# UNIT 1 : DIFFERENTIAL CALCULUS

## Unit - I

## Differential Calculus.

**Problems on angle between radius vector and tangent,**

1. Find the angle between radius vector and tangent to the curve $r^2 \sec 2\theta = a^2$.

soln:-

given:-

$$r^2 \sec 2\theta = a^2$$

differentiating w.r.to $\theta$

$$2r\frac{dr}{d\theta} \sec 2\theta + r^2 \tan 2\theta \sec 2\theta \times 2 = 0$$

$$2r \cdot \sec 2\theta \frac{dr}{d\theta} = 2r^2 \tan 2\theta \sec 2\theta$$

$$\frac{dr}{d\theta} = -r \tan 2\theta$$

angle b/w radius vector & tangent,

$$\tan\phi = \frac{r}{dr/d\theta}$$

$$= \frac{-r}{r \cdot \tan 2\theta}$$

$$\tan\phi = -\cot 2\theta$$

$$\boxed{\phi = \frac{\pi}{2} + 2\theta}$$

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a) Find the angle between the radius vector and tangent and also find the slope of tangent to the curve $r^2 = a^2 \sin 2\theta$.

Soln:-

<u>given</u> :-

$$r^2 = a^2 \sin 2\theta$$

differentiating w r to $\theta$.

$$\cancel{2}r \cdot \frac{dr}{d\theta} = a^2 \cos 2\theta \cdot \cancel{2}$$

$$\frac{dr}{d\theta} = \frac{a^2 \cos 2\theta}{r}$$

$$\tan\phi = \frac{r}{dr/d\theta}$$

$$\tan\phi = \frac{r}{\frac{a^2 \cos 2\theta}{r}}$$

$$= \frac{r^2}{a^2 \cos 2\theta} \quad [\text{as } r^2 = a^2 \sin 2\theta]$$

$$= \frac{a^2 \sin 2\theta}{a^2 \cos 2\theta}$$

$$\tan\phi = \tan 2\theta$$

$$\boxed{\phi = 2\theta}$$

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At the point where $\theta = \frac{\pi}{12}$, $\boxed{\phi = \frac{\pi}{6}}$

Now,

$$\psi = \theta + \phi = \frac{\pi}{12} + \frac{\pi}{6}$$

$$= \frac{3\pi}{12}$$

$$= \frac{\pi}{4}$$

Slope of the tangent at $\theta = \frac{\pi}{12}$ is $\tan\psi$ which is $\tan\frac{\pi}{4}$.

$$\tan(\theta+\phi) = \tan\left(\frac{\pi}{4}\right) = 1$$

3. Find the angle between the radius vector and tangent to the curve $r = \sin\theta + \cos\theta$.

Soln:-

given:- $r = \sin\theta + \cos\theta$

differentiating with respect to $\theta$.

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

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$$\tan\phi = \frac{r}{dr/d\theta}$$

$$= \frac{r}{\cos\theta - \sin\theta}$$

$$= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

$\div$ each by $\cos\theta$

$$= \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$= \frac{\tan\left(\frac{\pi}{4}\right) + \tan\theta}{1 - \tan\left(\frac{\pi}{4}\right)\tan\theta} \qquad \left[\tan\frac{\pi}{4} = 1\right]$$

$$\left[\begin{array}{l} \text{As} \\ \tan(a+b) \\ = \frac{\tan(a) + \tan b}{1 - \tan a \tan b} \end{array}\right]$$

$$\tan\phi = \tan\left(\frac{\pi}{4} + \theta\right)$$

$$\boxed{\phi = \frac{\pi}{4} + \theta}$$

4. Find the angle between the radius vector and tangent to the curve $r^m \cos m\theta = a^m$

soln:- Given:-

$$r^m \cos m\theta = a^m$$

differentiating wrt $\theta$

$$m r^{m-1} \cdot \frac{dr}{d\theta} \cdot \cos m\theta + r^m(-\sin m\theta) \cdot m = 0$$

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$\cancel{m} r^{m-1} \frac{dr}{d\theta} \cdot \cos m\theta = r^m \sin m\theta \cancel{m}$

$$\frac{dr}{d\theta} = \frac{\sin m\theta}{\cos m\theta} \cdot r^{m-(m-1)}$$

$$\frac{dr}{d\theta} = \tan m\theta \cdot r$$

$$\tan\phi = \frac{r}{dr/d\theta}$$

$$= \frac{\cancel{r}}{\tan m\theta \cdot \cancel{r}}$$

$$\tan\phi = \cot m\theta$$

$$\tan\phi = \tan\left(\frac{\pi}{2} - m\theta\right)$$

$$\boxed{\phi = \frac{\pi}{2} - m\theta}$$

5. Show that the tangents to the polar curve $r = a(1+\cos\theta)$ at the point $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are respectively parallel and perpendi

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$$\frac{dr}{d\theta} = r \cdot (\cot n\theta)$$

$$\tan \phi = \frac{r}{dr/d\theta}$$

$$= \frac{\cancel{r}}{\cancel{r} \cdot \cot n\theta}$$

$$\tan \phi = \tan n\theta$$

$$\phi = n\theta$$

when $n = \theta$, $\phi = n\theta$.

angle made by the tangent to the curve is, $\psi = \theta + \phi$

$$= \theta + n\theta$$

$$\psi = \theta(n+1)$$

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# Angle between two Curves

1. Find the angle of intersection of curves $r = a\sec^2\left(\frac{\theta}{2}\right)$ ; $r = b\csc^2\left(\frac{\theta}{2}\right)$

Ans → Given functions $r = a\sec^2\left(\frac{\theta}{2}\right)$ and $r = b\csc^2\left(\frac{\theta}{2}\right)$

diff $r = a\sec^2\left(\frac{\theta}{2}\right)$ w.r.t $\theta$

$$\frac{dr}{d\theta} = a\sec^2\left(\frac{\theta}{2}\right)\tan\frac{\theta}{2}$$

$$\therefore \tan\phi_1 = \frac{r}{dr/d\theta} = \frac{a\sec^2\left(\frac{\theta}{2}\right)}{a\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)}$$

$$= \cot\theta/2$$

$$\therefore \tan\phi_1 = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

diff $r = b\csc^2\frac{\theta}{2}$ w.r.t $\theta$

$$\frac{dr}{d\theta} = -b\csc^2\frac{\theta}{2}\cot\frac{\theta}{2}$$

$$\therefore \tan\phi_2 = \frac{r}{dr/d\theta} = \frac{b\csc^2\frac{\theta}{2}}{-b\csc^2\frac{\theta}{2}\cot\frac{\theta}{2}}$$

$$= -\tan\frac{\theta}{2}$$

$$= \tan\left(-\frac{\theta}{2}\right)$$

$$\Rightarrow \phi_2 = -\frac{\theta}{2}$$

$$\therefore |\phi_1 - \phi_2| = \frac{\pi}{2} - \frac{\theta}{2} + \frac{\theta}{2}$$

$$= \frac{\pi}{2}$$

$\therefore$ The angle between the two curves is $\frac{\pi}{2}$

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Find the angle of intersection of curves $r = a\theta$, $r = \frac{a}{\theta}$

→ Given $r = a\theta$ — (I) and $r = \frac{a}{\theta}$ — (II)

from (I) & (II)

$$a\theta = \frac{a}{\theta}$$

$$\Rightarrow \theta = +1$$

now, diff $r = a\theta$ w.r.t $\theta$

$$\frac{dr}{d\theta} = a$$

$$\therefore \tan\phi_1 = \frac{r}{dr/d\theta} = \frac{a\theta}{a} = \theta \quad \text{— (III)}$$

diff $r = \frac{a}{\theta}$ w.r.t. $\theta$

$$\frac{dr}{d\theta} = \frac{-a}{\theta^2}$$

$$\therefore \tan\phi_2 = \frac{r}{dr/d\theta} = \frac{a/\theta}{-a/\theta^2} = -\theta \quad \text{— (IV)}$$

Putting $\theta = 1$ in (III) & (IV)

$\tan\phi_1 = 1$ & $\tan\phi_2 = -1$

as, $\tan\phi_1 . \tan\phi_2 = -1$, these curves are orthogonal.

$\therefore$ The angle between curves is $90°$ or $\frac{\pi}{2}$.

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Show that the curves $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$ cut each other orthogonally.

→ Given $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$

diff $r = a(1+\cos\theta)$ w.r.t $\theta$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\phi_1 = \frac{r}{dr/d\theta} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta} = \frac{2\cos^2\theta/2}{-2\sin\theta/2\cos\theta/2}$$

$$= -\cot\theta/2 \quad \text{— (1)}$$

$$= \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \phi_1 = \frac{\pi}{2}+\frac{\theta}{2}$$

diff $r = b(1-\sin\theta)$ w.r.t $\theta$

$$\frac{dr}{d\theta} = b\sin\theta$$

$$\therefore \tan\phi_2 = \frac{r}{dr/d\theta} = \frac{b(1-\cos\theta)}{b\sin\theta} = \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

$$= \tan\theta/2 \quad \text{— (11)}$$

As $\tan\phi_1 \tan\phi_2 = \tan\theta/2\,(-\cot\theta/2) = -1$

The given curves cut orthogonally.

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Show that the curves $\frac{2a}{r} = 1+\cos\theta$ & $\frac{2a}{r} = 1-\cos\theta$ cut orthogonally each other.

Ans → Given $\frac{2a}{r} = 1-\cos\theta$

taking log on both sides

$$\log 2a - \log r = \log(1-\cos\theta)$$

diff w.r.t $\theta$

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{dr}{d\theta} = \frac{-r\sin\theta}{1-\cos\theta}$$

$$\therefore \tan\phi_1 = \frac{r}{dr/d\theta} = -\frac{(1-\cos\theta)}{\sin\theta}$$

$$= \frac{-2\sin^2\theta/2 \cancel{\cos}}{2\sin\theta/2\cos\theta/2}$$

$$= -\tan\theta/2 \quad \text{—(I)}$$

Given $\frac{2a}{r} = 1+\cos\theta$

taking log on both sides

$$\log 2a - \log r = \log(1+\cos\theta)$$

diff w.r.t $\theta$

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$$

$$\frac{dr}{d\theta} = \frac{r\sin\theta}{1+\cos\theta}$$

$$\therefore \tan\phi_2 = \frac{r}{dr/d\theta} = \frac{1+\cos\theta}{\sin\theta}$$

$$= \cot\theta/2 \quad \text{—(II)}$$

from (I) & (II)

$$\tan\phi_1 \tan\phi_2 = -\tan\theta/2(\cot\theta/2) = -1$$

$\therefore$ The 2 curves cut each other orthogonally.

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# PROBLEMS ON PEDAL EQUATION

1. Find the pedal equation of the curve $2a = r(1+\cos\theta)$.

↳ Soln:

given curve: $r(1+\cos\theta) = 2a$.

Differentiating with respect to $\theta$:

$$\frac{dr}{d\theta}(1+\cos\theta) - r\sin\theta = 0$$

$$\Rightarrow \frac{dr}{d\theta}\{1+\cos\theta) = r\sin\theta$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi = \tan\frac{\theta}{2} = \cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\Rightarrow \underline{\phi = \frac{\pi}{2} - \frac{\theta}{2}}$$

Now, pedal equation:

$$p = r\sin\phi$$

$$\Rightarrow p = r\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\Rightarrow p = r\cos\frac{\theta}{2} \longrightarrow (1)$$

In the given curve:

$$r(1+\cos\theta) = 2a$$

$$\Rightarrow r.\cancel{2}\cos^2\frac{\theta}{2} = \cancel{2}a$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{a}{r} \quad \Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{a}{r}} \longrightarrow (2)$$

substituting (2) in (1):

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$$p = r\sqrt{\frac{a}{r}} = \sqrt{ar}$$

$\Rightarrow \boxed{p^2 = ar}$ → This is the required pedal equation.

2. Find the pedal equation of the curve $\frac{l}{r} = 1 + e\cos\theta$.

↳ Soln:

given curve: $r(1+e\cos\theta) = l$

Differentiating with respect to $\theta$:

$$\frac{dr}{d\theta}(1+e\cos\theta) + r[-e\sin\theta] = 0$$

$$\Rightarrow \frac{dr}{d\theta}(1+e\cos\theta) = re\sin\theta$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{e.\sin\theta}{1+e\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$$

pedal equation: $\frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right] = \frac{1}{r^2}\left[\frac{(1+e\cos\theta)^2 + e^2\sin^2\theta}{(1+e\cos\theta)^2}\right]$$

$\Rightarrow$ substituting $1+e\cos\theta = \frac{l}{r}$ in the denominator:

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+e^2\cos^2\theta + 2e\cos\theta + e^2\sin^2\theta}{\left(\frac{l}{r}\right)^2}\right]$$

$$= \frac{1}{\cancel{r^2}}\left[\frac{1+2e\cos\theta + e^2(\sin^2\theta+\cos^2\theta)}{\frac{l^2}{\cancel{r^2}}}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{l^2}\left[1+2e\cos\theta+e^2\right]$$

From given curve:

$$1+e\cos\theta = \frac{l}{r}$$

$$\Rightarrow e\cos\theta = \frac{l}{r}-1 = \frac{l-r}{r}$$

$$2e\cos\theta = \frac{2(l-r)}{r}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{l^2}\left[1+\frac{2(l-r)}{r}+e^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{l^2}\left[\frac{r+2l-2r+e^2r}{r}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{[2l-r+e^2r]}{l^2r} \quad \Rightarrow \boxed{\frac{1}{p^2} = \frac{e^2r-r+2l}{l^2r}}$$

multiplying and dividing by 'r' on RHS (to get given answer)

$$\boxed{\frac{1}{p^2} = \frac{e^2r^2-r^2+2lr}{l^2r^2}}$$

3. Find the pedal equation of the curve $r^m = a^m\sin m\theta + b^m\cos m\theta$

↳ Done in class notes. Please refer.

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4. Show that the pedal equation of the curve $r\cos\left[\frac{1}{a}\sqrt{a^2 - b^2}\right]\theta = \sqrt{a^2-b^2}$ is $p^2 = \frac{a^2b^2}{r^2+b^2}$.

↳ Done in class notes [same as : $\frac{r}{\sqrt{a^2-b^2}} = \sec\left(\frac{\sqrt{a^2-b^2}}{a}\right)\theta$]

5. Find the pedal equation of the curve $r\left[1-\sin\frac{\theta}{2}\right]^2 = a$.

↳ given equation : $r\left[1-\sin\frac{\theta}{2}\right]^2 = a$

Differentiating with respect to $\theta$ :

$$\frac{dr}{d\theta}\left[1-\sin\frac{\theta}{2}\right]^2 + r.\cancel{2}\left(1-\sin\frac{\theta}{2}\right)\left(-\cos\frac{\theta}{2}\right).\frac{1}{\cancel{2}} = 0$$

$$\Rightarrow \frac{dr}{d\theta}\left(1-\sin\frac{\theta}{2}\right)^2 = r\left(1-\sin\frac{\theta}{2}\right).\cos\frac{\theta}{2}$$

$$\Rightarrow \frac{1}{r}.\frac{dr}{d\theta} = \cancel{\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}}{}} \ \frac{\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)}{\left(1-\sin\frac{\theta}{2}\right)^{\cancel{2}}}$$

$$\Rightarrow \cot\phi = \frac{\cos\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$$

The pedal equation :

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right] \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{\cos^2\frac{\theta}{2}}{\left(1-\sin\frac{\theta}{2}\right)^2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2} \mp 2\sin\frac{\theta}{2}}{\frac{a}{r}}\right]$$

[from given curve : $\left(1-\sin\frac{\theta}{2}\right)^2 = \frac{a}{r}$]

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$$\Rightarrow \frac{1}{p^2} = \frac{1}{ar}\left[2 \mp 2\sin\frac{\theta}{2}\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{ar}\left[2\left(1 \mp \sin\frac{\theta}{2}\right)\right]$$

from given curve,

$$\left(1 - \sin\frac{\theta}{2}\right) = \sqrt{\frac{a}{r}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{ar}\left[2\sqrt{\frac{a}{r}}\right] \Rightarrow \frac{1}{p^2} = \frac{2}{\sqrt{a}\cdot r^{3/2}}$$

squaring on both sides:

$$\frac{1}{p^4} = \frac{4}{ar^3} \Rightarrow \boxed{4p^4 = ar^3}$$ → This is required pedal equation.

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# PROBLEMS ON RADIUS OF CURVATURE

## CARTESIAN FORM

1. Find the radius of curvature of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point $(x, y)$.

↳ Given curve : $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating with respect to $x$:

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-\frac{1}{3}} \cdot y_1 = 0$$

$$\Rightarrow y^{-1/3} \cdot y_1 = -x^{-1/3} \Rightarrow y_1 = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\Rightarrow \boxed{y_1 = -\left(\frac{y}{x}\right)^{1/3}}$$

Differentiating $y_1$ with respect to $x$:

$$y_2 = -\frac{x^{\frac{1}{3}} \cdot \frac{1}{3} \cdot y^{-\frac{2}{3}} \cdot y' - y^{\frac{1}{3}} \cdot \frac{1}{3} \cdot x^{-\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$= -\left( \frac{x^{\frac{1}{3}} \cdot y^{-\frac{2}{3}}\left(-\frac{y^{1/3}}{x^{1/3}}\right) - y^{\frac{1}{3}} \cdot x^{-\frac{2}{3}}}{3x^{\frac{2}{3}}} \right) \Rightarrow \boxed{y_2 = \frac{y^{-\frac{1}{3}} + y^{\frac{1}{3}} \cdot x^{-\frac{2}{3}}}{3x^{\frac{2}{3}}}}$$

We know, radius of curvature, $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

substituting $y_1$ & $y_2$ in above:

$$\rho = \frac{\left(1 + \frac{y^{2/3}}{x^{2/3}}\right)^{3/2}}{y^{-1/3} + y^{\frac{1}{3}} \cdot x^{-\frac{2}{3}}} \times 3x^{\frac{2}{3}}$$

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$$\Rightarrow \frac{(x^{2/3}+y^{2/3})^{3/2} \cdot 3x^{2/3}}{x\left[y^{-1/3}+y^{1/3}\cdot x^{-2/3}\right]}$$

but, from given curve, $x^{2/3}+y^{2/3}=a^{2/3}$

$$\Rightarrow \rho = \frac{3a}{x^{1/3}(y^{-1/3}+y^{1/3}\cdot x^{-2/3})} = \frac{3a}{y^{-1/3}\cdot x^{1/3}+y^{1/3}\cdot x^{-1/3}}$$

$$= \frac{3a}{y^{-1/3}x^{-1/3}\left[x^{2/3}+y^{2/3}\right]} = \frac{3a}{y^{-1/3}x^{-1/3}\cdot a^{2/3}}$$

$$\Rightarrow \boxed{\rho = 3(axy)^{1/3}}$$

2. Show that the radius of curvature of the curve $y=4\sin x-\sin 2x$ at the point $x=\frac{\pi}{2}$ is $\frac{5\sqrt{5}}{4}$.

↳ Given curve: $y=4\sin x-\sin 2x$

Differentiating $y$ with respect to $x$:

$$y_1 = 4\cos x - 2\cos 2x$$

at $x=\frac{\pi}{2}$:

$$y_1 = 4\cos\frac{\pi}{2} - 2\cos\pi \Rightarrow \boxed{y_1 = 2}$$

Differentiating $y_1$ with respect to $x$:

$$y_2 = -4\sin x + 4\sin 2x$$

at $x=\frac{\pi}{2}$

$$y = -4\sin\frac{\pi}{2} + 4\sin\pi \Rightarrow \boxed{y_2 = -4}$$

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Radius of curvature in cartesian form, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$\Rightarrow \rho = \frac{(1+4)^{3/2}}{-4}$$

radius of curvature is not negative, hence we neglect the negative sign.

hence, $\rho = \frac{5^{3/2}}{4} = \frac{5\sqrt{5}}{4} \Rightarrow \boxed{\rho = \frac{5\sqrt{5}}{4}}$

Hence Proved.

3. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that the radius of curvature at an end point of the major axis is equal to the semi-latus rectum.

↳ Note: In this ~~[illegible]~~ ellipse, end point of major axis is $(a,0)$ or $(-a,0)$

Length of latus rectum is $\frac{2b^2}{a} \Rightarrow$ semi-latus rectum is $\frac{b^2}{a}$.

Given curve: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiating with respect to $x$:

$$\frac{\cancel{2}x}{a^2} + \frac{\cancel{2}y}{b^2} \cdot y_1 = 0$$

$$\Rightarrow y_1 = -\frac{x}{y}\frac{b^2}{a^2}$$

at point $(a,0)$ $\quad y_1 \to \infty$

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Therefore we rotate the axes by 90°, and find $\frac{dx}{dy}$

at $(a,0)$,

$$\therefore \frac{dx}{dy} = x_1 = \frac{-ya^2}{xb^2} \qquad \boxed{x_1 = 0}$$

Differentiating $x_1$ with respect to $y$:

$$x_2 = -\frac{a^2}{b^2}\left[\frac{x - y.x_1}{x^2}\right]$$

at $(a,0)$,

$$x_2 = -\frac{a^2}{b^2}\left[\frac{a-0}{a^2}\right] \Rightarrow \boxed{x_2 = -\frac{a}{b^2}}$$

now, since we have differentiated with respect to $x$:,

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

$$\Rightarrow \rho = \frac{1}{x_2} \qquad [\because x_1 = 0]$$

$$\Rightarrow \boxed{\rho = \frac{b^2}{a}}$$ → This is equal to semi-latus rectum

Hence Proved.

4. For the curve $y = \frac{ax}{a+x}$, prove that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$

→ Given curve: $y = \frac{ax}{a+x}$

Differentiating above equation with respect to $x$:

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$$y_1 = \frac{(a+x)(a) - ax(1)}{(a+x)^2} = \frac{a^2 + ax - ax}{(a+x)^2} \Rightarrow \boxed{y_1 = \frac{a^2}{(a+x)^2}}$$

Differentiating $y_1$ with respect to $x$:

$$\boxed{y_2 = \frac{-2a^2}{(a+x)^3}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \frac{a^4}{(a+x)^4}\right)^{3/2}}{\frac{2a^2}{(a+x)^3}}$$ [ '-' is neglected]

$$= \frac{\left[(a+x)^4 + a^4\right]^{3/2}}{2a^2(a+x)^3}$$

$$2\rho = \frac{\left[(a+x)^4 + a^4\right]^{3/2}}{a^2(a+x)^3}$$

now, $$\left(\frac{2\rho}{a}\right)^{2/3} = \left\{\frac{\left[(a+x)^4 + a^4\right]^{3/2}}{a^3(a+x)^3}\right\}^{2/3}$$

$$= \frac{(a+x)^4 + a^4}{a^2(a+x)^2} = \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2} \longrightarrow (1)$$

consider $y = \frac{ax}{a+x}$ $\quad \frac{y}{x} = \frac{a}{a+x}$

hence we can rewrite (1):

$$\boxed{\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2}$$

Hence Proved.

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Show that the radius of curvature ~~of the curve~~, at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $\rho=\frac{(CD)^3}{ab}$, where CD is semi-diameter conjugate to CP.

→ Two diameters of an ellipse are said to be conjugate if each bisects a chord parallel to the other.

Consider a point P on the ellipse, it is $(a\cos\theta, b\sin\theta)$

[ $\theta \to$ parameter ]

Then, if CP & CD are conjugate diameters,

D is $\left(a\cos\left(\theta+\frac{\pi}{2}\right), b\sin\left(\theta+\frac{\pi}{2}\right)\right)$

that is, D is $(-a\sin\theta, b\cos\theta)$.

Also C (0,0) is the centre of the ellipse.

$\therefore$ $CD=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ --- (I) (using distance formula)

at P, $x=a\cos\theta$
$y=b\sin\theta$

We can proceed by finding $\rho$ in parametric form, which will be less complicated than cartesian form.

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$$\therefore \quad x_1 = \frac{dx}{d\theta} = -a\sin\theta \longrightarrow (1)$$

$$x_2 = \frac{d^2x}{d\theta^2} = -a\cos\theta \longrightarrow (2)$$

$$y_1 = \frac{dy}{d\theta} = b\cos\theta \longrightarrow (3)$$

$$y_2 = \frac{d^2y}{d\theta^2} = -b\sin\theta \longrightarrow (4)$$

In parametric form,

$$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1y_2 - x_2y_1}$$

$$\Rightarrow \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

we know that $CD = (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}$ from (I):

hence,

$$\boxed{\rho = \frac{CD^3}{ab}} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

Hence Proved.

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## POLAR FORM

1. Show that the radius of curvature of the curve $r^n = a^n \sin n\theta$.

↳ given curve, $r^n = a^n \sin n\theta$

Here, it is easier to find pedal equation, and then find $\rho$.

differentiating on both sides with respect to $\theta$:

$$\cancel{n} \cdot r^{n-1} \cdot \frac{dr}{d\theta} = \cancel{n} \cdot a^n \cos n\theta$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^{n-1}}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \cot\phi = \frac{a^n \cos n\theta}{r^n} = \frac{a^n \cos n\theta}{a^n \sin n\theta} \quad [\because r^n = a^n \sin n\theta]$$

$$\Rightarrow \cot\phi = \cot n\theta$$

$$\Rightarrow \boxed{\phi = n\theta}$$

pedal equation,

$$p = r\sin\phi$$
$$= r \sin n\theta$$

from given curve,

$$\sin n\theta = \frac{r^n}{a^n}$$

$$\Rightarrow \boxed{p = \frac{r^{n+1}}{a^n}}$$

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now, differentiating above equation w.r.t r:

$$\frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$$

$$\Rightarrow \frac{dr}{dp} = \frac{a^n}{(n+1)r^n}$$

we know that

$$\rho = r \cdot \frac{dr}{dp}$$

$$\Rightarrow \rho = \frac{r \cdot a^n}{(n+1)r^n} \Rightarrow \boxed{\rho = \frac{a^n}{(n+1) \cdot r^{n-1}}}$$

2. Obtain the pedal equation of the curve $r = a\sec 2\theta$. Hence find the radius of curvature at any point $(r, \theta)$ on the curve.

↳ Given $r = a\sec 2\theta$

Differentiating w.r.t θ:

$$\frac{dr}{d\theta} = 2a\sec 2\theta \cdot \tan 2\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{2a\sec 2\theta \cdot \tan 2\theta}{a\sec 2\theta} \quad [r = a\sec 2\theta]$$

$$\Rightarrow \cot\phi = 2\tan 2\theta$$

Pedal equation:

$$\frac{1}{p^2} = \frac{1}{r^2}[1 + \cot^2\phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + 4\tan^2 2\theta\right] = \frac{1}{r^2}\left[1 + 4(\sec^2 2\theta - 1)\right]$$

$$= \frac{1}{r^2}\left[4\sec^2 2\theta - 3\right]$$

from $r = a\sec 2\theta$,

$$\sec 2\theta = \frac{r}{a} \Rightarrow \sec^2 2\theta = \frac{r^2}{a^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{4r^2}{a^2} - 3\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{4}{a^2} - \frac{3}{r^2} \cdots (1) = \frac{4r^2 - 3a^2}{a^2 r^2}$$

$$\Rightarrow \boxed{p^2(4r^2 - 3a^2) = a^2 r^2}$$ → This is pedal equation

Diff (1) w.r.t p:

$$-\frac{2}{p^3} = -\frac{6}{r^3}\cdot\frac{dr}{dp} \Rightarrow \frac{1}{p^3} = \frac{3}{r^3}\cdot\frac{dr}{dp} \Rightarrow r\cdot\frac{dr}{dp} = \frac{r^4}{3p^3}$$

$$\Rightarrow \boxed{\rho = \frac{r^4}{3p^3}}$$

3. Show that for the curve $r = a\sin n\theta$, the radius of curvature at the pole is $\frac{na}{2}$.

↳ Given curve: $\boxed{r = a\sin n\theta}$

Differentiating above equation w.r.t $\theta$: [w.r.t ⇒ with respect to]

$$r_1 = \frac{dr}{d\theta} = an\cos n\theta. \Rightarrow \boxed{r_1 = an\cos n\theta}$$

Again differentiating w.r.t $\theta$:

$$r_2 = \frac{d^2 r}{d\theta^2} = -an^2\sin n\theta. \Rightarrow \boxed{r_2 = -an^2\sin n\theta}$$

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Radius of curvature, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

at the pole, P, $(r, \theta) = (0,0)$

$\Rightarrow r = a\sin(n0) = 0 \Rightarrow \boxed{r = 0}$ at P

$r_1 = an\cos(0) = an \Rightarrow \boxed{r_1 = an}$ at P

$r_2 = -an^2\sin(0) = 0 \Rightarrow \boxed{r_2 = 0}$ at P

substituting these values in $\rho$ equation:

$$\rho = \frac{(a^2n^2)^{3/2}}{2a^2n^2} = \frac{a^3n^3}{2a^2n^2} \Rightarrow \boxed{\rho = \frac{an}{2}} \text{ at } P(0,0). \text{ (Pole)}$$

Hence proved.

4. Show that for the curve $r(1-\cos\theta) = 2a$, $\rho^2$ is proportional to $r^3$ where $\rho$ is the radius of curvature at any point.

→ Given curve: $r(1-\cos\theta) = 2a$

$\Rightarrow \cancel{r = \frac{2a}{1-\cos\theta}} \Rightarrow r\left(2\sin^2\frac{\theta}{2}\right) = 2a$

$\Rightarrow r = a\,\mathrm{cosec}^2\frac{\theta}{2}$ $\quad\left[1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$

Differentiating with respect to $\theta$:

$$\frac{dr}{d\theta} = -\cancel{2}a\,\mathrm{cosec}^{\cancel{2}}\frac{\theta}{2}\cdot\mathrm{cosec}\frac{\theta}{2}\cdot\cot\frac{\theta}{2}\cdot\frac{1}{\cancel{2}}$$

Dividing by '$r$' on both sides:

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$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-a\,\mathrm{cosec}^2\frac{\theta}{2}\cdot\cot\theta/2}{r} = \frac{-a\,\mathrm{cosec}^2\frac{\theta}{2}\cdot\cot\frac{\theta}{2}}{a\,\mathrm{cosec}^2\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi = -\cot\frac{\theta}{2}$$

$$\Rightarrow \boxed{\phi = -\frac{\theta}{2}}$$

The pedal equation, $p = r\sin\phi$

$$\Rightarrow p = r\sin\left(-\frac{\theta}{2}\right)$$

$$\Rightarrow p = -r\sin\frac{\theta}{2}$$

given $r\left(\sin^2\frac{\theta}{2}\right) = a$

$$\Rightarrow \sin\frac{\theta}{2} = \sqrt{\frac{a}{r}}$$

$\Rightarrow p = -r\sqrt{\frac{a}{r}}$ $\Rightarrow \boxed{p = -\sqrt{ar}}$ → This is pedal equation.

we know that radius of curvature, $\rho = r.\frac{dr}{dp}$

$p = -\sqrt{ar}$ $\Rightarrow p^2 = -ar$

$$\Rightarrow r = -\frac{p^2}{a}$$

Differentiating w.r.t $p$:

$$\frac{dr}{dp} = -\frac{2p}{a}$$

~~I we will neglect the -ve sign, because $\rho$ is never -ve~~

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$\Rightarrow \rho = r \cdot \left(\frac{-2p}{a}\right)$

$\Rightarrow \rho = \frac{2}{a} \cdot r(-p)$

from pedal equation

$p = -\sqrt{ar}$

$\Rightarrow -p = \sqrt{ar}$

$\Rightarrow \rho = \frac{2}{a}(r\sqrt{ar})$

$\Rightarrow \rho = \frac{2}{\sqrt{a}} \cdot r^{3/2}$

squaring on both sides:

$\rho^2 = \frac{2}{a} \cdot r^3$

$\Rightarrow \boxed{\rho^2 \propto r^3}$

Hence Proved.

5. Show that for the curve $r = a\sin^3\frac{\theta}{3}$, $\rho^3$ is proportional to $r^2$.

↳ Given curve: $r = a\sin^3\frac{\theta}{3}$

diff w.r.t $\theta$:

$\frac{dr}{d\theta} = \cancel{3}a\sin^2\frac{\theta}{3}.\cos\frac{\theta}{3} \times \frac{1}{\cancel{3}}$

$\Rightarrow \frac{dr}{d\theta} = a\sin^2\frac{\theta}{3} \cdot \cos\frac{\theta}{3}$

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Dividing by 'r' on both sides:

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{a\sin^2\frac{\theta}{3}\cdot\cos\frac{\theta}{3}}{a\sin^3\frac{\theta}{3}} = \frac{\cos\frac{\theta}{3}}{\sin\frac{\theta}{3}}$$

$$\Rightarrow \cot\phi = \cot\frac{\theta}{3}$$

$$\Rightarrow \boxed{\phi = \frac{\theta}{3}}$$

pedal equation: $p = r\sin\phi$

$$= p = r\sin\frac{\theta}{3}$$

given $r = a\sin^3\frac{\theta}{3} \Rightarrow \sin\frac{\theta}{3} = \left(\frac{r}{a}\right)^{1/3}$

$$\Rightarrow p = r\cdot\frac{r^{1/3}}{a^{1/3}} \Rightarrow \boxed{p = \frac{r^{4/3}}{a^{1/3}}}$$ → This is pedal equation.

We know that $\rho = r\cdot\frac{dr}{dp}$

differentiating pedal equation w.r.t $\underline{r}$:

$$\frac{dp}{dr} = \frac{\frac{4}{3}\cdot r^{1/3}}{a^{1/3}}$$

$$\Rightarrow \frac{dr}{dp} = \frac{3a^{1/3}}{4r^{1/3}}$$

$$\therefore \rho = r\cdot\frac{dr}{dp} = r\cdot\frac{3a^{1/3}}{r^{1/3}\,4} = r^{2/3}\cdot\frac{3a^{1/3}}{4} = r^{2/3}(K)$$

cubing ~~squaring~~ : $\rho^3 = r^2\cdot K \Rightarrow \boxed{\rho^3 \propto r^2}$ Hence Proved.

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## PARAMETRIC FORM

1. Find the radius of curvature of the curve $x = e^t \cos t$, $y = e^t \sin t$.

↳ given curve: $x = e^t \cos t$, $y = e^t \sin t$.

$$x_1 = \frac{dx}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$$

$$\Rightarrow \boxed{x_1 = e^t \cos t - e^t \sin t}$$

$$x_2 = \frac{d^2x}{dt^2} = e^t(\cos t - \sin t) + e^t(-\sin t - \cos t)$$

$$= e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t$$

$$\Rightarrow \boxed{x_2 = -2e^t \sin t}$$

$$y_1 = \frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$$

$$\Rightarrow \boxed{y_1 = e^t \sin t + e^t \cos t}$$

$$y_2 = \frac{d^2y}{dt^2} = e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t$$

$$\Rightarrow \boxed{y_2 = 2e^t \cos t}$$

We know that $\rho = \dfrac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - x_2 y_1}$

$$\Rightarrow \rho = \frac{\left[e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2\right]^{3/2}}{e^t(\cos t - \sin t) \cdot 2e^t \cos t + e^t(\sin t + \cos t) \cdot 2e^t \sin t}$$

$$\Rightarrow \rho = \frac{\left[e^{2t}\left(\cos^2 t + \sin^2 t - \cancel{2\sin t \cos t} + \sin^2 t + \cos^2 t + \cancel{2\sin t \cos t}\right)\right]^{3/2}}{2e^{2t}\left[\cos^2 t - \sin t \cos t + \sin^2 t + \sin t \cos t\right]}$$

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$$\Rightarrow \rho = \frac{\left[e^{2t}(2)\right]^{3/2}}{2e^{2t}} \quad [\because \sin^2 t + \cos^2 t = 1]$$

$$\Rightarrow \rho = \left[e^{2t}(2)\right]^{1/2}$$

$$\Rightarrow \rho = e^t\sqrt{2} \Rightarrow \boxed{\rho = \sqrt{2}e^t}$$ → This is required radius of curvature.

2. Find the radius of curvature of the curve $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$.

↳ Given curve: $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$

$x_1 = \frac{dx}{dt} = a(1+\cos\theta)$ | $y_1 = \frac{dy}{dt} = a\sin\theta$

$x_2 = \frac{d^2x}{dt^2} = -a\sin\theta$ | $y_2 = \frac{d^2y}{dt^2} = a\cos\theta$

$$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1y_2 - x_2y_1}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a(1+\cos\theta)a\cos\theta + a^2\sin^2\theta}$$

$$= \frac{\left[a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta\right]^{3/2}}{a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}$$

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$$= \frac{(2a^2 + 2a^2\cos\theta)^{3/2}}{a^2 + a^2\cos\theta} = \frac{[2a^2(1+\cos\theta)]^{3/2}}{a^2(1+\cos\theta)}$$

$$= \frac{[2a^2 \cdot 2\cos^2\frac{\theta}{2}]^{3/2}}{a^2 \cdot 2\cos^2\frac{\theta}{2}}$$

$\left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$

$$= \frac{8a^3\cos^3\frac{\theta}{2}}{2a^2\cos^2\frac{\theta}{2}}$$

$$\Rightarrow \boxed{\rho = 4a\cos\left(\frac{\theta}{2}\right)}$$ → This is the required radius of curvature.

3. Find the radius of curvature of the curve $x = a\sin 2\theta(1+\cos 2\theta)$, $y = a\cos 2\theta(1-\cos 2\theta)$.

↳ soln:

given curve: $x = a\sin 2\theta\,(1+\cos 2\theta)$, $y = a\cos 2\theta\,(1-\cos 2\theta)$

$$x = a\sin 2\theta + a\sin 2\theta\cos 2\theta$$
$$= a\sin 2\theta + \frac{a}{2}\sin 4\theta$$

$$x_1 = 2a\cos 2\theta + 4\frac{a}{2}\cos 4\theta$$
$$= 2a\cos 2\theta + 2a\cos 4\theta$$
$$= 2a[\cos 2\theta + \cos 4\theta] = \underline{4a\cos 3\theta \cdot \cos\theta}$$

$\left[\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)\right]$

$$x_2 = -2a[2\sin 2\theta + 4\sin 4\theta]$$
$$= -4a[\sin 2\theta + 2\sin 4\theta]$$

(using unsimplified $x_1$ for further differentiation to avoid product rule).

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$$y = a\cos 2\theta(1-\cos 2\theta)$$
$$= a\cos 2\theta - a\cos^2 2\theta$$

$$y_1 = -a\cdot 2\cdot \sin 2\theta + 2a\cdot \cos 2\theta \cdot \sin 2\theta \cdot 2$$

$$\Rightarrow y_1 = 2a\cdot 2\sin 2\theta\cdot\cos 2\theta - 2a\sin 2\theta$$

$$= 2a\cdot \sin 4\theta - 2a\sin 2\theta$$

$$y_1 = 2a[\sin 4\theta - \sin 2\theta] = 4a\sin\theta\cdot\cos 3\theta$$

$\left[\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cdot\cos\left(\frac{C+D}{2}\right)\right]$

$$y_2 = 2a[4\cos 4\theta - 2\cos 2\theta]$$
$$= 4a[2\cos 4\theta - \cos 2\theta]$$

$$x_1 = 4a\cdot\cos 3\theta\cdot\cos\theta$$
$$y_1 = 4a\cos 3\theta\cdot\sin\theta$$
$$x_2 = -4a[\sin 2\theta + 2\sin 4\theta]$$
$$y_2 = 4a[2\cos 4\theta - \cos 2\theta]$$

$$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - x_2 y_1}$$

$$\Rightarrow \rho = \frac{(16a^2\cos^2 3\theta\cdot\cos^2\theta + 16a^2\cos^2 3\theta\cdot\sin^2\theta)^{3/2}}{4a\cdot\cos 3\theta\cdot\cos\theta\cdot 4a(2\cos 4\theta - \cos 2\theta) + 4a\cos 3\theta\cdot\sin\theta\cdot 4a(\sin 2\theta + 2\sin 4\theta)}$$

$$= \frac{[16a^2\cos^2 3\theta(\cos^2\theta + \sin^2\theta)]^{3/2}}{16a^2\cos 3\theta[2\cos\theta\cdot\cos 4\theta - \cos\theta\cdot\cos 2\theta + \sin\theta\cdot\sin 2\theta + 2\sin\theta\cdot\sin 4\theta]}$$

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$$= \frac{64a^3\cos^3 3\theta}{16a^2\cos 3\theta\left[2(\cos\theta\cdot\cos 4\theta + \sin\theta\cdot\sin 4\theta) - (\cos\theta\cdot\cos 2\theta - \sin\theta\sin 2\theta)\right]}$$

$$= \frac{64a^3\cos^3 3\theta}{16a^2\cos 3\theta\left[2\cos 3\theta - \cos 3\theta\right]}$$

$$= \frac{64a^3\cos^3 3\theta}{16a^2\cos^2 3\theta}$$

$\Rightarrow \boxed{\rho = 4a\cos 3\theta}$ → This is radius of curvature.

4. ~~Show that~~ Find the radius of curvature at any point t on the curve $x = a\left[\cos\theta + \log\left(\tan\frac{\theta}{2}\right)\right]$, $y = a\sin\theta$.

↳ Given curve: $x = a\left[\cos\theta + \log\left(\tan\frac{\theta}{2}\right)\right]$, $y = a\sin\theta$

$$x_1 = a\left[-\sin\theta + \frac{1}{\tan\frac{\theta}{2}}\cdot\sec^2\frac{\theta}{2}\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin\theta + \frac{1}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right] = a\left[\frac{1}{\sin\theta} - \sin\theta\right]$$

$$= a\left[\frac{1-\sin^2\theta}{\sin\theta}\right] = a\left[\frac{\cos^2\theta}{\sin\theta}\right]$$

$\Rightarrow \cancel{\boxed{x_1 = a\cot\theta}}$ $\Rightarrow \boxed{x_1 = a\cot\theta\cdot\cos\theta}$

$\cancel{\boxed{x_2 = -a\,\mathrm{cosec}^2\theta}}$ $x_2 = \cot\theta(-\sin\theta) + \cos\theta(-\mathrm{cosec}^2\theta)$

$$\boxed{x_2 = a(-\cos\theta - \cot\theta\cdot\mathrm{cosec}\,\theta)}$$

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$$\boxed{y_1 = a\cos\theta}$$

$$\boxed{y_2 = -a\sin\theta}$$

$$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - x_2 y_1}$$

$$= \frac{\left(a^2\cot^2\theta\cdot\cos^2\theta + a^2\cos^2\theta\right)^{3/2}}{-a\cot\theta\cdot\text{cosec}\theta)\,a\sin\theta + a(\cos\theta + \cot\theta\cdot\text{cosec}\theta)\cdot a\cos\theta}$$

$$= \frac{\left[a^2\cos^2\theta\,(1+\cot^2\theta)\right]^{3/2}}{a^2\cos^2\theta + a^2\cos\theta\cdot\cot\theta\cdot\text{cosec}\theta - a^2\sin\theta\cdot\cot\theta\cdot\cos\theta}$$

$$= \frac{a^3\cos^3\theta\cdot\text{cosec}^3\theta}{\cancel{a^2\cos^2\theta} + a^2\cot^2\theta - \cancel{a^2\cos^2\theta}} = \frac{a^3\cos^3\theta\cdot\text{cosec}^3\theta}{a^2\cot^2\theta}$$

$$= \frac{a^3\cot^3\theta}{a^2\cot^2\theta}$$

$$\Rightarrow \rho = a\cot\theta$$

at point $t$:

$$\boxed{\rho = a\cot t} \rightarrow \text{This is radius of curvature}$$

5. Prove that the radius of curvature of a circle, $x^2+y^2=a^2$ is a constant.
[Hint : take $x=a\cos t$, $y=a\sin t$; $0\leq t\leq\pi$]

Solution:

given, $x^2+y^2=a^2$

let $x=a\cos t$

$y=a\sin t$

$x_1=-a\sin t$ $\quad$ $y_1=a\cos t$

$x_2=-a\cos t$ $\quad$ $y_2=-a\sin t$

we know that $\rho=\dfrac{(x_1^2+y_1^2)^{3/2}}{x_1y_2-x_2y_1}$

$$\Rightarrow \rho=\frac{(a^2\sin^2t+a^2\cos^2t)^{3/2}}{a^2\sin^2t+a^2\cos^2t}$$

$\Rightarrow \rho=\dfrac{a^3}{a^2}$ $\Rightarrow$ $\boxed{\rho=a}$

$\Rightarrow \boxed{\rho=\text{constant}}$

Since $a$ is a constant.

Hence Proved.

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# Pedal form

1. Write p-r equation of the polar curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ and hence find the radius of curvature at any point on the curve.

soln:-

$$\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$$

diff wrto $r$

$$\frac{d\theta}{dr} = \frac{1}{a} \times \frac{1}{\cancel{2}\sqrt{r^2-a^2}} \times \cancel{2}r - \frac{(-1)}{\sqrt{1-\frac{a^2}{r^2}}} \times \frac{-1}{r^2}$$

$$= \frac{r}{a} \times \frac{1}{\sqrt{r^2-a^2}} - \frac{\cancel{r}}{\sqrt{r^2-a^2}} \times \frac{a}{r^{\cancel{2}}}$$

$$= \frac{r}{a} \times \frac{1}{\sqrt{r^2-a^2}} - \frac{a}{r\sqrt{r^2-a^2}}$$

$$= \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r}{a} - \frac{a}{r}\right]$$

$$= \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r^2-a^2}{ar}\right]$$

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$$\frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\cot\phi = \frac{1}{r} \times \frac{dr}{d\theta}$$

$$= \frac{1}{\cancel{r}} \times \frac{a\cancel{r}}{\sqrt{r^2-a^2}}$$

$$\cot\phi = \frac{a}{\sqrt{r^2-a^2}}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \cot^2\phi\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{a^2}{r^2-a^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{\cancel{r^2}}\left[\frac{\cancel{r^2} - \cancel{a^2} + \cancel{a^2}}{r^2-a^2}\right]$$

$$p^2 = r^2 - a^2$$

$$p = \sqrt{r^2-a^2}$$

diff wr to $r$

$$\frac{dp}{dr} = \frac{1}{\cancel{2}\sqrt{r^2-a^2}} \times \cancel{2}r$$

$$= \frac{r}{\sqrt{r^2-a^2}}$$

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$$\rho = r \cdot \frac{dr}{dp}$$

$$= \cancel{r} \times \frac{\sqrt{r^2 - a^2}}{\cancel{r}}$$

$$\rho = \sqrt{r^2 - a^2}$$

(3) Find the radius of curvature of the curve $r^3 = 2ap^2$.

soln:-

given:-

$$r^3 = 2ap^2$$

$$p^2 = \frac{r^3}{2a}$$

$$p = \sqrt{\frac{r^3}{2a}}$$

$$= \frac{r\sqrt{r}}{\sqrt{2a}}$$

$$p = \frac{r^{3/2}}{\sqrt{2a}}$$

diff wr to r

$$\frac{dp}{dr} = \frac{3}{2} \frac{r^{1/2}}{\sqrt{2a}}$$

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$$\rho = r \cdot \frac{dr}{dp}$$

$$= r \times \frac{2}{3}\sqrt{2a} \times \frac{1}{\sqrt{r}}$$

$$= \frac{2}{3}\sqrt{2a} \times \sqrt{r}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

4. Find the radius of curvature of the curve $r^2 + 3p^2 = a^2$.

soln:-

$$r^2 + 3p^2 = a^2$$

~~diff wr to~~

$$P = \sqrt{\frac{a^2 - r^2}{3}}$$

diff wr to $r$

$$\frac{dp}{dr} = \frac{1}{\sqrt{3}}\left[\frac{1}{2\sqrt{a^2 - r^2}} \times 2r\right]$$

$$\frac{dp}{dr} = -\frac{r}{\sqrt{3}\sqrt{a^2 - r^2}}$$

$$r^2 + 3p^2 = a^2$$

$$3p^2 = a^2 - r^2$$

$$p = \sqrt{\frac{a^2 - r^2}{3}}$$

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diff wr to r

$$\frac{dp}{dr} = \frac{1}{\sqrt{3}} \times \frac{1}{\cancel{2}\sqrt{a^2-r^2}} \times -\cancel{2}r$$

$$\frac{dp}{dr} = -\frac{r}{\sqrt{3}\sqrt{a^2-r^2}}$$

$$\rho = r \cdot \frac{dr}{dp}$$

$$= \cancel{r} \times -\frac{\sqrt{3}\sqrt{a^2-r^2}}{\cancel{r}}$$

$$\rho = -\sqrt{3}\,(a^2-r^2)^{1/2}$$

5. Find the radius of curvature of the curve $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$

~~Soln~~
Soln:-

given

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

$$\frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}}$$

$$p = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}}}$$

diff wr to $r$

$$\frac{dp}{dr} = -\frac{1}{\left(\sqrt{\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}}\right)^2} \times \frac{-2r}{a^2b^2}$$

$\cancel{= 2}$

$$\frac{dp}{dr} = -\frac{1}{\left(\sqrt{\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}}\right)^2} \times \frac{1}{\cancel{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}}} \times \frac{-\cancel{2}r}{a^2b^2}$$

$$= \frac{r}{\left(\sqrt{\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}}\right)^3} \cdot \frac{1}{a^2b^2}$$

$$\int = r \cdot \frac{dr}{dp}$$

$$= \cancel{r} \times \frac{a^2b^2}{\cancel{r}}\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}\right)^{3/2}$$

$= \cancel{a^2b}$

$$\int = a^2b^2\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}\right)^{3/2}$$

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(2) If $\rho_1$ & $\rho_2$ are the radii of curvature at the extremities of a polar chord of the polar curve $r = a(1+\cos\theta)$, prove that $\rho_1^2 + \rho_2^2$

$$= \frac{16a^2}{9}.$$

Soln:- given:-

$$r = a(1+\cos\theta)$$

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# Taylor's & Maclaurin's Series

1. Expand $f(x) = \log(\sin x)$ in powers of $(x-2)$

↳ Soln.

Given function: $f(x) = \log(\sin x)$

Taylor's series: $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \infty$

comparing $(x-2)$ with the above series,

$$\boxed{a = 2}$$

$\Rightarrow f(a) = \log[\sin 2] = \underline{\underline{-3.36}}$

Differentiating $f(x)$ with respect to $x$:

$f'(x) = \frac{1}{\sin x} \cdot \cos x = \cot x \quad \Rightarrow f'(2) = \cot 2 = \underline{\underline{28.64}}$

$f''(x) = -\operatorname{cosec}^2 x \quad \Rightarrow f''(2) = -\operatorname{cosec}^2(2) = -821.03$

$$\boxed{\therefore f(x) = -3.36 + \frac{(x-2)}{1!}(28.64) - \frac{(x-2)^2}{2!}(821.03) + \dots \infty}$$

2. Find the Taylor's series expansion of the function $f(x) = \tan^{-1} x$ at $a = 1$.

↳

Given function: $f(x) = \tan^{-1} x$

Let $y = f(x) = \tan^{-1} x$.

Given $a = 1$.

$\therefore y(1) = \tan^{-1}(1) = \frac{\pi}{4} \quad \Rightarrow \boxed{y(1) = \frac{\pi}{4}}$

Differentiating $y$, with respect to $x$:

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$y_1 = \frac{1}{1+x^2}$ ... [where $y_1 \Rightarrow \frac{dy}{dx}$]

$\Rightarrow y_1(1) = \frac{1}{1+1} \Rightarrow \boxed{y_1(1) = \frac{1}{2}}$

Cross multiplying,

$(1+x^2)y_1 = 1$

Differentiating with respect to $x$, 'n' times using Lebnitz's theorem:

$$\frac{d^n}{dx^n}(\underset{u}{\underbrace{(1+x^2)}})(\underset{v}{\underbrace{y_1}}) = \frac{d^n}{dx^n}(1)$$

$$\Rightarrow (1+x^2)y_{n+1} + n.2x.y_n + \frac{n(n-1)}{\cancel{2}}.\cancel{2}.y_{n-1} = 0$$

$$\Rightarrow (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$$

at $x = 1$:

$2y_{n+1} + 2ny_n = 0$ [$\because n(n-1)y_{n-1} = 0$]

$\Rightarrow \boxed{y_{n+1} = -ny_n}$

here, $n = 0, 1, 2, 3, \ldots$

at $n = 1$,

$y_2(1) = -1(y_1) = -1(\frac{1}{2}) \Rightarrow \boxed{y_2(1) = -\frac{1}{2}}$

at $n = 2$,

$y_3(1) = -2[y_2(1)] = -2(-\frac{1}{2}) \Rightarrow \boxed{y_3(1) = 1}$

at $n = 3$,

$y_4(1) = -3[y_3(1)] = -3(1) \Rightarrow \boxed{y_4(1) = -3}$

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substituting these values in:

$$f(x) = f(a) + \frac{(x-a)}{1!} \cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

here, $a = 1$

$f(x) = y$

$f'(a) = y_1(1)$, etc.

Therefore,

$$y = \frac{\pi}{4} + \frac{(x-1)}{1!} \cdot \frac{1}{2} + \frac{(x-1)^2}{2!} \cdot \left(\frac{-1}{2}\right) + \frac{(x-1)^3}{3!}(1) + \frac{(x-1)^4}{4!}(-3) + \dots \infty$$

$$\Rightarrow \boxed{y = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \dots \infty}$$

3. Obtain the Maclaurin's series expansion of the function $f(x) = \sqrt{1+\sin 2x}$ up to fourth degree terms.

→ Given function, $f(x) = \sqrt{1+\sin 2x}$

since it is a Maclaurin's series, we know that $a = 0$.

let $y = f(x) = \sqrt{1+\sin 2x}$

$\Rightarrow y(0) = \sqrt{1} \Rightarrow \boxed{y(0) = 1}$

rewriting $f(x)$:

$f(x) = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}$ $[\because \sin^2 x + \cos^2 x = 1]$

$\Rightarrow f(x) = \sqrt{(\sin x + \cos x)^2}$

$\Rightarrow f(x) = \sin x + \cos x \longrightarrow (1)$

Differentiating (1) with respect to $x$:

$f'(x) = \cos x - \sin x \Rightarrow \boxed{f'(0) = 1}$

again differentiating with respect to $x$:

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$f''(x) = -\sin x - \cos x \Rightarrow \boxed{f''(0) = -1}$

Differentiating with respect to x:

$f'''(x) = -\cos x + \sin x \Rightarrow \boxed{f'''(0) = -1}$

Differentiating this with respect to x:

$f^{(4)}(x) = \sin x + \cos x \Rightarrow \boxed{f^{(4)}(0) = 1}$

The Maclaurin's series expansion is:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots \infty$$

substituting the values,

$$f(x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1) + \dots \infty$$

$$\Rightarrow \boxed{f(x) = x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \infty}$$

4. Obtain the Maclaurin's series expansion of the function $f(x) = e^{a\sin^{-1}(x)}$.

↳ DONE IN CLASS NOTES - REFER.

5. Obtain the Maclaurin's series expansion of the function $f(x) = \log(1+e^x)$.

↳ Given function, $f(x) = \log(1+e^x)$

let $y = \log(1+e^x)$

$\boxed{y(0) = \log 2}$

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Differentiating $y$ with respect to $x$:

$$y_1 = \frac{e^x}{1+e^x} \qquad y_1(0) = \frac{1}{1+1} \Rightarrow \boxed{y_1(0) = \frac{1}{2}}$$

Differentiating $y_1$ with respect to $x$:

$$y_2 = \frac{(1+e^x)(e^x) - e^x(e^x)}{(1+e^x)^2} = \frac{1+e^{2x}-e^{2x}}{(1+e^x)^2} = \frac{1}{(1+e^x)^2} = (1+e^x)^{-2} \Rightarrow \boxed{y_2(0) = \frac{1}{4}}$$

Differentiating $y_2$ with respect to $x$:

$$y_3 = -2(1+e^x)^{-3} \cdot e^x = \frac{-2e^x}{(1+e^x)^3} \qquad y_3(0) = \frac{-2(1)}{(1+1)^3} = \frac{-2}{2^3} = -\frac{1}{4}$$

$$\Rightarrow \boxed{y_3(0) = -\frac{1}{4}}$$

The Maclaurins series is:

$$f(x) = y = y(0) + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots \infty$$

$$\Rightarrow y = \log 2 + \frac{x}{1!} \times \frac{1}{2} + \frac{x^2}{2!} \times \frac{1}{4} + \frac{x^3}{3!}\left(-\frac{1}{4}\right) + \cdots \infty$$

$$\Rightarrow \boxed{y = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{24} + \cdots \infty}$$

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