

Principles of Point Estimation

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Maximum Likelihood Estimation

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Topics to be covered...



- √ The Method of Maximum Likelihood for Bernoulli,
- Binomial and Poisson Distributions.
- ✓ Pitfalls of Point Estimators

Bernoulli Distribution – Estimate Likelihood Function for (p)



Let $X_1, ..., X_n$ be a random sample from the population with Bernoulli (p)distribution.

The probability mass function is given by,

$$P(X = x_i) = p^{x_i} (1-p)^{1-x_i}$$

The likelihood function is,

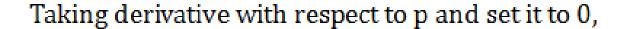
$$L = f(p \mid x_1, \dots, x_n) = p^{x_1}(1-p)^{1-x_1} * p^{x_2}(1-p)^{1-x_2} * \dots * p^{x_n}(1-p)^{1-x_n}$$

$$L = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} \sim p^{\sum x_i} (1-p)^{n-\sum x_i}$$

Take log of the likelihood function,

$$\ln L = \sum x_i \ln p + \left(n - \sum x_i\right) \ln(1 - p)$$

Bernoulli Distribution – Estimate Likelihood Function for (p)



$$\frac{d \ln L}{dp} = 0 \to \sum x_i \, \frac{d \ln p}{dp} + \left(n - \sum x_i \right) + \frac{d \ln(1-p)}{dp} = 0$$

$$\frac{\sum x_i}{p} + \left(n - \sum x_i\right) * \frac{-1}{1 - p} = 0$$

$$\sum x_i - p \sum x_i + p \sum x_i = 0$$

$$\sum_{i} x_i - np = 0$$

$$p = \frac{\sum x_i}{n} = \bar{X}$$

The MLE of p is $\hat{p} = \overline{X}$



Binomial Distribution – Estimate Likelihood Function



Let $X \sim Bin(n,p)$ where n is known and p is unknown

The probability mass function of X is
$$f(x; n, p) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

The likelihood function is given by,

$$L(p; n, x) = \frac{n!}{x! (n-x)!} p^{x} (1-p)^{n-x}$$

Takelog of the likelihood function,

$$\ln L(p; n, x) = \ln \left[\left(\frac{n!}{x! (n - x)!} \right) p^x (1 - p)^{n - x} \right]$$

$$= \ln \left(\frac{n!}{x! (n - x)!} \right) + x \ln p + (n - x) \ln(1 - p)$$

Binomial Distribution – Estimate Likelihood Function



Taking derivative with respect to p and set it to 0,

$$\frac{d}{dp}L(p;n,x) = \frac{d}{dp}\left(\ln n! - \ln x! - \ln(n-x)! + x\ln p + (n-x)\ln(1-p)\right)$$

$$= 0 + \frac{x}{p} - \frac{x}{1-p} - \frac{n}{1-p}$$
 Apply chain rule,

Multiply both sides by p(1-p)

$$0 = x - xp + xp - np$$

$$p = \frac{x}{n}$$

The MLE of p is
$$\hat{p} = \frac{X}{n}$$

Example – Binomial Distribution – Estimate Likelihood Function

Consider the following example,

 $X \sim Bin(20, p), p is unknown$

Suppose we observe X = 7. The probability mass function is,

$$f(7;p) = \frac{20!}{7! \, 13!} \, p^7 \, (1-p)^{13}$$

In the probability mass function it is as written f (7; p) rather than f (7). Here the data value 7 is constant.

When a probability mass function or probability density function is considered to be a function of parameters, it is called a **likelihood function**.



How can we maximise this likelihood function?



When \hat{p} is substituted for p, it maximizes the likelihood function Let's compute maximum likelihood function for f(7;p)

We could maximize this function by taking the derivative with respect to p and setting it equal to 0.

$$\ln f(7;p) = \ln 20! - \ln 7! - \ln 13! + 7 \ln p + 13 \ln(1-p)$$

We take the derivative with respect to p and set it equal to 0.

$$\frac{d}{dp}\ln f(7;p) = \frac{7}{p} - \frac{13}{1-p} = 0$$

The maximizing value is $\frac{7}{20}$

Therefore the maximum likelihood estimate is $\hat{p} = \frac{7}{20}$

MLE for Poisson Distribution (λ) – Estimating Likelihood



Let $X_1, ..., X_n$ be an independent random sample from an $Exp(\lambda)$, where λ is unknown.

The likelihood function is the joint probability density function of X_1, \dots, X_n considered as a function of the parameter λ .

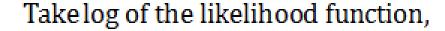
The probability function is given by,

$$f(x_1,...,x_n;\lambda) = \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

The likelihood function is,

$$L(x_i; \lambda) = \sum_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x!}$$

MLE for Poisson Distribution (λ) – Estimating Likelihood



$$\ln L(x_i;\lambda) = \sum_{i=1}^n \ln e^{-\lambda} + \sum_{i=1}^n x_i \ln \lambda - \sum_{i=1}^n \ln x_i$$

Taking derivative with respect to λ and set it to 0,

$$\frac{d}{d\lambda}L(x_i;\lambda)=0$$

$$\frac{d}{d\lambda} \left(\sum_{i=1}^{n} (-\lambda) + \sum_{i=1}^{n} x_i \ln(\lambda) - \sum_{i=1}^{n} \ln x_i \right) = 0$$



MLE for Poisson Distribution (λ) – Estimating Likelihood



$$= -n + \frac{1}{\lambda} \sum_{i=0}^{n} x_i + 0 = 0$$

$$\lambda = \frac{1}{\lambda} \sum_{i=1}^{n} x_i = \bar{X}$$

The MLE of
$$\lambda$$
 is $\hat{\lambda} = \overline{X}$

Example - Poisson Distribution

The following data are the observed frequencies of occurrence of domestic accidents: we have n = 647 data as follows

Number of Accidents	Frequency
0	447
1	132
2	42
3	21
4	3
5	2



Problem:

Example - Poisson Distribution

Solution:

$$\lambda = \frac{1}{\lambda} \sum_{i=1}^{n} x_i = \bar{X}$$

The MLE of λ is $\hat{\lambda} = \overline{X}$

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{X}$$

$$= \frac{(447*0) + (132*1) + (42*2) + (21*3) + (3*4) + (2*5)}{674}$$

$$= 0.465$$



Maximum Likelihood (MLE) – Desirable Properties

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Maximum likelihood is the most commonly used method of estimation.

The main reason for this is that in most cases that arise in practice, MLEs have two very desirable properties,

- 1. In most cases, as the sample size *n* increases, the bias of the MLE converges to 0.
- 2. In most cases, as the sample size *n* increases, the variance of the MLE converges to a theoretical minimum.

Pitfalls of Point Estimators

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- A point estimator is a single number which may vary from sample to sample.
- Certainly the point estimators are slightly different from true population parameter.
- It cannot be confidently claimed to be close to the actual parameter.
- This can be solved by estimating population parameters in the given intervals of values where point estimator can be centered. This interval is called **confidence interval**.



THANK YOU

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