HARMONIC ANALYSIS

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In many practical problems, the function $f(x)$ is specified either graphically or through a table of corresponding values of x & $f(x)$. In such problems, the integrals cannot be evaluated by usual methods of integration to find a_0, a_n & b_n; They can be evaluated using the following property of definite integrals.

Property: If M is the mean value of $\phi(x)$ over the interval (a,b), then

$$M = \frac{1}{(b-a)} \int_a^b \phi(x)\,dx$$

Now, using this property, alternative formulae for a_0, a_n & b_n are

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = 2\,\frac{\sum f(x)}{N} = \frac{2}{N}\sum f(x)$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{N}\sum f(x)\cos\left(\frac{n\pi x}{l}\right)$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{N}\sum f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Thus

$$a_0 = \frac{2}{N} \sum f(x)$$

$$a_n = \frac{2}{N} \sum f(x) \cos\left(\frac{n\pi x}{l}\right)$$

$$a_n = \frac{2}{N} \sum f(x) \sin\left(\frac{n\pi x}{l}\right)$$

NOTE :- where N is the number of subintervals

$\therefore$ The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

put $\frac{\pi x}{l} = \theta$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

$$f(x) = \frac{a_0}{2} + \left[a_1 \cos\theta + b_1 \sin\theta\right] + \left[a_2 \cos(2\theta) + b_2 \sin(2\theta)\right] + \dots\dots$$

Here

$[a_1 \cos\theta + b_1 \sin\theta] \rightarrow 1^{st}$ harmonic

$[a_2 \cos(2\theta) + b_2 \sin(2\theta)] \rightarrow 2^{nd}$ harmonic & so on

$\sqrt{a_1^2 + b_1^2} \rightarrow$ amplitude of 1^{st} harmonic

$\sqrt{a_2^2 + b_2^2} \rightarrow$ amplitude of 2^{nd} harmonic

$\frac{a_0}{2} \rightarrow$ Constant term

Express y as a Fourier series upto first harmonics.

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	7.9	7.2	3.6	0.5	0.9	6.8	7.9

Ans: By data $f(x)$ is defined in $(0, 2\pi)$

period $= 2l \Rightarrow 2l = 2\pi \Rightarrow l = \pi$, $\frac{n\pi x}{l} = nx$

& $N = 6$

$$\therefore f(x) = \frac{a_0}{2} + \sum [a_n \cos(nx) + b_n \sin(nx)]$$

Fourier series upto 1^{st} harmonic is

$$f(x) = \frac{a_0}{2} + [a_1 \cos x + b_1 \sin x]$$

Where $a_0 = \frac{2}{N}\sum f(x)$, $a_1 = \frac{2}{N}\sum f(x)\cos x$,

$b_1 = \frac{2}{N}\sum f(x)\sin x$. Now construct the foll table

x°	$f(x)$	$\sin x$	$\cos x$	$f(x)\sin x$	$f(x)\cos x$
0	7.9	0	1.0	0.0	7.9
60	7.2	0.866	0.5	6.2352	3.6
120	3.6	0.866	−0.5	3.1176	−1.8
180	0.5	0.0	−1.0	0.0	−0.5
240	0.9	−0.866	−0.5	−0.7794	−0.45
300	6.8	−0.866	0.5	−5.8888	3.4
	26.9			2.6846	12.15

* Since $f(0) = f(2\pi)$ Neglect any one of these in the table

$\therefore a_0 = \frac{2}{6} \times 26.9 = 8.9667$, $a_1 = \frac{2}{6}(12.15) = 4.05$, $b_1 = \frac{2}{6}(2.6846) = 0.8949$

$\therefore$ Fourier series upto 1^{st} harmonics is

$$f(x) = (8.9667)/2 + [(4.05)\cos x + (0.8949)\sin x]$$

2) Find the first three coefficients of cosine & two coefficients of sine terms in the Fourier series for the following data

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Ans: $f(x)$ is defined in the $(0,6)$

[* It is understood that $f(0)=f(6)=9$]

$2l=6 \Rightarrow l=3$, $\frac{n\pi x}{l}=\frac{n\pi x}{3}$, $N=6$

Here $n=0,1,2$

$\therefore$ The Fourier series upto 2^{nd} harmonics is

$$f(x)=\frac{a_0}{2}+\left[a_1\cos\left(\frac{\pi x}{3}\right)+b_1\sin\left(\frac{\pi x}{3}\right)\right]$$
$$+\left[a_2\cos\left(\frac{2\pi x}{3}\right)+b_2\sin\left(\frac{2\pi x}{3}\right)\right]$$

put $\frac{\pi x}{3}=\theta$,

$$f(x)=\frac{a_0}{2}+\left[a_1\cos\theta+b_1\sin\theta\right]+\left[a_2\cos(2\theta)+b_2\sin(2\theta)\right]$$

Where $a_0=\frac{2}{N}\sum f(x)$

$$a_1=\frac{2}{N}\sum f(x)\cos\theta$$

$$b_1=\frac{2}{N}\sum f(x)\sin\theta$$

$$a_2=\frac{2}{N}\sum f(x)\cos(2\theta)$$

$$b_2=\frac{2}{N}\sum f(x)\sin(2\theta)$$

x	$\frac{\pi x}{3}=\theta$	$\sin\theta$	$\cos\theta$	$f(x)$	$f(x)\sin\theta$	$f(x)\cos\theta$	$f(x)\sin(2\theta)$	$f(x)\cos(2\theta)$
0	0	0	1	9	0	9		
1	$\pi/3$	$\sqrt{3}/2$	$1/2$	18	$9\sqrt{3}$	9		
2	$2\pi/3$	$\sqrt{3}/2$	$-1/2$	24	$12\sqrt{3}$	-12		
3	π	0	-1	28	0	-28	HOME WORK	
4	$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	26	$-13\sqrt{3}$	-13		
5	$5\pi/3$	$-\sqrt{3}/2$	$1/2$	20	$-10\sqrt{3}$	10		
				125	$-2\sqrt{3}$	-25		

$$a_0 = \frac{2}{6} \times 125 = 41.6667$$

$$a_1 = \frac{2}{6} \times -25 = -8.3334$$

$$b_1 = \frac{2}{6} \times -2\sqrt{3} = -1.1548$$

$a_2 =$ HOME WORK

$b_2 =$ HOME WORK

Complex Fourier Series

The Complex Fourier series of a periodic function $f(x)$ in the interval $(-l, l)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\left(\frac{in\pi x}{l}\right)}$$

Where $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\left(\frac{in\pi x}{l}\right)} dx$$

(1) Find the Complex form of Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$

Ans: Here $2l = 1-(-1) = 2 \Rightarrow l = 1$, $\frac{n\pi x}{l} = n\pi x$

$\therefore$ Complex Fourier series for $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}, \text{ where } C_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x} dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x} dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} = \frac{1}{-2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= \frac{1}{-2(1+in\pi)}\left[e^{-1} e^{-in\pi} - e^{1} e^{in\pi}\right]$$

$$= \frac{1}{-2(1+in\pi)}\left[e^{-1}\{\cos n\pi - i\cancel{\sin(n\pi)}^{0}\} - e\{\cos(n\pi) + i\cancel{\sin n\pi}^{0}\}\right]$$

$$= \frac{1}{2(1+in\pi)}\left[(-1)^n\{e^{1} - e^{-1}\}\right] = \frac{(-1)^n[1-in\pi]\sinh 1}{1+n^2\pi^2}$$

$$\therefore \boxed{e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n[1-in\pi]\sinh 1}{1+n^2\pi^2} e^{in\pi x}}$$

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