Half-Range Series

Date. 02/04/2020

Can you find a Fourier Series for the function $f(x)$ in $(0, l)$?

Ans: NO. $\because$ $f(x)$ is not satisfies the conditions of Fourier series [Dirichlet's Conditions]

$\therefore$ We have to extend the function to cover the range $(-l, l)$, (its width is $2l$), then the new function may be even or odd.

Two ways to extend the function to cover $(-l, l)$

(1) Extend $f(x)$ such that $f(-x) = f(x)$
New function is even. So that $b_n = 0$
$\therefore$ Fourier Series contains only constant term ie a_0 & a_n ie cosine term, which is called as Half-Range Cosine Series.

(2) Extend $f(x)$ such that $f(-x) = -f(x)$.
New function is odd. So that $a_0 = a_n = 0$
$\therefore$ Fourier Series contains only sine terms ie b_n which is called as Half-range Sine series

Important points

(1) Both Cosine & Sine Series defined in $(0, l)$ ie 50 % of Full Range.

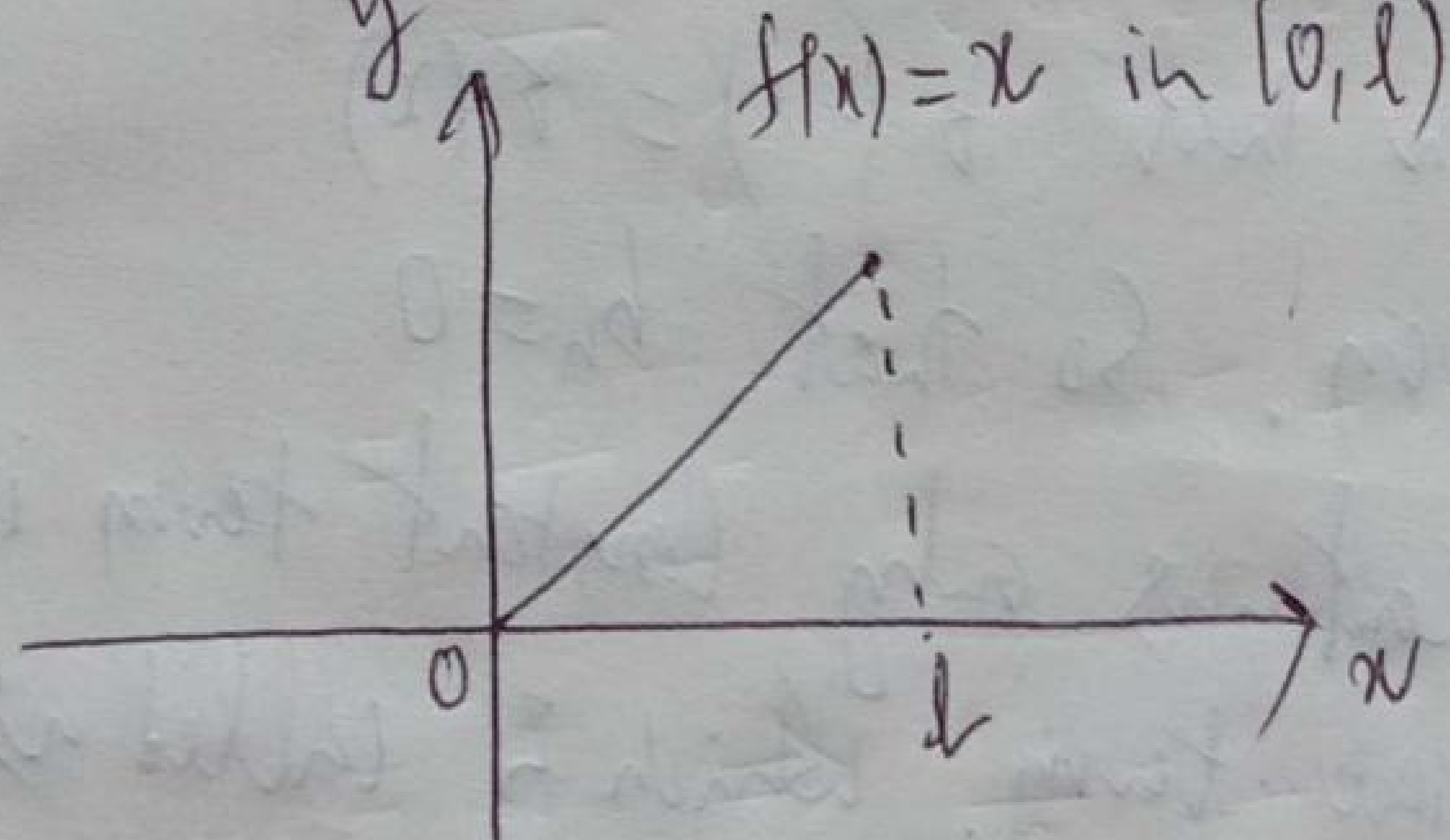
(2) while finding Half-range Series, do not Verify whether the given function $f(x)$ is odd or even.

(3) To find l compare the given interval with $(0, l)$

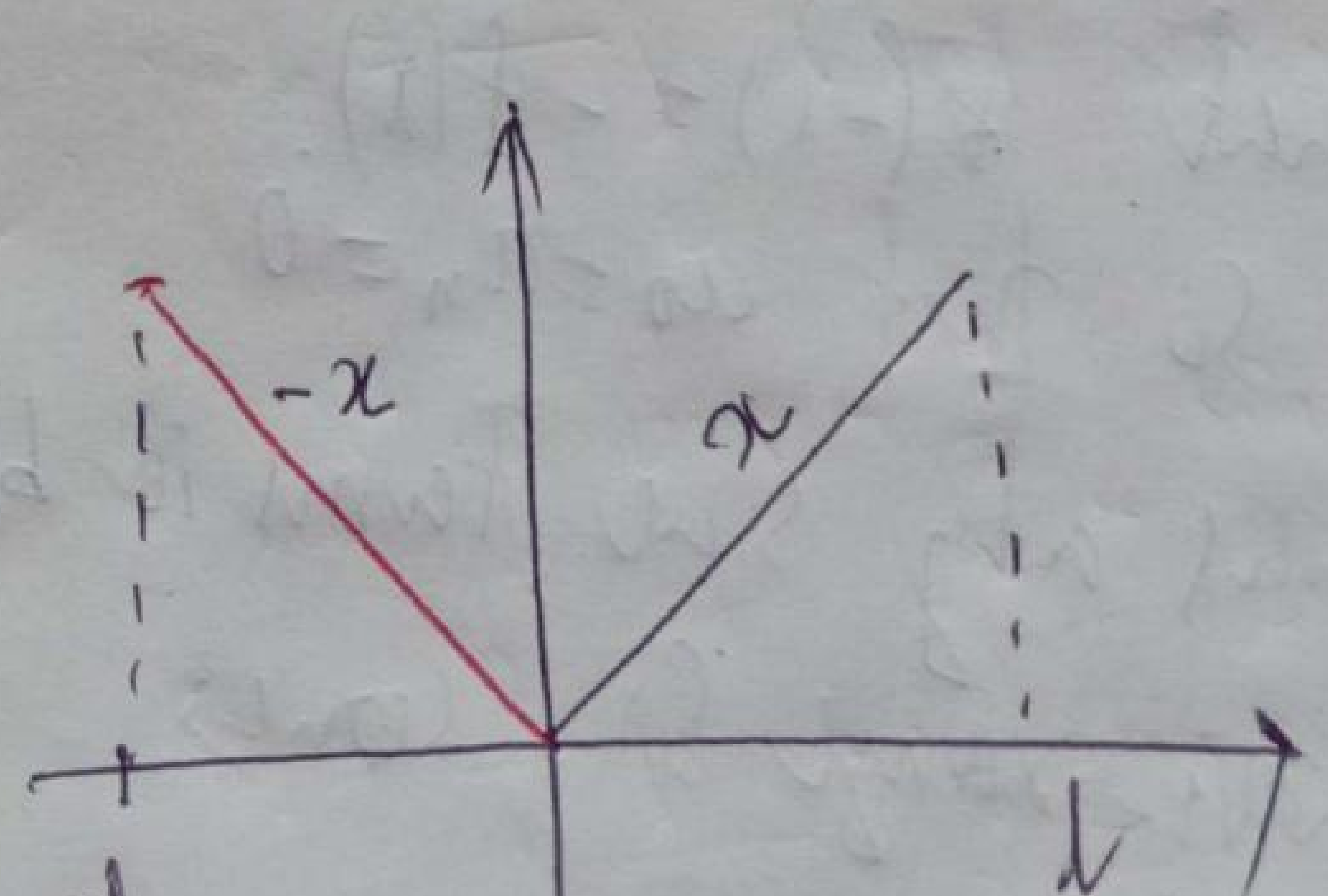
(4) Cosine Series Contains only a_0 & a_n

(5) Sine Series Contains only b_n

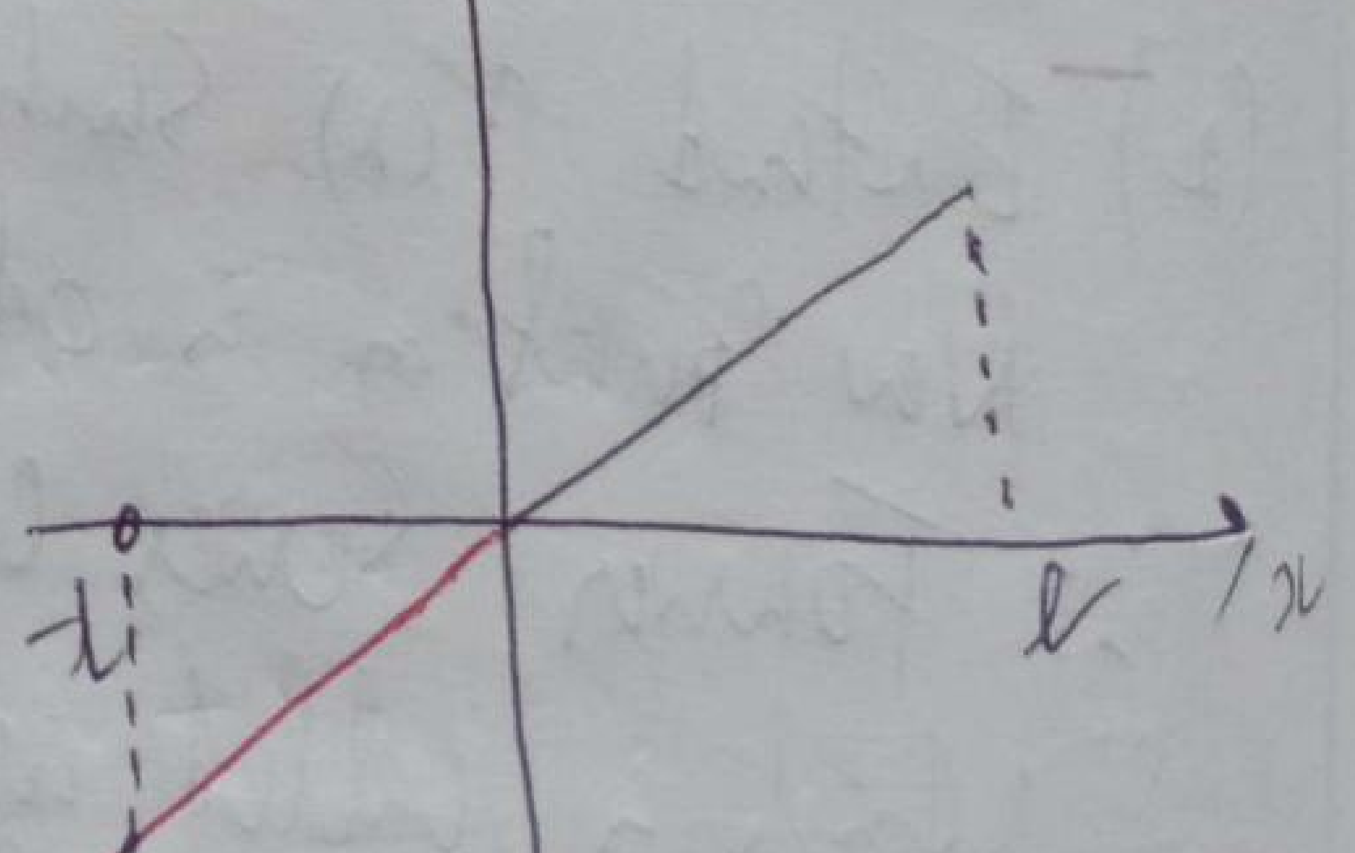
$f(x) = x$ in $(0, l)$



Extensions of $f(x)$



$f(-x) = f(x)$



$f(-x) = -f(x)$

Half-Range Fourier Cosine Series for $f(x)$ in $(0,l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Half-Range Fourier Sine Series for $f(x)$ in $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

(i) Find the half-range cosine series for $f(x) = x \sin x$ in $(0, \pi)$ & hence deduce that

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - + - \cdots$$

Ans: By data $f(x) = x \sin x$ in $(0, \pi)$

$$(0, l) = (0, \pi) \Rightarrow l = \pi \qquad \therefore \frac{n\pi x}{l} = \frac{n\pi x}{\pi} = nx$$

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad — ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad \text{(Ref: formula)}$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \,dx = \frac{2}{\pi}\left[x(-\cos x) - \cancel{(-\sin x)\cdot 1}^{\,0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\{\pi \cos \pi - 0\}\right]$$

$$a_0 = 2 \Rightarrow \boxed{\frac{a_0}{2} = 1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (x \sin x) \cos(nx)\,dx \quad — (*)$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \frac{1}{2}\left[\sin(n+1)x + \sin(1-n)x\right]dx$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\cdot x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx, \quad n \neq 1$$

$$a_n = \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\cancel{\frac{\sin(n+1)x}{n+1}}^{0} + \cancel{\frac{\sin(n-1)x}{n-1}}^{0}\right\}\cdot 1\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[-\frac{1}{n+1}\left\{x\cos(n+1)x\right\}_{x=0}^{\pi} + \frac{1}{n-1}\left\{x\cos(n-1)x\right\}_{x=0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\left\{\pi\cos(n+1)\pi - 0\right\} + \frac{1}{n-1}\left\{\pi\cos(n-1)\pi - 0\right\}\right]$$

$$= \frac{\pi}{\pi}\left[-\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}\right]$$

$$a_n = \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} = (-1)^n\left[\frac{n-1-(n+1)}{n^2-1}\right]$$

$$\boxed{a_n = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1}}$$ Impt $n \neq 1$

$n > 1$

We need to find a_1, put $n=1$ in $(*)$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{x\sin(2x)}{2}dx$$

$$a_1 = \frac{1}{\pi}\left[x\left\{-\frac{\cos(2x)}{2}\right\} - \left\{-\cancel{\frac{\sin(2x)}{4}}^{0}\right\}\cdot 1\right]_0^{\pi} = -\frac{1}{2}$$

$$\boxed{a_1 = -1/2}$$

$\therefore$ Eq^n ① becomes

$$x\sin x = 1 + \left\{-\frac{1}{2}\right\}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos(nx) \rightarrow ②$$

To deduce the given result, put $x = \frac{\pi}{2}$ in ②

$$\frac{\pi}{2}\cdot\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\left(\frac{\pi}{2}\right) + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{-1}{2^2-1}\cos\pi + \frac{1}{9-1}\cos\left(\frac{3\pi}{2}\right) - \frac{1}{16-1}\cos(2\pi) + \cdots\right]$$

$$\frac{\pi-2}{2} = 2\left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots\right]$$

$$\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - + \cdots\cdots$$

(2) Find the Fourier half-range (a) cosine series (b) sine series of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \end{cases}$$

Ans: $f(x)$ is defined in $(0, 2)$

$$\therefore (0, l) = (0, 2) \Rightarrow l = 2 \;\&$$

$$\frac{n\pi x}{l} = \frac{n\pi x}{2} = \frac{n\pi x}{2} = \frac{n\pi x}{2}$$

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad — (1)$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx = \frac{2}{\pi}\int_0^{2} f(x)\,dx$$

$$a_0 = \frac{2}{2}\left[\int_0^1 x\,dx + \int_1^2 (2-x)\,dx\right]$$

$$a_0 = \frac{2}{2}\left[\frac{x^2}{2}\Big|_{x=0}^{1} + \frac{(2-x)^2}{-2}\Big|_{x=1}^{2}\right]$$

$$a_0 = \frac{2}{2}\left[\frac{1}{2}(1-0) - \frac{1}{2}(0-1)\right] = \frac{2}{\pi}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{2}{\pi}$$

$$a_0 = \frac{2}{2} \Rightarrow \boxed{\frac{a_0}{2} = \frac{1}{2}} \Rightarrow \boxed{\frac{a_0}{2} = \frac{1}{2}}$$

And $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{\pi}\int_0^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$

$$a_n = \frac{2}{2}\left[\int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{2}{2}\left[x\left\{\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right\} - \left\{-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right\}\cdot 1\,\Bigg|_{x=0}^{1}\right.$$

$$\left. + (2-x)\left\{\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}\right\} - \left\{-\frac{\cos(n\pi x/2)}{\frac{n^2\pi^2}{4}}\right\}(-1)\,\Bigg|_{x=1}^{2}\right]$$

$$= \frac{2}{2}\left[\frac{2}{n\pi}\left\{\sin(n\pi/2) - 0\right\} + \frac{4}{n^2\pi^2}\left\{\cos\left(\frac{n\pi}{2}\right) - 1\right\}\right.$$

$$\left. + \frac{2}{n\pi}\left\{0 - \sin(n\pi/2)\right\} - \frac{4}{n^2\pi^2}\left\{\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$b_n = \frac{2}{2}\left[\frac{4}{n^2\pi^2}\left\{2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right\}\right]$$

$$b_n = \frac{4}{\pi^2 n^2}\left\{2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right\}$$

$\therefore$ Eq^n (1) becomes

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left\{2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right\} \cos\left(\frac{n\pi x}{2}\right)$$

<u>Sine Series</u>

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \quad (\because l = 2)$$

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2-x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= x\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right\} - \left\{\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right\} \Bigg|_{x=0}^{1}$$

$$+ (2-x)\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right\} - \left\{\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right\}(-1) \Bigg|_{x=1}^{2}$$

$$= \frac{-2}{n\pi}\left\{\cos\left(\frac{n\pi}{2}\right) - 0\right\} + \frac{4}{n^2\pi^2}\left\{\sin\left(\frac{n\pi}{2}\right) - 0\right\}$$

$$- \frac{2}{n\pi}\left\{0 - \cos\left(\frac{n\pi}{2}\right)\right\} - \frac{4}{n^2\pi^2}\left\{0 - \sin\left(\frac{n\pi}{2}\right)\right\}$$

$$b_n = \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\therefore\ f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$$

Parsevals formula

Date-03/04/2020

(1) Parseval's formula for Fourier series for $f(x)$ in the interval $(-l, l)$ is

$$\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

(2) Parseval's formula for cosine series for $f(x)$ in the interval $(0, l)$ is

$$\int_{0}^{l} [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

$$= \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots\right]$$

(3) Parseval's formula for sine series for $f(x)$ in the interval $(0, l)$ is

$$\int_{0}^{l} [f(x)]^2 dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2$$

$$= \frac{l}{2}\left[b_1^2 + b_2^2 + b_3^2 + \cdots\right]$$

3) By using the sine series for $f(x) = 1$ in $0 < x < \pi$. Show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \cdots = \frac{\pi^2}{8}$.

Ans: By data $f(x) = 1$ in $(0, \pi)$

Here $l = \pi$, $\frac{n\pi x}{l} = \frac{n\pi x}{\pi} = nx$

$\therefore$ The sine series in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{— ①}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{\pi}\int_0^{\pi} 1 \sin(nx) dx$

$$b_n = \frac{2}{\pi}\left\{\frac{-\cos(nx)}{n}\right\}_{x=0}^{\pi} = -\frac{2}{\pi n}\left[(-1)^n - 1\right]$$

$$b_n = \frac{2}{\pi n}\left[1 - (-1)^n\right]$$

Eqn ① becomes
$$\boxed{1 = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 - (-1)^n\right] \sin(nx)} \quad \text{— ②}$$

Using this we can't prove the given result, so consider the parsevals formula for sine series.

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[b_1^2 + b_2^2 + b_3^2 + \cdots\right]$$

$$\therefore \int_0^{\pi} [1]^2 dx = \frac{\pi}{2}\left[b_1^2 + b_2^2 + b_3^2 + \cdots + \infty\right]$$

$$\int_0^{\pi} dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + 0 + \left(\frac{4}{3\pi}\right)^2 + 0 + \left(\frac{4}{5\pi}\right)^2 + \cdots\right]$$

$$x\Big|_0^{\pi} = \left(\frac{\pi}{2}\right)\left(\frac{4}{\pi}\right)^2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\pi = \frac{8}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

or $$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

4) Obtain the Fourier Series for the function $f(x) = x^2$ in $-\pi < x < \pi$. Hence S.T $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Ans: By data $f(x) = x^2$ in $(-\pi, \pi)$

Since $f(x) = x^2$ is even, $\boxed{b_n = 0}$ & $l = \pi$

$\therefore$ Fourier Series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{--- (1)}$$

Write directly

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$a_0 = \left[\frac{2}{\pi}\frac{x^3}{3}\right]_0^{\pi} \Rightarrow a_0 = \frac{2}{3}\frac{\pi^3}{\pi} = \frac{2\pi^2}{3} \Rightarrow \boxed{\frac{a_0}{2} = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} \underset{\text{I}}{x^2}\,\underset{\text{II}}{\cos(nx)}\,dx$$

$$a_n = \frac{2}{\pi}\left[x^2\left\{\cancel{\frac{\sin(nx)}{n}}^{0}\right\} - \left\{\frac{-\cos(nx)}{n^2}\right\}(2x) - \left\{-\cancel{\frac{\sin(nx)}{n^3}}^{0}\right\}2\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{2}{n^2}\left\{x\cos(nx)\right\}\right]_0^{\pi}$$

$$a_n = \frac{4}{\pi n^2}\left[\pi\cos(n\pi) - 0\right] \Rightarrow \boxed{a_n = \frac{\pi 4(-1)^n}{\pi n^2} = \frac{4(-1)^n}{n^2}}$$

$\therefore$ Eq (1) becomes

$$\boxed{x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)} \quad \text{——— (2)}$$

To prove the given result, Consider parsevals formula for Fourier Series

$$\boxed{\begin{array}{l}\frac{a_0}{2} = \frac{\pi^2}{3} \Rightarrow \frac{a_0^2}{2} = \frac{2\pi^4}{9} \\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0\end{array}}$$

$$\int_{-l}^{l}[f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

$$\int_{-\pi}^{\pi} x^4\,dx = \pi\left[\frac{1}{2}\left\{\frac{4\pi^4}{9}\right\} + \sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n}\right]$$

$$\left.\frac{x^5}{5}\right|_{x=-\pi}^{\pi} = \pi\left[\frac{2\pi^2}{9} + \frac{16}{1^4} + \frac{16}{2^4} + \frac{16}{3^4} + \cdots\cdots\right]$$

$$\frac{1}{5}(\pi^5 + \pi^5) = \pi\left[\frac{2\pi^4}{9}\right] + 16\pi\left(\frac{1}{1^4} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right)$$

$$\frac{2\pi^5}{5} - \frac{2\pi^4}{9} = 16\pi\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\cdots = \frac{\pi^4}{90}$$