UNIT-1 (MATRICES AND GAUSSIAN ELIMINATION)

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Q1) Equations given are:

$$2x + y + 5z + u = 5$$
$$x + y - 3z - 4u = -1$$
$$3x + 6y - 2z + u = 8$$
$$2x + 2y + 2z - 3u = 2$$

Representing them in $Ax = b$ form

$$\begin{bmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 8 \\ 2 \end{bmatrix}$$

The Augmented Matrix is of the form

$$[A \ b] = \begin{bmatrix} 2 & 1 & 5 & 1 & 5 \\ 1 & 1 & -3 & -4 & -1 \\ 3 & 6 & -2 & 1 & 8 \\ 2 & 2 & 2 & -3 & 2 \end{bmatrix}$$

Performing row operations:

$R_2 \longrightarrow R_2 - \left(\frac{1}{2}\right) R_1$ $\quad$ $R_3 \rightarrow R_3 - \left(\frac{3}{2}\right) R_1$ $\quad$ $R_4 \rightarrow R_4 - \left(\frac{2}{2}\right) R_1$

$$[A \ b] \sim \begin{bmatrix} 2 & 1 & 5 & 1 & 5 \\ 0 & 1/2 & -11/2 & -9/2 & -7/2 \\ 0 & 9/2 & -19/2 & -1/2 & 1/2 \\ 0 & 1 & -3 & -4 & -3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \left(\frac{9/2}{1/2}\right) R_2$ $\quad$ $R_4 \rightarrow R_4 - \left(\frac{1}{1/2}\right) R_2$

$$[A \ b] \sim \begin{bmatrix} 2 & 1 & 5 & 1 & 5 \\ 0 & 1/2 & -11/2 & -9/2 & -7/2 \\ 0 & 0 & 40 & 40 & 32 \\ 0 & 0 & 8 & 5 & 4 \end{bmatrix}$$

$R_4 \rightarrow R_4 - \left(\frac{8}{40}\right) R_3$

$$[A\ b] \sim \begin{bmatrix} 2 & 1 & 5 & 1 & 5 \\ 0 & 1/2 & -11/2 & -9/2 & -7/2 \\ 0 & 0 & 40 & 40 & 32 \\ 0 & 0 & 0 & -3 & -12/5 \end{bmatrix} \sim [U\ c]$$

Back substitution:

$$-3u = -12/5$$

$$\therefore \boxed{u = 4/5 = 0.8}$$

$$40(z) + 40\left(\frac{4}{5}\right) = 32$$

$$\therefore \boxed{z = 0}$$

$$y - 11(0) - 9\left(\frac{4}{5}\right) = -7$$

$$\therefore \boxed{y = 1/5}$$

$$2(x) + 1\left(\frac{1}{5}\right) + 5(0) + 1\left(\frac{4}{5}\right) = 5$$

$$\therefore \boxed{x = 2}$$

$\therefore$ The given system of equations is consistent and have a unique solution.

Q2) Equations of the plane are:

$$x + 2y + z = 4$$
$$y - z = 1$$
$$x + 3y = 0$$

Representing in $Ax = b$ form

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

Augmented system is of the form:

$$[A\ b] = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix}$$

Row transformation

$$R_3 \rightarrow R_3 - \left(\frac{1}{1}\right) R_1$$

$$[A\ b] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{1}{1}\right) R_2$

$$[U\ c] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

From this, we obtain that $0 = -5$, which is absurd. So there is no intersection of planes and the given system of equations is inconsistent.

If the last equation is changed to $x + 3y = 5$

$$[A\ b] = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 5 \end{bmatrix}$$

Row transformations:

$R_3 \to R_3 - \left(\frac{1}{1}\right) R_1$

$$[A\ b] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{1}{1}\right) R_2$

$$[U\ c] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can see that the last row is reduntant. In this case we have 2 equations and 3 variables

$\therefore$ There will be infinite solutions.

Let's assume $\boxed{z = K}$ (constant)

$1(y) - 1(K) = 1$

$\therefore \boxed{y = 1+K}$

$1(x) + 2(1+K) + 1(K) = 4$

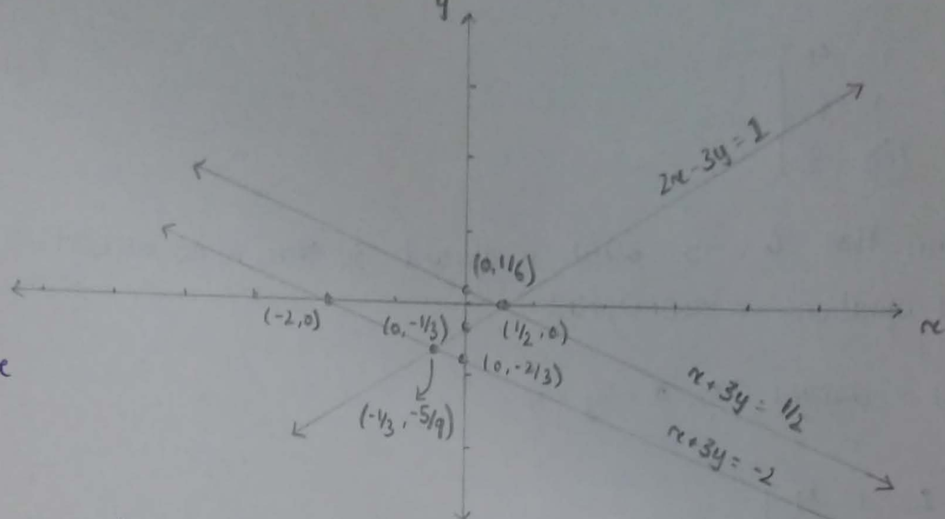
$\therefore \boxed{x = 2-3K}$

$\therefore$ The given system of equations is now consistent.

93) Given system of equations:

$2x - 3y = 1$
$x + 3y = -2$

Plotting both the equations on the graph.



Row picture

From the above graph, we see that the lines $2x-3y=1$ and $x+3y=-2$ intersect at the points $x=-1/3$ and $y=-5/9$. Therefore, that is the unique solution of the equation.

Given set of equations:

$2x - 3y = 1$
$x + 3y = 1/2$

From the above graph, we see that the lines intersect at ~~x=0~~ $x=1/2$ and $y=0$. Therefore that is the unique sol^n of equation.

94) A 3x3 Identity matrix will have (3!) 6 possible permutations

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{12} I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{13} I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{23} I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{23} P_{12} I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{23} P_{13} I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Upon careful observation, we see that $P_{23} P_{12}$ is the inverse of $P_{12} P_{13}$ and vice-versa.

The other matrices are self inverses.

Lets assume P matrix to be: $P_{23} P_{12} I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$I - P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$[I-P]x = 0 \quad \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x - y = 0 \qquad y - z = 0 \qquad z - x = 0$

$\therefore \boxed{x = y} \qquad \therefore \boxed{y = z} \qquad \therefore \boxed{z = x}$

From this we obtain: $x = y = z$.

So a set of sol^n is $(1,1,1)$.

$\det(I-P) = 1(1) + 1(-1) + 0$
$= 0$

$\therefore$ I-P is a singular matrix and non-invertible.

★ Note: If P chosen was (i) P_{12}, we would get $x = y$ and $z \in R$

(ii) P_{13}, we would get $x = z$ and $y \in R$

(iii) P_{23}, we would get $y = z$ and $x \in R$

(iv) $P_{23} P_{13}$, we would get $x = y = z$

$\therefore$ taking the intersection, we arrive at the conclusion that $x = y = z$ is the best solution for $[I-P]x = 0$.

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q5) A system of equations is called singular if determinant of the co-efficient matrix is equal to 0.

$$|A| = 0$$

Singular cases:

① consider the following system of equations

$$2x + 2y = 1$$
$$x + y = 5$$

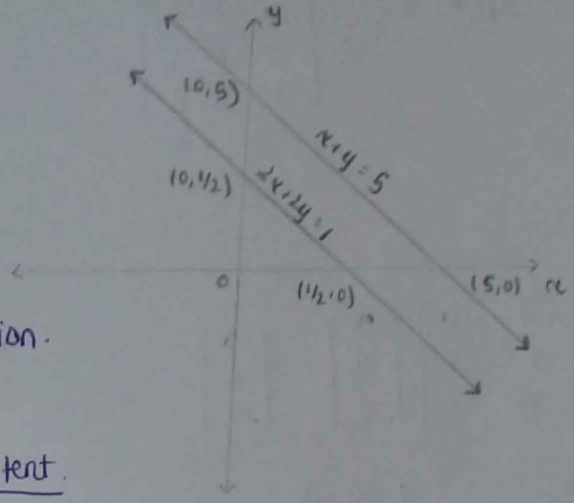
Row picture

we observe that the following set of equations result in parallel lines.

There is no - point of intersection.

This results in no solution

The given system is inconsistent.



② Consider the following set of equations

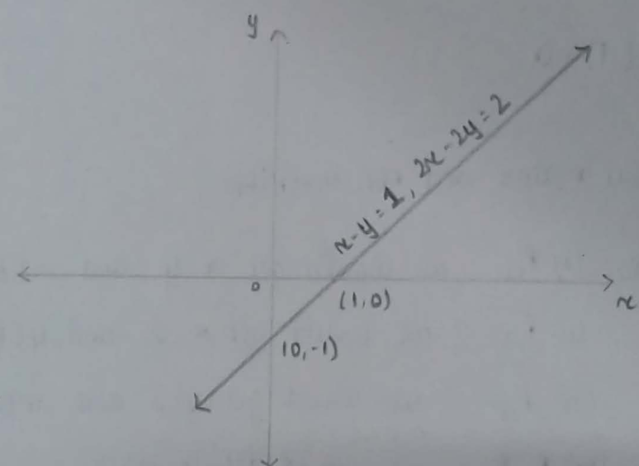
$$x - y = 1$$
$$2x - 2y = 2$$

Row picture

We observe that the following set of equations result in one single line.

This results in ifinite solutions.

The given system is consistent.



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q6) The given set of equations is:

$$x - 2y - 3z = 0$$
$$y + z = -8$$
$$-x + y + 2z = 3$$

Representing in the form $Ax = b$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 3 \end{bmatrix}$$

Augmented matrix is:

$$[A\ b] = \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{bmatrix}$$

Applying row transformations:

$R_3 \rightarrow R_3 - \left(\frac{-1}{1}\right) R_1$

$$[A \; b] \sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \left(\frac{-1}{1}\right) R_2$

$$[U \; c] \sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

As we can see, the given system is inconsistent.

We have been given unique sol^n of $z = 1$.

For that, 3rd pivot should have been equal to -5.

$a_{33}'' = a_{33}' + a_{23}$ (from 2nd row transformation)

$-5 = a_{33}' + 1$

$\therefore a_{33}' = -6$.

$a_{33}' = a_{33} + \quad a_{13}$ (from 1st row transformation)

$-6 = a_{33} - 3$

$\therefore a_{33} = -3$.

Therefore the augmented matrix should be:

$$[A \; b] = \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ -1 & 1 & -3 & 3 \end{bmatrix}$$

$\therefore$ the 3rd equation is $-x + y - 3z = 3$

$$\text{New } [U \; c] \sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & -5 & -5 \end{bmatrix}$$

The system is now consistent, with unique solution.

$\therefore \boxed{z = 1}$

$\therefore y + 1(1) = -8 \qquad \therefore \boxed{y = -9}$

$\therefore 1(x) - 2(-9) - 3(1) = 0 \qquad \therefore \boxed{x = -15}$

Q7) $A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix}$

Applying row transformations

$R_1 \longleftrightarrow R_3$

$$P_{13} A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$P_{23} P_{13} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{bmatrix} = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

$$\therefore \quad U = P_{23} P_{13} A = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

Q8) The equations are:

$$x + y + z = 2$$
$$x + 2y + 3z = 1$$
$$y + 2z = 0$$

Representing in the form $Ax = b$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Augmented matrix is:

$$[A \ b] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Applying row transformations: $R_2 \rightarrow R_2 - \left(\frac{1}{1}\right) R_1$ $\quad R_3 \rightarrow R_3 - (0) R_1$

$$[A \ b] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \left(\frac{1}{1}\right) R_2$

$$[U \; c] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ This system is inconsistent and has no solution.

It is singular in nature.

Let us assume the third eq^n to be:

$$y + 2z = a$$

So from our row transformed matrix, we can conclude that the system would only be consistent if ~~ott~~ we have a ZERO ROW.

For that we need $\frac{1}{1} = \frac{2}{2} = \frac{-1}{a}$

$$\therefore a = -1$$

Equation should be $y + 2z = -1$.

If that is our equation, then $[U \; c] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

After backward substitution, we get:

$$y + 2(z) = -1$$

Let's assume $z = K$ (constant) $K \in R$.

$$\therefore y = -1 - 2K.$$

$$x(1) + y(1) + z(1) = 2$$

$$x + (-1 - 2K) + K = 2$$

$$\therefore x = 3 + K$$

So a solution is $(3+K, -1-2K, K)$. Substituting $K = 1$ (any random number)

A solution is $(4, -3, 1)$.

Q9) The equations are:

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + qz = t$$

Representing in the $Ax = b$ form:

$$\begin{bmatrix} 1 & 4 & -2 \\ 1 & 7 & -6 \\ 0 & 3 & q \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ t \end{bmatrix}$$

The augmented matrix $[A\ b]$ is

$$[A\ b] = \begin{bmatrix} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{bmatrix}$$

Applying row transformations: $R_2 \to R_2 - \left(\frac{1}{1}\right) R_1$ $\quad R_3 \to R_3 - (0) R_1$

$$[A\ b] \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{bmatrix}$$

for infinite solutions, we should have a ZERO ROW.

$$\frac{3}{3} = \frac{-4}{q} = \frac{5}{t}$$

This yields the solution that $q = -4$ and $t = 5$.

Applying row transformation (with updated values of q and t): $R_3 \to R_3 - \left(\frac{3}{3}\right) R_2$

$$[U\ c] = \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After backward substitution, we get:

$$3y - 4z = 5$$

Let's assume $z = K$ (constant) [in this case, $z = 1$]

$$\therefore y = [5 + 4(1)]/3 = 3$$

$$x(1) + 4(3) - 2(1) = 1$$

$$\therefore x = -9$$

The solution with $z = 1$ is $(-9, 3, 1)$.

Q10) Given $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$

First, we will find U, (which will also help in finding pivots.

Applying row transformation: $R_2 \to R_2 - \left(\frac{3}{1}\right)R_1$ $R_3 \to R_3 - \left(\frac{5}{1}\right)R_1$

$\downarrow$ contributes to L_{21} $\downarrow$ contributes to L_{31}

$$A \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

Row transformation: $R_3 \to R_3 - \left(\frac{3}{3}\right)R_2$

$\downarrow$ contributes to L_{32}

$$U = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$$

$A = LU$

we still need to further decompose U to DU

$$U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/1 & 3/1 & 5/1 \\ 0 & 3/3 & 3/3 \\ 0 & 0 & 2/2 \end{bmatrix}}_{U}$$

$$\therefore A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$\therefore$ Since A is a symmetric L and U are transposes of each other.

Q11) $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$

Applying row transformations: $R_2 \to R_2 - \underbrace{\left(\frac{-4}{2}\right)}_{\text{contributes to } L_{21}} R_1$ $\quad R_3 \to R_3 - \underbrace{\left(\frac{2}{2}\right)}_{\text{contributes to } L_{31}} R_1$ $\quad R_4 \to R_4 - \underbrace{\left(\frac{-6}{2}\right)}_{\text{contributes to } L_{41}} R_1$

$A \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$

Applying row transformations: $R_3 \to R_3 - \underbrace{\left(\frac{-9}{3}\right)}_{\text{contributes to } L_{32}} R_2$ $\quad R_4 \to R_4 - \underbrace{\left(\frac{12}{3}\right)}_{\text{contributes to } L_{42}} R_2$

$A \sim \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & +7 \end{bmatrix} \end{array}$

Since C_3 has 0 in both the last row, we use C_4 to get multiplier.

Applying row transformation : $R_4 \to R_4 - \underbrace{\left(\frac{4}{2}\right)}_{\text{contributes to } L_{43}} R_3$

$U = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Since, there is no zero row, $r(A) = 4$.

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$ $\therefore A = LU$

(decomposition is complete)

Q12) Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$

First, we will use Gaussian elimination to find U.

Applying row transformations: $R_2 \to R_2 - \underbrace{\left(\frac{4}{2}\right)}_{L_{21}} R_1$ $\quad R_3 \to R_3 - \underbrace{\left(\frac{8}{2}\right)}_{L_{31}} R_1$ $\quad R_4 \to R_4 - \underbrace{\left(\frac{6}{2}\right)}_{L_{41}} R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

Applying row transformations: $R_3 \to R_3 - \underbrace{\left(\frac{3}{2}\right)}_{L_{32}} R_2$ $\quad R_4 \to R_4 - \underbrace{\left(\frac{4}{2}\right)}_{L_{42}} R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 3.5 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Applying row transformation: $R_4 \to R_4 - \underbrace{\left(\frac{2}{2}\right)}_{L_{43}} R_3$

$$U = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 7/2 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 3/2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

Decomposition of U into DU

$$U = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} 2/2 & 1/2 & 1/2 & 0/2 \\ 0 & 2/2 & 2/2 & 1/2 \\ 0 & 0 & 2/2 & 7/4 \\ 0 & 0 & 0 & (5/2)/(5/2) \end{bmatrix}$$

$$\therefore A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 3/2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 7/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

Q13) given $A = \begin{bmatrix} 1 & a & b \\ 1 & a & 2 \\ 1 & 0 & b \end{bmatrix}$

To find Inverse, we need to augment A with a 3x3 identity matrix

$$[A\ I] = \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 1 & a & 2 & 0 & 1 & 0 \\ 1 & 0 & b & 0 & 0 & 1 \end{array}\right]$$

For Gauss-Jordan method, we need to apply row transformation on A such that we can can convert it to identity matrix.

Applying row transformations: $R_2 \to R_2 - \left(\frac{1}{1}\right) R_1$ $\quad R_3 \to R_3 - \left(\frac{1}{1}\right) R_1$

$$[A\ I] \sim \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 0 & 2-b & -1 & 1 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \end{array}\right]$$

Applying row transformations: $R_3 \leftrightarrow R_2$

$$[A\ I] \sim \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

Applying row transformation: $R_1 \to R_1 - \left(\frac{b}{2-b}\right) R_3$ $\quad R_2 \to R_2 - \left(\frac{0}{2-b}\right) R_3$

$$[A\ I] \sim \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1+\frac{b}{2-b} & \frac{-b}{2-b} & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

Applying row transformation $R_1 \to R_1 - \left(\frac{-a}{a}\right) R_2$

$$[A\ I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \left(\frac{2}{2-b}-1\right) & \frac{-b}{2-b} & 1 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

Multiplication on each row: $R_1/1$, $R_2/(-a)$ $\quad R_3/(2-b)$

$$[I\ A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{b}{2-b} & \frac{-b}{2-b} & 1 \\ 0 & 1 & 0 & 1/a & 0 & -1/a \\ 0 & 0 & 1 & -1/2-b & 1/2-b & 0 \end{array}\right]$$

we observe that $A^{-1} = \begin{bmatrix} \frac{b}{2-b} & \frac{-b}{2-b} & 1 \\ 1/a & 0 & -1/a \\ \frac{-1}{2-b} & \frac{1}{2-b} & 0 \end{bmatrix}$

Comparing with $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

we find: ① $\frac{1}{a} = 1$ and $\frac{-1}{a} = -1$ $\therefore a = 1$

② $\frac{-1}{2-b} = -1$ and $\frac{1}{2-b} = +1$ $\therefore b = 1$

Q14) Given matrices are:

$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Converting in ~~equation~~ augmented form:

~~$10x + 20y = 1$~~

$\begin{bmatrix} 10 & 20 & 1 \\ 20 & 50 & 0 \end{bmatrix}$ $\begin{bmatrix} 10 & 20 & 0 \\ 20 & 50 & 1 \end{bmatrix}$

$R_2 \rightarrow R_2 - \left(\frac{20}{10}\right)R_1$ $R_2 \rightarrow R_2 - \left(\frac{20}{10}\right)R_1$

$\begin{bmatrix} 10 & 20 & 1 \\ 0 & 10 & -2 \end{bmatrix}$ $\begin{bmatrix} 10 & 20 & 0 \\ 0 & 10 & 1 \end{bmatrix}$

Backward substition yields:

$y = -1/5$ $v = 1/10$

$\Rightarrow 10(x) + 20(-1/5) = 1$ $\Rightarrow 10(u) + 20(v) = 0$

$\therefore x = 1/2$ $\therefore u = -1/5$

Substituting values in A^{-1}, we get $A^{-1} = \begin{bmatrix} 1/2 & -1/5 \\ -1/5 & 1/10 \end{bmatrix}$

Given equation $A^{-1}B = I$

Pre-multiplication of A on both sides:

$(AA^{-1})B = AI$ (we know $AA^{-1} = I$)

$\therefore B = A$

We know A^{-1} So $A = (A^{-1})^{-1}$

$$\therefore A = \frac{1}{\det(A^{-1})}\begin{bmatrix} 1/10 & 1/5 \\ 1/5 & 1/2 \end{bmatrix}$$

$$\det(A^{-1}) = \left(\frac{1}{2}\times\frac{1}{10}\right) - \left(\frac{-1}{5}\times\frac{-1}{5}\right)$$
$$= \frac{1}{20} - \frac{1}{25}$$
$$= \frac{1}{100}$$

$$\therefore B = 100\begin{bmatrix} 1/10 & 1/5 \\ 1/5 & 1/2 \end{bmatrix}$$

Q15) Given $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$

We will augment A with a 4×4 identity matrix.

Applying row transformations: $R_2 \to R_2 - \left(\frac{1/4}{1}\right)R_1$ $\quad R_3 \to R_3 - \left(\frac{1/3}{1}\right)R_1$ $\quad R_4 \to R_4 - \left(\frac{1/2}{1}\right)R_1$

$$\cancel{A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1 \end{bmatrix}}$$

$$[A\ I] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 & -1/3 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1 & -1/2 & 0 & 0 & 1 \end{bmatrix}$$

Applying row transformations: $R_3 \to R_3 - \left(\frac{1/3}{1}\right)R_2$ $\quad R_4 \to R_4 - \left(\frac{1/2}{1}\right)R_2$

$$[A\ I] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 1/2 & 1 & -3/8 & -1/2 & 0 & 0 \end{bmatrix}$$

Applying row transformations: $R_4 \to R_4 - \left(\frac{1/2}{1}\right)R_3$

$$[A^{-1}\ I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & -1/3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1/4 & -1/3 & -1/2 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ -1/4 & -1/3 & 1 & 0 \\ -1/4 & -1/3 & -1/2 & 1 \end{bmatrix}$$

Q16) It is a polynomial of degree 2.

$$y = ax^2 + bx + c$$

It passes through the points (1,6), (2,3) and (3,2)

So, we generate 3 equations:

$$a + b + c = 6$$
$$4a + 2b + c = 3$$
$$9a + 3b + c = 2$$

Writing this in the form $Ax = b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

writing the augmented matrix

$$[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 3 \\ 9 & 3 & 1 & 2 \end{bmatrix}$$

Permorming Gaussian elimination, using row transformations:

$R_2 \to R_2 - \left(\frac{4}{1}\right) R_1$ $\quad$ $R_3 \to R_3 - \left(\frac{9}{1}\right) R_1$

$$[A\ b] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -21 \\ 0 & -6 & -8 & -52 \end{bmatrix}$$

Row transformation: $R_3 \to R_3 - \left(\frac{-6}{-2}\right) R_2$

$$[U\ c] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -21 \\ 0 & 0 & 1 & 11 \end{bmatrix}$$

Using backward substitution, we get

$\therefore c = 11$

$-2(b) - 3(11) = -21$

$\therefore b = -6$

$a + 1(-6) + 1(11) = 6$

$\therefore a = 1$

The required equation is $x^2 - 6x + 11$.

Q17) Chemical equation:

$$a\, C_3H_8 + b\, O_2 \longrightarrow c\, CO_2 + d\, H_2O$$

Comparing no. of carbon molecules on both sides:

$3a = c$

$\therefore 3a - c = 0 \rightarrow (1)$

Comparing no. of oxygen molecules on both sides:

$2b = 2c + d$

$\therefore 2b - 2c - d = 0 \rightarrow (2)$

Comparing no. of hydrogen molecules on both sides:

$8a = 2d$

$\therefore 4a - d = 0 \rightarrow (3)$

We have 3 equations and 4 variables, so its clear that we will have infinite solutions.

Writing in $Ax = b$ form:

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 4 & 0 & 0 & -1 \end{bmatrix}_{3\times4} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$$

The Augmented matrix is

$$[A\ b] = \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 4 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Performing Gaussian elimination:

$R_2 \to R_2 - \left(\frac{0}{3}\right)R_1$ $\quad\quad$ $R_3 \to R_3 - \left(\frac{4}{3}\right)R_1$

$$[A \ b] \sim \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \end{bmatrix} : [U \ c]$$

★ Since these are gas molecules, (a, b, c, d) have to be a set of natural numbers.

Backward substitution yields:

$$\frac{4}{3}c - d = 0 \quad\quad (\times 3 \text{ for natural numbers})$$

$$4c - 3d = 0$$

Let's assume $d = K$ (constant) ~~KER~~ $K \in N$

$$\therefore \ c = \frac{3K}{4}.$$

$$2(b) - 2\left(\frac{3K}{4}\right) - 1(K) = 0$$

$$\therefore b = \frac{5K}{4}.$$

$$3(a) + (-1)\left(\frac{3K}{4}\right) = 0$$

$$a = \frac{K}{4}$$

$\therefore$ ~~a, b,~~ The solⁿ for the equations is $\left(\frac{K}{4}, \frac{5K}{4}, \frac{3K}{4}, K\right)$

Since they have to be natural numbers, let's assume $K = 4$.

$$\therefore (a, b, c, d) = (1, 5, 3, 4)$$

The gas equation is:

$$C_3H_8 + 5O_2 \longrightarrow 3CO_2 + 4H_2O.$$

EOF