Uncertainties in Least Squares Coefficients

Consider Bivariate data $(x_{i,}\ y_i)$ for i=1,2,3,…………….n

$$y = \beta_0 + \beta_1 x$$

The line $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, ε_i is the error, that best fits the data in the sense of minimizing the sum of the squared errors. It is called the least squares regression line

$\widehat{\beta_0}$, $\widehat{\beta_1}$ are estimates of β_0 , β_1 .

$\widehat{y_l} = \widehat{\beta_0} + \widehat{\beta_1} x_i$ = Fitted line

If ε_i tend to be large then $(x_{i,}\ y_i)$ are widely scattered around the line.

If ε_i tend to be small then $(x_{i,}\ y_i)$ are tightly clustered around the line.

The line $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ has Normal distribution with

$$\mu_{y_i} = \beta_0 + \beta_1 x_i$$

$$\sigma_{yi}{}^2 = \sigma^2$$

$$\widehat{\beta_1} = \sum_{i=1}^{n} \left[\frac{x_i - \bar{x}}{\sum_{i-1}^{n} (x_i - \bar{x})^2} \right] yi$$

$$\widehat{\beta_0} = \sum_{i=1}^{n} \left[\frac{1}{n} - \frac{\bar{x}\,(x_i - \bar{x})}{\sum_{i-1}^{n} (x_i - \bar{x})^2} \right] yi$$

Mean of the estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ are

$\mu_{\widehat{\beta_0}} = \beta_0$ $\qquad \mu_{\widehat{\beta_1}} = \beta_1$

Uncertainty in the estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ are

$$\sigma_{\widehat{\beta_0}} = \sigma \sqrt{\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i-1}^{n} (x_i - \bar{x})^2} \right]}$$

$$\sigma_{\widehat{\beta_1}} = \sigma \sqrt{\left[\frac{1}{\sum_{i-1}^{n} (x_i - \bar{x})^2} \right]}$$

Since the value of σ is unknown it is approximated with *s*

$$s_{\widehat{\beta_0}} = s\sqrt{\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

$$s_{\widehat{\beta_1}} = s\sqrt{\left[\frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

Where s is the estimate of the error standard deviation σ and

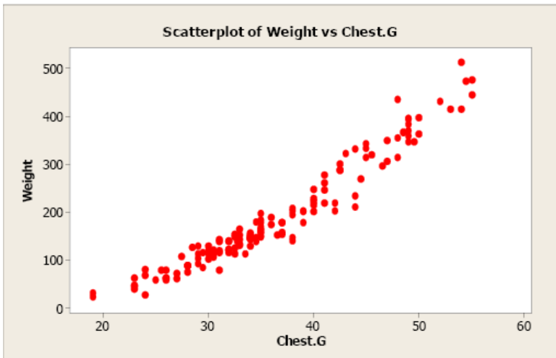
$$s = \sqrt{\frac{(1-r^2)\sum(y_i-\bar{y})^2}{n-2}}$$

$$s_{\widehat{\beta_0}} = s\sqrt{\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]} \qquad s_{\widehat{\beta_1}} = s\sqrt{\left[\frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}\right]}$$

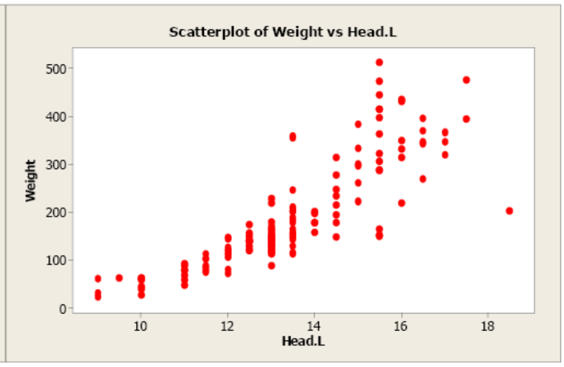
$$s_{\widehat{\beta_1}} \;\; \alpha \;\; \frac{1}{\sum_{i-1}^{n}(x_i - \bar{x})^2}$$

If x – values are more spread then the uncertainty of estimates $\widehat{\beta_0}$, $\widehat{\beta_1}$ are Smaller.

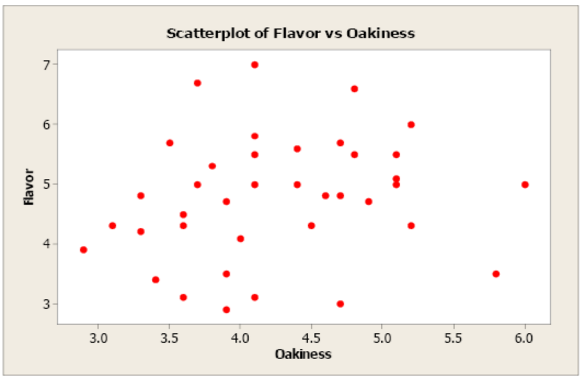
The standard deviation of x is more.



Strong positive relationship
r = 0.96



Moderate positive relationship
r = 0.67



Very weak positive relationship
r = 0.07

Problem: Two engineers are conducting independent experiments to estimate spring constant for a particular spring. The first engineer suggests measuring the length of the spring with no load, then applying loads of 0,1,2,3,& 4 lb. The second engineer suggests using loads of 0, 2, 4, 6 & 8 lb. Which will be more precise?

Sol: X ----- 0, 1, 2, 3, 4 Y-------- 0, 2, 4, 6, 8

σ_y is twice as great as σ_x .

Uncertainty of X is twice as large as the uncertainty of Y. Hence, the Engineer, Y 's estimate is twice as precise.