

UE19CS251

DESIGN AND ANALYSIS OF ALGORITHMS

UNIT 5: Limitations of Algorithmic Power and  
Coping with the Limitations

Decision Trees

PES University

## Outline

### Concepts covered

- Decision Trees
  - Smallest of three numbers
  - Sorting
  - Searching

## 1 Problem Types: Optimization and Decision

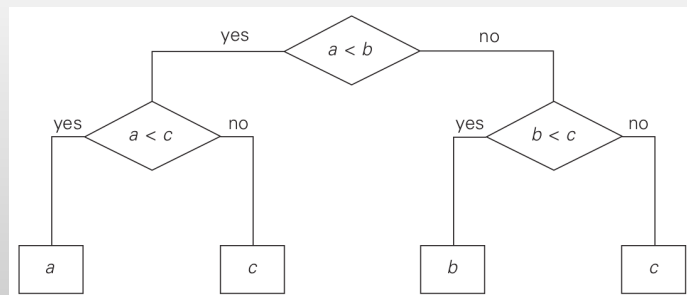
- Optimization problem: find a solution that maximizes or minimizes some objective function
- Decision problem: answer yes/no to a question
- Many problems have decision and optimization versions
  - Ex: traveling salesman problem
  - optimization: find Hamiltonian cycle of minimum length
  - decision: find Hamiltonian cycle of length  $m$
- Decision problems are more convenient for formal investigation of their complexity

## 2 Introduction

- Many important algorithms, especially those for sorting and searching, work by comparing items of their inputs
- We can study the performance of such algorithms with a device called the decision tree

## 3 Example: Decision tree for minimum of three numbers

Decision tree for a determining the minimum of three numbers



## 4 Central Idea

- The central idea behind this model lies in the observation that a tree with a given number of leaves, which is dictated by the number of possible outcomes, has to be tall enough to have that many leaves
- Specifically, it is not difficult to prove that for any binary tree with  $l$  leaves and height  $h$

$$h \geq \lceil \log_2 l \rceil$$

- A binary tree of height  $h$  with the largest number of leaves has all its leaves on the last level
- Hence, the largest number of leaves in such a tree is  $2^h$
- In other words,  $2^h \geq l$  which implies  $h \geq \lceil \log_2 l \rceil$

## 5 Decision Trees for Sorting Algorithms

- Most sorting algorithms are comparison-based, i.e., they work by comparing elements in a list to be sorted
- By studying properties of binary decision trees, for comparison-based sorting algorithms, we can derive important lower bounds on time efficiencies of such algorithms
- We can interpret an outcome of a sorting algorithm as finding a permutation of the element indices of an input list that puts the list's elements in ascending order
- For example, for the outcome  $a < c < b$  obtained by sorting a list  $a, b, c$
- The number of possible outcomes for sorting an arbitrary  $n$ -element list is equal to  $n!$

## 6 Decision Trees for Sorting Algorithms

- The height of a binary decision tree for any comparison-based sorting algorithm and hence the worst -case number of comparisons made by such an algorithm cannot be less than

$$C_{worst}(n) \geq \lceil \log_2 n! \rceil$$

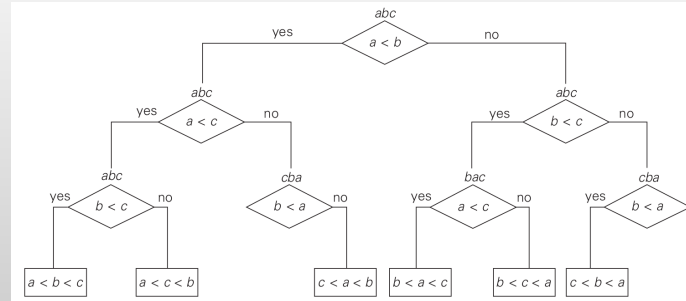
- Using Stirling's formula:

$$\lceil \log_2 n! \rceil \approx \log_2 \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = n \log_2 n - n \log_2 e + \frac{\log_2 n}{2} + \frac{\log_2 \pi}{2} \approx n \log_2 n$$

- About  $n \log_2 n$  comparisons are necessary to sort an arbitrary  $n$ -element list by any comparison-based sorting algorithm

## 7 Decision Trees for Sorting Algorithms

### Decision Tree for Three Element Selection Sort



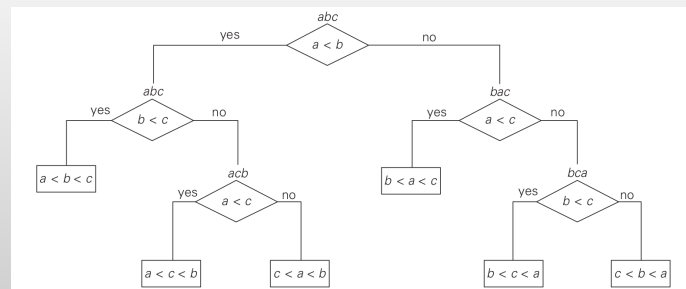
## 8 Decision Trees for Sorting Algorithms

- We can also use decision trees for analyzing the average-case behavior of a comparison based sorting algorithm
- We can compute the average number of comparisons for a particular algorithm as the average depth of its decision tree's leaves, i.e., as the average path length from the root to the leaves
- For example, for the three-element insertion sort this number is:

$$\frac{2 + 3 + 3 + 2 + 3 + 3}{6} = 2\frac{2}{3}$$

## 9 Decision Trees for Sorting Algorithms

### Decision Tree for Three Element Insertion Sort



## 10 Decision Trees for Sorting Algorithms

- Under the standard assumption that all  $n!$  outcomes of sorting are equally likely, the following lower bound on the average number of

comparisons  $C_{avg}$  made by any comparison-based algorithm in sorting an  $n$ -element list has been proved

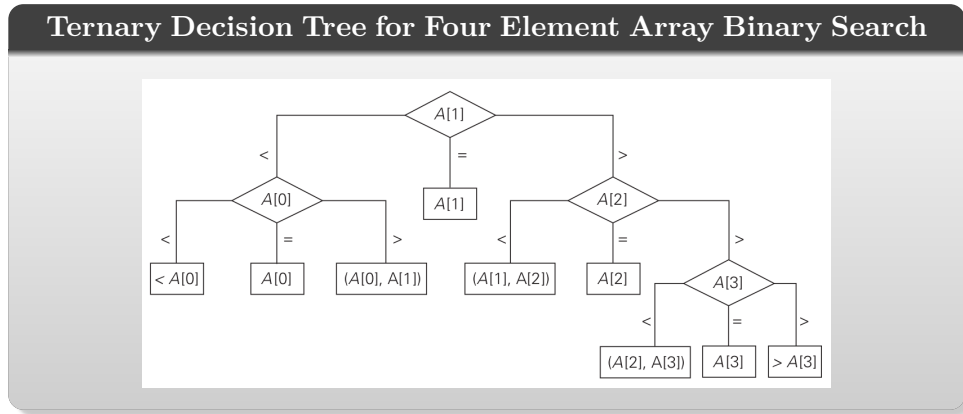
$$C_{avg}(n) \geq \log_2 n!$$

## 11 Decision Trees for Searching Algorithms

- Decision trees can be used for establishing lower bounds on the number of key comparisons in searching a sorted array of  $n$  keys:  $A[0] < A[1] < \dots < A[n-1]$
- The number of comparisons made by binary search in the worst case:

$$C_{worst}^{bs}(n) = \lfloor \log_2 n \rfloor + 1 = \lceil \log_2(n+1) \rceil$$

## 12 Decision Trees for Searching Algorithms



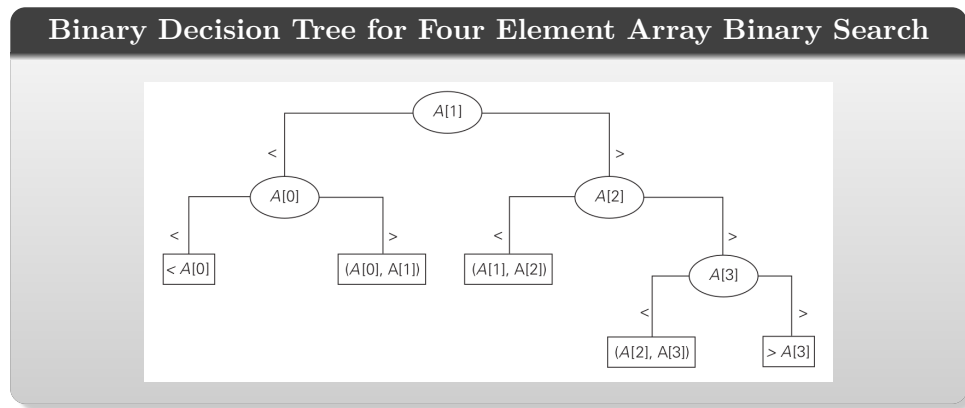
## 13 Decision Trees for Searching Algorithms

- For an array of  $n$  elements, all such decision trees will have  $2n+1$  leaves ( $n$  for successful searches and  $n+1$  for unsuccessful ones)
- Since the minimum height  $h$  of a ternary tree with  $l$  leaves is  $\text{floor}(\log_3 l)$ , we get the following lower bound on the number of worst-case comparisons:

$$C_{worst}(n) \geq \lceil \log_3(2n+1) \rceil$$

- This lower bound is smaller than  $\lceil \log_2(n+1) \rceil$ , the number of worst-case comparisons for binary search
- Can we prove a better lower bound, or is binary search far from being optimal?

## 14 Decision Trees for Searching Algorithms



- The binary decision tree is simply the ternary decision tree with all the middle subtrees eliminated

$$C_{worst}(n) \geq \lceil \log_2(n + 1) \rceil$$

## 15 Think About It

- Consider the problem of finding the median of a three-element set  $a$ ,  $b$ ,  $c$  of orderable items
  - What is the information-theoretic lower bound for comparison-based algorithms solving this problem?
  - Draw a decision tree for an algorithm solving this problem
  - Is the above bound tight?