Q1. Solve using LU decomposition

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 2 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2}$$

$$U = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 2 \\ 0 & 0 & -10 & 5 \end{array}\right]$$

$$L = \begin{array}{l} l_{21} = 4 \\ l_{31} = 3 \\ l_{32} = -2 \end{array} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$z_1 = 1$ $\qquad$ $4 + z_2 = 6$ $\qquad$ $3 - 4 + z_3 = 4$

$z_2 = 2$ $\qquad$ $z_3 = 5$

$$Ux = z$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$-10\,z = 5$ $\qquad$ $-y + 5/2 = 2$ $\qquad$ $x + 1/2 - 1/2 = 1$

$x_3 = -1/2$ $\qquad$ $x_2 = 1/2$ $\qquad$ $x_1 = 1$

$$(1,\ 1/2,\ -1/2)$$

Q2. Find LU factorisation of

$$A = \begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 4 & -4 & -1 & 4 & 11 \\ 2 & -5 & -2 & 2 & -1 \\ 0 & 2 & 1 & 0 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & -2 & -1 & 0 & -4 \\ 0 & 2 & 1 & 0 & 4 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$\therefore$ L is 4x4 matrix

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$A = LU$$

Q3. Find LU factorisation of

$$A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$$

$R_2 \to R_2 + 1/2 R_1$
$R_3 \to R_3 + 1/2 R_1$

$$\begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 3 & 1 \end{bmatrix}$$

Q4. Find LU and LDU factorisation of

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 1/3 R_1]{R_2 \to R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & -1/3 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 + 5/11 R_2$$

$$\begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 1 & 7/11 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow[R_2 \to -3/11 R_2,\ R_3 \to -33/46 R_3]{R_1 \to 1/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -46/33 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix}$$

LU factorisation:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -46/33 \end{bmatrix}}_{U}$$

LDU factorisation:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & -11/3 & 0 \\ 0 & 0 & -46/33 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 1 & 7/11 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Qs. Find LDU factorisation

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Need row exchanges with permutation matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 - 2R_1$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xleftarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$P_{12} A = LU$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Q6. LDU factorisation

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\text{Dummy matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$P_{34} \cdot P_{23} \cdot A = LDU$$