CLASS 4 : CONTENT

- Linear independence and dependence of vectors
- Basis
- Dimension
- Span of set of vectors

LINEAR COMBINATION

Let V be a vector space and $v_1, v_2, v_3, \ldots\ldots v_n$ be the vectors in V. Then the form $c_1v_1 + c_2v_2 + \ldots. + c_nv_n$, where $c_1, c_2, \ldots\ldots. c_n$ are scalars, is called a *linear combination of the vectors* .

Linear combination of vectors involve scalar multiplication and vector addition of vectors .

To decide on linear independence of vectors we need to look for their linear combination .

The trivial combination with all scalars 'c'=0 ,produces $0v_1 + 0v_2 + \ldots. + 0v_n = 0$

If this is the only way to produce zero ,given vectors are independent

if any other combination produces zero then vectors are dependent

LINEAR INDEPENDENCE AND DEPENDENCE

A set of vectors $\{v_1, v_2, v_3, \ldots\ldots v_n\}$ of a vector space, is said to be linearly independent if the linear combination $c_1v_1 + c_2v_2 + \ldots. + c_nv_n = 0$,

where $c_1 = c_2 = c_3 = \ldots\ldots. = c_n = 0$

A set of vectors $\{v_1, v_2, v_3, \ldots\ldots v_n\}$ of a vector space, is said to be linearly dependent if there exists scalars $c_1, c_2, \ldots\ldots.. c_n \in R,$ not all zero such that

$c_1v_1 + c_2v_2 + \ldots. + c_nv_n = 0$ either all $c_1, c_2, \ldots\ldots.. c_n$ are nonzero of few scalars are zero and few are non zero .

(A set of vectors $\{v_1, v_2, v_3, \ldots\ldots v_n\}$ is said to be linearly independent if one vector cannot be written as the combination of the other vectors.

LINEAR INDEPENDENCE AND DEPENDENCE

To decide whether given set of vectors are independent or dependent apply the following procedure :

1) Place the vectors $\{v_1, v_2, v_3, ...v_n\}$ of the given set of vectors as columns of matrix A .
2) Apply Gauss Elimination on the Matrix A
3) If all the columns of the matrix is with pivot then the set of vectors are linearly independent
4) If certain columns of the matrix A do not hold pivot then the set of vectors are linearly dependent
5) If $\rho(A) = n =$ number of columns then vectors are linearly independent

LINEAR COMBINATION

E.g.: Check whether $\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}1\\1\\2\\3\end{bmatrix},\begin{bmatrix}1\\3\\7\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$.

Solution

$$\begin{bmatrix}1&1&1\\2&1&3\\1&2&7\\3&3&1\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&-1&1\\0&1&6\\0&0&-2\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&-1&1\\0&0&7\\0&0&-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1\\0&-1&1\\0&0&7\\0&0&0\end{bmatrix};\rho(A)=3=n$$

Hence the given vectors are linearly independent.

LINEAR COMBINATION

E.g.: Check whether $\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}4\\3\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$.

Solution

$$A=\begin{bmatrix}1&3&4\\2&1&3\\1&0&1\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&-5&-5\\0&-3&-3\end{bmatrix}\sim\begin{bmatrix}1&3&4\\0&-5&-5\\0&0&0\end{bmatrix}$$

$$\rho(A)=2, n=3$$

Therefore they are dependent

LINEARLY INDEPENDENT VECTORS

Note : 1. The columns of a square invertible matrix are always independent.

2. The columns of a matrix A of order m x n with $m < n$ are always dependent.

3. The columns of A are independent exactly when N(A) = Z

(Z means 0)

4. The ‘r ‘nonzero rows of an echelon matrix U and a reduced matrix R are always independent and so are the ‘r’ columns that contain the pivots.

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VECTOR SPACES

BASIS

A subset $S = \{v_1, v_2, v_3, ...v_n\}$ of a vector space is called a basis for

Vector space V if

i. S is a linearly independent set.

ii. S $spans\ the\ vector\ space\ V$.

The *dimension of a vector space* is the number of basis vector.

VECTOR SPACES

BASIS

A Basis for Vector space V is a sequence of vectors having properties :

1. The vectors are linearly independent (not too many vectors)
2. They span the space V (not too few vectors) .
3. every vector in the space is a combination of the basis vectors ,because they span (s.t. combination is unique)
4. if $\{v_1, v_2, v_3, ...v_n\}$ and $\{w_1, w_2, w_3, ...w_n\}$ are both the bases for the same vector space then m=n ,the number of vectors is same .
5. basis is maximal independent set and minimal spanning set.

VECTOR SPACES

BASIS

6. If The vectors $\{v_1, v_2, v_3, ...v_n\}$ and $\{w_1, w_2, w_3, ...w_n\}$ are both bases for the same vector space ,then m = n , i.e. the number of vectors is the same .

7. Any linearly independent set in V can be extended to a basis ,by adding more vectors if necessary ,any spanning set in V can be reduced to a basis ,by discarding vectors if necessary .

8. there exists one and only one way to write any vector ‘v’ in Vector space V as a combination of the bases vectors of that vector space

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BASIS

Examples of Basis :

1. Basis of R^2 vector space is {(1,0) ,(0,1)} .

2. Basis of R^3 vector space is {(1,0,0) ,(0,1,0),(0,0,1)} .

3. Basis of R^nvector space is {(1,0,0,0,…….,0) ,(0,1,0,……,0),(0,0,……,1)}

4.Marices $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is the basis for vector space of 2*2 matices .

5. The set $\left\{1,t,t^2,..........,t^n\right\}$ is a basis of space of Polynomials P_n

SPAN OF A SET

Let $W = \{v_1, v_2, v_3, ...v_n\}$ be a set of vectors belonging to vector space V,

then *span* of W is the set of all linear combinations of vectors of W .

i.e., span of $W = span(W) = c_1 v_1 + c_2 v_2 + ... + c_n v_n$

= subspace of V

E.g. : What do these vectors $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\4\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}$ span

Solution

The given vectors span a 2D subspaces of $\mathbb{R}^2$.

$$A=\begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3\\0 & 0 & -1\end{bmatrix};\quad \rho(A)=2$$

THANK YOU

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