LINEAR ALGEBRA AND ITS APPLICATIONS

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MATRICES AND GAUSSIAN ELIMINATION

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MATRICES AND GAUSSIAN ELIMINATION:

Course Content: Triangular Factors

❖ Consider the system of equations Ax=b. Using elementary row transformations we
reduce A to U an Upper triangular matrix. While doing so Ax=b reduces to Ux=c(i.e [b]
reduces to [c]) which can be solved by back substitution to obtain x. The steps involved
are $Ax = b \Rightarrow E_{32}E_{31}E_{21}Ax = Ux = c\left(\because E_{32}E_{31}E_{21}A = U\right)$
To undo the steps of elimination we must trace back the steps instead of subtracting we
must add . For this we need inverses of Elementary Matrices i.e $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1}$
E_{21}^{-1} can be obtained by changing –l to l in the transformation. Similarly
E_{31}^{-1} and E_{32}^{-1} can also be obtained. But while going from U to A the inverses should
be in reverse direction i.e., $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = A$

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MATRICES AND GAUSSIAN ELIMINATION:

Triangular Factorization A=LU:

❖ Any square matrix A can be factored as A=LU where

➢ **L is a Lower Triangular Matrix**

➢ **With 1's on the diagonal**

➢ **Having multiplier's l_{ij} below the diagonal in their respective positions.**

➢**And**

➢ **U is an Upper Triangular Matrix**

➢ **Having pivots on the diagonal ($u_{11}, u_{22}. u_{33}$)**

➢i.e
$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

Consider the matrix $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[R_3 - 1/3R_1]{R_2 - 2/3R_1} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{pmatrix}$

$\xrightarrow{R_3 + (5/11)R_2} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix} = U \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{pmatrix}$

$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix}$

MATRICES AND GAUSSIAN ELIMINATION:

Triangular Factorization A=LDU:

❖ **A=LU** factorization is Unsymmetric as L has 1's on the diagonal whereas U has pivots on

the diagonal. In order to make this factorization symmetric we do **A=LDU** factorization

➢ **D is a Diagonal Matrix with pivots d$_1$,d$_2$.d$_3$ on the diagonal**

➢ **L is a Lower Triangular Matrix**

➢ **With 1's on the diagonal**

➢ **Having multiplier's l_{ij} below the diagonal in their respective positions.**

➢ **U is an Upper Triangular Matrix with 1's on the diagonal obtained by dividing each**

row by its pivot.

i.e

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \quad D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2` & 0 \\ 0 & 0 & d_3 \end{pmatrix} \quad U = \begin{pmatrix} 1 & u_{12}/u_{11} & u_{13}/u_{11} \\ 0 & 1 & u_{23}/u_{22} \\ 0 & 0 & 1 \end{pmatrix}$$
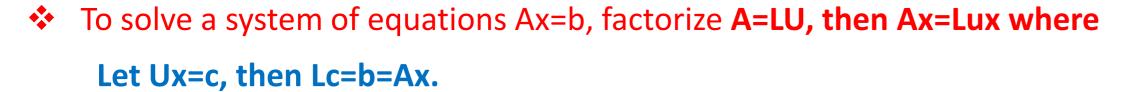
MATRICES AND GAUSSIAN ELIMINATION:

Factorise A=LU and hence A=LDU.

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 + (1/2)R_1} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_3 + (2/3)R_2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_4 + (3/4)R_3} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} = U \quad A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

❖ To solve a system of equations Ax=b, factorize **A=LU, then Ax=Lux where**

    Let Ux=c, then Lc=b=Ax.

    Solve Lc=b, using Forward elimination and then find x using Ux=c by Backward

    substitution which gives x.

  Ex: Solve as triangular systems without multiplying LU to find A.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \qquad \text{Let } Lc = b \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -2 \\ c_3 = 0 \end{cases} \text{ then } Ux = c \Rightarrow \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x + 4y + 4z = 2 \\ y + 2z = -2 \\ z = 0 \end{cases}$$

$$(x, y, z) = (5, -2, 0)$$

MATRICES AND GAUSSIAN ELIMINATION:

❖ Find A=LU and LDU factorization given

$$A = \begin{pmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{pmatrix} \xrightarrow[\substack{R_3 + 2R_1 \\ R_4 + R_1}]{R_2 - 1/2 R_1} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{pmatrix} \xrightarrow[R_4 - R_2]{R_3 + 2R_2} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{pmatrix}$$

$$\xrightarrow{R_4 + 2R_3} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{pmatrix} = U \quad A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & 5/2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1/3 & -2/3 & 2/3 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

THANK YOU

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