VECTOR SPACES

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CLASS 3 : CONTENT

- Problems on Echelon form ,free and pivot variables

ECHELON FORM OF A MATRIX

Problem 1 :

For each of the following matrices find

1. Echelon Form 'U'
2. Row reduced Echelon Form 'R'
3. Rank of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \qquad E = \begin{pmatrix} -2 & 3 & 1 \end{pmatrix}$$

ECHELON FORM OF A MATRIX

Solution :-

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix} = U = \text{Echelon Form}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix} \xrightarrow{R_2/8} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1-3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R \text{ Rowreduced Echelonform}$$

Rank of matrix $A = \rho(A) = 2$ (Number of non zero rows in echelon form of the matrix)

ECHELON FORM OF A MARIX

$$B = \begin{pmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{pmatrix} \xrightarrow{R_2 - \frac{3}{2} R_1} \begin{pmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{pmatrix} = U = \text{Echelon form}$$

$$\begin{pmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -11 & 11 \end{pmatrix} \xrightarrow{R_2/(-11)} \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_1 - 3R_2} R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

(Row reduced Echelon form)

Rank of $B = \rho(B) = 2$

ECHELON FORM OF A MATRIX

$$C = \begin{pmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1 \text{ and}} \begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix} = U \quad \text{Echelon form}$$

$$\begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2/5} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Also $\rho(C) = 2$.

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{R_1 + R_2}$$

ECHELON FORM OF A MATRIX

$$D = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \xrightarrow{R_2 + \frac{3}{2} R_1} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = U$$

$$U \xrightarrow{R_1/(-2)} R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\rho(D) = 1$$

$$E = \begin{pmatrix} -2 & 3 & 1 \end{pmatrix} \xrightarrow{R_1/-2} \begin{pmatrix} 1 & -3/2 & -1/2 \end{pmatrix} = U = R$$

$$\rho(E) = 1$$

PIVOT VARIABLE AND FREE VARIABLE

Problem 2:

Solve the following system of linear eqns by identifying pivot variables and free variables:

$$x + 2y + 3z = 9$$
$$2x - 2z = -2$$
$$3x + 2y + z = 7$$

Solution: Apply Gauss Elimination on the augmented matrix

$$[A : b] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right]$$

PIVOT AND FREE VARIABLE

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

$$U = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -2 & -8 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_3 - R_2}$$

Pivots $= (1, -2)$;

$\rho(A) = 2$

$\rho(A:b) = 2$

$n = 3$

$\rho(A:b) = \rho(A) < n \rightarrow$ Infinitely Many Solution

System of Linear eqns given has Many solution

PIVOT AND FREE VARIABLES

Solving system of Eqns further we obtain

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -20 \\ 0 \end{pmatrix}$$

$$x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -20 \\ 0 \end{pmatrix}$$

$x, y \rightarrow$ Pivot variables $\rightarrow$ associated to columns with pivots

$z \longrightarrow$ free variable $\rightarrow$ associated to column without pivot

Solution $\{k-1,\ 5-2k,\ k\}$ where $z = k \in \mathbb{R}$

SPECIAL SOLUTIONS

Problem 3 :

Reduce the matrix $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{pmatrix}$ to its Echelonform and hence find special solution.

Solution:

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \sim R_3} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

if $\underline{c=1}$ then $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$

Solving for $Rx = 0 \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

SPECIAL SOLUTIONS

$$Rx = 0 \quad \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Pivot var (x)

Free variables (y, z, t)

$$x + y + 2z + 2t = 0 \Rightarrow x = -y - 2z - 2t$$

Sol.

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \sim \begin{pmatrix} -y - 2z - 2t \\ y \\ z \\ t \end{pmatrix} \sim y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Where $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is the special solution

SPECIAL SOLUTIONS

if $c \neq 1$ then

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - \frac{1}{c-1} R_2} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 / c-1} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Pivot var (x, y)

Free var (z, t)

Solving for $Rx = 0$

$$x + 2z + 2t = 0 \Rightarrow x = -2z - 2t, \quad y = 0$$

SPECIAL SOLUTIONS

Special Solutions $\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Note : Echelon form is sufficient to look for back substition to obtain solution, but need for Row reduced Echelon form R is the matrix R readily gives Special solutions.

Given $A \longrightarrow U \longrightarrow R$ (to obtain special solutions)

THANK YOU

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