

Unit II: Vector Spaces

1. Define column space and Null space of the matrix with example
2. Explain the special solution of the system of linear equations.
3. Describe (geometrically) the column space and null space of the following matrices:

i) $[0]$, ii) $[0 \ 1]$, iii) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, iv) $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

Ans: i) $C(A) = z$, $N(A) = R^1$; ii) $C(A) = R^1$, $N(A) = \text{line in } R^2$

iii) $C(A) = R^2$, $N(A) = \text{line in } R^3$ iv) $C(A) = R^2$, $N(A) = z \text{ in } R^2$

4. Write all the subspaces of R^3

5. Find the column space and null space of $A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix}$.

Give an example of a matrix whose column space is the same as that of A but null space is different.

Answer: $C(A)$ is a 2d plane in R^3 and $N(A)$ is the origin in R^2 . The matrix

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{bmatrix}$ has $C(A)$ but its $N(A)$ is a line in R^3 passing through $(1, 1, -1)$

6. For which vectors $b = (b_1, b_2, b_3)$ does the following system $Ax = b$ have a solution?

i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}$ iv) $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ -1 & -2 \end{bmatrix}$

Answers: i) all b ii) $b_3=0$ iii) $b_2=2b_1$ iv) $b_3=-b_1$

7. Decide the dependence or independence of the vectors

i) $(1, 3, 2), (2, 1, 3), (3, 2, 1)$

ii) $(4, 2, 2), (2, 4, 2), (4, 8, 2)$

iii) $(0, 0, 0), (1, 2, 5), (-1, 2, 3)$

Ans: i) dependent ii) independent iii) dependent

8. If v_1, v_2, v_3 are linearly independent, determine whether the vectors

$v_1 - v_2, v_2 - v_3, v_3 - v_1$ are linearly independent or dependent.

9. Reduce the following matrices to row echelon form and then reduced row echelon form.

$$\text{i) } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 13 & 20 & 27 \\ 9 & 26 & 43 & 62 \end{bmatrix} \quad \text{ii) } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

10. Choose the number q so that (if possible) the ranks are (i) 1 (ii) 2 (iii) 3

$$\text{i) } A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{ii) } A = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$$

Ans: For A, $q = 3$ gives rank 1 and every other q gives rank 2. For B, $q = 6$ gives rank 1 and every other q gives rank 2.

11. Define span of a set with example

12. Define Basis with example.

13. Find the row space and left null space of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

Ans: Row Space: (1, 1, 1); Left null space: (-1, 1, 0), (-2, 0, 1)

14. Find the four fundamental subspaces, their dimensions and bases given

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 & -3 \\ -2 & 0 & 0 & 1 & 2 \\ 0 & 3 & 1 & -1 & 6 \\ -1 & -2 & -3 & 3 & 9 \end{bmatrix}$$

Ans: Answer : Basis for $C(A)$ is columns 1, 2, 3 ; Basis for $C(A^T)$ is rows 1, 2, 3
Basis for $N(A)$ is $\{ (7, 1, 0, 3, 0), (-8, -5, -3, 0, 3) \}$; $N(A^T) = \{ c(1, 0, 1, 1) \}$

15. Obtain the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Ans: Column space : (1, 2, 3, 4); Row space: all the four rows;

Null space: (0, -1, -1, 0, 1) ; Left Null space : z

- 16.** Construct a matrix whose column space contains the vectors (1,1,5), (0,3,1) whose null space contains (1, 1, 2)

$$\text{Ans: } A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

- 17.** Find a left or right inverse for the following matrices

$$\text{i) } A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{ii) } A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{Ans: i) } A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \quad \text{ii) } A = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$