

UNIT IV- Orthogonalization, Eigenvalues and Eigenvectors:

NOTES:

Contents:

- Orthogonal Bases,
- The Gram- Schmidt Orthogonalization,
- Introduction to Eigenvalues and Eigenvectors,
- Properties of Eigenvalues and Eigenvectors,
- Symmetric Matrices, Diagonalization of a Matrix.
- Powers and Products of the matrices.

Orthogonalization and the Gram- Schmidt Process

Definition :

In an **orthogonal basis**, every vector is perpendicular to every other vector.

The coordinate axes are mutually orthogonal.

Mutually perpendicular unit vectors are called **Orthonormal** vectors.

Examples :

For the vector space \mathbb{R}^2 ,

1. The set $(2, 0)$, $(0, 2)$ is an orthogonal basis.
2. The set $(1, -2)$, $(2, 1)$ is an orthogonal basis.
3. The set $(1, 0)$, $(0, 1)$ is an orthonormal basis.

Definition :

- A matrix with Orthonormal columns will be called Q.
- A square matrix with Orthonormal columns is called an **Orthogonal matrix** denoted by Q.

Examples:

Rotation matrix, any permutation matrix.

Note : The size of Q has to be square or tall.

Properties of Q:

- If Q (square or rectangular) has orthonormal columns, then $Q^T Q = I$.
- An orthogonal matrix is a square matrix with orthonormal columns. Then Q^T is Q^{-1} .
- If Q is rectangular then Q^T is left inverse of Q.
- Multiplication by any Q preserves length. The norms of x and Qx are equal.
- Also, Q preserves inner products and angles, since

$$(Q^T x)^T (Qy) = x^T Q^T Qy = x^T y.$$

- Since Q preserves lengths and inner products it preserves angle between two vectors.
- If q_1, q_2, \dots, q_n are orthonormal basis of \mathbb{R}^n then any vector b from \mathbb{R}^n can be expressed as
 $b = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$ -----**(1)**

Multiply both sides by q_1^T .

$$\text{Then } x_1 = q_1^T b.$$

$$\text{Similarly, } x_2 = q_2^T b, \dots, x_n = q_n^T b.$$

$$\text{Hence, } b = (q_1^T b)q_1 + (q_2^T b)q_2 + \dots + (q_n^T b)q_n$$

= sum of one dimensional projections on to q_i 's.

The matrix form of equation (1) is $Qx = b$ and the solution of this system of equations is

$$x = Q^{-1}b = Q^T b$$

Note: The rows of a square matrix are orthonormal whenever the columns are.

Example:

Orthonormal columns
Orthonormal rows

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}.$$

Rectangular Matrices with Orthonormal Columns

- If Q has orthonormal columns, the least-squares problem becomes easy.
- $Q^T Q = I$ are the normal equations for the best solution -in which $Q^T Q = I$.
- $p = Qx$, the projection of b is $(q_1^T b)q_1 + \dots + (q_n^T b)q_n$
- $p = QQ^T b$, the projection matrix is $P = QQ^T$.

1) The vectors $q_1 = (1, 0, 0)$, $q_2 = (0, 3/5, 4/5)$ and $q_3 = (0, 4/5, -3/5)$ form an orthonormal basis for R^3 . Express the vector $v = (7, -5, 10)$ as a linear combination of the q 's.

$$\text{Solution: } \begin{pmatrix} 7 \\ -5 \\ 10 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$$

$$7 = c_1 + 0 + 0$$

Solving the equations $-5 = 0c_1 + \frac{3}{5}c_2 + \frac{4}{5}c_3$, we get $c_1 = 7, c_2 = 5, c_3 = -10$

$$10 = 0c_1 + \frac{4}{5}c_2 - \frac{3}{5}c_3$$

Therefore $v = 7q_1 + 5q_2 - 10q_3$

2) Find a third column so that the matrix $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \text{---} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \text{---} \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \text{---} \end{bmatrix}$ is orthogonal.

Solution: Let $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & x \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & y \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & z \end{bmatrix}$ i. e, third columns elements are (x, y, z)

Since Q has to be orthogonal then

$$a^T c = 0 \Rightarrow \left(\frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x + y + z = 0 \dots (1)$$

$$b^T c = 0 \Rightarrow \left(\frac{1}{\sqrt{14}} \frac{2}{\sqrt{14}} \frac{-3}{\sqrt{14}} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x + 2y - 3z = 0 \dots (2)$$

Solving equation (1) and equation (2), by taking $z = 1$ (because z is the free variable) we get, $x = -5$ and $y = 4$.

$$\text{Therefore } c = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} \text{ OR } c = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$$

$$\text{Let } q_3 = \pm \begin{pmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{pmatrix} \text{ (After Normalization i.e., } \sqrt{(-5)^2 + (4)^2 + 1^2} = \sqrt{42})$$

$$\text{Hence } Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{42}} \end{bmatrix} \text{ OR } Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \frac{+5}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{-4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{42}} \end{bmatrix}$$

3) Let $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$, $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$. Verify that

(i) $Q^T Q = I$ (ii) $\|Qx\| = \|x\|$, $\|Qy\| = \|y\|$ (iii) $(Qx)^T(Qy) = x^T y$

Solution: To Prove that $Q^T Q = I$:

$$Q^T Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(ii) To prove that $\|Qx\| = \|x\|$, $\|Qy\| = \|y\|$

$$\begin{aligned} \text{Consider } Qx &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \\ \|Qx\| &= \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11} \\ \|x\| &= \sqrt{(\sqrt{2})^2 + 3^2} = \sqrt{11} \end{aligned}$$

Hence $\|Qx\| = \|x\|$

$$\begin{aligned} \text{Consider } Qy &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}, \\ \|Qy\| &= \sqrt{1^2 + (-7)^2 + 2^2} = \sqrt{54} \\ \|y\| &= \sqrt{(-3\sqrt{2})^2 + 6^2} = \sqrt{54} \end{aligned}$$

Hence $\|Qy\| = \|y\|$

(iii) $(Qx)^T(Qy) = x^T y$

$$Qx = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad Qy = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix},$$

$$(Qx)^T = [3, -1, 1] \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} = 12$$

4) If W is a subspace spanned by the orthogonal vectors $(2, 5, -1)$ and $(-2, 1, 1)$ find the point in W that is closest to $(1, 2, 3)$.

Solution: Let $\omega = \left\{ C_1 \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $A = \begin{bmatrix} 2 & -2 \\ -5 & 1 \\ 1 & 1 \end{bmatrix}$

Normalizing A we get $A = \begin{bmatrix} \frac{2}{\sqrt{2^2+5^2+(-1)^2}} = \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{(-2)^2+1^2+1^2}} = \frac{-2}{\sqrt{6}} \\ \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix} = Q$

We have $P = A\hat{x}$, $\hat{x} = \frac{Q^T b}{Q^T Q}$ and $b = (1, 2, 3)$

$$\hat{x} = \frac{Q^T b}{Q^T Q} = \frac{\begin{bmatrix} \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} (1, 2, 3)}{\begin{bmatrix} \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{6}} \\ \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}} = \begin{bmatrix} \frac{9}{\sqrt{30}} \\ \frac{3}{\sqrt{6}} \end{bmatrix}$$

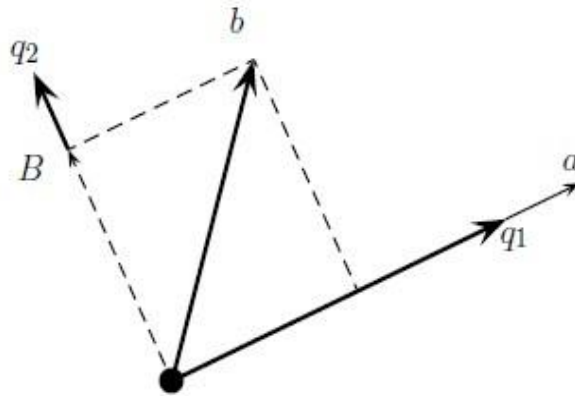
$$P = A\hat{x} = \begin{bmatrix} \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{6}} \\ \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{9}{\sqrt{30}} \\ \frac{3}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{5} \\ \frac{5}{2} \\ \frac{1}{5} \end{bmatrix}$$

The Gram-Schmidt Process

- This is a process of converting a set of linearly independent vectors into a set of orthonormal vectors. The number of vectors given is always equal to the number of vectors produced.
- Consider any 3 independent vectors a, b, c . Then the first orthonormal $q_1 = a/\text{norm}(a)$.
- If ' b ' is perpendicular to the vector ' a ' then $q_2 = b/\text{norm}(b)$ otherwise we subtract the component of b in q_1 direction to get

$$B = b - (q_1^T b)q_1$$

- $q_2 = B/\text{norm}(B)$.



- The third vector c is not in the plane of a and b (or q_1 and q_2). If ' c ' is perpendicular to the plane spanned by the vectors a and b then $q_3 = c/\text{norm}(c)$
- Otherwise
 $C = c - (q_1^T c)q_1 - (q_2^T c)q_2$
 $q_3 = C/\text{norm}(C)$.

This is the one idea of the whole Gram-Schmidt process, to subtract from every new vector its components in the directions that are already settled. That idea is used over and over again. When there is a fourth vector, we subtract away its components in the directions of q_1, q_2, q_3 .

A = QR factorization

Where Q -Orthonormal columns and R is the upper triangular matrix(square).

The Factorization $A=QR$

- We started with a matrix A , whose columns were a, b, c .
- We ended with a matrix Q , whose columns are q_1, q_2, q_3 .
- A and Q are of order m by n .

To find a relation between A and Q we express a, b, c as linear combinations of q_1, q_2, q_3

If suppose A is the 3x3 matrix then $R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}$

How is this $A = QR$ factorization useful?

It simplifies the least squares problem $Ax = b$

$$A^T A = R^T Q^T Q R = R^T R.$$

The fundamental equation $A^T A \hat{x} = A^T b$ simplifies to a triangular system:

$$R^T R \hat{x} = R^T Q^T b \quad \text{or} \quad R \hat{x} = Q^T b.$$

5) What multiple of $a_1 = (1, 1)$ should be subtracted from $a_2 = (4, 0)$ to make the result orthogonal to a_1 ? Factorize $A = [a_1, a_2]$ into QR.

Solution: Form the given data, we can consider, $a_2 - C_1 a_1$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} - C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 - C_1 \\ -C_1 \end{pmatrix}$$

Now this must be orthogonal to a_1 , therefore $(1, 1) \begin{pmatrix} 4 - C_1 \\ -C_1 \end{pmatrix} = 0$
 $\Rightarrow 4 - C_1 - C_1 = 0$

$$\Rightarrow C_1 = 2$$

To find $A = QR$, $A = [a_1, a_2] = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e = a_2 - (q_1^T a_2) q_1$$

$$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} (1, 1) \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{4}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|} = \frac{\begin{pmatrix} 2 \\ -2 \end{pmatrix}}{\sqrt{8}}$$

$$Q = [q_1 q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \\ \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{8}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{8}} \end{bmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{8}} & \frac{4}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{8}} & \frac{4}{\sqrt{2}} \end{bmatrix}$$

6) Find an orthonormal set q_1, q_2, q_3 for which q_1 and q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}. \text{ which fundamental subspace contains } q_3? \text{ What is the least}$$

squares solution of $Ax = b$ if $b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$?

$$\text{Solution: (i) Let } a = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \text{ and } b_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{9}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$e = b - (q_1^T b)q_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|}, \|e\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{7}$$

$$q_2 = \frac{e}{\|e\|} = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$e_1 = b_1 - (q_1^T b_1)q_1 - (q_2^T b_1)q_2$$

$$= \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} - \left(\begin{bmatrix} \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \right) \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \\ = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$q_3 = \frac{e_1}{\|e_1\|} = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \|e_1\| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$\text{Therefore } Q = \begin{pmatrix} q_1 & q_2 & q_3 \\ \frac{1}{3} & \frac{2}{\sqrt{7}} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{7}} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{2}{\sqrt{7}} & \frac{1}{3} \end{pmatrix}$$

(ii) $q_3 \in N(A^T)$, Since $N(A^T) \perp C(A)$ and $q_3 \perp C(A)$

$$(iii) \text{ We have } A=QR, R = Q^T A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$$

Also we have $R\hat{x} = Q^T b_1$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

This is in matrix form $A\hat{x} = b$, let $\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

$$\Rightarrow 3x_1 - 3x_2 = -3$$

$$0x_1 + 3x_2 = 6$$

Solving $x_1 = 1, x_2 = 2$,

Therefore, the least squares solution $\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

7) Use the Gram – Schmidt process to find a set of orthonormal vectors from the independent vectors $a_1 = (1, 1, 1)$, $a_2 = (0, 1, 1)$ and $a_3 = (0, 0, 1)$. Also find the $A = QR$ factorization where $A = [a_1 \ a_2 \ a_3]$.

Solution: Let $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$e = b - (q_1^T b)q_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{3}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|}, \|e\| = \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}$$

$$q_2 = \frac{e}{\|e\|} = \frac{3}{\sqrt{6}} \begin{pmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$e_1 = c - (q_1^T c)q_1 - (q_2^T c)q_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left(\begin{bmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \begin{pmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$q_3 = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 2 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \|e_1\| = \sqrt{(0)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

Therefore $Q = \begin{pmatrix} q_1 & q_2 & q_3 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

TO find R:

We have $A=QR$,

$$R = Q^T A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{OR } R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Eigen values and Eigen vectors

Definition :

Let A be a square matrix of order n. If there exists a real or complex number λ and a non zero vector x such that $Ax = \lambda x$ then x is called the Eigenvector of A and λ is its corresponding Eigen value.

Procedure to find eigenvalues and eigenvectors of A

1. Find the characteristic equation, that is determinant of $A - \lambda I = 0$.
2. This gives an equation of degree n. It starts with $(-\lambda)^n$.
3. Find the roots of this equation. The n roots are the eigenvalues of A.
4. For each eigenvalue λ , solve the equation $(A - \lambda I)x = 0$. Since the determinant of $A - \lambda I$ is zero, there are solutions other than $x = 0$. Those are the eigenvectors.

Note :Corresponding to ‘n’ distinct Eigen values we get ‘n’ independent Eigen vectors. But when 2 or more eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to repeated roots.

Properties of Eigen Values and Eigen vectors

- If λ is an Eigen value of A with x as the corresponding Eigen vector then λ^2 is an Eigen value of A^2 with the same Eigen vector x.
- For a given Eigen vector x, there corresponds only one Eigen value λ .
- For a given Eigen value there corresponds infinitely many Eigen vectors.
- $\lambda = 0$ is an Eigen value of A , if and only if A is singular i.e $\det(A)=0$.
- If λ is an Eigen value of A with x as the Eigen vector then $1/\lambda$ is an Eigen value of A^{-1} provided A^{-1} exists.
- A and its transpose A^T have the same Eigen values.
- The Eigen values of a diagonal matrix are just the diagonal elements of the matrix.
- The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.
- The product of the Eigen values of a matrix A is equal to its determinant.

The Cayley-Hamilton Theorem

Statement:

Every square matrix satisfies its own characteristic equation .

Example: Let the characteristic equation is:

$\det(A-tI) = t^2-4t+2 = 0$ and hence it can be verified that

$$A^2- 4A +2I = 0$$

Note: If a matrix is invertible then we can find its inverse using Cayley- Hamilton Theorem.

Example: For the Matrix above Cayley Hamilton theorem, $A^2-4A+2I= 0$.

Therefore $A^{-1} =(4I-A)/2$.

8) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, shift A to $A - 7I$ what are the eigenvalues and eigenvectors and how are they related to those of A ?

Solution: To find the eigen values of A :

Consider the characteristic equation $|A - \lambda I| = 0$

$$\text{i.e., } |A - \lambda I| = \begin{vmatrix} 4 - \lambda & 3 \\ 1 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (4 - \lambda)(2 - \lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda = 1, 5$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 4 - \lambda & 3 \\ 1 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x + 3y = 0, \quad x + y = 0 \Rightarrow x = -y$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = k_1$ be the free variable, then $x = -k_1$,

Therefore, the eigen vector is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} k_1$$

Case 2: When $\lambda = 5$.

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 3y = 0, \quad x - 3y = 0 \Rightarrow x = 3y$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = k_2$ be the free variable, then $x = 3k_2$,

Therefore, the eigen vector is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} k_2$$

Trace of A = sum of the elements of principal diagonal of $A = 4 + 2 = 6$.

Sum of the eigen values = $1 + 5 = 6$.

Trace of A = Sum of the eigen values.

Determinant of $A = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$ and Product of the eigen values $= 1 \times 5 = 5$.

Therefore, Determinant of $A = \text{Product of the eigen values}$.

Now we shift A to $A - 7I$ i.e., $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & -5 \end{bmatrix} = A - 7I = B(\text{say})$

The eigen values of B :

Consider the characteristic equation $|B - \lambda I| = 0$

$$\begin{aligned} \text{i.e., } |B - \lambda I| &= \begin{vmatrix} -3 - \lambda & 3 \\ 1 & -5 - \lambda \end{vmatrix} = 0 \\ &\Rightarrow (-3 - \lambda)(-5 - \lambda) - 3 = 0 \\ &\lambda = -2, -6 \end{aligned}$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} -3 - \lambda & 3 \\ 1 & -5 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = -2$.

$$\begin{aligned} \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -x + 3y &= 0, \quad x - 3y = 0 \Rightarrow x = 3y \end{aligned}$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = k_1$ be the free variable, then $x = 3k_1$,

Therefore, the eigen vector is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} k_1$$

Case 2: When $\lambda = -6$.

$$\begin{aligned} \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 3x + 3y &= 0, \quad x + y = 0 \Rightarrow x = -y \end{aligned}$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = k_2$ be the free variable, then $x = -k_2$,

Therefore, the eigen vector is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} k_2$$

Therefore, we can say that Eigen vectors are same for A and $B = A - \lambda I$ and eigen values are been shifted λ to $\lambda - 7$

9) Find the eigenvalues of A , A^2 , A^{-1} and $A + 4I$ if

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\text{i.e., } |A - \lambda I| = \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (2 - \lambda)(2 - \lambda) - 2 = 0$$

$$\lambda = 1, 3$$

Eigen values of A are 1, 3.

We know that the eigen values of A^{-1} is $\frac{1}{\lambda}$, if eigen values of A is λ .

Hence the eigen values of $A^{-1} = 1, \frac{1}{3}$

The eigen values of A^2 is $1^2, 3^2 = 1, 9$.

Eigen values of $A + 4I$: Here we need to shift λ to $\lambda + 4$

Eigen value of $A + 4I$: $1 + 4 = 4, 3 + 4 = 7$.

10) Write three different 2×2 matrices for which the eigenvalues are 4, 5 and determinant is 20.

Solution: $\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}$ (For upper and lower triangular matrices the principal diagonal given the eigen values).

11) Find the eigenvalues and the corresponding eigenvectors of

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1 - \lambda & 1 & 3 \\ 1 & 5 - \lambda & -3 \\ 3 & 1 & 1 - \lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\lambda^3 - (\text{sum of element of diagonal of } A)\lambda^2 + (\text{sum of the mionors of } A)\lambda - \text{Determinant of } A$$

Solving we get $\lambda = -2, 3, 6$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Case 1: When $\lambda = -2$.

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$3x + y + 3z = 0$$

$$x + 7y + z = 0$$

$$3x + y + 3z = 0$$

Taking any two different equations say

$$3x + y + 3z = 0$$

$$x + 7y + z = 0$$

($ax + by + cz = 0$ and $dx + ey + fz = 0$, then $\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$)

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20}$$

$$X_1(\text{Eigen vector}) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Case 2: When $\lambda = 3$.

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + 3z = 0$$

$$x + 2y + z = 0$$

$$3x + y - 2z = 0$$

Taking any two different equations say

$$-2x + y + 3z = 0$$

$$x + 2y + z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x}{-5} = -\frac{y}{-5} = \frac{z}{-5}$$

$$X_2(\text{Eigen vector}) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 6$.

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-5x + y + 3z = 0$$

$$x - y + z = 0$$

$$3x + y - 5z = 0$$

Taking any two different equations say

$$-5x + y + 3z = 0$$

$$x - y + z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x}{4} = -\frac{y}{-8} = \frac{z}{4}$$

$$X_3(\text{Eigen vector}) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

12) Use the Cayley – Hamilton's theorem to find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

Put $\lambda = A$, we get $A^3 - A\lambda^2 - A + 9 = 0$

Multiply by A^{-1} , we get $A^2 - 3A - I + 9A^{-1} = 0$,

($AA^{-1} = I$ and $IA^{-1} = A^{-1}$)

$$A^{-1} = \frac{1}{9}[3A + I - A^2]$$

$$\frac{1}{9} \left[3 \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{9} & \frac{-7}{9} \\ \frac{1}{3} & \frac{-1}{9} & \frac{-1}{9} \end{bmatrix}$$

Symmetric Matrices:

A symmetric matrix is a matrix A such that $A^T = A$, such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries in pairs- on opposite sides of the main diagonal

For eg: $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

NOTE: (i) If A is symmetric, then any two eigen vectors from different eigen values are orthogonal

(ii) An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

Diagonalization of a Matrix

Key Idea : The eigenvectors diagonalize a matrix.

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S , then $S^{-1}AS$ is a **diagonal matrix**. The eigenvalues of A are on the diagonal of Λ .

The matrix S is called an **eigenvector matrix**.

- If the matrix A has no repeated eigenvalues then its n eigenvectors are automatically independent.
- Therefore any matrix with distinct Eigen values can be diagonalized.
- The diagonalizing matrix S is not unique. An eigenvector x can be multiplied by a constant, and remains an eigenvector.
- Diagonalizability of A depends on enough eigenvectors.
- Invertibility of A depends on non zero eigen values.

Powers and Products

If A is diagonalizable then $A = S \Lambda S^{-1}$. So $A^K = S \Lambda^K S^{-1}$

13) If possible, diagonalize the $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 6 - \lambda & -2 & -1 \\ -2 & 6 - \lambda & -1 \\ -1 & -1 & 5 - \lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 17\lambda^2 - 90\lambda + 144 = 0$$

Solving we get $\lambda = 8, 3, 6$

Consider $(A - \lambda I)X = 0$

$$\begin{vmatrix} 6 - \lambda & -2 & -1 \\ -2 & 6 - \lambda & -1 \\ -1 & -1 & 5 - \lambda \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Case 1: When $\lambda = 8$.

The eigen vector $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Case 2: When $\lambda = 6$.

The eigen vector $v_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

Case 1: When $\lambda = 3$.

The eigen vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Normalizing we get

$$u_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix},$$

$$\text{Let } P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{Then } A = PDP^{-1}$$

Since P is the square $P^{-1} = P^T$

$$\text{i.e., } A = PDP^T$$

14) Factor $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and hence compute A^{100} .

Solution: To find the eigen values of A:

Consider the characteristic equation $|A - \lambda I| = 0$

$$\text{i.e., } |A - \lambda I| = \begin{vmatrix} 4 - \lambda & 3 \\ 1 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (4 - \lambda)(2 - \lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda = 1, 5$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 4 - \lambda & 3 \\ 1 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x + 3y = 0, \quad x + y = 0 \Rightarrow x = -y$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = 1$ be the free variable, then $x = -1$,

Therefore, the eigen vector is

$$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 5$.

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 3y = 0, \quad x - 3y = 0 \Rightarrow x = 3y$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = 1$ be the free variable, then $x = 3$,

Therefore, the eigen vector is

$$X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\text{Therefore } S = [X_2, X_1] = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

We have $A = S \Lambda S^{-1}$ and $A^{100} = S \Lambda^{100} S^{-1}$

$$\begin{aligned} &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}^{100} \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5^{100} & 0 \\ 0 & 1^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

15) Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots, a matrix R such that $R^2 = A$. How many such square root matrices are there?

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{aligned} \text{i.e., } |A - \lambda I| &= \begin{vmatrix} 5 - \lambda & 4 \\ 4 & 5 - \lambda \end{vmatrix} = 0 \\ &\Rightarrow (5 - \lambda)(5 - \lambda) - 16 = 0 \\ &(\lambda - 9)(\lambda - 1) = 0 \\ &\lambda = 1, 9 \end{aligned}$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5 - \lambda & 4 \\ 4 & 5 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{aligned} \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 4x + 4y &= 0, \quad 4x + 4y = 0 \Rightarrow x = -y \end{aligned}$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = 1$ be the free variable, then $x = -1$,

Therefore, the eigen vector is

$$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 9$.

$$\begin{aligned} \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -4x + 4y &= 0, \quad 4x - 4y = 0 \Rightarrow x = y \end{aligned}$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = 1$ be the free variable, then $x = 1$,

Therefore, the eigen vector is

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore $S = [X_1, X_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$,

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \quad S^{-1} = \frac{-1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

We have $A = S \Lambda S^{-1}$ and

$$A^{\frac{1}{2}} = S \Lambda^{\frac{1}{2}} S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{\frac{1}{2}} & 0 \\ 0 & 9^{\frac{1}{2}} \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

We have 4 square roots in $\Lambda^{\frac{1}{2}}$, i.e., $\sqrt{1} = \pm 1, \sqrt{9} = \pm 3$. We get different eigen values for different values.

One square root is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. There are 4 of them.

16) Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write two different diagonalizing matrices S.

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda^3 - (\text{sum of element of diagonal of } A)\lambda^2 + (\text{sum of the mionors of } A)\lambda - \text{Determinant of } A$$

Solving we get $\lambda = 0, 0, 3$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Case 1: When $\lambda = 0$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y + z = 0$$

$$x + y + z = 0$$

$$x + y + z = 0$$

All the equations are same, $3(\text{unknowns}) - 1(\text{equation}) = 2$ free variable.

Let y and z be the free variable with $y = 1$ and $z = 0$, then $x = -1$,

If we take $y = 0$ and $z = 1$, then $x = -1$.

Therefore, the eigen vector are $X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Similarly, if we take $y = 1$ and $z = 1$, then $x = -2$, then eigen vector we can take $X_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $y = 0$ and $z = 1$ we have $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case 2: $\lambda = 3$.

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + z = 0$$

$$x - 2y + z = 0$$

$$x + y - 2z = 0$$

Taking any two different equations say

$$-2x + y + z = 0$$

$$x - 2y + z = 0$$

($ax + by + cz = 0$ and $dx + ey + fz = 0$, then $\frac{x}{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ d & f \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}})$

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x}{3} = \frac{-y}{-3} = \frac{z}{3}$$

$$X_3(\text{Eigen vector}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore $S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ can be diagonalizing matrices

17) Find the matrices Λ and S to diagonalize $A = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$. What are limits of Λ^k and $S \Lambda^k S^{-1}$ as $k \rightarrow \infty$.

Solution: Consider the characteristic equation $|A - \lambda I| = 0$

$$\begin{aligned} \text{i.e., } |A - \lambda I| &= \begin{vmatrix} 0.6 - \lambda & 0.4 \\ 0.4 & 0.6 - \lambda \end{vmatrix} = 0 \\ &\Rightarrow \lambda^2 - 1.2\lambda + 0.2 = 0 \\ &\lambda = 1, 0.2 \end{aligned}$$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 0.6 - \lambda & 0.4 \\ 0.4 & 0.6 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$.

$$\begin{aligned} \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -0.4x + 0.4y &= 0, \quad 0.4x - 0.4y = 0 \Rightarrow x = y \end{aligned}$$

Since both equations are same 2(unknowns)-1(equation)=1 free variable. Let $y = 1$ be the free variable, then $x=1$,

Therefore, the eigen vector is

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 0.2$.

$$\begin{aligned} \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0.4x + 0.4y &= 0, \quad 0.4x + 0.4y = 0 \Rightarrow x = -y \end{aligned}$$

Since both equations are same $2(\text{unknowns}) - 1(\text{equation}) = 1$ free variable. Let $y = 1$ be the free variable, then $x = -1$,

Therefore, the eigen vector is

$$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Therefore $S = [X_1, X_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We have $A = S \Lambda S^{-1}$ and

$$A^k = S \Lambda^k S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0.2^k \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + 0.2^k & 1 - 0.2^k \\ 1 - 0.2^k & 1 + 0.2^k \end{bmatrix},$$

$$\Lambda^k \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A^k = S \Lambda^k S^{-1} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ as } k \rightarrow \infty$$

