Fourier Series

Introduction:

In mathematics, infinite series are very important. They are used extensively in calculators and computers for evaluating values of many functions.

Fourier series are used in the analysis of **periodic** functions. Many of the phenomena studied in engineering and science are periodic in nature, for example, the current and voltage in an alternating current circuit.

These periodic functions can be analysed into their constituent components (fundamentals and harmonics) by a process called **Fourier analysis**. Fourier series are used in the analysis of **periodic** functions.

Fourier series is an infinite series representation of a periodic function in terms of trignometric sines and cosines functions.

Fourier series are used in applied mathematics, and especially in physics and electronics, to express periodic functions such as those that comprise communications signal waveforms.

The main **advantage of Fourier** analysis is that very little information is lost from the signal during the transformation.

Condition for Fourier series expansion

The Fourier series corresponding to a given function f(x) may not converge in all cases and if it converges its sum may not be f(x). For the Fourier expansion of a given function f(x) to be possible, its sufficient that f(x) satisfies the following conditions:

1. f(x) is periodic, single valued and finite.
2. f(x) can have only a finite number of finite discontinuities in any one period.

3. f(x) has at the most a finite number of maxima and minima in any one period.

The above conditions are known as “Dirichlet’s conditions”. These are sufficient but not necessary.

Euler's formulae

Let f(x) be a function defined in the interval $(\alpha, \alpha + 2\Pi)$ and let $f(x+2\Pi) = f(x)$, i.e., f(x) is periodic with period 2π.

Assuming that f(x) can be represented by a series as given below:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then $a_0 = \frac{1}{\Pi} \int_{\alpha}^{\alpha+2\Pi} f(x)dx$

$$a_n = \frac{1}{\Pi} \int_{\alpha}^{\alpha+2\Pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\Pi} \int_{\alpha}^{\alpha+2\Pi} f(x)\sin nxdx$$

In the Fourier expansion of f(x),

$a_1 \cos x + b_1 \sin x$ is called the first harmonic or

fundamental mode, whose frequency is $\frac{1}{2\Pi}$

The term $a_n \cos nx + b_n \sin nx$ is called the nth

harmonic whose frequency is $\frac{n}{2\Pi}$, that is, n

times the fundamental frequency.

Special cases:

(i) Putting α=0, the interval (0,2π) and Euler formulae become

$$a_0 = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\,dx$$

$$a_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\sin nx\,dx$$

(ii) Putting α=-π, the interval becomes (-π, π) and Euler formulae reduce to

$$a_0 = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x) dx$$

$$a_n = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x) \sin nx dx$$

Fourier series of Even and Odd functions:
Fourier series of an even function consists of cosine terms only.
Let f(x) be an even periodic function defined in (-π,π) with f(x+2π)=f(x). Then

$$a_0 = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)dx = \frac{2}{\Pi}\int_{0}^{\Pi} f(x)dx$$

$$a_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\cos nxdx = \frac{2}{\Pi}\int_{0}^{\Pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x) \sin nx dx = 0$$

Therefore, we have the Fourier cosine series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Fourier series of an odd function consists of sine terms only.

Let f(x) be an even periodic function defined in (-π,π) with f(x+2π)=f(x). Then

$$a_0 = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)dx = 0$$ and $$a_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\cos nx dx = 0$$

$$b_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\sin nx dx = \frac{2}{\Pi}\int_{0}^{\Pi} f(x)\sin nx dx$$

Therefore, we have the Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier series in other intervals

Consider the periodic function f(x) of period 2l defined in (α, α+2l). Put $z = \frac{\Pi x}{l}$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\Pi}{l}x + b_n \sin\frac{n\Pi}{l}x \right)$$

$$a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2\Pi} f(x)\,dx; \quad a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\frac{n\Pi}{l}dx$$

$$b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\frac{n\Pi}{l}dx$$

Corollary 1: If f(x) is defined in (0,2l), then the Fourier coefficients are:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\Pi}{l}dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\Pi}{l}dx$$

Corollary 2: If f(x) is defined in (-l,l), then the Fourier coefficients are:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\Pi}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\Pi}{l}dx$$

Half Range Series

In many physical problems we require the Fourier series expansion of a function f(x) defined in a finite interval (0,l) [which is half of the interval (-l,l)] in a series of cosines only or a series of sines only. For this purpose we make periodic extensions of f(x).

Half Range Fourier Cosine series

We extend the function f(x) defined in (o, l) to the other half (-l, 0) in such a way that the resulting function f1(x) is an even periodic function of period 2l and also f1(x)= f(x) in (0,l). The function f1(x) is called the even periodic extension of f(x) of period 2l. As f1(x) is an even function defined in (-l,l), its Fourier expansion consists of cosine terms only.

Thus, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\Pi}{l}x\right)$

where $a_0=\frac{2}{l}\int_0^l f(x)dx$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\Pi}{l}dx$$

Half Range Fourier Sine series

Extend the function f(x) defined in (o, l) to the other half (-l, 0) in such a way that the resulting function f2(x) is an odd periodic function of period 2l and also f2(x)= f(x) in (0,l). The function f2(x) is called the odd periodic extension of f(x) of period 2l. As f2(x) is an odd function defined in (-l,l), its Fourier expansion consists of sine terms only.

Then, $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\Pi}{l} x$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin \frac{n\Pi}{l} dx$

Special case: If f(x) is defined in (0,π), Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x)dx = \frac{2}{\Pi} \int_{0}^{\Pi} f(x)dx$

$$a_n = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x)\cos nx dx = \frac{2}{\Pi} \int_{0}^{\Pi} f(x)\cos nx dx$$

Fourier sine series corresponding to f(x) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{1}{\Pi} \int_{-\Pi}^{\Pi} f(x) \sin nx dx = \frac{2}{\Pi} \int_{0}^{\Pi} f(x) \sin nx dx$$

Harmonic Analysis

The Fourier coefficients $a_0, a_n, b_n (n = 1,2,....)$ are evaluated by integration when the function f(x) is given explicitly by an analytical expression. When the function is given by a graph or a table of values, this integration cannot be performed. But approximate values of the first few coefficients can be obtained by numerical techniques. Normally first few terms of a Fourier series dominate.

Consider a periodic function y=f(x) of a period 2π defined in [0,2π]. Suppose that the values of the function corresponding to a given set of equi-spaced values $x_0(=0), x_1, x_2, \ldots\ldots, x_m(=2\Pi)$ of x are tabulated and let the corresponding values of f(x) be $y_0, y_1, y_2, \ldots\ldots, y_m$, respectively. Then the interval [0,2π] is denoted into m equal parts and the spacing is $h = \frac{2\Pi}{m}$.

Fourier series is

$$f(x)=\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+....$$

We have $a_0=2[Mean\ value\ of\ \ f(x)=y\ in\ (0,2\Pi)]$

Similarly, $a_n=2[Mean\ value\ of\ \ f(x)\cos nx\ in\ (0,2\Pi)]$

$$b_n=2[Mean\ value\ of\ \ f(x)\sin nx\ in\ (0,2\Pi)]$$

Complex Fourier Series

This is a form which is commonly used in fields such as signal analysis. The Fourier series of a periodic function f(x) of period 2l defined in (-l,l) is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\Pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{\frac{-in\Pi x}{l}} dx, \quad n = 0, \pm 1, \pm 2, \ldots.$$