UNIT-5

Fourier Series

Introduction

What is Fourier series?

Why we need Fourier series?

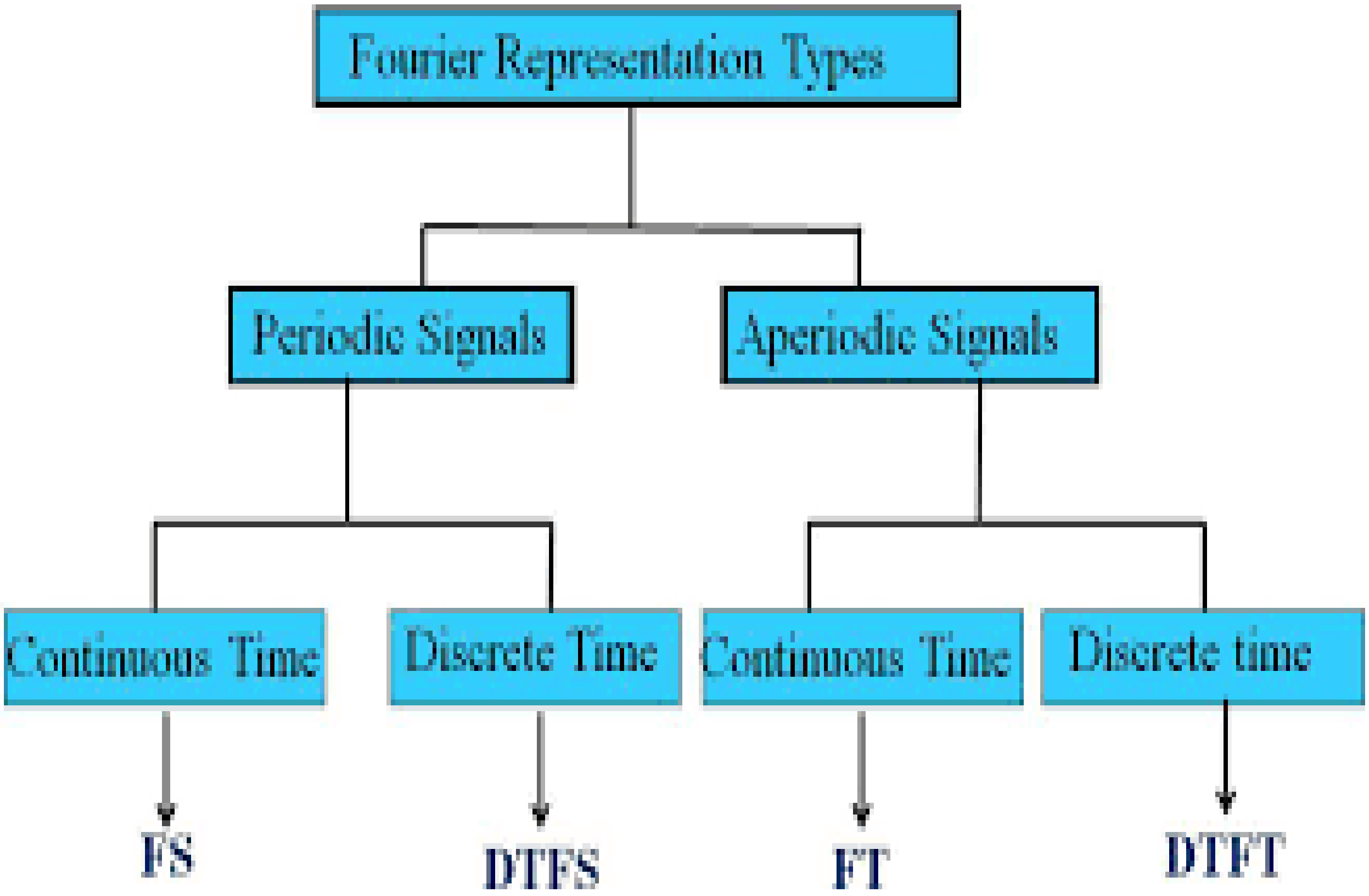
What are the different Types of Fourier Series?

What is Fourier series?

- Fourier series ,Fourier transforms & there applications are given by Joseph Fourier

- Fourier series expansion is used for periodic signals to expand them in terms of their harmonics which are sinusoidal or orthogonal to one another
- It is used for the analysis purpose

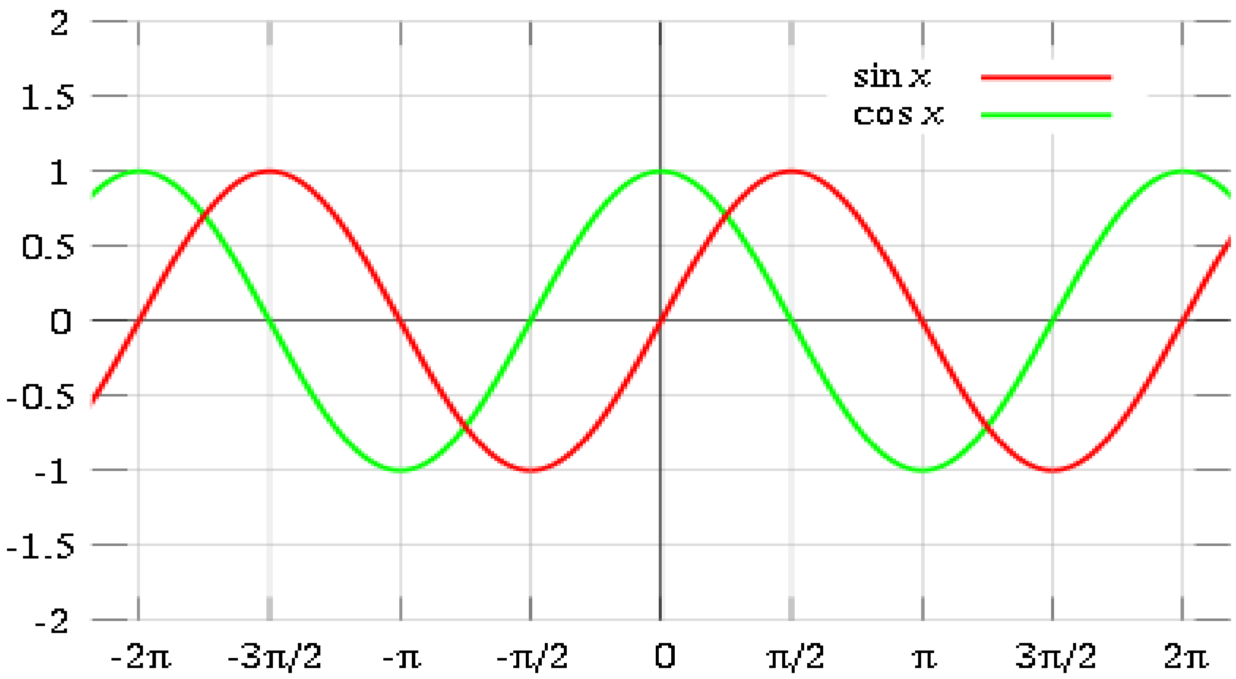
Different types of Fourier series



Applications

- Signal Processing
- Image processing
- Heat distribution mapping
- Wave simplification
- Light Simplication(Interference , Deffraction etc.)
- Radiation measurements etc.

Periodic function



Periodic function

A function $f(x)$ is said to be periodic if $f(x+T) = f(x)$ for all real x and for some positive number T. T is known as the period of the $f(x)$.

Fundamental period of $f(x)$ is the smallest period of f

Eg. $cos x, sin x$ ...are periodic functions with period 2π.

Result1: If T is the period of $f(x)$. Then nT is also period of f for any integer n.

Periodic function

Result2:The function $h(x) = af(x) + bg(x)$ has period T if $f(x) and\ g(x)$ have period T.

Result 3: If $f(x)$ is a periodic function of period T, then $f(ax) with\ a \neq 0$, is a periodic function with period $\frac{T}{a}$.

Result 4: The period of a sum of a number of periodic function is the least common multiple of the periods.

Result 5: A constant is periodic for any period T.

Trigonometric series

Trigonometric series is a fundamental series of the form $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ where constants a_0, a_n, b_n are called the coefficients.

Euler's (Fourier-Euler) Formulae

Let $f(x)$, a periodic function with period 2π defined in the interval $(\alpha, \alpha + 2\pi)$, be the sum of a trigonometric series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (1)$$

Then $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$. $\qquad (2)$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)cosnxdx \;\; for\; n = 1,2,3 \ldots (3)$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)sinnxdx \quad forn = 1,2,3 \ldots (4)$$

.The formula (2),(3),(4) are known as Euler formulae gives the coefficient's a_0, a_n, b_n which are known as Fourier coefficients of $f(x)$.

Fourier Series

Fourier series of a periodic function $f(x)$ with period 2π is the trigonometric series (1) with the Fourier coefficients a_0, a_n, b_n given by the Euler formulae (2),(3),(4).

The individual terms in Fourier series is known as harmonics.

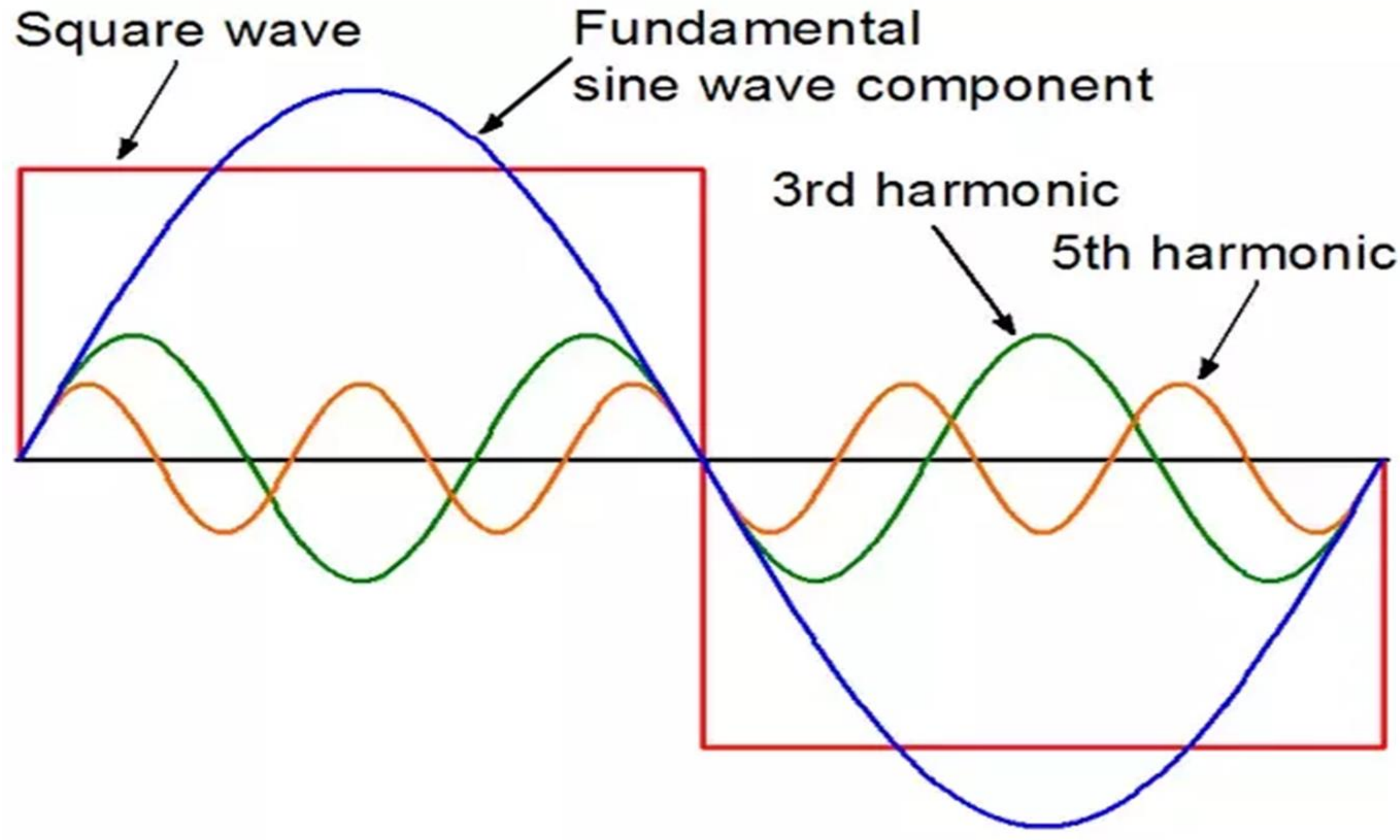
Dirichlet conditions:

Let $f(x)$be periodic function with period 2π. Let $f(x)$ be a piecewise continuous, and bounded in the interval $(\alpha, \alpha + 2\pi)$with finite number of extrema. Then

1. at the point of continuity Fourier series of $f(x)$ (RHS) $converges\, to$ $f(x)(LHS)$.

2. At the point of discontinuity Fourier series of $f(x)$ converges to the arithmetic mean of left and right hand limits of $f(x)$.

Harmonics?


Square wave
Fundamental
sine wave component
3rd harmonic
5th harmonic

1. Obtain the Fourier series to represent e^{-ax} $from\ x = -\pi\ \ to\ \ x = \pi$ and hence derive series for $\frac{\pi}{sinh\pi}$

Soln: Given $f(x) = e^{-ax}$

We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx$$

$$a_0 = \frac{1}{\pi}[e^{-ax}/-a]_{-\pi}{}^{\pi}$$

$\frac{e^{a\pi}-e^{-a\pi}}{a\pi} = \frac{2sinha\pi}{a\pi}$...............(2)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}cosnxdx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-acosnx + nsinnx)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\frac{e^{-a\pi}}{a^2+n^2}(-acosn\pi + nsinn\pi)\text{-}\frac{1}{\pi}\frac{e^{a\pi}}{a^2+n^2}(-acosn\pi - nsinn\pi)$$

$$a_n = \frac{1}{\pi}\frac{e^{a\pi}}{a^2+n^2}(a(-1)^n)\text{-}\frac{e^{-a\pi}}{a^2+n^2}(a(-1)^n)$$

$$a_n = \frac{1}{\pi}\ \frac{a(-1)^n}{a^2+n^2}2sinha\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (e^{-ax}) sinnxdx$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-asinnx - ncosnx)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\frac{e^{-a\pi}}{a^2+n^2}(-asinn\pi - ncosn\pi) \text{-} \frac{1}{\pi}\frac{e^{a\pi}}{a^2+n^2}(asinn\pi - ncosn\pi)$$

$$b_n = \frac{e^{-a\pi}}{a^2+n^2}\frac{1}{\pi}(-n(-1)^n) \text{-} \frac{e^{a\pi}}{a^2+n^2}(-n(-1)^n)$$

$$b_n = \frac{1}{\pi}\frac{n(-1)^n}{a^2+n^2} 2sinha\pi$$

$f(x) = e^{-ax} =$
$\frac{sihaπ}{a}\frac{1}{π} +$
$\sum_{n=1}^{\infty} \frac{1}{\pi} \left(\frac{a(-1)^n}{a^2+n^2} 2sinha\pi \ \cos nx + \frac{n(-1)^n}{a^2+n^2} 2sinha\pi \ \sin nx \right)$...(5)

When $x = 0$ &$a = 1$ we get
$1 = \frac{sih\pi}{1}\frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi} \left(\frac{(-1)^n}{1+n^2} 2sinh\pi \right)$

$$\frac{\pi}{sinh\pi} = 2\left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \cdots \right)$$

2. Find the Fourier series of $f(x) = x + x^2 \ in\ (-\pi, \pi)$.

Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Given $f(x) = x + x^2$

We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2 dx$

$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx$ (x is an odd function in the interval)

$a_0 = \frac{2}{\pi}\left(\frac{x^3}{3}\right)^{\pi}$=$\frac{2\pi^2}{3}$……………………………………….(2)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) cosnxdx$$

$$a_n = \frac{1}{\pi}\left[(x + x^2)\left(\frac{sinnx}{n}\right) - (1 + 2x)\left(\frac{-cosnx}{n^2}\right) + (2)(\frac{-sinnx}{n^3}\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[(1 + 2\pi)\left(\frac{cosn\pi}{n^2}\right) - (1 - 2\pi)\left(\frac{cosn\pi}{n^2}\right)\right]$$

$$a_n = \frac{4}{n^2} cosn\pi = \frac{4}{n^2}(-1)^n \text{..(3)}$$

Solution

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) sinnxdx$$

$$b_n = \frac{1}{\pi}\left[(x + x^2)\left(\frac{-cosnx}{n}\right) - (1 + 2x)\left(\frac{-sinnx}{n^2}\right) + 2\frac{cosnx}{n^2}\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[(\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) - (-\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) + 2\frac{cosn\pi}{n^2} - 2\frac{cosn\pi}{n^2}\right]$$

$$b_n = \frac{-2}{n} cosn\pi = \frac{-2}{n}(-1)^n \text{……………………………………(4)}$$

Solution

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n \cos nx + \frac{-2}{n}(-1)^n \sin nx\right) \ldots (5)$$

Solution

Since the function is not defined at π we have

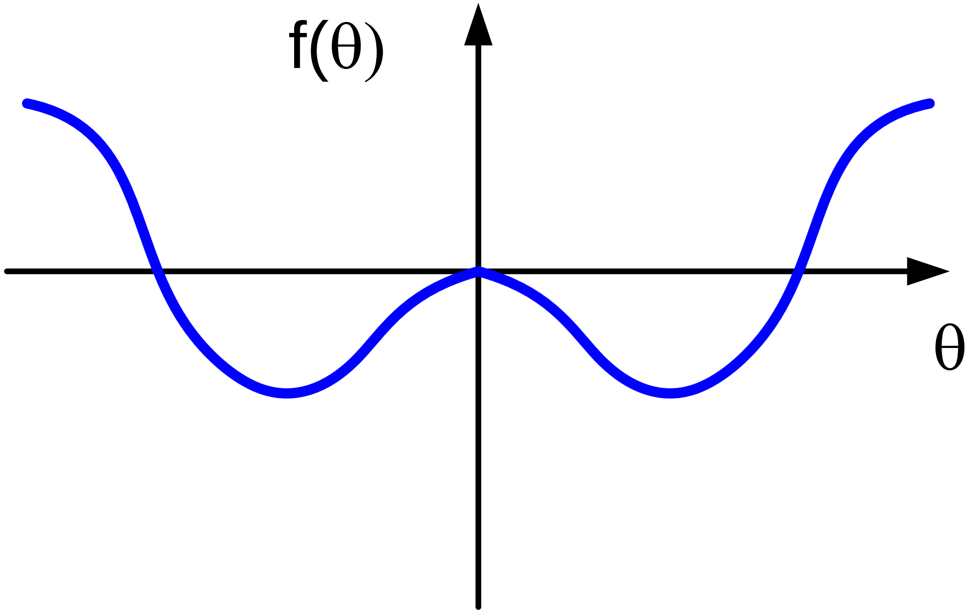
$$f(\pi) = \frac{(f(\pi+)+f(\pi-))}{2}$$

$$= \frac{(f(-\pi +) + f(\pi-))}{2} = \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

Fourier Series
– Odd & Even Functions

Even Functions

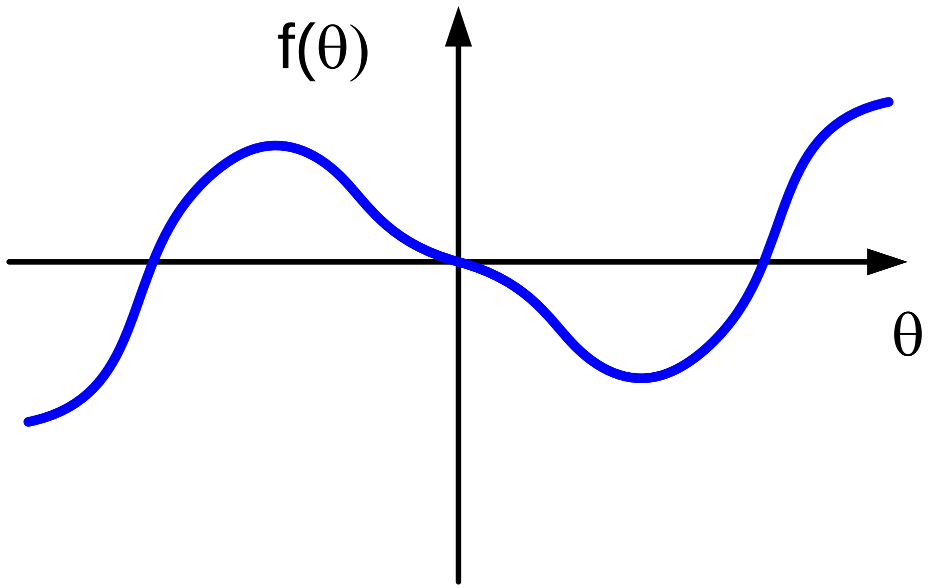


The value of the function would be the same when we walk equal distances along the X-axis in opposite directions.

Mathematically -

$$f(-\theta) = f(\theta)$$

Odd Functions



The value of the function would change its sign but with the same magnitude when we walk equal distances along the X-axis in opposite directions.

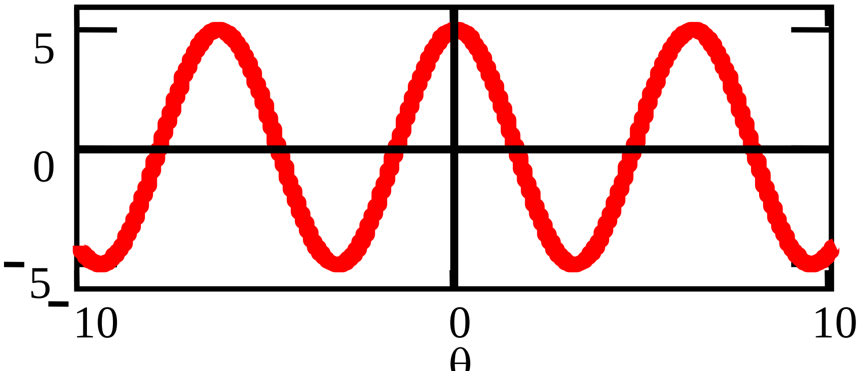
Mathematically -

$$f(-\theta) = -f(\theta)$$

1.Even functions can be represented by cosine waves because, cosine waves are even functions.

2.A sum of even functions is another even function.

3. Product of two even function is even

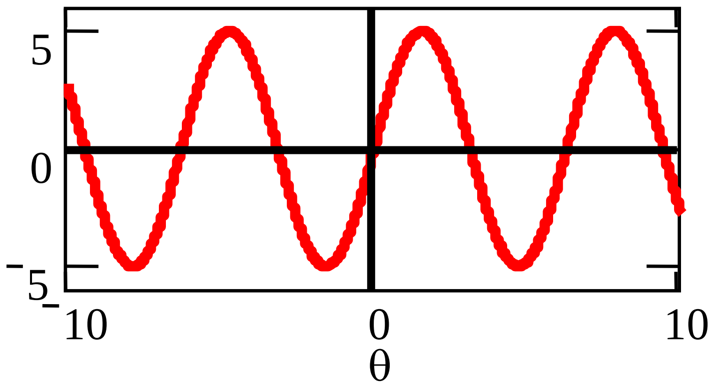


1. **Odd functions can be represented by sine waves because, sine waves are odd functions.**

2. **A sum of odd functions is another odd function.**

3. **Product of an odd function and even function is odd**

4. **Product of two odd functions is even**



Recall!!!

We know that, for a periodic function $f(x)$*of period* π , *defined over* $(-\pi, \pi)$, *the Fourier series expansion*

is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\, cos\, nx + b_n \sin nx) \quad \ldots\ldots\ldots (1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cosnx\, dx\ , \qquad n = 1,2,3 \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sinnx\, dx\ , \qquad n = 1,2,3 \ldots\ .$$

Fourier Series expansion of an even function

Case 1 : Suppose $f(x)$ *is an even function in* $(-\pi, \pi)$*, then the Fourier series expansion contains only cosine terms and is known as* ***Fourier cosine series*** *given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots \dots \dots (2)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \boldsymbol{\frac{2}{\pi}\int_{0}^{\pi} f(x)dx}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \boldsymbol{\frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx \, , n = 1, 2, 3 \dots}$$

Fourier Series expansion of an odd function

Case 2 : Suppose $f(x)$ is an odd function in $(-\pi, \pi)$, the Fourier series expansion contains only sine terms and is known as ***Fourier sine series*** *given by*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots (3)$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sinnx\, dx = \boldsymbol{\frac{2}{\pi}\int_{0}^{\pi} f(x)\, sinnx\, dx\,, n = 1, 2, 3 \ldots}$$

Find the Fourier Series expansion of period $\boldsymbol{2\pi}$ for the function

$$\boldsymbol{f(x) = \begin{cases} x(\pi - x), -\pi \leq x \leq 0 \\ x(\pi + x), 0 \leq x \leq \pi \end{cases}}$$

$$\mathrm{f(x)} = \begin{cases} \varphi_1(x) & x \geq 0 \\ \varphi_2(x) & x < 0 \end{cases} \qquad \varphi_1(-x) = \begin{cases} \varphi_2(x) & even \\ -\varphi_2(x) & odd \end{cases}$$

Solution : Since $f(x)$ is an odd function the Fourier Series expansion of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} b_n sinnx$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) sinnxdx = \boldsymbol{\frac{2}{\pi}\int_0^{\pi} f(x)\ sinnx\ dx}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} \boldsymbol{x(\pi + x)} sinnxdx\ \right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} (\boldsymbol{\pi x} sinnx + x^2 sinnx)dx\ \right\}$$

$$b_n = \frac{2}{\pi}\left\{ \int_0^{\pi}(\boldsymbol{\pi x sinnx} + \boldsymbol{x^2 sinnx})dx \right\}$$

$$= \frac{2}{\pi}\left\{\left[\boldsymbol{\pi x}.\frac{(-\boldsymbol{cosnx})}{\boldsymbol{n}} - \boldsymbol{\pi(1)}.\frac{(-\boldsymbol{sinnx})}{\boldsymbol{n^2}}\right]_{\boldsymbol{0}}^{\boldsymbol{\pi}}\right\}$$

$$+\frac{2}{\pi}\left[\boldsymbol{x^2}.\frac{(-\boldsymbol{cosnx})}{\boldsymbol{n}} - \boldsymbol{(2x)}.\frac{(-\boldsymbol{sinnx})}{\boldsymbol{n^2}} + \boldsymbol{(2)}.\frac{(\boldsymbol{cosnx})}{\boldsymbol{n^3}}\right]_{\boldsymbol{0}}^{\boldsymbol{\pi}}$$

$$= \frac{2}{\pi}\left\{-\frac{\boldsymbol{\pi^2(-1)^n}}{\boldsymbol{n}} - \frac{\boldsymbol{\pi^2(-1)^n}}{\boldsymbol{n}} + \frac{\boldsymbol{2(-1)^n}}{\boldsymbol{n^3}} - \frac{\boldsymbol{2}}{\boldsymbol{n^3}}\right\}$$

$$= \frac{4}{\pi}\left\{\frac{(-1)^n-1}{n^3} - \frac{\pi^2(-1)^n}{n}\right\}$$

Therefore, $f(x) = \sum_{n=1}^{\infty} b_n sinnx$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi}\left\{\frac{(-1)^n-1}{n^3} - \frac{\pi^2(-1)^n}{n}\right\} sinnx$$

Obtain the Fourier Series to represent the function
$$\boldsymbol{f(x) = |x|, -\pi < x < \pi.}$$

Solution : Since $f(x)$ is an even function the Fourier Series

expansion of $f(x)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$ where

$$|x| = \begin{cases} x \geq 0 \\ x < 0 \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} xdx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \qquad = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x cosnx dx$$

$$= \frac{2}{\pi}\left[x.\frac{(sinnx)}{n} - (1).\frac{(-cosnx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Therefore the Fourier Series of

$$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1]cosnx$$

Fourier series of functions having period $2L$

Let $f(x)$ be aperiodic function with arbitrary period $2L$ defined in an interval $c < x < c + 2L$. Then the Fourier series expansion of $f(x)$ is $f(x) = a_0/2 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

Where $a_0 = \frac{1}{L}\int_c^{c+2L} f(x)dx$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos(\frac{n\pi x}{L})dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Fourier series for even and odd functions defined in $(-L, L)$

Fourier series of an even function $f(x) in\ (-L, L)$ contains only cosine terms and is known as Fourier cosine series given by

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{L}\right)$$

Where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_{0}^{L} f(x)dx$

And $a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)cos\frac{n\pi x}{L}dx$

Fourier series for even and odd functions defined in $(-L, L)$

Fourier series of an odd function $f(x) in\ (-L, L)$ contains only sine terms and is known as Fourier sine series given by

$$f(x) = \sum_{n=1}^{\infty}\left(b_n sin\frac{n\pi x}{L}\right)$$

Where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x) sin\frac{n\pi x}{L}dx$$

1. Find the Fourier series expansion of $f(x) = x(1-x)(2-x)\ in\ (0,2)$. Deduce the sum of the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$

Solution

$$f(x) = x(1-x)(2-x)$$

We have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$.....(1)

$a_0 = \frac{1}{1}\int_0^2 f(x)dx$

$$a_0 = \int_0^2 x(1-x)(2-x)\, dx$$

$a_0 = 0$..........................(2)

Solution

$$a_n = \int_0^2 f(x)\cos\frac{n\pi x}{L}\,dx$$

$$a_n = \int_0^2 x(1-x)(2-x)\ \cos n\pi x\,dx$$

$$a_n = \int_0^2 (x^3 - 3x^2 + 2x)\cos n\pi x\,dx$$

$$a_n = \left[(x^3 - 3x^2 + 2x)\left(\frac{sin n\pi x}{n\pi}\right) - (3x^2 - 6x + 2)\left(\frac{-cos n\pi x}{n^2\pi^2}\right) + (6x - 6)\left(\frac{-sin n\pi x}{n^3\pi^3}\right) - (6)\left(\frac{cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$a_n = \left[\frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right] - 6\left[\frac{1}{n^4\pi^4} - \frac{1}{n^4\pi^4}\right]$$

a_n=0

Solution

$$b_n = \int_0^2 f(x) \sin \frac{n\pi x}{L} \, dx$$

$$b_n = \int_0^2 x(1-x)(2-x) \ \sin n\pi x \, dx$$

$$b_n = \int_0^2 (x^3 - 3x^2 + 2x) \ \sin n\pi x \, dx$$

$$b_n = \left[(x^3 - 3x^2 + 2x)\left(\frac{-cosn\pi x}{n\pi}\right) - (3x^2 - 6x + 2)\left(\frac{-sinn\pi x}{n^2\pi^2}\right) + (6x - 6)\left(\frac{cosn\pi x}{n^3\pi^3}\right) - (6)\left(\frac{sinn\pi x}{n^3\pi^3}\right) \right]_0^2$$

$$b_n = [0 - 0] + 6\left[\frac{1}{n^3\pi^3} + \frac{1}{n^3\pi^3}\right]$$

$$b_n = \frac{12}{n^3\pi^3}$$

Solution

$$f(x) = x(1-x)(2-x) = \sum_{n=1}^{\infty}\left(\frac{12}{n^3\pi^3}\sin n\pi x\right)$$

When $x = \frac{1}{2}$

$$1 - \frac{3}{4} + \frac{1}{8} = \sum_{n=1}^{\infty}\left(\frac{12}{n^3\pi^3}\sin\frac{n\pi}{2}x\right)$$

$$\frac{3}{8X12} = \frac{1}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$\frac{\pi^3}{32} = \left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

Parseval's identity

$$\int_{-l}^{l}[f(x)]^2dx = l\left[\frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2)\right]$$

For even functions

$$2\int_0^l[f(x)]^2dx = l\left[\frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}{a_n}^2\right]$$

For Odd functions

$$2\int_0^l[f(x)]^2dx = l\left[\sum_{n=1}^{\infty}{b_n}^2\right]$$

- 2.Obtain the Fourier Series for the function
- f(x) = x^2 , $-\pi$ < x < π. Hence show that
- $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$
- Solution:
- Here f(x)=f(-x). So x^2 is an even function.
- Hence we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, cosnx\, dx \;, \quad b_n=0$$

- and the Fourier expansion of f(x) over ($-$π, π) is
- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Calculation of Fourier coefficients

- $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx,$
- $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx,$
- $= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$
    - $= \frac{2\pi^2}{3}$
-

Calculation of Fourier coefficients:

- $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ cosnx\ dx$
- $a_n = \frac{2}{\pi}\int_0^{\pi} (x^2)cosnx\ dx$
- $= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$
- $= \frac{2}{\pi}\left[(2x)\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$
- $= \frac{2}{\pi n^2}[(2\pi)(\cos n\pi - 0]$
- $= \frac{1}{\pi n^2}\cos n\pi(4\pi) = \frac{4}{n^2}(-1)^n$

- The required Fourier series is given by
- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx) \quad \dots \dots \dots (1)$
  - $x^2 = \frac{2\pi^2}{3}\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n \cos nx\right)$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n \cos nx\right) \quad \dots \dots (2)$$

Deduction

- Consider, $x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n \cos nx\right)$ .... (2)
- To deduce, we will use Parseval's identity for even functions.

  - $2\int_0^l [f(x)]^2 dx = l\left[\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} {a_n}^2\right]$

- Therefore,

$$2\int_0^{\pi} [x^2]^2 dx = \pi\left[\frac{1}{2}\frac{\pi^4}{9} + \sum_{n=1}^{\infty}(\frac{4}{n^2}(-1)^n)^2\right]$$

- $2\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^5}{18} + 16\,\pi \sum_{n=1}^{\infty}\frac{1}{n^4}$
- $2\,\frac{\pi^5}{5} - \frac{\pi^5}{18} = 16\,\pi \sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum \frac{1}{n^4} = \frac{\pi^4}{90}$

3. $f(t) = \begin{cases} 0 \quad if\ -2 \le t \le -1 \\ 1+t \quad if\ -1 \le t \le 0 \\ 1-t \qquad if\ 0 \le t \le 1 \\ 0 \qquad if\ 1 \le t \le 2 \end{cases}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \, cos \frac{n\pi t}{L} \right) \quad \dots \dots \dots (1)$$

$$2L = 4 \quad so\ L = 2$$

$$a_0 = \frac{2}{L} \int_0^2 f(t)dt = \frac{2}{2} \int_0^2 f(t)dt =$$

$$= \int_0^1 1 - t\ dt = \frac{1}{2}$$

$$a_n = \frac{2}{L}\int_0 f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \int_o^1 (1-t)cos\frac{n\pi t}{2}dt$$

$$= \frac{4}{n^2\pi^2}\left[1 - cos\frac{n\pi}{2}\right]$$

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1 - cos\frac{n\pi}{2}\right]cos\frac{n\pi t}{2}$$

4. $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right) \quad \dots \dots \dots (1)$$

$2L = 2 \quad so\ L = 1$

$$a_0 = \frac{1}{L} \int_0^2 f(x)dx = \int_0^1 \pi x dx + \int_1^2 \pi(2 - x)dx = \pi$$

$$a_n = \int_0^1 \pi x \, cos n\pi x dx + \int_1^2 \pi(2-x) cos n\pi x dx$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \int_0^1 \pi x \ sinn\pi x dx + \int_1^2 \pi(2-x) sinn\pi x dx$$

$=0$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]cosn\pi x$$

5. $f(x) = \begin{cases} 0 & -2 < x < 1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

$$a_0 = \frac{1}{2}\int_{-1}^{1} k dx = k$$

$$a_n = \frac{1}{2}\int_{-1}^{1} k\, cos\frac{n\pi x}{2}\, dx$$

$$a_n = \frac{2k}{n\pi} sin\frac{n\pi}{2}$$

$$b_n = \int_{-1}^{1} k \; sin \frac{n\pi x}{2} dx$$

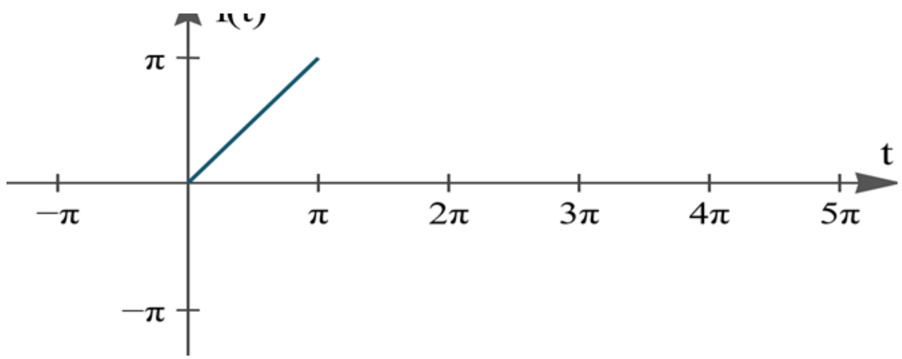
$$b_n = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{n\pi} \sum_{n=1}^{\infty} sin\frac{n\pi}{2} cos\frac{n\pi x}{2}$$

Half range Fourier Series

Example:

In the figure below , the graph of F(t) = t is sketched from t=0 to t= $\boldsymbol{\pi}$

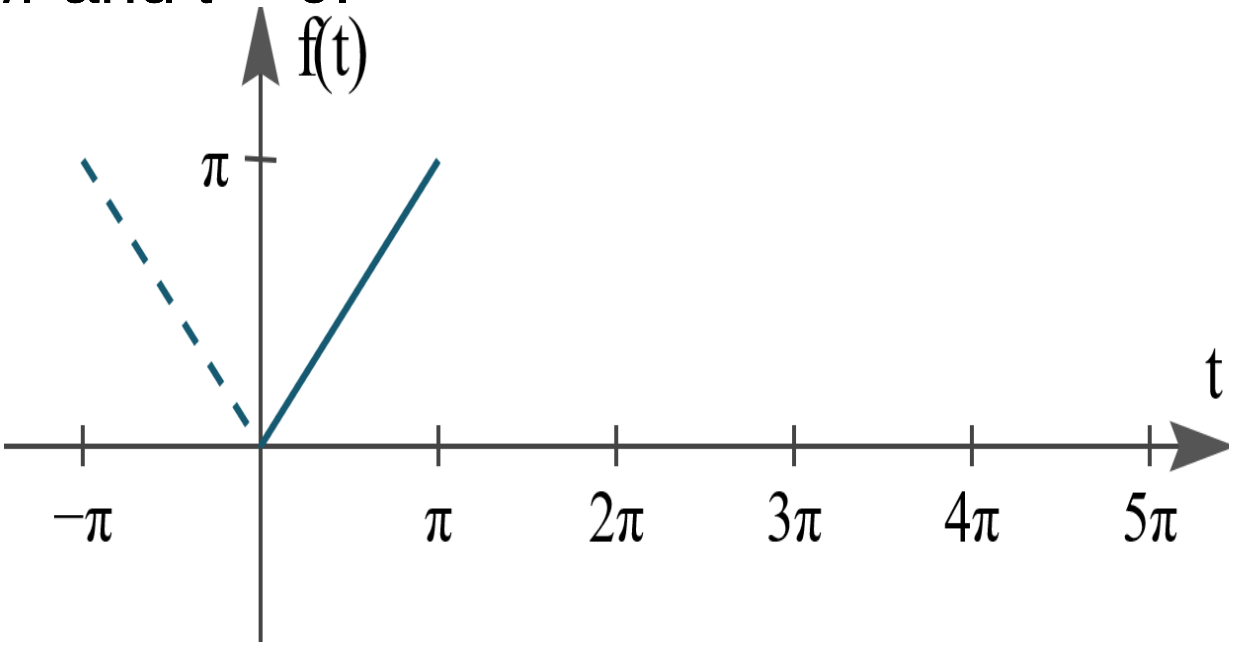


Graph of F(t), $0<t<\boldsymbol{\pi}$

Cosine Series

An even function means that it must be symmetrical about the $f(t)$ axis and this is shown in the following figure by the broken line between

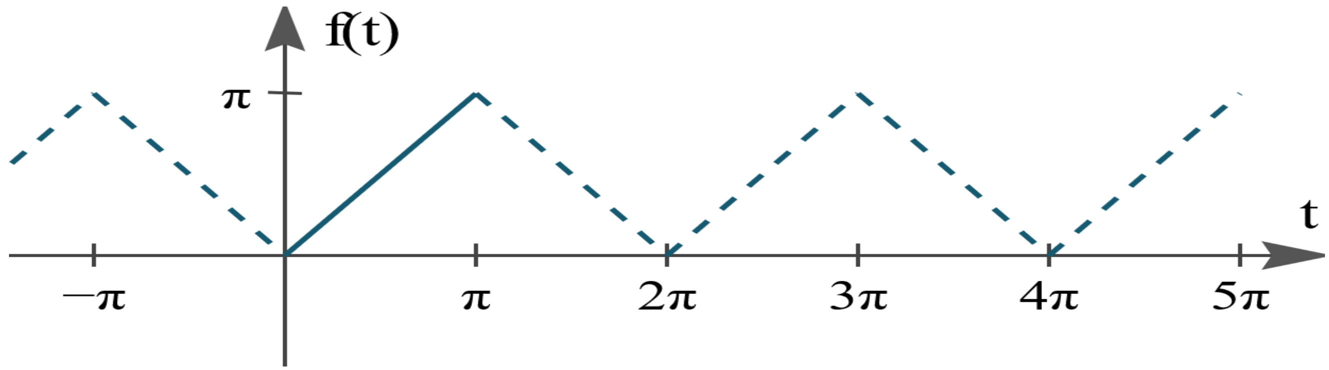
$t = -\pi$ and $t = 0$.



Graph of f(t), illustrating it is an even function

Cosine Series

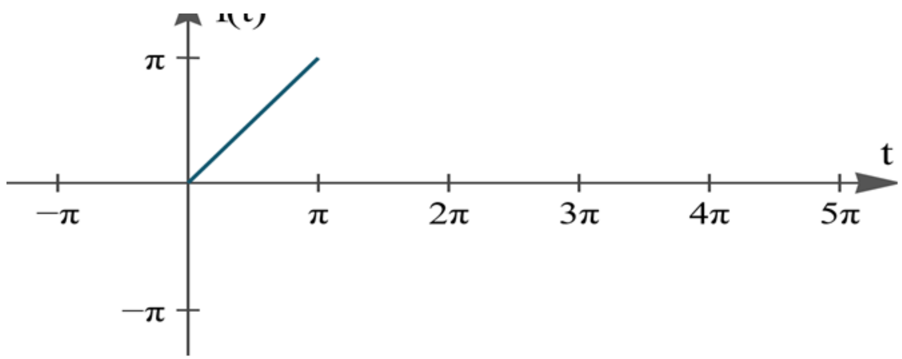
It is then assumed that the "triangular wave form" produced is periodic with period 2π outside of this range as shown by the dotted lines.



Graph of $f(t)$, a triangular waveform.

Example:

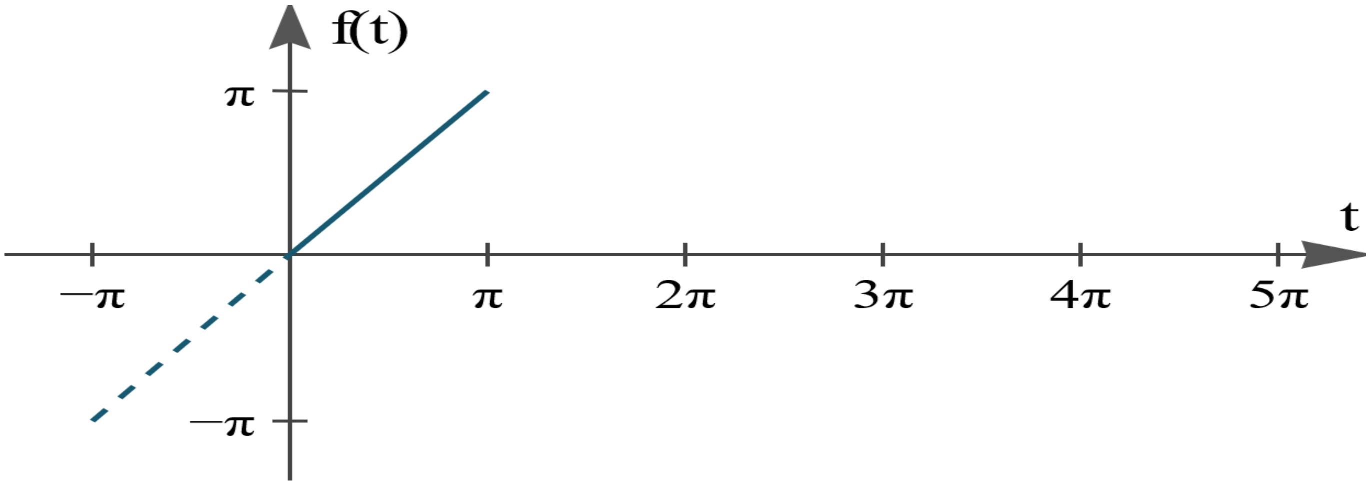
In the figure below , the graph of F(t) = t is sketched from t=0 to t= ***π***



Graph of F(t), 0<t<***π***

Sine Series

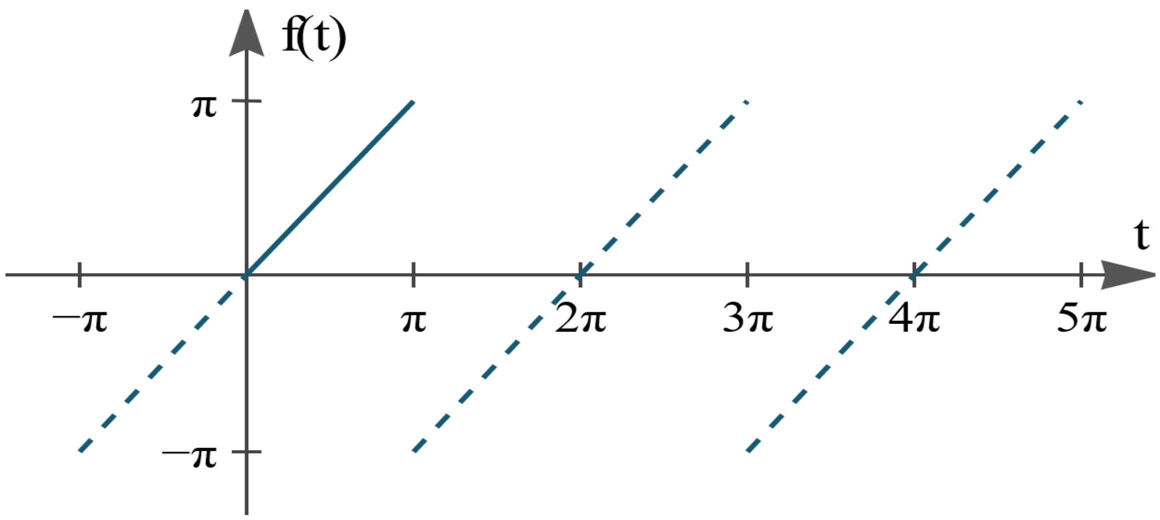
An odd function means that it is symmetrical about the *origin* and this is shown in the following figure by the broken line between $t = -\pi$ and $t = 0$.



Graph of f(t), illustrating it is an odd function

Sine Series

It is then assumed that the waveform produced is periodic of period 2π outside of this range as shown by the dotted lines.



Graph of $f(t)$,a periodic odd function.

Half Range Fourier series of f(x) in(0,$\boldsymbol{\pi}$)

Cosine series	Sine Series
• $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ • Fourier Coefficients: • $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx.$ • $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$	• $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ • Fourier Coefficients: • $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx =$

Half Range Fourier series of f(x) in(0,*l*)

Cosine series	Sine Series
• $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ • <u>Fourier Coefficients:</u> • $a_0 = \frac{2}{l}\int_0^l f(x)dx.$ • $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$	• $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ • <u>Fourier Coefficients:</u> • $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

- 1. Find the Fourier half range a) Cosine series
- b) Sine series of $\mathbf{f}(\boldsymbol{x}) = \begin{cases} \boldsymbol{x}, & \mathbf{0} < \boldsymbol{x} < \mathbf{1} \\ \mathbf{2} - \boldsymbol{x}, & 1 < \boldsymbol{x} < \mathbf{2} \end{cases}$
- Solution:
- a) Cosine series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{l}$
- where
- $a_0 = \frac{2}{2}\int_0^2 f(x)dx$
- $= \int_0^1 x\, dx + \int_1^2 (2-x)\mathrm{dx}$
- $= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = 1$

- $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$
- $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$
- $= \int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}\,dx$
- $=\left[x.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^1 - \left[-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_0^1 + \left[(2-x).\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_1^2 - \left[\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_1^2$
- $= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$
- $-\frac{4}{n^2\pi^2}\left[\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right]$
- $= \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left[1 + (-1)^n\right]$

- b) Sine series : $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$
- $where\ \ b_n = \ \boldsymbol{\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx}$
- $b_n = \frac{2}{2}\int_0^2 f(x) sin\frac{n\pi x}{2} dx$
- $= \int_0^1 x\ sin\frac{n\pi x}{2} dx + \int_1^2 (2-x) sin\frac{n\pi x}{2} dx$
- $= \left[x.\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^1 - \left[-\frac{s\ in\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_0^1 + \left[(2-x).\frac{-cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_1^2 - \left[\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_1^2$
- $= \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\left[sin\left(\frac{n\pi}{2}\right)\right] + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$
- $-\frac{4}{n^2\pi^2}\left[0 - sin\left(\frac{n\pi}{2}\right)\right]$
- $= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

Half- Range Series

- Cosine series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{2}$

- $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}[1 + (-1)^n]cos\frac{n\pi x}{2}$

- Sine series :$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$
- $f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\frac{n\pi x}{2}$

Harmonic Analysis

- It is the process of finding the constant term and
- the first few cosine and sine terms numerically.
- Data:

x	x_1	x_2	x_3		…….		x_N
y	y_1	y_2	y_3		…….		y_N

Period	Fourier Coefficients		
2π	$a_0 = \frac{2}{N}\sum y$	$a_n = \frac{2}{N}\sum y\cos nx$	$b_n = \frac{2}{N}\sum y \sin nx$
2 l	$a_0 = \frac{2}{N}\sum y$	$a_n = \frac{2}{N}\sum y\cos\left(\frac{n\pi x}{l}\right)$	$b_n = \frac{2}{N}\sum y \sin\left(\frac{n\pi x}{l}\right)$

- Example Problem: Express y as a Fourier series up to first harmonics.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
y	7.9	7.2	3.6	0.5	0.9	6.8

Solution: Here, length of interval is $2\boldsymbol{\pi}$.

Here N= 6 . $a_0 = \frac{2}{N}\sum y = \frac{2}{6}\sum y = \frac{1}{3}(26.9) = 8.9667$

$$\boldsymbol{a_n = \frac{2}{N}\sum y\,\mathbf{cos}nx} \qquad\qquad \boldsymbol{b_n = \frac{2}{N}\sum y\,\mathbf{sin}nx}$$

$\boldsymbol{a_1} = \frac{2}{6}\sum y \cos x$

$= \frac{1}{3}\sum y \cos x$

$\boldsymbol{b_1} = \frac{2}{6}\sum y \sin x$

$= \frac{1}{3}\sum y \sin x$

x	y	cosx	ycosx	sinx	ysinx
0	7.9	1	7.9	0	0
$\pi/3$	7.2	0.5	3.6	0.866	6.2352
$2\pi/3$	3.6	-0.5	-1.8	0.866	3.1176
π	0.5	-1	-0.5	0	0
$4\pi/3$	0.9	-0.5	-0.45	-0.866	-0.7794
$5\pi/3$	6.8	0.5	3.4	-0.866	-5.8888
Total	26.9		12.15		2.6846

- $a_0 = \frac{2}{N}\sum y$ =
- $\frac{2}{6}\sum y$= $\frac{1}{3}(26.9) = 8.9667$
- $a_1 = \frac{\mathbf{1}}{\mathbf{3}}\sum y$ cosx =
- $\frac{\mathbf{1}}{\mathbf{3}}[\mathbf{12.15}] = 4.05$
- $b_1 = \frac{1}{\mathbf{3}}\sum y$ sinx =
- $\frac{1}{\mathbf{3}}[\mathbf{2.6846}] = 0.8949$
- **Therefore Fourier series up to first harmonic is given by**
- **f(x)**$= \frac{a_0}{2} + a_1 cosx + b_1 sinx$
- y = 4.48335 +4.05 cos x +0.8949sin x

- 14. Determine the first 2 coefficients of cosine and two coefficient of sine terms in the Fourier series
- for the following data

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$
y	0	9.2	14.4	17.8	17.3	11.7

- Solution: Here, length of interval is $\boldsymbol{\pi}$. That is, $l = \frac{\pi}{2}$

Here N= 6 . $a_0 = \frac{2}{N}\sum y = \frac{2}{6}\sum y =$

$$\frac{1}{3}(70.4) = 23.46$$

$$a_n = \frac{2}{N}\sum y\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2}{6}\sum y\cos\left(\frac{n\pi x}{\pi/2}\right)$$

$$= \frac{1}{3}\sum y\cos(2nx)$$

$$b_n = \frac{2}{N}\sum y\sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{2}{6}\sum y\cos\left(\frac{n\pi x}{\pi/2}\right)$$

$$= \frac{1}{3}\sum y\sin(2nx)$$

$$a_n = \frac{1}{3}\sum y\cos(2nx)$$

x	y	cos2x	ycos2x	cos4x	ycos4x
0	0	1	0	1	0
$\pi/6$	9.2	0.5	4.6	-0.5	-4.6
$\pi/3$	14.4	-0.5	-7.2	-0.5	-7.2
$\pi/2$	17.8	-1	-17.8	1	17.8
$2\pi/3$	17.3	-0.5	-8.65	-0.5	-8.65
$5\pi/6$	11.7	0.5	5.85	-0.5	-5.85
Total			-23.2		-8.5

Cosine terms

- $a_n = \frac{1}{\mathbf{3}} \sum y$ cos2nx
- $a_1 = \frac{1}{\mathbf{3}} \sum y$ cos2x = $\frac{1}{\mathbf{3}} [-\mathbf{23.2}] = -7.73$
- $a_2 = \frac{1}{\mathbf{3}} \sum y$ cos4x = $\frac{1}{\mathbf{3}} [-\mathbf{8.5}] = -2.83$

$$b_n = \frac{1}{3}\sum y \sin 2nx$$

x	y	sin2x	ysin2x	sin4x	ysin4x
0	0	0	0	0	0
$\pi/6$	9.2	0.866	7.9672	0.866	7.9672
$\pi/3$	14.4	0.866	12.4704	-0.866	-12.4704
$\pi/2$	17.8	0	0	0	0
$2\pi/3$	17.3	-0.866	-14.9818	0.866	14.9818
$5\pi/6$	11.7	-0.866	-10.1322	-0.866	-10.1322
Total			-4.6764		-3454

Sine terms

- $b_n = \frac{1}{\mathbf{3}} \sum y$ sinnx
- $b_1 = \frac{1}{\mathbf{3}} \sum y$ sin2x = $\frac{1}{\mathbf{3}} [-\mathbf{4.6764}] = -1.2124$
- $b_2 = \frac{1}{\mathbf{3}} \sum y$ sin4x = $\frac{1}{\mathbf{3}} [\mathbf{0.3464}] = 0.1155$

- **Therefore Fourier series is given by**
- **f(x)**$= \frac{a_0}{2} + a_1 cos2x + b_1 sin2x + a_2 cos4x + b_2 sin4x$
- y = 11.733 - 7.733cos2x - 1.2124 sin2x - 2.833cos4x + 0.116sin 4x

Complex form of Fourier series

The complex exponential form of the Fourier series is

$\boldsymbol{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}}$ **where**

$\boldsymbol{c_n = \frac{1}{2L}\int_{c}^{c+2L} f(x) e^{\frac{-in\pi x}{L}} dx}$.

1. Find the Complex exponential form of the Fourier Series of $f(x) = e^{-x} \quad -1 < x < 1$

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}, C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}} dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\, e^{\frac{-in\pi x}{1}} dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx = \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\,\Bigg|_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)} - \frac{e^{(1+in\pi)x}}{-(1+in\pi)}\right]$$

$$= \frac{1}{2(1+in\pi)}\left[ee^{in\pi} - e^{-1}e^{in\pi}\right]$$

$$= \frac{1}{2(1+in\pi)}\left[e(cosn\pi + isinn\pi) - e^{-1}(cosn\pi - isinn\pi)\right]$$

$$= \frac{1}{2(1+in\pi)}(e - e^{-1})(-1)^n$$

$$= \frac{(1 - in\pi)}{(1 - n^2\pi^2)2}(e - e^{-1})(-1)^n$$

$$= \frac{(-1)^n(1 - in\pi)sinh1}{(1 - n^2\pi^2)}$$

$$f(x) = e^{-x} = \sum_{-\infty}^{\infty} \frac{(-1)^n(1 - in\pi)sinh1}{(1 - n^2\pi^2)} e^{in\pi x}$$

2. Find the Complex exponential form of the Fourier Series of $f(x) = cosax \quad -\pi < x < \pi$

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}, C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}} dx$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} cosax\, e^{\frac{-in\pi x}{\pi}} dx$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} cosax\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-n\pi}}{a^2 - n^2}\{-incosax + asinax\}^{\pi}{}_{-\pi}\right]$$

$$= \frac{-incosa\pi}{2\pi(a^2 - n^2)}\left[e^{-in\pi} - e^{in\pi}\right]$$

$$+ \frac{asina\pi}{2\pi(a^2 - n^2)}\left[e^{-in\pi} + e^{in\pi}\right]$$

$$= \frac{asina\pi}{2\pi(a^2 - n^2)} 2(-1)^n$$

$$f(x) = cosax = \sum_{-\infty}^{\infty} \frac{asina\pi}{\pi(a^2 - n^2)} (-1)^n e^{inx}$$