22C 131 Homework 2 : Solutions

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1. (Problem 5.13) Let $USELESS_{TM} = \{< M > | M$ is a TM with one or more useless states$\}$. Show that A_{TM} reduces to $USELESS_{TM}$. Assume for the sake of contradiction that TM R decides $USELESS_{TM}$. Construct TM S that uses R to decide A_{TM}. The new TM has a useless state exactly when M doesn't accept w. For this purpose, we use the universal turing machine.

   TM S : On input $< M, w >$

   TM T : On input x :

   a. Replace x on the input by the string $< M, w >$.

   b. Run the universal TM U to simulate $< M, w >$.

   (Note that this TM uses all it's states)

   If U accepts, enter a special state q_A and *accept*.

   1. Run R on T to determine whether T has any useless states.
   2. If R rejects, *accept*. Otherwise *reject*.

   If M accepts w, then T enters all states, but if M doesn't accept w, then T avoids q_A. So T has a useless state, q_A iff M doesn't accept w.

2. (Problem 5.15) Let $LM_{TM} = \{< M, w > | M$ ever moves left while computing $w\}$. LM_{TM} is decidable. We construct a TM $LEFT_{TM}$ that decides LM_{TM}. To construct this TM we claim that any TM that ever makes a left move must do so in at most $w + n_M + 1$ steps, where n_M is the number of states of M, and w is the size of the input (we use w to denote both the input and it's size, but the usage should be clear from context). To see this, assume TM M makes a LEFT move, and consider the shortest computation path $p = q_0, q_1 \cdots, q_s$ of M ending in a LEFT move. First note that since M has only been scanning blanks from state q_w, we may remove any cycles in the computation path and still be left with a legal computation path ending in

a left move. Hence, p does not contain cycles, and can have length at most $w + n_M + 1$. Now we can construct $LEFT_{TM}$ as follows :

TM $LEFT_{TM}$: On input $< M, w >$
1. Simulate M on w for $n_M + w + 1$ steps.
2. if M ever makes a left move *accept*. Otherwise *reject*.

3. (Problem 5.17) The PCP over a unary alphabet is decidable. We describe a TM M that decides unary PCP. Given a unary PCP instance,

   TM M: On input $< P >$
   1. Check if $a_i = b_i$ for some i. If so, *accept*.
   2. Check if there exist i, j such that $a_i > b_i$ and $a_j < b_j$.
      If so, *accept*, else *reject*.

   In the first stage, M checks for a single domino which forms a match. In the second stage, M looks for two dominos which form a match. If it finds such a pair, it can construct a match by picking $(b_j - a_j)$ copies of the i^{th} domino, putting them together with $(a_i - b_i)$ copies of the j^{th} domino. This construction has $a_i(b_j - a_j) + a_j(a_i - b_i) = a_i b_j - a_j b_i$ $1's$ on top, and $b_i(b_j - a_j) + b_j(a_i - b_i) = a_i b_j - a_j b_i$ $1's$ on the bottom. If neither stages of M accept, the problem instance contains dominos with all upper parts having more/less $1's$ than the lower parts. In such a case, no match exists and therefore M rejects.

4. (Problem 5.33) First we show that $A_{TM} \leq_M \overline{S}$. This shows S is not turing recognizable. The function f can be described as follows.

   f: On input $< M, w >$,
   1. Construct machine M_1 that does the following :
      M_1 : On input x,
         Run M on w. If M accepts w, *reject*.
         Otherwise if $x =< M_1 >$ *accept*.
   2. Output $< M_1 >$.

If M accepts w, then $L(M_1) = \emptyset$. Hence, $< M_1 >$ is in $\overline{S}$. Conversely, if M does not accept w, then $L(M_1) = \{< M_1 >\}$, and hence $< M_1 > \in S$. This shows S is not turing recognizable. We now show that $\overline{S}$ is not turing recognizable by reducing from A_{TM} to S.

g: On input $< M, w >$,
1. Construct machine M_2 that does the following :
   M_2 : On input x
   Run M on w. If M accepts w,
   check if $x =< M_2 >$. If it is, then *accept*.
   Otherwise *reject*.
2. Output $< M_2 >$.

In this case, if M accepts w, $L(M_2) =< M_2 >$, and hence, $< M_2 > \in S$. Otherwise $L(M_2) = \emptyset$ and $< M_2 > \in \overline{S}$. Hence, $\overline{S}$ is not turing recognizable.

5. (Problem 5.35) Let $X = \{< M, w > | M$ is a single-tape TM that never modifies the portion of the tape that contains the input $w\}$. We show that X is undecidable by reducing from A_{TM}. Let R be a TM that decides X. We use R to construct a TM S that decides A_{TM}.

TM S: On input $< M, w >$
TM M_X: On input $< M, w >$
   a. Mark the right end of the input with a symbol $\$ \notin \Gamma_M$.
   b. Copy w to the part of the tape after \$. Call this part w'
   c. Simulate M on w'.
   d. If M accepts, write any character on the first cell of the input tape and *accept*.
   e. Otherwise *reject*.
1. Input $< M_X, w >$ to R.
2. If R accepts, *accept*. Otherwise *reject*.

Note that M_X ever modifies it's input iff M accepts w. Hence, we have decided A_{TM}, a contradiction.