Design and Analysis of Algorithms

Transitive Closure (Warshall's Algorith

Reetinder Sidhu

Department of Computer Science and Engineering

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Transitive Closure (Warshall's Algorithm)

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PES UNIVERSITY ONLINE

- Dynamic Programming
  - Computing a Binomial Coefficient
  - The Knapsack Problem
  - Memory Functions
  - **Warshall's and Floyd's Algorithms**
- Limitations of Algorithmic Power
  - Lower-Bound Arguments
  - Decision Trees
  - P, NP, and NP-Complete, NP-Hard Problems
- Coping with the Limitations
  - Backtracking
  - Branch-and-Bound. Architecture (microprocessor instruction set)

Concepts covered

- Transitive Closure (Warshall's Algorithm)
  - Motivation
  - Algorithm
  - Example

Transitive Closure

- Computes the transitive closure of a relation
- Alternatively: existence of all nontrivial paths in a digraph (directed graph)
- Example of transitive closure:

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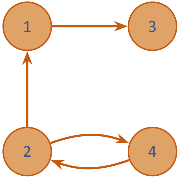

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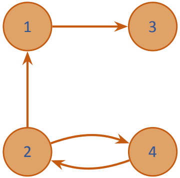

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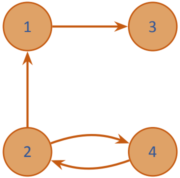

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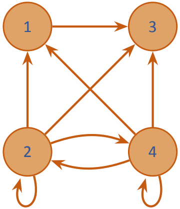

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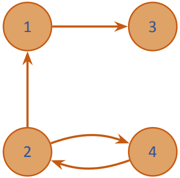

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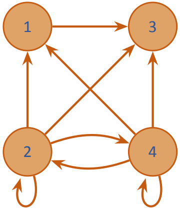

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Warshall's Algorithm

- Constructs transitive closure T as the last matrix in the sequence of $n \times n$ matrices $R^{(0)}, \ldots, R^{(k)}, \ldots, R^{(n)}$ where $R^{(k)}[i,j] = 1$ iff there is nontrivial path from i to j with only first k vertices allowed as intermediate vertices
  - $R^{(0)} = A$ (adjacency matrix), $R^{(n)} = T$ (transitive closure)
- On the k^{th} iteration, the algorithm computes $R^{(k)}$

$$R^{(k)}[i,j] = \begin{cases} 1 & \text{if path from } i \text{ to } k \text{ and } k \text{ to } j \text{, i.e., } R^{(k-1)}[i,k] = R^{(k-1)}[k,j] = 1 \\ R^{(k-1)}[i,j] & \text{otherwise} \end{cases}$$

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$$R^{(k)}[i,j] = R^{(k-1)}[i,j] \ \textbf{or} \ (R^{(k-1)}[i,k] \ \textbf{and} \ R^{(k-1)}[k,j])$$

Transitive Closure (Warshall's Algorithm)

Algorithm

Transitive Closure (Warshall's Algorithm)

procedure WARSHALL(()$A[1 \ldots n, 1 \ldots n]$)
    ▷ **Input:** The adjacency matrix A of a digraph with n vertices
    ▷ **Output:** The transitive closure of the digraph
    $R^{(0)} \leftarrow A$
    for $k \leftarrow 1$ **to** n **do**
        for $i \leftarrow 1$ **to** n **do**
            for $j \leftarrow 1$ **to** n **do**
                $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **or** $(R^{(k-1)}[i,k]$ **and** $R^{(k-1)}[k,j])$;
    return R

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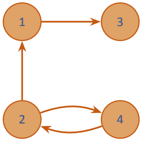

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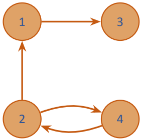

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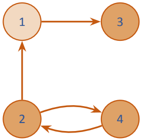

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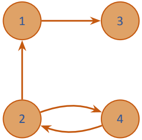

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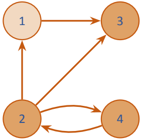

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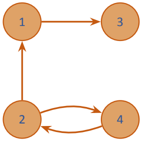

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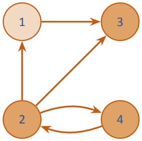

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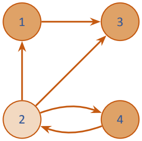

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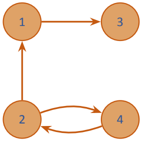

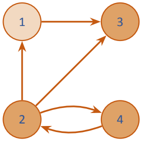

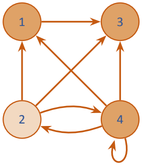

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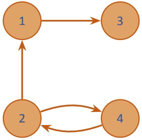

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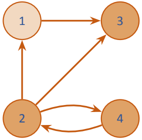

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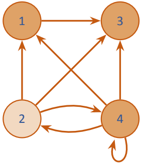

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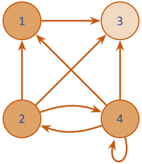

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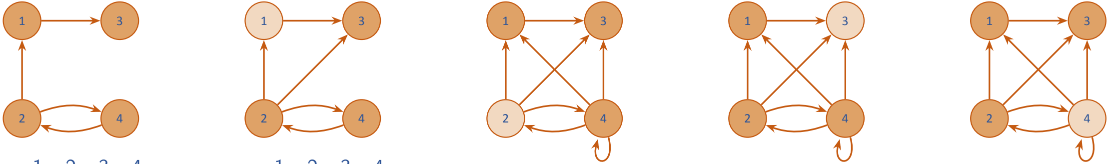

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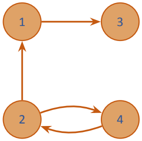

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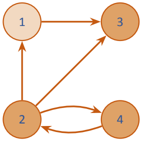

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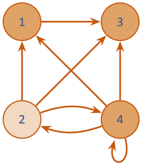

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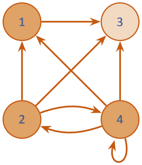

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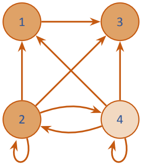

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Think About It

- Is Warshall's algorithm efficient for sparse graphs? Why / why not?
- Can Warshall's algorithm be used to determine if a graph is a DAG (Directed Acyclic Graph)? If so, how?