## Unit 3 – Linear Transformations and Orthogonality

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| Linear Transformations, Orthogonal Vectors and Subspaces, Cosines and Projections onto Lines, Projections and Least Squares.<br>**Self Learning Component:** Inner Products and Cosines |

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| 30-31 | Linear Transformations , Examples |
| 32-33 | Transformations Represented by Matrices |
| 34-35 | Rotations, Reflections and Projections |
| 36-38 | Orthogonal Vectors and Subspaces |
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| 41-42 | Projections and Least Squares |
| 43 | **Scilab Class Number 6-Projections by Least Squares** |

**Class work Problems:**

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| 1 | Find the matrix that rotates every vector in $R^2$ through $90^o$ in the positive sense and then projects the result onto the x axis. Find also the matrix that projects onto x axis and then rotates every vector through $90^o$. Compare the results. |
| 2 | Let S and T be two linear transformations on $R^2$ defined by T ( x , y ) = ( x , 0 ) and S ( x , y ) = ( 0, x ) for all ( x , y ) in $R^2$. Show that ST ≠ TS. |
| 3 | Suppose T is reflection across $45^o$ line and S is reflection across the y axis. Find in general the products ST and TS. |
| 4 | Give an example of a non linear transformation T such that T ( 0 ) = 0. |
| 5 | Which of these transformations is not linear ? The input vector is $v = ( v_1, v_2 )$.<br>(i) $T ( v ) = ( v_2 , v_1 )$ (ii) $T ( v ) = ( v_1, v_1 )$<br>(iii) $T ( v ) = ( 0, v_1 )$ (iv) $T ( v ) = ( 0, 1 )$<br>Answer : All are linear except (iv) |
| 6 | For each of the following linear transformations T, find a basis and the dimension of the range and kernel of T:<br>(i) $T : R^2 \rightarrow R^3$ defined by T ( x , y ) = ( x + y , x – y , y )<br>(ii) $T : R^2 \rightarrow R^2$ defined by T ( x , y ) = ( y, 0 )<br>Answer : (i) Range is a 2-d plane spanned by ( 1, 1, 0 ) and ( 1, -1, 1 ) and kernel is the origin in $R^2$.<br>(ii) Range and kernel are x axis. |
| 7 | Find the matrix of the linear transformation T on $R^3$ defined by<br>T ( x , y, z ) = ( 2y + z , x – 4y , 3x ) with respect to<br>(i) the standard basis ( 1, 0, 0 ) , ( 0, 1 , 0 ) , ( 0, 0, 1 ) and<br>(ii) the basis ( 1, 1, 1 ) , ( 1, 1, 0 ) , ( 1, 0, 0 )<br>Answer : (i) $\begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$ |

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| 8 | Let V be the vector space of all 2 x 2 real matrices and T be a linear transformation on V that sends each matrix X onto AX where A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.Find the matrix of T with respect to the standard basis of $M_{2x2}$.<br>Answer : $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ |
| 9 | On the space $P_3$ of cubic polynomials, what matrix represents $\frac{d^2}{dt^2}$ ? Find its null space and column space. What do they mean in terms of polynomials ? |
| 10 | From the cubics $P_3$ to the fourth degree polynomials $P_4$, what matrix represents multiplication by 3t – 5 ? |
| 11 | Suppose all vectors x in the unit square 0 ≤ x , y ≤ 1 are transformed to Ax where A is a 2 x 2 matrix.<br>(i) What is the shape of the transformed region when A is nonsingular ?<br>(ii) For which matrices A is that region a square ?<br>(iii) For which A is it a line ?<br>(iv) For which A is the new area still unity ?<br>Answer : (i) Parallelogram with one corner at ( 0, 0 )<br>(ii) A is orthogonal (iii) A is singular (iv) Area of parallelogram = det A |
| 12 | Describe the linear transformations of the xy plane that are represented with the standard basis by the matrices<br>(i) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$<br>Answer : (i) ref on x axis (ii) shear that sends ( x , y ) to ( x , 2x + y ) . It leaves y axis unchanged and transforms x axis to the line y = 2x (iii) rotation thro ‘ – $90^o$ |
| 13 | Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$ Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space and a vector z orthogonal to the null space.<br>Answer : x = ( 2, -1 , 0 ) , y = ( 1, 1, -1 ) , z = ( 1 , 2, 1 ) |
| 14 | Find all vectors in $R^3$ that are orthogonal to ( 1, 1, 1 ) and ( 1, -1 , 0 ). Produce an orthonormal basis from these vectors.<br>Answer : all multiples of ( 1, 1, -2 ). Normalize the three independent vectors. |
| 15 | Let P be the plane in $R^4$ with equation x - 2y + 3z - t = 0. Find a vector perpendicular to P. What matrix has the plane P as its null space ? What is a basis for P?<br>Answer : ( 1 , -2 , 3, -1 ) is perpendicular to P. The matrix A = [ 1 -2 3 1 ] has P= N(A) . A basis for P is ( 2 , 1, 0, 0 ) , ( -3 , 0, 1, 0 ) , ( 1 , 0, 0, 1 ). |

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| 16 | Suppose S is spanned by ( 1 , 2, 2, 3 ) and ( 1, 3, 3, 2 ). Find a basis for $S^{\perp}$.<br>Answer : ( 0, 1, -1, 0 ) , ( -5 , 0, 1, 1 ) |
| 17 | By choosing the correct vector b in the Schwarz inequality, prove that<br>$(a_1 + a_2 + ........ + a_n)^2 \leq n\,( a_1^{\;2} + a_2^{\;2} + .......... + a_n^{\;2} )$ . When does equality hold ? |
| 18 | Verify that the length of the projection is $\|p\| = \|b\| \cos\theta$. |
| 19 | What multiple of a = ( 1, 1, 1 ) is closest to b = ( 2, 4, 4 ) ? Find also the point on the line through b that is closest to a.<br>Answer : 10/3 ( 1, 1, 1 ) , ( 5/9, 10/9, 10/9 ) |
| 20 | Find the matrix that projects every point in $R^3$ onto the line of intersection of the planes x + y + z = 0 and x – z = 0. What are the column space and row space of this matrix ?<br>Answer : $\frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$ |
| 21 | Project b = ( 1, 2, 2 ) onto the line through a = ( 1, 1, 1 ). Check that e is perpendicular to a.<br>Answer : p = 5/3 ( 1, 1, 1 ). |
| 22 | Project b = ( 1, 0, 0 ) onto the lines through $a_1$ = ( -1 , 2, 2 ) , $a_2$ = ( 2, 2, -1 ) and $a_3$ = ( 2 , -1 , 2 ). Add the three projections to get the sum b. Also find the corresponding projection matrices $P_1$, $P_2$ and $P_3$. Check that their sum is I and the product is 0. |
| 23 | Find $\|E\|^2 = \|Ax - b\|^2$ and set to zero its derivatives with respect to u and v if<br>A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, x = \begin{bmatrix} u \\ v \end{bmatrix}\ ,\ b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$<br>Compare the resulting equations with $A^T A\hat{x} = A^T b$ to confirm that Calculus and Geometry give the same Normal equations. Find the solution $\hat{x}$ and the projection p = $A\hat{x}$ . Why is p = b ?<br>Solution: u = 1 , v = 3. Since Ax = b is solvable, p = b. |
| 24 | Find the projection of b = ( 1, 2, 7 ) onto the column space of A spanned by ( 1 , 1, -2 ) and ( 1 , -1 , 4 ). Split b into p + q with p in C(A) and q in N($A^T$).<br>Answer : p = ( 3, 0, 6 ). |
| 25 | If V is the subspace spanned by ( 1, 1, 0, 1 ) , ( 0, 0, 1, 0 ) find<br>(i) a basis for $V^{\perp}$ (ii) the projection matrix P onto V<br>(iii) the vector in V closest to b = ( 0, 1, 0, -1 ) in $V^{\perp}$<br>Answer : (i) ( -1 , 1, 0, 0 ) , ( -1 , 0, 0, 1 ) (iii) p = 0 |
| 26 | Find a basis for the orthogonal complement of the row space of A = $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$ . Split the vector ( 3, 3, 3 ) into a row space component xr and a null space component xn.<br>Answer : basis is ( -2 , -2 , 1 ), ( 3, 3, 3 ) = ( 1, 1, 4 ) + ( 2, 2, -1 ) |

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<tr><td>27</td><td>Use the method of least squares to fit the best line to the data $b = 4, 3, 1, 0$ at<br>$t = -2, -1, 0, 2$ respectively. Find the projection of $b = (4, 3, 1, 0)$ onto the column space of A. Calculate the error vector e and check that e is orthogonal to the columns of A.<br>Answer : $b = 61/35 - 36t/35$. $P = 1/35\ (133, 95, 61, -11)$</td></tr>
<tr><td>28</td><td>( optional ) An ice- cream vendor records the number of hours of sun shine (x) versus the number of ice- creams sold in an hour ( y ) at his shop from Monday to Friday and found the following data :<br>
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<tr><td>x:</td><td>2</td><td>3</td><td>5</td><td>7</td><td>9</td></tr>
<tr><td>y:</td><td>4</td><td>5</td><td>7</td><td>10</td><td>15</td></tr>
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Find the best values of m and c that suit the equation $y = mx + c$. If there is a weather forecast that says there would be 8 hours of sun shine the next day, estimate the number of ice- creams that he expects to sell on that day.<br>Answer : $y = 1.518x + 0.305$, 13</td></tr>
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