08/03/2018

UNIT - 4 : INVERSE LAPLACE TRANSFORMS

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$\Rightarrow$ $f(t) = L^{-1}[F(s)]$ is called Inverse Laplace Transform of $F(s)$.

→ Inverse Laplace transforms of Standard Functions.

(i) $L^{-1}\left[\frac{a}{s}\right] = a.$

(ii) $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

(iii) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}.$

(iv) $L^{-1}\left[\frac{a}{a^2+s^2}\right] = \sin at$

$L^{-1}\left[\frac{1}{a^2+s^2}\right] = \frac{\sin at}{a}.$

(v) $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

(vi) $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a} \sinh at.$

$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at.$

(vii) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at.$

(viii) $L^{-1}\left[\frac{1}{s-\log a}\right] = a^t$.

(ix) $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}$

$L[t^n] = \frac{n!}{s^{n+1}}$

(x) $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$.

$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

~~(xi)~~ Eg: $L^{-1}\left[\frac{1}{s^{3/2}}\right] = \frac{t^{1/2}}{1/2\,\Gamma(1/2)} = \frac{2\sqrt{t}}{\sqrt{\pi}}$

$L^{-1}\left[\frac{1}{2s+1}\right] = L^{-1}\left[\frac{1}{2[s+1/2]}\right] = \frac{1}{2}e^{-1/2t}$.

$L^{-1}\left[\frac{1}{s^2+2}\right] = \frac{1}{\sqrt{2}}\sin\sqrt{2}t$

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Properties

1) $L^{-1}[aF(s) + bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]$.

$\Rightarrow$ Linearity property

2) If $L[f(t)] = F(s)$, then $L[e^{at}f(t)] = F(s-a)$.

$\Rightarrow L^{-1}[F(s-a)] = e^{at}f(t)$.

$\Rightarrow L^{-1}[F(s+a)] = e^{-at}f(t)$.

Numericals

1) $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = e^{at} L^{-1}\left[\frac{1}{s^2+b^2}\right]$

$= e^{at} \frac{\sin bt}{b}$

2) $L^{-1}\left[\frac{(s-a)}{(s-a)^2+b^2}\right]$

$= e^{at} L^{-1}\left[\frac{s}{s^2+b^2}\right] = e^{at} \cos bt$

3) $L^{-1}\left[\frac{1}{(s+a)^2-b^2}\right] = e^{-at} L^{-1}\left[\frac{1}{s^2-b^2}\right]$

$= e^{-at} \frac{\sinh bt}{b}$

4) $L^{-1}\left[\frac{(s+a)}{(s+a)^2-b^2}\right] = e^{-at} L^{-1}\left[\frac{s}{s^2-b^2}\right]$

$= e^{-at} \cosh bt.$

5) $L^{-1}\left[\frac{1}{(s-a)^{n+1}}\right]$

$= e^{at} L^{-1}\left[\frac{1}{s^{n+1}}\right] = e^{at} \frac{t^n}{n!}$ $n = 1, 2, 3 \ldots$

6) $L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$

7) $L^{-1}\left[\frac{1}{s+6}\right] = e^{-6t}$

8) $L^{-1}\left[\frac{1}{3s+7}\right]$

$= L^{-1}\left[\frac{1}{3} \times \frac{1}{s+7/3}\right]$

$= \frac{1}{3} e^{-\frac{7}{3}t}$ //

9) $L^{-1}\left[\frac{1}{s^2+6}\right]$

$= \frac{\sin\sqrt{6}t}{\sqrt{6}}$ //

10) $L^{-1}\left[\frac{1}{s^6}\right] = \cancel{\frac{t^5}{6!}} \quad \frac{t^5}{5!}$

$= \frac{t^5}{120}$ //

11) $L^{-1}\left[\frac{1}{s^{5/2}}\right]$

$= \frac{t^{3/2}}{\Gamma(5/2)}$ // $= \frac{t^{3/2}}{3/2 \cdot 1/2 \times \sqrt{\pi}} = \frac{4}{3}\frac{t^{3/2}}{\sqrt{\pi}}$ //

12) $L^{-1}\left[\frac{1}{(s+2)^{3/2}}\right]$

$= e^{-2t} L^{-1}\left[\frac{1}{s^{3/2}}\right]$

$= e^{-2t} \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} = \frac{2e^{-2t}t^{1/2}}{\sqrt{\pi}}$

$= 2e^{-2t}\sqrt{\frac{t}{\pi}}$ //

Inverse Laplace transforms by Partial Fractions

13) $L^{-1}\left[\dfrac{1}{s^2-2s+5}\right]$

$\Rightarrow \quad s^2-2s+5 = (s-1)^2+4$

$$L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right] = e^{t}\,\frac{\sin 2t}{2}$$

14) $L^{-1}\left[\dfrac{s}{(s+3)^2+36}\right]$

$$\cancel{= x\left[e^{-3t}\,\frac{\cos 6t}{6}\right]}$$

$$= L^{-1}\left[\frac{s+3-3}{(s+3)^2+6^2}\right]$$

$$= L^{-1}\left[\frac{(s+3)}{(s+3)^2+6^2}\right] - L^{-1}\left[\frac{3}{(s+3)^2+6^2}\right]$$

$$= e^{-3t}\left[\cos 6t - \frac{3\sin 6t}{6}\right]$$

$$= e^{-3t}\left[\cos 6t - \frac{\sin 6t}{2}\right]$$

Inverse Laplace Transforms by Partial Fractions

Numericals

1) $\dfrac{s^2+s-2}{s(s+3)(s-2)} = F(s)$

$$\rightarrow \quad = \frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{(s-2)}$$

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(x) Dr of F(s):

$$s^2+s-2 = A(s+3)(s-2) + B(s)(s-2) + C(s)(s+3)$$

$s=0$: $-2 = A(3)(-2) = -6A$

$\Rightarrow A = \frac{1}{3}$

$s=2$: $4 = C(2)(5) = 10C$

$\Rightarrow C = \frac{2}{5}$

$s=-3$: $4 = B(-3)(-5) = +15B$

$\Rightarrow B = \frac{+4}{15}$

$$\therefore F(s) = \frac{1}{3s} + \frac{2}{5(s+3)} + \frac{4}{15(s-2)}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{3s}\right] + L^{-1}\left[\frac{2}{5(s+3)}\right] + L^{-1}\left[\frac{4}{15(s-2)}\right]$$

$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t} \quad //$$

2) $L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right]$

$$\Rightarrow F(s) = \frac{s}{(s-3)(s^2+4)} = \frac{A}{(s-3)} + \frac{Bs+C}{(s^2+4)}$$

$$s = A(s^2+4) + (Bs+C)(s-3)$$

$s=3$: $3 = A(9+4) = 13A$

$A = \frac{3}{13}$

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Equate co-efficient of ~~x^2~~ s^2 and s on both sides.

$$s = A(s^2+4) + Bs^2 + Cs - 3Bs - 3C.$$

Co-eff of s^2: $0 = A + B \Rightarrow B = -A = \cancel{2}$

$$B = -\frac{3}{13}.$$

Coeff of s; $1 = C - 3B \Rightarrow C = 1 + 3B$

$$= 1 + 3\left(\frac{-3}{13}\right) = \frac{1\cancel{3}-9}{13}$$

$$C = \frac{4}{13}$$

$$\therefore F(s) = \frac{3}{13(s-3)} + \frac{\left(\frac{-3}{13}s + \frac{4}{13}\right)}{(s^2+4)}.$$

$$L^{-1}[F(s)] = \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t.$$

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3) $\dfrac{4s+5}{(s+1)^2(s+2)}$

$A=3$
$B=1$
$C=-3$

$$\Rightarrow F(s) = \frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{(s+1)^2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

$$4s+5 = A(s+1)(s+2) + B(s+2)^2 + C(s+1)2(s+2)$$

$A = 3$; $B = 1$; $C = -3$.

$$F(s) = \frac{3}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+2)}$$

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4) $\dfrac{s^2+5}{(s^2+1)(s^2+2s+2)}$

$\Rightarrow \quad = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{(s^2+2s+2)}$

$\Rightarrow \cancel{\dfrac{s^2+5}{(s^2+1)(s^2+2s+2)}} \quad s^2+5 = (As+B)(s^2+2s+2) + ((Cs+D)(s^2+1))$

Equating co-efficients: $A = \frac{3}{5}$, $B = \frac{1}{5}$, $C = \frac{-3}{5}$, $D = \frac{-2}{5}$

$F(s) = \dfrac{3/5\, s}{s^2+1} + \dfrac{1/5}{(s^2+1)} \dfrac{-3/5\, s}{(s^2+2s+2)} - \dfrac{2/5}{s^2+2s+2}.$

$\Rightarrow s^2+2s+2 = (s+1)^2 + 2 - (1)^2$
$= (s+1)^2 + 1^2$

$L^{-1}[F(s)] = \frac{3}{5}\cos t + \frac{1}{5}\sin t - \frac{3}{5} L^{-1}\left[\dfrac{(s+1)-1}{(s+1)^2+1}\right]$
$- \frac{2}{5} L^{-1}\left[\dfrac{1}{(s+1)^2+1}\right]$

$= \frac{3}{5}\cos t + \frac{1}{5}\sin t - \frac{3}{5} e^{t}\cos t - \sin t - \frac{2}{5} e^{t} \sin t.$

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Note: * If $L[f(t)] = F(s)$, then,

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)$$

$$\Rightarrow L^{-1}[F(s)/s] = \int_0^t f(t)\,dt$$

$$= \int_0^t L^{-1}[F(s)]\,dt$$

* $$L^{-1}[F(s)/s^2] = \int_0^t\int_0^t f(t)\,dt\,.dt$$

5) $$L^{-1}\left[\frac{1}{s(s+2)^2}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$$

$$\Rightarrow F(s) = \frac{1}{(s+2)^2}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}\,t = f(t).$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt.$$

$$= \int_0^t \underset{\text{II}}{e^{-2t}}\ \underset{\text{I}}{t}\,dt.$$

$$= t\left(\frac{e^{-2t}}{-2}\right) - \left(\frac{e^{-2t}}{4}\right)\Big]_0^t$$

$$= \frac{-1}{2}(te^{-2t} - 0) - \frac{1}{4}(e^{-2t} - 1).$$

$$= \frac{-1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}$$

6) $$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$$

$$\Rightarrow F(s) = \frac{1}{(s^2+4)}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+2^2}\right] = \frac{1}{2}\sin 2t = f(t)$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

$$= \frac{1}{2}\int_0^t \sin 2t \, dt$$

$$= \frac{-1}{2}\frac{\cos 2t}{2}\Big]_0^t = \frac{-1}{4}[\cos 2t - 1] //$$

Note:- * If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n F^n(s)$$
$$= (-1)^n \frac{d^n}{ds^n}[F(s)].$$

$$\Rightarrow L^{-1}[(-1)^n F^n(s)] = t^n f(t).$$

Where $f(t) = L^{-1}[F(s)]$.

If $n=1$; then :- $L^{-1}[-F'(s)] = t f(t)$
If $n=2$; then :- $L^{-1}[F''(s)] = t^2 f(t)$.

* If $L[f(t)] = F(s)$ & $f(0)=0$, then,

$$L[f'(t)] = sF(s) - f(0) \quad ; \text{but } f(0)=0$$

$$\Rightarrow L[f'(t)] = sF(s)$$

$$\Rightarrow L^{-1}[sF(s)] = f'(t) = \frac{d}{ds}f(t).$$

where $f(t) = L^{-1}[F(s)]$

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t)

Numericals

1) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

$\Rightarrow$ WKT: $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

i.e. $L^{-1}[F(s)] = \frac{1}{a}\sin at = f(t)$.

WKT: $L^{-1}[-F'(s)] = t\,f(t)$

i.e. $L^{-1}[-F'(s)] = \cancel{t} \times f(t)$

$L^{-1}\left[-\frac{d}{ds}\left[\frac{1}{s^2+a^2}\right]\right] = t\,f(t)$.

$L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = f(t) \times t$.

$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2} f(t)$

$= \frac{t\sin at}{2a}$.

2) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

$\Rightarrow = L^{-1}\left[s\left(\frac{s}{(s^2+a^2)^2}\right)\right]$

$= L^{-1}[s\,F(s)]$

$$F(s) = \frac{s}{(s^2+a^2)^2}$$

WKT: $L^{-1}[s\,F(s)] = f'(t)$

$$= \frac{d}{ds}[f(t)] \quad \text{——— (1)}$$

$$f(t) = L^{-1}[F(s)]$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t \sin at}{2a}$$

From (1): $L^{-1}[s\,F(s)] = \frac{d}{dt}\left[\frac{t \sin at}{2a}\right]$

$$L^{-1}[s\,F(s)] = \frac{1}{2a}\left[ta \cos at + \sin at\right].$$

$$= \frac{\sin at + at \cdot \cos at}{2a}.$$

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3) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

$$\Rightarrow \quad = L^{-1}\left[\frac{s}{s(s^2+a^2)^2}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$$

where $F(s) = \frac{s}{(s^2+a^2)^2}$

WKT: $L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t \int_0^t L^{-1}[F(s)]\,dt\,dt$.

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a} = f(t)$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

$$= \int_0^t \frac{t\sin at}{2a}\,dt$$

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) - \left(-\frac{\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[-\frac{1}{a}t\cos at + \frac{1}{a^2}\sin at - (0+0)\right]$$

$$= \frac{1}{2a}\left[\frac{1}{a^2}\sin at - \frac{t}{a}\cos at\right]$$

Property:

Multiplication by e^{-as}

If $L[f(t)] = F(s)$, then:

$$L[f(t-a)\,u(t-a)] = e^{-as}L[f(t)] = e^{-as}F(s).$$

$$\Rightarrow L^{-1}[e^{-as}F(s)] = f(t-a)\,u(t-a).$$

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Numericals

1) $\frac{e^{-\pi s}}{s^2+1} = e^{-\pi s}\frac{1}{s^2+1} = e^{-\pi s} F(s)$

$\Rightarrow$ $L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = f(t)$

$L^{-1}[e^{-\pi s} F(s)] = f(t-\pi)u(t-\pi)$

$= \sin(t-\pi)\ u(t-\pi)$ //

2) $\frac{se^{-2\pi s}}{s^2+1}$

$\Rightarrow$ $= e^{-2\pi} F(s)$

$L^{-1}[F(s)] = \cos t = f(t)$

$L^{-1}[e^{-2\pi s} F(s)] = f(t-2\pi)\ u(t-2\pi)$

$= \cos(t-2\pi)\ u(t-2\pi)$ //

3) $\frac{se^{-2s}}{s^2+8s+16}$

$\Rightarrow$ $F(s) = \frac{s}{s^2+8s+16} = \frac{s}{(s+4)^2}$

$= \frac{s+4-4}{(s+4)^2} = \frac{s+4}{(s+4)^2} - \frac{4}{(s+4)^2}$

$= \frac{1}{s+4} - \frac{4}{(s+4)^2}$

$L^{-1}[F(s)] = e^{-4t} - 4e^{-4t} \times t$

$= e^{-4t}(1-4t).$

$L^{-1}[e^{-2s} F(s)] = f(t-2)\ u(t-2)$

$= e^{-4(t-2)}(1-4(t-2))\ u(t-2)$ //

$= e^{8-4t}(9-4t)\ u(t-2)$

4) $\frac{e^{-4s}}{(s-2)^4}$

$\Longrightarrow$

$$F(s) = \frac{1}{(s-2)^4}$$

$$L^{-1}[F(s)] = e^{2t}\frac{t^3}{6}$$

$$L^{-1}[e^{-4s}F(s)] = f(t-4)\,u(t-4)$$

$$= e^{2(t-4)}\frac{(t-4)^3}{6}\,u(t-4)$$

Inverse Laplace Transforms of Logorithmic Functions & Inverse Trignometric Functions

If $L[f(t)] = F(s)$, then,

①: $L^{-1}[-F'(s)] = t f(t)$

②: $L^{-1}[F''(s)] = t^2 f(t)$

In case of logarithmic fn, we apply properties of logarithm & then differentiate WRT s to obtain $F'(s)$. Then, multiply by '-1' & take inverse on both sides.

If a log fn persists in $F'(s)$, we differentiate again WRT s to obtain $F''(s)$. & use the property given above.

In case of Inverse trig fns, we simply differentiate the given fn $F(s)$ & ~~use~~ use the result ①.

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Numericals.

1) $\log\left[\frac{s+a}{s+b}\right]$

$\Longrightarrow$ $F(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$-F'(s) = \frac{1}{s+b} - \frac{1}{s+a}$

$L^{-1}[-F'(s)] \Rightarrow \frac{e^{-bt} - e^{-at}}{t} = f(t)$ //

2) $\log\left[\frac{s^2+4^2}{(s-3)^2}\right]$

$\Longrightarrow$ $F(s) = \log(s^2+4^2) - \log(s-3)^2$

$F'(s) = \frac{2s}{s^2+4^2} - \frac{2}{s-3}$

$-F'(s) = \frac{2}{s-3} - \frac{2s}{s^2+4^2}$

$L^{-1}[-F'(s)] = 2e^{3t} - 2\cos 4t$

$t f(t) = 2e^{3t} - 2\cos 4t$

$f(t) = \frac{2(e^{3t} - \cos 4t)}{t}$ //

3) $\cot^{-1}(s+1)$

$\Longrightarrow$ $F'(s) = -\frac{1}{1+(s+1)^2}$

$-F'(s) = \frac{1}{1+(s+1)^2}$

$$L[-F'(s)] = L^{-1}\left[\frac{1}{1+(s+1)^2}\right] = \cancel{t}\, \frac{e^{-t}\sin t}{\cancel{t}}$$

$$t f(t) = e^{-t}\sin t$$

$$f(t) = \frac{e^{-t}\sin t}{t} \;//$$

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4) $s \log\left[\frac{s}{\sqrt{s^2+1}}\right]$

$\Rightarrow$

$$F(s) = s\left[\log s - \frac{1}{2}\log(s^2+1)\right]$$

$$F'(s) = s\left[\frac{1}{s} - \frac{\cancel{2}s}{\cancel{2}(s^2+1)}\right] + \left[\log s - \frac{1}{2}\log(s^2+1)\right]$$

$$= 1 - \frac{s^2}{s^2+1} + \log s - \frac{1}{2}\log(s^2+1)$$

$$F''(s) = -\left[\frac{(s^2+1)2s - s^2(2s)}{(s^2+1)^2}\right] + \frac{1}{s} - \frac{1}{2(s^2+1)}$$

$$= \frac{2s^3 - 2s^3 - 2s}{(s^2+1)^2} + \frac{1}{s} - \frac{1}{2(s^2+1)}$$

$$\cancel{= \frac{1}{\cancel{2}s} \quad \frac{-1}{(s^2+1)^2} \quad - \frac{\cancel{2}s}{\cancel{2}(s^2+1)}}$$

$$= \frac{1}{s} - \frac{2s}{(s^2+1)^2} - \frac{s}{(s^2+1)}$$

$$L^{-1}[F''(s)] = L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{s}{(s^2+1)^2}\right] - L^{-1}\left[\frac{s}{s^2+1}\right]$$

WKT: $L^{-1}\left[\frac{s}{(s^2+a)^2}\right] = \frac{t\sin at}{2a}$.

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$$t^2 f(t) = 1 - \frac{t\sin t}{2} - \cos t$$

$$\times \quad f(t) = \frac{1-\cos t}{t^2} - \frac{\sin t}{2t}$$

$$t^2 f(t) = 1 - 2\frac{t\sin t}{2} - \cos t$$

$$f(t) = \frac{1-\cos t}{t^2} - \frac{\sin t}{t}$$

5) $\tan^{-1}\left(\frac{2}{s^2}\right)$

$$\Rightarrow \quad \cancel{F'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2} \times 2 \times \frac{-2}{s^3} = \frac{s^2}{(s^4+2^2)} \times \frac{-2\times 2}{s^3}}$$

$$\cancel{-F'(s) = \frac{+2s^3}{s^2+2^2} \quad \times \frac{2}{s(s^4+2^2)}}$$

$$F'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2} \times 2\left(\frac{-2}{s^3}\right)$$

$$F'(s) = \frac{-4s}{s^4+4}$$

$$-F'(s) = \frac{4s}{s^4+4}.$$

$$L^{-1}[-F'(s)] = 4L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= 4\, L^{-1}\left[\frac{s}{(s^2)^2 + 2^2}\right] \Rightarrow \text{TBC}$$

6) $\cot^{-1}\left(\frac{s+3}{2}\right)$

$$\Rightarrow \quad F'(s) = \frac{-1}{1 + \left(\frac{s+3}{2}\right)^2} = \frac{-4}{4 + (s+3)^2} \times \frac{1}{2}$$

$$-F'(s) = \frac{\cancel{4}\,2}{4 + (s+3)^2} = \cancel{2}\left[\frac{2}{2^2 + (s+3)^2}\right]$$

$$L^{-1}[-F'(s)] = \cancel{2e^{-3t}\cos 2t} \quad 2\, \frac{e^{-3t}\sin 2t}{2\cancel{t}}$$

$\cancel{f(t)}$

$$t\, f(t) = \cancel{2}\, e^{-3t} \sin 2t$$

$$f(t) = \frac{e^{-3t}\sin 2t}{t}.$$

7) $\frac{1}{s^2 - 4s + 5}$ $\qquad e^{2t}\sin t$

$$\Rightarrow \quad \frac{1}{(s-2)^2 + 5 - 2^2} = \frac{1}{(s-2)^2 + 1}$$

$$L^{-1}\left[\frac{1}{(s-2)^2 + 1}\right] = e^{2t}\sin t$$

8) $\frac{s+2}{s^2 + 4s - 21}$

$$\Rightarrow \quad \frac{s+2}{(s+2)^2 - 21 - 4} = \frac{s+2}{(s+2)^2 - 5^2}$$

$$L^{-1}\left[\frac{(s+2)}{(s+2)^2-5^2}\right] = e^{-2t}\cosh 5t$$

can also do by partial fraction.

9) $\dfrac{1}{4s^2-25}$

$$\Rightarrow \frac{1}{4\left(s^2-\left(\frac{5}{2}\right)^2\right)}$$

$$\frac{1}{4}L^{-1}\left[\frac{1}{s^2-(5/2)^2}\right] = \frac{1}{4}\sinh\frac{5t}{2}\times\frac{2}{5}$$

$$= \frac{1}{10}\sinh\frac{5t}{2}$$

10) $\dfrac{3s}{2s+9}$

$$\Rightarrow \frac{3}{2}\left(\frac{s}{s+9/2}\right) = \frac{3}{2}\left[\frac{s+9/2-9/2}{s+9/2}\right]$$

$$= \frac{3}{2}\left[1-\frac{9}{2}\left(\frac{1}{s+9/2}\right)\right]$$

$$\cancel{\frac{3}{2}L^{-1}\left[1-\frac{9}{2}\left(\frac{1}{s+9/2}\right)\right]} = \frac{3}{2}-\frac{27}{4}\frac{1}{(s+9/2)}$$

$$L^{-1}\left[\cancel{\frac{3}{2}}\frac{27}{4}\left(\frac{1}{s+9/2}\right)\right] = \frac{27}{4}e^{-\frac{9}{2}t}$$

$$\Rightarrow \frac{3}{2}-\frac{27}{4}e^{-9/2t}$$

11) $\dfrac{s+1}{s^2-6s+25}$

$\Longrightarrow \dfrac{s+1}{s^2-6s+9+25-9} = \dfrac{(s+1)}{(s-3)^2+4^2}$

$= \cancel{\dfrac{s}{(s}} \quad \dfrac{(s-3)+4}{(s-3)^2+4^2}$

$= \dfrac{(s-3)}{(s-3)^2+4^2} + \dfrac{4}{(s-3)^2+4^2}$

$= e^{3t}[\cos 4t + \sin 4t]$ //

12) $\dfrac{s^2+3}{s(s^2+9)}$

$\Longrightarrow \dfrac{s}{(s^2+9)} + \dfrac{3}{s(s^2+9)}$

$L^{-1}\left[\dfrac{F(s)}{s}\right] = \int_0^t f(t)\,dt.$

$= \int_0^t L^{-1}[F(s)]\,dt$

$* \; L^{-1}\left[\dfrac{1}{s}\dfrac{3}{(s^2+9)}\right] = \int_0^t L^{-1}\left[\dfrac{3}{s^2+9}\right] dt.$

$L^{-1}\left[\dfrac{3}{s^2+3^2}\right] = \sin 3t.$

$\Rightarrow \int_0^t \sin 3t\,dt = \dfrac{-1}{3}\cos 3t.$

$* \Rightarrow L^{-1}\left[\dfrac{s}{s^2+9}\right] = \cos 3t$

$\Rightarrow$ Ans: $\cos 3t - \frac{1}{3}\cos 3t$

$= \frac{2}{3}\cos 3t$ //

13) $\dfrac{s+4}{s(s-1)(s^2+4)}$ $\Rightarrow -1 + e^t - \frac{1}{2}\sin 2t$

$\Rightarrow$ $F(s) = \dfrac{A}{s} + \dfrac{B}{(s-1)} + \dfrac{Cs+D}{s^2+4}$

$s+4 = A(s-1)(s^2+4) + B(s)(s^2+4) + (Cs+D)(s)(s-1)$

$s=0$: $4 = A(-1)(4) = -4A$

$A = -1$

$s=1$: $5 = B(1)(5) = 5B$

$B = 1$

~~S+4 = A~~

$s+4 = A(s^3+4s-s^2-4) + B(s^3+4s)$

$+ C(s^3-s^2) + D(s^2-s)$

Equating co-efficients of s:

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14) $\dfrac{4s}{s^4+4}$

$\Longrightarrow$ $t\,f(t) = L^{-1}\left[\dfrac{4s}{s^4+4}\right]$

$s^4+4 = s^4+4+4s^2-4s^2$

$= (s^2+2)^2-(2s)^2$

$= (s^2+2+2s)(s^2+2-2s)$.

$4s = (s^2+2+2s)-(s^2+2-2s)$.

$t\,f(t) = L^{-1}\left[\dfrac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)(s^2+2-2s)}\right]$

$= L^{-1}\left[\dfrac{1}{(s^2+2-2s)} - \dfrac{1}{(s^2+2+2s)}\right]$

$= L^{-1}\left[\dfrac{1}{(s^2-1)+1} - \dfrac{1}{(s^2+1)+1}\right]$

$= e^t \dfrac{\sin t}{\cancel{2}} - e^{-t}\sin t = \cancel{2}\sin t(e^t-e^{-t})$.

$\therefore f(t) = \dfrac{\sin t\,(e^t-e^{-t})}{t} \times \dfrac{2}{2}$

$= \dfrac{2\sin t\,\sinh t}{t}$

15) $\dfrac{1}{s^4+4}$

$\Longrightarrow$ $t\,f(t) = L^{-1}\left[\dfrac{1}{s^4+4}\right]$

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$$s^4 + 4 = (s^2+2s+2)(s^2-2s+2).$$

$$\frac{1}{s^4+4} = \frac{1}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2-2s+2}.$$

$$1 = (As+B)(s^2-2s+2) + (Cs+D)(s^2+2s+2).$$

$$= As^3 + Bs^2 - 2As^2 - 2Bs + 2As + 2B$$
$$+ Cs^3 + Ds^2 + 2Cs^2 + 2Ds + 2Cs + 2D.$$

Coeff of s^3: $\quad 0 = A + C \Rightarrow A = -C. \quad$ —— (1)

Coeff of s^2: $\quad 0 = B - 2A + 2C + D \quad$ —— (2)

Coeff of s: $\quad 0 = -2B + 2A + 2D + 2C$
$$\Rightarrow A - B + C + D = 0. \quad \text{—— (3)}$$

Constants: $\quad 1 = 2B + 2D \Rightarrow B + D = \frac{1}{2} \quad$ —— (4).

Using (1) in (2): $\qquad B = \frac{1}{2} - D.$

$$0 = B + D + 2C + 2C \Rightarrow B + D + 4C = 0 \quad \text{—— (5)}$$

Using (4) in (5): $\quad \frac{1}{2} + 4C = 0$

$$\cancel{C = 2x - \frac{4}{2} = -2}$$

$$4C = \frac{-1}{2} \Rightarrow C = \frac{-1}{8}$$

$$\therefore A = \frac{1}{8}$$

(3): $A - B + C + D = 0$; $B = \frac{1}{2} - D$

$$A + D - \frac{1}{2} + C + D = 0.$$

$$\frac{1}{8} + 2D - \frac{1}{2} - \frac{1}{8} = 0$$

$$2D = \frac{1}{2} \Rightarrow \cancel{D} \; D = \frac{1}{4}$$

(4): $B + \frac{1}{4} = \frac{1}{2} \Rightarrow B = \frac{1}{\cancel{2}4}$

$$\therefore \frac{1}{s^4+4} = \frac{\frac{s}{8} + \frac{1}{4}}{(s^2+2s+2)} + \frac{-\frac{s}{8} + \frac{1}{4}}{(s^2-2s+2)}$$

$$= \frac{1}{8}\frac{s}{(s^2+2s+2)} + \frac{1}{4}\frac{1}{(s^2+2s+2)} - \frac{1}{8}\frac{s}{(s^2-2s+2)} + \frac{1}{4}\frac{1}{(s^2-2s+2)}$$

$$= \frac{1}{8}\frac{s+1-1}{(s+1)^2+1^2} + \frac{1}{4}\frac{1}{(s+1)^2+1^2} - \frac{1}{8}\frac{s+1-1}{(s-1)^2+1^2} + \frac{1}{4}\frac{1}{(s-1)^2+1^2}$$

$$= \frac{e^{-t}}{8}[\cos t - \sin t] + \frac{e^{-t}}{4}\sin t - \frac{e^{t}}{8}[\cos t + \sin t] + \frac{e^{t}}{4}\sin t.$$

$= \frac{1}{8}e^{-t}\,[\cos t - \sin t + 2\sin t] + \frac{e^{t}}{8}[\cos t + \sin t - 2\sin t]$

$= \frac{e^{-t}}{8}[\cos t + \sin t] + \frac{e^{t}}{8}[\cos t - \sin t].$

$= \frac{1}{4}[\sin t \cosh t - \cos t \sinh t].$

16) $\frac{s}{s^4+s^2+1}$

$\Longrightarrow$ $s^4+s^2+1 = s^4+s^2+1+s^2-s^2.$

$= (s^2+1)^2 - s^2.$

$= (s^2+s+1)(s^2-s+1).$

$\frac{s}{s^4+s^2+1} = \frac{As+B}{(s^2+s+1)} + \frac{Cs+D}{(s^2-s+1)}$

$S = As+B\,(\cancel{s}^2$

$s = \frac{(s^2+s+1)-(s^2-s+1)}{2}$

$\Rightarrow F(s) = \frac{1}{2}\left[\frac{(s^2+s+1)-(s^2-s+1)}{(s^2+s+1)(s^2-s+1)}\right]$

$= \frac{1}{2}\left[\frac{1}{(s^2-s+1)} - \frac{1}{(s^2+s+1)}\right]$

$= \frac{1}{2}\left[\frac{1}{(s-1/2)^2+3/4} - \frac{1}{(s+1/2)^2+3/4}\right]$

$\therefore L^{-1}[F(s)] = \frac{2\times 1}{\sqrt{3}\,2}\left[e^{1/2 t}\sin\sqrt{3}/2\,t - e^{-1/2 t}\sin\sqrt{3}/2\,t\right]$

$$\frac{s^2}{s^4-a^4}$$

$$= \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left[e^{1/2 t} - e^{-1/2 t} \right]$$

$$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) \sinh\left(\frac{1}{2} t\right) //$$

17) $$\frac{s+2}{(s^2+4s+5)^2}$$

$$\Longrightarrow \quad (s^2+4s+5)^{\cancel{2}} = s^2+4s+\cancel{2}4 + 1$$

$$= (s+2)^2 + 1^2$$

$$\frac{s+2}{((s+2)^2+1^2)^2}$$

WKT: $$L^{-1}\left[\frac{s}{(s+a^2)^2}\right] = \frac{t \sin at}{2a}.$$

$$\Rightarrow \quad e^{-2t} \frac{t \sin 2t}{4} //$$

18) $$\frac{s^2}{s^4-a^4}$$

$$\Longrightarrow \quad s^4 - a^4 = (s^2+a^2)(s^2-a^2).$$

$$s^2 = \frac{(s^2+a^2)+(s^2-a^2)}{2}.$$

$$\cancel{L^{-1}\left[\frac{s^2}{s^4-a^4}\right]} =$$

$$F(s) = \frac{1}{2} \frac{(s^2+a^2)+(s^2-a^2)}{(s^2+a^2)(s^2-a^2)}.$$

$$= \frac{1}{2}\left[\frac{1}{s^2-a^2} + \frac{1}{s^2+a^2}\right]$$

$$L^{-1}[F(S)] = \frac{1}{2}[\sinh at + \sin at]$$

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Convolution Theorem

Convolution of 2 functions $f(t)$ & $g(t)$ is denoted by $f(t) * g(t)$ & is defined by

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du.$$

$$f(t) * g(t) = g(t) * f(t)$$

$$L[f(t) * g(t)] = L[f(t)]\ L[g(t)]$$

Proof! [Not for exam]

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$L[f(t) * g(t)] = \int_0^\infty e^{-st} \int_0^t f(u)\, g(t-u)\, du\, dt$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} f(u)\, g(t-u)\, du\, dt$$

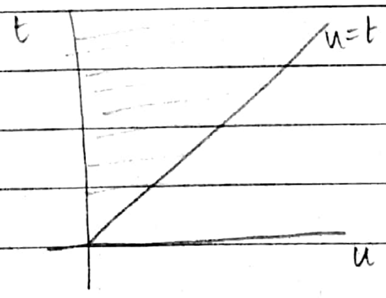
$u : 0 \to t$

$t : 0 \to \infty$

Changing limits:

$u : 0 \to \infty$

$t : \cancel{0\to} \ u \to \infty$



$\therefore L[f(t)*g(t)] = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u) g(t-u)\, dt\, du.$

Put $t-u = x \Rightarrow dt = dx$.

When $t = u$; $x = 0$

$t = \infty$; $x = \infty$

$$\int_{u=0}^{\infty}\int_{x=0}^{\infty} e^{-s(x+u)} f(u)\, g(x)\, dx\, du.$$

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-su} e^{-sx} f(u)\, g(x)\, dx\, du.$$

$$= \int_0^{\infty} e^{-su} f(u)\, du \times \int_0^{\infty} \cancel{g} e^{-sx} g(x)\, dx$$

$$= L[f(u)] \times L[g(x)].$$

$$L[f(t)*g(t)] = L[f(t)]\, L[g(t)]$$

Numericals

Verify Convolution Theorem for the following

1) $f(t) = t$; $g(t) = \sin t$.

$\Longrightarrow L[f(t)*g(t)] = L[f(t)]\, L[g(t)]$.

RHS: $L[t]\, L[\sin t] = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)}$.

LHS: $f(t)*g(t) = \int_0^t f(u)\, g(t-u)\, du.$

$= \int_0^t \underset{\text{I}}{u}\, \underset{\text{II}}{\sin(t-u)}\, du.$ $\quad \int_0^t u \cos(t-u)\, du$

$\Longrightarrow L[F(s)] = \frac{1}{s(s^2+1)}$

By parts:

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$$= u\left(\frac{-\cos(t-u)}{-1}\right) - \frac{\sin(t-u)}{1}(1) \Big]_0^t$$

$$= u\cos(t-u) - \sin(t-u)$$

$$= t\cos(0) - 0 \mp 0 \mp \sin t.$$

$$= t + \sin t$$

$$L[f(t) * g(t)] = L[t - \sin t]$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$= \frac{s^2+1-s^2}{s^2(s^2+1)} = \frac{1}{s^2(s^2+1)} = RHS.$$

2) $f(t) = t^2$; $g(t) = t e^{-2t}$

$\Longrightarrow$ RHS: $L[f(t)]\ L[g(t)]$

$$= L[t^2]\ L[t e^{-2t}]$$

$$= \frac{2}{s^3}\ \frac{1}{(s+2)^2}$$

LHS: $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

$\times \left[= \int_0^t u^2 (t-u) e^{-2(t-u)}\, du. \right.$

$$= \int_0^t g(u)\, f(t-u)\, du$$

$$= \int_0^t u e^{-2u} (t-u)^2\, du$$

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$$= \int_0^t u(u^2+t^2-2ut)\, e^{-2u}\, du.$$

$$= \int_0^t \underset{\substack{\mathrm{I} \\ \text{diff}}}{(u^3+ut^2-2u^2t)}\ \underset{\substack{\mathrm{II} \\ \text{int}}}{e^{-2u}}$$

Bernoulli's Rule:

$$= \left[(u^3+ut^2-2u^2t)\frac{e^{-2u}}{-2}\right]_0^t - \left[\frac{e^{-2u}}{4}(3u^2 - 4ut+t^2)\right]_0^t$$

$$+ \left[\frac{e^{-2u}}{-8}(6u-4t)\right]_0^t - \left[\frac{e^{-2u}}{16}(6).\right]$$

$$= (t^3+t^3-2t^3)\frac{e^{-2t}}{-2} - 0 - \frac{e^{-2t}}{4}(3t^2-4t^2+t^2)$$

$$+ \cancel{e^{-2t}}\frac{t^2}{4} + \frac{e^{-2t}}{-8}(6t-4t) - \frac{4t}{8} - \frac{6}{16}(e^{-2t}-1)$$

$$= \frac{t^2}{4} + \frac{e^{-2t}}{\cancel{-8}\,4}(\cancel{2}t) - \frac{\cancel{4t}\,2}{\cancel{8}\,2}\ /\!\!8\,4 - \frac{\cancel{6}\,3\,e^{-2t}}{\cancel{16}\,8} + \frac{\cancel{6}\,3}{\cancel{16}\,8}$$

$$= \frac{1}{4}\left[t^2 - \frac{\cancel{2}te^{-2t}}{4} - 2t - \frac{3e^{-2t}}{2} + \frac{3}{2}\right]$$

$$L[LHS] = \frac{1}{4}\left[\frac{2}{s^3} - \frac{1}{4}\frac{1}{(s+2)^2} - \frac{2}{s^2} - \frac{3}{2}\frac{1}{(s+2)} + \frac{3}{2s}\right]$$

$$= \frac{2}{s^3}\frac{1}{(s+2)^2}$$ //

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Inverse by Convolution Theorem

If $L^{-1}[F(s)] = f(t)$ & $L^{-1}[G(s)] = g(t)$, then,

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t g(u)\,f(t-u)\,du$$

Numericals.

Find L^{-1} using convolution theorem.

1) $\dfrac{s}{(s^2+a^2)^2}$

$\Longrightarrow$

$$= \frac{1}{(s^2+a^2)}\,\frac{s}{(s^2+a^2)} = F(s)\,G(s).$$

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(u)\,g(t-u)\,du.$$

$$f(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at.$$

$$g(t) = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at.$$

$$\therefore L^{-1}[F(s)\,G(s)] = \int_0^t \frac{1}{a}\sin au\,\cos a(t-u)\,du.$$

$$= \frac{1}{a}\int_0^t \sin au\,\cos(at-au)\,du.$$

$\frac{1}{2}(\sin(a+b) + \sin(a-b))$

$$= \frac{1}{2a}\int_0^t \sin(at) + \sin(2au-at)\,du$$

$$= \frac{1}{2a}\left[\sin at\,(u) - \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left(t\sin at - \frac{1}{2a}\cos at\right) - \left(0 - \frac{1}{2a}\cos at\right)$$

$$= \frac{1}{2a} t\sin at \;//$$

2) $\dfrac{1}{(s^2+a^2)^2}$

$$\Longrightarrow \quad = \frac{1}{(s^2+a^2)}\,\frac{1}{(s^2+a^2)} \Rightarrow f(t) = g(t).$$

$$f(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = g(t).$$

$$L^{-1}[F(s)\,G(s)] = \int_0^t \left(\frac{1}{a}\right)\sin au \left(\frac{1}{a}\right)\sin a(t-u)\, du.$$

$-\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

$$= \frac{-1}{2a^2}\int_0^t \cos \cancel{2au}(at) - \cos(2au-at)\, du.$$

$$\times \quad = \frac{-1}{2a^2}\left[t\cos at - \frac{1}{2a}\cos(2au-at)\right]_0^t$$

$$= \frac{-1}{2a^2}\left[u\cos at - \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{-1}{2a^2}\left[t\cos at - \frac{1}{2a}\sin at\right] - \frac{1}{2a^2}\left[0 - \frac{\sin at}{2a}\right]$$

$$= \frac{-1}{2a^2}\left[t\cos at - \frac{\sin at}{a}\right] //$$

3) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ $\quad \dfrac{a\sin at - b\sin bt}{a^2-b^2}$ $\quad a \neq b.$

$\longrightarrow \quad = \dfrac{s}{s^2+a^2} \quad \dfrac{s}{s^2+b^2} = F(s)\ G(s).$

$f(t) = \cos at \ ; \quad g(t) = \cos bt$

$L^{-1}[F(s)\,G(s)] = \int_0^t \cos au \ \cos b(t-u)\ du$

$\frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$= \int_0^t \cos au \ \cos(bt - bu)\ du.$

$= \dfrac{1}{2} \int_0^t \cos(au + bt - bu) + \cos(au - bt + bu)\, du$

$= \dfrac{1}{2}\left[\dfrac{\sin(au+bt-bu)}{-b} + \dfrac{\sin(au-bt+bu)}{b}\right]_0^t$

$= \cancel{\dfrac{1}{2b}}\,\cancel{[2\sin at]} = \dfrac{1}{2b}[-\sin at + \sin at]$

$\cancel{= 0}$

$= \dfrac{1}{2}\left[\dfrac{\sin(au+bt-bu)}{a-b} + \dfrac{\sin(au+bu-bt)}{a+b}\right]_0^t$

$= \dfrac{1}{2}\left[\dfrac{1}{(a-b)}(\sin at - \sin bt) + \dfrac{1}{(a+b)}(\sin at + \sin bt)\right]$

$= \dfrac{a\sin at - b\sin bt}{a^2-b^2}.$ //

4) $\dfrac{1}{s^3(s^2+4)}$

$\longrightarrow \quad \dfrac{1}{s^3} \cdot \dfrac{1}{(s^2+2^2)} = F(s)\ G(s).$

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$$f(t) = \frac{t^2}{2} \quad ; \quad g(t) = \frac{1}{2}\sin 2t$$

$$FL^{-1}[F(s)\,G(s)] = \int_0^t \frac{1}{2}\sin 2u \,\frac{(t-u)^2}{2}\,du.$$

$$= \frac{1}{4}\int_0^t (t-u)^2 \sin 2u\,du.$$

$$L^{-1}[F(s)\,G(s)] = \int_0^t \frac{u^2}{2}\,\frac{\sin 2(t-u)}{2}\,du.$$

$$= \frac{1}{4}\int_0^t u^2 \sin 2(t-u)\,du.$$

$$= \frac{1}{4}\left[u^2\left(\frac{-\cos 2(t-u)}{-2}\right) - \left(\frac{\sin(t-u)}{4}\right)(2u)\right]_0^t$$

$$= \frac{1}{8}\left[u^2\cos 2(t-u) - 2u(\sin(t-u))\right]_0^t$$

$$= \cancel{\frac{1}{8}}\cancel{[u^2} \quad \frac{1}{8}\left[(u^2-0) - (u^2\cos 2t - 2u\sin t)\right]$$

$$= \frac{1}{8}\left(u^2 - u^2\cos 2t\right.$$

$$= \frac{1}{8}\left[t^2 - 0 - u^2\cos 2t + 2u\sin t\right]$$

$$= \frac{1}{4}\left[(t-u)^2\left(-\frac{\cos 2u}{2}\right) - \left(-\frac{\sin 2u}{4}\right)(-2(t-u)) + \frac{\cos 2u}{8}(2)\right]_0^t$$

$$= \frac{1}{4}\left[\frac{t^2}{2} - \frac{1}{4} + \frac{\cos 2t}{4}\right]$$

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5) $\frac{1}{s(s+1)^3}$ $\qquad 1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)$

$\longrightarrow \quad = \frac{1}{s} \quad \frac{1}{(s+1)^3} = F(s)\ G(s).$

$f(t) = 1 \quad ; \quad g(t) = e^{-t}\left(\frac{t^2}{2}\right)$

$$L^{-1}[F(s)G(s)] = \int_0^t e^{-u}\frac{u^2}{2}\,du.$$

$$\times \left[\cancel{= e^{-u}\left(\frac{u^3}{6}\right) - \frac{u^4}{24}\left(\frac{e^{-u}}{-u}\right)} \right.$$

$$= e^{-u}\left(\frac{u^3}{6}\right) - \frac{u^4}{24}\left[-u e^{-u-1}\right)\Big]_0^t$$

$$\left. = e^{-t}\frac{t^3}{6} \right]$$

$$= \frac{1}{2}\left[\frac{u^2}{\cancel{}}\ \frac{e^{-u}}{-u} - \frac{e^{-u}}{u^2}(2u)\right]_0^t$$

$$= \frac{1}{2}\left[t e^{-t} - \frac{2e^{-t}}{t} - 0 - 0\right]$$

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Laplace transforms of Derivatives

If $L[y(t)] = Y(s)$, then

* $L\left[\frac{dy}{dt}\right] = L[y'(t)] = sY(s) - y(0).$

* $L\left[\frac{d^2y}{dt^2}\right] = L[y''(t)] = s^2Y(s) - sy(0) - y'(0).$

$$* \; L\left[\frac{d^3y}{dt^3}\right] = L[y'''(t)] = s^3Y(s) - s^2y(0) - sy'(0) - y''(0).$$

Numericals

Solve the following diff eqns.

1) $$\frac{d^2y}{dt^2} + 4y = \sin t \; u(t-2\pi) \quad ; \quad y(0) = y'(0) = 0.$$

$\Longrightarrow$ $$y''(t) + 4y(t) = \sin t \; u(t-2\pi).$$

$$L[y''(t)] + 4L[y(t)] = L[\sin t \; u(t-2\pi)].$$

WKT: $$L[f(t-a)\,u(t-a)] = e^{-as} \, L[f(t)].$$

$$f(t-2\pi) = \sin t.$$
$$f(t) = \sin(t+2\pi) = \sin t$$
$$L[f(t)] = \frac{1}{s^2+1}$$

$$L[\sin t \; u(t-2\pi)] = \frac{e^{-2\pi s}}{s^2+1}$$

LHS: $$s^2Y(s) - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0} + 4Y(s) - \cancel{4y(0)}^{0} = \frac{e^{-2\pi s}}{s^2+1}$$

~~Both~~ $$Y(s)[s^2+4] = \frac{e^{-2\pi s}}{s^2+1}$$

$$Y(s) = \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

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$$L^{-1}[Y(s)] = L^{-1}\left[\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right]$$

$y(t)$ ×

$$L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right]$$

$$\Rightarrow \frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$1 = (As+B)(s^2+4) + ((Cs+D)(s^2+1))$$

$$1 = As^3 + 4As^2 + Bs^2 + 4 + Cs^3 + Ds^2 + Cs + D$$

s^3 :- $A + C = 0$

$A = -C$

s^2 :- $B + D = 0 \Rightarrow B = -D$

check

s :- $4A + C = 0$ $\cancel{A = -\frac{C}{4}}$

const :- $4 + D = 1 \Rightarrow D = -3.$

$B = 3$

$A = 0;\ C = 0;$

$$\Rightarrow F(s) = \frac{3}{s^2+1} - \frac{3}{s^2+4}$$

$$L^{-1}[F(s)] = 3\sin t - \frac{3\sin 2t}{2}$$

$$L^{-1}[F(s)] = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

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$$L^{-1}[e^{-2\pi s} F(s)] = f(t-2\pi)\, u(t-2\pi)$$

$$= \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-2\pi) \right] u(t-2\pi)$$

2) $\dfrac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t \; ; \quad y(0) = 0$

$$\Longrightarrow \quad y'(t) + 2y(t) + \int_0^t y(t)\,dt = \sin t$$

$$L[y'(t)] + 2L[y(t)] + L\left[\int_0^t y(t)\,dt\right] = L[\sin t]$$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$\rightarrow \quad sY(s) - \cancel{y(0)}^{\,0} + 2Y(s) + \frac{Y(s)}{s} = \frac{1}{s^2+1}$$

$$Y(s)\left[s + 2 + \frac{1}{s}\right] = \frac{1}{s^2+1}$$

$\frac{3s}{2} + 2$

$$Y(s)\left[\frac{3s+4}{2}\right] = \frac{1}{s^2+1}$$

$\frac{3s+4}{2}$

$$Y(s) = \frac{2}{(3s+4)(s^2+1)}$$

$$L^{-1}[Y(s)] = 2\, L^{-1}\left[\frac{1}{(3s+4)(s^2+1)}\right]$$

$$\frac{1}{(3s+4)(s^2+1)} = \frac{A}{(3s+4)} + \frac{Bs+C}{(s^2+1)}$$

$$1 = A(s^2+1) + (Bs+C)(3s+4)$$

$\frac{1}{2}\sin t - e^{-t}t$

$$Y(S)\left[\frac{S^2+2S+1}{S}\right] = \frac{1}{(S^2+1)}$$

$$Y(S) = \frac{S}{(S+1)^2(S^2+1)}$$

$$L^{-1}[Y(S)] = L^{-1}\left[\frac{S}{(S+1)^2(S^2+1)}\right] = y(t).$$

$= 0$

$= -1/2$

$= 0$

$= 1/2$

$$\frac{S}{(S+1)^2(S^2+1)} = \frac{A}{S+1} + \frac{B}{(S+1)^2} + \frac{CS+D}{(S^2+1)}$$

$$S = A(S+1)(S^2+1) + B(S^2+1) + ((S+D)(S+1)^2$$

S^2+2S+1

$$S=-1; \quad -1 = 2B \Rightarrow B = \frac{-1}{2}$$

$$S = AS^4 + AS^2 + AS + A + BS^2 + B + CS^3 + 2CS^2 + 2CS + DS^2 + DS + 1.$$

$\Rightarrow A = 0 ; B = -1/2 ; C = 0 ; D = 1/2.$

OR: By factors of denominator:

$$y(t) = \frac{1}{2} L^{-1}\left[\frac{(S+1)^2 - (S^2+1)}{(S^2+1)(S+1)^2}\right]$$

$$\Rightarrow Y(S) = \frac{-1}{2(S+1)^2} + \frac{1}{2(S^2+1)}$$

$$= \frac{1}{2}[\sin t - e^{-t}t]$$

__/__/____

3) $\frac{d^3y}{dt^3} - \frac{3d^2y}{dt^2} + \frac{3dy}{dt} - y = t^2e^t$

$y(0) = 1 \; ; \; y'(0) = 0 \; ; \; y''(0) = -2$

$\Rightarrow y'''(t) - 3y''(t) + 3y'(t) - y = t^2e^t.$

$L[y'''(t) - 3L[y''(t)] + 3L[y'(t)] - L[y(t)] = L[t^2e^t]$

LHS: $s^3Y(s) - s^2y(0) - \cancel{sy'(0)}^0 - y''(0) - 3[s^2Y(s)$
$- sy(0) - \cancel{y'(0)}] + 3[sY(s) - y(0)] - Y(s)$

$= s^3Y(s) - s^2 + 2 - 3s^2Y(s) + 3s + 3sY(s) - 3 - Y(s)$

$= Y(s)[s^3 - 3s^2 + 3s - 1] - s^2 - s + 2$

RHS $= L[t^2e^t] = \frac{2}{(s-1)^3}$

$\Rightarrow Y(s)\underbrace{[s^3 - 3s^2 + 3s - 1]}_{(s-1)^3} - (s^2 + s - 2) = \frac{2}{(s-1)^3}$

$Y(s) = \frac{2 + (s^2 + s - 2)(s-1)^3}{(s-1)^3 \; (s-1)^3}$

$Y(s) = \frac{2}{(s-1)^6} + \frac{s^2 + s - 2}{(s-1)^3}$

3 $L^{-1}[Y(s)] = y(t) = \frac{2e^t t^5}{5!} + L^{-1}\left[\frac{s^2 + s - 2}{(s-1)^3}\right]$

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4) $[tD^2 + (1-2t)D - 2]y = 0; \quad y(0)=1, \; y'(0)=2$

$\Longrightarrow \quad t\,y''(t) + (1-2t)\,y'(t) - 2y(t) = 0$

$L[t\,y''(t)] + L[(1-2t)\,y'(t)] - 2L[y(t)] = 0$

$\Rightarrow L[t\,y''(t)] + L[y'(t)] - 2L[t\,y'(t)] - 2L[y(t)] = 0$

WKT: $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n} F(s)$

$$\Rightarrow (-1)\frac{d}{ds}L[y''(t)] + sY(s) - y(0) + 2\frac{d}{ds}L[y'(t)] - 2Y(s) = 0.$$

$$\Rightarrow -\frac{d}{ds}[s^2Y(s) - s\,y(0) - y'(0)] + sY(s) - 1 + 2\frac{d}{ds}[sY(s) - y(0)] - 2Y(s) = 0.$$

$$\Rightarrow -\frac{d}{ds}[s^2Y(s) - s - 2] + sY(s) - 1 + 2\frac{d}{ds}(sY(s) - 1) - 2Y(s) = 0$$

$$\Rightarrow (-s^2Y'(s) - 2sY(s) + 1 + 0) + sY(s) + 2[sY'(s) + Y(s)] - 2Y(s) = 0.$$

$$\Rightarrow Y'(s)[-s^2 + 2s] + Y(s)[-2s + s + \cancel{2} - \cancel{2}] = 0.$$

$\div s: \quad Y'(s)[2-s] = Y(s)$

$$\frac{Y'(s)}{Y(s)} = \frac{1}{s-2}$$

$$\int \frac{Y'(s)}{Y(s)}\,ds = \log Y(s)$$

$$\int \frac{1}{s-2} = -\log(s-2) + \log c$$

$$\Rightarrow \log Y(s) = \log\left(\frac{c}{s-2}\right)$$

$$Y(s) = \frac{c}{s-2}$$

$$L^{-1}[Y(s)] = ce^{2t}$$

$$y(t) = ce^{2t}$$

$$y(0) = 1 \Rightarrow ce^{2(0)} = 1 \Rightarrow c = 1.$$

$$y(t) = e^{2t}$$

---

5) $\frac{dx}{dt} + y = \sin t$; $\frac{dy}{dt} + x = \cos t$

Given $x = 1,\ y = 0$ when $t = 0$.

$$\Rightarrow \cancel{x'(t)}\ x'(t) + y(t) = \sin t \quad \text{--- (1)}$$

$$y'(t) + x(t) = \cos t \quad \text{--- (2).}$$

$$(1):\ L[x'(t)] + L[y(t)] = L[\sin t].$$

$$\cancel{s^2X(s) - sx'(0) - x(0) + sY(s) - y(0)}$$

$$= \frac{1}{s^2+1}.$$

$$sX(s) - \cancel{s} x(0) + Y(s) = \frac{1}{s^2+1}$$

But $x(0) = 1$

$$\Rightarrow sX(s) - 1 + Y(s) = \frac{1}{s^2+1}$$

$$sX(s) + Y(s) = \frac{1}{s^2+1} + 1 \quad \text{--- (3)}$$

(2): $y'(t) + x(t) = \cos t$

$$L[y'(t)] + L[x(t)] = L[\cos t]$$

$$sY(s) - y(0) + X(s) = \frac{s}{s^2+1}$$

But $y(0) = 0$

$$\Rightarrow sY(s) + X(s) = \frac{s}{s^2+1} \quad \text{--- (4)}$$

Solve (1) & (2):

(1) - (2)s : $sX(s) + y(s) = \frac{1}{s^2+1} + 1$

$$\underset{(-)}{} s^2Y(s) \underset{(-)}{+} sX(s) = \underset{(-)}{} \frac{s^2}{s^2+1}$$

$$Y(s)[s^2+1] = \frac{2}{s^2+1}$$

$$y(s) = \frac{\cancel{+}2}{(s^2+1)(1-s^2)}$$

$$y(s) = \frac{-2}{(s^2+1)(s^2-1)}$$

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$y(t) = L^{-1}[Y(s)] = \sin t - \sinh t$

Using this in ②:

$$\frac{d}{dt}(\sin t - \sinh t) + x(t) = \cos t.$$

$$\cos t - \cosh t + x(t) = \cos t.$$

$$x(t) = \cosh t$$

6) $\frac{dx}{dt} - y = e^t; \quad \frac{dy}{dt} + x = \sin t.$

Given: $x(0) = 1; \quad y(0) = 0.$

$\Rightarrow$ $\frac{dx}{dt} - y = e^t$

$$x'(t) - y(t) = e^t.$$

$$x''(t) - y'(t) = e^t$$

But $y'(t) = \sin t - x(t)$

$\Rightarrow x''(t) + x(t) - \sin t = e^t.$

$$x''(t) + x(t) = e^t + \sin t.$$

Put $t=0$ in ①; $x'(0) - y(0) = 1.$

$$x'(0) = 1.$$

$$L[x''(t)] + L[x(t)] = L[e^t] + L[\sin t]$$

$$s^2X(s) - s\,\cancel{x(0)}^{\,s} - \cancel{x'(0)}^{\,1} + X(s) = \frac{1}{s-1} + \frac{1}{s^2+1}$$

$$X(s)[s^2+1] - s = \frac{1}{(s-1)} + \frac{1}{(s^2+1)}$$

$$X(s)(s^2+1) = \frac{1}{(s-1)} + \frac{1}{(s^2+1)} + s$$

$$X(s) = \frac{1}{(s^2+1)(s-1)} + \frac{1}{(s^2+1)^2} + \frac{s}{s^2+1}$$

$$L^{-1}[X(s)] = L^{-1}\left[\frac{1}{(s^2+1)(s-1)}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

~~x(t)=~~

WKT: $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{-1}{2a^2}[t\cos at - \sin at]$

$$L^{-1}[X(s)] = L^{-1}\left[\frac{1}{(s^2+1)(s-1)}\right] - \frac{1}{2}(t\cos at - \sin at) + \cos t$$

$L^{-1}\left[\frac{1}{(s^2+1)(s-1)}\right]$:

$$\frac{1}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$1 = (As+B)(s-1) + C(s^2+1)$$

$$1 = As^2 + Bs - As - B + C(s^2+1)$$

$s=1$: $2C = 1 \Rightarrow C = \frac{1}{2}$

s: $Bs - As = 0 \rightarrow A = B$

~~s^2: $A + C = 0 \Rightarrow B = 0$~~

s^2: $A + C = 0 \Rightarrow A = -C = -\frac{1}{2}$

~~Mt~~

$\therefore A = \frac{-1}{2}$; $B = \frac{-1}{2}$; $C = \frac{1}{2}$

$$L^{-1}\left[\;\right] \Rightarrow \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}}{(s-1)}$$

$$= \frac{-1}{2}\left(\frac{s-1}{s^2+1}\right) + \frac{1}{2}\frac{1}{(s-1)}$$

$$\frac{-1}{2}L\left[\frac{s}{s^2+1}\right] + \frac{1}{2}L\left[\frac{1}{s^2+1}\right] + \frac{1}{2}L\left[\frac{1}{s-1}\right]$$

$$= \frac{1}{2}\left[\sin t - \cos t + e^t\right]$$

$$\therefore x(t) = -\frac{1}{2}(t\cos t - \sin t) + \cos t + \frac{1}{2}(\sin t - \cos t - e^t).$$

$$y(t) = \frac{t\sin t}{2} + \frac{1}{2}\left[\cos t - \sin t - e^t\right]$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

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7) Solve the problem of resonance damped vibration of a spring. Equation is:

$$\frac{md^2y}{dt^2} + c\frac{dy}{dt} + ky = 0.$$

$$\Longrightarrow \quad my''(t) + cy'(t) + ky(t) = 0.$$

Assume $y(0) = k_1$, $y'(0) = k_2$.

$\div m$:- $\quad y''(t) + \frac{c}{m}y'(t) + \frac{k}{m}y(t) = 0$

$$\frac{c}{m} = A \; ; \quad \frac{k}{m} = B$$

$$y''(t) + Ay'(t) + By(t) = 0$$

$$L[y''(t)] + A\,L[y'(t)] + B\,L[y(t)] = 0.$$

$$s^2Y(s) - sy(0) - y'(0) + A[sY(s) - y(0)] + BY(s) = 0$$

$$Y(s)[s^2 + As + B] - sk_1 - k_2 - Ak_1 = 0.$$

$$Y(s) = \frac{k_1 s + k_2 + k_1 A}{s^2 + As + B} \qquad [k_3 = k_2 + k_1 A]$$

$$= \frac{k_1 s}{s^2+As+B} + \frac{k_3}{s^2+As+B}$$

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{k_1 s}{s^2+As+B}\right] + k_3 L^{-1}\left[\frac{1}{s^2+As+B}\right]$$

$$s^2 + As + B = \left(s + \frac{A}{2}\right)^2 + B - \frac{A^2}{4}$$

$$= \left(s + \frac{A}{2}\right)^2 + \frac{4B - A^2}{4}$$

$$= \left(s + \frac{A}{2}\right)^2 + C^2 \qquad \left[C = \frac{4B - A^2}{4}\right]$$

$$\therefore L^{-1}[Y(s)] = k_1 L^{-1}\left[\frac{s}{(s+A/2)^2 + C^2}\right] + k_3 L^{-1}\left[\frac{1}{(s+A/2)^2 + C^2}\right]$$

$$= \cancel{k_1 e^{-\frac{A}{2}t} \cos at}$$

$$= k_1 L^{-1}\left[\frac{s + A/2 - A/2}{(s+A/2)^2 + C^2}\right] + k_3 L^{-1}\left[\frac{1}{(s+A/2)^2 + C^2}\right]$$

$$y(t) = k_1 e^{-\frac{A}{2}t}\left[\cos ct - \frac{A}{2c}\sin ct\right] + k_3 e^{-\frac{A}{2}t}\frac{\sin ct}{c}$$

8) A resistance R in series with inductance L is connected to emf $E(t)$. $[= H(t)]$. The current i is given by $L\frac{di}{dt} + Ri = H(t)$

If the switch is connected at $t=0$ and disconnected at $t=a$, find the current i in terms of time, t.

$\Longrightarrow$ $\cancel{H(t)}$ & $H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

$\Rightarrow H(t) = \begin{cases} 1 & \text{at } t = 0 \text{ to } a. \\ 0 & \text{otherwise} \end{cases}$

$\cancel{L\frac{di}{dt} + Ri = X}$

$$L[H(t)] = \int_0^\infty e^{-st} H(t)\,dt$$

$$= \int_0^a e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right)_0^a$$

$$= \frac{1}{s}[1-e^{-as}].$$

Given: $i(0) = 0.$

$$L[L\,i'(t)] + R\,L[i(t)] = L[H(t)]$$

$$= L(sI(s) - \cancel{i(0)}^{0}) + RI(s) = \frac{1}{s}[1-e^{-as}].$$

$$\& \; I(s)[Ls+R] = \frac{1}{s}[1-e^{-as}]$$

$$I(s) = \frac{(1-e^{-as})}{s(Ls+R)}$$

$$L^{-1}[I(s)] = L^{-1}\left[\frac{(1-e^{-as})}{s(Ls+R)}\right]$$

$$= L^{-1}\left[\frac{1}{s(Ls+R)}\right] - L^{-1}\left[\frac{e^{-as}}{s(Ls+R)}\right] \quad \text{—— (1)}$$

* Consider : $\frac{1}{s(Ls+R)} = \frac{1}{Ls(s+\frac{R}{L})} = \frac{1}{L} \frac{F(s)}{s}$

WKT : $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\, dt.$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{(s+R/L)}\right] = e^{-\frac{R}{L}t}$$

$$\Rightarrow \int_0^t e^{-\frac{R}{L}t}\, dt = \left.\frac{Le^{-\frac{R}{L}t}}{-R}\right]_0^t$$

$$= -\frac{L}{R}\left[e^{-\frac{R}{L}t} - 1\right]$$

$$L^{-1}\left[\frac{1}{Ls(s+R/L)}\right] = \frac{1}{L} \times \frac{L}{R}(1-e^{-\frac{R}{L}t})$$

$$f(t) = \frac{1}{R}(1-e^{-\frac{R}{L}t})$$

* Consider ~~F(s)~~ $\frac{e^{-as}}{s(Ls+R)} = e^{-as} F(s)$

$$F(s) = \frac{1}{s(Ls+R)} \Rightarrow L^{-1}[F(s)] = \frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

$$= f(t)$$

$$L[e^{-as}F(s)] = f(t-a)\,u(t-a).$$

$$= \frac{1}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right] u(t-a).$$

① $\Rightarrow L^{-1}[I(s)] = \frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right) - \frac{1}{R}\left(1-e^{-\frac{R}{L}(t-a)}\right)u(t-a)$

$$= i(t)$$

q) The current i & charge q in a series circuit containing inductance L & capacitance C, emf, E ~~u(t)~~ satisfying the diff eqn

$$L\frac{di}{dt} + \frac{q}{C} = \cancel{u(t)}.E \; ; \quad i = \frac{dq}{dt}$$

Express i & q in terms of t given that L, C, E are constants & the values of i & q are 0 initially.

$\Longrightarrow$

$$L\frac{di}{dt} + \frac{q}{C} = \cancel{u(t)}\,E$$

$$L\frac{d}{dt}\left(\frac{dq}{dt}\right) + \frac{q}{C} = E.$$

$$L\,q''(t) + \frac{q(t)}{C} = E.$$

$$L\left[s^2Q(s) - s\,\cancel{q'(0)}^{\,0} - \cancel{q(0)}^{\,0}\right] + \frac{1}{C}[Q(s)] = E.$$

$$Q(s)\left[Ls^2 + \frac{1}{C}\right] = E$$

$$Q(s) = \frac{E}{Ls^2 + 1/c}$$

$$L^{-1}[Q(s)] = q(t) = L^{-1}\left[\frac{E}{Ls^2 + 1/c}\right]$$

$$\frac{E}{Ls^2 + 1/c} = \frac{E}{L}\left[\frac{1}{s^2 + 1/cL}\right]$$

$$\cancel{L^{-1}\left[\frac{1}{s^2 + 1/cL}\right]} \neq$$

$$\Rightarrow q(t) = EC\left[1 - \cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

$$i = \frac{dq}{dt} = q'(t) = EC\left[0 + \sin\left(\frac{t}{\sqrt{LC}}\right) \times \frac{1}{\sqrt{LC}}\right]$$

$$= \frac{EC}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$= E\sqrt{\frac{C}{L}} \sin\left(\frac{t}{\sqrt{LC}}\right) //$$

23/03/2018

REVISION

Find Laplace transforms

1m 1) $e^{-2t}(2\cos 5t - 5\sin t)$ 2m 13) $\int_0^t \int_0^t e^{-t} \cos t \, dt \, dt$.

2m 2) $e^{-4t} \, t^{-5/2}$

1m 3) $e^{-2t} \sinh 4t$ 1m 14) $\int_0^\infty t e^{-2t} \sin 4t \, dt$.

1m 4) $\frac{\delta(t-2)}{t}$

1m 5) $2\delta(t-1) + 3\delta(t+3)$

2m 6) $t^2 \delta(t-3)$

2m 7) $t^2 u(t-3)$

2m 8) $f(t) = \begin{cases} e^t ; & 0<t<1 \\ 0 ; & t>1 \end{cases}$

4m 9) $t e^{-3t} \cos 2t$

3m 10) $\frac{\sin^2 t}{t}$

5m 11) $f(t) = t^2, \; 0<t<2$ & $f(t-2) = f(t)$

6m 12) $\frac{\sin^2 t}{t^2}$

1) $e^{-2t} 2\cos 5t - e^{-2t} 5\sin t$.