UNIT - 2

VECTOR SPACES

Vector Spaces & Subspaces

Definition :

A ***real vector space*** is a *nonempty* set V of vectors together with rules for *vector addition* and *multiplication by scalars*.

Addition and multiplication must produce vectors in the space and they must satisfy the following conditions:

For all x , y, z $\in$ V and c, c_1, c_2 $\in$ R,

1. Closure : $x + y \in V$ for all $x, y \in V$
2. Commutativity : $x + y = y + x$
3. Associativity : $x + (y + z) = (x + y) + z$
4. Identity: There exists a unique *zero vector* "0" such that $x + 0 = 0 + x$
5. Inverse : For each x there is a unique vector $-x$ such that $x + (-x) = 0$

6. Closure : $c.x \in V$

7. $1.x = x$

8. $(c_1 c_2)x = c_1(c_2 x)$

9. $c(x + y) = cx + cy$

10. $(c_1 + c_2)x = c_1 x + c_2 x$

Precisely ,

*We can add any two vectors and we can multiply all vectors by scalars. In other words, we can take **<u>linear combinations</u>**.*

Few examples :

1. R = the set of all real numbers
2. $R^2 = \{ (x, y) \ / \ x, y \in R \}$
3. $R^3 = \{ (x, y, z) \ / \ x, y, z \in R \}$
4. $R^n = \{ (x_1, x_2, \ldots, x_n) \ / \ x_i \in R \}$
5. $R^\infty = \{ (x_1, x_2, \ldots,) \ / \ x_i \in R \}$
6. The space of all m x n matrices

Definition :

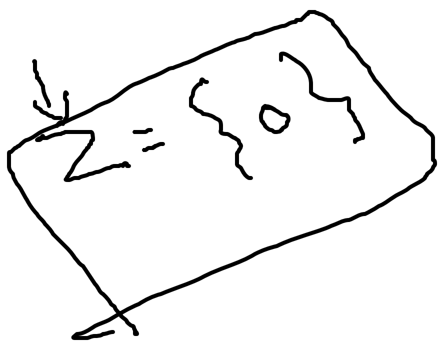
A ***subspace*** S of a vector space V is a nonempty subset that satisfies the following two conditions :

For all x, y $\in$ S and c $\in$ R

(i) $x + y \in S$

(ii) $cx \in S$

The smallest subspace Z contains only one vector, the zero element. It is the zero dimensional space containing only the point at the origin. At the other extreme, the largest subspace is the whole of the original space.



The possible subspaces are

Z = set of all integers

(0,0)

The only possible subspace is itself.

Few examples…

1. For V = R , the set of reals, the possible subspaces are

Set itself. i.e. $S = \{0\}$.

In ~~linear algebra~~

(i) Z = { 0 } zero space. Z = It contains only the identity element ✓ need not be zero.

(ii) R itself (R) set of all real numbers. Z, It can be anything.

2. For V = R^2, the possible subspaces are

$R^2 = \{(x, y) / x, y \in R\}$ 0-origin alone.

(i) Z = { (0 , 0) }

(ii) all straight lines passing through (0, 0) take any 2 points on x-axis

(iii) R^2 itself — largest R^2 given set itself

$c X \in$ set.

(5,0) {Larger than origin smaller than R^2

(2,0) set is closed w.r.t. to addition

3. For $V = R^3$, the possible subspaces are

(i) $Z = \{ (0, 0, 0) \}$

(ii) all lines passing through (0, 0, 0)

(iii) all planes passing through (0, 0, 0)

(iv) R^3 itself

In general, if $V = R^n$, the possible subspaces are Z , lines through origin, 2-d planes through origin, 3-d planes through origin, ….. , (n-1)- d planes through origin and the space R^n itself.

The column Space

Definition:

Let A be a m x n matrix. The ***column space of A*** is the set of all linear combinations of the columns of A denoted by C(A). Thus,

$$C(A) = \{ b \in R^m \,/\, Ax = b \text{ is solvable} \}$$

Note : C(A) is a subspace of R^m.

Few examples....

1. The smallest possible column space comes from the zero matrix A = 0. The only combination of the columns is b = 0.
2. If A is a 5 x 5 identity matrix then C(A) is the whole of R^5, the 5 columns of A can combine to produce any 5 dimensional vector b. In fact, any 5 x 5 nonsingular matrix A will have R^5 as its column space !!

3\. Let $$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$$

Then C(A) is the subspace of R^3 consisting of vectors b that are linear combinations of the vectors (1, 5, 2) and (0, 4, 3). Geometrically the subspace is a 2- d plane.

4. Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$

Then C(B) is the subspace of R^3 consisting of vectors b that are linear combinations of the vectors (1, 5, 2) , (0, 4, 3) and (1, 9, 5).

Note : The column spaces of A and B are same though the matrices are different. This is because the new column is a linear combination of the other two columns. Hence, appending a dependent column does not alter the column space of a matrix .

The Null Space

Definition :

Let A be a matrix of order m x n. The ***null space of A*** is the set of all solutions of the homogeneous system of equations Ax = 0 denoted by N (A) . Thus,

$$N(A) = \{\ x \in R^n \ /\ Ax = 0\ \}$$

Note : N(A) is a subspace of R^n.

Example :

Let A = $\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$

Then $$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives x = y = 0 as the only solution. The null space of this matrix thus contains only the zero vector (0, 0).

Now, if a third column is appended then

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives infinitely many solutions (c , c , –c) all of which lie on a line that obviously passes through the origin.

Note :

The matrices

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$$

have the same column space but different null space !!

Echelon Form of a Matrix

Definition :

A matrix A of order m x n is said to be in ***echelon form U*** if

(i) Pivots are the first nonzero entries in their rows

(ii) Below each pivot is a column of zeros

(iii) Each pivot lies to the right of the pivots in the rows above

(iv) Zero rows, if any, come last

Row Reduced Form of a Matrix

Definition :

Let A be a matrix of order m x n and U be its echelon form. Then the matrix A is said to be in ***row reduced echelon form R*** if in U

(i) the pivots are all 1 and

(ii) there are zeros above the pivots

Rank of a Matrix

Definition:

The ***rank of a matrix*** A is the number of nonzero rows in the echelon form U of A and is denoted by $\rho(A)$ or simply r.

Note :

If A is a matrix of order m x n then its rank $r \leq \min(m, n)$.

Pivot variables & Free Variables

$$Let\ \ A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Then the row reduced form of A is given by

$$R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solutions of Rx = 0 (or Ux = 0 or Ax = 0) are

$$x = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -3y+t \\ y \\ -t \\ t \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The variables x and z whose columns contain the pivots are called ***pivot variables*** and the remaining variables y and t are called ***free variables***. The vectors (-3, 1, 0, 0) and (1, 0, -1, 1) are called the ***special solutions*** of Ax = 0. All the other solutions are linear combinations of these two.

Note :

If $Ax = 0$ has more unknowns than the equations ($n > m$) it has at least one special solution. There are more solutions than the trivial $x = 0$. In other words, the null space of A is larger than Z.

Linear Independence, Basis and Dimension

Definition :

A set of vectors $v_1, v_2, ..., v_k$ of a vector space V is said to be ***linearly independent*** if the equation

$$c_1v_1 + c_2v_2 + ... + c_kv_k = 0 \,,\; c_i \in R$$

holds if and only if $c_i = 0$ for all i. If any of the $c_i \neq 0$ then the set is linearly dependent.

Few examples.....

1. The set containing only the zero vector is dependent. For, we choose some c ≠ 0.
2. The columns of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$$

are linearly independent whereas the columns of $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$ are dependent.

Note :

1. The columns of a square invertible matrix are always independent.
2. The columns of a matrix A of order m x n with $m < n$ are always dependent.
3. The columns of A are independent exactly when $N(A) = Z$.
4. The r nonzero rows of an echelon matrix U and a reduced matrix R are always independent and so are the r columns that contain the pivots.

Definition :

A set of vectors $v_1, v_2, \ldots, v_k$ of a vector space V is said to ***span*** V if every v Є V is a linear combination of these vi's.

Note : (i) The columns of A span C(A).

(ii) The columns (rows) of a square invertible matrix A of order n x n span the whole of R^n.

Definition :

A ***basis*** for a vector space V is a set of vectors having the following two properties at once:

(i) the vectors are linearly independent

(ii) the vectors span the space V

Note :

1. Every vector v in V is a unique combination of the base vectors.
2. A basis for V is not unique.
3. The columns of A that contain the pivots form a basis for C(A).
4. A basis for V is a maximal independent set and also a minimal spanning set.

Definition:

Any two bases for V have the same number of vectors. This number which is common to all the bases is called the ***dimension*** of the vector space V.

Note : The dimension of a vector space is unique!!

The Four Fundamental Subspaces

Definition:

Let A be a matrix of order m x n. The following are called the ***four fundamental subspaces*** of A

1. The column space of A denoted by C(A)
2. The null space of A denoted by N(A)
3. The row space of A denoted by $C(A^T)$
4. The left null space of A denoted by $N(A^T)$

Note :

1. The row space of A_{mxn} is the column space of A^T. It is spanned by the rows of A.
2. The left null space contains all vectors y for which $A^T y = 0$.
3. N(A) and $C(A^T)$ are subspaces of R^n.
4. $N(A^T)$ and C(A) are subspaces of R^m.
5. Dim C(A) = Dim $C(A^T)$ = r = rank of A.
6. Dim N(A) = n-r and Dim $N(A^T)$ = m- r.
7. The dimension of the null space of a matrix is called its ***nullity***.

continued.....

8. The rank- nullity theorem :

For any matrix A_{mxn} ,

$\dim C(A) + \dim N(A) =$ no. of columns

i.e $\quad r \quad + \quad (n-r) \quad = n$

This law applies to A^T as well. Hence,

$\dim C(A^T) + \dim N(A^T) = m$

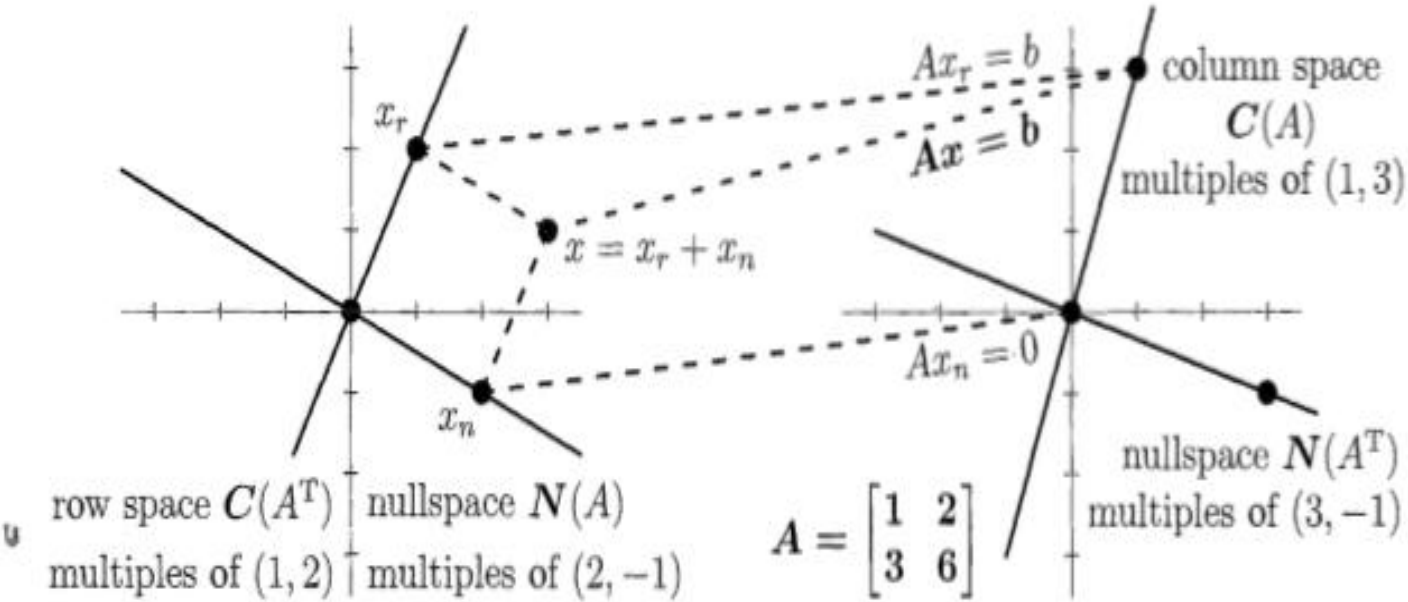
i. e $\quad r \quad + \quad (m-r) \quad = m$

Example :

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Then, m = n= 2 and rank r = 1.

1. C(A) is the line through (1, 3)
2. C(A^T) is the line through (1, 2)
3. N(A) is the line through (-2 , 1)
4. N(A^T) is the line through (-3, 1)


x_r
$Ax_r = b$
column space
$\boldsymbol{C}(A)$
multiples of $(1, 3)$
$\mathbf{Ax = b}$
$x = x_r + x_n$
$Ax_n = 0$
x_n
nullspace $\boldsymbol{N}(A^{\mathrm{T}})$
multiples of $(3, -1)$
row space $\boldsymbol{C}(A^{\mathrm{T}})$
multiples of $(1, 2)$
nullspace $\boldsymbol{N}(A)$
multiples of $(2, -1)$
$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Existence of Inverses

Definition:

Let A_{mxn} ($m \le n$) be a matrix such that $r = m$. Then $Ax = b$ has at least one solution x for every b if and only if the columns span R^m. In this case, A has a ***right inverse*** C such that $AC= I_m$.

Let A_{mxn} ($m \ge n$) be a matrix such that $r = n$. Then $Ax = b$ has at most one solution x for every b if and only if the columns are linearly independent. In this case, A has a ***left inverse*** B such that $BA= I_n$.

Note :

1. The right (left) inverse of a matrix is not unique.
2. When m= n, the matrix A has a unique inverse (B = C).
3. The best one sided inverses can be found using $B = \left(A^T A\right)^{-1} A^T$, $C = A^T \left(AA^T\right)^{-1}$

Example:

Let $$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Then, a right inverse of A is

$$C = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}$$

Since the third row is arbitrary, there are infinitely many right inverses for A.

Matrices Of Rank One

When the rank of a matrix is as small as possible, a complicated system of equations can be broken into simple pieces. Those simple pieces are matrices of rank one. The matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix}$$

has rank r = 1.

We can write such matrices as a column times row. That is

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$