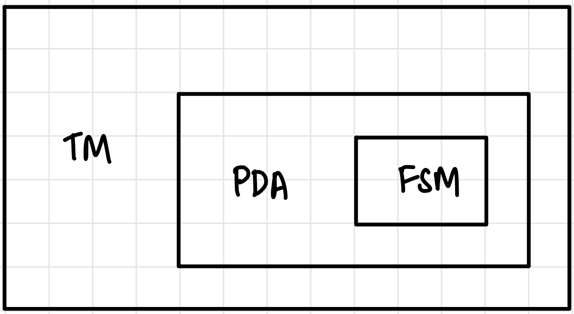
I/P loaded from, O/P written to

Chomsky Hierarchy

Turing recognisable, semi-decidable



Languages



Machines

Recursively Enumerable Set

These languages are accepted by TM

a) a set is recursively enumerable if there exists an algorithm/ procedure that can enumerate all members of the set

b) makes sense to talk about the i^{th} element of a set

c) can generate an element of the set in finite time

d) if the set can be mapped with the set of natural numbers N in one-to-one correspondence

e) finite no. of elements between two elements in the set

Question 25

Prove that the following are RE languages (write TM to enumerate strings)

1) Set of N
2) Set of even no.s (E)

] write each element onto tape with 1-1 correspondance to natural no.s

- countable infinity

3) Σ^*

- set of all strings over the alphabet Σ
- countably infinite
- suppose $\Sigma = \{a, b, c\}$
- generate strings of increasing length

works

λ length 0

a, b, c length 1

aa, ab, ac, ba, bb, bc, ca, cb, cc length 2

⋮

can print string in finite time

- alphabetical does not work

λ, a, aa, aaa ... never reaches b

4) Set of all TMs are enumerable

- every TM can be described as binary string
- length $|\langle M \rangle|$ is finite — finite binary strings
- set of TM $\subset (0+1)^*$ (Σ^*)

finite states

- proper subset (not all encodings are valid)
- enumerate like prev. example
- makes sense to talk about i^{th} TM

5) $Q = \{ \frac{m}{n} \mid m, n \in N \}$ is a set of +ve rational no.s

Prove countably ∞

	1	2	3	4	5	
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	. . .
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	. . .
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	. . .
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	. . .
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	. . .
	⋮	⋮	⋮	⋮	⋮	

check for duplicates in prog.

- count diagonally; not row/column wise

6) Enumerate all strings in RE language

- parallel computation, one step at a time
- no full string processed at once

Non-Recursively Enumerable

- cantor – pair elements of different sets
- Diagonalisation

Question 26

Set of all real no.s

1 & 2

infinite no.s between

if precision limited, becomes recursively enumerable

- cannot enumerate all real no.s
- uncountably infinite
- cannot talk about ith element (real no.)

Formal Proof

Diagonalisation – proof by contradiction

Assumption: R is enumerable

	f(n) ⇒ func. that lists all real no.s
1	3 . 1 4 2 3 4
2	7 . 8 9 1 4 0
3	8 . 2 3 6 7 1
4	9 . 1 1 7 9 2
5	2 . 2 3 0 1 3

r_d = .19693

complement / add / sub / alter

$\overline{r_d}$ = .08582

- $\overline{r_d}$ cannot be #1 $\because$ it differs in the first dec.
- cannot be #2 $\because$ it differs in the second dec.
- differs from i^{th} row in i^{th} place

- $\overline{r_d}$ is new number that has not been enumerated (contradiction)

- $\therefore$ it is not recursively enumerable

Question 27

Set of all languages

any language $\subseteq \Sigma^*$ $\leftarrow$ enumerable

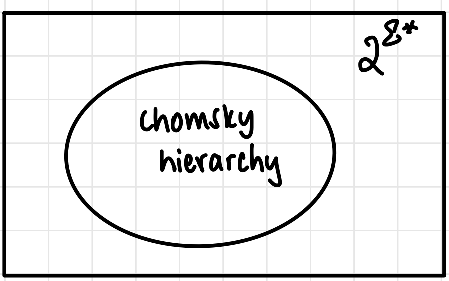
$\{L_1, L_2 \ldots\}$

all strings

Total no. of subsets that can be formed using the set Σ^*

= Power set of Σ^*

$\mathcal{L} \in P(\Sigma^*)$ $\quad 2^{\Sigma^*}$ languages



Formal Proof

Assumption: Set of all languages is enumerable

TMs		s_1	s_2	s_3	s_4	s_5	s_6	... (all strings - enum)
M_1	L_1	1	0	0	1	1	0	$\mathcal{L}_1 = \{s_1, s_4, s_5 \dots$
M_2	L_2	0	1	0	0	0	1	$\mathcal{L}_2 = \{s_2, s_6 \dots$
M_3	L_3	1	0	1	1	0	1	$\mathcal{L}_3 = \{s_1, s_3, s_4, s_6 \dots$
M_4	L_4	1	1	0	0	0	1	$\mathcal{L}_4 = \{s_1, s_2, s_6 \dots$
M_5	L_5	0	1	1	0	0	1	$\mathcal{L}_5 = \{s_2, s_3, s_6 \dots$
⋮	⋮	0	1	1	0	1	1	$\mathcal{L}_6 = \{s_2, s_3, s_5, s_6 \dots$

(1 – string is in language)

$\mathcal{L}_{diag} = \{s_1, s_2, s_3, s_6\}$
↓
could be enumerated

find $\mathcal{L}$'s complement

$\mathcal{L}_{\text{NON-RE}} = \{s_4, s_5 \dots\}$
↓
not same as any enumerated string

- differs from i^{th} row in i^{th} place
- ∴ there are more languages than Turing Machines
- TMs are RE and finite
- There are more problems than solutions (TMs)
- Most $\mathcal{L}$ are non recursively enumerable (do not have direct description)

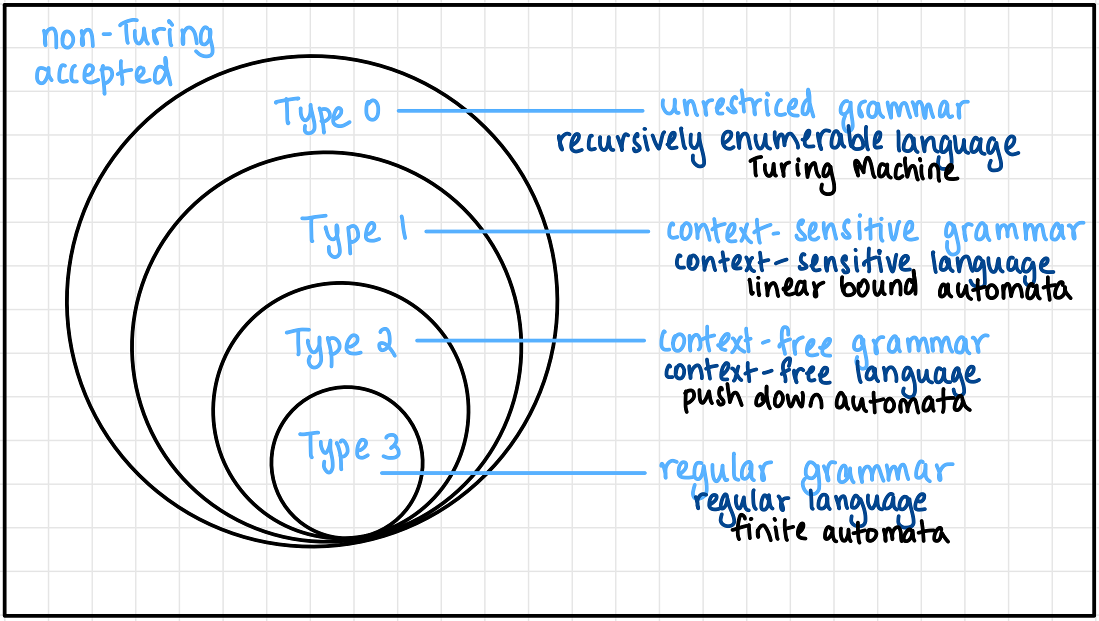
- There exist RE languages that are not decidable as well as non RE languages that are full undecidable

Recursive	Recursively Enumerable
• recursive ⊂ RE	
• enumeration procedure (Turing Machine)	• enumeration procedure (Turing Machine)
• Membership function	• No membership function (can enter ∞ loop)
• TM deciders	• TM acceptors

undecidable → non-RE (R, set of all langs)

undecidable → RE but not decidable (PCP, empty intersection, halting problem, acceptance problem)

Chomsky Hierarchy



Undecidability

- Exploring limits of algorithmic solvability; unsolvability
- Non-RE → no TM
- RE but not decidable → no algorithm for all instances
- Try to simplify / alter problem if unsolvable

POST CORRESPONDANCE PROBLEM

- Undecidable decision problem $\Rightarrow$ RE but not decidable
- Emil Post in 1946
- Given two sets of strings A & B framed over the same alphabet $\Sigma = \{a, b\}$

could be infinite; same no. of strings in A & B

	A	B
1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
$\vdots$	$\vdots$	$\vdots$
n	a_n	b_n

- Come up with a sequence such that if strings of A & B are concatenated in that sequence, the resulting strings are equal
- eg: sequence 3, 2, 1

$$a_3 a_2 a_1 = b_3 b_2 b_1$$

should hold true

- Programming approach: if list is finite and repetitions are not allowed, all permutations can be listed and it is solvable
- If infinite or repetitions allowed, unsolvable

Question 28

PCP for the given sets of strings

	A	B
1	10	10110
2	111	01
3	110110	1010

- cannot start with 2 as it differs in first place
- cannot start with 3 as it differs in second place
- can start with 1

	1	3	
A	10	110 110	×
B	10 110	1010	

- this set of A & B has no solution

Question 29

	A	B
1	a	baa
2	ab	aa
3	bba	bb

PCP : solve

- cannot start with 1 or 2

	3	2	3	1
A	bba	ab	bba	a
B	bb	aa	bb	baa

bbaabbbaa

solvable

Question 30

Solve PCP

	A	B
1	bb	bbb
2	baa	aab
3	bbb	bb

1)

	1	3
A	bb	bbb
B	bbb	bb

$(13+31)^*$ any combination

2)

	1	2	3
A	bb	baa	bbb
B	bbb	aab	bb

Question 31

Solve PCP

	A	B
1	10	101
2	011	11
3	101	011

	1	3	1
A	10	101	10
B	101	011	101

×

	1	3	3	
A	10	101	101	. . .
B	101	011	011	. . .

never halts

- If able to prove A: undecidable (PCP), can prove B is undecidable

Idea of Reduction

subroutine

$A \leq_m B$ assume if B can be solved, so can A

(reduce A to B)

know undecidable (PCP)

Question 32

∞ loop

(Problem of empty intersection) Given 2 CFLs $\mathcal{L}_1$ & $\mathcal{L}_2$, can we find out if $\mathcal{L}_1 \cap \mathcal{L}_2$ is ϕ

- Reduce PCP to $\mathcal{L}_1 \cap \mathcal{L}_2 = \phi$; change I/P

$$PCP \leq_m \mathcal{L}_1 \cap \mathcal{L}_2 = \phi ?$$

PCP

	A	B
a	1	111
b	10111	10
c	10	0

convert

↓

2 grammars

List A: $S_A \rightarrow 1S_A a \mid 10111 S_A b \mid 10 S_A c \mid -$

List B: $S_B \rightarrow 111 S_B a \mid 10 S_B b \mid 0 S_B c \mid -$

	b	a	a	c
A	10111	1	1	10
B	10	111	111	0

- try to generate w= 10 111 1110 with grammars

1) $S_A \rightarrow 1S_A a \mid 10111 S_A b \mid 10 S_A c \mid -$ (baac)

$$
\begin{aligned}
S_A &\overset{lm}{\Rightarrow} 10111\, S_A\, b \\
&\Rightarrow 101111\, S_A\, ab \\
&\Rightarrow 1011111\, S_A\, aab \\
&\Rightarrow 101111110\, S_A\, caab \\
&\Rightarrow 101111110 - caab
\end{aligned}
$$

sequence kept in check

reversed because it is nested

2) $S_B \rightarrow 111 S_B a \mid 10 S_B b \mid 0 S_B c \mid -$

$$
\begin{aligned}
S_B &\overset{lm}{\Rightarrow} 10\, S_B\, b \\
&\Rightarrow 10111\, S_B\, ab \\
&\Rightarrow 10111111\, S_B\, aab \\
&\Rightarrow 101111110\, S_B\, caab \\
&\Rightarrow 101111110 - caab
\end{aligned}
$$

undecidable

- However, PCP is unsolvable ⇒ solution to empty intersection is unsolvable (no algorithm)

- One particular PCP is solvable but not all

Question 33

$A \le_m B$
$B \le_m C$
$D \le_m C$

A is RE but not decidable, C is recursive.

Can this statement be true?

No, it is false ∵ A is UD ⇒ B is UD ⇒ C is UD

Acceptance Problem of TMs & Halting Problem of TMs

- A_{TM} and $HALT_{TM}$
- RE languages but not decidable $\Rightarrow$ undecidable languages
- $A_{TM} = \{ \langle M, w\rangle \mid \text{if } M \text{ is a TM and } M \text{ accepts } w \}$

  pair
- $HALT_{TM} = \{ \langle M, w\rangle \mid \text{if } M \text{ is a TM and } M \text{ halts on } w \}$
- Diagonalisation $\rightarrow$ reduced $A_{TM} \leq_m HALT_{TM}$ ← subroutine

  undecidable

Proving that $\mathcal{L} = A_{TM}$ is Undecidable

$$A_{TM} = \{ \langle M, w\rangle \mid \text{if } M \text{ is a TM and } M \text{ accepts } w \}$$

- Assume there exists a TM 'H' that can decide any A_{TM} (takes $\langle M, w\rangle$)
- Define new TM 'D' that uses 'H' as a subroutine
- I/P : M, $\langle M\rangle$ $\Rightarrow$ O/P : accept if M rejects description
  reject if M accepts description

Machine H

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	. .	$\langle D\rangle$
M_1	A	R	A	R		A
M_2	A	A	A	A	. . .	R
M_3	R	R	R	R		A
M_4	A	A	R	R		R
⋮		⋮				
D	R	R	A	A		?

can be anything

must be its contradiction ?

- $\therefore$ D & H cannot exist
- A_{TM} is undecidable

$HALT_{TM}$

- $HALT_{TM} = \{ \langle M, w \rangle \mid$ if M is a TM and M halts on w $\}$
- Halt by accepting or rejecting
- $A_{TM} \leq_m HALT_{TM}$
- Assume there exists TM 'R' that decides the language $HALT_{TM}$

  subroutine
- TM that decides A_{TM} = 'H'
- If R decides halting, then definitely H decides A_{TM}; but H cannot exist