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Unit-2 Introductory problems:

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1. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

(i) $z = x^2y - x\sin(xy)$

$$\frac{\partial z}{\partial x} = 2xy - [\sin(xy) + x\cos(xy) \times y]$$

$$= 2xy - \sin xy - xy\cos(xy)$$

$$\frac{\partial z}{\partial x} = xy(2 - \cos(xy)) - \sin(xy)$$

$$\frac{\partial z}{\partial y} = x^2 - x^2\cos(xy) = x^2(1 - \cos xy)$$

(ii) $z = \cos^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = \frac{-1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \times \left(\frac{1}{y}\right) = \frac{-1}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \times \left(\frac{-x}{y^2}\right) = \frac{x}{y\sqrt{y^2 - x^2}}$$

(iii) $z = \tan^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\frac{x^2 + y^2}{x + y}\right)^2} \times \left(\frac{2x(x+y) - (x^2 + y^2)}{(x+y)^2}\right)$$

$$= \frac{1}{(x+y)^2 + (x^2 + y^2)^2}\left(x^2 - y^2 + 2xy\right)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{x^2 - y^2 + 2xy}{(x^2 + y^2)^2 + (x + y)^2}}$$

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$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2}\left[\frac{2y(x+y)-(x^2+y^2)}{(x+y)^2}\right]$$

$$\frac{\partial z}{\partial y} = \frac{y^2-x^2+2xy}{(x+y)^2+(x^2+y^2)^2}$$

2. If $x+y = x^2+y^2$

Let $z = \frac{x^2+y^2}{(x+y)} = \frac{x^2+y^2+2xy-2xy}{x+y}$

$$z = (x+y) - \frac{2xy}{(x+y)}$$

$$\frac{\partial z}{\partial x} = 1 - \frac{2y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{2x^2}{x+y}$$

R.H.S

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = \left[\frac{2y^2}{(x+y)^2}+\frac{2x^2}{(x+y)^2}-1\right]\times 4$$

$$= \left[\frac{x-y}{x+y}\right]^2 \times 4$$

$$= \left[\left(\frac{\cancel{2x^2}}{x+y}\right)^2 - \frac{2y}{x+y}\right]$$

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = \left(\frac{x-y}{x+y}\right)^2 \times 4 \quad \text{—— (1)}$$

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L.H.S

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left[1-\frac{2y^2}{(x+y)^2}+\frac{2x^2}{(x+y)^2}-1\right]^2$$

$$=\left[2\left(\frac{x^2-y^2}{(x+y)^2}\right)\right]^2$$

$$=2^2\left(\frac{\cancel{(x+y)}(x-y)}{(x+y)^{\cancel{2}}}\right)^2$$

$$=\left[2\left(\frac{x-y}{x+y}\right)\right]^2$$

$$=4\times\left[\frac{x-y}{x+y}\right]^2 \quad -(2)$$

By Eq^n (1) & Eq^n (2)

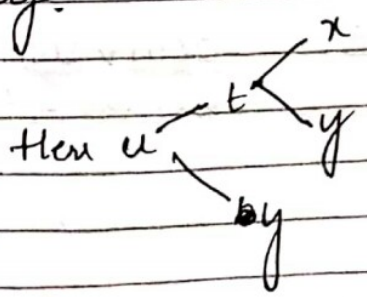
$$\boxed{L.H.S = R.H.S}$$

3. $u = e^{(ax+by)} * f(ax-by)$

let $ax-by=t$.

Then $ax=t+by$.

$$u = e^{(t+by+by)} * f(t)$$



$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t}\times\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial t}\times(a) \Rightarrow \boxed{\frac{\partial u}{\partial x}=a\frac{\partial u}{\partial t}}$$

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$$\frac{\partial u}{\partial y} = e^{(t+by)} \times b \times f(t) + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = \cancel{2}b\ u - b\frac{\partial u}{\partial t}$$

LHS

$$b\frac{\partial u}{\partial x} + \cancel{2}a\frac{\partial u}{\partial y}$$

$$\Rightarrow ba\frac{\partial u}{\partial t} + \cancel{2}a(b)u - ab\frac{\partial u}{\partial t}$$

$$\Rightarrow 2abu = R.H.S$$

4. If $u = \log(\tan x + \tan y + \tan z)$

$$(\sin 2x)\frac{\partial u}{\partial x} + \sin 2y\left(\frac{\partial u}{\partial y}\right) + \sin 2z\frac{\partial u}{\partial z} = 2$$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 z$$

$$LHS = \frac{2\sin x \sec}{\tan x + \tan y + \tan z} = \frac{2\tan x}{\tan x + \tan y + \tan z} + \frac{2\tan y}{\tan x + \tan y + \tan z} + \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$L.H.S = 2\left(\frac{\tan x + \tan y + \tan z}{\tan x + \tan y + \tan z}\right)$$

$$= 2 \times 1 = 2$$

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5. $u = f(r)$

where $r = \sqrt{x^2+y^2+z^2}$

As

$$u \longrightarrow r \longrightarrow x,\; y,\; z$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{f'(r) \times x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f''(r) \times x^2}{x^2+y^2+z^2} + f'(r)\left[\frac{\sqrt{x^2+y^2+z^2} - \frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right]$$

$$\text{①} \leftarrow \frac{\partial^2 u}{\partial x^2} = \frac{f''(r) \times x^2}{x^2+y^2+z^2} + f'(r)\left[\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

Similarly:

As x, y, z have symettry

$$\text{②} \longleftarrow \frac{\partial^2 u}{\partial y^2} = \frac{f''(r)\, y^2}{x^2+y^2+z^2} + f'(r)\left[\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\text{③} \longleftarrow \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)\, z^2}{x^2+y^2+z^2} + f'(r)\left[\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

① + ② + ③

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r^3}\left[r^2\right] f'(r)$$

As $x^2+y^2+z^2 = r^2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r).$$

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6. If $u = \log(x^3+y^3+z^3-3xyz)$

S.T

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

L.H.S

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} (3y^2-3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3z^2-3xy)$$

L.H.S

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\frac{(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xy}$$

(As $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$)

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{x+y+z} = R.H.S$$

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7.) At what rate is area of the rectangle changing if its length is 15mts and increasing at 3mts/sec while its width is 6mts and increasing at 2mts/sec

let

Length of rectangle = 'x'

Breadth of rectangle = 'y'

$\frac{dl}{dt} = 15 \text{ mts/sec} = \frac{dx}{dt}$ } Given.

$\frac{dy}{dt} = 2 \text{ mts/sec}$

Area of Rectangle $= xy$

$$\frac{dA}{dt} = \frac{dx}{dt} y + x \frac{dy}{dt}.$$

$$= 15 \times 2 + 6 \times 3$$

$$\boxed{\frac{dA}{dt} = 48 \ m^2/s}$$

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Total derivatives

1> Given $u = \sin\left(\frac{x}{y}\right)$ where $x = e^t, y = t^2$, find the total derivative of u w.r.t t.

Sol

$$x = e^t \quad , \quad y = t^2$$

$$\frac{dx}{dt} = e^t \quad , \quad \frac{dy}{dt} = 2t$$

$$\frac{\partial u}{\partial x} = \frac{\cos\left(\frac{x}{y}\right)}{y} \quad , \quad \frac{\partial u}{\partial y} = \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$$

by total derivatives

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{\cos}{y}\left(\frac{x}{y}\right) e^t + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)(2t)$$

putting $x = e^t, y = t^2$

$$= \frac{\cos}{t^2}\left(\frac{e^t}{t^2}\right) e^t - \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^4}\right)(2t^2)$$

$$= \left(1 - \frac{2}{t}\right)\frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)$$

2) If $u = x^2 + y^2 + z^2$, where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$; find the total derivative of u w.r.t t.

Sol

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

$$\frac{dz}{dt} = e^t \cos t - e^t \sin t$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z$$

by total derivatives

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2e^{2t} + 2e^{2t}\sin t(\sin t + \cos t) + 2e^{2t}\cos t(\cos t - \sin t)$$

$$= 2e^{2t} + 2e^{2t}(\sin^2 t + \cos^2 t)$$

$$= 4e^{2t}$$

3) The height of a tree increases at a rate of 2ft per year and the radius increases at 0.1 ft per year. What rate is volume of increasing when the height is 20 ft and radius 1.5 ft?

Sol.

$$V = \pi r^2 h$$

$$\frac{dv}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

given $\frac{dr}{dt} = 0.1$ and $\frac{dh}{dt} = 2$

& $r = 1.5$ and $h = 20$

$$\therefore \frac{dv}{dt} = 2\pi\left(\frac{3}{2}\right)(20)\left(\frac{1}{10}\right) + \pi\left(\frac{3}{2}\right)^2(2)$$

$$= 32.97 \text{ ft}^3/\text{year}.$$

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2) If $u = x^2 + y^2 + z^2$, where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$; find the total derivative of u w.r.t t.

Sol

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

$$\frac{dz}{dt} = e^t \cos t - e^t \sin t$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z$$

by total derivatives

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2e^{2t} + 2e^{2t}\sin t(\sin t + \cos t) + 2e^{2t}\cos t(\cos t - \sin t)$$

$$= 2e^{2t} + 2e^{2t}(\sin^2 t + \cos^2 t)$$

$$= 4e^{2t}$$

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$$= 32.97 \text{ ft}^3/\text{year}.$$

4) If $u = e^x \sin(yz)$, where $x = t^2$, $y = t-1$, $z = \frac{1}{t}$, find $\frac{du}{dt}$ at $t=1$

Sol.

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 1 \quad \frac{dz}{dt} = -\frac{1}{t^2}$$

$$\frac{\partial u}{\partial x} = e^x \sin(yz), \quad \frac{\partial u}{\partial y} = e^x \cos(yz) z$$

$$\frac{\partial u}{\partial z} = e^x \cos(yz)\, y$$

$$\frac{du}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

$$= 2t\, e^x \sin(yz) + e^x \cos(yz) z - \frac{1}{t^2} \cdot e^x \cos(yz) y$$

putting $x = t^2$, $y = t-1$, $z = \frac{1}{t}$

and $t=1$

$$= 2e \sin(0) + e\cos(0) - e\cos(0)(0)$$

$$= \underline{\underline{e}}$$

5) If $u = \tan^{-1}\left(\frac{y}{x}\right)$ where $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$

find $\frac{du}{dt}$.

Sol

$$\frac{dx}{dt} = e^t + e^{-t}, \quad \frac{dy}{dt} = e^t - e^{-t}$$

$$\frac{\partial u}{\partial x} = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{du}{dt} = (e^t + e^{-t})\left(\frac{-y}{x^2+y^2}\right) + (e^t - e^{-t})\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{-(e^t+e^{-t})(e^t+e^{-t})}{(e^t-e^{-t})^2 + (e^t+e^{-t})^2} + \frac{(e^t-e^{-t})(e^t-e^{-t})}{(e^t-e^{-t})^2 + (e^t+e^{-t})^2}$$

$$= \frac{-2}{e^{2t} + e^{-2t}}$$

Partial differentiation

Problems on composite functions.

If $u = f\left(xz, \frac{y}{z}\right)$, prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$.

$$u = f\left(xz, \frac{y}{z}\right)$$

Let $xz = v$
and $w = y/z$

Now, acc to composite function

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} \quad -(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} \quad -(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z} \quad -(3)$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} z + \frac{\partial u}{\partial w}(0)$$

$$\frac{\partial u}{\partial \quad} = \frac{\partial u}{\quad} z$$

$$x\frac{\partial u}{\partial x} = xz\frac{\partial u}{\partial v} \quad -(4)$$

(2) $\Rightarrow$

$$\frac{\partial u}{\partial y} = 0 + \frac{\partial u}{\partial w}\left(\frac{1}{z}\right)$$

xly $-y$ on b-s

$$-y\frac{\partial u}{\partial y} = -\frac{y}{z}\frac{\partial u}{\partial w} \quad -(5)$$

(3) $$\frac{\partial u}{\partial z} = x\cdot\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\left(\frac{-y}{z^2}\right)$$

xly $-z$ on b-s

$$-z\frac{\partial u}{\partial z} = -zx\frac{\partial u}{\partial v} + \frac{y}{z}\frac{\partial u}{\partial w} \quad (6)$$

add - (4) + (5) + (6)

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = xz\frac{\partial u}{\partial v} - \frac{y}{z}\frac{\partial u}{\partial w} - zx\frac{\partial u}{\partial v} + \frac{y}{z}\frac{\partial u}{\partial w}$$

$$\therefore\ x\frac{\partial u}{\partial w} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$$

2. If $u = f(x^2-y^2, y^2-z^2, z^2-x^2)$ prove that
$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\cdot\frac{\partial u}{\partial z} = 0$$

Soln:-

$$u = f(x^2-y^2, y^2-z^2, z^2-x^2)$$

Let $r = x^2-y^2$

$s = y^2-z^2$

$t = z^2-x^2.$

Now acc to composite function,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \quad -(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \quad -(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} \quad -(3)$$

$(1) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}[2x] + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}[-2x]$

$$\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial r} - 2x\frac{\partial u}{\partial t}$$

$$\frac{1}{2x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad -(4)$$

$(2) \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(-2y) + \frac{\partial u}{\partial s}(2y) + \frac{\partial u}{\partial s}(0)$

$$\frac{\partial u}{\partial y} = -2y\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s}$$

$$\frac{1}{2y}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad -⑤$$

③ $\Rightarrow$ $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-2z) + \frac{\partial u}{\partial t}(2z)$$

$$\frac{1}{2z}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} \quad -⑥$$

add ④, ⑤, ⑥

$$\frac{1}{2x}\frac{\partial u}{\partial x} + \frac{1}{2y}\frac{\partial u}{\partial y} + \frac{1}{2z}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s}$$

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

4 If $z = f(u, v)$, $u = \log(x^2+y^2)$, $v = \frac{y}{x}$,

show that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$.

soln:

$$u = \log(x^2+y^2)$$

$v = \frac{y}{x}$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad - (1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad - (2)$$

$\Rightarrow (1)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{1}{x^2+y^2} \times 2x + \frac{\partial z}{\partial v}\frac{-y}{x^2}$$

$$= \frac{\partial z}{\partial u}\frac{2x}{x^2+y^2} - \frac{\partial z}{\partial v}\frac{y}{x^2} \quad - \cancel{(3)}$$

xly $-y$ on b-s

$\Rightarrow \cancel{(2)}$

$$-y\frac{\partial z}{\partial x} = -y\frac{\partial z}{\partial u}\frac{2x}{x^2+y^2} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} \quad - (3)$$

$\Rightarrow (2)$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{1}{x^2+y^2} \times 2y + \frac{\partial z}{\partial v}\frac{1}{x}$$

xly x ~~x~~ b.s

$$x\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{2xy}{x^2+y^2} + \frac{\partial z}{\partial v} \quad - (4)$$

add ~~2~~ ~~3~~ (4) and (3).

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \cancel{-\frac{\partial z}{\partial u}\frac{2xy}{x^2+y^2}} + \frac{y^2}{x^2}\frac{\partial z}{\partial v}$$

$$+ \cancel{\frac{\partial z}{\partial u}\frac{2xy}{x^2+y^2}} + \frac{\partial z}{\partial v}$$

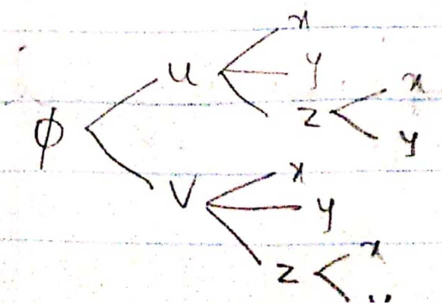
$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}\left(1+\frac{y^2}{x^2}\right)$$

~~from~~ we know that $v = \frac{y}{x}$,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}\left(1+v^2\right)$$

Q3. If $\phi(cx-az, cy-bz)=0$, show that $ap+bq=c$, where $p=\frac{\partial z}{\partial x}$; $q=\frac{\partial z}{\partial y}$.

↳



Problems on implicit functions

Q1) For the curve $xe^y + ye^x = 0$ find the equation of the tangent line at the origin.

given:- $xe^y + ye^x = 0$

$(x, y) = (0, 0)$

Let $z = xe^y + ye^x$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$0 = e^y + ye^x + [xe^y + e^x]\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{(e^y + ye^x)}{(xe^y + e^x)}$$

$$\left.\frac{dy}{dx}\right|_{(0,0)} = -\frac{(e^0 + 0)}{(0 + e^0)}$$

$$= -1$$

equation of the tangent line,

$y - y_1 = m(x - x_1)$

$y - 0 = -1(x - 0)$

$y = -x$

$y + x = 0$

5) If $x^3+3x^2y+6xy^2+y^3=1$, find $\frac{dy}{dx}$

soln:-

given :-

$$x^3+3x^2y+6xy^2+y^3=1$$

Let $z = x^3+3x^2y+6xy^2+y^3-1$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$0 = 3x^2+6xy+6y^2 + (3x^2+12xy+3y^2) \times \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-(3x^2+6xy+6y^2)}{3x^2+12xy+3y^2}$$

$$\frac{dy}{dx} = \frac{-3}{3}\frac{(x^2+2xy+2y^2)}{(x^2+4xy+y^2)}$$

$$\frac{dy}{dx} = -\frac{(x^2+2xy+2y^2)}{(x^2+4xy+y^2)}$$

2) Find $\frac{dy}{dx}$ when i) $x^y + y^x = c$

soln:-

$$x^y + y^x = c$$

$\cancel{x^y + y^x}$ $x^y + y^x - c = 0$

Let $z = x^y + y^x - c$

$$x^y + y^x - c = 0$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$0 = yx^{y-1} + y^x \log y + \left(x^y \log x + xy^{x-1}\right) \times \frac{dy}{dx}$$

$$\left(x^y \log x + xy^{x-1}\right)\frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y\right)$$

$$\frac{dy}{dx} = -\frac{\left(yx^{y-1} + y^x \log y\right)}{\left(x^y \log x + xy^{x-1}\right)}$$

ii) $(\cos x)^y = (\sin y)^x$

given:- $(\cos x)^y = (\sin y)^x$

applying log on b.s

$$y \log_e \cos x = x \log_e \sin y$$

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$$x \log_e \sin y - y \log_e \cos x = 0$$

Let $z = x \log_e \sin y - y \log_e \cos x$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$0 = \log_e \sin y \; \cancel{+} - \left[y \times \frac{1}{\cos x} \times -\sin x \; \cancel{+ \log_e \cos x} \right]$$

$$+ \left[x \times \frac{1}{\sin y} \times \cos y - 0 \right] \frac{dy}{dx}$$

$$0 = \cancel{\log_e \sin y} + [y \tan x] + [x \cot y$$

$$+ \left[x \times \frac{1}{\sin y} \times \cos y - \log_e \cos x \right] \frac{dy}{dx}$$

$$\log_e \sin y + y \tan x = -[x \cot y - \log_e \cos x] \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\log_e \sin y + y \tan x}{\log_e \cos x - x \cot y}$$

3. If $z = \sin\left(\frac{x}{y}\right)$ and $x^2 + y^2 = a^2$, find $\frac{dz}{dx}$

soln:-

given:- $z = \sin\left(\frac{x}{y}\right)$

$x^2 + y^2 = a^2$

$x^2 + y^2 = a^2$

$y = \sqrt{a^2 - x^2}$ — ①

diff w r to x

$$\frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{a^2 - x^2}} \times -\cancel{2}x = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$z = \cancel{\sin\left(\frac{x}{\frac{-x}{\sqrt{a^2-x^2}}}\right)} \quad \sin\left(\frac{x}{\sqrt{a^2 - x^2}}\right)$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$= \cos\left(\frac{x}{\sqrt{a^2 - x^2}}\right) \frac{\sqrt{a^2 - x^2} - x \times \frac{1}{\cancel{2}\sqrt{a^2 - x^2}} \times -\cancel{2}x}{\left(\sqrt{a^2 - x^2}\right)^2}$$

$+ 0$

$$= \cos\left(\frac{x}{\sqrt{a^2 - x^2}}\right)\left[\frac{a^2 - \cancel{x^2} + \cancel{x^2}}{\sqrt{a^2 - x^2}\; a^2 - x^2}\right]$$

$$= \frac{a^2}{\sqrt{a^2 - x^2}} \cos\left(\frac{x}{\sqrt{a^2 - x^2}}\right)$$

$$\frac{dz}{dx} = \frac{a^2}{y^3} \cos\left(\frac{x}{y}\right) \quad - \text{from } (1)$$

4. If $u = \sin(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$, find $\frac{du}{dx}$.

Soln:- given:-

$$u = \sin(x^2 + y^2)$$

$$ax^2 + by^2 = c^2$$

$$y^2 = \frac{c^2 - a^2x^2}{b^2}$$

$$y = \frac{\sqrt{c^2 - a^2x^2}}{b} \quad -(1)$$

diff w.r.t.o x

$$\frac{dy}{dx} = \frac{1}{b} \times \frac{1}{\cancel{2}\sqrt{c^2 - a^2x^2}} - \times a^2 \times \cancel{2}x$$

$$= -\frac{a^2x}{b\sqrt{c^2 - ax^2}}$$

$$\therefore u = \sin\left(x^2 + \frac{c^2 - a^2x^2}{b^2}\right)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = \cos\left(x^2 + \frac{c^2 - a^2x^2}{b^2}\right)\left[2x + \frac{1}{b^2} \times -2a^2x\right]$$

$$= \cos\left(x^2 + y^2\right) 2x\left[1 - \frac{a^2}{b^2}\right]$$

$$= 2\left(1 - \frac{a^2}{b^2}\right) x \cos\left(x^2 + y^2\right)$$

Problems on Euler's theorem and extension of Euler's theorem.

(1) Verify Euler's theorem for the following functions

i) $u = x^2yz - 4y^2z^2 + 2xz^3$ —— (1)

Checking weather u is a homogenous function

$$u(\lambda x, \lambda y, \lambda z) = (\lambda x)^2(\lambda y)(\lambda z) - 4(\lambda y)^2(\lambda z)^2 + 2(\lambda x)(\lambda z)^3$$
$$= \lambda^4 u$$

$\therefore$ u is a homogenous function of degree 4

Differentiating u partially w.r.t x

$$\frac{\partial u}{\partial x} = 2xyz + 2z^3.$$

Differentiating u partially w.r.t y

$$\frac{\partial u}{\partial y} = x^2z - 8yz^2$$

Differentiating u partially w.r.t z.

$$\frac{\partial u}{\partial z} = x^2y - 8y^2z + 6xz^2$$

According to Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \qquad (n = 4)$$

LHS

$$= x(2xyz + 2z^3) + y(x^2z - 8yz^2) + z(x^2y - 8y^2z + 6xz^2)$$
$$= 2x^2yz + 2xz^3 + x^2yz - 8y^2z^2 + x^2yz - 8y^2z^2 + 6xz^3$$
$$= 4(x^2yz - 4y^2z^2 + 2xz^3) = 4u$$

ii) $u = y^n \log\left(\frac{x}{y}\right)$

Checking weather u is a homogenous function

$u(\lambda x, \lambda y) = (\lambda y)^n \log\left(\frac{\lambda x}{\lambda y}\right)$

$= \lambda^n u$

$\therefore$ u is a homogenous function of degree n

Differentiating u partially w.r.t x

$$\frac{\partial u}{\partial x} = y^n \times \frac{1}{\frac{x}{y}} \times \frac{1}{y} = \frac{y^n}{x}$$

Differentiating u partially w.r.t y

$$\frac{\partial u}{\partial x} = y^n \times \frac{1}{\frac{x}{y}} \times \frac{-x}{y^2} + n y^{n-1} \log\left(\frac{x}{y}\right)$$

According to Euler's theorem.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u$$

LHS

$= x\left(\frac{y^n}{x}\right) + y\left(-y^{n-1} + n y^{n-1} \log\left(\frac{x}{y}\right)\right)$

$= y^n - y^n + n y^n \log\left(\frac{x}{y}\right)$

$= n u$

(2-Q) If $u = \dfrac{x^3y^3z^3}{x^2+y^2+z^2} + \cos\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = \dfrac{7x^3y^3z^3}{x^2+y^2+z^2}$

~~2-(Q).~~

(2) Given

$$u = \frac{x^3y^3z^3}{x^2+y^2+z^2} + \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$$

Let

$$v = \frac{x^3y^3z^3}{x^2+y^2+z^2} \; ; \; w = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$$

$$v(\lambda x, \lambda y, \lambda z) = \frac{(\lambda x)^3(\lambda y)^3(\lambda z)^3}{(\lambda x)^2+(\lambda y)^2+(\lambda z)^2} = \lambda^7 v.$$

$\therefore$ v is a homogeneous function of degree 7

$$w(\lambda x, \lambda y, \lambda z) = \cos\left(\frac{(\lambda x)(\lambda y)+(\lambda y)(\lambda z)+(\lambda z)(\lambda x)}{(\lambda x)^2+(\lambda y)^2+(\lambda z)^2}\right)$$

$$= w.$$

$\therefore$ w is a homogeneous function of degree 0

According to Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 7v. \quad -(1)$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0. \quad -(2)$$

(1) + (2)

$$\Rightarrow x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) + z\frac{\partial}{\partial z}(v+w) = 7v$$

$\because u = v+w$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{7x^3y^3z^3}{x^2+y^2+z^2}$$

3. (a) If $u = \frac{x^3+y^3}{y\sqrt{x}} + \frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ at $x=1, y=2$

3 Given

$$u = \frac{x^3+y^3}{y\sqrt{x}} + \frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$$

Let

$$v = \frac{x^3+y^3}{y\sqrt{x}}, \quad w = \frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$$

$$v(\lambda x, \lambda y) = \frac{(\lambda x)^3+(\lambda y)^3}{(\lambda y)\sqrt{\lambda x}} = \lambda^{3/2} v.$$

$\therefore$ v is a homogenous function of degree 3/2

$$w(\lambda x, \lambda y) = \frac{1}{(\lambda x)^7}\sin^{-1}\left(\frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2+2(\lambda x)(\lambda y)}\right)$$
$$= \lambda^{-7} w.$$

$\therefore$ w is a homogenous function of degree -7

According to Euler's theorem and extension of Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{3}{2}v. \quad —(1)$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \frac{3}{2}\left(\frac{1}{2}\right)v = \frac{3}{4}v. \quad —(2)$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -7w. \quad —(3)$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = -7(-8)w = 56w \quad —(4).$$

(1) + (2) + (3) + (4)

$$\Rightarrow x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) + x^2\frac{\partial^2}{\partial x^2}(v+w) + 2xy\frac{\partial^2}{\partial x \partial y}(v+w)$$
$$+ y^2\frac{\partial^2}{\partial y^2}(v+w) = \frac{9v}{4} + 49w.$$

4

$\therefore u = v + w$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

$$= \frac{9}{4}\left(\frac{x^3+y^3}{y\sqrt{x}}\right) + 49\left(\frac{1}{x^7} \sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)\right)$$

At $x = 1, y = 2$

$$= \frac{9}{4}\left(\frac{1+8}{2}\right) + 49\left(\frac{1}{(1)^7} \sin^{-1}\left(\frac{1+4}{1+4}\right)\right)$$

$$= \frac{81}{4} + \frac{49\pi}{2}.$$

(4) If $u = \sin^{-1}\left(\frac{x-y}{x+y}\right)^{1/2}$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

Checking weather u is homogenous.

$$u(\lambda x, \lambda y) = \sin^{-1}\left(\frac{\lambda x - \lambda y}{\lambda x + \lambda y}\right)^{1/2} = u$$

$\therefore u$ is homogenous of degree 0.

$$\sin u = \left(\frac{x-y}{x+y}\right)^{1/2} \Rightarrow \sin^2 u = \frac{x-y}{x+y}.$$

Let $z = \sin^2 u = \frac{x-y}{x+y}$

$$z(\lambda x, \lambda y) = \frac{\lambda x - \lambda y}{\lambda x + \lambda y} = z.$$

$\therefore z$ is homogenous of degree 0.

~~Applying~~

Differentiating z w.r.t x partially

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x} = 2 \sin u \cos u \frac{\partial u}{\partial x}$$

But $\frac{\partial z}{\partial x} = \frac{x+y-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial y} = 2\sin u \cos u \frac{\partial u}{\partial y}$$

But $\frac{\partial z}{\partial y} = \frac{(x+y)(-1)-(x-y)(1)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$

Consider

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(2\sin u \cos u \frac{\partial u}{\partial x}\right) + y\left(2\sin u \cos u \times \frac{\partial u}{\partial y}\right)$$

$$x\left(\frac{2y}{(x+y)^2}\right) + y\left(\frac{-2x}{(x+y)^2}\right) = 2\sin u \cos u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$0 = 2\sin u \cos u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

5.Q If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$

5) Given

$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

Let $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$.

$$z(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\sqrt{\lambda x} + \sqrt{\lambda y}} = \lambda^{1/2} z$$

$\therefore$ z is homogenous of degree $\frac{1}{2}$.

According to the extension of Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2\frac{\partial^2 y}{\partial x^2}+2xy\frac{\partial^2 y}{\partial x\partial y}+y^2\frac{\partial^2 y}{\partial y^2}=\frac{1}{2}\left(\frac{-1}{2}\right)(\sin u) \quad -(1)$$

$$\cancel{x^2\frac{\partial^2 y}{\partial u^2}\left(\frac{\partial^2 u}{\partial x^2}\right)}+\cancel{2xy\frac{\partial^2 y}{\partial u^2}\left(\frac{\partial^2 u}{\partial x\partial y}\right)}+\cancel{y^2\frac{\partial^2 y}{\partial u^2}\left(\frac{\partial^2 u}{\partial y^2}\right)}$$

$$=-\frac{1}{4}\sin u$$

$$\frac{\partial^2 y}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\sin u)\right)$$

$$=\frac{\partial}{\partial x}\left(\cos u\frac{\partial u}{\partial x}\right)$$

$$=\cos u\frac{\partial^2 u}{\partial x^2}-\sin u\left(\frac{\partial u}{\partial x}\right)^2 \quad -(2)$$

Similarly

$$\frac{\partial^2 y}{\partial y^2}=\cos u\frac{\partial^2 u}{\partial x^2}-\sin u\left(\frac{\partial u}{\partial x}\right)^2 \quad -(3)$$

~~According to Eul~~

$$\frac{\partial^2 y}{\partial x\partial y}=\cos u\frac{\partial^2 u}{\partial x\partial y}-\sin u\frac{(\partial u)^2}{\partial x\partial y} \quad -(4)$$

According to Euler's theorem

$$x\frac{\partial y}{\partial x}+y\frac{\partial y}{\partial y}=\frac{1}{2}\sin u$$

Squaring

$$x^2\left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+y^2\left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2$$

$$+2xy\left(\frac{\partial y}{\partial u}\right)^2\left(\frac{(\partial u)^2}{\partial x\partial y}\right)=\frac{1}{4}\sin^2 u$$

$$\cos^2 u\left(x^2\left(\frac{\partial u}{\partial x}\right)^2+y^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{(\partial u)^2}{\partial x\partial y}\right)=\frac{1}{4}\sin^2 u$$

$$x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{(\partial u)^2}{\partial x\,\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{4}\tan^2 u \qquad -(5)$$

Substituting (2), (3), (4) in (1)

$$\cos u\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$$

$$- \sin u\left(x^2\left(\frac{\partial u}{\partial x}\right)^2 + \frac{2xy(\partial u)^2}{\partial x\,\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2\right)$$

$$\cos u\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) = -\frac{\sin u}{4}$$

$$= -\frac{\sin u}{4} + \frac{1}{4}\tan^2 u \sin u$$

$$= \frac{1}{4}\sin u\left(\frac{\sin^2 u}{\cos^2 u} - 1\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\frac{\sin u(-\cos 2u)}{\cos^3 u}$$

$$= -\frac{\sin u \cos 2u}{4\cos^3 u}$$

(6) Using Euler's theorem, show that if $u = \tan^{-1}(x^2+2y^2)$ then

i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\sin u\cos 3u$

i) Let $z = \tan u = x^2+2y^2$

$z(\lambda x, \lambda y) = (\lambda x)^2 + 2(\lambda y)^2 = \lambda^2 z$.

$\therefore$ z is a homogenous function of degree 2

According to Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

$$x\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right) + y\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\sec^2 u = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u = \sin 2u \quad -(1).$$

ii) $(1)^2 \Rightarrow$ $x^2\left(\frac{\partial u}{\partial x}\right)^2 + y^2\left(\frac{\partial u}{\partial y}\right)^2 + 2xy\frac{(\partial u)^2}{\partial x\partial y} = \sin^2 2u \quad -(2).$

According to Extension of Euler's theorem

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} = n(n-1)z = 2\tan u \quad -(6)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\tan u)\right)$$

$$= \frac{\partial}{\partial x}\left(\sec^2 u\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 y}{\partial x^2} = \operatorname{Sec}^2 u \frac{\partial^2 u}{\partial x^2} + 2\operatorname{Sec}^2 u \operatorname{Tan} u \left(\frac{\partial u}{\partial x}\right)^2 \quad -③$$

Similarly

$$\frac{\partial^2 y}{\partial y^2} = \operatorname{Sec}^2 u \frac{\partial^2 u}{\partial y^2} + 2\operatorname{Sec}^2 u \operatorname{Tan} u \left(\frac{\partial u}{\partial x}\right)^2 \quad -④$$

$$\frac{\partial^2 y}{\partial x \partial y} = \operatorname{Sec}^2 u \cdot \frac{\partial^2 u}{\partial x \partial y} + 2\operatorname{Sec}^2 u \operatorname{Tan} u \frac{(\partial u)^2}{\partial x \partial y}. \quad -⑤$$

Substituting ③, ④, ⑤ in 6.

$$\operatorname{Sec}^2 u\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^3 u}{\partial y^2}\right)$$
$$+ 2\operatorname{Sec}^2 u \operatorname{Tan} u \left(x^2 \left(\frac{\partial u}{\partial x}\right)^2 + 2xy \frac{(\partial w)^2}{\partial x \partial y} + y^2 \left(\frac{\partial u}{\partial y}\right)^2\right) = 2 \operatorname{Tan} u$$

$$\operatorname{Sec}^2 u\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right) = 2 \operatorname{Tan} u - 2\operatorname{Sec}^2 u \operatorname{Tan} u \times \operatorname{Sin}^2 2u$$

$$= \frac{2 \operatorname{Sin} u}{\operatorname{Cos} u} - \frac{8 \operatorname{Sin}^3 u \operatorname{Cos}^2 u}{\operatorname{Cos}^3 u}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \operatorname{Sin} u \operatorname{Cos} u - 8 \operatorname{Sin}^3 u \operatorname{Cos} u$$
$$= 2 \operatorname{Sin} u (\operatorname{Cos} u - 4 \operatorname{Cos} u (1 - \operatorname{Cos}^2 u))$$
$$= 2 \operatorname{Sin} u (4 \operatorname{Cos}^3 u - 3 \operatorname{Cos} u)$$
$$= 2 \operatorname{Sin} u \operatorname{Cos} 3u$$

7.Q If $u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$, find the values of i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

7) Given

$$u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$$

Let $z = e^u = \frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}$.

$$z(\lambda x, \lambda y) = \frac{(\lambda x)^2+(\lambda y)^2}{\sqrt{\lambda x}+\sqrt{\lambda y}} = \lambda^{3/2} z.$$

$\therefore$ z is a homogenous function of degree $3/2$

(i) According to Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{3}{2}z.$$

$$x\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right) = \frac{3}{2}e^u$$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = \frac{3}{2}e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2} \quad -(1)$$

ii) $(1)^2$

$$\Rightarrow x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{(\partial u)^2}{\partial x\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2 = \frac{9}{4} \quad -(2)$$

According to extension of Euler's theorem

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = \frac{3}{2}\left(\frac{1}{2}\right)e^u \quad -(3)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(e^u)\right)$$

$$= \frac{\partial}{\partial x}\left(e^u\frac{\partial u}{\partial x}\right)$$

u

$$\frac{\partial^2 z}{\partial x^2} = e^u \frac{\partial^2 u}{\partial x^2} + e^u\left(\frac{\partial u}{\partial x}\right)^2 = e^u\left(\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right) \quad -④$$

Similarly

$$\frac{\partial^2 z}{\partial y^2} = e^u\left(\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right) \quad -⑤$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = e^u\left(\frac{\partial^2 u}{\partial x\,\partial y} + \frac{(\partial u)^2}{\partial x\,\partial y}\right) \quad -⑥$$

Substituting ④, ⑤, ⑥ in ③.

$$e^u\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$$

$$+ e^u\left(x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{(\partial u)^2}{\partial x\,\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2\right) = \frac{3e^u}{4}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{3}{4} - \frac{9}{4} = \frac{-6}{4} = -\frac{3}{2}$$

(8) If $z = \log(x^2+y^2) + \frac{x^2+y^2}{x+y} - 2\log(x+y)$, find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

Let $u = \log(x^2+y^2)$, $v = \frac{x^2+y^2}{x+y}$, $w = -2\log(x+y)$

Let $p = e^u = x^2+y^2$, $q = e^{-w/2} = x+y$.

$$v(\lambda x, \lambda y) = \frac{(\lambda x)^2+(\lambda y)^2}{\lambda x + \lambda y} = \lambda v$$

$\therefore$ v is a homogenous function of degree 1

$$P(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2 = \lambda^2 P.$$

$\therefore$ P is a homogenous function of degree 2.

$$q(\lambda x, \lambda y) = \lambda x + \lambda y = \lambda q$$

$\therefore$ q is a homogenous function of degree 1

According to Euler's theorem

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = 2P.$$

$$x\frac{\partial P}{\partial u}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial P}{\partial u}\left(\frac{\partial u}{\partial y}\right) = 2P$$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 2e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \quad \text{—(1)}$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{x^2+y^2}{x+y} \quad -(2)$$

$$x\frac{\partial q}{\partial x}+y\frac{\partial q}{\partial y}=q$$

$$x\frac{\partial q}{\partial w}\left(\frac{\partial w}{\partial x}\right)+y\frac{\partial q}{\partial w}\left(\frac{\partial w}{\partial y}\right)=e^{-w/2}$$

$$x\, e^{-w/2}\left(\frac{-1}{2}\right)\frac{\partial w}{\partial x}+y\, e^{-w/2}\left(\frac{-1}{2}\right)\frac{\partial w}{\partial y}=e^{-w/2}$$

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=-2 \quad -(3)$$

(1) + (2) + (3)

$$\Rightarrow x\frac{\partial}{\partial x}(u+v+w)+y\frac{\partial}{\partial y}(u+v+w)=2+\frac{x^2+y^2}{x+y}-2$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{x^2+y^2}{x+y}.$$

PROBLEMS ON MAXIMA AND MINIMA FOR A FUNCTION OF 2 VARIABLES

1. Discuss the maxima and minima of $f(x,y) = xy(a-x-y)$, $a>0$.

↳ Given curve: $f(x,y) = xy(a-x-y)$

$$\frac{\partial f}{\partial x} = y(a-x-y) + xy(-1) = y(a-x-y) - xy = y[a-2x-y]$$

$$\frac{\partial f}{\partial y} = x(a-x-y) - xy = x[a-x-2y]$$

At a critical point, $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow y(a-x-y) - xy = 0 \quad \text{and} \quad x(a-x-y) - xy = 0$$

$$\Rightarrow y(a-x-y) = xy \quad \text{and} \quad x(a-x-y) = xy$$

$$\Rightarrow y(a-x-y) = x(a-x-y)$$

$$\Rightarrow \underline{x=y}$$

$\cancel{r = f_{xx} = \frac{\partial^2 f}{\partial x^2}}$

$$\Rightarrow y(a-y-y) - y^2 = 0$$

$$\Rightarrow ay - 2y^2 - y^2 = 0$$

$$\Rightarrow ay - 3y^2 = 0$$

$$\Rightarrow y(a-3y) = 0$$

$$\Rightarrow y = 0 \quad \text{or} \quad y = \frac{a}{3}$$

and $x = 0$ or $x = \frac{a}{3}$

$\therefore$ the critical points are $(0,0)$ and $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[y(a-x-y) - xy\right] = \frac{\partial}{\partial x}\left[y(a-2x-y)\right]$$

$$\Rightarrow r = y(-2) \Rightarrow \boxed{r = -2y}$$

$$s = f_{xy} = \frac{\partial f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[y(a-2x-y)\right]$$

$$= (a-2x-y) - y$$

$$\boxed{s = a-2x-2y}$$

$$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[x(a-x-2y)\right]$$

$$\boxed{t = -2x}$$

here at critical point $(0,0)$:

$r = 0$ ~~$\Rightarrow rt - s^2$~~

Hence, nothing can be said about the curve at the point $(0,0)$.

at critical point $\left(\frac{a}{3}, \frac{a}{3}\right)$:

$r = -\frac{2a}{3}$, $s = a - \frac{2a}{3} - \frac{2a}{3} = -\frac{a}{3}$ $t = -\frac{2a}{3}$

$r < 0$

$rt - s^2$

$= \frac{4a^2}{9} + \frac{a}{3} > 0$

hence, $rt - s^2 > 0$ and $r < 0$

$\Rightarrow$ the curve $f(x,y)$ has a <u>maximum</u> at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$.

$f_{max} = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right)$

$= \frac{a^2}{9}\left(\frac{a}{3}\right) \Rightarrow \boxed{f_{max} = \frac{a^3}{27}}$

2. Find the extreme values of the function $f(x,y) = \sin x . \sin y . \sin(x+y)$.

↳ Given function $f(x,y) = \sin x . \sin y . \sin(x+y)$

$\frac{\partial f}{\partial x} = \sin y \left[\cos x . \sin(x+y) + \sin x . \cos(x+y)\right]$

$\Rightarrow f_x = \sin y . \sin(2x+y)$ $\quad [\because \sin A . \cos B + \cos A \sin B = \sin(A+B)]$

$\frac{\partial f}{\partial y} = \sin x \left[\sin y \cos(x+y) + \cos y \sin(x+y)\right]$

$\Rightarrow f_y = \sin x . \sin(x+2y)$

at a critical point, $f_x = 0$ & $f_y = 0$

$\Rightarrow \sin y . \sin(2x+y) = 0$

$\sin x . \sin(x+2y) = 0$

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here $x=0,\ y=0$

or

$2x+y=0 \rightarrow (1)$

& $x+2y=0 \rightarrow (2)$

consider (1) & (2) :

$2x+y=0$

$(x+2y=0)\times 2$

$\Rightarrow\ 2x+y=0$

$(-)\ 2x+4y=0$

or $x=\pi,\ y=\pi$

or $2x+y=\pi \rightarrow (1)$ & $x+2y=\pi \rightarrow (2)$

consider (1) & (2)

$(2x+y=\pi)\times 2$

$(x+2y=\pi)$.

$\Rightarrow\ 4x+2y=2\pi$

$(-)\ x+2y=\pi$

$3x=\pi$

$\Rightarrow\ x=\frac{\pi}{3}$

$\frac{2\pi}{3}+y=\pi$

$\Rightarrow\ y=\pi/_3$

hence, the critical points are $(0,0)$ & $(\pi/_3, \pi/_3)$

nothing can be said about curve at $(0,0)$.

Hence, we consider $(\frac{\pi}{3}, \frac{\pi}{3})$ only.

$r = f_{xx}$

$\Rightarrow r = \frac{\partial}{\partial x}[\sin y \cdot \sin(2x+y)]$

$\boxed{r = 2\sin y \cdot \cos(2x+y)}$

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$$s = f_{xy} = \frac{\partial}{\partial y}\left[\sin y \cdot \sin(2x+y)\right]$$

$$= \sin y \cdot \cos(2x+y) + \cos y \cdot \sin(2x+y)$$

$$\Rightarrow \boxed{s = \sin(2x+2y)}$$

$$t = f_{yy} = \frac{\partial}{\partial y}\left[\sin x \cdot \sin(x+2y)\right]$$

$$\boxed{t = 2\sin x \cos(x+2y)}$$

$\therefore$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$$r = 2\sin\frac{\pi}{3} \cdot \cos\pi = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$s = \sin\left(\frac{4\pi}{3}\right) = \sin\left(\pi + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$t = 2\sin\frac{\pi}{3}\cos\pi = -\sqrt{3}$$

$$\boxed{r < 0}$$

$$rt - s^2$$

$$= 3 - \frac{3}{4} = \frac{9}{4} > 0 \quad \& \quad \boxed{rt - s^2 > 0}$$

$\therefore$ $f(x,y)$ has a maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

and $f_{max} = \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{3} \cdot \sin\left(\frac{2\pi}{3}\right)$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$\Rightarrow \boxed{f_{max} = \frac{3\sqrt{3}}{8}}$$

3. In a plane triangle ABC, find the maximum value of $\cos A \cdot \cos B \cdot \cos C$.

↳ Given function : $f = \cos A \cdot \cos B \cdot \cos C$

But, in a Δle, $A + B + C = 180$

$\Rightarrow C = 180 - (A+B)$

$\therefore f(A,B) = \cos A \cdot \cos B \cdot \cos[180 - (A+B)]$

$f(A,B) = -\cos A \cdot \cos B \cdot \cos(A+B)$

now,

$$\frac{\partial f}{\partial A} = -\cos B[-\sin A \cdot \cos(A+B) - \cos A \cdot \sin(A+B)]$$

$$\boxed{f_A = \cos B \cdot \sin(2A+B)}$$

$$\frac{\partial f}{\partial B} = -\cos A[-\sin B \cdot \cos(A+B) - \cos B \cdot \sin(A+B)]$$

$$\Rightarrow \boxed{f_B = \cos A \cdot \sin(A+2B)}$$

at a critical point,

$f_A = 0$ & $f_B = 0$

$\Rightarrow 2A + B = \pi$ & $A + 2B = \pi$

$\Rightarrow 2A + 4B = 2\pi$

$\Rightarrow 2A + B = \pi$

$(-)\ 2A + 4B = 2\pi$

$-3B = -\pi \quad \Rightarrow \boxed{B = \frac{\pi}{3}}$

& $\boxed{A = \frac{\pi}{3}}$

$\therefore$ critical point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$r = f_{AA}$

$\boxed{r = 2\cos B \cdot \cos(2A+B)}$

$s = f_{BA}$

$= \frac{\partial}{\partial A}\left[\cos A \cdot \sin(A+2B)\right]$

$= -\sin A \cdot \sin(A+2B) + \cos A \cdot \cos(A+2B)$

$\boxed{s = \cos(2A+2B)}$

$t = f_{BB}$

$= \frac{\partial}{\partial B}\left[\cos A \cdot \sin(A+2B)\right]$

$\boxed{t = 2\cos A \cdot \cos(A+2B)}$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$r = 2\cos\frac{\pi}{3} \cdot \cos(\pi)$

$\boxed{r = -1} \Rightarrow \boxed{r < 0}$

$s = \cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right) = -\frac{1}{2}$

$\boxed{t = -1}$

$rt - s^2$

$= -1 \times -1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} > 0$

$\therefore$ f has maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

and $f_{max} = -\cos\frac{\pi}{3} \cdot \cos\frac{\pi}{3} \cdot \cos\left(\frac{2\pi}{3}\right)$

$\boxed{f_{max} = \frac{1}{8}}$

4\. Show that $f(x,y) = x^3 + y^3 - 3axy$ has a maximum at the point (a,a) if $a<0$ and a minimum at the point (a,a) if ~~a<~~ $a>0$.

↳ Given curve, $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay = 3(x^2 - ay)$

$f_y = 3y^2 - 3ax = 3(y^2 - ax)$

at a critical point,

$f_x = 0$ & $f_y = 0$

$\Rightarrow x^2 - ay = 0 \rightarrow (1)$ and $y^2 - ax = 0 \rightarrow (2)$

$\Rightarrow ax = y^2$

& $x = \frac{y^2}{a}$.

substituting x in (1):

$\frac{y^4}{a^2} - ay = 0$

$y^4 - a^3 y = 0$

$\Rightarrow y(y^3 - a^3) = 0$

$\Rightarrow y = 0$ or $y^3 = a^3$

$\Rightarrow \boxed{y = a}$

$\Rightarrow x = \frac{a^2}{a} = a \Rightarrow \boxed{x = a}$ or $x = 0$.

We consider critical point as (a,a).

$r = f_{xx} = 3(2x) = 6x$

$t = f_{yy} = 3(2y) = 6y$

$s = f_{yx} = \frac{\partial}{\partial y}\left[3(x^2 - ay)\right] = -3a$

<u>Case (i)</u> if <u>$a<0$</u>:

at <u>(a, a)</u>:

$r = 6a \Rightarrow \boxed{r<0}$

$t = 6a$

$s = -3a$

$rt - s^2 = 36a^2 + 9a^2 = \underline{45a^2 > 0} \Rightarrow \boxed{rt - s^2 > 0}$

$\therefore$ f has maximum at (a, a)

<u>Case (ii)</u> if $a > 0$:

at (a, a)

$r = 6a \Rightarrow \boxed{r>0}$

$t = 6a$

$s = -3a$

$rt - s^2 = 45a^2 > 0 \Rightarrow \boxed{rt - s^2 > 0}$

$\therefore$ f has minimum at (a, a)

Hence Proved.

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Langrange's method of multiplier

1) Find the maximum and minimum distance from point (1,2,2) to the sphere $x^2+y^2+z^2 = 36$

→ let the maximum and minimum distances of the point (1,2,2) from sphere $x^2+y^2+z^2=36$

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

given $\phi(x,y,z) = x^2+y^2+z^2 = 36$

$$F = f + \lambda\phi$$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-36)$$

At a critical point

$$\frac{\partial F}{\partial x} = 0 \, , \quad \frac{\partial F}{\partial y} = 0 \, , \quad \frac{\partial F}{\partial z} = 0$$

$$2(x-1) + \lambda 2x = 0 \, , \quad 2(y-2) + \lambda 2y = 0, \quad 2(z-2) + \lambda 2z = 0$$

$$\lambda = \frac{x-1}{x} = \frac{y-2}{y} = \frac{z-2}{z}$$

$$\underline{2x = y = z}$$

$$\therefore \frac{y^2}{4} + y^2 + y^2 = 36 \Rightarrow y = \pm 4$$

$\therefore$ $x, y, z = 2, 4, 4$ & $-2, -4, -4$

using value x, y, z and $f(x,y,z)$

$\therefore$ maximum $= 9$ and minimum $= 3$.

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2) A rectangular box open at the top is to have a volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

→ Let (x, y, z) be sides of box.

$$f(x,y,z) = xy + 2yz + 2zx$$

$$\phi(x,y,z) = xyz$$

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$= xy + 2yz + 2zx + \lambda xyz$$

$$F_x = y + 2z + \lambda yz = 0$$

$$\lambda = \frac{-y-2z}{yz}$$

$$F_y = x + 2z + \lambda xz = 0$$

$$\lambda = \frac{-x-2z}{xz}$$

$$F_z = 2y + 2x + \lambda xy = 0$$

$$\lambda = \frac{-2y-2x}{xy}$$

$$\therefore \frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2y+2x}{xy}$$

$$\Rightarrow x = y = 2z$$

using $x = y = 2z$ and $xyz = 32$

we get $x = 4, y = 4, z = 2$

3) The temperature T at a point (x,y,z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$

$\rightarrow$ $f(x,y,z) = 400xyz^2$

$\phi(x,y,z) = x^2+z^2+y^2 = 1$

$F = f + \lambda\phi = 400xyz^2 + \lambda(x^2+y^2+z^2)$

$F_x = 400yz^2 + 2\lambda x = 0$

$\lambda = \dfrac{-200yz^2}{x}$ — (i)

$F_y = 400xz^2 + 2\lambda y = 0$

$\lambda = \dfrac{-200xz^2}{y}$ — (ii)

$F_z = 800xyz + 2\lambda z = 0$

$\lambda = -400xy$ — (iii)

from (i), (ii), (iii)

$2x^2 = 2y^2 = z^2$ — (iv)

from (iv) & $x^2+y^2+z^2=1$

$x = \pm\frac{1}{2} = y$

$z = \pm\frac{1}{\sqrt{2}}$

for maximum T $x = \frac{1}{2} = y$ $z = \frac{1}{\sqrt{2}}$

$T = 50$

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4) A closed rectangular box has length twice its breadth and has constant volume V. Determine the dimension of the box requiring the least surface area.

→ Given x, y, z sides of rectangular box and $x = 2y$

$f(x,y,z) = 2(xy + yz + zx)$

using $x = 2y$

$f(y,z) = 2(2y^2 + 3yz)$

$\phi = 2y^2 z$

$F = f + \lambda\phi \Rightarrow 2(2y^2 + 3yz) + \lambda 2y^2 z$

$F_y = 2(4y + 3z) + 4\lambda yz$

$\lambda = \dfrac{-4y - 3z}{2yz}$ — (i)

$F_z = 2(3y) + \lambda 2y^2$

$\lambda = -\dfrac{3}{y}$ — (ii)

from (i) & (ii)

$z = \dfrac{4}{3}y$

puting in $V = 2y^2 z$

$y = 2\left(\dfrac{V}{3}\right)^{1/3}$

$z = \left(\dfrac{8}{9}V\right)^{1/3} = 2\left(\dfrac{V}{9}\right)^{1/3}$

$x = 4\left(\dfrac{V}{3}\right)^{1/3}$

5) find the volume of greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Sol → let the edges of the greatest parallelopiped be $2x, 2y$ & $2z$ which are parallel to axes. ∴ $V = 8xyz$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) \quad \text{—①}$$

$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right)=0$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right)=0 \quad , \quad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right)=0$$

from equating λ we get $x^2/a^2 = y^2/b^2$

similarly $y^2/b^2 = z^2/c^2$

putting these values in ① we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$$

$$\Rightarrow x = a\sqrt{3}, \; y = b/\sqrt{3}, \; z = c/\sqrt{3}$$

∴ the greatest volume is

$$\frac{8abc}{3\sqrt{3}}$$

6> Prove that of all rectangular parallelopipeds of the same volume, the cubes has the least surface area.

Sol-

$$\phi = xyz \ (v)$$

$$f = 2(xy + yz + zx)$$

$$F = 2(xy + yz + zx) + \lambda(xyz)$$

$$\frac{\partial F}{\partial x} = 2y + 2z + \lambda yz = 0$$

$$\lambda = \frac{-2y - 2z}{yz} \quad \text{--- (i)}$$

$$\lambda = -\frac{2z - 2z}{xz} \quad \text{--- (ii)}$$

$$\lambda = \frac{-2y - 2(x)}{yx} \quad \text{--- (iii)}$$

from (i), (ii), (iii) we get

$$x = y = z$$

$\therefore$ volume $= x^3$

$$x = (v)^{1/3}$$

similarly

$$y = (v)^{1/3}$$

$$z = (v)^{1/3}$$

$\therefore$ The cube of side $x = (v)^{1/3}$ has least surface area.

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ERRORS & APPROXIMATIONS

1. Find the percentage error in calculating the volume & surface area of a sphere due to an error of $x\%$ in the radius.

→ Volume of sphere $(V) = \frac{4\pi r^3}{3}$

~~Volume of~~

Surface area, $S = 4\pi r^2$

$$\delta V = \frac{\partial V}{\partial r} . \delta r = 4\pi r^2 . \delta r$$

$$\Rightarrow \frac{\delta V}{V} = \frac{4\pi r^2 . \delta r}{\frac{4\pi r^3}{3}} = 3 . \frac{\delta r}{r}$$

$$\Rightarrow \boxed{\frac{\delta V}{V} \% = 3x\%}$$

$$\delta S = \frac{\partial S}{\partial r} . \delta r = 8\pi r . \delta r$$

$$\Rightarrow \frac{\delta S}{S} = \frac{8\pi r . \delta r}{4\pi r^2} = 2 . \frac{\delta r}{r}$$

$$\Rightarrow \boxed{\frac{\delta S}{S} \% = 2x\%}$$

2. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the volume & lateral surface.

4 Volume, $V = \pi r^2 h$ & Lateral surface area $= 2\pi r h = \pi d h$

$= \frac{\pi d^2 h}{4}$ (S)

$\underline{d = 4cm}$ given ~~$r = \frac{4}{2} = 2cm$~~ $\delta d = 0.1 cm$

& $\underline{h = 6cm}$ $\delta h = 0.1 cm$

$$\delta V = \frac{\partial V}{\partial d} \cdot \delta d + \frac{\partial V}{\partial h} \cdot \delta h$$

$$= \frac{2 \cancel{2}\pi d h}{4}(0.1) + \frac{\pi d^2}{4}(0.1)$$

$$= \cancel{2 \times \pi \times 2 \times 6 \times 0.1 + \pi \times 2 \times 2 \times 0.1}$$

~~$\delta V = 8$~~

$$= \frac{\pi \times 4 \times 6 \times 0.1}{2} + \frac{\pi \times 4 \times 4 (0.1)}{4} = 0.4\pi + 1.2\pi$$

$$\Rightarrow \boxed{\delta V = 1.6\pi cm^3}$$

$$\delta S = \frac{\partial S}{\partial d} \delta d + \frac{\partial S}{\partial h} \cdot \delta h$$

$$= \cancel{2\pi h (0.1) + 2\pi r (0.1)}$$

$$= \cancel{2 \times \pi \times 6 (0.1) + 2 \times \pi \times 2 \times 0.1}$$

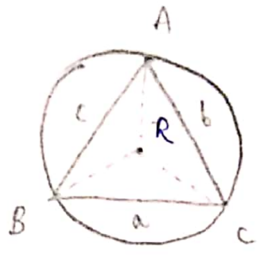
$$= \pi h (0.1) + \pi d (0.1)$$

$$= \pi (6)(0.1) + \pi (4)(0.1)$$

$$\Rightarrow \delta S = 0.6\pi + 0.4\pi$$

$$\Rightarrow \boxed{\delta S = \pi cm^2}$$

3. If the sides of a plane triangle ABC vary in such a way that its circum radius remains constant, prove that $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$



↳ The circumradius R of ΔABC is given by $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$

$\Rightarrow a = 2R\sin A$

$\Rightarrow da = 2R\cos A \cdot dA$

similarly, $db = 2R\cos B \cdot dB$ & $dc = 2R\cos C \cdot dC$

$\Rightarrow \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(dA + dB + dC)$

In a triangle,

$A + B + C = \pi$

$\Rightarrow$ ~~differentiating~~ Taking differentials on both sides:

$d(A + B + C) = d\pi$

$\Rightarrow dA + dB + dC = 0$

$\therefore \boxed{\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0}$

Hence Proved

4. The power 'P' required to propel a steamer of length 'l' at a speed 'u', is given by $P = \lambda u^3 l^3$ where λ is a constant. If 'u' is increased by 3% and l is decreased by 1%, find the corresponding increase in 'P'.

↳ $P = \lambda u^3 l^3$

taking log on both sides:

$$\log P = \log \lambda + 3\log u + 3\log l$$

$$\Rightarrow \frac{\delta P}{P} = 3 \cdot \frac{\delta u}{u} + 3 \cdot \frac{\delta l}{l}$$

$$= 3(0.03) - 3(0.01)$$

$$= 0.09 - 0.03$$

$$\Rightarrow \frac{\delta P}{P} = 0.06$$

$$\therefore \boxed{\frac{\delta P}{P}\% = 6\%}$$, i.e, P increases by 6%

5. In estimating the cost of a pile of bricks measured as 2m × 15m × 1.2m, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cu. m and bricks cost ₹530 per 1000, find the approximate error in the cost.

↳ Let volume of ~~each~~ pile of bricks, $V = l \times b \times h$

$$\Rightarrow V = lbh$$

Now, for 1 cu m ⟶ 450 bricks

$\Rightarrow$ for V cu.m $\longrightarrow$ V × 450 bricks

cost of 1000 brick $\rightarrow$ ₹ 530.

$\therefore$ cost of V × 450 bricks, $C = \frac{V \times 450 \times 530}{1000}$

$$\Rightarrow C = \frac{V \times 45 \times 53}{10}$$

$$\Rightarrow C = kV \qquad [k = 238.5)$$

$$\Rightarrow \cancel{C = kV} \; C = k.lbh$$

$$\therefore \Rightarrow \delta C = \frac{\partial C}{\partial l} \cdot \delta l + \frac{\partial C}{\partial b} \cdot \delta b + \frac{\partial C}{\partial h} \cdot \delta h$$

$$\frac{\delta C}{\cancel{C}} = kbh.\delta l + klh.\delta b + klb.\delta h$$

$$\frac{\delta C}{C} = \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta h}{h}$$

$$= 0.01 + 0.01 + 0.01$$

$$\Rightarrow \frac{\delta C}{C} = 0.03 \qquad \Rightarrow \frac{\delta C}{C}\% = 3\%.$$

now, from measurements, cost,

$$C = 238.5 \times 2 \times 1.5 \times 1.2 = 8586.$$

$\therefore$ error $= 3\%$ of C

$$= \frac{3}{100} \times 8586 \rightarrow \boxed{\text{error} = ₹257.58}$$

$\therefore$ He pays this much extra, & hence it is a loss for him

6. The period T of a simple pendulum of length 'l' is given by $T = 2\pi\sqrt{\frac{l}{g}}$. Find the error and the percentage error made in computing T by using $l = 2m$, $g = 10m/s^2$, if the true values are $l = 2.1m$ & $g = 9.8m/s^2$.

↳ Given, $T = 2\pi\sqrt{\frac{l}{g}}$

$$\Rightarrow T = 2\pi . l^{1/2} . g^{-1/2}$$

$$\delta T = \frac{\partial T}{\partial l} . \delta l + \frac{\partial T}{\partial g} . \delta g$$

given, $\delta l = 0.1m$

& $\delta g = 0.2 m/s^2$

$$\Rightarrow \delta T = \left(2\pi g^{-1/2} . \frac{1}{\cancel{2}} . l^{-1/2}\right) . (0.1) - \cancel{2}\pi l^{1/2} . \frac{1}{\cancel{2}} . g^{-3/2} . (0.2)$$

$$= \cancel{2}\pi \left[(10)^{-1/2} . (2)^{-1/2} \cancel{\}} 0.1 - (2)^{1/2} . (10)^{-3/2} . 0.2 \right]$$

$$\Rightarrow \delta T = \pi \left[0.0224 - 0.0089 \right]$$

$$= 3.14 (0.0135) = \underline{0.0421}$$

$$\frac{\delta T}{T} = \frac{1}{2} . \frac{\delta l}{l} - \frac{1}{2} \frac{\delta g}{g} = \frac{1}{2} \left(\frac{0.1}{2} - \frac{0.2}{10} \right)$$

$$= \frac{1}{2} (0.03) = 0.015$$

$$\frac{\delta T}{T} \times 100 = \underline{1.5\%}$$