AUTOMATA, FORMAL LANGUAGES AND LOGIC

MODULE 5

Propositional Logic

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Outline

- Proof by resolution
  - Unit resolution
  - Conjunctive normal form
  - Resolution algorithm

Proof by Resolution - Conjunctive Normal Form (CNF)

- **Conjunctive Normal Form (CNF)**

every sentence of propositional logic is logically equivalent to a conjunction of clauses.

A sentence expressed as a **conjunction** of clauses **(disjunction of literals)** is said to be in **conjunctive normal form or CNF**.

Existing Knowledge Base

R_1: $\neg P_{1,1}$

R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

R4: $\neg B_{1,1}$

R5: $B_{2,1}$

$R_6 : (B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

$R_7 : ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

$R_8 : (\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1}))$

$R_9 : \neg(P_{1,2} \vee P_{2,1})$

R10 : ¬ P1,2 ∧ ¬P2,1

R11 : ¬B1,2

R12 : B1,2 ⇔ (P1,1 ∨ P2,2 ∨ P1,3)

R13 : ¬P2,2

R14 : ¬P1,3

R15: P1,1 ∨ P2,2 ∨ P3,1

R16: P1,1 ∨ P3,1

R17: P3,1

Proof by Resolution - Conjunctive Normal Form (CNF)

R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

converting the sentence B1,1 ⇔ (P1,2 ∨ P2,1) into CNF. The steps are as follows:

1. Eliminate ⇔, replacing α ⇔ β with (α ⇒ β) ∧ (β ⇒ α). Biconditional Elimination

(B1,1 ⇒ (P1,2 ∨ P2,1)) ∧ ((P1,2 ∨ P2,1) ⇒ B1,1) .

2. Eliminate ⇒, replacing α ⇒ β with ¬α ∨ β:

(¬B1,1 ∨ P1,2 ∨ P2,1) ∧ (¬(P1,2 ∨ P2,1) ∨ B1,1) .

3. CNF requires ¬ to appear only in literals

¬(¬α) ≡ α (double-negation elimination)

¬(α ∧ β) ≡ (¬α ∨ ¬β) (De Morgan)

¬(α ∨ β) ≡ (¬α ∧ ¬β) (De Morgan)

In the example, we require just one application of the last rule:

(¬B1,1 ∨ P1,2 ∨ P2,1) ∧ **((¬P1,2 ∧ ¬P2,1)** ∨ B1,1) .

4. Now we have a sentence containing nested ∧ and ∨ operators applied to literals. We apply the distributivity law

(¬B1,1 ∨ P1,2 ∨ P2,1) ∧ (¬P1,2 ∨ B1,1) ∧ (¬P2,1 ∨ B1,1) .

Proof by Resolution - Conjunctive Normal Form (CNF)

Empty Clause

A disjunction of no disjuncts – is equivalent to False because a disjunction is true only if at least one of its disjuncts is true.

Another way to see empty clause represents a contradiction is to observe that it arises only from resolving two complementary unit clauses such as P and ¬P.

Resolving two singleton clauses leads to the *empty clause*; i.e. the clause consisting of no literals at all, as shown below. The derivation of the empty clause means that the database contains a contradiction.

$$\frac{\begin{array}{c}\{p\}\\ \{\neg p\}\end{array}}{\{\}}$$

Proof by Resolution

A Resolution Algorithm

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  $clauses \leftarrow$ the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  $new \leftarrow \{\}$
  loop do
    for each C_i, C_j **in** *clauses* **do**
      $resolvents \leftarrow$ PL-RESOLVE(C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      $new \leftarrow new \cup resolvents$
    if $new \subseteq clauses$ **then return** *false*
    $clauses \leftarrow clauses \cup new$

Proof by Resolution

Example :-

- Given

$KB = (B_{11} \Leftrightarrow (P_{12} \vee P_{21})) \wedge \neg B_{11}$, prove $\neg P_{12}$

Resolution: Convert to CNF and $\alpha = \neg P_{12}$

show $KB \wedge \neg\alpha$ is unsatisfiable

$\boxed{\neg P_{21} \vee B_{11}}$ $\boxed{\neg B_{11} \vee P_{12} \vee P_{21}}$ $\boxed{\neg P_{12} \vee B_{11}}$ $\boxed{\neg B_{11}}$ $\boxed{P_{12}}$

Proof by Resolution

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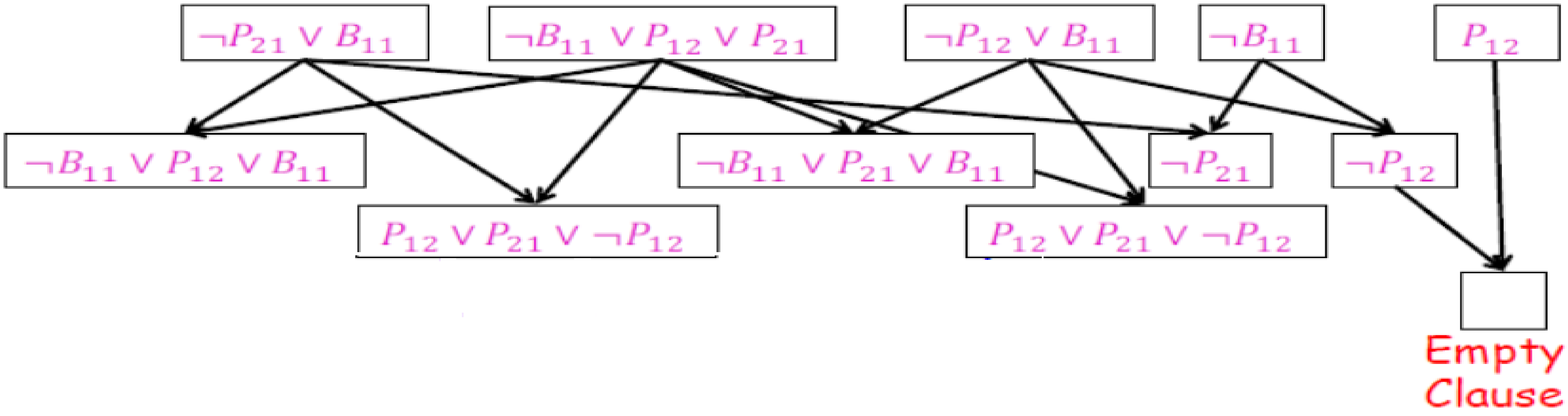

Exercise

- Given the premises (p ⇒ q) and (r ⇒ s), use Propositional Resolution to prove the conclusion (p V r ⇒ q V s).
- Ans:

1.	{~p,q}	Premise
2.	{~r,s}	Premise
3.	{p,r}	Goal
4.	{~q}	Goal
5.	{~s}	Goal
6.	{~p}	Resolution: 1, 4
7.	{~r}	Resolution: 2, 5
8.	{r}	Resolution: 3, 6
9.	{}	Resolution: 8, 7

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Exercise

In each of the following questions, say which of the answers best characterizes the result of applying resolution to the clauses shown.

1. $\{p, q, \neg r\}$ and $\{r, s\}$
   - $\{p, q, s\}$
   - $\{p, q, r, s\}$
   - $\{p, q, \neg r, s\}$
   - There are no resolvents.

2. $\{p, q, r\}$ and $\{r, \neg s, \neg t\}$
   - $\{p, q, r, \neg s, \neg t\}$
   - $\{p, q, \neg s, \neg t\}$
   - There are no resolvents.

Exercise

In each of the following questions, say which of the answers best characterizes the result of applying resolution to the clauses shown.

3. $\{q, \neg q\}$ and $\{q, \neg q\}$
   - (selected) $\{q, \neg q\}$
   - $\{q\}$
   - $\{\neg q\}$
   - $\{\}$
   - There are no resolvents.

Exercise

A Propositional 2-CNF expression is a conjunction of clauses, where each clause contains exactly 2-literals.

(A ∨ B) ∧ (~A ∨ C) ∧ (~B ∨ D) ∧ (~C ∨ G) ∧ (~D ∨ G)

Prove using Resolution that the above sentence entails G.

Ans:

Given "KB ∧ ~ Alpha" is

R1: (A ∨ B) , R2: (~A ∨ C), R3: (~B ∨ D), R4: (~C ∨ G),

R5:(~D ∨ G) and R6: ~G

Resolving R1 and R2: R7 = B ∨ C

Resolving R3 and R7: R8 = C ∨ D

Resolving R4 and R8: R9 = D ∨ G

Resolving R5 and R9: R10 = G

Resolving R6 and R10: R11 = {}

Hence Contradiction, So Alpha is TRUE and the given Statement entails G.

THANK YOU

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