Chapter 15

Vector Differential Calculus: Gradient, Divergence and Curl

INTRODUCTION

Principal application of vector functions is the analysis of motion in space. The gradient defines the normal to the tangent plane, the directional derivative gives the rate of change in any given direction. If $\overline{F}$ is the velocity field of a fluid flow, then divergence of $\overline{F}$ at a point $P(x, y, z)$ (flux density) is the rate at which fluid is (diverging) piped in or drained away at P, and the curl $\overline{F}$ (or circulation density) is the vector of greatest circulation in flow. We express grad, div and curl in general curvilinear coordinates and in cylindrical and spherical coordinates which are useful in engineering, physics or geometry involving a cylinder or cone or a sphere.

In this Chapter 15, vector differential calculus is considered, which extends the basic concepts of (ordinary) differential calculus to vector functions, by introducing derivative of a vector function and the new concepts of gradient, divergence and curl.

15.1 VECTOR DIFFERENTIATION

Definitions

Scalar function

Scalar function of a scalar variable t is a function $F = F(t)$ which uniquely associates a scalar $F(t)$ for every value of the scalar t in an interval $[a, b]$.

Scalar field

Scalar field is a region in space such that for every point P in this region, the scalar function f associates a scalar $f(P)$.

Scalar function of a vector variable $\overline{u}$ is a function $F = F(\overline{u})$ which uniquely associates a scalar $F(\overline{u})$ for every vector $\overline{u}$.

Vector function

Vector function of a scalar variable t is a function $\overline{F} = \overline{F}(t)$ which uniquely associates a vector $\overline{F}$ for each scalar t.

Vector field

Vector field is a region in space such that with every point P in that region, the vector function $\overline{V}$ associates a vector $\overline{V}(P)$.

Vector function

Vector function of a vector variable $\overline{u}$ is $\overline{F} = \overline{F}(\overline{u})$ if for every $\overline{u}$ a unique vector $\overline{F}(\overline{u})$ is associated.

Derivative

Derivative of a vector function $\overline{F}(u)$ with respect to a scalar variable u is denoted by and is defined as

$$\frac{d\overline{F}}{du} = \lim_{\Delta u \to 0} \frac{\overline{F}(u + \Delta u) - \overline{F}(u)}{\Delta u}.$$

Let $\overline{i}, \overline{j}, \overline{k}$ be three mutually orthogonal unit vectors in the direction of the x, y, z-coordinate axes such that $\overline{i}, \overline{j}, \overline{k}$ form a right handed triad (i.e., $i \cdot i = 1, i \cdot j = 0, \overline{i} \cdot k = 0, j \cdot j = 1, \ldots$ etc.).

Derivative in the Component Form

Let $\overline{F}(u) = F_1(u)\overline{i} + F_2(u)\overline{j} + F_3(u)\overline{k}$ in the component form with $F_1(u)$, $F_2(u)$ and $F_3(u)$ as components of $\overline{F}$ in the x, y, z-coordinate axes.

Then
$$\frac{d\overline{F}}{du} = \frac{dF_1}{du}\overline{i} + \frac{dF_2}{du}\overline{j} + \frac{dF_3}{du}\overline{k}.$$

Thus the derivative of a vector function $\overline{F}$ with respect to a scalar variable u is the vector whose components are the derivatives of the components F_1, F_2, F_3 of $\overline{F}$ with respect to u.

Results: Most of the basic rules of differentiation that are true for a scalar function of a scalar variable hold good for vector function of a scalar variable, provided the order of factors in vector products is maintained.

1. $\frac{d\overline{C}}{du} = 0$ $\quad(\overline{C} = \text{constant vector})$
2. $\frac{d}{du}[\overline{F}(u) \pm \overline{G}(u)] = \frac{d\overline{F}}{du} \pm \frac{d\overline{G}}{du}$
3. $\frac{d}{du}[\alpha(u)\overline{F}(u)] = \alpha(u)\frac{d\overline{F}}{du} + \overline{F}\frac{d\alpha}{du}$
4. $\frac{d}{du}[\overline{F}(u)\cdot\overline{G}(u)] = \frac{d\overline{F}}{du}\cdot\overline{G} + \overline{F}\cdot\frac{d\overline{G}}{du}$
5. $\frac{d}{du}[\overline{F}(u) \times \overline{G}(u)] = \overline{F} \times \frac{d\overline{G}}{du} + \frac{d\overline{F}}{du} \times \overline{G}$
6. $\frac{d}{du}[\overline{A}(u)\cdot\overline{B}(u) \times \overline{C}(u)] = \overline{A}\cdot\overline{B} \times \frac{d\overline{C}}{du} + \overline{A}\cdot\frac{d\overline{B}}{du} \times \overline{C} + \frac{d\overline{A}}{du}\cdot\overline{B} \times \overline{C}.$
7. $\frac{d}{du}[\overline{A} \times (\overline{B} \times \overline{C})] = \overline{A} \times \left(\overline{B} \times \frac{d\overline{C}}{du}\right) + \overline{A} \times \left(\frac{d\overline{B}}{du} \times \overline{C}\right) + \frac{d\overline{A}}{du} \times (\overline{B} \times \overline{C}).$

Velocity and Acceleration

Let $\overline{r}$ be the position vector of a point P $(x(t), y(t), z(t))$ in space where t is the scalar time. Then $\overline{r}$ in the component form is

$$\overline{r} = \overline{r}(t) = x(t)\overline{i} + y(t)\overline{j} + z(t)\overline{k}$$

Derivative of $\overline{r}$ denoted by $\dot{\overline{r}}$ is defined as

$$\frac{d\overline{r}}{dt} = \dot{\overline{r}} = \lim_{\Delta t \to 0} \frac{\overline{r}(t + \Delta t) - \overline{r}(t)}{\Delta t} = \frac{dx}{dt}\overline{i} + \frac{dy}{dt}\overline{j} + \frac{dz}{dt}\overline{k}$$

$\dot{\overline{r}}$ and $\ddot{\overline{r}}$ denote the velocity and acceleration of a particle with position vector $\overline{r}$.

Unit Tangent Vector

Let s be the arc length reckoned (measured) from a fixed point M_0 of a space curve c whose equation is $\overline{r} = \overline{r}(s)$. Then the unit tangent vector of c is

$$\frac{d\overline{r}}{ds} = \frac{dx}{ds}\overline{i} + \frac{dy}{ds}\overline{j} + \frac{dz}{ds}\overline{k}$$

such that

$$\left|\frac{d\overline{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1.$$

Partial Derivatives of a Vector Function $\overline{F}$

which depends on more than one scalar variables, u, v, w: The partial derivative of $\overline{F}$ with respect to u is

$$\frac{\partial\overline{F}}{\partial u} = \lim_{\Delta u \to 0} \frac{\overline{F}(u + \Delta u, v, w) - \overline{F}(u, v, w)}{\Delta u}$$

In the component form, if $\overline{F}(u,v,w) = F_1(u,v,w)i + F_2(u,v,w)j + F_3(u,v,w)k$ then the partial derivative of $\overline{F}$ with respect to say u is obtained by taking the partial derivatives of the components F_1, F_2, F_3 of $\overline{F}$ with respect to u. i.e.,

$$\frac{\partial\overline{F}}{\partial u} = \frac{\partial F_1}{\partial u}\overline{i} + \frac{\partial F_2}{\partial u}\overline{j} + \frac{\partial F_3}{\partial u}\overline{k}$$

Higher order partial derivatives can be obtained similarly.

WORKED OUT EXAMPLES

Example 1: Find the magnitude of the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time $t > 0$. Find unit tangent vector to the curve.

Solution: The position vector $\overline{r}$ of the particle is

$$\overline{r}(t) = x(t)i + y(t)j + z(t)\overline{k}$$
$$\overline{r}(t) = 2\sin 3t\overline{i} + 2\cos 3tj + 8t\overline{k}$$
$$\text{Velocity} = \overline{V} = \frac{d\overline{r}}{dt} = 6\cos 3t\, i - 6\sin 3t\, j + 8\overline{k}$$
$$\text{Acceleration} = \overline{a} = \dot{\overline{V}} = \frac{d\ddot{r}}{dt}$$
$$= -18\sin 3t\, i - 18\cos 3t\, j + 0$$
$$|\overline{V}| = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64}$$
$$= \sqrt{36 + 64} = 10$$
$$|\overline{a}| = \sqrt{18^2\sin^2 3t + 18^2\cos^2 3t} = 18.$$
$$\text{Unit tangent vector} = \frac{d\overline{r}}{dt} \Big/ \left|\frac{d\overline{r}}{dt}\right|$$
$$= \frac{1}{10}[6\cos 3ti - 6\sin 3t\overline{j} + 8\overline{k}]$$

Example 2: If $\bar{A} = t^2 i - tj + (2t+1)\bar{k}$, $\bar{B} = (2t-3)i + j - tk$ find (a) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$ (b) $\frac{d}{dt}(\bar{A}\times\bar{B})$ (c) $\frac{d}{dt}|A+B|$ (d) $\frac{d}{dt}\left(\bar{A}\times\frac{dB}{dt}\right)$ at $t=1$.

Solution:

a. $\bar{A}\cdot B = t^2(2t-3) - t + (2t+1)(-t)$

$$\frac{d}{dt}(\bar{A}\cdot\bar{B}) = 6t^2 - 6t - 1 - 4t - 1\Big|_{\text{at } t=1} = -6$$

b. $$\bar{A}\times\bar{B} = \begin{vmatrix} i & j & k \\ t^2 & -t & (2t+1) \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$\bar{A}\times\bar{B} = i(t^2 - 2t - 1) + j(+t^3 + 4t^2 - 4t - 3) + \bar{k}(3t^2 - 3t)$$

$$\frac{d}{dt}(\bar{A}\times\bar{B}) = (2t-2)\bar{i} + (3t^2 + 8t - 4)\bar{j} + (6t-3)\bar{k}$$

$$\text{At } t=1,\ \frac{d}{dt}(\bar{A}\times\bar{B}) = 7\bar{j} + 3\bar{k}$$

c. $$\bar{A}+\bar{B} = (t^2 + 2t - 3)\bar{i} + (1-t)j + (t+1)\bar{k}$$

$$|\bar{A}+\bar{B}| = \sqrt{(t^2+2t-3)^2 + (1-t)^2 + (t+1)^2}$$

$$= \sqrt{t^4 + 4t^3 - 12t + 11}$$

$$\frac{d}{dt}|\bar{A}+\bar{B}| = \frac{4t^3 + 12t^2 - 12}{2\sqrt{t^4 + 4t^3 - 12t + 11}} \text{ at } t = 1 \text{ is } 1.$$

d. $\frac{dB}{dt} = 2\bar{i} + 0 - \bar{k}$

$$\bar{A}\times\frac{d\bar{B}}{dt} = \begin{vmatrix} i & j & k \\ t^2 & -t & (2t+1) \\ 2 & 0 & -1 \end{vmatrix}$$

$$= t\bar{i} + (t^2 + 4t - 2)j + 2t\bar{k}$$

$$\frac{d}{dt}\left(\bar{A}\times\frac{d\bar{B}}{dt}\right) = i + (2t+4)\bar{j} + 2\bar{k} \text{ at } t = 1$$

$$\text{is } \bar{i} + 6j + 2k$$

Alter: $$\frac{d}{dt}\left(\bar{A}\times\frac{d\bar{B}}{dt}\right) = \frac{d\bar{A}}{dt}\times\frac{d\bar{B}}{dt} + \bar{A}\times\frac{d^2\bar{B}}{dt^2}$$

$$\frac{d\bar{A}}{dt} = 2ti - j + 2\bar{k}$$

$$\frac{d\bar{A}}{dt}\times\frac{d\bar{B}}{dt} = \begin{vmatrix} i & j & k \\ 2t & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= i + (2t+4)j + 2k$$

Also $$\frac{d^2\bar{B}}{dt^2} = 0 + 0 + 0$$

so that $$\bar{A}\times\frac{d^2\bar{B}}{dt^2} = 0$$

Thus $$\frac{d}{dt}\left(\bar{A}\times\frac{d\bar{B}}{dt}\right) = i + (2t+4)j + 2k \text{ at } t = 1$$

is $i + 6j + 2k$.

Example 3: If $\bar{A} = \cos xy\, i + (3xy - 2x^2)j - (3x+2y)\bar{k}$ find

$$\frac{\partial\bar{A}}{\partial x}, \frac{\partial\bar{A}}{\partial y}, \frac{\partial^2 A}{\partial x^2}, \frac{\partial^2 A}{\partial y^2}, \frac{\partial^2 A}{\partial x\partial y}, \frac{\partial^2 A}{\partial y\partial x}.$$

Solution:

$$\frac{\partial\bar{A}}{\partial x} = -y\sin xy\, i + (3y - 4x)j - (3\bar{k})$$

$$\frac{\partial\bar{A}}{\partial y} = -x\sin xy\bar{i} + (3xj) - 2\bar{k}$$

$$\frac{\partial^2\bar{A}}{\partial x^2} = -y^2\cos xy\, i - 4j$$

$$\frac{\partial^2 A}{\partial y^2} = -x^2\cos xy\, i$$

$$\frac{\partial^2 A}{\partial x\partial y} = (-\sin xy - xy\cos xy)i + 3j.$$

Example 4: Prove that $\bar{A}\cdot\frac{d\bar{A}}{dt} = 0$ if $\bar{A}$ is a constant vector.

Solution: For any vector $\bar{A}$, $\bar{A}\cdot\bar{A} = A^2$.
Differentiating w.r.t., t

$$\frac{d}{dt}(\bar{A}\cdot\bar{A}) = \bar{A}\cdot\frac{d\bar{A}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{A} = 2A\frac{dA}{dt}$$

$$2\bar{A}\cdot\frac{d\bar{A}}{dt} = 2A\frac{dA}{dt}$$

If A is a vector of constant magnitude $\frac{dA}{dt} = 0$ so that

$$\bar{A}\cdot\frac{d\bar{A}}{dt} = 0.$$

EXERCISE

1. If $\bar{A} = 5t^2 i + tj - t^3 k$ and $\bar{B} = \sin ti - \cos tj$, find (a) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$, (b) $\frac{d}{dt}(\bar{A}\times\bar{B})$ (c) $\frac{d}{dt}(\bar{A}\cdot\bar{A})$.

Ans. **a.** $(5t^2-1)\cos t + 11t\sin t$
b. $(t^3\sin t - 3t^2\cos t)i - (t^3\cos t + 3t^2\sin t)j + (5t^2\sin t - 11t\cos t - \sin t)\overline{k}$
c. $100t^3 + 2t + 6t^5$

2. Find (a) $\frac{d}{du}(\overline{A}\cdot\overline{B})$ (b) $\frac{d}{du}(\overline{A}\times\overline{B})$ if $\overline{A}(u) = 2ui - 3u^2j + u^3k$, $\overline{B}(u) = \sin ui - uk$.

Ans. $2\sin u + 2u\cos u - 4u^3; 9u^2i + (u^3\cos u + 3u^2\sin u + 4u)j + (3u^2\cos u + 6u\sin u)\overline{k}$

3. If $\overline{A} = 2ti - t^2j + t^3k$, $\overline{B} = -ti + t^2k$ and $\overline{C} = t^3j - 2tk$ find $\frac{d}{dt}(\overline{A}\cdot\overline{B}\times\overline{C})$ at $t=1$.

Ans. $-12t^5 + 8t^3 - 7t^6, -11$

4. If $\overline{A} = \sin ui + \cos uj + uk$, $\overline{B} = \cos ui - \sin uj - 3k$ and $\overline{C} = 2i + 3j - k$ find $\frac{d}{du}(\overline{A}\times(\overline{B}\times\overline{C}))$ at $u=0$.

Ans. $7i + 6j - 6k$

5. If $\overline{A} = (2x^2y - x^4)i + (e^{xy} - y\sin x)j + (x^2\cos y)\overline{k}$ find (a) $\frac{\partial A}{\partial x}$, (b) $\frac{\partial A}{\partial y}$, (c) $\frac{\partial^2 A}{\partial x^2}$, (d) $\frac{\partial^2 \overline{A}}{\partial y^2}$, (e) $\frac{\partial^2 \overline{A}}{\partial x\partial y}$, (f) $\frac{\partial^2 \overline{A}}{\partial y\partial x}$.

Ans. **a.** $(4xy - 4x^3)i + (ye^{xy} - y\cos x)j + 2x\cos yk$
b. $2x^2i + (xe^{xy} - \sin x)j - x^2\sin y\overline{k}$
c. $(4y - 12x^2)\overline{i} + (y^2e^{xy} + y\sin x)j + 2\cos yk$
d. $\overline{0} + x^2e^{xy}j - x^2\cos y\overline{k}$
e, f. $4xi + (xye^{xy} + e^{xy} - \cos x)j - 2x\sin yk$

6. Find $\frac{\partial^3}{\partial x^2\partial z}(f\overline{A})$ at the point $(2, -1, 1)$ if

$$f = xy^2z, \overline{A} = xzi - xy^2j + yz^2k.$$

Ans. $4y^2zi - 2y^4j; 4i - 2j$

7. Find $\frac{\partial^2(\overline{A}\times\overline{B})}{\partial x\partial y}$ at $(1, 0, -2)$ if $\overline{A} = x^2yzi - 2xz^3j + xz^2\overline{k}$, $\overline{B} = 2zi + yj - x^2k$.

Ans. $-4i - 8j$

8. Prove that $\overline{A}\times\frac{d\overline{A}}{dt} = 0$ if $\overline{A}(t)$ has constant (fixed) direction.

Hint: Take $\overline{A} = a(t)\overline{B}(t)$ where $a(t) = |\overline{A}|$ and $\overline{B}(t)$ is a unit vector in the direction of $\overline{A}$ so that $\frac{d\overline{B}}{dt} = 0$.

9. Given the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$ find the unit tangent vector at the point $t = 2$.

Ans. $ti + 2j + (2t-3)k/\left(\sqrt{5t^2 - 12t + 13}\right)$; $(2i + 2j + k)/3$

10. Find the angle between the tangents to the curve $\overline{r} = t^2i + 2tj - t^3k$ at the points $t = \pm 1$.

Hint: $\overline{T}_1\cdot\overline{T}_2 = T_1T_2\cos\theta$.

Ans. $\theta = \cos^{-1}(9/17)$

11. Determine the magnitude of velocity and acceleration at $t = 0$ of a particle moving along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$; where t is the time.

Ans.
$$\overline{V} = -e^{-t}i - 6\sin 3t\, j + 6\cos 3t\, k$$
$$\overline{a} = e^{-t}i - 18\cos 3t\, j - 18\sin 3t\, k$$
$|\overline{V}|$ at $t=0$ is $\sqrt{37}$; $|\overline{a}|$ at $t=0$ is $\sqrt{325}$

12. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\overline{i} - 3\overline{j} + 2\overline{k}$.

Hint: Component of $\overline{V}$ = dot product of $\overline{V}$ with unit vector in the direction of $i - 3j + 2k$.

Ans. $8\sqrt{14}/7; -\sqrt{14}/7$

13. If a, b, w are constants show that the acceleration of a particle with displacement vector $\overline{r} = a\cos wt + b\sin wt$ is always directed towards the origin.

Hint: $\overline{a} = \ddot{\overline{r}} = -w^2\overline{r}$.

14. Find the angle between the directions of the velocity and acceleration vectors at time t of a particle with position vector $\overline{r} = (t^2+1)i - 2tj + (t^2-1)k$.

Ans. $\text{arc}\cos t\sqrt{2}/\sqrt{2t^2+1}$

15. Prove that $\frac{d}{du}(\overline{A}\times\overline{B}) = \overline{C}\times(\overline{A}\times\overline{B})$ if

$$\frac{d\overline{A}}{du} = \overline{C}\times\overline{A} \quad \text{and} \quad \frac{d\overline{B}}{du} = \overline{C}\times\overline{B}$$

Hint:

$$\frac{d}{du}(\overline{A}\times\overline{B}) = \overline{A}\times\frac{d\overline{B}}{du} + \frac{d\overline{A}}{du}\times\overline{B} = \overline{A}\times(\overline{C}\times\overline{B}) + (\overline{C}\times\overline{A})\times\overline{B}$$

$$= (A\cdot B)C - (A\cdot C)B - (\overline{B}\cdot\overline{A})C + (B\cdot C)A$$

$$= \overline{C} \times (\overline{A} \times \overline{B}).$$

15.2 DIRECTIONAL DERIVATIVE, GRADIENT OF A SCALAR FUNCTION AND CONSERVATIVE FIELD

In vector differential calculus, it is very convenient to introduce the symbolic linear vector differential "Hamiltonian" operator **del** defined and denoted as

$$\nabla = \overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z} \tag{1}$$

This operator read as del (or nabla) is *not* a vector (neither has magnitude nor direction) but combines both differential and vectorial properties analogous to those of ordinary vectors.

Directional Derivative

If $f = f(x, y, z)$ then the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the derivatives (rates of change) of f in the "direction" of the coordinate axes OX, OY, OZ respectively. This concept can be extended to define a derivative of f in a "given" direction $\overline{PQ}$ (Fig. 15.1).

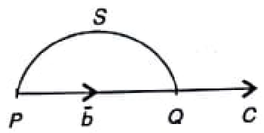

Fig. 15.1

Let P be a point in space and $\overline{b}$ be a unit vector from P in the given direction. Let s be the arc lengths measured from P to another point Q along the ray C in the direction of $\overline{b}$. Now consider

$$f(s) = f(x, y, z) = f((s), y(s), z(s))$$

Then
$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \tag{2}$$

The *directional derivative* of f at the point P in the given direction $\overline{b}$ is $\frac{df}{ds}$ given by (2). $\frac{df}{ds}$ gives the rate of change of f in the direction of $\overline{b}$.

Since
$$\frac{dx}{ds}\overline{i} + \frac{dy}{ds}\overline{j} + \frac{dz}{ds}\overline{k} = \overline{b} = \text{unit vector} \tag{3}$$

Using the del operator defined by (1) $\frac{df}{ds}$ given by (2) can be rewritten as

$$\frac{df}{ds} = \left(\overline{i}\frac{\partial f}{\partial x} + \overline{j}\frac{\partial f}{\partial y} + \overline{k}\frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}\overline{i} + \frac{dy}{ds}\overline{j} + \frac{dz}{ds}\overline{k}\right)$$

$$\frac{df}{ds} = \left[\left(\overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}\right) f\right]\cdot\overline{b} = \nabla f\cdot\overline{b} \tag{4}$$

Thus the directional derivative of f at P is the component (dot product) of ∇f in the direction of (with) unit vector $\overline{b}$.

The directional derivative in the direction of any (non unit) vector $\overline{a}$ is

$$\frac{df}{ds} = \nabla f\cdot\left(\frac{\overline{a}}{|\overline{a}|}\right) \tag{5}$$

Equation (4) introduces the vector quantity, the ***gradient of a scalar function*** $f(x, y, z)$ or gradient f denoted by ∇f and defined as

$$\nabla f = \left(\overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}\right) f$$

$$= \overline{i}\frac{\partial f}{\partial x} + \overline{j}\frac{\partial f}{\partial y} + \overline{k}\frac{\partial f}{\partial z} = \text{grad } f = \text{vector}$$

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right].$$

Properties of gradient

1. Projection of ∇f in any direction is equal to the derivative of f in that direction.
2. The gradient of f is in the direction of the normal to the level surface $f(x, y, z) = C =$ constant. So, the angle between any two surfaces $f(x, y, z) = C_1$ and $g(x, y, z) = C_2$ is the angle between their corresponding normals given by ∇f and ∇g respectively.
3. The gradient at P is in the direction of maximum increase of f at P.
4. The modulus of the gradient is equal to the largest directional derivative at a given point P.

i.e., $\max \left.\frac{df}{ds}\right|_P = \left|\nabla f\right|_P$

$$= \left.\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\right|_{\text{at } P.}$$

These properties thus state that the vector gradient f indicates the direction and magnitude of maximum change of a scalar function f at a given point.

Normal derivative

$\frac{df}{dn} = \nabla f \cdot \bar{n}$ where $\bar{n}$ is the unit normal to the surface f = constant.

Conservative

A vector function $\bar{A}$ is said to be a conservative vector field if $\bar{A} = \nabla f$ i.e., $\bar{A}$ is the gradient of a scalar function f. In this case f is known as the *potential function* of $\bar{A}$.

Instead of applying (operating) on a scalar function f, if del is applied to a vector function $\bar{A}$, we get divergence and curl: (see Sections 15.3, 15.4)

Worked Out Examples

Example 1: If $\bar{A} = 2x^2 i - 3yzj + xz^2 k$ and $f = 2z - x^3 y$ find (i) $A \cdot \nabla f$ and (ii) $A \times \nabla f$ at the point $(1, -1, 1)$.

Solution: Here $\frac{\partial f}{\partial x} = -3x^2 y$, $\frac{\partial f}{\partial y} = -x^3$, $\frac{\partial f}{\partial z} = 2$ so that

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\nabla f = -3x^2 yi - x^3 j + 2k$$

i.

$$\bar{A} \cdot \nabla f = (2x^2 i - 3yzj + xz^2 k) \cdot (-3x^2 yi - x^3 j + 2k) = -6x^4 y + 3x^3 yz + 2xz^2$$

$$\bar{A} \cdot \nabla f\Big|_{1,-1,1} = 6 - 3 + 2 = 5$$

ii.

$$\bar{A} \times \nabla f = \begin{vmatrix} i & j & k \\ 2x^2 & -3yz & xz^2 \\ -3x^2 y & -x^3 & 2 \end{vmatrix}$$

Expanding the determinant

$$\bar{A} \times \nabla f = (-6yz + x^4 z^2)i - j(4x^2 + 3x^3 yz^2) + (-2x^5 - 9x^2 y^2 z)\bar{k}$$

$$\bar{A} \times \nabla f\Big|_{1,-1,1} = 7i - j - 11\bar{k}.$$

Example 2: Evaluate

i. ∇r^n iv. ∇r

ii. $\nabla |\bar{r}|^3$ v. $\nabla(\ln r)$

iii. $\nabla(3r^2 - 4\sqrt{r} + 6r^{-\frac{1}{3}})$ vi. $\nabla(r^{-1})$

Solution:

i. $\nabla r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z}$

$$\frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r}$$

$$\frac{\partial r^n}{\partial x} = nr^{n-2}x$$

Similarly, $\frac{\partial r^n}{\partial y} = nr^{n-2}y$, and

$$\frac{\partial r^n}{\partial z} = nr^{n-2}z.$$

Then $\nabla r^n = nr^{n-2}(xi + yj + zk) = nr^{n-2}\bar{r}$

ii. Put $n = 3$ in the result (i) above

$$\nabla r^3 = 3r^{3-2}\bar{r} = 3r\bar{r}$$

iii. $\nabla(3r^2 - 4\sqrt{r} + 6r^{-\frac{1}{3}})$

$$= 3\nabla r^2 - 4\nabla r^{\frac{1}{2}} + 6\nabla r^{-\frac{1}{3}}$$

Applying result (i) above with $n = 2, \frac{1}{2}, -\frac{1}{3}$, we get

$$= 3(2r^{2-2}\bar{r}) - 4\left(\frac{1}{2}r^{\frac{1}{2}-2}\bar{r}\right) + 6\left(-\frac{1}{3}r^{-\frac{1}{3}-2}\bar{r}\right)$$

$$= \left(6 - 2r^{-\frac{3}{2}} - 2r^{-\frac{7}{3}}\right)\bar{r}$$

iv. $\nabla r = 1r^{1-2}\bar{r} = \frac{\bar{r}}{r}$

v. $\nabla f = \nabla \ln r = \frac{1}{r}\nabla r = \frac{1}{r}\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$

vi. $\nabla(r^{-1}) = -1 \cdot r^{-1-2}\bar{r} = -\bar{r}/r^3$.

Example 3: Find the directional derivative of $f(x, y, z) = 4e^{2x-y+z}$ at the point $(1, 1, -1)$ in the direction toward the point $(-3, 5, 6)$.

Solution:

$$\nabla f = 4e^{2x-y+z}(2i - j + k)$$

$$\nabla f\Big|_{(1,1,-1)} = 4(2i - j + k)$$

A unit vector $\hat{a}$ from the point $(1, 1, -1)$ in the direction toward the point $(-3, 5, 6)$ is

$$\hat{a} = \frac{-4i + 4j + 7k}{\sqrt{16 + 16 + 49}} = \frac{-4i + 4j + 7k}{9}$$

The required directional derivative is

$$\nabla f\Big|_{(1,1,-1)} \cdot \hat{a} = 4(2i - j + k) \cdot \frac{(-4i + 4j + 7k)}{9} = -\frac{20}{9}.$$

Example 4: Find the values of the constants a, b, c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

Solution: Since $\overline{k}$ is a unit vector parallel to the z-axis, the maximum of magnitude of the directional derivative of f at $(1, 2, -1)$ in the direction parallel to z-axis is given by

$$\nabla f_{\text{at }(1,2,-1)} \cdot \overline{k}$$

Here
$$\nabla f = (ay^2 + 3x^2cz^2)i + (2axy + bz)j + (by + 2czx^3)k$$

So that

$$\nabla f_{\text{at }(1,2-1)} = (4a + 3c)i + (4a - b)\overline{j} + (2b - 2c)\overline{k}$$

$$\begin{aligned} \text{Maximum} &= \nabla f_{\text{at }(1,2,-1)} \cdot k \\ &= [(4a + 3c)i + (4a - b)j + (2b - 2c)k] \cdot k \\ &= (2b - 2c) \end{aligned}$$

It is given in the problem that this maximum is 64. Thus

$$2b - 2c = 64$$

or
$$b - c = 32 \tag{1}$$

Since ∇f is in the direction of z-axis, it is perpendicular to the x and y-axes

Thus

$$\nabla f\Big|_{(1,2,-1)} \cdot i = [(4a + 3c)i + (4a - b)j + (2b - 2c)k] \cdot i$$

$$= 4a + 3c = 0 \tag{2}$$

Similarly,

$$\nabla f \cdot j = 4a - b = 0 \tag{3}$$

Solving the Equations (1), (2), (3), we get

$$a = 6, b = 24, c = -8.$$

Example 5: Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

Solution: The given surfaces

$$f = ax^2 - byz - (a + 2)x = 0 \tag{1}$$

and
$$g = 4x^2y + z^3 = 4 \tag{2}$$

are orthogonal at the point P $(1, -1, 2)$ provided the normals to (1) and (2) at P are at right angles. The normal to surface (1) is given by ∇f,

$$\nabla f\Big|_{1,-1,2} = [2ax - (a + 2)]\overline{i} - bz\overline{j} - by\overline{k}\ \Big|_{1,-1,2}$$

$$\nabla f = (a - 2)i - 2bj + bk \tag{3}$$

and normal to surface (2) by ∇g,

$$\nabla g\Big|_{1,-1,2} = 8xyi + 4x^2j + 3z^2k\ \Big|_{1,-1,2}$$

$$\nabla g = -8i + 4j + 12k \tag{4}$$

The orthogonality condition is

$$0 = \nabla f \cdot \nabla g = [(a - 2)i - 2bj + bk] \cdot [-8i + 4j + 12k]$$

$$0 = -2a + 4 + b \tag{5}$$

Since $(1, -1, 2)$ lies on the surface (1), we have

$$a + 2b - (a + 2) = 0$$

i.e.,
$$b = 1.$$

So from (5), $a = \frac{5}{2}$.

Example 6: If $\nabla f = (y^2 - 2xyz^3)i + (3 + 2xy - x^2z^3)j + (6z^3 - 3x^2yz^2)\overline{k}$, find f if $f(1, 0, 1) = 8$.

Solution: Since

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = \nabla f = (y^2 - 2xyz^3)i + (3 + 2xy - x^2z^3)j + (6z^3 - 3x^2yz^2)k$$

We have

$$\frac{\partial f}{\partial x} = y^2 - 2xyz^3 \quad (1)$$

$$\frac{\partial f}{\partial y} = 3 + 2xy - x^2z^3 \quad (2)$$

$$\frac{\partial f}{\partial z} = 6z^3 - 3x^2yz^2 \quad (3)$$

Integrating (1), (2), (3) partially w.r.t., x, y, z respectively, we get

$$f = xy^2 - x^2yz^3 + c_1(y, z) \quad (4)$$

$$f = 3y + xy^2 - x^2yz^3 + c_2(x, z) \quad (5)$$

$$f = \frac{6}{4}z^4 - x^2yz^3 + c_3(x, y) \quad (6)$$

where c_1, c_2, c_3 arbitrary functions of the variables indicated.

To find $c_1(y, z)$, differentiate (4) partially w.r.t. z and equate it with (3). Thus

$$0 - 3x^2yz^2 + \frac{\partial c_1}{\partial z} = \frac{\partial f}{\partial z} = 6z^3 - 3x^2yz^2$$

So $$\frac{\partial c_1}{\partial z} = 6z^3 \quad (7)$$

Integrating (7) partially w.r.t. 'z', we get

$$c_1(y, z) = \frac{6}{4}z^4 + c_4(y) \quad (8)$$

where c_4 is a function of y alone.

Substituting (8) in (4), we have

$$f = xy^2 - x^2yz^3 + \frac{3}{2}z^4 + c_4(y) \quad (9)$$

To find c_4, differentiate (9) partially w.r.t. y and equate it with (2), we get

$$2xy - x^2z^3 + 0 + \frac{dc_4}{dy} = \frac{\partial f}{\partial y} = 3 + 2xy - x^2z^3$$

So $$\frac{dc_4}{dy} = 3 \quad (10)$$

Integrating (10) w.r.t. y, we have

$$c_4(y) = 3y + c_5 \quad (11)$$

where c_5 is a pure arbitrary constant.

Substituting (11) in (9), we get the required

$$f(x, y, z) = xy^2 - x^2yz^3 + \frac{3}{2}z^4 + 3y + c_5$$

Since $\quad 8 = f(1, 0, 1) = 1 - 0 + 0 + 3 + c_5$

$$\therefore \quad c_5 = 4.$$

Hence $\quad f = xy^2 - x^2yz^3 + \frac{3}{2}z^4 + 3y + 4.$

Similar result can be obtained by starting from (5) or (6).

Example 7: Find $f(r)$ such that $\nabla f = \frac{\bar{r}}{r^5}$ and $f(1) = 0$.

Solution: It is given that

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = \nabla f = \frac{\bar{r}}{r^5} = \frac{xi + yj + zk}{r^5}$$

so $$\frac{\partial f}{\partial x} = \frac{x}{r^5}, \frac{\partial f}{\partial y} = \frac{y}{r^5}, \text{ and } \frac{\partial f}{\partial z} = \frac{z}{r^5}$$

We know that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \frac{x}{r^5}dx + \frac{y}{r^5}dy + \frac{z}{r^5}dz$$

$$df = \frac{xdx + ydy + zdz}{r^5} = \frac{rdr}{r^5} = r^{-4}dr$$

Integrating $$f(r) = \frac{r^{-3}}{-3} + c$$

Since $$0 = f(1) = -\frac{1}{3} + c$$

so $$c = \frac{1}{3}$$

Thus $$f(r) = \frac{1}{3} - \frac{1}{3}\frac{1}{r^3}.$$

EXERCISE

1. Find ∇f if $f = \ln(x^2 + y^2 + z^2)$.

Ans. $2(xi + yj + zk)/(x^2 + y^2 + z^2)$

2. If $f(x, y, z) = 3x^2y - y^3z^2$, find ∇f and $|\nabla f|$ at $(1, -2, -1)$.

Ans. $\nabla f = -12i - 9j - 16k, |\nabla f| = \sqrt{481}$

3. If $f = 2xz^4 - x^2y$, find ∇f and $|\nabla f|$ at $(2, -2, -1)$.

Ans. $10i - 4j - 16k, 2\sqrt{93}$

4. Find ∇f when $f = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2+y^2+z^2}}$.

Ans. $(2 - r)e^{-r}\bar{r}$

5. If $U = 3x^2y$, $V = xz^2 - 2y$ evaluate $\nabla[\nabla U . \nabla V]$.

Ans. $(6yz^2 - 12x)\bar{i} + 6xz^2\bar{j} + 12xyz\bar{k}$

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Ans. $\pm\frac{1}{3}(i - 2j - 2k)$

7. Find the unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3, 1, -4)$.

Ans. $(2i + j - 2k)/3$

8. Determine a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

Ans. $\pm(\bar{i} + 3\bar{j} - \bar{k})/\sqrt{11}$.

9. What is the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \ln z - y^2 = -4$ at $(-1, 2, 1)$.

Ans. $\frac{15}{\sqrt{17}}$

10. Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\bar{i} - \bar{j} - 2\bar{k}$.

Ans. $\frac{37}{3}$

11. Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $i + 2j + 2k$ at the point $(1, 2, 0)$.

Ans. $\frac{10}{3}$

12. Determine the directional derivative of $f = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$.

Ans. $-\frac{11}{3}$

13. Find the maximal directional derivative of x^3y^2z at $(1, -2, 3)$.

Ans. $4\sqrt{91}$

14. **a.** In what direction from the point $(2, 1, -1)$ is the directional derivative of $f = x^2yz^3$ a maximum?

b. What is the magnitude of this maximum?

Ans. **a.** The directional derivative is a maximum in the direction of $\nabla f = -4i - 4j + 12k$.

b. The magnitude of this maximum is $4\sqrt{11}$.

15. Find the direction in which temperature changes most rapidly with distance from the point $(1, 1, 1)$ and determine the maximum rate of change if the temperature at any point is given by $f(x, y, z) = xy + yz + zx$.

Ans. Maximum direction is $2i + 2j + 2k$, maximum: $2\sqrt{3}$.

16. In what direction from $(3, 1, -2)$ is the directional derivative of $f = x^2y^2z^4$ maximum. Find also the magnitude of the maximum.

Ans. $96(\bar{i} + 3j - 3k)$; $96\sqrt{19}$

17. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Ans. The acute angle $= \cos^{-1}\left(\frac{8\sqrt{21}}{63}\right) = 54° \cdot 25$.

18. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z = 47$ at $(4, -3, 2)$.

Ans. $\theta = \cos^{-1}(19/29)$

19. Determine the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.

Ans. $\cos^{-1}(1/\sqrt{22})$

20. Calculate the angle between the normals to the surface $2x^2 + 3y^2 = 5z$ at the points $(2, -2, 4)$ and $(-1, -1, 1)$.

Ans. $\theta = \cos^{-1}(65/(\sqrt{233}\sqrt{77}))$

21. If $\nabla f = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$, find $f(x, y, z)$ if $f(1, -2, 2) = 4$.

Ans. $f = x^2yz^3 + 20$

22. Find f given $\nabla f = 2xi + 4yj + 8z\bar{k}$.

Ans. $f = x^2 + 2y^2 + 4z^2$

23. **a.** Determine f when $\nabla f = (zy\, i + xz\, j - xy\, k)/z^2$.

b. If $\nabla f = xy\, i + 2xy\, j$ find f.

Ans. **a.** $f = xy/z$

b. f does not exist

24. Prove that $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$.

Hint: Use quotient formula for derivative of f/g.

15.3 DIVERGENCE

Divergence of a vector function $\overline{A}(x, y, z)$ is written as divergence of $\overline{A}$ or div of $\overline{A}$ and denoted by $\nabla \cdot \overline{A}$ is defined as

$$\nabla \cdot \overline{A} = \left(i\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (\overline{A})$$

If $\overline{A} = A_1 i + A_2 j + A_3 k$, then

$$\nabla \cdot \overline{A} = \left(i\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (A_1 i + A_2 j + A_3 k)$$

$$\nabla \cdot \overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = \text{a scalar quantity}$$

Note that $\nabla \cdot \overline{A} \neq A \cdot \nabla$ because L.H.S. $\nabla \cdot \overline{A}$ is a scalar quantity, whereas the R.H.S. $\overline{A} \cdot \nabla = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}$ is a scalar differential operator.

Physically the divergence of $\overline{A}$ at point P constitutes the volume density of the flux of $\overline{A}$ at P. i.e., divergence measures outflow minus inflow.

A point P in a vector field $\overline{A}$ is said to be a source (sink) if divergence $\overline{A} > (<) 0$.

Solenoidal Function

$\overline{A}$ is said to be solenoidal if divergence $\overline{A} = 0$ (at all points of function).

WORKED OUT EXAMPLES

Example 1: Evaluate divergence of $(2x^2 z\overline{i} - xy^2 zj + 3yz^2 k)$ at the point (1, 1, 1).

Solution: Divergence of $\overline{A}$

$$= \text{div } \overline{A} = \nabla \cdot \overline{A}$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Here $\quad \overline{A} = 2x^2 z\overline{i} - xy^2 z\overline{j} + 3yz^2 k$

$$= A_1 i + A_2 j + A_3 k$$

so that $\quad \frac{\partial A_1}{\partial x} = \frac{\partial}{\partial x}(2x^2 z) = 4xz$

$$\frac{\partial A_2}{\partial y} = \frac{\partial}{\partial y}(-xy^2 z) = -2xyz$$

$$\frac{\partial A_3}{\partial z} = \frac{\partial}{\partial z}(3yz^2) = 6yz$$

Thus $\quad \nabla \cdot \overline{A} = 4xz - 2xyz + 6yz,$

$$\nabla \cdot \overline{A}\Big|_{(1,1,1)} = 8.$$

Example 2: Determine the constant b such that

$\overline{A} = (bx + 4y^2 z)i + (x^3 \sin z - 3y)j - (e^x + 4\cos x^2 y)\overline{k}$

is solenoidal.

Solution: $\nabla \cdot \overline{A} = b - 3 = 0 \quad \therefore b = 3.$

Example 3: Find the directional derivative of $\nabla \cdot \overline{U}$ at the point (4, 4, 2) in the direction of the corresponding outer normal of the sphere $x^2 + y^2 + z^2 = 36$ where $\overline{U} = x^4 i + y^4 j + z^4 k$.

Solution: Let $f = \nabla \cdot \overline{U} = \nabla \cdot (x^4 i + y^4 j + z^4 \overline{k}) = 4(x^3 + y^3 + z^3)$

$$\nabla f\Big|_{(4,4,2)} = 12\ (x^2 i + y^2 j + z^2 k)\Big|_{4,4,2}$$

$$= 48(4i + 4j + k)$$

Normal to the sphere $g = x^2 + y^2 + z^2 = 36$ is

$$\nabla g\Big|_{4,4,2} = 2(xi + yj + zk)\ \Big|_{\text{at } 4,4,2} = 4(2i + 2j + k)$$

$$\overline{a} = \text{unit normal} = \frac{\nabla g}{|\nabla g|} = \frac{4(2i + 2j + k)}{\sqrt{64 + 64 + 16}}$$

$$= \frac{2i + 2j + k}{3}$$

The required directional derivative is

$$\nabla f \cdot \overline{a} = 48(4i + 4j + k) \cdot \frac{(2i + 2j + k)}{3}$$

$$= 16(8 + 8 + 1) = 272.$$

Example 4: $\nabla \cdot (r^3 \overline{r})$.

Solution: Since $\nabla \cdot (f\overline{A}) = f\nabla \cdot \overline{A} + \nabla f \cdot \overline{A}$

$$\nabla \cdot (r^3 \overline{r}) = r^3 \nabla \cdot \overline{r} + \overline{r} \cdot \nabla r^3$$

$$= 3r^3 + \overline{r} \cdot [3r^{3-2}\overline{r}]$$

$$= 3r^3 + 3r\overline{r} \cdot \overline{r}$$

$$= 3r^3 + 3r\, r^2 = 6r^3.$$

Example 5: If f and g are solutions of the Laplace equation show that

$$\nabla \cdot (f\nabla g - g\nabla f) = 0$$

Solution:

$$\nabla \cdot (f\nabla g - g\nabla f) = \nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f)$$

$$= f\nabla \cdot \nabla g + \nabla f \cdot \nabla g$$

$$-g\nabla\cdot\nabla f - \nabla g\cdot\nabla f$$
$$= f\nabla^2 g + \nabla f\cdot\nabla g$$
$$-g\nabla^2 f - \nabla g\cdot\nabla f = 0$$

Note that $\nabla\cdot\nabla f = \nabla^2 f$.
Since f and g satisfy Laplace's equation we have $\nabla^2 f = 0$ and $\nabla^2 g = 0$, and also $\nabla f\cdot\nabla g = \nabla g\cdot\nabla f$ by commutative property.

Example 6: Find $\nabla(\nabla\cdot\overline{A})$ where $\overline{A} = \overline{r}/r$.

Solution: Consider

$$\nabla\cdot\overline{A} = \nabla\cdot\left(\frac{\overline{r}}{r}\right) = r^{-1}\nabla\cdot\overline{r} + \overline{r}\cdot\nabla r^{-1}$$
$$= 3r^{-1} + \overline{r}\cdot(-r^{-1-2}\overline{r})$$
$$= 3r^{-1} - r^{-3}\overline{r}\cdot\overline{r}$$
$$= 3r^{-1} - r^{-3}r^2 = 2r^{-1}$$

So $$\nabla(\nabla\cdot A) = \nabla\left(\nabla\cdot\frac{\overline{r}}{r}\right) = \nabla(2r^{-1}) = 2\nabla r^{-1}$$
$$= 2(-1)r^{-1-2}\overline{r} = -2r^{-3}\overline{r}.$$

EXERCISE

1. Prove that $\nabla\cdot\overline{r} = 3$.
2. Find $\nabla\cdot\overline{A}$ when $\overline{A} = (x\overline{i} + y\overline{j} + z\overline{k})/r$ where $r = \sqrt{x^2+y^2+z^2}$.

Ans. $2/r$

3. Calculate $\nabla\cdot(3x^2i + 5xy^2j + xyz^3k)$ at the point $(1, 2, 3)$.

Ans. 80

4. If $\overline{A} = 3xyz^2\overline{i} + 2xy^3\overline{j} - x^2yz\overline{k}$ and $f = 3x^2 - yz$ find (i) $\nabla\cdot\overline{A}$ (ii) $\overline{A}\cdot\nabla f$ (iii) $\nabla\cdot(f\overline{A})$ (iv) $\nabla\cdot\nabla f$.

Ans. (i) 4 (ii) -15 (iii) 1 (iv) 6

5. Find $\nabla\cdot\left[(e^y \sin x\cos z)i + e^{-x}\sin y\cos z\overline{j} + z^2e^z\overline{k}\right]$.

Ans. $e^y\cos x\cdot\cos z + e^{-x}\cdot\cos y\cos z + (z^2 + 2z)e^z$

6. Show that $\overline{A} = 3y^4z^2\overline{i} + 4x^3z^2j - 3x^2y^2\overline{k}$ is solenoidal.
7. Prove that $\overline{A} = (2x^2 + 8xy^2z)i + (3x^3y - 3xy)\overline{j} - (4y^2z^2 + 2x^3z)\overline{k}$ is not solenoidal but $\overline{B} = xyz^2\overline{A}$ is solenoidal.
8. Determine the constant b such that $\overline{A} = (bx^2y + yz)\overline{i} + (xy^2 - xz^2)\overline{j} + (2xyz - 2x^2y^2)\overline{k}$ has zero divergence (i.e., $\nabla\cdot\overline{A} = 0$).

Ans. $b = -2$

9. Evaluate $\nabla\cdot[r\nabla(1/r^3)]$.

Ans. $3r^{-4}$

10. Find most general $f(r)$ such that $f(r)\overline{r}$ is solenoidal.

Ans. $f(r) = c/r^3$ where c is an arbitrary constant

11. Show that $\nabla f\times\nabla g$ is solenoidal.

Hint: $\nabla\cdot(\nabla f\times\nabla g) = \nabla g\cdot(\nabla\times\nabla f) - \nabla f\cdot(\nabla\times\nabla g) = 0$.

12. Prove that $\overline{A} = (y^2 - z^2 + 3yz - 2x)\overline{i} + (3xz + 2xy)\overline{j} + (3xy - 2xz + 2z)\overline{k}$ is both solenoidal and irrotational.
13. Show that $\nabla\cdot(f\overline{A}) = 5f$ where $f = x^2 + y^2 + z^2$ and $\overline{A} = x\overline{i} + y\overline{j} + z\overline{k}$.
14. Find the directional derivative of $\nabla\cdot\overline{U}$ at the point $(4, 4, 2)$ in the direction of the corresponding outer normal of the sphere $x^2 + y^2 + z^2 = 36$ where $\overline{U} = xz\overline{i} + yx\overline{j} + zy\overline{k}$.

Ans. 5/3

15. Show that the vector field $\overline{V} = \frac{a(xi+yj)}{x^2+y^2}$ is a "source" or "sink" field according as $a > 0$ or $a < 0$.

Hint: If $\nabla\cdot\overline{V} > 0$ then $\overline{V}$ is a source field and if $\nabla\cdot\overline{V} < 0$ it is a sink field.

16. If $f = x^2yz$ and $g = xy - 3z^2$, calculate $\nabla(\nabla f\cdot\nabla g)$.

Ans. $2(y^3 + 3x^2y - 6xy^2)z\overline{i} + 2(3xy^2 + x^3 - 6x^2y)z\overline{j} + 2(xy^2 + x^3 - 3x^2y)y\overline{k}$.

15.4 CURL

Curl of $\overline{A}$, denoted by $\nabla\times\overline{A}$ also known as rotation $\overline{V}$ or rot of $\overline{V}$ is defined as

$$\text{curl of } \overline{A} = \nabla\times\overline{A} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$$
$$\times(A_1i + A_2j + A_3k)$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + j\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$\nabla \times \overline{A} =$ a vector quantity.

Irrotational Field

A vector point function $\overline{A}$ is said to be irrotational, if curl of $\overline{A}$ is zero at every point where $\overline{A}$ is defined. Otherwise it is said to be rotational. The curl of any vector point function, in general, gives the measure of the angular velocity at any point of the vector field.

Worked Out Examples

Example 1: Find the curl of $\overline{V} = e^{xyz}(\overline{i} + \overline{j} + \overline{k})$ at the point (1, 2, 3).

Solution: Curl of $\overline{V}$

$$= \nabla \times \overline{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}.$$

$$= e^{xyz}\left[(xz-xy)\overline{i} - (yz-xy)j + (yz-xz)\overline{k}\right]$$

$$\nabla \times \overline{V}\Big|_{1,2,3} = e^6\left[\overline{i} - 4j + 3\overline{k}\right].$$

Example 2: Prove that $\nabla \times \nabla f = 0$ for any $f(x, y, z)$.

Solution:

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \overline{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = 0$$

since $f_{yz} = f_{zy}$, $f_{xz} = f_{zx}$ and $f_{xy} = f_{yx}$.

Note: Since $\overline{V} = \nabla\phi$ for a conservative field, $\nabla \times \overline{V} = \nabla \times \nabla\phi = 0$. Thus for a conservative field $\overline{V}$, we have $\nabla \times \overline{V} = 0$.

Example 3: If $f(r)$ is differentiable and $r = \sqrt{x^2 + y^2 + z^2}$ show that $f(r)\overline{r}$ is irrotational. Hence deduce that (i) $r^n\overline{r}$ is irrotational (ii) $\nabla \times \overline{r} = 0$.

Solution: Here $f(r)\overline{r} = f(r)[xi + yj + zk]$

$$\nabla \times (f(r)\overline{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= i\left(\frac{z\partial f}{\partial y} - \frac{y\partial f}{\partial z}\right) - j\left(\frac{z\partial f}{\partial x} - \frac{x\partial f}{\partial z}\right) + k\left(\frac{y\partial f}{\partial x} - \frac{x\partial f}{\partial y}\right)$$

Here $\dfrac{\partial f(r)}{\partial y} = \dfrac{\partial}{\partial r} f(r) \cdot \dfrac{\partial r}{\partial y} = f'(r) \cdot \dfrac{y}{r}$

since $\dfrac{\partial r}{\partial y} = \dfrac{\partial}{\partial y}\sqrt{x^2 + y^2 + z^2}$

$$= \frac{1}{2}\frac{2y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r}$$

Similarly, $\dfrac{\partial f}{\partial x} = f'\dfrac{x}{r}$ and $\dfrac{\partial f}{\partial z} = f'\dfrac{z}{r}$

Substituting these values

$$\nabla \times (f\overline{r}) = i\left[zf'\frac{y}{r} - yf'\frac{z}{r}\right] - j\left[zf'\frac{x}{r} - xf'\frac{z}{r}\right] + k\left[yf'\frac{x}{r} - xf'\frac{y}{r}\right]$$

$$= 0$$

i. with $f(r) = r^n$.

$$\nabla \times (r^n\overline{r}) = \nabla(f(r)\overline{r}) = 0$$

follows from the above result.

ii. with $n = 0$, $\nabla \times \overline{r} = 0$ from above result (i).

Example 4: Prove that $\overline{A} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational. Find a scalar function $f(x, y, z)$ such that $\overline{A} = \nabla f$.

Solution:

$$\nabla \times \overline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= i[-1 - (-1)] - j[3z^2 - 3z^2] + k[6x - 6x] = 0$$

Therefore $\overline{A}$ is irrotational.

To find f: $\overline{A} = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

Comparing components of i, j, k on either side

$$\frac{\partial f}{\partial x} = 6xy + z^3 \quad (1)$$

$$\frac{\partial f}{\partial y} = 3x^2 - z \quad (2)$$

$$\frac{\partial f}{\partial z} = 3xz^2 - y \quad (3)$$

Integrating (1) partially w.r.t. x, we get

$$f = 3x^2 y + xz^3 + c_1(y, z) \quad (4)$$

Differentiating (4) partially w.r.t. y and equating it with (2), we get

$$3x^2 + 0 + \frac{\partial c_1}{\partial y} = \frac{\partial f}{\partial y} = 3x^2 - z$$

i.e.,

$$\frac{\partial c_1}{\partial y} = -z \quad (5)$$

Integrating (5) partially w.r.t. y, we have

$$c_1(y, z) = -zy + c_2(z) \quad (6)$$

Substituting (6) in (4)

$$f = 3x^2 y + xz^3 - zy + c_2(z) \quad (7)$$

Differentiating (7) partially w.r.t. z and equating it with (3), we get

$$0 + 3xz^2 - y + \frac{dc_2}{dz} = 3xz^2 - y$$

So

$$\frac{dc_2}{dz} = 0$$

i.e., c_2 = a pure constant (independent of z)

Thus the required scalar function

$$f = 3x^2 y + xz^3 - zy + c_2.$$

Example 5: Find curl curl of $\overline{A} = x^2 y\overline{i} - 2xzj + 2yz\overline{k}$ at the point $(1, 0, 2)$.

Solution:

$$\nabla \times \overline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$$

$$\nabla \times \overline{A} = i[2z + 2x] - j[0 - 0] + \overline{k}[-2z - x^2]$$

Now

$$\nabla \times (\nabla \times \overline{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2(z + x) & 0 & -(2z + x^2) \end{vmatrix}$$

$$= i[0 - 0] - j[-2x - 2] + k[0 - 0]$$

$$= 2(x + 1)\overline{j}$$

$$\nabla \times (\nabla \times A)\Big|_{\text{at } (1,0,2)} = 2(1 + i)j = 4\overline{j}.$$

Exercise

1. Prove that $\nabla \times \overline{r} = 0$.
2. Find the curl of $yzi + 3zxj + z\overline{k}$ at $(2, 3, 4)$.

Ans. $-6i + 3j + 8k$

3. Find $\nabla \times \left[(yz - 2x^2 y)i + x(y^2 - z^2)j + 2xy(z - xy)\overline{k}\right]$ at the point $(1, 1, 1)$.

Ans. $4x(z - xy)\overline{i} + (y - 2yz + 4xy^2)j + (2x^2 + y^2 - z^2 - z)\overline{k}$; $3\overline{j} + \overline{k}$

4. If $f = x^2 yz$, $g = xy - 3z^2$, calculate $\nabla \cdot (\nabla f \times \nabla g)$.

Ans. zero

5. Determine curl of $xyz^2 i + yzx^2 j + zxy^2\overline{k}$ at the point $(1, 2, 3)$.

Ans. $xy(2z - x)\overline{i} + yz(2x - y)j + zx(2y - z)\overline{k}$; $10\overline{i} + 3\overline{k}$

6. If $\overline{A}$ and $\overline{B}$ are irrotational show that $\nabla \cdot (\overline{A} \times \overline{B}) = 0$.
7. Determine the constants a and b such that curl of $(2xy + 3yz)\overline{i} + (x^2 + axz - 4z^2)j + (3xy + 2byz)k = 0$.

Ans. $a = 3, b = 4$.

8. Find the value of constant b such that

$$\overline{A} = (bxy - z^3)\overline{i} + (b - 2)x^2 j + (1 - b)xz^2 k$$

has its curl identically equal to zero.

Ans. $b = 4$

9. Evaluate $\nabla \times (\overline{r}r^{-2})$. Find f such that $\overline{r}r^{-2} = -\nabla f$ with $f(a) = 0$ where $a > 0$.

Ans. $f = \ln(a/r)$

10. Determine the constants a, b, c so that

$$\overline{A} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)\overline{k}$$

is irrotational. Find a scalar function $f(x, y, z)$ such that $\overline{A} = \nabla f$.

Ans. **i.** $a = 4, b = 2, c = -1$
ii. $f = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$

11. Prove that $\overline{A} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ is irrotational and find the scalar potential f such that $\overline{A} = \nabla f$.

Ans. $f(x, y, z) = \frac{x^3+y^3+z^3}{3} - xyz$

12. Show that $\nabla \times (\nabla \times (\nabla \times (\nabla \times \overline{A}))) = \nabla^4 \overline{A}$ where $\overline{A}$ is a solenoidal vector.

13. Prove that $(y^2 - z^2 + 3yz - 2x)\overline{i} + (3xz + 2xy)j + (3xy - 2xz + 2z)$ is both solenoidal and irrotational.

14. Prove that $\nabla \cdot (\nabla \times \overline{A}) = 0$.

15.5 RELATED PROPERTIES OF GRADIENT, DIVERGENCE AND CURL OF SUMS

The gradient, divergence and curl are distributive with respect to the sum and difference of functions:

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$

2. $\nabla \cdot (\overline{A} \pm \overline{B}) = (\nabla \cdot \overline{A}) \pm (\nabla \cdot \overline{B})$

3. $\nabla \times (\overline{A} \pm \overline{B}) = (\nabla \times \overline{A}) \pm (\nabla \times \overline{B})$.

The above results follow, since derivative of sum or difference of scalars or vectors is sum or difference of the derivatives of scalars or vectors. For example,

$$\begin{aligned}
\nabla \cdot (\overline{A} \pm \overline{B}) &= \nabla \cdot ((A_1 \pm B_1)i + (A_2 \pm B_2)j + (A_3 \pm B_3)k) \\
&= \frac{\partial}{\partial x}(A_1 \pm B_1) + \frac{\partial}{\partial y}(A_2 \pm B_2) + \frac{\partial}{\partial z}(A_3 \pm B_3) \\
&= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \pm \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) \\
&= \nabla \cdot \overline{A} \pm \nabla \cdot \overline{B}.
\end{aligned}$$

Gradient, Divergence and Curl of Products

1. $\nabla(fg) = f\nabla g + g\nabla f$

2. $\nabla \cdot (f\overline{A}) = f\nabla \cdot \overline{A} + (\nabla f) \cdot \overline{A}$

3. $\nabla \times (f\overline{A}) = f\nabla \times \overline{A} + (\nabla f) \times \overline{A}$

4. $\nabla(\overline{A} \cdot \overline{B}) = (B \cdot \nabla)\overline{A} + (A \cdot \nabla)\overline{B} + \overline{B} \times (\nabla \times A) + A \times (\nabla \times B)$

5. $\nabla \cdot (\overline{A} \times \overline{B}) = \overline{B} \cdot (\nabla \times \overline{A}) - \overline{A} \cdot (\nabla \times \overline{B})$

6. $\nabla \times (\overline{A} \times \overline{B}) = (B \cdot \nabla)\overline{A} - B(\nabla \cdot A) - (A \cdot \nabla)B + A(\nabla \cdot B)$.

The results 1, 2, 3 follow from the fact that the derivative of a product of scalar functions is the product of the derivatives of the scalar functions.

For example,

$$\begin{aligned}
\nabla \times (f\overline{A}) &= \nabla \times (f(A_1 i + A_2 j + A_3 k)) \\
&= \nabla \times (fA_1 i + fA_2 j + fA_3 k) \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_1 & fA_2 & fA_3 \end{vmatrix} \\
&= i\left[\frac{\partial(fA_3)}{\partial y} - \frac{\partial}{\partial z}(fA_2)\right] \\
&\quad - j\left[\frac{\partial}{\partial x}(fA_3) - \frac{\partial}{\partial z}(fA_1)\right] \\
&\quad + k\left[\frac{\partial}{\partial x}(fA_2) - \frac{\partial}{\partial y}(fA_1)\right]
\end{aligned}$$

Expanding the product of the derivatives and rearranging the terms, we get

$$\begin{aligned}
&= f\left[i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - j\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\right. \\
&\quad \left. + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right] + i\left(A_3\frac{\partial f}{\partial y} - A_2\frac{\partial f}{\partial z}\right) \\
&\quad - j\left(A_3\frac{\partial f}{\partial x} - A_1\frac{\partial f}{\partial z}\right) + k\left(A_2\frac{\partial f}{\partial x} - A_1\frac{\partial f}{\partial y}\right)
\end{aligned}$$

$$\nabla \times (f\overline{A}) = f\nabla \times \overline{A} + (\nabla f) \times \overline{A}.$$

Example 1: Prove that $\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$.

Solution:

$$\bar{A}\times\bar{B}=\begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\bar{A}\times\bar{B}=i(A_2B_3-A_3B_2)-j(A_1B_3-A_3B_1)+k(A_1B_2-A_2B_1)$$

$$\nabla\cdot(\bar{A}\times\bar{B})=\frac{\partial}{\partial x}(A_2B_3-A_3B_2)-\frac{\partial}{\partial y}(A_1B_3-A_3B_1)+\frac{\partial}{\partial z}(A_1B_2-A_2B_1).$$

Expanding the derivatives of the products and rearranging the 12 terms in to 2 groups of 6 terms each, we get

$$\nabla\cdot(\bar{A}\times\bar{B})=\left[B_1\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+B_2\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+B_3\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\right]-\left[A_1\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)+A_2\left(\frac{\partial B_1}{\partial z}-\frac{\partial B_3}{\partial x}\right)+A_3\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)\right]$$

$$\nabla\cdot(\bar{A}\times\bar{B})=\bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B}).$$

Example 2: Prove that

$$\nabla\times(\bar{A}\times\bar{B})=(B\cdot\nabla)\bar{A}-\bar{B}(\nabla\cdot\bar{A})-(\bar{A}\cdot\nabla)\bar{B}+\bar{A}(\nabla\cdot\bar{B}).$$

Solution:

$$\nabla\times(\bar{A}\times\bar{B})=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\times(\bar{A}\times\bar{B})$$
$$=i\times\frac{\partial}{\partial x}(\bar{A}\times\bar{B})+j\times\frac{\partial}{\partial y}(\bar{A}\times\bar{B})+\bar{k}\times\frac{\partial}{\partial z}(\bar{A}\times\bar{B})$$

Expanding the derivative of the products, we get 6 terms,

$$=\left[i\times\left(\frac{\partial\bar{A}}{\partial x}\times B\right)+i\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)\right]+\left[\bar{j}\times\left(\frac{\partial\bar{A}}{\partial y}\times\bar{B}\right)+\bar{j}\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial y}\right)\right]+\left[\bar{k}\times\left(\frac{\partial\bar{A}}{\partial z}\times\bar{B}\right)+\bar{k}\times\left(\bar{A}\times\frac{\partial B}{\partial z}\right)\right] \quad (1)$$

Since $\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$,

$$\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times B\right)=(\bar{i}\cdot\bar{B})\frac{\partial\bar{A}}{\partial x}-\left(\bar{i}\cdot\frac{\partial\bar{A}}{\partial x}\right)\bar{B}$$
$$=(\bar{B}\cdot i)\frac{\partial\bar{A}}{\partial x}-\left(\bar{i}\cdot\frac{\partial}{\partial x}\bar{A}\right)\bar{B} \quad (2)$$

We get similar results for the 3rd and 5th terms in the R.H.S. of (1). Collecting these 3 terms from the R.H.S. of (1), namely 1st, 3rd and 5th terms and using the summation notation with respect to i (and x), we get

$$i\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)+j\times\left(\frac{\partial A}{\partial y}\times B\right)+\bar{k}\times\left(\frac{\partial A}{\partial z}\times\bar{B}\right)$$
$$=\sum\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)$$
$$=\sum\left(B\cdot i\frac{\partial}{\partial x}\right)\bar{A}-\sum\left(i\frac{\partial}{\partial x}\cdot A\right)\bar{B}$$

Since the summation is with respect to i, we get

$$=\left(\bar{B}\cdot\sum i\frac{\partial}{\partial x}\right)\bar{A}-\left\{\left(\sum i\frac{\partial}{\partial x}\right)\cdot\bar{A}\right\}\bar{B}$$

$$\sum\bar{i}\times\left(\frac{\partial\bar{A}}{\partial x}\times\bar{B}\right)=(\bar{B}\cdot\nabla)\bar{A}-(\nabla\cdot\bar{A})\bar{B} \quad (3)$$

In a similar manner, interchanging the roles of $\bar{A}$ and $\bar{B}$, for the remaining 3 terms namely 2nd, 4th and 6th terms of (1), we get

$$\sum i\times\left(\bar{A}\times\frac{\partial\bar{B}}{\partial x}\right)=(\nabla\cdot\bar{B})\bar{A}-(\bar{A}\cdot\nabla)\bar{B} \quad (4)$$

Adding (3) and (4) the required result is obtained.

Example 3: Prove that

$$\nabla(\bar{A}\cdot\bar{B})=\bar{A}\times(\nabla\times\bar{B})+\bar{B}\times(\nabla\times\bar{A})+(\bar{A}\cdot\nabla)\bar{B}+(\bar{B}\cdot\nabla)\bar{A}.$$

Solution: $\nabla(\bar{A}\cdot\bar{B}) = i\frac{\partial}{\partial x}(\bar{A}\cdot\bar{B}) + j\frac{\partial}{\partial y}(\bar{A}\cdot\bar{B}) + k\frac{\partial}{\partial z}(\bar{A}\cdot\bar{B})$.

Expanding the derivative of the product terms and rearranging the 6 terms, we get

$$\nabla(\bar{A}\cdot\bar{B}) = \sum i\frac{\partial}{\partial x}(\bar{A}\cdot\bar{B})$$

$$= \left(\sum i\frac{\partial}{\partial x}\bar{A}\right)\cdot\bar{B} + \bar{A}\cdot\sum i\frac{\partial\bar{B}}{\partial x} \quad (1)$$

Consider

$$\bar{A}\times(\nabla\times\bar{B}) = \bar{A}\times\left(\left(\sum i\frac{\partial}{\partial x}\right)\times\bar{B}\right)$$

$$= \bar{A}\times\left(\sum i\times\frac{\partial\bar{B}}{\partial x}\right)$$

Using triple cross product result

$$= \sum\left(\bar{A}\cdot\frac{\partial\bar{B}}{\partial x}\right)i - \left(\bar{A}\cdot\sum i\right)\frac{\partial\bar{B}}{\partial x}$$

$$= \sum i\left(\bar{A}\cdot\frac{\partial\bar{B}}{\partial x}\right) - \left(\bar{A}\cdot\sum i\frac{\partial}{\partial x}\right)\bar{B}$$

$$\bar{A}\times(\nabla\times B) = \sum i\left(\frac{\partial\bar{B}}{\partial x}\cdot\bar{A}\right) - (\bar{A}\cdot\nabla)\bar{B}$$

Rewriting, we have

$$\bar{A}\times(\nabla\times\bar{B}) + (\bar{A}\cdot\nabla)\bar{B} = \left(\sum i\frac{\partial\bar{B}}{\partial x}\right)\cdot\bar{A}$$

$$= \bar{A}\cdot\left(\sum i\frac{\partial\bar{B}}{\partial x}\right) \quad (2)$$

Similarly (interchanging the roles of $\bar{A}$ and $\bar{B}$), we get

$$\bar{B}\times(\nabla\times\bar{A}) + (\bar{B}\cdot\nabla)\bar{A} = \bar{B}\cdot\left(\sum i\frac{\partial\bar{A}}{\partial x}\right) \quad (3)$$

Addition of (2) and (3) gives the desired result.

15.6 SECOND-ORDER DIFFERENTIAL OPERATOR

It is a two-fold application of the operator ∇ to function.

Laplacian Operator ∇^2

$$\text{div grad } f = \nabla\cdot(\nabla f)$$

$$= \nabla\cdot\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$

$$= \nabla^2 f = \Delta f$$

Thus the scalar differential operator (read as "nabla squared" or "delta")

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is known as the Laplacian operator.

Thus we have following second order differential operators:

1. $\nabla\cdot\nabla f$ = div grad $f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2. $\nabla\times\nabla f$ = curl grad $f = 0$
3. $\nabla\cdot\nabla\times\bar{A}$ = div cur $\bar{A} = 0$
4. $\nabla\times(\nabla\times\bar{A})$ = curl curl $\bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A}$
   (4) may be rewritten as
5. $\nabla(\nabla\cdot\bar{A})$ = grad div $\bar{A} = \nabla\times(\nabla\times\bar{A}) + \nabla^2\bar{A}$

The possible combinations of second order differential operators are tabulated below:

	Scalar field f	Vector field $\bar{A}$	
	grad	**div**	**curl**
grad	—	grad div$\bar{A}$	—
div	div grad $f = \Delta f$	—	div curl $\bar{A} = 0$
curl	curl grad $f = 0$	—	curl curl $\bar{A}$ = grad div $\bar{A} - \Delta\bar{A}$

Example 1: Prove that $\nabla\times\nabla f = 0$ for any scalar function f.

Solution: $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} \overline{k}$ so

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial z}\right)$$

$$= 0 + 0 + 0 = \overline{0}$$

Since $f_{yz} = f_{zy}$, $f_{xz} = f_{zx}$, $f_{xy} = f_{yx}$.

Note: Gradient field describing a motion, in this case, is known as "irrotational".

If gradient field is not a velocity field, then it is known as "conservative".

Example 2: Prove that $\nabla \cdot (\nabla \times \overline{A}) = 0$. for any vector function $\overline{A}$.

Solution:

$$\nabla \times \overline{A} = i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - j\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

So that

$$\nabla \cdot (\nabla \times \overline{A}) = \frac{\partial}{\partial x}\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right] + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} = 0.$$

Example 3: Show that $\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A}$.

Solution:

$$\nabla \times (\nabla \times \overline{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

Expanding the determinant, we have

$$\nabla \times (\nabla \times \overline{A}) = \left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial z \partial x}\right)\overline{i} + \left(\frac{\partial^2 A_3}{\partial z \partial y} - \frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_1}{\partial x \partial y}\right)\overline{j} + \left(\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\overline{k}$$

Rearranging the 12 terms into 2 groups of 6 terms each, we get

$$\nabla \times (\nabla \times \overline{A}) = \left[i\frac{\partial}{\partial x}\left\{\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right\} - i\left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]$$
$$+ \left[j\frac{\partial}{\partial y}\left\{\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right\} - j\left(\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2}\right)\right]$$
$$+ \left[\overline{k}\frac{\partial}{\partial z}\left\{\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right\} - k\left(\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_3}{\partial z^2}\right)\right]$$

$$= \left(\overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}\right) \times \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1\overline{i} + A_2\overline{j} + A_3\overline{k})$$

$$\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A}.$$

Example 4: Prove that $\nabla \times (f\nabla g) = \nabla f \times \nabla g = -\nabla \times (g\nabla f)$ and deduce that $\nabla \times (f\nabla f) = 0$

Solution: $\nabla \times (f\nabla g) = \nabla f \times \nabla g + f\nabla \times \nabla g = \nabla f \times \nabla g$, also

$$-\nabla \times (g\nabla f) = -\nabla g \times \nabla f - g\nabla \times \nabla f$$

$$= \nabla f \times \nabla g - 0$$

since $\nabla \times \nabla g = 0$.

Taking $f = g$, $\nabla \times (f\nabla f) = \nabla f \times \nabla f = 0$.

Example 5: Prove that $\nabla \cdot (f\nabla g \times g\nabla f) = 0$.

Solution: Since $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$

$$\begin{aligned}\nabla \cdot (f\nabla g \times g\nabla f) &= g\nabla f \cdot (\nabla \times (f\nabla g)) \\ &\quad - f\nabla g \cdot (\nabla \times (g\nabla f)) \\ &= 0\end{aligned}$$

Since $\nabla \times (f\nabla g) = \nabla f \times \nabla g$ and $\nabla \times (g\nabla f) = -\nabla f \times \nabla g$, from just above example.

Example 6: Prove that

$$\nabla \cdot (f\nabla \times \bar{A}) = \nabla f \cdot (\nabla \times \bar{A}).$$

Solution:

$$\begin{aligned}\nabla \cdot (f\nabla \times \bar{A}) &= \nabla f \cdot (\nabla \times \bar{A}) + f\nabla \cdot (\nabla \times \bar{A}) \\ &= \nabla f \cdot (\nabla \times \bar{A})\end{aligned}$$

Since $\nabla \cdot (\nabla \times \bar{A}) = 0$.

Worked Out Examples

Laplacian ∇^2

Example 1: Calculate $\nabla^2 f$ when $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$ at the point (1, 1, 0).

Solution:

$$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \times \left(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5\right)$$

Consider

$$\frac{\partial}{\partial x}(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5) = 6xz + 12x^2y + 2$$

$$\frac{\partial^2}{\partial x^2}(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5) = 6z + 24xy.$$

Similarly,

$$\frac{\partial^2}{\partial y^2} f = -2z^3$$

and

$$\frac{\partial^2}{\partial z^2} f = -6y^2z$$

Thus substituting these values, we have

$$\nabla^2 f = 6z + 24xy - 2z^3 - 6y^2z$$

$\nabla^2 f$ at the point (1, 1, 0) is $0 + 24 \cdot 1 \cdot 1 + 0 + 0 = 24$.

Example 2: Prove that

a. $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$

b. Find $f(r)$ such that $\nabla^2 f(r) = 0$.

Solution:

a. $\nabla^2 f(r) = \nabla \cdot \nabla f(r)$

Since $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r}$$

$$\nabla f(r) = \frac{\partial f}{\partial r}(xi + yj + zk)\frac{1}{r}$$

i.e., $$\nabla f(r) = \frac{\bar{r}}{r}\frac{df}{dr} = \bar{r}\left(\frac{f'(r)}{r}\right) \quad (1)$$

Using (1), we have

$$\nabla^2 f(r) = \nabla \cdot (\nabla f(r)) = \nabla \cdot \left(\bar{r}\frac{f'}{r}\right) = \nabla \cdot \left(\bar{r}\left(\frac{f'}{r}\right)\right)$$

Applying the result

$$\nabla \cdot (\bar{A}f) = f(\nabla \cdot \bar{A}) + (\nabla f) \cdot \bar{A}$$

$$\nabla^2 f = \nabla \cdot \left(\bar{r}\left(\frac{f'}{r}\right)\right) = \frac{f'}{r}\nabla \cdot \bar{r} + \left(\nabla \frac{f'}{r}\right) \cdot \bar{r} \quad (2)$$

Consider

$$\nabla\left(\frac{f'}{r}\right) = \nabla(f'r^{-1}) = r^{-1}\nabla f' + f'\nabla r^{-1}$$

Using (1) for f', we get

$$\nabla f' = \bar{r}\frac{f''}{r}$$

and $$\nabla r^{-1} = -1 \cdot r^{-1-2} \cdot \bar{r} = -\frac{\bar{r}}{r^3}$$

We have

$$\nabla\left(\frac{f'}{r}\right) = r^{-1}\frac{f''}{r}\bar{r} + f'\left(-\frac{\bar{r}}{r^3}\right) \quad (3)$$

$$= \left(\frac{f''}{r^2} - \frac{f'}{r^3}\right)\bar{r} \quad (4)$$

Also $$\nabla \cdot \bar{r} = 3$$

Substituting (3) and (4) in (2), we get

$$\nabla^2 f = 3\frac{f'}{r} + \left(\frac{f''}{r^2} - \frac{f'}{r^3}\right)\bar{r}\cdot\bar{r}$$

$$= 3\frac{f'}{r} + \left(\frac{f''}{r^2} - \frac{f'}{r^3}\right)r^2$$

$$\nabla^2 f(r) = f'' + \frac{2}{r}f' = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

b. Since $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0$

This is a 2nd order homogeneous equation which is separable in f'

$$\frac{df'}{dr} + \frac{2}{r}f' = 0$$

with solution $f' = \frac{c}{r^2}$.

Integrating w.r.t. r

$$f(r) = B + \frac{A}{r}$$

where A and B are arbitrary constants.

Example 3: Prove that

i. $\nabla^2 r^n = n(n+1)r^{n-2}$ where n is a constant

ii. $\nabla^2 r^2 = 6$

iii. $\nabla^2\left(\frac{1}{r}\right) = 0$

iv. $\nabla^2 \ln r = \frac{1}{r^2}$

vi. $\nabla^2(gh) = g\nabla^2 h + h\nabla^2 g$.

Solution:

i. With $f(r) = r^n$, from the result from above Example 3 (i), we get

$$\nabla^2 f = \nabla^2 r^n = \frac{d^2}{dr^2}(r^n) + \frac{2}{r}\frac{d}{dr}(r^n)$$

$$= \frac{d}{dr}(n\cdot r^{n-1}) + \frac{2}{r}\cdot nr^{n-1}$$

$$= n\cdot(n-1)r^{n-2} + 2nr^{n-2}$$

$$= nr^{n-2}[n-1+2] = n(n+1)r^{n-2}$$

ii. Put $n = 2$ in (i) $\nabla^2 r^2 = 2(2+1)r^{2-2} = 6$

iii. Put $n = -1$ in (i)

$$\nabla^2\left(\frac{1}{r}\right) = (-1)(-1+1)r^{-1-2} = 0$$

iv. With $f(r) = \ln r$

$$\nabla^2 f = \nabla^2 \ln r = \frac{d^2}{dr^2}\ln r + \frac{2}{r}\frac{d}{dr}\ln r$$

$$= -\frac{1}{r^2} + \frac{2}{r}\cdot\frac{1}{r} = \frac{1}{r^2}$$

v. With $f = gh$

$$\nabla^2 f = \nabla^2(gh) = \frac{d^2}{dr^2}(gh) + \frac{2}{r}\frac{d}{dr}(gh)$$

$$= 2\frac{dg}{dr}\frac{dh}{dr} + g\frac{d^2 h}{dr^2} + h\frac{d^2 g}{dr^2} + \frac{2}{r}\left\{g\frac{dh}{dr} + h\frac{dg}{dr}\right\}$$

$$= 2\frac{dg}{dr}\frac{dh}{dr} + g\left[\frac{d^2 h}{dr^2} + \frac{2}{r}\frac{dh}{dr}\right] + h\left[\frac{d^2 g}{dr^2} + \frac{2}{r}\frac{dg}{dr}\right]$$

$$= 2\nabla g\cdot\nabla h + g\nabla^2 h + h\nabla^2 g$$

where we have used result of above Example 2(a).

Aliter: The above examples can also be solved directly. For example consider

$$\nabla^2 \ln r = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\ln r$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\ln\sqrt{(x^2+y^2+z^2)} \quad (1)$$

Consider

$$\frac{\partial}{\partial x}\ln\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot 2x$$

$$= \frac{x}{r^2}$$

$$\frac{\partial^2}{\partial x^2}\ln\sqrt{x^2+y^2+z^2}$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{r^2}\right] = \frac{r^2\cdot 1 - x\cdot 2r\frac{\partial r}{\partial x}}{r^4}$$

$$= \frac{r^2 - 2xr\frac{x}{r}}{r^4} = \frac{r^2 - 2x^2}{r^4} \quad (2)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\ln\sqrt{x^2+y^2+z^2} = \frac{r^2 - 2y^2}{r^4} \quad (3)$$

and

$$\frac{\partial^2}{\partial z^2}\ln\sqrt{x^2+y^2+z^2} = \frac{r^2 - 2z^2}{r^4} \quad (4)$$

Substituting (2), (3), (4) in (1), we get

$$\nabla^2 \ln r = \frac{r^2-2x^2}{r^4} + \frac{r^2-2y^2}{r^4} + \frac{r^2-2z^2}{r^4}$$

$$= \frac{3r^2-2(x^2+y^2+z^2)}{r^4} = \frac{3r^2+2r^2}{r^4} = \frac{r^2}{r^4} = \frac{1}{r^2}.$$

Example 4: Prove that

$$\nabla^2(fg) = f\nabla^2 g + 2\nabla g \cdot \nabla f + g\nabla^2 f.$$

Solution:

$$\nabla^2(fg) = \nabla \cdot \nabla(fg)$$

$$= \nabla \cdot [f\nabla g + g\nabla f]$$

$$= \nabla \cdot (f\nabla g) + \nabla \cdot (g\nabla f)$$

$$= [f\nabla \cdot \nabla g + \nabla f \cdot \nabla g] + [g\nabla \cdot \nabla f + \nabla g \cdot \nabla f]$$

$$\nabla^2 fg = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f.$$

Example 5: Find the directional derivative of $\nabla \cdot (\nabla f)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $f = 2x^3y^2z^4$.

Solution: Let

$$U(x, y, z) = \nabla \cdot \nabla f = \nabla^2 f = \nabla^2(2x^3y^2z^4)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(2x^3y^2z^4)$$

$$= 2 \cdot [6xy^2z^4 + 2x^3z^4 + 12x^3y^2z^2]$$

Normal to the surface $g = xy^2z - 3x - z^2 = 0$ is

$$\nabla g = (y^2z - 3)\overline{i} + (2xyz)\overline{j} + (xy^2 - 2z)\overline{k}$$

$$\nabla g\Big|_{\text{at}(1,-2,1)} = \overline{i} - 4\overline{j} + 2\overline{k}$$

Unit vector $\hat{a}$ in the direction of normal at the point P $(1, -2, 1)$ is

$$\hat{a} = \frac{\nabla g}{|\nabla g|} = \frac{i-4j+2k}{\sqrt{1+16+4}} = \frac{i-4j+2k}{\sqrt{21}}$$

Consider

$$\nabla U = \nabla(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\overline{i}$$

$$+ (24xyz^4 + 48x^3yz^2)j$$

$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\overline{k}$$

$$\nabla U\Big|_{\text{at}(1,-2,1)} = 348i - 144j + 400\overline{k}$$

Thus the required directional derivative is

$$\nabla U \cdot \hat{a} = (348\overline{i} - 144\overline{j} + 400\overline{k}) \cdot \frac{(i-4j+2k)}{\sqrt{21}} = \frac{1724}{\sqrt{21}}.$$

Example 6: Evaluate $\nabla^2\left[\nabla \cdot \left(\frac{\overline{r}}{r^2}\right)\right]$.

Solution: Consider

$$\nabla \cdot \left(\frac{\overline{r}}{r^2}\right) = \nabla \cdot (r^{-2}\overline{r})$$

$$= r^{-2}\nabla \cdot \overline{r} + \overline{r} \cdot \nabla r^{-2}$$

$$= 3r^{-2} + \overline{r} \cdot (-2r^{-4}\overline{r})$$

$$= 3r^{-2} - 2r^{-4}\overline{r} \cdot \overline{r}$$

$$\nabla \cdot \left(\frac{\overline{r}}{r^2}\right) = 3r^{-2} - 2r^{-4}r^2 = r^{-2}$$

Now

$$\nabla^2\left[\nabla \cdot \left(\frac{\overline{r}}{r^2}\right)\right] = \nabla^2(r^{-2})$$

Applying result (i) of Example 3 above with $n = -2$

$$= -2(-2+1)r^{-2-2} = 2r^{-4}.$$

EXERCISE

Laplacian ∇^2

1. Show that $\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2. Calculate $\nabla^2 f$ when $f = 4x^2 + 9y^2 + z^2$

Ans. 28

3. Find $\nabla^2 f$ at the point (2, 3, 1) when $f = xy/z$

Ans. $2xy/z^3$; 12

4. If $\overline{F} = r^a\overline{r}$ prove that

$$\nabla^2\overline{F} = a(a+3)r^{n-2}\overline{F}$$

5. Show that ∇f is both solenoidal and irrotational if $\nabla^2 f = 0$.

15.7 CURVILINEAR COORDINATES: CYLINDRICAL AND SPHERICAL COORDINATES

It is more convenient in many problems to define the position of a point P in space by three numbers (q_1, q_2, q_3) instead of the three cartesian coordinates (x, y, z). Then q_1, q_2, q_3 are known as "curvilinear coordinates" of the point P. The three surfaces $q_1 = c_1, q_2 = c_2$ and $q_3 = c_3$, (refer Fig. 15.2)where c_1, c_2, c_3 are constants, are known as "coordinate surfaces" of the system of curvilinear coordinates. On these coordinates surfaces, say $q_1 = c_1$, one of the coordinates, here q_1, remains constant.

The "coordinate curves (lines) (axis)" are the curves (or lines) of intersection of any two coordinate surfaces. Thus on the coordinate curve say which is the intersection of $q_2 = c_2$ and $q_3 = c_3$, only q_1 varies, while q_2 and q_3 remain constant. Suppose the rectangular coordinates (x, y, z) of any point P in space be expressed as functions of (q_1, q_2, q_3) so that

$$x = x(q_1, q_2, q_3),\ y = y(q_1, q_2, q_3),$$
$$z = z(q_1, q_2, q_3) \tag{1}$$

Solving (1) for q_1, q_2, q_3 in terms of x, y, z, we get

$$q_1 = q_1(x, y, z),\ q_2 = q_2(x, y, z),$$
$$q_3 = q_3(x, y, z) \tag{2}$$

The set of Equations (1) and (2) are known as "transformation of coordinates".

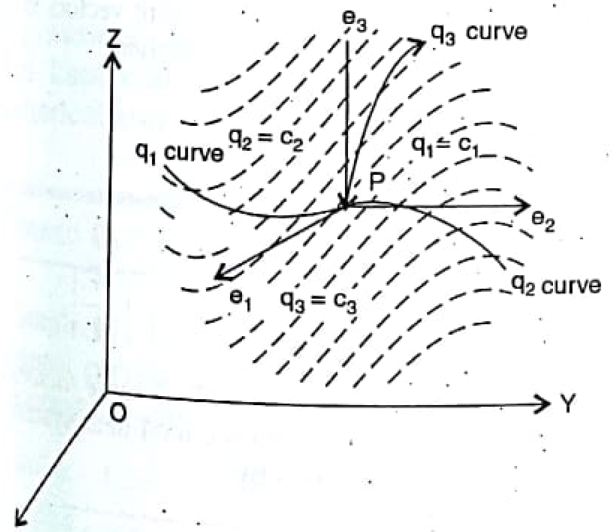

Fig. 15.2

If the coordinate surfaces intersect at right angles (and therefore the coordinate lines are at right angles), then the curvilinear coordinate system is known as "orthogonal curvilinear system of coordinates".

Let $\overline{e}_1, \overline{e}_2, \overline{e}_3$ be unit vectors directed along the tangents to the coordinates axes q_1, q_2, q_3 at the point P in the direction of increasing q_1, q_2, q_3 respectively, such that $\overline{e}_1, \overline{e}_2, \overline{e}_3$ form a right-handed trihedral (triad).

Example: Rectangular cartesian coordinate system x, y, z, where the three coordinate surfaces are planes $x = c_1, y = c_2, z = c_3$ which are mutually at right angles.

Note: The basic difference between curvilinear coordinates and cartesian coordinates is that the unit vectors $\overline{i}, \overline{j}, \overline{k}$ in the cartesian coordinate system remain constant and are same for all points of space, while in any other system the unit vectors $\overline{e}_1, \overline{e}_2, \overline{e}_3$, generally speaking, are not constant i.e., change their directions when passing from one point P to the other.

Example: Cylindrical coordinates (refer Fig. 15.3)

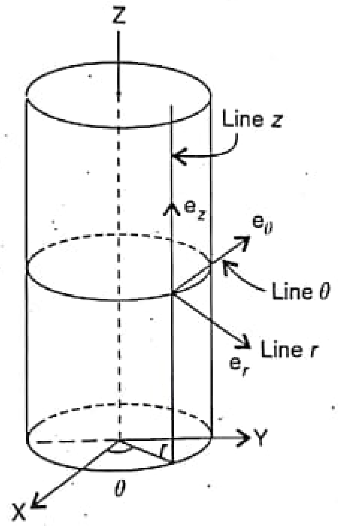

Fig. 15.3

$$\left.\begin{array}{ll} q_1 = r, & 0 \le r < \infty \\ q_2 = \theta, & 0 \le \theta < 2\pi \\ q_3 = z, & -\infty < z < +\infty \end{array}\right\} \tag{3}$$

coordinate surfaces are

$r =$ constant: circular cylinders coaxial with z-axis

$\theta =$ constant: half plane, adjoining z-axis, through z-axis

$z =$ constant: plane perpendicular to z-axis coordinate lines (or axes) are

r: rays with origin on z-axis and perpendicular to z-axis

θ: circles with centre on z-axis and lying in planes perpendicular to z-axis

z: straight lines parallel to the z-axis

The transformation that relate cartesian coordinates to cylindrical coordinates are

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \qquad (4)$$

$r = \sqrt{x^2+y^2}$, $\theta = \text{arc } \tan\frac{y}{x}$, $z = z$

Example: Spherical coordinates (see Fig. 15.4)

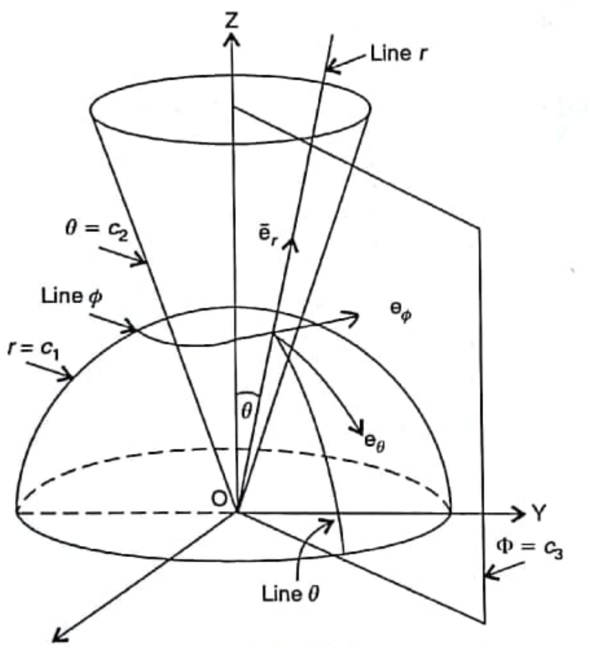

Fig. 15.4

$$\begin{aligned} q_1 &= r, & 0 \le r < +\infty \\ q_2 &= \theta, & 0 \le \theta \le \pi \\ q_3 &= \phi, & 0 \le \phi < 2\pi \end{aligned} \qquad (5)$$

The coordinate surfaces are

$r = c_1$, spheres centred at origin 0

$\theta = c_2$, circular half-angle cones with z-axis with vertex at origin.

$\phi = c_3$, (half) planes adjoining the z-axis through z-axis.

coordinate lines are:

r: rays emanating from origin 0

θ: meridians on a sphere

ϕ: parallel on a sphere

cartesian coordinates are related to spherical coordinates as follows:

$$\left.\begin{aligned} x &= r\cos\phi\ \sin\theta \\ y &= r\sin\phi\ \sin\theta \\ z &= r\cos\theta. \end{aligned}\right\} \qquad (6)$$

Unit Vectors in Curvilinear System

Suppose $\overline{r} = \overline{r}(q_1, q_2, q_3)$ be the position vector of a point P. A tangent vector of the q_1 curve at P (for which q_2 and q_3 are constants) is $\frac{\partial\overline{r}}{\partial q_1}$. Then a unit tangent vector $\overline{e}_1$ in this direction is

$$\overline{e}_1 = \frac{\partial\overline{r}}{\partial q_1} \Big/ \left|\frac{\partial\overline{r}}{\partial q_1}\right|$$

so that
$$\frac{\partial\overline{r}}{\partial q_1} = h_1\overline{e}_1 \qquad (7)$$

where
$$h_1 = \left|\frac{\partial\overline{r}}{\partial u_1}\right|.$$

Similarly if $\overline{e}_2$ and $\overline{e}_3$ are unit tangent vectors to the u_2 and u_3 curves at P respectively then

$$\frac{\partial\overline{r}}{\partial q_2} = h_2\overline{e}_2 \qquad (8)$$

and
$$\frac{\partial\overline{r}}{\partial q_3} = h_3\overline{e}_3 \qquad (9)$$

where
$$h_2 = \left|\frac{\partial\overline{r}}{\partial q_2}\right|, \quad h_3 = \left|\frac{\partial\overline{r}}{\partial q_3}\right|.$$

The quantities h_1, h_2, h_3 are called *scale factors* or *Lame coefficients* of the given curvilinear system of coordinates, and are given by

$$h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial z_i}\right)^2}; \qquad (10)$$

$$i = 1, 2, 3$$

Thus

$$d\overline{r} = \frac{\partial\overline{r}}{\partial q_1}dq_1 + \frac{\partial\overline{r}}{\partial q_2}dq_2 + \frac{\partial\overline{r}}{\partial q_3}dq_3$$

$$d\overline{r} = h_1 dq_1 \overline{e}_1 + h_2 dq_2 \overline{e}_2 + h_3 dq_3 \overline{e}_3 \qquad (11)$$

Example: In rectangular coordinate system (q_1, q_2, q_3) is replaced by (x, y, z). Here

$$h_1 = h_2 = h_3 = 1, \quad \overline{e}_1 = \overline{i}, \quad \overline{e}_2 = \overline{j}, \quad \overline{e}_3 = \overline{k}$$

Example: Cylindrical coordinates

$$q_1 = r, \quad q_2 = \theta, \quad q_3 = z$$

By virtue of (10)

$$h_1 = H_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1$$

$$h_2 = H_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r$$

$$h_3 = H_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1$$

Example: Spherical coordinates

$$q_1 = r, q_2 = \theta, q_3 = \phi$$

Using (10), we get

$$h_1 = H_r = 1, h_2 = H_\theta = r, h_3 = H_\phi = r\sin\theta.$$

Expressions of Gradient, Divergence, Curl and Laplacian in Orthogonal Curvilinear, Spherical and Cylindrical Coordinates

Worked Out Examples

Example 1: Derive an expression for ∇f in orthogonal curvilinear coordinates. Hence deduce ∇ in rectangular cartesian coordinates.

Solution: Let

$$\nabla f = f_1 \overline{e}_1 + f_2 \overline{e}_2 + f_3 \overline{e}_3 \qquad (1)$$

where f_1, f_2, f_3 are unknowns to be determined.

Since

$$d\overline{r} = \frac{\partial \overline{r}}{\partial q_1} dq_1 + \frac{\partial \overline{r}}{\partial q_2} dq_2 + \frac{\partial \overline{r}}{\partial q_3} dq_3$$

$$d\overline{r} = h_1 \overline{e}_1 dq_1 + h_2 \overline{e}_2 dq_2 + h_3 \overline{e}_3 dq_3 \qquad (2)$$

By taking dot product of (1) and (2), we have

$$df = \nabla f \cdot d\overline{r} = h_1 f_1 dq_1 + h_2 f_2 dq_2 + h_3 f_3 dq_3 \qquad (3)$$

But by definition of differential,

$$df = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial q_3} dq_3 \qquad (4)$$

Equating (3) and (4), we get

$$f_1 = \frac{1}{h_1}\frac{\partial f}{\partial q}, f_2 = \frac{1}{h_2}\frac{\partial f}{\partial q_2}, f_3 = \frac{1}{h_3}\frac{\partial f}{\partial q_3} \qquad (5)$$

Substituting (5) in (1), we get

$$\nabla f = \frac{\overline{e}_1}{h_1}\frac{\partial f}{\partial q_1} + \frac{\overline{e}_2}{h_2}\frac{\partial f}{\partial q_2} + \frac{\overline{e}_3}{h_3}\frac{\partial t}{\partial q_3}$$

Thus the nabla operator ∇ in orthogonal curvilinear coordinates is

$$\nabla = \frac{\overline{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\overline{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\overline{e}_3}{h_3}\frac{\partial}{\partial q_3} \qquad (6)$$

Putting

$$h_1 = h_2 = h_3 = 1, \quad \overline{e}_1 = \overline{i},$$
$$\overline{e}_2 = \overline{j}, \quad \overline{e}_3 = \overline{k}$$

and $\quad q_1 = x, \quad q_2 = y, \quad q_3 = z$

(6) reduces to

$$\nabla = \overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}.$$

Example 2: Show that $\overline{e}_1 = h_2 h_3 \nabla q_2 \times \nabla q_3$.

Solution: From Example 1 with $f = q_1$, we have

$$\nabla f = \nabla q_1 = \frac{\overline{e}_1}{h_1}\frac{\partial q_1}{\partial q_1} + 0 + 0 = \frac{\overline{e}_1}{h_1}$$

Similarly, $\quad \nabla q_2 = \frac{\overline{e}_2}{h_2}$

and $\quad \nabla q_3 = \frac{\overline{e}_3}{h_3}$

Now

$$\nabla q_2 \times \nabla q_3 = \frac{\overline{e}_2}{h_2} \times \frac{\overline{e}_3}{h_3} = \frac{1}{h_2 h_3}\overline{e}_2 \times \overline{e}_3 = \frac{\overline{e}_1}{h_2 h_3}$$

So $\quad \overline{e}_1 = h_2 h_3 \nabla q_2 \times \nabla q_3.$

In a similar way, we get

$$\overline{e}_2 = h_3 h_1 \nabla q_3 \times \nabla q_1$$
$$\overline{e}_3 = h_1 h_2 \nabla q_1 \times \nabla q_2.$$

Example 3: Derive an expression for $\nabla \cdot \overline{A}$ in orthogonal curvilinear coordinates. Deduce $\nabla \cdot \overline{A}$ in rectangular coordinates

Solution: Let

$$\bar{A} = A_1\bar{e}_1 + A_2\bar{e}_2 + A_3\bar{e}_3$$

So that

$$\nabla \cdot \bar{A} = \nabla \cdot (A_1\bar{e}_1 + A_2\bar{e}_2 + A_3\bar{e}_3)$$
$$= \nabla \cdot (A_1\bar{e}_1) + \nabla \cdot (A_2\bar{e}_2) + \nabla \cdot (A_3\bar{e}_3) \quad (1)$$

Consider,

$$\nabla \cdot (A_1\bar{e}_1) = \nabla \cdot (A_1 h_2 h_3 \nabla q_2 \times \nabla q_3)$$

Using result of above Example 2

$$= \nabla(A_1h_2h_3) \cdot \nabla q_2 \times \nabla q_3 + A_1h_2h_3 \nabla \cdot (\nabla q_2 \times \nabla q_3)$$
$$= \nabla(A_1h_2h_3) \cdot \frac{\bar{e}_2}{h_2} \times \frac{\bar{e}_3}{h_3} + 0$$
$$= \nabla(A_1h_2h_3) \cdot \frac{\bar{e}_1}{h_2h_3}.$$

Using ∇f result from above Example 1.

$$\nabla \cdot (A_1\bar{e}_1) = \left[\frac{\bar{e}_1}{h_1}\frac{\partial}{\partial q_1}(A_1h_2h_3) + \frac{\bar{e}_2}{h_2}\frac{\partial}{\partial q_2}(A_1h_2h_3) + \frac{\bar{e}_3}{h_3}\frac{\partial}{\partial q_3}(A_1h_2h_3)\right] \cdot \frac{\bar{e}_1}{h_2h_3}$$

$$\nabla \cdot (A_1\bar{e}_1) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q_1}(A_1h_2h_3) + 0 + 0 \quad (2)$$

Similarly, we get

$$\nabla \cdot (A_2\bar{e}_2) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q_2}(A_2h_3h_1) \quad (3)$$

$$\nabla \cdot (A_3\bar{e}_3) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q_3}(A_3h_1h_2) \quad (4)$$

Adding (2), (3), (4) and using (1), we get the required result as

$$\nabla \cdot \bar{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3) + \frac{\partial}{\partial q_2}(A_2h_3h_1) + \frac{\partial}{\partial q_3}(A_3h_1h_2)\right].$$

Putting $h_1 = h_2 = h_3 = 1$

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$
$$\nabla \cdot \bar{A} = \tfrac{\partial A_1}{\partial x} + \tfrac{\partial A_2}{\partial y} + \tfrac{\partial A_3}{\partial z}$$

Example 4: Derive an expression for $\nabla \times \bar{A}$ in orthogonal curvilinear coordinates. Deduce $\nabla \times \bar{A}$ for cartesian coordinates.

Solution: If

$$\bar{A} = A_1\bar{e}_1 + A_2\bar{e}_2 + A_3\bar{e}_3$$

then

$$\nabla \times \bar{A} = \nabla \times (A_1\bar{e}_1 + A_2\bar{e}_2 + A_3\bar{e}_3)$$
$$\nabla \times \bar{A} = \nabla \times (A_1\bar{e}_1) + \nabla \times (A_2\bar{e}_2) + \nabla \times (A_3\bar{e}_3) \quad (1)$$

Consider

$$\nabla \times (A_1\bar{e}_1) = \nabla \times (A_1h_1\nabla q_1)$$

Since

$$\bar{e}_1 = h_1\nabla q_1$$
$$\nabla \times (A_1\bar{e}_1) = \nabla(A_1h_1) \times \nabla q_1 + A_1h_1\nabla \times \nabla q_1$$
$$= \nabla(A_1h_1) \times \frac{\bar{e}_1}{h_1} + 0$$

Substituting gradient value from above Example 1

$$\nabla \times (A_1\bar{e}_1) = \left[\frac{\bar{e}_1}{h_1}\frac{\partial}{\partial q_1}(A_1h_1) + \frac{\bar{e}_2}{h_2}\frac{\partial}{\partial q_2}(A_1h_1) + \frac{\bar{e}_3}{h_3}\frac{\partial}{q_3}(A_1h_1)\right] \times \frac{\bar{e}_1}{h_1}$$
$$= 0 - \frac{\bar{e}_3}{h_2h_1}\frac{\partial}{\partial q_2}(A_1h_1) + \frac{\bar{e}_2}{h_3h_1}\frac{\partial}{\partial q_3}(A_1h_1) \quad (2)$$

In a similar way, we get

$$\nabla \times (A_2\bar{e}_2) = \frac{\bar{e}_3}{h_1h_2}\frac{\partial}{\partial q_1}(A_2h_2) - \frac{\bar{e}_1}{h_2h_3}\frac{\partial}{\partial q_3}(A_2h_2) \quad (3)$$

and

$$\nabla \times (A_3\bar{e}_3) = \frac{\bar{e}_1}{h_2h_3}\frac{\partial}{\partial q_2}(A_3h_3) - \frac{\bar{e}_2}{h_3h_1}\frac{\partial}{\partial q_1}(A_3h_3) \quad (4)$$

Adding (2), (3), (4), we get the required expression for which can be written as

$$\nabla\times\bar{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\bar{e}_1 & h_2\bar{e}_2 & h_3\bar{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}.$$

For cartesian system,

$$\nabla\times A=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Example 5: Express $\nabla^2 f$ in orthogonal curvilinear coordinates. Deduce for cartesian system

Solution: From Example 1,

$$\nabla f=\frac{\bar{e}_1}{h_1}\frac{\partial f}{\partial q_1}+\frac{\bar{e}_2}{h_2}\frac{\partial f}{\partial q_2}+\frac{\bar{e}_3}{h_3}\frac{\partial f}{\partial q_3}$$

If $\bar{A}=A_1\bar{e}_1+A_2\bar{e}_2+A_3\bar{e}_3=\nabla f$

Then equating coefficients of $\bar{e}_1,\bar{e}_2,\bar{e}_3$, we get

$$A_1=\frac{1}{h_1}\frac{\partial f}{\partial q_1},\ A_2=\frac{1}{h_2}\frac{\partial f}{\partial q_2},\ A_3=\frac{1}{h_3}\frac{\partial f}{\partial q_3}$$

Thus

$$\nabla^2 f=\nabla\cdot\nabla f=\nabla\cdot\bar{A}$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3)+\frac{\partial}{\partial q_2}(A_2h_3h_1)+\frac{\partial}{\partial q_3}(A_3h_1h_2)\right]$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

For cartesian system,

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

Example 6: Express (a) ∇f, (b) $\nabla\cdot\bar{A}$, (c) $\nabla\times\bar{A}$, (d) $\nabla^2 f$ in cylindrical coordinates (r,θ,z).

Solution: For cylindrical coordinates (r,θ,z), we know that

$$q_1=r,\ q_2=\theta,\ q_3=z,\ \bar{e}_1=\bar{e}_r,\ \bar{e}_2=\bar{e}_\theta,\ \bar{e}_3=\bar{e}_z$$

and

$$h_1=h_r=1,\quad h_2=h_\theta=r,\quad h_3=h_z=1$$

a. From Example 1

$$\nabla f=\frac{\bar{e}_1}{h_1}\frac{\partial f}{\partial q_1}+\frac{\bar{e}_2}{h_2}\frac{\partial f}{\partial q_2}+\frac{\bar{e}_3}{h_3}\frac{\partial f}{\partial q_3}$$

$$\nabla f=\frac{1}{1}\frac{\partial f}{\partial r}\bar{e}_r+\frac{1}{r}\frac{\partial f}{\partial \theta}\bar{e}_\theta+\frac{1}{1}\frac{\partial f}{\partial z}\bar{e}_z$$

b. From Example 3

$$\nabla\cdot\bar{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1)+\frac{\partial}{\partial q_2}(h_3h_1A_2)+\frac{\partial}{\partial q_3}(h_1h_2A_3)\right]$$

$$=\frac{1}{1\cdot r\cdot 1}\left[\frac{\partial}{\partial r}(r\cdot 1\cdot A_r)+\frac{\partial}{\partial \theta}(1\cdot 1\cdot A_\theta)+\frac{\partial}{\partial z}(1\cdot r\cdot A_z)\right]$$

$$\nabla\cdot\bar{A}=\frac{1}{r}\left[\frac{\partial}{\partial r}(r\,A_r)+\frac{\partial}{\partial \theta}A_\theta+\frac{\partial}{\partial z}(r\,A_z)\right]$$

c.

$$\nabla\times\bar{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\bar{e}_1 & h_2\bar{e}_2 & h_3\bar{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$=\frac{1}{1\cdot r\cdot 1}\begin{vmatrix} \bar{e}_r & r\bar{e}_\theta & \bar{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}.$$

d. From Example 5,

$$\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

$$=\frac{1}{1\cdot r\cdot 1}\left[\frac{\partial}{\partial r}\left(\frac{r\cdot 1}{1}\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1\cdot 1}{r}\frac{\partial f}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(\frac{1\cdot r}{1}\frac{\partial f}{\partial z}\right)\right]$$

$$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{\partial^2 f}{\partial z^2}.$$

Example 7: Express (a) ∇f, (b) $\nabla\cdot\bar{A}$, (c) $\nabla\times\bar{A}$, (d) $\nabla^2 f$ in spherical curvilinear coordinates.

Solution: Here

$$q_1=r,\quad q_2=\theta,\quad q_3=\phi,$$
$$\bar{e}_1=\bar{e}_r,\quad \bar{e}_2=\bar{e}_\theta,\quad \bar{e}_3=\bar{e}_\phi$$
$$h_1=h_r=1,\quad h_2=h_\theta=r,\quad h_3=h_\phi=r\sin\theta.$$

a.
$$\nabla f=\frac{\bar{e}_1}{h_1}\frac{\partial f}{\partial q_1}+\frac{e_2}{h_2}\frac{\partial f}{\partial q_2}+\frac{\bar{e}_3}{h_3}\frac{\partial f}{\partial q_3}$$
$$\nabla f=\frac{\bar{e}_r}{1}\frac{\partial f}{\partial r}+\frac{\bar{e}_\theta}{r}\frac{\partial f}{\partial\theta}+\frac{\bar{e}_3}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

b.
$$\nabla\cdot\bar{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1)+\frac{\partial}{\partial q_2}(h_3h_1A_2)+\frac{\partial}{\partial q_3}(h_1h_2A_3)\right]$$
$$=\frac{1}{1\cdot r\cdot r\cdot\sin\theta}\left[\frac{\partial}{\partial r}(r\cdot r\sin\theta A_r)+\frac{\partial}{\partial\theta}(r\sin\theta\cdot 1\cdot A_\theta)+\frac{\partial}{\partial\phi}(1\cdot r\cdot A_\phi)\right]$$
$$\nabla\cdot\bar{A}=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r)+\frac{\partial}{\partial\theta}(r\sin\theta A_\theta)+\frac{\partial}{\partial\phi}(rA_\phi)\right]$$
$$\nabla\cdot\bar{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(A_\phi).$$

c.
$$\nabla\times\bar{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\bar{e}_1 & h_2\bar{e}_2 & h_3\bar{e}_3\\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3}\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}$$
$$\nabla\times\bar{A}=\frac{1}{1\cdot r\cdot r\sin\theta}\begin{vmatrix}e_r & re_\theta & r\sin\theta e_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & r\,A_\theta & r\sin\theta A_\phi\end{vmatrix}$$
$$=\frac{1}{r^2\sin\theta}\left[\left\{\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)-\frac{\partial}{\partial\phi}(rA_\theta)\right\}\bar{e}_r-\left\{\frac{\partial}{\partial r}(r\sin\theta A_\phi)-\frac{\partial}{\partial\phi}(A_r)\right\}r\bar{e}_\theta+\left\{\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial}{\partial\theta}(A_r)\right\}r\sin\theta\bar{e}_\phi\right].$$

d.
$$\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$
$$\nabla^2 f=\frac{1}{1\cdot r\cdot r\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r\cdot r\sin\theta}{1}\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta\cdot 1}{r}\frac{\partial f}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1\cdot r}{r\cdot\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$$
$$\nabla^2 f=\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\right].$$

Example 8: Prove that a spherical coordinate system is orthogonal.

Solution: The position vector of any point in spherical coordinates is

$$\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}=\rho\sin\theta\cos\phi\bar{i}+\rho\sin\theta\sin\phi\bar{j}+\rho\cos\theta\bar{k}$$

The tangent vectors to the ρ,θ,ϕ curves are given respectively by $\frac{\partial\bar{r}}{\partial\rho},\frac{\partial\bar{r}}{\partial\theta},\frac{\partial\bar{r}}{\partial\phi}$ where

$$\frac{\partial\bar{r}}{\partial\rho}=\sin\theta\cos\phi\bar{i}+\sin\theta\sin\phi\bar{j}+\cos\theta\bar{k}$$
$$\frac{\partial\bar{r}}{\partial\theta}=\rho\cos\theta\cos\phi\bar{i}+\rho\cos\theta\sin\phi\bar{j}-\rho\sin\theta\bar{k}$$
$$\frac{\partial\bar{r}}{\partial\phi}=-\rho\sin\theta\sin\phi\bar{i}+\rho\sin\theta\cdot\cos\phi\bar{j}+0$$

The unit vectors in these directions are

$$\bar{e}_1=\bar{e}_\rho=\frac{\partial\bar{r}/\partial\rho}{|\partial\bar{r}/\partial\rho|}=\frac{\sin\theta\cos\phi\bar{i}+\sin\theta\sin\phi\bar{j}+\cos\theta\bar{k}}{\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta}=\frac{\sin\theta\cos\phi i+\sin\phi\sin\theta j+\cos\theta\bar{k}}{1}$$

$$\bar{e}_2=\bar{e}_\theta=\frac{\partial\bar{r}/\partial\theta}{|\partial\bar{r}/\partial\theta|}=\frac{\rho\cos\theta\cos\phi\bar{i}+\rho\cos\theta\sin\phi\bar{j}-\rho\sin\theta\bar{k}}{\rho}$$

$$\bar{e}_3 = \bar{e}_\phi = \frac{\partial \bar{r}/\partial \phi}{|\partial \bar{r}/\partial \phi|} = \frac{-\rho \sin\theta \sin\phi \bar{i} + \rho \sin\theta \cos\phi \bar{j}}{\rho \sin\theta}$$

Then

$$\bar{e}_1 \cdot \bar{e}_2 = \sin\theta \cdot \cos\theta \cdot \cos^2\phi + \sin\theta \cdot \cos\theta \cdot \sin^2\phi - \cos\theta \sin\theta = 0$$

$$\bar{e}_1 \cdot \bar{e}_3 = -\sin\theta \cdot \cos\phi \sin\phi + \sin\theta \cdot \sin\phi \cdot \cos\phi = 0$$

$$\bar{e}_2 \cdot \bar{e}_3 = -\cos\theta \cdot \cos\phi \sin\phi + \cos\theta \sin\phi \cos\phi = 0.$$

So $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are mutually perpendicular and the spherical coordinate system is orthogonal.

Example 9: Find the Jacobian of x, y, z with respect to the orthogonal curvilinear coordinates q_1, q_2, q_3.

Solution:

$$J\left(\frac{x, y, z}{q_1, q_2, q_3}\right) = \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2} \\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3} \end{vmatrix}$$

This determinant is triple scalar product given by

$$J = \left(\frac{\partial x}{\partial q_1}\bar{i} + \frac{\partial y}{\partial q_1}\bar{j} + \frac{\partial z}{\partial q_1}\bar{k}\right) \cdot \left(\frac{\partial x}{\partial q^2}\bar{i} + \frac{\partial y}{\partial q_2}\bar{j} + \frac{\partial z}{\partial q_2}\bar{k}\right) \times \left(\frac{\partial x}{\partial q_3}\bar{i} + \frac{\partial y}{\partial q_3}\bar{j} + \frac{\partial z}{\partial q_3}\bar{k}\right)$$

$$= \frac{\partial \bar{r}}{\partial q_1} \cdot \frac{\partial \bar{r}}{\partial q_2} \times \frac{\partial \bar{r}}{\partial q_3} = h_1 \bar{e}_1 \cdot h_2 \bar{e}_2 \times h_3 \bar{e}_3$$

Jacobian $J = h_1 h_2 h_3 \bar{e}_1 \cdot \bar{e}_2 \times \bar{e}_3 = h_1 h_2 h_3.$

Note: If $J = 0$, $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are coplanar and coordinate transformation breaks.

Example 10: Find the Jacobian $J\left(\frac{x, y, z}{q_1, q_2, q_3}\right)$ for

a. cylindrical,
b. spherical coordinates.

Solution:

a. $J = h_1 h_2 h_3 = 1 \cdot r \cdot 1 = r$

b. $J = h_1 h_2 h_3 = 1 \cdot r \cdot r \cdot \sin\theta = r^2 \sin\theta.$

Example 11: Find

$$\frac{\partial \bar{r}}{\partial q_1}, \frac{\partial \bar{r}}{\partial q_2}, \frac{\partial \bar{r}}{\partial q_3}, \nabla q_1, \nabla q_2, \nabla q_3$$

in cylindrical coordinates.

Solution:

$\bar{r} = xi + yj + zk = \rho\cos\theta i + \rho \sin\theta j + zk$

Then

$$\frac{\partial \bar{r}}{\partial q_1} = \frac{\partial \bar{r}}{\partial \rho} = \cos\theta \bar{i} + \sin\theta \bar{j}$$

$$\frac{\partial \bar{r}}{\partial q_2} = \frac{\partial \bar{r}}{\partial \theta} = -\rho \sin\theta \bar{i} + \rho\cos\theta \bar{j}$$

$$\frac{\partial \bar{r}}{\partial q_3} = \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

$$\nabla q_1 = \nabla \rho = \frac{1}{h_1}\frac{\partial \rho}{\partial \rho}\bar{e}_\rho + \frac{1}{h_2}\frac{\partial \rho}{\partial \theta}\bar{e}_\theta + \frac{1}{h_3}\frac{\partial \rho}{\partial z}\bar{e}_z$$

$$= \frac{1}{1} \cdot 1 \cdot \bar{e}_\rho = \cos\theta \bar{i} + \sin\theta \bar{j}$$

Similarly,

$$\nabla q_2 = \nabla \theta = \frac{1}{h_2}\frac{\partial \theta}{\partial \theta}\bar{e}_\theta = \frac{1}{\rho}\bar{e}_\theta = \frac{-\sin\theta i + \cos\theta j}{\rho}$$

$$\nabla q_3 = \nabla z = \frac{1}{h_3}\frac{\partial z}{\partial z}\bar{e}_z = \bar{e}_z = \bar{k}.$$

Example 12: Prove that the surface area of a given region R of the surface $\bar{r} = \bar{r}(u, v)$ is $\int\int_R \sqrt{EG - F^2}\, du\, dv$. Use this to determine the surface area of a sphere.

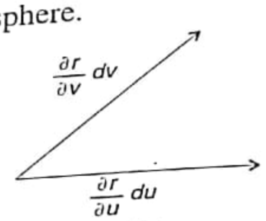

Fig. 15.5

Solution: Element of area is given by

$$dS = \left|\left(\frac{\partial \bar{r}}{\partial u}du\right) \times \frac{\partial \bar{r}}{\partial v}dv\right| = \left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right| du\, dv$$

$$= \sqrt{\left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right)}\, du\, dv$$

Using

$$(\overline{A}\times\overline{B})\cdot(\overline{C}\times\overline{D})=(\overline{A}\cdot\overline{C})(\overline{B}\cdot\overline{D})-(\overline{A}\cdot\overline{D})(\overline{B}\cdot\overline{C}),$$

we have

$$dS=\sqrt{\left(\frac{\partial\overline{r}}{\partial u}\cdot\frac{\partial\overline{r}}{\partial u}\right)\left(\frac{\partial\overline{r}}{\partial v}\cdot\frac{\partial\overline{r}}{\partial v}\right)-\left(\frac{\partial\overline{r}}{\partial u}\cdot\frac{\partial\overline{r}}{\partial v}\right)\left(\frac{\partial\overline{r}}{\partial v}\cdot\frac{\partial\overline{r}}{\partial u}\right)}\times du\,dv$$

$$dS=\sqrt{EG-F^2}\,du\,dv$$

where

$$E=\frac{\partial\overline{r}}{\partial u}\cdot\frac{\partial\overline{r}}{\partial u},\quad G=\frac{\partial\overline{r}}{\partial v}\cdot\frac{\partial\overline{r}}{\partial v},\quad F=\frac{\partial\overline{r}}{\partial u}\cdot\frac{\partial\overline{r}}{\partial v},$$

Integrating over the region R, the surface area of the given region R is given by

$$S=\int\int_R\sqrt{EG-F^2}\,du\,dv$$

To find the surface area of a sphere of radius b the position vector $\overline{r}$ of any point on the surface is

$$\overline{r}=xi+yj+zk$$

In spherical coordinates

$$\overline{r}=b\sin\theta\cos\phi\overline{i}+b\sin\theta\sin\phi j+b\cos\theta k$$

$$\overline{r}=\overline{r}(\theta,\phi)$$

Differentiating partially w.r.t. θ and ϕ, we get

$$\frac{\partial\overline{r}}{\partial\theta}=b\cos\theta\cos\phi\overline{i}+b\cos\theta\sin\phi\overline{j}-b\sin\theta\overline{k}$$

$$\frac{\partial\overline{r}}{\partial\phi}=-b\sin\theta\sin\phi\overline{i}+b\sin\theta\cdot\cos\phi\overline{j}$$

$$E=\frac{\partial\overline{r}}{\partial\theta}\cdot\frac{\partial\overline{r}}{\partial\theta}=b^2,G=\frac{\partial\overline{r}}{\partial\phi}\cdot\frac{\partial\overline{r}}{\partial\phi}=b^2\sin^2\theta,$$

$$\overline{F}=\frac{\partial\overline{r}}{\partial\phi}\cdot\frac{\partial\overline{r}}{\partial\theta}=0$$

$$S=\int\int_R\sqrt{EG-F^2}d\theta\,d\phi$$

$$=\int_0^{2\pi}\int_0^{\pi}\sqrt{b^2\cdot b^2\sin^2\theta-0}\,d\theta\,d\phi$$

$$=2\pi b^2(-1-1)=4\pi b^2.$$

Example 13: Prove that curl of gradient $f=0$ in any orthogonal curvilinear coordinate system.

Solution: In any orthogonal curvilinear coordinate system

$$\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial q_1}\overline{e}_1+\frac{1}{h_2}\frac{\partial f}{\partial q_2}\overline{e}_2+\frac{1}{h_3}\frac{\partial f}{\partial q_3}\overline{e}_3$$

Then

$$\nabla\times\nabla f=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\overline{e}_1 & h_2\overline{e}_2 & h_3\overline{e}_3\\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3}\\ h_1\frac{1}{h_1}\frac{\partial f}{\partial q_1} & h_2\frac{1}{h_2}\frac{\partial f}{\partial q_2} & h_3\frac{1}{h_3}\frac{\partial f}{\partial q_3}\end{vmatrix}$$

$$=\frac{1}{h_1h_2h_3}\left[h_1\overline{e}_1\left\{\frac{\partial^2 f}{\partial q_2\partial q_3}-\frac{\partial^2 f}{\partial q_3\partial q_2}\right\}-h_2\overline{e}_2\left\{\frac{\partial^2 f}{\partial q_1\partial q_3}-\frac{\partial^2 f}{\partial q_3\partial q_1}\right\}+h_3\overline{e}_3\left\{\frac{\partial^2 f}{\partial q_1\partial q_2}-\frac{\partial^2 f}{\partial q_2\partial q_1}\right\}\right]=0.$$

Example 14: Find the square of the element of arc length in cylindrical coordinates and determine the corresponding scale factors.

Solution: The position vector is

$$\overline{r}=\rho\cos\theta\overline{i}+\rho\sin\theta\overline{j}+z\overline{k}$$

Then

$$d\overline{r}=\frac{\partial\overline{r}}{\partial\rho}d\rho+\frac{\partial\overline{r}}{\partial\theta}d\theta+\frac{\partial\overline{r}}{\partial z}dz$$

$$=(\cos\theta i+\sin\theta j)d\rho+(-\rho\sin\theta i+\rho\cos\theta j)d\theta+k\,dz$$

$$d\overline{r}=(\cos\theta d\rho-\rho\sin\theta d\theta)i+(\sin\theta d\rho+\rho\cos\theta d\theta)j+k\,dz$$

Thus

$$ds^2=d\overline{r}\cdot d\overline{r}=(\cos\theta d\rho-\rho\sin\theta d\theta)^2+(\sin\theta d\rho+\rho\cos\theta d\theta)^2+(dz)^2$$

$$=(d\rho)^2+\rho^2(d\theta)^2+(dz)^2$$

Here

$$h_1=h_\rho=1,h_2=h_\theta=\rho,h_3=h_z=1$$

are the scale factors.

Example 15: A vector field is given in cylindrical coordinates as

$$\overline{A}(P) = \rho_\rho + \theta \overline{e_\theta}$$

Find the vector lines of the field.

Solution: It is given that $a_1 = 1, a_2 = \theta, a_3 = 0$.

So that

$$\frac{d\rho}{1} = \frac{P\,d\theta}{\theta} = \frac{dz}{0}$$

whence

$$z = c_1$$

$$\rho = c_2\theta$$

which are Archimedean spirals lying in planes parallel to the xy-plane (i.e., $z =$ constant).

Example 16: Compute the gradient of the scalar field $f = \rho + z\cos\theta$ specified in cylindrical coordinates (ρ, θ, z).

Solution:

$$\nabla f = \frac{\partial f}{\partial \rho}\overline{e_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\overline{e_\theta} + \frac{\partial f}{\partial z}\overline{e_z}$$

$$= 1 \cdot \overline{e_\rho} + \frac{1}{\rho}(-z\sin\theta)\overline{e_\theta} + \cos\theta\,\overline{e_z}.$$

Example 17: Compute the curl of $\overline{A}$ specified in cylindrical coordinates where

$$\overline{A} = \sin\theta\,\overline{e_\rho} + \frac{\cos\theta}{\rho}e_\theta - \rho z\overline{e_z}$$

Solution: Since

$$\nabla \times \overline{A} = \begin{vmatrix} \frac{1}{\rho}\overline{e_\rho} & \overline{e_\theta} & \frac{1}{\rho}\overline{e_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} \frac{1}{\rho}\overline{e_\rho} & \overline{e_\theta} & \frac{1}{\rho}\overline{e_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \sin\theta & \cos\theta & -\rho z \end{vmatrix}$$

$$= \frac{1}{\rho}\overline{e_\rho}(0-0) - \overline{e_\theta}(-z-0) + \frac{1}{\rho}\overline{e_z}(0-\cos\theta)$$

$$\nabla \times \overline{A} = z\overline{e_\theta} - \frac{\cos\theta}{\rho}\overline{e_z}$$

Example 18: Show that the vector field $\overline{A}$ in spherical coordinates

$$\overline{A} = \frac{2\cos\theta}{r^3}\overline{e_r} + \frac{\sin\theta}{r^3}\overline{e_\theta}$$

is solenoidal.

Solution: We know that divergence in spherical coordinates

$$\nabla \cdot \overline{A} = \frac{1}{\rho^2}\frac{\partial(\rho^2 a_1)}{\partial \rho} + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_2) + \frac{1}{\rho\sin\theta}\frac{\partial a_3}{\partial \phi}$$

$$\nabla \cdot \overline{A} = \frac{1}{\rho^2}\frac{\partial}{\partial r}\left(\rho^2 \cdot \frac{2\cos\theta}{\rho^3}\right) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\sin\theta}{\rho^3}\right) + 0$$

$$= \frac{1}{\rho^2}\left(-\frac{2\cos\theta}{\rho^2}\right) + \frac{1}{\rho^4\sin\theta}\cdot 2 \cdot \sin\theta\cos\theta = 0$$

wherever $r \neq 0$, which means that the vector field $\overline{A}$ is solenoidal at all points except at $r = 0$.

Example 19: Find the potential of

$$\overline{A} = \frac{1}{\rho}e^{\theta\phi}\overline{e_r} + \frac{\theta\ln\rho}{r\sin\theta}e^{\theta\phi}\overline{e_\phi} + \frac{\ln\rho}{\rho}\phi e^{\theta\phi}\overline{e_\theta}$$

given in spherical coordinates.

Solution: In spherical coordinates

$$\nabla \times \overline{A} = \frac{1}{\rho^2\sin\theta}\begin{vmatrix} \overline{e_r} & \rho\overline{e_\theta} & r\sin\theta\overline{e_\phi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{\rho}e^{\theta\phi} & \phi\ln r e^{\theta\phi} & \theta\ln r e^{\theta\phi} \end{vmatrix} = 0.$$

Thus $\overline{A}$ is a potential field in the region where $r > 0, \theta \neq n\pi (n = 0, \neq 1, \cdot s)$.

Let $\quad \overline{A} = \nabla f = \frac{\partial f}{\partial \rho}\overline{e_\rho} + \frac{\partial f}{\partial \theta}\overline{e_\theta} + \frac{\partial f}{\partial \phi}\overline{e_\phi}$

where $f = f(\rho, \theta, \phi)$ is the desired potential function, which is the solution of the following system of differential equations.

$$\frac{\partial f}{\partial \rho} = \frac{1}{\rho}e^{\theta\phi} \tag{1}$$

$$\frac{\partial f}{\partial \theta} = \phi e^{\theta\phi}\ln\rho \tag{2}$$

$$\frac{\partial f}{\partial \phi} = \theta e^{\theta\phi}\ln r \tag{3}$$

Integrating (1) w.r.t. ρ we get

$$f = e^{\theta\phi}\ln\rho + c_1(\phi, \theta) \tag{4}$$

Differentiating (4) w.r.t. 'θ' and equating it with (2)

$$\phi e^{\theta\phi}\ln r = \frac{\partial f}{\partial\theta} = \phi e^{\theta\phi}\ln r + \frac{\partial c_1}{\partial\theta}$$

i.e., $\frac{\partial c_1}{\partial \theta} = 0$ whence

$$c_1(\phi, \theta) = c_2(\phi) \tag{5}$$

Substituting (5) in (4)

$$f = e^{\theta\phi} \ln \rho + c_2(\phi) \tag{6}$$

Differentiating (6) w.r.t. ϕ and equating it with (3) we obtain

$$\theta e^{\theta\phi} \ln r = \frac{\partial f}{\partial \phi} = \theta e^{\theta\phi} \ln r + \frac{dc_2}{d\phi}$$

so $\frac{dc_2}{d\phi} = 0$ i.e., $c_2 = c =$ constant. The desired potential is

$$f(r, \theta, \phi) = e^{\theta\rho} \ln r + c$$

Example 20: Find all the solutions of the Laplace's equation $\nabla^2 f = 0$ that depend solely on the distance ρ.

Solution: Laplace's equation is spherical coordinates with spherical symmetry (f must not depend on θ or ϕ, i.e., $f = f(r)$). We have

$$\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial f}{\partial \rho} \right) = 0$$

so that

$$\rho^2 \frac{\partial f}{\partial r} = c_1 = \text{const.}$$

whence

$$f = \frac{c_1}{\rho} + c_2$$

where c_1 and c_2 are constants.

Example 21: Find the Laplacian of

$$f(\rho, \theta, z) = \rho^2\theta + z^2\theta^3 - \rho\theta z$$

Solution:

$$\nabla^2 f = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(\rho \frac{\partial f}{\partial z} \right) \right]$$

Here

$$\frac{\partial f}{\partial \rho} = 2\rho\theta + 0 - \theta z, \frac{\partial^2 f}{\partial \rho^2} = 2\theta$$

$$\frac{\partial f}{\partial \theta} = \rho^2 + 3z^2\theta^2 - \rho z, \frac{\partial^2 f}{\partial \theta^2} = 6z^2\theta$$

$$\frac{\partial f}{\partial z} = 2z\theta^3 - \rho\theta, \frac{\partial^2 f}{\partial z^2} = 2\theta^3$$

$$\nabla^2 f = \frac{1}{\rho} \left[\frac{\partial f}{\partial \rho} + \rho \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial^2 f}{\partial \theta^2} + \rho \frac{\partial^2 f}{\partial z^2} \right]$$

Substituting the above partial derivatives

$$\nabla^2 f = \frac{1}{\rho} [2\overline{\rho}\theta - \theta\overline{z} + \rho\overline{2}\theta + \frac{1}{\rho} 6\overline{z^2}\theta + \rho 2\overline{\theta^3}]$$

$$\nabla^2 f = 4\overline{\theta} - \frac{\theta z}{\rho} + \frac{6\theta z^2}{\rho^2} + 2\theta^3$$

Example 22: (a) Find the unit vectors $\overline{e}_\rho, \overline{e}_\theta, \overline{e}_\phi$ in spherical coordinate system in terms of i, j, k. (b) Solve for i, j, k in terms of e_r, e_θ, e_ϕ.

Solution: From Example 8

$$\overline{e_\rho} = \sin\theta \cos\phi \overline{i} + \sin\theta \sin\phi \overline{j} + \cos\theta \overline{k} \tag{1}$$

$$\overline{e_\theta} = \cos\theta \cos\phi \overline{i} + \cos\theta \sin\phi \overline{j} - \sin\theta \overline{k} \tag{2}$$

$$\overline{e_\phi} = -\sin\phi \overline{i} + \cos\phi \overline{j} \tag{3}$$

Solving (1), (2), (3) simultaneously

(1) $\times \sin\theta$: $\sin\theta \overline{e_\rho}$

$$= \sin^2\theta(\cos\phi i + \sin\phi j) + \sin\theta\cos\theta\overline{k}$$

(2) $\times \cos\theta$: $\cos\theta \overline{e_\theta}$

$$= \cos^2\theta(\cos\phi \overline{i} + \sin\phi \overline{j}) - \sin\theta\cos\theta\overline{k}$$

Adding

$$\sin\theta \overline{e_\rho} + \cos\theta \overline{e_\theta} = \cos\phi i + \sin\phi \overline{j} \quad (4) \times \sin\phi$$

$$\overline{e_\phi} = -\sin\phi \overline{i} + \cos\phi \overline{j} \quad (5) \times \cos\phi$$

Adding

$$\overline{j} = (\sin\theta \overline{e_\rho} + \cos\theta \overline{e_\theta}) \sin\phi + \cos\phi \overline{e_\phi} \tag{6}$$

Substituting (6) in (3)

$$\overline{i} = \frac{1}{\sin\phi} (\cos\phi J - \overline{e_\phi})$$

$$\overline{i} = \frac{1}{\sin\phi} \Big[\{(\sin\theta \overline{e_\rho} + \cos\theta \overline{e_\theta}) \sin\phi + \cos\phi \overline{e_\phi}\} \times \cos\phi - \overline{e_\phi} \Big]$$

$\overline{i} = \{\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta + \cot\phi\overline{e}_\phi\}$
$\times \cos\phi - \text{cosec}\phi\overline{e}_\phi$ (7)

Substituting (6) and (7) in (2)

$\overline{i}_t = \cos\theta \cdot \cos\phi\,[\{\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta + \cot\phi\overline{e}_\phi\}$
$\times \cos\phi - \text{cosec}\phi\overline{e}_\phi] + \cos\theta\sin\phi$
$\times [(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\sin\phi + \cos\phi\overline{e}_\phi] - \sin\theta\overline{k}$

Solving for $\overline{k}$,

$\overline{k} = \cot\theta \cdot \cos\phi\,[\{\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta + \cot\phi\overline{e}_\phi\}$
$\times \cos\phi - \text{cosec}\phi\overline{e}_\phi] + \cot\theta\sin\phi\,[(\sin\theta\overline{e}_\rho$
$+ \cos\theta\overline{e}_\theta)\sin\phi + \cos\phi\overline{e}_\phi] - \text{cosec}\theta \cdot \overline{e}_\theta$ (8)

Example 23: Represent the vector $\overline{A} = 2yi - zj + 3xk$ in spherical coordinates and determine A_r, A_θ, A_ϕ.

Solution: Substituting (7), (6), (8) in $\overline{A} = 2y\,i - zj + 3x\overline{k}$ from the above Example 22, we get A in spherical coordinates as

$\overline{A} = [\{\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta + \cot\phi\overline{e}_\phi\}\cos\phi - \text{cosec}\phi\overline{e}_\phi]$
$\times 2r\sin\theta\sin\phi - [(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\sin\phi + \cos\phi\overline{e}_\phi]$
$\times r\cos\theta + 3r\sin\theta\cot\theta\cos\phi\,[\{\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta$
$+ \cot\phi\overline{e}_\phi\}\cos\phi - \text{cosec}\phi\overline{e}_\rho] + 3r\sin\theta \cdot \cos\phi$
$\times \cot\theta \cdot \sin\phi\,[(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\sin\phi + \cos\phi\overline{e}_\phi]$
$-3r\sin\theta \cdot \cos\phi\,\text{cosec}\,\theta\,\overline{e}_\theta.$

Collecting the coefficients of $\overline{e}_\rho, \overline{e}_\theta$ and $\overline{e}_\phi$, we rewrite

$\overline{A} = [2r\sin^2\theta \cdot \sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi$
$+3r\sin^2\theta\cot\theta\cos^2\phi\cos\phi$
$+3r\sin^2\theta\cot\theta \cdot \cos\phi\sin\phi \cdot \sin\phi]\overline{e}_\rho$
$+[\cos\theta \cdot \cos\phi \cdot 2r \cdot \sin\theta \cdot \sin\phi - \cos\theta \cdot \sin\phi r\cos\theta$
$+3r\sin\theta \cdot \cos\phi \cdot \cot\theta \cdot \cos\phi\cos\theta \cdot \cos\phi$
$+3r\sin\theta \cdot \cos\phi \cdot \cot\theta \cdot \sin\phi\cos\theta\sin\phi$
$-3r\sin\theta \cdot \cos\phi \cdot \text{cosec}\theta]\overline{e}_\theta + \Big[(\cot\phi\cos\phi$
$-\text{cosec}\phi)2r\sin\theta\sin\phi - \cos\phi r\cos\theta$
$+(\cot\phi \cdot \cos\phi - \text{cosec}\phi\}3r\sin\theta \cdot \cot\theta \cdot \cos^2\phi$
$+\cos\phi \cdot 3r\sin\theta \cdot \cot\theta \cdot \cos\phi \cdot \sin\phi\Big]e_\phi$

Simplifying the result, we get

$$A_r = 2r\sin^2\theta\sin\phi \cdot \cos\phi - r\sin\theta \cdot \cos\theta \cdot \sin\phi$$
$$+3r\sin\theta\cos\theta \cdot \cos\phi$$
$$A_\theta = 2r\sin\theta\cos\theta \cdot \sin\phi\cos\phi - r\cos^2\theta \cdot \sin\phi$$
$$-3r\sin^2\theta \cdot \cos\phi$$
$$A\phi = -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

Example 24: Express the velocity $\overline{v}$ and acceleration $\overline{a}$ of a particle in spherical coordinates.

Solution: In rectangular coordinates the position vector, velocity and acceleration vectors are

$$\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$$
$$\overline{V} = \frac{d\overline{r}}{dt} = \dot{x}i + \dot{y}j + \dot{z}k$$
$$\overline{a} = \frac{d^2\overline{r}}{dt^2} = \ddot{x}i + \ddot{y}j + \ddot{z}k$$

In spherical coordinates

$$\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$$
$$= r\sin\theta\cos\phi\overline{i} + r\sin\theta\sin\phi\overline{j} + r\cos\theta\overline{k}$$

Substituting $\overline{i}, \overline{j}, \overline{k}$ from (7), (6), (8) of the previous Example 22, we get

$\overline{r} = r\sin\theta\cos\phi\,[(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\cos\phi - \sin\phi\overline{e}_\phi]$
$+r\sin\theta\sin\phi\,[(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\sin\phi + \cos\phi\overline{e}_\phi]$
$+r\cos\theta\,[\cot\theta \cdot \cos\phi\,\{(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\cos\phi$
$-\sin\phi\overline{e}_\phi\}] + r\cos\theta\,[\cot\theta\sin\phi\,\{(\sin\theta\overline{e}_\rho$
$+\cos\theta\overline{e}_\theta)\sin\phi + \cos\phi\overline{e}_\phi\}] - r\cos\theta \cdot \text{cosec}\theta\overline{e}_\theta$

Collecting the coefficients of $\overline{e}_\rho, \overline{e}_\theta, \overline{e}_\phi$

$\overline{r} = [r\sin^2\theta \cdot \cos^2\phi + r\sin^2\theta \cdot \sin^2\phi$
$+r\cos\theta \cdot \cot\theta \cdot \sin\theta \cdot \cos^2\phi$
$+r\cos\theta \cdot \cot\theta \cdot \sin^2\phi \cdot \sin\theta]\overline{e}_\rho$
$+\overline{e}_\theta[r\sin\theta \cdot \cos^2\phi \cdot \cos\theta + r\sin\theta\sin^2\phi \cdot \cos\theta$
$+r\cos\theta \cdot \cot\theta \cdot \cos^2\phi\cos\theta$
$+r\cos\theta \cdot \cot\theta \cdot \sin^2\phi\cos\theta - r\cos\theta\cos\theta]$
$+[-r\sin\theta \cdot \cos\phi\sin\phi + r\sin\theta \cdot \sin\phi\cos\phi$
$-r\cos\theta\cot\theta \cdot \cos\phi \cdot \sin\phi$
$+r\cos\theta \cdot \cot\theta \cdot \sin\phi\cos\phi]\overline{e}_\phi$

Simplifying, we get

$$\overline{r} = r\overline{e}_\rho + 0 \cdot \overline{e}_\theta + 0\overline{e}_\phi \qquad (1)$$

Differentiating (1) w.r.t. 't'

$$\text{Velocity} = \overline{V} = \frac{d\overline{r}}{dt} = \frac{dr}{dt}\overline{e}_\rho + r\frac{d}{dt}\overline{e}_\rho \qquad (2)$$

Here

$$\frac{d}{dt}\overline{e}_\rho = \frac{d}{dt}(\sin\theta\cos\phi\overline{i} + \sin\theta\sin\phi\overline{j} + \cos\theta\overline{k})$$

$$= \cos\theta \cdot \dot{\theta}\cos\phi i - \sin\theta\sin\phi \cdot \dot{\phi}\overline{i}$$

$$+ \cos\theta \cdot \dot{\theta} \cdot \sin\phi\overline{j} + \sin\theta \cdot \cos\phi\dot{\phi}\overline{j} - \sin\theta\dot{\theta}\overline{k}$$

$$= \dot{\theta}(\cos\theta \cdot \cos\phi\overline{i} + \cos\theta\sin\phi\overline{j} - \sin\theta\overline{k})$$

$$+ \dot{\phi}\sin\theta(-\sin\phi\overline{i} + \cos\phi\overline{j})$$

$$\frac{d}{dt}(\overline{e}_\rho) = \dot{\theta}\overline{e}_\theta + \dot{\phi}\sin\theta\overline{e}_\phi \qquad (3)$$

Substituting (3) in (1)

$$\overline{V} = \overline{r}\overline{e}_\rho + r[\dot{\theta}\overline{e}_\theta + \dot{\phi}\sin\theta\overline{e}_\phi]$$

$$\overline{V} = v_\rho\overline{e}_\rho + v_\theta\overline{e}_\theta + v_\phi\overline{e}_\phi \qquad (4)$$

where

$$v_\rho = \overline{r}, v_\theta = \overline{r}\dot{\theta}, v_\phi = \overline{r}\dot{\phi}\sin\theta$$

Differentiating (4) w.r.t. 't'

$$\text{Acceleration} = \overline{a} = \frac{d\overline{V}}{dt}$$

$$= \frac{d\overline{v}_\rho}{dt}\overline{e}_\rho + v_\rho\frac{d}{dt}\overline{e}_\rho + \frac{dv_\theta}{dt} \cdot \overline{e}_\theta$$

$$+ v_\theta \cdot \frac{d\overline{e}_\theta}{dt} + \frac{dv_\phi}{dt}\overline{e}_\phi + v_\phi \cdot \frac{d}{dt}\overline{e}_\phi \qquad (5)$$

Here

$$\frac{dv_\rho}{dt} = \frac{d}{dt}\overline{r} = \ddot{r},$$

$$\frac{dv_\theta}{dt} = \frac{d}{dt}(\overline{r}\dot{\theta}) = \dot{r}\dot{\theta} + r\ddot{\theta}$$

$$\frac{dv_\phi}{dt} = \frac{d}{dt}(r\dot{\phi}\sin\theta) = \dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta$$

$$\frac{d\overline{e}_\theta}{dt} = -\sin\theta \cdot \dot{\theta}\cos\phi\overline{i} - \cos\theta \cdot \sin\phi\dot{\phi}\overline{i}$$

$$- \sin\theta\dot{\theta}\sin\phi\overline{j} + \cos\theta \cdot \cos\phi\dot{\phi}j - \cos\theta\dot{\theta}\overline{k}$$

$$= -\dot{\theta}(\sin\theta \cdot \cos\phi\overline{i} + \sin\theta\sin\phi\overline{j} + \cos\theta\overline{k})$$

$$+ \dot{\phi}\cos\theta(-\sin\phi\overline{i} + \cos\phi\overline{j})$$

$$\frac{d\overline{e}_\theta}{dt} = -\dot{\theta}\overline{e}_\rho + \dot{\phi}\cos\theta\overline{e}_\phi$$

$$\frac{d\overline{e}_\phi}{dt} = \frac{d}{dt}(-\sin\phi i + \cos\phi j) = -\cos\phi\dot{\phi}i - \sin\phi\dot{\phi}j$$

$$= -\dot{\phi}(\cos\phi i + \sin\phi j)$$

Substituting these values in (5), we get and replacing $\overline{i}$ and $\overline{j}$ by (7) and (6) of previous Example 22

$$\overline{a} = \ddot{r}\overline{e}_\rho + \dot{r}(\dot{\theta}\overline{e}_\theta + \dot{\phi}\sin\theta\overline{e}_\phi) + (\dot{r}\dot{\theta} + r\ddot{\theta})\overline{e}_\theta$$

$$+ r\dot{\theta}(-\dot{\theta}\overline{e}_\rho + \dot{\phi}\cos\theta\overline{e}_\phi) + (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta$$

$$+ r\dot{\phi}\dot{\theta}\cos\theta)\overline{e}_\phi - r\dot{\phi}^2\sin\theta \cdot \cos\phi\,[(\sin\theta\overline{e}_\rho$$

$$+ \cos\theta\overline{e}_\theta)\cos\phi - \sin\phi\overline{e}_\phi] - r\dot{\phi}^2\sin\theta\sin\phi$$

$$\times [(\sin\theta\overline{e}_\rho + \cos\theta\overline{e}_\theta)\sin\phi + \cos\rho\overline{e}_\phi]$$

Rearranging the terms

$$\overline{a} = \left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta(\cos^2\phi + \sin^2\phi)\right]\overline{e}_\rho$$

$$+ \left[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta \cdot \cos\theta \cdot (\cos^2\phi\right.$$

$$\left. + \sin^2\rho)\right]\overline{e}_\theta + [\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + \dot{r}\dot{\phi}\sin\theta$$

$$+ r\ddot{\phi}\sin\theta + r\dot{\phi}^2\sin\theta \cdot \cos\phi\sin\phi$$

$$- r\dot{\phi}^2\sin\theta \cdot \sin\phi\cos\phi]\overline{e}_\phi$$

Thus

$$\overline{a} = a_r\overline{e}_r + a_\theta\overline{e}_\theta + a_\phi\overline{e}_\phi$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$$

$$a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2$$

$$a_\phi = \frac{1}{r\sin\theta} \cdot \frac{d}{dt}(r^2\sin^2\theta\dot{\phi}).$$

EXERCISE

Find the equations of the vector fields $\overline{A}$ where:

1. $\overline{A} = \overline{e}_\rho + \frac{1}{\rho}\overline{e}_\theta + \overline{e}_z$ (cylindrical)

Ans. $\rho = \theta + c_1, \rho = z + c_2$

2. $\overline{A} = 2\alpha\frac{\cos\theta}{\rho^3}\overline{e}_r + \frac{\alpha\sin\theta}{\rho^3}\overline{e}_\theta, \alpha = \text{constant}$ (spherical)

Ans. $\phi = c_1, \rho = c_2\sin^2\theta$

Find the gradient of the scalar fields f:

3. $f = \rho z \cos\theta$ (cylindrical)

Ans. $(z\cos\theta)\overline{e}_\rho - (z\sin\theta)\overline{e}_\theta + (\rho\cos\theta)\overline{e}_z$

4. $f = \rho^2 \sin 2\theta \sin\phi$ (spherical)

Ans. $(2\rho\sin 2\theta\sin\phi)\overline{e}_\rho + (2\rho\cos 2\theta\sin\phi)\overline{e}_\theta + (2\rho\cos\theta\cos\phi)\overline{e}_\phi$

5. $f = xyz$ in (a) cylindrical (b) spherical coordinates

Ans. a. $(\rho z\sin 2\theta)\overline{e}_\rho + (\rho z\cos 2\theta)\overline{e}_\theta + \left(\frac{\rho^2}{2}\sin 2\theta\right)\overline{e}_z$

b. $(3\rho^2\sin^2\theta\cos\theta \cdot \sin\phi\cos\phi)\overline{e}_\rho + \frac{\rho^2}{2}\sin 2\phi\{-\sin^3\theta + 2\sin\theta\cos^2\theta\}\overline{e}_\theta + (\rho^2\sin\theta\cos\theta\cos 2\theta)\overline{e}_\phi$

6. $f = \rho^2 + 2\rho\cos\theta - e^z\sin\theta$ (in cylindrical)

Ans. $2(\rho + \cos\theta)\overline{e}_\rho - (2\sin\theta + \frac{1}{\rho}e^z\cos\theta)e_\theta - e^z\sin\theta e_z$

7. $f = 3\rho^2\sin\theta + e^\rho\cos\phi - r$ (in spherical)

Ans. $(6\rho \cdot \sin\theta + e^\rho\cos\phi - 1)\overline{e}_\rho + 3\rho\cos\theta\overline{e}_\theta - \frac{e^\rho\sin\phi}{\rho\sin\theta}e_\phi$

Compute the divergence of $\overline{A}$:

8. $\overline{A} = \theta \arctan\rho\overline{e}_\rho + 2\overline{e}_\theta - z^2e^z\overline{e}_z$ (in cylindrical)

Ans. $\frac{\theta}{\rho}\arctan\rho + \frac{\theta}{1+\rho^2} - (z^2 + 2z)e^z$

9. $\overline{A} = \rho^2\overline{e}_\rho - 2\cos^2\phi\overline{e}_\theta + \frac{\phi}{\rho^2+1}\overline{e}_\phi$ (in spherical)

Ans. $4\rho - \frac{2}{\rho}\cos^2\phi\cot\theta + \frac{1}{\rho(\rho^2+1)\sin\theta}$

Compute the curl of $\overline{A}$:

10. $\overline{A} = (2\rho + \alpha\cos\phi)\overline{e}_\rho - \alpha\sin\theta\overline{e}_\theta + \rho\cos\theta\overline{e}_\phi$, $\alpha =$ constant (in spherical)

Ans. $\frac{\cos 2\theta}{\sin\theta}\overline{e}_r - (2\cos\theta + \frac{\alpha\sin\phi}{\rho\sin\theta})\overline{e}_\theta - \frac{\alpha\sin\theta}{\rho}\overline{e}_\phi$

11. $\overline{A} = \cos\theta\overline{e}_\rho - \frac{\sin\theta}{\rho}\overline{e}_\theta + \rho^2\overline{e}_z$ (in cylindrical)

Ans. $-2\rho\overline{e}_\theta + \frac{\sin\theta}{\rho}\overline{e}_z$

12. a. Show that $\overline{A} = z\left\{(\sin\theta)\overline{e}_\rho + \cos\theta\overline{e}_\theta\right\} - \rho\cos\theta\overline{e}_z$ (in cylindrical) is solenoidal.

Hint: $\nabla \cdot \overline{A} = 0$

b. Show that $\overline{A} = (\rho\, z\sin 2\theta)\overline{e}_\rho + (\rho z\cos 2\theta)\overline{e}_\theta + \frac{\rho^2\sin^2\theta}{2}\overline{e}_z$ is irrotational.

Hint: $\nabla \times \overline{A} = 0$.

13. Show that $\overline{A}$ is a potential field where $\overline{A} = \frac{2\cos\theta}{\rho^3}\overline{e}_r + \frac{\sin\theta}{\rho^3}e_\theta$ (in spherical)

14. Show that $\overline{A} = f(\rho)\overline{e}_\rho$ is a potential field where f is any differentiable function.

Hint: $\overline{e}_1 = \cos\theta\overline{i} + \sin\theta\overline{j}$, $\overline{e}_2 = -\sin\theta\overline{i} + \cos\theta\overline{k}$, $\overline{e}_3 = \overline{k}$ prove that $\overline{e}_1 \cdot \overline{e}_2 = \overline{e}_2 \cdot \overline{e}_3 = \overline{e}_1 \cdot \overline{e}_3 = 0$.

15. In cylindrical coordinate ρ, θ, z, show that $\nabla(\log\rho)$ and $\nabla\theta$ are solenoidal vectors (if $\rho \neq 0, \theta \neq 0$).

16. Show that $\nabla^2 f = 2\rho^2\cos 2\theta$ when $f = \rho^2 z^2\cos 2\theta$ (in cylindrical)

17. Prove that $\nabla^2 f = 2\sin 2\theta + 2\cot\theta\cos 2\theta - \frac{2}{\rho^2}\operatorname{cosec}^2\theta\cos 2\phi$ if $f = \rho^2\sin 2\theta + \cos^2\phi$ (in spherical)

18. Represent the vector $\overline{A} = zi - 2xj + yk$ in cylindrical coordinates. Determine A_ρ, A_θ, A_z.

Ans. $A_\rho = z\cos\theta - 2\rho\cos\theta \cdot \sin\theta$

$A_\theta = -z\sin\theta - 2\rho\cos^2\theta$

$A_z = \rho\sin\theta$

19. Represent the vector $\overline{A} = xy\overline{i} - zj + xz\overline{k}$ in spherical coordinate system.

Ans. $A_\rho = \rho^2\sin^3\theta\sin\phi\cos^2\phi - \rho\sin\theta\cos\theta\sin\phi + \rho^2\sin\theta\cos^2\theta\cos\phi$

$A_\theta = \rho^2\sin^2\theta\cos\theta\sin\phi\cos^2\phi - \rho\cos^2\theta \cdot \sin\phi - \rho^2\sin^2\theta \cdot \cos\theta\cos\phi$

$A_\phi = -\rho^2\sin^2\theta\sin^2\phi\cos\phi - \rho\cos\theta\cos\phi$

20. Express $\overline{A} = 2y\overline{i} - z\overline{j} + 3x\overline{k}$ in spherical polar coordinate system.

Ans. $A_\rho = 2\rho\sin^2\theta \cdot \sin\phi\cos\phi - \rho\sin\theta\cos\theta \cdot \sin\phi + 3\rho\sin\theta\cos\theta\cos\rho$

$A_\theta = 2\rho\sin\theta \cdot \cos\theta\sin\phi\cos\phi$

	Cartesian Coordinates (x, y, z)	Curvilinear Coordinates	Spherical Coordinates (r, θ, ϕ)	Cylindrical Coordinates (ρ, θ, z)
1. Equations of transformation of coordinates	$x = x$ $y = y$ $z = z$	$q_1 = q_1(x, y, z)$ $q_2 = q_2(x, y, z)$ $q_3 = q_3(x, y, z)$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$x = \rho\cos\theta$ $y = \rho\sin\theta$ $z = z$
2. Scale factors	$h_1 = 1, h_2 = 1, h_3 = 1$	$h_1 h_2 h_3$	$h_1 = 1, h_2 = r, h_3 = r\sin\theta$	$h_1 = 1, h_2 = \rho, h_3 = 1$
3. Base vectors	$\bar{i}, \bar{j}, \bar{k}$	$\bar{e}_1, \bar{e}_2, \bar{e}_3$	$\bar{e}_r = (\sin\theta\cos\phi)i$ $+ (\sin\theta\sin\phi)j + (\cos\theta)k$ $e_\theta = (\cos\theta\cos\phi)i +$ $(\cos\theta\sin\phi)j - (\sin\theta)k$ $e_\phi = (-\sin\phi)i + (\cos\phi)j$	$e_\rho = (\cos\theta)i + (\sin\theta)j$ $e_\theta = (-\sin\theta)i + (\cos\theta)j$ $e_z = k$
4. Jacobian (J)	$\frac{\partial(x,y,z)}{\partial(x,y,z)} = 1$	$J\left(\frac{x, y, z}{q_1, q_2, q_3}\right) = h_1 h_2 h_3$	$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$	$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \rho$
5. (Arc Length)^2	$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$	$(ds)^2 = h_1^2 dq_1^2 + h_2^2 dq^2 + h_3^2 dq_3^2$	$(ds)^2 = (dr)^2 + r^2(d\theta)^2 +$ $r^2\sin^2\theta(d\phi)^2$	$ds^2 = (d\rho)^2 + \rho^2(d\theta)^2 +$ $(dz)^2$
6. Area elements on the coordinate surfaces	$ds_1 = dy\,dz$ $ds_2 = dz\,dx$ $ds_3 = dx\,dy$	$dA_1 = h_2 h_3 dq_2 dq_3$ $dA_2 = h_1 h_3 dq_1 dq_3$ $dA_3 = h_1 h_2 dq_1 dq_2$	$ds_r = r^2\sin\theta\,d\theta\,d\phi$ $ds_\theta = r\sin\theta\,d\phi\,dr$ $ds_\phi = r\,dr\,d\theta$	$ds_\rho = \rho d\theta\,dz$ $ds_\theta = dz\,d\rho$ $ds_z = \rho d\rho d\theta$
7. Volume element (dv)	$dv = dx\,dy\,dz$	$dv = h_1 h_2 h_3 dq_1 dq_2 dq_3$	$dv = r^2\sin\theta dr d\theta d\phi$	$dv = \rho d\rho d\theta dz$
8. Grad f	$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$	$\frac{1}{h_1}\frac{\partial f}{\partial q_1}\bar{e}_1 + \frac{1}{h_2}\cdot\frac{\partial f}{\partial q_2}\bar{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\bar{e}_3$	$\nabla f = \frac{\partial f}{\partial r}e_1 + \frac{1}{r}\frac{\partial f}{\partial \theta}e_\theta$ $+ \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}e_\phi$	$\nabla f = \frac{\partial f}{\partial \rho}e_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}e_\theta +$ $\frac{\partial f}{\partial z}e_z$
9. Div A	$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$	$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_3 h_1 A_2)\right.$ $\left.+ \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right]$	Div A $= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_1)\right.$ $\left.+ \frac{\partial}{\partial \theta}(r\sin\theta A_2) + \frac{\partial}{\partial \phi}(rA_3)\right]$	$\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_1) + \frac{\partial}{\partial \theta}(A_2)\right.$ $\left.+ \frac{\partial}{\partial z}(\rho A_3)\right]$
10. Curl A	$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$	$\frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\bar{e}_1 & h_2\bar{e}_2 & h_3\bar{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$	$\frac{1}{r^2\sin\theta}\begin{vmatrix} e_r & re_\theta & (r\sin\theta)e_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_1 & rA_2 & (r\sin\theta)A_3 \end{vmatrix}$	$\frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial/\partial\rho & \partial\partial\theta & \partial/\partial_z \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$
11. Laplacian ($\nabla^2 f$)	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\right.$ $\left.\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$	$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta}$ $+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$	$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$

$-\rho\cos^2\theta\sin\phi - 3\rho\sin^2\theta\cos\phi$

$A_\phi = -2\rho\sin\theta\sin^2\phi - \rho\cos\theta\cos\rho$

21. Prove that a cylindrical coordinate system is orthogonal.

22. Find the square of the element of arc length in spherical coordinates and determine the corresponding scale factors.

Ans. $(ds)^2 = (dr)^2 + \rho^2(d\theta)^2 + \rho^2\sin^2\theta(d\phi)^2$

$h_1 = h_\rho = 1, h_2 = h_\theta = \rho, h_3 = h_\phi = \rho\sin\theta$

23. Prove that in any orthogonal curvilinear coordinate system, $\nabla\cdot\nabla\times\bar{A} = 0$

24. If q_1, q_2, q_3 are general coordinates, show that $\frac{\partial\bar{r}}{\partial q_1}, \frac{\partial\bar{r}}{\partial q_2}, \frac{\partial\bar{r}}{\partial q_3}$ and $\nabla q_1, \nabla q_2, \nabla q_3$ are reciprocal system of vectors.

Hint: Use $\nabla q_1\cdot d\bar{r} = dq_1 = (\nabla q_1\cdot\frac{\partial\bar{r}}{\partial q_1})dq_1 + (\nabla q_1\cdot\frac{\partial\bar{r}}{\partial q_2})dq_2 + (\nabla q_1\cdot\frac{\partial\bar{r}}{\partial q_3})dq_3$

25. Prove that

$$\left\{\frac{\partial\bar{r}}{\partial q_1}\cdot\frac{\partial\bar{r}}{\partial q_2}\times\frac{\partial\bar{r}}{\partial q_3}\right\}\{\nabla q_1\cdot\nabla q_2\times\nabla q_3\} = 1.$$

Hint: $V\left(\frac{1}{V}\right) = 1$ or $J\left(\frac{x,y,z}{q_1,q_2,q_3}\right)\cdot J\left(\frac{q_1,q_2,q_3}{x,y,z_1}\right) = 1$

26. In cylindrical coordinate system ρ,θ,z prove that

$$\nabla\rho^n = n\rho^{n-1}\bar{e}_\rho$$

$$\nabla^2(\rho^n\cos n\theta) = 0.$$

27. In spherical coordinate system ρ,θ,ϕ, prove that

28. $\nabla\cdot[\bar{e}_\rho\cdot\cot\phi - 2\bar{e}_\phi] = 0$

29. $\nabla^2\left[\left(\rho+\frac{1}{\rho^2}\right)cos\phi\right] = 0$

30. Find $\frac{\partial\bar{r}}{\partial q_1}, \frac{\partial\bar{r}}{\partial q_2}, \frac{\partial\bar{r}}{\partial q_3}, \nabla q_1, \nabla q_2, \nabla q_3$ in spherical coordinate system

Ans. $\frac{\partial\bar{r}}{\partial\rho} = \sin\theta\cos\phi i + \sin\theta\sin\phi j + \cos\theta k$

$\frac{\partial\bar{r}}{\partial\theta} = \rho\cos\theta\cos\rho i + \rho\cos\theta\sin\phi j - \rho\sin\theta k$

$\frac{\partial\bar{r}}{\partial\phi} = -\rho\sin\theta\sin\phi\bar{i} + \rho\sin\theta\cos\phi j$

$\nabla\rho = \sin\theta\cos\phi\bar{i} + \sin\theta\sin\phi\bar{j} + \cos\theta\bar{k}$

$\nabla\theta = \frac{\cos\theta\cdot\cos\phi\bar{i} + \cos\theta\cdot\sin\phi\bar{j} - \sin\theta\bar{k}}{\rho}$

$\nabla\phi = \frac{-\sin\phi\bar{i} + \cos\phi j}{\rho\sin\theta}$

31. Express the velocity $\bar{V}$ and acceleration $\bar{a}$ of a particle in cylindrical coordinates.

Ans. $\bar{r} = \rho\bar{e}_\rho + z\bar{e}_z$

$\bar{V} = \dot{\rho}\bar{e}_\rho + \rho\dot{\theta}e_\theta + \dot{z}e_z$

$\bar{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\bar{e}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\bar{e}_\theta + \ddot{z}\bar{e}_z$

Vector Integral Calculus

INTRODUCTION

Vector integral calculus extends the concepts of (ordinary) integral calculus to vector functions. It has applications in fluid flow, design of underwater transmission cables, heat flow in stars, study of satellites. Line integrals are useful in the calculation of work done by variable forces along paths in space and the rates at which fluids flow along curves (circulation) and across boundaries (flux). In this chapter we consider three important integral theorems. Green's theorem, a great theorem of calculus, which converts line integrals to double integrals, evaluates flow and flux integrals across closed plane curves in non-conservative vector fields. Stokes theorem states that the circulation of a vector field around the boundary of a surface in space equals the integral of the normal component of the curl of the field over the surface. Gauss divergence theorem, which is important in electricity, magnetism and fluid flow, says that the outward flux of a vector field across a closed surface equals the triple integral of the divergence of the field over the region enclosed by the surface.

16.1 VECTOR INTEGRATION: INTEGRATION OF A VECTOR FUNCTION OF A SCALAR ARGUMENT

Definitions

Primitive

A vector function $\overline{F}(u)$ is the primitive of the vector function $\overline{f}(u)$ if

$$\frac{d\overline{F}(u)}{du} = f(u)$$

Indefinite integral

Indefinite integral of the vector function $\overline{f} = \overline{f}(u)$ of a scalar argument u is the collection of all primitive functions of $\overline{f}(u)$ and is denoted by

$$\int \overline{f}(u)du = \int \frac{d\overline{F}(u)}{du}du = \overline{F}(u) + \overline{c}$$

where c is an arbitrary constant vector.

Properties

1. $\int \alpha \overline{f}(u)du = \alpha \int \overline{f}(u)du$,
   (α = numerical constant)
2. $\int [\overline{f}(u) \pm \overline{g}(u)]du = \int \overline{f}(u)du \pm \int \overline{g}(u)du$
3. If $\overline{f}(u) = f_1(u)i + f_2(u)j + f_3(u)k$ then
   $\int \overline{f}(u)du = i \int f_1(u)du + j \int f_2(u)du + k \int f_3(u)du$

Note: The integration of a vector function reduces to the evaluation of three ordinary real (scalar) integrals.

Definite integral between limits $u = a$ and $u = b$ is

$$\int_a^b \overline{f}(u)du = \overline{F}(u) + \overline{c}\Big|_a^b = \overline{F}(b) - \overline{F}(a)$$

WORKED OUT EXAMPLES

Example 1: Evaluate

a. $\int \overline{A}(u)\,du$ and

b. $\int_2^4 \overline{A}(u)\,du$

if $\overline{A}(u) = (3u^2 - u)i + (2 - 6u)j - 4uk$.

Solution:

a. $$\int \overline{A}(u)du = \int \left[(3u^2 - u)i + (2 - 6u)j - 4uk\right] du$$
$$= i\int (3u^2 - u)du + j\int (2 - 6u)du$$
$$+k\int -4u du$$
$$= \left(u^3 - \frac{u^2}{2}\right)i + (2u - 3u^2)j$$
$$-2u^2k + \overline{c}$$

b. $$\int_2^4 \overline{A}(u)du = \left(u^3 - \frac{u^2}{2}\right)i$$
$$+(2u - 3u^2)j - 2u^2\overline{k}\Big|_2^4$$
$$= 50i - 32j - 24k.$$

Example 2: If $\overline{A}(u) = ui - u^2j + (u-1)k$ and $\overline{B}(u) = 2u^2i + 6uk$ then evaluate $\int_0^2 \overline{A} \times \overline{B}\,du$.

Solution:

$$\overline{A} \times \overline{B} = \begin{vmatrix} i & j & k \\ u & -u^2 & (u-1) \\ 2u^2 & 0 & 6u \end{vmatrix}$$

$$\overline{A} \times \overline{B} = -6u^3i + (2u^3 - 8u^2)j + 2u^4k$$

$$\int_0^2 \overline{A} \times \overline{B}du = \int_0^2 \left[-6u^3i + (2u^3 - 8u^2)j + 2u^4k\right] du$$
$$= -\frac{6u^4}{4}i + \left(\frac{2u^4}{4} - \frac{8u^3}{3}\right)j + \frac{2u^5}{5}k\Big|_0^2$$
$$= -24i - \frac{40}{3}j + \frac{64}{5}k.$$

Example 3: Let $\overline{A} = ti - 3j + 2tk$, $\overline{B} = i - 2j + 2k$, $\overline{D} = 3i + tj - k$. Evaluate $\int_1^2 \overline{A} \cdot \overline{B} \times \overline{D}\,dt$.

Solution:

$$\overline{A} \cdot \overline{B} \times \overline{D} = \begin{vmatrix} t & -3 & 2t \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix}$$
$$= t(2 - 2t) + 3(-1 - 6) + 2t(t + 6)$$
$$= +14t - 21$$

$$\int_1^2 (\overline{A} \cdot \overline{B} \times \overline{D})dt = \int_1^2 (14t - 21)dt = (7t^2 - 21t)\Big|_1^2$$
$$= 0.$$

Example 4: The acceleration $\overline{a}$ of a particle at any time $t \geq 0$ is given by

$$\overline{a}(t) = e^{-t}i - 6(t+1)j + 3\sin t\,k$$

If the velocity $\overline{v}$ and displacement $\overline{r}$ are zero at $t = 0$, find $\overline{v}$ and $\overline{r}$ at any time t.

Solution: $\overline{a} = \frac{d\overline{v}}{dt} = \overline{e}^t i - 6(t+1)j + 3\sin tk$.

Integrating with respect to t

$$\overline{v}(t) = i\int e^{-t}dt - j\int 6(t+1)dt + 3\overline{k}\int \sin t\,dt$$
$$= -e^{-t}i - j(3t^2 + 6t) - 3\overline{k}\cos t + \overline{c}$$

Given that $\overline{v} = 0$ when $t = 0$

so $\quad 0 = -i - 3\overline{k} + \overline{c}$

or $\quad \overline{c} = \overline{i} + 3\overline{k}$

Thus

$$\overline{v}(t) = (1 - e^{-t})i - j(3t^2 + 6t) + 3k(1 - \cos t)$$

Integrating $\overline{v} = \frac{d\overline{r}}{dt}$ with respect to t

$$\overline{r}(t) = \int \overline{v}(t)dt = i\int (1 - e^{-t})dt - j\int (3t^2 + 6t)dt$$
$$+3k\int (1 - \cos t)dt$$
$$\overline{r} = i(t + e^{-t}) - j(t^3 + 3t^2) + 3k(t - \sin t) + \overline{c_1}$$

since $\overline{r} = 0$ at $t = 0$, we have

$0 = i + \overline{c_1}$ so $\overline{c_1} = -\overline{i}$

Thus

$$\overline{r}(t) = i(-1 + t + e^{-t}) - j(t^3 + 3t^2) + (3t - 3\sin t)\overline{k}.$$

Example 5: If $\overline{A}(u) = ui + u^2j + u^3k$, $\overline{B}(u) = u^3i + u^2j + uk$

find $$\int_1^2 \left[\overline{A}(u) \times \frac{d\overline{B}}{du} + \frac{d\overline{A}}{du} \times \overline{B}(u)\right] du$$

Solution:

$$I = \int_1^2 \left[A \times \frac{dB}{du} + \frac{dA}{du} \times B\right] du$$

$$= \int_1^2 \frac{d}{du}(\overline{A} \times \overline{B})du = \overline{A} \times \overline{B} + \overline{c} \Big|_1^2$$

since $\frac{d}{du}(\overline{A} \times \overline{B}) = \overline{A} \times \frac{d\overline{B}}{du} + \frac{d\overline{A}}{du} \times \overline{B}$.

Here $\overline{A} \times \overline{B} = \begin{vmatrix} i & j & k \\ u & u^2 & u^3 \\ u^3 & u^2 & u \end{vmatrix}$

$$= (u^3 - u^5)i + (u^6 - u^2)j + (u^3 - u^5)k$$

$$I = -24i + 60j - 24k.$$

Example 6: Evaluate $\int_2^3 \overline{A} \cdot \frac{d\overline{A}}{dt}$ if

$\overline{A}(2) = 2i - j + 2k$ and $\overline{A}(3) = 4i - 2j + 3k$.

Solution: Since $\overline{A} \cdot \frac{d\overline{A}}{dt} = \frac{A\,dA}{dt}$ where $A = |\overline{A}|$

$$\int_2^3 \overline{A} \cdot \frac{d\overline{A}}{dt} dt = \int_2^3 A \frac{dA}{dt} dt = \int_2^3 A\,dA$$

$$= \frac{A^2}{2} \Big|_2^3 = \frac{1}{2}[A^2(3) - A^2(2)]$$

$$= \frac{1}{2}[29 - 9] = 10$$

since

$$A(3) = |A|(3)| = \sqrt{16+4+9} = \sqrt{29},$$

$$A(2) = |\overline{A(2)}| = \sqrt{4+1+4} = \sqrt{9}.$$

EXERCISE

1. Find (a) $\int \overline{A}(u)du$ and (b) $\int_1^2 \overline{A}(u)du$ if $\overline{A}(u) = (u - u^2)i + 2u^3 j - 3k$.

Ans. a. $\left(\frac{u^2}{2} - \frac{u^3}{3}\right)i + \frac{u^4}{2}j - 3uk + \overline{c}$

b. $\frac{-5}{6}i + \frac{15}{2}j - 3k$

2. Evaluate $\int_0^{\pi/2}(3 \sin ui + 2 \cos uj)du$.

Ans. $3i + 2j$

3. Find $\int \overline{A}(u)du$ where $\overline{A}(u) = (3u^2 - \sin u)i + (e^u + \cos u)j + 4u^3k$.

Ans. $(u^3 + \cos u)i + (e^u + \sin u)j + u^4k + c$

4. Find $\int_0^1 t\overline{F}(t)dt$ when $\overline{F}(t) = 2ti - t^2j + t^3k$.

Ans. $\frac{2}{3}u^3i - \frac{u^4}{4}j + \frac{u^5}{5}k \Big|_0^1 = \frac{2}{3}i - \frac{1}{4}j + \frac{1}{5}k$

5. Evaluate $\int_0^2 \overline{A} \cdot \overline{B}\,dt$ where $\overline{A} = ti - t^2j + (t-1)k$ and $B = 2t^2i + 6tk$.

Ans. 12

6. Let $\overline{A} = ti - 3j + 2tk$, $\overline{B} = i - 2j + 2k$, $\overline{D} = 3i + tj - k$ then evaluate

$$\int_1^2 \overline{A} \times (\overline{B} \times \overline{D})\,dt.$$

Ans. $\frac{-87}{2}i - \frac{44}{3}j + \frac{15}{2}k$

7. Evaluate

$$\int \overline{A} \times \frac{d^2\overline{A}}{dt^2}dt$$

Hint:

$$\frac{d}{dt}\left(\overline{A} \times \frac{d\overline{A}}{dt}\right) = A \times \frac{d^2A}{dt^2} + \frac{dA}{dt} \times \frac{dA}{dt}$$

$$= A \times \frac{d^2A}{dt^2}.$$

Ans. $\overline{A} \times \frac{d\overline{A}}{dt} + \overline{c}$

8. Let $\overline{A}(t) = e^{-t} \sin t\ i + e^{-t} \cos tj + t^2k$. Evaluate

$$\int_1^2 \overline{A} \cdot \frac{d\overline{A}}{dt}dt.$$

Hint: see Worked Out Example 6, on page 16.3, above.

Ans. $\frac{1}{2}\left[15 + e^{-4} - e^{-2}\right]$

9. The acceleration of a particle at any time $t \geq 0$ is given by $\overline{a} = 12 \cos 2ti - 8 \sin 2tj + 16tk$. If the velocity $\overline{V}$ and displacement $\overline{r}$ are zero at $t = 0$, find $\overline{V}$ and $\overline{r}$ at any time t.

Ans. $\overline{V} = 6 \sin 2ti + (4 \cos 2t - 4)j + 8t^2k$

$\overline{r} = (3 - 3\cos 2t)i + (2 \sin 2t - 4t)j + 8\frac{t^3}{3}k$

10. Find the velocity and displacemant of a particle having acceleration $4t^3i - 5t^4j + 3t^2k$ at any time t, given that acceleration and velocity are initially $i - j$ and $i + j + k$ respectively.

Ans. Velocity: $(t^4 + 1)i - (t^4 + 1)j + t^3k$

displacement:

$$\left(\frac{t^5}{5} + t + 1\right)i - \left(\frac{t^6}{6} + t - 1\right)j + \left(\frac{t^4}{4} + 1\right)k$$

11. Find $\overline{A}$ if $\frac{d^2\overline{A}}{du^2} = 6ui - 12u^2 j + 4\cos uk$ and given that $\frac{d\overline{A}}{du} = -1 - 3k$ and $\overline{A} = 2i + j$ when $u = 0$.

Ans. $(u^3 - u + 2)i + (1 - u^4)j + (4 - 4\cos u - 3u)k$

12. Find the areal velocity of a particle which moves along the path $\overline{r} = a\cos wti + b\sin wt\overline{j}$ where a, b, w are constants and t is time.

Hint: Areal velocity $= \frac{1}{2}\overline{r} \times \overline{V}$, where $\overline{V}$ = velocity.

Ans. $\frac{1}{2}a\, bw\overline{k}$

16.2 LINE INTEGRALS: WORK DONE, POTENTIAL, CONSERVATIVE FIELD AND AREA

For the ordinary definite integral $\int_a^b f(x)dx$ the region of integration is an interval $a \le x \le b$ on the x-axis. i.e., we integrate along the x-axis from a to b.

This concept can be generalized to define a definite integral evaluated along a curve.

Line Integrals

Let c be curve defined from A to B with corresponding arc lengths $s = a$ and $s = b$ respectively. Divide c into n arbitrary portions (see Fig. 16.1).

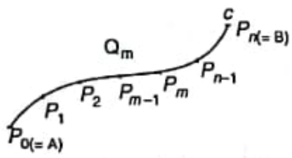

Fig. 16.1

Evaluate the given function f at a point in each of these portions and form the sum

$$J_n = \sum_{m=1}^{n} f(Q_m)\Delta s_m$$

where $\Delta s_m = s_m - s_{m-1}$. The limit of this sum as $n \to \infty$ is known as the line integral of f along c from A to B and is denoted by

$$\int_c f(p)ds = \int_a^b f(s)ds = \int_c f(x, y, z)ds \quad (1)$$

when P has coordinates $x(s), y(s), z(s)$. The line integral (1) is also known as curve integral or curvilinear integral.

Thus in a line integral, the integrand f is integrated (evaluated) along a curve (line). The curve c is known as path of integration. Its end points a and b are called the initial and terminal points.

The direction along the curve c from a to b is called the sense of integration.

Curve is said to be a closed curve (path) when the end points coincide. In such case the line integral is denoted as $\oint_c$.

Let the parametric equation of the curve c be

$$\overline{r}(t) = x(t)\overline{i} + y(t)\overline{j} + z(t)\overline{k}, \quad a \le t \le b \quad (2)$$

Properties of line integrals

Let $\overline{F} = \overline{F}(\overline{r}) = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$ be a vector function. Then a line integral of $\overline{F}(\overline{r})$ along (taken over) the curve c is defined as

$$\int_c \overline{F}(\overline{r}) \cdot d\overline{r} = \int_c F_1dx + F_2dy + F_3dz$$
$$= \int_c F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}$$
$$= \int_a^b \overline{F}_1(\overline{r}(t)) \cdot \frac{d\overline{r}}{dt}dt \quad (3)$$

Observation

Evaluation of a line integral reduces to evaluation of an ordinary integral along a coordinate axis (say x-axis).

For the line integral (3) the following properties follow from integral calculus:

1. $\int_c k\overline{F} \cdot d\overline{r} = k\int_c \overline{F} \cdot d\overline{r}$, k = constant
2. $\int_c(\overline{F} \pm \overline{G}) \cdot d\overline{r} = \int_c \overline{F} \cdot d\overline{r} \pm \int_c \overline{G} \cdot d\overline{r}$
3. $\int_c \overline{F} \cdot d\overline{r} = \int_{c_1} \overline{F} \cdot d\overline{r} + \int_{c_2} \overline{F} \cdot d\overline{r}$

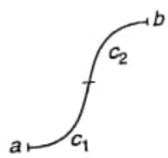

Fig. 16.2

where c is the sum of two curves c_1 and c_2 (see Fig. 16.2)

4. $\int_a^b \overline{F} \cdot d\overline{r} = -\int_b^a \overline{F} \cdot d\overline{r}$.

A. Work done by a force (work integral)

A natural application of the line integral is to define the work done by a force $\overline{F}$ in moving (displacing) a particle along a curve c from point P_1 to point P_2 as

$$\text{Work done} = \int_{P_1}^{P_2} \overline{F} \cdot d\overline{r} \tag{4}$$

When $\overline{F}$ denotes velocity of a fluid, then the *circulation* of $\overline{F}$ around a closed curve c is defined by

$$\text{Circulation} = \oint_c \overline{F} \cdot d\overline{r}$$

B. Independence of path; conservative field and scalar potential

If $\overline{F} = \nabla\phi$ then the line integral from P_1 and P_2 is independent of path joining P_1 to P_2 (Fig. 16.3)

$$\int_{P_1}^{P_2} \overline{F} \cdot d\overline{r} = \int_{P_1}^{P_2} \nabla\phi \cdot d\overline{r}$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right)$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$$

Thus the line integral depends only on the end points P_1 and P_2 and not on the path joining them. Recall that when $\overline{F} = \nabla\phi$, then $\nabla \times \overline{F} = \nabla \times \nabla\phi = 0$. In such a case, $\overline{F}$ is called a conservative vector field and ϕ is called its scalar potential.

Note: That a conservative force field is also irrotational (since $\nabla \times \overline{F} = 0$).

Result 1: The work done in a conservative force field in moving a particle from P_1 to P_2 is independent of the path joining P_1 and P_2, but depends only on the end points P_1 and P_2. In such cases a scalar potential ϕ exists such that force field $\overline{F} = \nabla\phi$ and thus the work done from P_1 to $P_2 = \phi(P_2) - \phi(P_1)$ (without the need to evaluate the work integral).

Result 2: In a conservative field $\overline{F}$

$$\oint_c \overline{F} \cdot d\overline{r} = 0$$

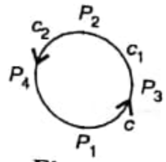

Fig. 16.3

along any closed curve c because

$$\oint_c \overline{F} \cdot d\overline{r} = \int_{P_1P_3P_2P_4} \overline{F} \cdot d\overline{r} = \int_{P_1P_3P_2} + \int_{P_2P_4P_1}$$

$$= \int_{P_1P_3P_2} \overline{F} \cdot d\overline{r} - \int_{P_1P_4P_2} \overline{F} \cdot d\overline{r} = 0.$$

which follows from the independence of path.

C. Test for exact differential

For $\overline{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$, the necessary and sufficient condition that

$$F_1dx + F_2dy + F_3dz$$

be an exact differential is that $\overline{F}$ must be conservative i.e., $\nabla \times \overline{F} = 0$. When $\nabla \times \overline{F} = 0$, there exists a scalar ϕ such that $\overline{F} = \nabla\phi$. Then

$$F_1dx + F_2dy + F_3dz = \overline{F} \cdot d\overline{r} = \nabla\phi \cdot d\overline{r} = d\phi$$

$$= \text{Exact differential}$$

D. Area A of a regular region D

Bounded by a curve c (Fig. 16.4):

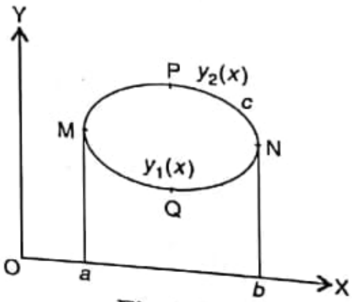

Fig. 16.4

$$A = \int_a^b y_2(x)d_x - \int_a^b y_1(x)dx$$

$$= -\int_{NPM} y\,dx - \int_{MQN} y\,dx = -\int_c ydx$$

Similarly, $$A = \int_c x\,dy$$

Thus $$A = \frac{1}{2}\int_c (x\,dy - y\,dx)$$

Worked Out Examples

Example 1: Evaluate $\int_c y^2dx - 2x^2dy$ along the parabola $y = x^2$ from (0, 0) to (2, 4).

Solution:

$$\int_c (y^2dx - 2x^2dy) = \int_0^2 (x^2)^2dx - 2x^2d(x^2)$$
$$= \int_0^2 x^4dx - 4x^3dx$$
$$= \frac{x^5}{5} - 4\frac{x^4}{4}\Big|_0^2 = -\frac{48}{5}.$$

Example 2:. Evaluate the line integral

$$\int_c x^{-1}(y+z)ds$$

where c the arc of circle $x^2 + y^2 = 4, z = 0$ from (2, 0, 0) to $(\sqrt{2}, \sqrt{2}, 0)$ in the counterclockwise direction.

Solution: Equation of circle in parametric form is

$$x = 2\cos t,\ y = 2\sin t$$

when $x = 2$, then $t = 0$ and when $x = \sqrt{2}$, then $t = \frac{\pi}{4}$

$$\overline{r} = xi + yj + zk = 2\cos ti + 2\sin tj + 0$$
$$\frac{d\overline{r}}{dt} = -2\sin ti + 2\cos tj$$
$$\dot{\overline{r}} \cdot \dot{\overline{r}} = \frac{d\overline{r}}{dt} \cdot \frac{d\overline{r}}{dt} = 4\sin^2 t + 4\cos^2 t = 4$$
$$\frac{ds}{dt} = \sqrt{\dot{\overline{r}} \cdot \dot{\overline{r}}} = \sqrt{4} = 2$$

Along c: $z = 0$ and $ds = 2dt$ so that

$$\int_c x^{-1}(y+z)ds = \int \frac{y+0}{x}ds = \int_e \frac{2\sin t}{2\cos t} \cdot 2dt$$
$$= 2\int_0^{\pi/4} \tan t \cdot dt = 2\ln\sec t\Big|_0^{\frac{\pi}{4}}$$
$$= 2\ln\sqrt{2} = \ln 2.$$

Example 3: If $\overline{F} = (2x + y^2)i + (3y - 4x)\overline{j}$ evaluate $\oint_c \overline{F} \cdot d\overline{r}$ around a triangle ABC in the xy-plane with $A(0, 0)$, $B(2, 0)$, $C(2, 1)$ (refer Fig. 16.5). (a) in the counterclockwise direction (b) what is the value in the opposite direction?

Solution: **a.** In the counterclockwise direction:

$$I = \oint_c \overline{F} \cdot d\overline{r} = \int_{c_1} \overline{F} \cdot d\overline{r} + \int_{c_2} \overline{F} \cdot d\overline{r} + \int_{c_3} \overline{F} \cdot d\overline{r}$$
$$= I_1 + I_2 + I_3$$

Along c_1: The straight line from $A(0, 0)$ to $B(2, 0)$, $y = 0$, $z = 0$ and x varies from 0 to 2

Thus $$\overline{r} = xi,\ d\overline{r} = \overline{i}dx,\ dy = 0,$$

So with $y = 0$

$$I_1 = \int_{c1} \overline{F} \cdot d\overline{r} = \int_0^2 (2xi) \cdot idx = \int_0^2 2xdx = x^2\Big|_0^2 = 4$$

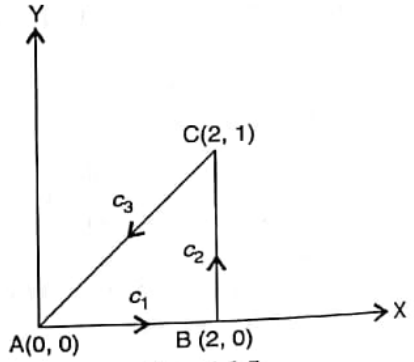

Fig. 16.5

Along c_2: The straight line from $B(2, 0)$ to $C(2, 1)$, $x = 2$, $z = 0$, y varies from 0 to 1

Thus $$\overline{r} = 2i + yj,\ d\overline{r} = \overline{j}dy$$

So with $x = 2$

$$I_2 = \int_{c_2} \overline{F} \cdot d\overline{r} = \int_0^1 \left[(4 + y^2)i + (3y - 8)j\right] \cdot [jdy]$$
$$= \int_0^1 (3y - 8)dy$$
$$= \frac{3y^2}{2} - 8y\ \Big|_0^1 = \frac{3}{2} - 8 = -\frac{13}{2}$$

Along c_3: Straight line from C(2,1) to A(0, 0) is $y = \frac{1}{2}x$, $dy = \frac{1}{2}dx$, x varies from 2 to 0

Thus $$\overline{r} = xi + \frac{1}{2}x\overline{j},\quad d\overline{r} = \left(i + \frac{1}{2}j\right)dx$$

So with $y = \frac{x}{2}$

$$I_3 = \int_{c_3} \overline{F} \cdot d\overline{r} =$$

$$= \int_2^0 \left[\left(2x + \frac{x^2}{4}\right) i + \left(\frac{3x}{2} - 4x\right) j\right] \cdot \left[i + \frac{j}{2}\right] dx$$

$$= \int_2^0 \left(2x + \frac{x^2}{4} + \frac{3x}{4} - \frac{4x}{2}\right) dx = \frac{x^3}{12} + \frac{3x^2}{8} \Big|_2^0$$

$$= -\left(\frac{8}{12} + \frac{12}{8}\right) = -\frac{13}{6}$$

The required line integral in the counterclockwise direction

$$I = \oint_c \overline{F} \cdot d\overline{r} = I_1 + I_2 + I_3 = 4 - \frac{13}{2} - \frac{13}{6} = -\frac{14}{3}$$

b. Line integral value in the opposite direction is $\frac{14}{3}$.

Example 4: Evaluate $\int_c f d\overline{r}$ where $f = 2xy^2z + x^2y$ and c is the curve $x = t, y = t^2, z = t^3$ from $t = 0$ to 1.

Solution:

$$\overline{r} = xi + yj + zk = ti + t^2 j + t^3 \overline{k}$$

$$d\overline{r} = (i + 2tj + 3t^2 \overline{k})dt$$

Along c:

$$f = 2t \cdot (t^2)^2(t^3) + (t^2) \cdot t^2 = 2t^8 + t^4$$

$$\int_c f d\overline{r} = \int_0^1 \left(2t^8 + t^4\right)\left(\overline{i} + 2t\overline{j} + 3t^2\overline{k}\right) dt$$

$$= \overline{i} \int_0^1 (2t^8 + t^4)dt + \overline{j} \int_0^1 (4t^9 + 2t^5)dt$$

$$+ \overline{k} \int_0^1 (6t^{10} + 3t^6)dt$$

$$= i\left(\frac{2t^9}{9} + \frac{t^5}{5}\right)\Big|_0^1 + \overline{j}\left(\frac{4t^{10}}{10} + \frac{2t^6}{6}\right)\Big|_0^1$$

$$+ \overline{k}\left(\frac{6t^{11}}{11} + \frac{3t^7}{7}\right)\Big|_0^1$$

$$= \frac{19}{45}\overline{i} + \frac{11}{15}\overline{j} + \frac{75}{77}\overline{k}.$$

Example 5: Find the work done in moving a particle in the force field $\overline{F} = 3x^2\overline{i} + (2xz - y)j + z\overline{k}$ along

a. straight line from $A(0, 0, 0)$ to $B(2, 1, 3)$

b. space curve c: $x = 2t^2, y = t, z = 4t^2 - t$ from $t = 0$ to $t = 1$

c. curve c: defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$.

Solution: **Work done along a curve c is $\int_c \overline{F} \cdot d\overline{r}$:**

a. $\overline{r} = 2ti + tj + 3tk$

$d\overline{r} = (2i + j + 3k)dt$

$\overline{F} = 3x^2\overline{i} + (2xz - y)\overline{j} + z\overline{k}$

$= 12t^2\overline{i} + (12t^2 - t)\overline{j} + 3t\overline{k}$

work done by $\overline{F}$ in moving along the straight line from $A(0, 0, 0)$ to $B(2, 1, 3)$

$$= \int_A^B \overline{F} \cdot d\overline{r}$$

$$= \int_0^1 \left[12t^2 i + (12t^2 - t)j + 3tk\right] \cdot [2i + j + 3k]dt$$

$$= \int_0^1 \left(24t^2 + 12t^2 - t + 9t\right) dt$$

$$= 12t^3 + 4t^2 \Big|_0^1 = 16$$

b. $\overline{r} = xi + yj + z\overline{k}$

$\overline{r} = 2t^2 i + t\overline{j} + (4t^2 - t)\overline{k}$

$d\overline{r} = 4ti + j + (8t - 1)\overline{k}$

$\overline{F} = 3(2t^2)^2 i + \left[2 \cdot (2t^2)(4t^2 - t) - t\right]\overline{j} + (4t^2 - t)k$

Work done

$$= \int_c \overline{F} \cdot d\overline{r} = \int_0^1 \left[12t^4 i + \left[4t^2(4t^2 - t) - t\right] j\right.$$

$$\left. + (4t^2 - t)k\right] \cdot [4t\, i + j + (8t - 1)k]$$

$$= \int_0^1 \left[48t^5 + (16t^4 - 4t^3 - t) + (8t - 1)(4t^2 - t)\right] dt$$

$$= 8t^6 + 16\frac{t^5}{5} + 7t^4 - 4t^3 \Big|_0^1 = 8 + \frac{16}{5} + 7 - 4$$

$$= 14.2$$

c. $\overline{r} = xi + y\overline{j} + z\overline{k} = xi + \frac{x^2}{4}j + \frac{3}{8}x^3\overline{k}$

$$= ti + \frac{t^2}{4}j + \frac{3}{8}t^3k$$

$$d\overline{r} = \left(\cdot i + \frac{t}{2}j + \frac{9}{8}t^2\overline{k}\right)$$

$$\overline{F} = 3x^2i + \left(2 \cdot x \cdot \frac{3}{8}x^3 - \frac{x^2}{4}\right)j + \frac{3}{8}x^3\overline{k}$$

Work done

$$= \int \overline{F} \cdot d\overline{r} = \int_0^2 \left[3t^2 + \frac{t}{2}\left(\frac{3}{4}t^4 - \frac{t^2}{4}\right) + \frac{27}{64}t^5\right] dt$$

$$= t^3 + \frac{t^6}{16} - \frac{t^4}{32} + \frac{27t^6}{384}\Big|_0^2 = 16$$

Example 6: If $\overline{A} = (y - 2x)i + (3x + 2y)j$, compute the circulation of $\overline{A}$ about a circle c in the xy plane with centre at the origin and radius 2, if c is traversed in the positive direction.

Solution: c : circle: $x^2 + y^2 = 4$

In parametric form $x = 2\cos t$, $y = 2\sin t$ with t varying 0 to 2π

$$\overline{A} = (2\sin t - 2(2\cos t))i + (3(2\cos t) + 2(2\sin t))\overline{j}$$

$$d\overline{r} = d(xi + yj) = dx\overline{i} + dy\overline{j}$$

$$d\overline{r} = (-2\sin t\ i + 2\cos t\ j)dt$$

By definition

circulation along curve $c = \int_c \overline{F} \cdot d\overline{r}$

$$= \int_0^{2\pi} [(2\sin t - 4\cos t)i$$
$$+(6\cos t + 4\sin t)j] \cdot [-2\sin t\ i + 2\cos t\ j]dt$$

$$= 4\int_0^{2\pi} \left[-\sin^2 t + 2\sin t\cos t + 3\cos^2 t + 2\sin t\cos t\right]dt$$

$$= 16\int_0^{2\pi} \sin t\, d(\sin t) - 4\int_0^{2\pi} \frac{1 - \cos 2t}{2}dt$$
$$+12\int_0^{2\pi} \frac{1 + \cos 2t}{2}dt$$

$$= 8\pi.$$

Example 7: Prove that $\overline{F} = (y^2\cos x + z^3)\overline{i} + (2y\sin x - 4)j + (3xz^2 + 2)\overline{k}$ is (a) conservative field (b) find scalar potential of $\overline{F}$ (c) find work done in moving an object in this field from $P_1(0, 1, -1)$ to $P_2\left(\frac{\pi}{2}, -1, 2\right)$.

Solution:

a. $\overline{F}$ is conservative if $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= i(0 - 0) - j(3z^2 - 3z^2)$$
$$+k(2y\cos x - 2y\cos x)$$
$$= 0$$

Hence $\overline{F}$ is conservative.

b. Let f be the scalar potential such that $\overline{F} = \nabla f$ then comparing the components of $\overline{i}, \overline{j}, \overline{k}$, we get

$$\frac{\partial f}{\partial x} = y^2\cos x + z^3 \quad (1)$$

$$\frac{\partial f}{\partial y} = 2y\sin x - 4 \quad (2)$$

$$\frac{\partial f}{\partial z} = 3xz^2 + 2 \quad (3)$$

Integrating (1) partially w.r.t. x,

$$f = y^2\sin x + xz^3 + g(y, z) \quad (4)$$

Differentiating (4) partially w.r.t. y and using (2)

$$2y\sin x + 0 + \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} = 2y\sin x - 4$$

Integrating w.r.t. y

$$g(y, z) = -4y + c_1(z) \quad (5)$$

Substituting (5) in (4)

$$f = y^2\sin x + xz^3 - 4y + c_1(z) \quad (6)$$

Differentiating (6) partially w.r.t. z and using (3)

$$0 + 3xz^2 - 0 + \frac{dc_1}{dz} = \frac{\partial f}{\partial z} = xz^2 + 2$$

Integrating w.r.t. z

$$c_1(z) = z^2 + c \quad (7)$$

Substituing (7) in (6)

$$f(x, y, z) = y^2 \sin x + xz^3 - 4y + z^2 + c$$

c. **Work done**

$$= f(P_2) - f(P_1) = f(\frac{\pi}{2}, -1, 2) - f(0, 1, -1)$$
$$= 12 + 4\pi.$$

Example 8: Show that $(z - e^{-x} \sin y)dx + (1 + e^{-x} \cos y)dy + (x - 8z)dz$ is an exact differential of a function f and find f.

Solution: Let

$$\overline{A} = Ai + A_2\overline{j} + A_3\overline{k} = (z - e^{-x} \sin y)\overline{i} + (1 + e^{-x} \cos y)\overline{j} + (x - 8z)\overline{k}$$

Then

$$\nabla \times \overline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - e^{-x} \sin y & 1 + e^{-x} \cos y & x - 8z \end{vmatrix}$$
$$= i(0 - 0) - j(1 - 1) + k(-e^{-x} \cos y - (-e^{-x} \cos y)) = 0$$

Since $\nabla \times \overline{A} = 0$ (see Section 16.2 C)
$A_1dx + A_2dy + A_3dz$ will be an exact differential

$$(z - e^{-x} \sin y)dx + (1 + e^{-x} \cos y)dy + (x - 8z)dz = df$$

Regrouping

$$(zdx + xdz) - 8zdz + dy + (e^{-x} \cos y dy - e^{-x} \sin y\, dx) = df$$

$\because$ $df = d(xz) - d(4z^2) + dy + d(e^{-x} \sin y)$

$\therefore$ $f = xz - 4z^2 + y + e^{-x} \sin y.$

Example 9: If $\overline{A} = (x - y)\overline{i} + (x + y)j$ evaluate $\oint_c \overline{A} \cdot d\overline{r}$ around the curve c consisting of $y = x^2$ and $y^2 = x$

Solution:

$\because$ $I = \oint_c \overline{F} \cdot d\overline{r} = \int_{c_1} \overline{F} \cdot d\overline{r} + \int_{c_2} \overline{F} \cdot d\overline{r} = I_1 + I_2$

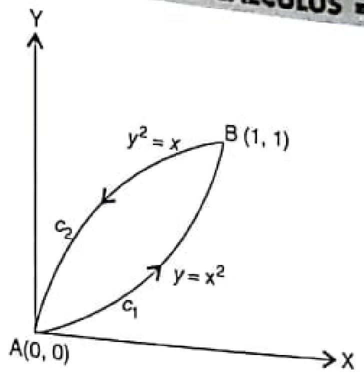

Fig. 16.6

where c_1: $y = x^2$ and $c_2 : y^2 = x$ as shown in Fig. 16.6 meet at points $A(0, 0)$ and $B(1, 1)$.

Along the curve $c_1 : y = x^2$, so that,

$$\overline{r} = xi + yj = xi + x^2 j = ti + t^2 j$$

$d\overline{r} = (i + 2tj)dt$ with t varying from 0 to 1

$$I_1 = \int_{c_1} (\overline{A} \cdot d\overline{r})$$
$$= \int_0^1 \left[(t - t^2)\overline{i} + (t + t^2)j\right] \cdot [i + 2tj]dt$$
$$= \int_0^1 \left[(t - t^2) + 2t(t + t^2)\right] dt = \int_0^1 (2t^3 + t^2 + t)dt$$
$$I_1 = \frac{2t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} \Big|_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = 1 + \frac{1}{3} = \frac{4}{3}$$

Along the curve $c_2 : y^2 = x, y = \sqrt{x}$

$$\overline{r} = x\overline{i} + yj = xi + \sqrt{x} j = ti + \sqrt{t} j$$
$$d\overline{r} = (i + \frac{1}{2\sqrt{t}} j)dt$$

with t varying from 1 to 0

$$I_2 = \int_{c_2} \overline{A} \cdot d\overline{r}$$
$$= \int_1^0 \left[(t - \sqrt{t})\overline{i} + (t + \sqrt{t})j\right] \cdot \left[i + \frac{1}{2\sqrt{t}} j\right] dt$$
$$= \int_1^0 \left[(t - \sqrt{t}) + \frac{1}{2}(\sqrt{t} + 1)\right] dt$$
$$= \int_1^0 \left(t - \frac{\sqrt{t}}{2} + \frac{1}{2}\right) dt$$
$$= \frac{t^2}{2} - \frac{t^{\frac{3}{2}}}{3} + \frac{1}{2}t \Big|_1^0 = -\frac{1}{2} + \frac{1}{3} - \frac{1}{2} = -\frac{2}{3}$$

Line integral $I = I_1 + I_2 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$.

Example 10: Compute the area of the region bounded by one arch of a cycloid $x = a(t - \sin t)$,

$y = a(1-\cos t)$ and the x-axis.

Solution: Area $A = \frac{1}{2}\int_c x dy - y dx$

$$A = \frac{1}{2}\int_0^{2\pi} a(t-\sin t)\cdot[a\sin t dt]$$
$$-a(1-\cos t)[a(1-\cos t)dt]$$
$$= \frac{a^2}{2}\int_0^{2\pi}(t\sin t - \sin^2 t - 1 - \cos^2 t + 2\cos t)dt$$
$$= \frac{a^2}{2}\int_0^{2\pi}(-2 + t\sin t + 2\cos t)dt$$
$$= \frac{a^2}{2}[-4\pi - 2\pi + 0] = -\frac{-6\pi a^2}{2}.$$

Exercise

Evaluate the following line integrals:

1. $\int_c xy^3 ds$ where c is the segment of the line $y = 2x$ in the xy plane from $A(-1,-2,0)$ to $B(1,2,0)$.

Ans. $16/\sqrt{5}$.

2. $\int_c (x^2+xy)dx + (x^2+y^2)dy$; c: square: $x \pm 1, y = \pm 1$

Ans. 0

3. $\int_c x^2 y dx + (x-z)dy + xyz\, dz$ where c is the arc of parabola $y = x^2$ in plane $z = 2$ from $A(0,0,2)$ to $B(1,1,2)$.

Ans. $-17/15$

4. $\int_c (x^2+y^2+z^2)^2 ds$ where c is the arc of circular helix $r(t) = \cos t\, i + \sin t\, j + 3t\, k$ from $A(1,0,0)$ to $B(1,0,6\pi)$.

Ans. $\sqrt{10}(2\pi + 6(2\pi)^3 + \frac{81}{5}(2\pi)^5)$

5. If $\overline{A}(t) = t\overline{i} - t^2 j + (t-1)k$, $\overline{B}(t) = 2t^2\overline{i} + 6t\overline{k}$ evaluate $\int_0^2 \overline{A}\cdot\overline{B}dt$.

Ans. 12

6. If $\overline{A}(t) = ti - 3j + 2t\overline{k}$, $\overline{B}(t) = i - 2j + 2k$, $\overline{c}(t) = 3i + tj - k$ then evaluate

   a. $\int_1^2 A\cdot(\overline{B}\times\overline{C})dt$

   b. $\int_1^2 (\overline{A}\times(\overline{B}\times\overline{C}))dt$

Ans. a. 0 b. $-\frac{87}{2}i - \frac{44}{3}j + \frac{15}{2}\overline{k}$

7. If $\overline{A}(2) = 2i - j + 2k$, $\overline{A}(3) = 4i - 2j + 3k$ then evaluate $\int_2^3 \overline{A}\cdot\frac{dA}{dt}dt$.

Ans. 10

8. Evaluate $\int_c \overline{A}\times d\overline{r}$ where $\overline{A} = 2y\overline{i} - z\overline{j} + x\overline{k}$ and c is the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $\pi/2$.

Ans. $i(2 - \frac{\pi}{4}) + j(\pi - \frac{1}{2})$

9. Evaluate $\int_c \overline{A}\cdot d\overline{r}$ where

   a. $\overline{A} = 2xi + 4yj - 3zk$, c: curve: $\overline{r}(t) = \cos t\,\overline{i} + \sin t\, j + t\, k$ from $t = 0$ to π

   b. $\overline{A} = yi + zj + xk$ c: circle $y^2 + z^2 = 1, x = 0$

   c. $\overline{A} = yzi + zxj + xy\overline{k}$ c: curve from $(0,0,0)$ to $(1,1,0)$ along the curve $x = y^2, z = 0$ in xy-plane, followed by the straight line path from $(1,1,0)$ to $(1,1,1)$.

Ans. a. $-3\pi^2/2$ b. $-\pi$ c. $3/4$

10. Determine whether the force field $\overline{F} = 2xz\overline{i} + (x^2-y)\overline{j} + (2z - x^2)\overline{k}$ is conservative or not.

Ans. $\nabla\times\overline{F} \neq 0$ so non-conservative

11. a. Prove that $\overline{F} = (4xy - 3x^2z^2)\overline{i} + 2x^2 j - 2x^3 z\overline{k}$ is a conservative field.

    b. Find its scalar potential f.

    c. Also find the work done in moving an object in this field from $(1,1,1)$ to $(0,0,0)$.

Ans. a. $\nabla\times\overline{F} = 0$, so conservative

b. scalar potential $f = 2x^2y - x^3z^2 + c$.

c. work done $= f(1,1,1) - f(0,0,0) = 1$.

12. If $\overline{A} = (2xy + z^3)\overline{i} + x^2\overline{j} + 3xz^2\overline{k}$

    a. Prove that the line integral $\int_c \overline{A}\cdot d\overline{r}$ is independent of the curve c joining two given points $P_1(1,-2,1)$ and $P_2(3,1,4)$.

    b. Show that there exists a scalar function f such that $\overline{A} = \nabla f$ and find f.

    c. Also find the work done in moving an object from P_1 to P_2.

Ans. a. $\nabla\times\overline{A} = 0$, $\overline{A}$ is conservative, so line integral is independent of path

b. $x^2y + xz^3 +$ constant

c. work done: 202

13. Find b such that the force field $\overline{A} = (e^x z - bxy)\overline{i} + (1 - bx^2)j + (e^x + bz)\overline{k}$ is conservative. Find the scalar potential f of $\overline{A}$ when $\overline{A}$ is conservative.

Ans. $b = 0,\ f = y + ze^x + c$

14. Find the scalar potential f of $\overline{F} = (z + \sin y)i + (-z + x\cos y)j + (x - y)\overline{k}$.

Ans. $f = xz + x\sin y - yz + c$

15. Find the total work done in moving a particle in a force field $\overline{A} = 3xy\overline{i} - 5zj + 10xk$ along the curve $x = t^2 + 1,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$.

Ans. 303

16. Calculate the work done in a force field given by $\overline{A} = (2y + 3)i + xzj + (yz - x)k$ when an object is moved from the point $P_1(0, 0, 0)$ to $P_2(2, 1, 1)$ along the curve $x = 2t^2$, $y = t,\ z = t^3$.

Ans. $\frac{288}{35}$

17. If $\overline{A} = (2x - y + 2z)\,\overline{i} + (x + y - z)j + (3x - 2y - 5z)\overline{k}$ calculate the circulation of $\overline{A}$ along the circle in the xy-plane of radius 2 and centre at origin.

Ans. Circulation $= \int \overline{A} \cdot d\overline{r} = 8\pi$.

18. Determine the circulation of $\overline{A} = yi + zj + xk$ around the curve $x^2 + y^2 = 1,\ z = 0$.

Ans. $-\pi$

19. If $\int_{P_1}^{P_2} \overline{A} \cdot d\overline{r}$ is independent of the path joining any two given points P_1 and P_2 in a given region then $\oint_c \overline{A} \cdot d\overline{r} = 0$ for all closed paths in the region passing through P_1 and P_2.

Hint: $P_1BP_2DP_1$ be any closed curve c

$$\oint_c = \int_{P_1BP_2DP_1} = \int_{P_1BP_2} + \int_{P_2DP_1}$$

$$= \int_{P_1BP_2} - \int_{P_1DP_2} = 0.$$

20. Prove that the work done in moving an object from P_1 to P_2 in a conservative force field $\overline{F}$ is independent of the path joining the two points P_1 and P_2.

Hint: Since $\overline{F}$ is conservative, $\overline{F} = \nabla f$

$$\int_{P_1}^{P_2} \overline{F} \cdot d\overline{r} = \int_{P_1}^{P_2} \nabla f \cdot d\overline{r}$$

$$= \int_{P_1}^{P_2} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \int_{P_1}^{P_2} df = f(P_2) - f(P_1)$$

Hint: Use $\frac{1}{2}\int_c xdy - ydx$ to compute area enclosed by c.

21. Compute the area of the ellipse $x = a\cos t$, $y = b\sin t$.

Ans. πab

22. Find the area under of one arch of the astroid $x = a\cos^3 t,\ y = a\sin^3 t$.

Ans. $3\pi a^2/8$

23. Find the area of the loop of the folium of descartes $x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$

Ans. $3a^2/2$.

16.3 SURFACE INTEGRALS: SURFACE AREA AND FLUX

The concept of surface integral is a simple and natural generalization of a double integral

$$\iint_R f(x, y)dxdy$$

taken over a plane region R. In a surface integral $f(x, y)$ is integrated over a curved surface.

Let S be a two-sided surface with one side of S taken arbitrarily as the positive side (the outer side if S is closed) (refer Fig. 16.7). A unit normal $\overline{n}$ at any point of the positive side of S is known as positive outward drawn unit normal.

In the xyz-space, the equation of a surface S is

$$g(x, y, z) = 0$$

with unit normal $\overline{n} = \frac{\nabla g}{|\nabla g|}$

When S is represented in parametric form as

$$\overline{r}(u, v) = x(u, v)\overline{i} + y(u, v)j + z(u, v)\overline{k}$$

with the two parameters u and v varying in a region R of uv-plane, then the unit normal $\overline{n}$ to S at P is given by

$$\overline{n} = \frac{N}{|N|}$$

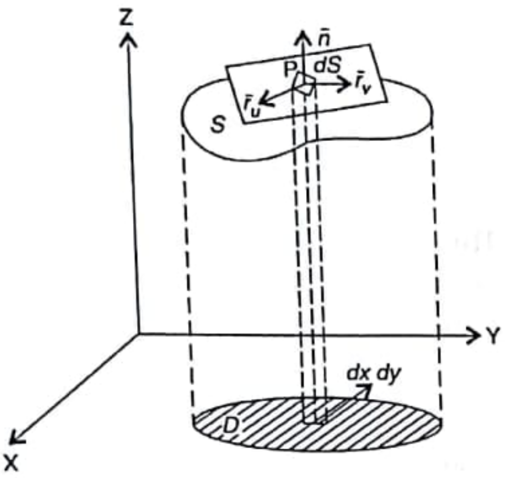

Fig. 16.7

where

$$\overline{N} = \overline{r}_u \times \overline{r}_v$$

The surface integral of a given vector function $\overline{F}$ taken over a surface S is defined as

$$\iint_S \overline{F} \cdot \overline{n}\, dS = \iint_S \overline{F} \cdot d\overline{S}$$

$$= \iint_R \overline{F}(\overline{r}(u, v)) \cdot \overline{N}(u, v) du dv$$

In the component form, where

$$\overline{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$$
$$\overline{n} = \cos\alpha\overline{i} + \cos\beta\overline{j} + \cos\gamma\overline{k}$$
$$\overline{N} = N_1\overline{i} + N_2\overline{j} + N_3\overline{k}$$

the surface integral takes the form

$$\iint_S \overline{F} \cdot \overline{n}\, dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma) dS$$

$$= \iint_R (F_1N_1 + F_2N_2 + F_3N_3) du\, dv$$

Here α, β, γ are the angles between $\overline{n}$ and the positive directions of the coordinate axes (i.e, $\overline{n} \cdot \overline{i} = |\overline{n}||i|\cos\alpha = \cos\alpha$, etc.)

Apart from the normal surface integral

$$\iint_S \overline{F}(\overline{r}) \cdot \overline{n}\, dS$$

the other types of surface integrals are

$$\iint_S \phi\, dS, \iint_S \phi\, \overline{n}\, dS, \iint_S \overline{A} \times \overline{n}\, dS$$

where ϕ is a scalar function.

Evaluation of a Surface Integral

A surface integral is evaluated by reducing it to a double integral by projecting the given surface S onto one of the coordinate planes. Let D be the projection of S onto the xy-plane (see Fig. 16.7). Then,

$$dS = \frac{dxdy}{|\overline{n} \cdot \overline{k}|}$$

Then,

$$\iint_S \overline{F} \cdot \overline{n}\, dS = \iint_D \overline{F}.\overline{n}\frac{dx\, dy}{|\overline{n} \cdot \overline{k}|}$$

where $\overline{n}$ is unit outward drawn normal to S. The R.H.S. double integral in x, y over the plane region D is evaluated as an two-fold iterated integral. In a similar way the surface integral can be evaluated by projecting S onto the Yz-plane as D_1 and Xz-plane as D_2 as follows

$$\iint_S \overline{F} \cdot \overline{n}\, dS = \iint_{D_1} \overline{F} \cdot \overline{n}\frac{dy\, dz}{|\overline{n} \cdot \overline{i}|}$$

$$\iint_S \overline{F} \cdot \overline{n}\, dS = \iint_{D_2} \overline{F} \cdot \overline{n}\frac{dx\, dz}{|\overline{n} \cdot \overline{j}|}$$

Surface Area of a Curved Surface

Let S be a surface represented by the equation

$$F(x, y, z) = 0 \quad (1)$$

Then the unit normal to the surface S is given by

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{F_x\overline{i} + F_y\overline{j} + F_z\overline{k}}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$$

where F_x, F_y, F_z are partial derivatives of F w.r.t. x, y, z respectively. Let D be the projection of S onto the xy-plane. Then

$$\text{Surface area of } S = \iint_S dS = \iint_D \frac{dx\, dy}{|\overline{n} \cdot \overline{k}|}$$

$$= \iint_D \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} dxdy$$

since $$\overline{n} \cdot \overline{k} = \frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$$

Corollary 1: If the equation of the surface S is $z = f(x, y)$ then

$$\text{Surface area} = \iint \sqrt{1 + z_x^2 + z_y^2}\, dxdy.$$

Flux

The normal component $\overline{F}\cdot\overline{n}$ is a scalar. Let ρ be the density, $\overline{V}$ be the velocity of a fluid and $\overline{F}=\rho\overline{V}$. Then flux of $\overline{F}$ represents the total quantity of fluid flowing in unit time through (across) the surface S in the positive direction. The flux of $\overline{F}$ across S is given by the flux integral

$$\text{Flux of } \overline{F} \text{ across } S=\iint_S \overline{F}\cdot\overline{n}\,dS.$$

WORKED OUT EXAMPLES

Example 1: Evaluate $\iint_S \overline{A}\cdot\overline{n}\,dS$ over the entire surface S of the region bounded by the cylinder $x^2+z^2=9$, $x=0$, $y=0$, $z=0$ and $y=8$ where $\overline{A}=6z\overline{i}+(2x+y)\overline{j}-x\overline{k}$ (see Fig. 16.8).

Solution: Here the entire surface S consists of 5 surfaces namely S_1 the curved (lateral) surface of the cylinder, $ABDCA$, $S_2: AOEC$, $S_3: OBDE$, $S_4: OAB$, $S_5: CDE$. Thus

$$\iint_S \overline{A}\cdot\overline{n}\,dS=\iint_{S_1+S_2+\cdots+S_5}$$
$$=\iint_{S_1}+\iint_{S_2}+\cdots+\iint_{S_5}$$
$$=SI_1+SI_2+SI_3+SI_4+SI_5$$

For the curved (lateral) surface S_1 of the cylinder

$$\text{unit normal } \overline{n}=\frac{\nabla(x^2+z^2)}{|\nabla(x^2+z^2)|}=\frac{2xi+2zk}{\sqrt{4x^2+4z^2}}$$
$$=\frac{xi+zk}{3}$$

so

$$\overline{A}\cdot\overline{n}=(6zi+(2x+y)j-xk)\cdot\left(\frac{xi+zk}{3}\right)=\frac{5}{3}xz$$

$$\overline{n}\cdot\overline{k}=\frac{z}{3}$$

$$SI_1=\iint_{S_1}\overline{A}\cdot\overline{n}\,dS=\iint\frac{\overline{A}\cdot\overline{n}}{|\overline{n}\cdot\overline{k}|}dxdy$$
$$=\iint\frac{5}{3}\frac{xz}{(z/3)}dxdy$$
$$=5\int_0^8\int_0^3 x\,dxdy=180$$

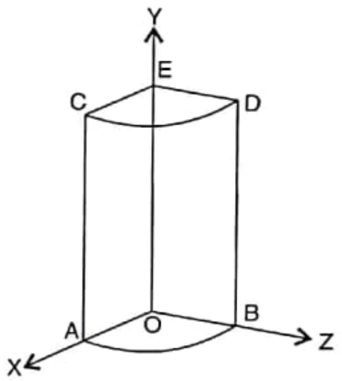

Fig. 16.8

On the plane $S_2: AOEC: z=0, \overline{n}=-k$, $\overline{A}\cdot\overline{n}=x$

$$SI_2=\iint_{S_2}\overline{A}\cdot\overline{n}\,dS=\int_0^8\int_0^3 x\,dx\,dy=36$$

On the plane S_3: $OBDE: x=0, \overline{n}=-\overline{i}$, $\overline{A}\cdot\overline{n}=-6z$

$$SI_3=\iint_{S_3}\overline{A}\cdot\overline{n}\,dS=\int_0^8\int_0^3 -6z\,dz\,dy=-216$$

On the sector $S_4: OAB: y=0, n=-j$, $\overline{A}\cdot\overline{n}=-(2x+y)=-2x$

$$SI_4=\iint_{S_4}\overline{A}\cdot\overline{n}\,dS=\iint_{OAB}-2x\,dxdz$$

In polar coordinates

$$SI_4=\int_0^{\frac{\pi}{2}}\int_0^3 -2\cdot r\cos t\cdot r\,dr\,dt=-18$$

On the sector $S_5: CDE: y=8, \overline{n}=j$, $\overline{A}\cdot\overline{n}=2x+y=2x+8$

$$SI_5=\iint_{S_5}\overline{A}\cdot\overline{n}\,dS=\iint_{CDE}(2x+8)dxdz$$

In polar coordinates

$$SI_5=\int_0^{\frac{\pi}{2}}\int_0^3(2r\cos t+8)r\,dr\,dt=18+18\pi$$

Thus the required surface integral is

$$SI=(180)+(36)+(-216)+(-18)+(18+18\pi)$$
$$=18\pi.$$

Example 2: Evaluate

a. $\iint_S(\nabla\times\overline{F})\cdot\overline{n}dS$ and

b. $\iint \phi \bar{n} dS$ if $\bar{F} = (x+2y)\bar{i} - 3zj + xk$, $\phi = 4x + 3y - 2z$ and S is the surface of the plane $2x + y + 2z = 6$ bounded by the coordinate planes $x = 0$, $y = 0$ and $z = 0$ (Fig. 16.9).

Solution: The unit normal $\hat{n}$ to the surface S is

$$\hat{n} = \frac{\nabla(2x+y+2z)}{|\nabla(2x+y+2z)|} = \frac{2i+j+2k}{\sqrt{4+1+4}}$$

$$\hat{n} = \frac{2i+j+2k}{3}$$

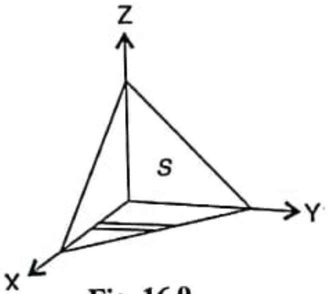

Fig. 16.9

a. $$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & -3z & x \end{vmatrix} = 3i - j - 2k$$

so $(\nabla \times \bar{F}) \cdot \hat{n} = (3i - j - 2k) \cdot \left(\frac{2i+j+2k}{3}\right) = \frac{1}{3}$

$$\text{SI} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n} dS = \frac{1}{3} \iint_S dS = \frac{1}{3} \iint \frac{dxdy}{|n \cdot k|}$$

$$= \frac{1}{3} \cdot \int_{x=0}^{3} \int_{y=0}^{6-2x} \frac{dydx}{2/3} = \frac{1}{2} \int_0^3 (6-2x)dx = \frac{9}{2}$$

b. $\text{SI} = \iint \phi \hat{n} dS = \iint (4x + 3y - 2z) \frac{(2i+j+2k)}{3} dS$

Eliminate z using, $z = \frac{6-2x-y}{2}$

$$\text{SI} = \frac{2i+j+2k}{3} \cdot \int_{x=0}^{3} \int_{y=0}^{6-2x} (6x + 4y - 6) \frac{dydx}{2/3}$$

$$= (2i + j + 2k) \int_0^3 [3(x-1)(6-2x) + (6-2x)^2] dx$$

$$\text{SI} = 72i + 36j + 72k.$$

Example 3: Find the surface area of the plane $x + 2y + 2z = 12$ cut off by $x = 0$, $y = 0$, and $x^2 + y^2 = 16$ (refer Fig. 16.10).

Solution: Rewriting equation of plane

$$z = \frac{12 - x - 2y}{2}$$

we have $z_x = -\frac{1}{2}$, $z_y = -1$

$$\text{Surface area} = \iint_R \sqrt{1 + z_x^2 + z_y^2} dxdy$$

$$= \iint \sqrt{1 + 1 + \frac{1}{4}} dxdy$$

$$= \frac{3}{2} \iint dxdy$$

In polar coordinates $= \frac{3}{2} \int_0^{\frac{\pi}{2}} \int_0^4 r\, dr\, d\theta = 6\pi.$

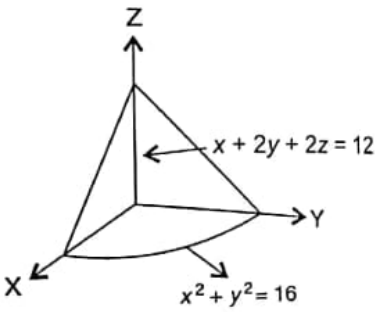

Fig. 16.10

Aliter: $F(x, y, z) = x + 2y + 2z - 12 = 0$, so $F_x = 1$, $F_y = 2$, $F_z = 2$

$$\sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{1+4+4} = 3$$

$$\text{Surface area} = \iint \sqrt{\frac{F_x^2 + F_y^2 + F_z^2}{|F_z|}} dxdy$$

$$= \iint \frac{3}{2} dx\, dy = 6\pi.$$

Flux

Example 4: Find the flux of the vector field $\bar{A} = (x - 2z)\bar{i} + (x + 3y + z)\bar{j} + (5x + y)\bar{k}$ through the upper side of the triangle ABC with vertices at the points $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ (see Fig. 16.11).

Solution: Equation of the plane in which the triangle ABC lies is

$$x + y + z = 1$$

Unit normal $\hat{n}$ to ABC is

$$\frac{\nabla(x+y+z-1)}{|\nabla(x+y+z-1)|} = \frac{i+j+k}{\sqrt{3}}$$

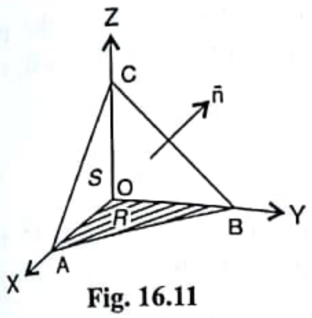

Fig. 16.11

$$\bar{A}\cdot\bar{n} = \frac{1}{\sqrt{3}}[(x-2z)+(x+3y+z)+(5x+y)]$$

$$= \frac{7x+4y-z}{\sqrt{3}}$$

Let AOB be the projection of ABC onto the xy-plane

Then

$$dS = \frac{dx\,dy}{|n\cdot k|} = \sqrt{3}dxdy$$

Flux across the triangle $ABC = \iint_S \bar{A}\cdot\bar{n}\,dS$

$$= \iint_{AOB} \frac{7x+4y-z}{\sqrt{3}}\sqrt{3}dxdy$$

Replace z by $1-x-y$

$$= \iint [7x+4y-(1-x-y)]\,dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}(8x+5y-1)dydx = \frac{5}{3}.$$

Exercise

1. If S is the surface $2x+y+2z=6$ bounded by $x=0, x=1, y=0, y=2$, evaluate (a) $\iint(\nabla\times\bar{F})\cdot\bar{n}dS$ and (b) $\iint_S \phi\bar{n}dS$.

Ans. **a.** 1 **b.** $2i+j+2k$

2. Evaluate $\iint_S \bar{A}\cdot\bar{n}\,dS$ where $\bar{A}=18zi-12j+3yk$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.

Ans. 24

3. Evaluate $\iint_S(\nabla\times\bar{F})\cdot\bar{n}dS$ where $\bar{F}=yi+(x-2xz)j-xyk$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

Ans. 0

4. If S is the entire surface of the cube bounded by $x=0, x=b, y=0, y=b, z=0,$ and $z=b$ and $\bar{A}=4xz\bar{i}-y^2j+yz\bar{k}$ then evaluate $\iint_S \bar{F}\cdot\bar{n}dS$.

Ans. $3b^4/2$

5. Let S be the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$. Evaluate $\iint_S \bar{A}\cdot\bar{n}dS$ where $\bar{A}=zi+xj-3y^2zk$.

Ans. 90

6. For the surface S defined in the previous problem 5, evaluate $\iint_S \phi\bar{n}dS$ where $\phi=\frac{3}{8}xyz$.

Ans. $100(i+j)$.

7. Find the surface integral over the parallelopiped $x=0, y=0, z=0, x=1, y=2, z=3$ when $\bar{A}=2xyi+yz^2j+xz\bar{k}$.

Ans. 33

8. If S is the surface of the sphere $x^2+y^2+z^2=d^2$ and $\bar{A}=axi+byj+czk$, evaluate $\iint_S \bar{F}\cdot\bar{n}dS$.

Hint: Project S onto xoy-plane and use symmetry.

Ans. $2\cdot\frac{2\pi d^3}{3}(a+b+c)$

9. Let S be the surface of the cylinder $x^2+y^2=a^2$ in the first octant between the planes $z=0$ and $z=h$. Evaluate $\iint \bar{A}\cdot\bar{n}dS$ where $\bar{A}=zi+xj-3zy^2k$.

Ans. $ah(a+h)/2$

Flux

10. Calculate the flux of water through the parabolic cylinder $y=x^2$, between the planes $x=0, z=0, x=3, z=2$ if the velocity vector is $\bar{A}=yi+2j+xzk$ m/sec.

Hint: Flux of $\bar{F}$ across S is $\iint_S \bar{F}\cdot\bar{n}dS$.

Ans. 69 m^3/sec

11. Find the flux across the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4$ and $z=6$ when the velocity vector $\bar{V}=2yi-zj+x^2k$.

Ans. 132

12. Find the flux of $\overline{A} = i - j + xyz\overline{k}$ through the circular region S obtained by cutting the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$ (take the side of S facing the positive side of the x-axis).

    Hint: S is bounded by the ellipse $2x^2 + z^2 = a^2$, $\hat{n} = (i - j)/\sqrt{2}$, $dS = \sqrt{2}dx\, dz$, area of the ellipse with semi axis $a/\sqrt{2}$ and a is $\pi a^2/\sqrt{2}$.

Ans. $\sqrt{2}\pi a^2$

13. Compute the flux of the vector field $\overline{A} = xi + yj + \sqrt{x^2+y^2-1}\; k$ through the outer side of the hyperboloid of one sheet $z = \sqrt{x^2+y^2-1}$ bounded by the planes $z = 0$ and $z = \sqrt{3}$.

    Hint: $\hat{n} = \frac{xi+yj}{\sqrt{x^2+y^2-1}} - k$, $\overline{A}\cdot\overline{n} = \frac{1}{\sqrt{x^2+y^2-1}}$ with polar coordinates, flux $= \int_0^{2\pi}\int_1^2 \frac{r\,dr\,d\theta}{\sqrt{r^2-1}}$ $= 2\sqrt{3}\pi$.

Ans. $2\sqrt{3}\pi$

14. Evaluate $\iint_S \overline{F}\cdot\overline{n}dS$ where $\overline{F} = \overline{r}/r^3$ and S is the sphere $x^2 + y^2 + z^2 = b^2$.

Ans. 4π

15. Calculate the surface integral of the vector function $\overline{A} = xi + yj$ over the portion of the surface of the unit sphere S: $x^2 + y^2 + z^2 = 1$ above the xy-plane $z \geq 0$.

Ans. $\frac{4\pi}{3}$

16. If S is the triangular surface with vertices $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 4)$ and $\overline{A} = xi + (z^2 - zx)j - xy\overline{k}$ then evaluate $\iint_S \overline{F}\cdot\overline{n}\,dS$.

Ans. $-\frac{22}{3}$

Surface area

17. What is the surface area of the surface S whose equation is $F(x, y, z) = 0$?

Ans. $$\iint_R \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}dxdy$$

where R is the projection of S on xy-plane.

18. Find the surface area of the plane $x + 2y + 2z = 12$ cut off by $x = 0$, $y = 0$, $x = 1$, $y = 1$.

Ans. $\frac{3}{2}$

19. Find the surface area of $z = x^2 + y^2$ included between $z = 0$ and $z = 1$.

Ans. $\frac{\pi}{6}\left(\sqrt{125} - 1\right)$

20. Find the surface area of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Ans. $16\,a^2$.

16.4 VOLUME INTEGRALS

Let V be a region in space enclosed by a closed surface $\overline{r} = \overline{r}(u, v)$. Let $\overline{F}(\overline{r})$ be a vector point function. Then the triple integral

$$\iiint_V \overline{F}(\overline{r})dV \quad \text{or briefly} \quad \iiint_V \overline{F}dV$$

is known as volume integral or space integral.

In the component form

$$\iiint_V \overline{F}dV = \overline{i}\iiint_V \overline{F}_1 dxdydz + \overline{j}\iiint_V F_2 dxdydz + \overline{k}\iiint_V F_3 dxdydz$$

$\iiint_V \phi dV$ is another form of a volume integral. These integrals are evaluated as three-fold iterated integrals.

Worked Out Examples

Example 1: Evaluate $\iiint_V f\,dV$ where $f = 2x + y$, V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$ (see Fig. 16.12).

Solution: This closed region is covered if x and z

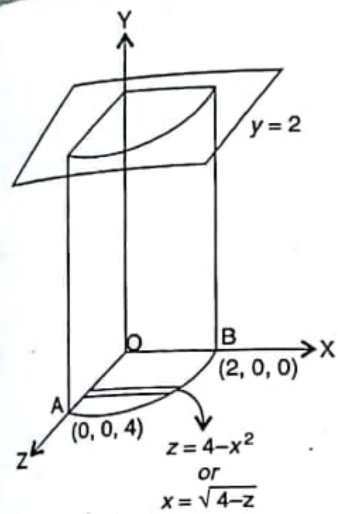

Fig. 16.12

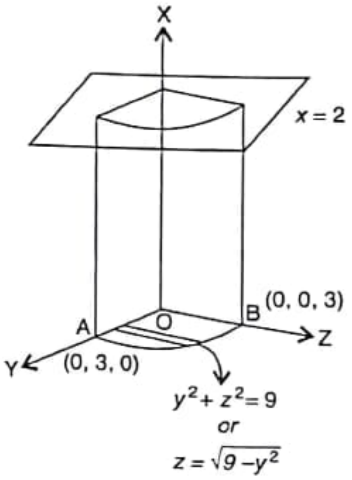

Fig. 16.13

varies covering the area OAB and y varies from 0 to 2. Thus

$$\iiint_V (2x+y)dV = \int_{y=0}^{2}\int_{z=0}^{4}\int_{x=0}^{\sqrt{4-z}}(2x+y)dxdzdy$$

$$= \int_0^2\int_0^4 (x^2+xy)\Big|_0^{\sqrt{4-z}} dzdy$$

$$= \int_0^2\int_0^4 \left[(4-z)+y\sqrt{4-z}\right]dzdy$$

$$= \int_0^2 4z - \frac{z^2}{2} - \frac{2}{3}y(4-z)^{\frac{3}{2}}\Big|_0^4$$

$$= \int_0^2 \left(8+\frac{16}{3}y\right)dy$$

$$= 8y + \frac{16}{6}y^2\Big|_0^2$$

$$= \frac{80}{3}.$$

Example 2: If V is the region in the first octant bounded by $y^2+z^2=9$ and the plane $x=2$ and $\bar{F}=2x^2yi-y^2j+4xz^2k$. Then evaluate

$$\iiint_V (\nabla\cdot\bar{F})dV.$$

Solution: $\nabla\cdot\bar{F} = 4xy-2y+8xz$

The volume V of the solid region is covered by covering the plane region OAB while x varies from 0 to 2 (Fig. 16.13). Thus

$$\iiint_V (\nabla\cdot\bar{F})dV$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}(4xy-2y+8xz)dzdydx$$

$$= \int_0^2\int_0^3 4xyz-2yz \;\; +4xz^2\Big|_0^{\sqrt{9-y^2}} dydx$$

$$= \int_0^2\int_0^3\left[(4xy-2y)\sqrt{9-y^2}+4x(9-y^2)\right]dydx$$

$$= \int_0^2 (4x-2)(-\frac{1}{3}(9-y^2)^{\frac{3}{2}} \;+4x\left(9y-\frac{y^3}{3}\right)\Big|_0^3 dx$$

$$= \int_0^2 [9(4x-2)+72x]dx$$

$$= \; 18x^2-18x+36x^2\Big|_0^2 = 180.$$

Example 3: Evaluate $\iiint_V \nabla\times\bar{A}dV$ where $\bar{A}=(x+2y)i-3zj+x\bar{k}$ and V is the closed region in the first octant bounded by the plane $2x+2y+z=4$

Solution: The solid region is covered by covering the plane region OAB in the xy-plane while z is varying from 0 to the plane $2x+2y+z=4$ (Fig. 16.14).

Thus $\quad z$ varies from 0 to $4-2x-2y$,

$\quad y$ varies from 0 to $2-x$

and $\quad x$ varies from 0 to 2.

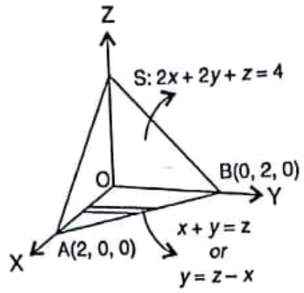

Fig. 16.14

Here

$$\nabla \times \overline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & -3z & x \end{vmatrix} = 3i - j - 2k$$

$$\iiint_V \nabla \times \overline{A} dV$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (3i - j + 2k) dz\, dy\, dx$$

$$= (3i - j + 2k) \int_0^2 \int_0^{2-x} (4 - 2x - 2y) dy\, dx$$

$$= 2(3i - j + 2k) \int_0^2 \left((2-x)^2 - \frac{(2-x)^2}{2} \right) dx$$

$$= (3i - j + 2k) \left[4x + \frac{x^3}{3} - 2x^2 \right]_0^2$$

$$= \frac{8}{3}(3i - j + 2k).$$

Example 4: Find the volume enclosed between the two surfaces $S_1 : z = 8 - x^2 - y^2$ and $S_2 : z = x^2 + 3y^2$ (see Fig. 16.15).

Solution: Eliminating z from the given two surfaces S_1 and S_2, we get $8 - x^2 - y^2 = z = x^2 + 3y^2$ i.e., $x^2 + 2y^2 = 4$. Thus the given two surfaces S_1 and S_2 intersect on the elliptic cylinder $x^2 + 2y^2 = 4$.

So the solid region between S_1 and S_2 is covered when

z varies from $x^2 + 3y^2$ to $8 - x^2 - y^2$,

y varies from $-\sqrt{\frac{4-x^2}{2}}$ to $\sqrt{\frac{4-x^2}{2}}$ and

x varies from -2 to 2.

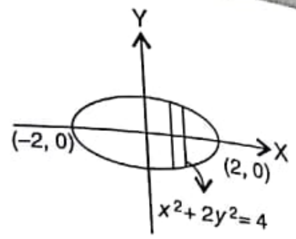

Fig. 16.15

So the required volume V enclosed between the two surfaces S_1 and S_2 is

$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{(4-x^2)}{2}}}^{\sqrt{\frac{(4-x^2)}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{(4-x^2)}{2}}}^{\sqrt{\frac{(4-x^2)}{2}}} (8 - 2x^2 - 4y^2) dy dx$$

$$= \int_{-2}^{2} \left[2(8 - 2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}} \right] dx$$

$$V = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{\frac{3}{2}} dx = 8\pi\sqrt{2}.$$

Exercise

1. Evaluate $\iiint_V f\, dV$ where $f = 45x^2y$ and V denotes the closed region bounded by the planes $4x + 2y + z = 8$, $x=0$, $y = 0$, $z = 0$.

Ans. 128

2. If $\overline{A} = (2x^2 - 3z)\overline{i} - 2xyj - 4x\overline{k}$ and V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$, evaluate $\iiint_V (\nabla \times \overline{A}) dV$.

Ans. $\frac{8}{3}(j - k)$

3. Evaluate $\iiint_V \overline{A} dV$ where $\overline{A} = xi + yj + 2zk$ and V is the volume enclosed by the planes $x = 0, y = 0, y = a, z = b^2$ and the surface $z = x^2$.

Ans. $\frac{ab^4}{4} i + \frac{a^2b^3}{3} j + \frac{4ab^5}{5} k$

4. Evaluate $\iiint_V \overline{B}\, dV$ where V is the region bounded by the surfaces $x = 0, y = 0$, $y=6$, $z=x^2$, $z = 4$ and $\overline{B} = 2xz\overline{i} - xj + y^2k$.

Ans. $128i - 24j + 384k$

5. If $\bar{A} = (x^3 - yz)\bar{i} - 2x^3yj + 2k$, evaluate $\iiint_V (\nabla \cdot \bar{A}) dV$ over the volume of a cube of side b.

Ans. $\frac{1}{3}b^3$

6. Evaluate $\iiint_V (\nabla \cdot \bar{B}) dV$ over the solid region of the sphere $x^2 + y^2 + z^2 = a^2$ when $\bar{B} = pxi + qyj + rzk$ where p, q, r are constants.

Ans. $\frac{4}{3}\pi a^2(p+q+r)$

Volume

7. Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Ans. $16a^3/3$

8. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2y$.

Ans. $\frac{\pi}{2}$

9. Find the volume cut from the sphere $x^2 + y^2 + z^2 = 4a^2$ by the cylinder $x^2 + y^2 = a^2$.

Ans. $4\pi a^3(8 - 3\sqrt{3})/3$

10. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.

Ans. $(8\sqrt{2} - 7)\pi a^3/6$.

16.5 GREEN'S* THEOREM IN PLANE: TRANSFORMATION BETWEEN LINE INTEGRAL AND DOUBLE INTEGRAL AREA IN CARTESIAN AND POLAR COORDINATES

If R is a closed region in the xy-plane bounded by a simple closed curve c and if $M(x, y)$ and $N(x, y)$ are continuous functions of x and y having continuous derivatives in R, then

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

*George Green (1793–1841) English mathematician.

where c is traversed in the positive direction (refer Fig. 16.16).

Proof: Let the equations of the curves AEB and AFB by $y = Y_1(x)$ and $y = Y_2(x)$ respectively, consider

$$\iint_R \frac{\partial M}{\partial y} dx\,dy = \int_{x=a}^{b} \int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y} dy\,dx$$

$$= \int_a^b [M(x, Y_2) - M(x, Y_1)] dx$$

$$= -\int_b^a M(x, Y_2) dx - \int_a^b M(x, Y_1) dx$$

$$= -\int_{BFA} M(x, y) dx - \int_{AEB} M(x, y) dx$$

$$= -\int_{BFAEB} M(x, y) dx$$

$$= -\oint_c M(x, y) dx \qquad (1)$$

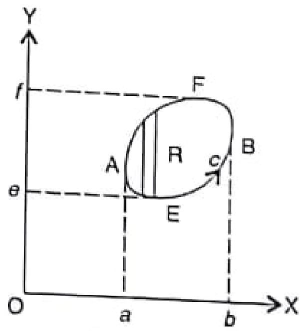

Fig. 16.16

Similarly let the equations of the curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\iint_R \frac{\partial N}{\partial x} dx\,dy = \int_{y=e}^{f} \int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x} dx\,dy$$

$$= \int_e^f [N(X_2, y) - N(X_1, y)] dy$$

$$= \int_e^f N(X_2, y) dy + \int_f^e N(X_1, y) dy$$

$$= \oint_c N(x, y) dy \qquad (2)$$

Adding (1) and (2), we get

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

Corollary 1: **Vector notation of Green's theorem**

Let $\overline{A} = Mi + Nj$ and $\overline{r} = x\overline{i} + y\overline{j}$ so that

$$\overline{A} \cdot d\overline{r} = M\,dx + N\,dy$$

$$\nabla \times \overline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & O \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\overline{k}$$

Thus

$$\oint_c \overline{A} \cdot d\overline{r} = \iint_R (\nabla \times \overline{A}) \cdot \overline{k}\, dR$$

where $dR = dxdy$.

Corollary 2: **Area A of the plane region R bounded by the simple closed curve c**

Let $N = x$, $M = -y$ so that

$$\oint_c x\,dy - y\,dx = \iint_R (1+1)dxdy$$

$$= 2\iint_R dxdy = 2A$$

Thus

$$A = \frac{1}{2}\oint_c x\,dy - y\,dx.$$

Corollary 3: Area A in polar coordinates

Let $x = r\cos t$, $y = r\sin t$, so that

$$dx = \cos t\,dr - r\sin t\,dt$$

$$dy = \sin t\,dr + r\sin t\,dt$$

Thus

$$A = \frac{1}{2}\int_c r^2\,dt$$

Corollary 4: Green's theorem is valid for a doubly (multiply) connected domain R where c is the boundary of the region R consisting of c_1 and c_2 (several) curves all traversed in the positive direction.

Corollary 5: If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then by Green's theorem

$$\oint_c M\,dx + N\,dy = 0.$$

Worked Out Examples

Green's theorem in plane

Example: Verify Green's theorem in plane for

$$\oint_c (x^2 - 2xy)dx + (x^2y + 3)dy$$

where c is the boundary of the region defined by $y^2 = 8x$ and $x = 2$ (refer Fig. 16.17).

Solution: Green's theorem states that

Line integral = Double integral.

a. The L.H.S. of the Green's theorem result is the line integral

$= LI = \oint_c (x^2 - 2xy)dx + (x^2y + 3)dy.$

Here c consists of the curves OA, ADB, BO, so

$$LI = \oint_c = \int_{OA+ADB+BO}$$

$$= \int_{OA} + \int_{ADB} + \int_{BO} = LI_1 + LI_2 + LI_3$$

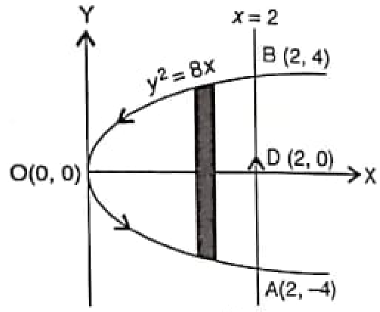

Fig. 16.17

Along OA: $y = -2\sqrt{2}\sqrt{x}$, so $dy = -\sqrt{\frac{2}{x}}dx$

$$LI_1 = \int_{OA} (x^2 - 2xy)dx + (x^2y + 3)dy$$

$$= \int_0^2 [x^2 - 2x(-2\sqrt{2}\sqrt{x})]dx$$

$$+[x^2(-2\sqrt{2}\sqrt{x}) + 3]\left(-\sqrt{\frac{2}{x}}\right)dx$$

$$= \int_0^2 (5x^2 + 4\sqrt{2}x^{\frac{3}{2}} - 3\sqrt{2}x^{-\frac{1}{2}})dx$$

$$= \frac{5x^3}{3} + 4\sqrt{3}\cdot\frac{2}{5}\cdot x^{\frac{5}{2}} - 3\sqrt{2}\cdot 2\sqrt{x}\Big|_0^2$$

$$= \frac{40}{3} + \frac{64}{5} - 12$$

Along ADB: $x = 2$, $dx = 0$

$$LI_2 = \int_{ADB} (x^2 - 2xy)dx + (x^2y + 3)dy$$

$$= \int_{-4}^{4} (4y + 3)dy = 24$$

Along BO: $y = 2\sqrt{2}\sqrt{x}$ with $x : 2$ to 0,

$dy = \sqrt{\frac{2}{x}}dx$

$$LI_3 = \int_{BO} (x^2 - 2xy)dx + (x^2y + 3)dy$$

$$= \int_2^0 (5x^2 - 4\sqrt{2}x^{\frac{3}{2}} + 3\sqrt{2}x^{-\frac{1}{2}})dx$$

$$= -\frac{40}{3} + \frac{64}{5} - 12$$

$$LI = LI_1 + LI_2 + LI_3 = \left(\frac{40}{3} + \frac{64}{5} - 12\right) + (24)$$

$$+\left(-\frac{40}{3} + \frac{64}{3} - 12\right) = \frac{128}{5}$$

b. Here

$$M = x^2 - 2xy,\ N = x^2y + 3,$$

$$\frac{\partial M}{\partial y} = -2x,\ \frac{\partial N}{\partial x} = 2xy.$$

So the R.H.S. of the Green's theorem is the double integral given by

$$\text{DI} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R [(2xy - (-2x))]dxdy$$

The region R is covered with y varying from $-2\sqrt{2}\sqrt{x}$ of the lower branch of the parabola to its upper branch $2\sqrt{2}\sqrt{x}$ while x varies from 0 to 2. Thus

$$\text{DI} = \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)dydx$$

$$= \int_0^2 xy^2 + 2xy \Big|_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= 8\sqrt{2} \int_0^2 x^{\frac{3}{2}} dx = \frac{128}{5}$$

Since L.I. = D.I. the Green's theorem is thus verified.

Area of a plane region

Example 2: Using Green's theorem, find the area of the region in the first quadrant bounded by the curves $y = x$, $y = \frac{1}{x}$, $y = \frac{x}{4}$. (see Fig. 16.18)

Solution: By Green's theorem area A of the region bounded by a closed curve c is given by

$$A = \frac{1}{2} \oint_c xdy - ydx$$

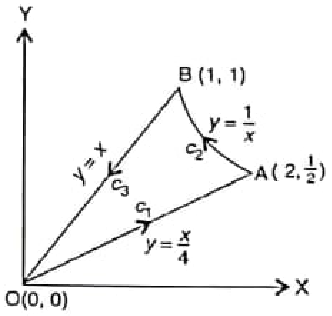

Fig. 16.18

Here c consists of the curves c_1: $y = \frac{x}{4}$, c_2: $y = \frac{1}{x}$ and c_3: $y = x$. So

$$A = \frac{1}{2}\oint_c = \frac{1}{2}\left[\int_{c_1} + \int_{c_2} + \int_{c_3}\right] = \frac{1}{2}\left[I_1 + I_2 + I_3\right]$$

Along c_1: $y = \frac{x}{4}$, $dy = \frac{1}{4}dx$, x: 0 to 2

$$I_1 = \int_{c_1} xdy - ydx = \int_{c_1} x\frac{1}{4}dx - \frac{x}{4}dx = 0$$

Along c_2: $y = \frac{1}{x}$, $dy = -\frac{1}{x^2}dx$, x: 2 to 1

$$I_2 = \int_{c_2} xdy - ydx = \int_2^1 x \cdot \left(-\frac{1}{x^2}\right) dx - \frac{1}{x}dx$$

$$= -2\ln x\Big|_2^1 = 2\ln 2$$

Along c_3: $y = x$, $dy = dx$; x: 1 to 0

$$I_3 = \int_{c_3} xdy - ydx = \int xdx - xdx = 0$$

$$A = \frac{1}{2}(I_1 + I_2 + I_3) = \frac{1}{2}(0 + 2\ln 2 + 0) = \ln 2.$$

Example 3: Find the area bounded by the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ with $a > 0$ (see Fig. 16.19).

Solution: Parametric equations of the hypocycloid are

$$x = a\cos^3 t,\ y = a\sin^3 t$$

$$dx = -3a\cos^2 t \sin t dt,$$

$$dy = 3a\sin^2 t \cos t dt$$

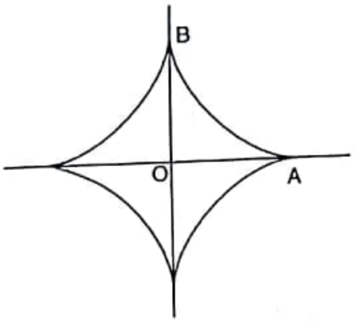

Fig. 16.19

Area bounded by the hypocycloid

$= 4 \cdot$ area under one leaf AB

$= 4$ area of the region AOB

Area of region $AOB = \frac{1}{2}\int_{ABOA} xdy - ydx$

$$= \frac{1}{2}\int_{AB} + \int_{BO} + \int_{OA}$$

$$= \frac{1}{2}\int_{AB} + 0 + 0$$

since $x = 0$ along BO and $y = 0$ along OA

$$= \frac{1}{2}\int_0^{\pi/2} \Big[a\cos^3 t \cdot (3a\sin^2 t\cos t dt)$$

$$-a\sin^3 t(-3a)\cos^2 t\sin t dt\Big]$$

$$= \frac{3a^2}{2}\int_0^{\pi/2} \sin^2 t\cos^4 t dt + \cos^2 t\sin^4 t dt$$

$$= \frac{3a^2}{2}\left[\frac{1\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] + \frac{3a^2}{2}\left[\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2}\right] = 3\frac{\pi a^2}{32}$$

Area bounded by hypocycloid $= 4\cdot\frac{3\pi a^2}{32} = \frac{3\pi a^2}{8}$.

Doubly connected region

Example 4: Verify Green's theorem in the plane for

$$\oint_c (2x - y^3)dx - xydy$$

where c is the boundary of the annulus (doubly connected) region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ (refer Fig. 16.20).

Solution: Here $M = 2x - y^3, N = xy$ so that $\frac{\partial M}{\partial y} = -3y^2$, $\frac{\partial N}{\partial x} = y$

Thus R.H.S. of Green's theorem is

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R (y + 3y^2)dxdy$$

where R is the annulus region.

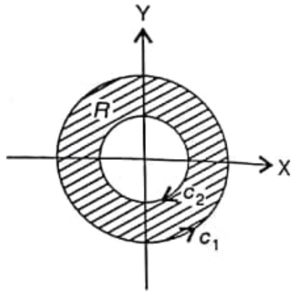

Fig. 16.20

Put $x = r\cos t$, $y = r\sin t$, so that t varies from 0 to 2π and r from 1 to 3

$$\text{R.H.S.} = \int_0^{2\pi}\int_1^3 (r\sin t + 3r^2\sin^2 t)r\, drdt$$

$$= \frac{26}{3}\int_0^{2\pi}\sin t dt + 60\int_0^{2\pi}\frac{1-\sin 2t}{2}dt = 60\pi$$

$$\text{L.H.S.} = \int_c Mdx + Ndy = \int_{c_1+c_2}(2x - y^3)dx - xydy$$

Changing to polar coordinate r, t

$$= \int (2r\cos t - r^3\sin^3 t)(-r\sin t dt) - \int r^3\cos^2 t\sin t dt$$

$$= r^4\frac{3\pi}{4}\Big|_1^3 = 60\pi.$$

EXERCISE

Use Green's theorem to evaluate the line integral $\oint_c Mdx + Ndy$ when $Mdx + Ndy$ equals to:

1. $-y^3dx + x^3dy$ where c: circle $x^2 + y^2 = 1$

Ans. $\frac{3\pi}{2}$

2. $x^{-1}e^y dx + (e^y \ln x + 2x)dy$ where c: the boundary of the region bounded by $y = 2$, $y = x^4 + 1$,

Ans. $\frac{16}{5}$

3. $(\cos x \sin y - xy)dx + \sin x \cos y \cdot dy$ where c: circle $x^2 + y^2 = 1$

Ans. 0

4. $(x^2 - \cosh y)dx + (y + \sin x)dy$ where c: the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1$

Ans. $\pi(\cosh 1 - 1)$

5. $(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c: boundary of the region defined by $x = 0$, $y = 0, x + y = 1$.

Ans. $\frac{5}{3}$

6. $e^{-x}(\sin y dx + \cos y dy)$ where c: rectangle with vertices at $(0, 0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$

Ans. $2(e^{-\pi} - 1)$

Verify **Green's theorem** or evaluate the line integral $\oint_c Mdx + Ndy$ (a) directly (b) using Green's theorem, where $Mdx + Ndy$ is:

7. $(xy + y^2)dx + x^2 dy$ with c: closed curve of the region bounded by $y = x$ and $y = x^2$

Ans. common value: $-\frac{1}{20}$

8. $(3x^2 - 8y^2)dx + (4y - 6xy)dy$ with c: boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$

Ans. common value: $\frac{3}{2}$

9. $(2x - y^3)dx - xydy$ with c: boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$

Ans. common value: 60π

10. $(3x + 4y)dx + (2x - 3y)dy$ with c: $x^2 + y^2 = 4$

Ans. common value: -8π

Area using Green's theorem

11. Find the area of the region bounded by $y = x^2$ and $y = x + 2$.

Ans. $\frac{9}{2}$

12. Calculate the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Deduce the area bounded by the circle $x^2 + y^2 = a^2$.

Hint: Put $x = a \cos t$, $y = b \sin t$.

Ans. Area of ellipse πab

put $a = b$, area of circle: πa^2

13. Find the area of the loop of the folium of descartes $x^3 + y^3 = 3axy$, $a > 0$.

Hint: Put $y = tx$, t : 0 to ∞.

Ans. $A = \frac{1}{2}\int x^2 dt = \frac{9}{2}\int_0^\infty \frac{a^2 t^2}{(1+t^3)^2} dt = \frac{3a^2}{2}$

14. Find the area of a loop of the four-leafed rose $\rho = 3 \sin 2\phi$.

Hint: $A = \frac{1}{2}\int_0^{\pi/2} \rho^2 d\phi = \frac{9\pi}{8}$.

15. Find the area of the cardioid $\rho = a(1 - \cos\theta)$, with $0 \le \theta \le 2\pi$.

Ans. $\frac{3\pi a^2}{2}$

16. Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$ and the x-axis.

Ans. $3\pi a^2$

17. Evaluate $\int_c \overline{A} \cdot d\overline{r}$ where

$$\overline{A} = \alpha[-3a \sin^2 t \cos t i + a(2 \sin t - 3 \sin^3 t)j + b \sin 2tk]$$

and the curve c is given by

$$\overline{r} = a \cos t i + a \sin t j + btk$$

and t varying from $\pi/4$ to $\pi/2$.

Ans. $\frac{\alpha}{2}(a^2 + b^2)$

18. Show that $\int_c f dg = \iint_R \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)dxdy$ where R is the region bounded by the simple closed curve c.

Hint: Use Green's theorem with $M = f\frac{\partial g}{\partial x}$ and $N = f\frac{\partial g}{\partial y}$.

19. Prove that $\int_c \frac{dF}{dn} dS = \iint_R \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dxdy$ where $\frac{dF}{dn}$ is the directional derivative of F in the direction of the outer normal $\overline{n}$ to the curve c bounding the region R.

Hint: Choose $M = -\frac{\partial F}{\partial y}$, $N = \frac{\partial F}{\partial x}$ and note that $\overline{n} = \frac{dy}{ds} i - \frac{dx}{ds} j$.

20. If $\nabla^2 f = 0$ in R, show that

$$\iint_R \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy = \int_c f \frac{\partial f}{\partial n} dS.$$

Hint: Take $M = -f\frac{\partial f}{\partial y}$, $N = f\frac{\partial f}{\partial x}$ and note that $\overline{n} = \frac{dy}{dS}i - \frac{dx}{dS}\overline{j}$.

21. Show that Green's theorem can be written in the form $\int_c \overline{F}\cdot\overline{n}ds = \iint_R \nabla\cdot\overline{F}dxdy$ where $\overline{F} = Mi - Nj$ and $\overline{n}$ is the outer unit normal to the curve c.

16.6 STOKES'* THEOREM

Transformation between line integral and surface integral. Let $\overline{A}$ be a vector function, having continuous first partial derivatives in a domain in space containing an open two sided surface S bounded by a simple closed curve c then

$$\iint_S (\nabla\times\overline{A})\cdot\overline{n}dS = \oint_c \overline{A}\cdot d\overline{r} \quad (1)$$

where $\overline{n}$ is a unit normal of S and c is traversed in the positive direction.

Proof: See Fig. 16.21. Assume that S can be represented as $z = f(x, y)$ or $x = g(y, z)$, or $y = h(x, z)$ where f, g, h are continuous, differentiable functions. Also assume that projections of S on the xy, yz, zx planes are regions bounded by simple closed curves. If $\overline{A} = A_1 i + A_2 j + A_3 k$ then the result of Stokes' theorem (1) can be written as

$$\iint_S (\nabla\times\overline{A})\cdot\overline{n}dS$$

$$= \iint_S (\nabla\times(A_1 i + A_2 j + A_3 k))\cdot\overline{n}dS = \oint_c \overline{A}\cdot d\overline{r}$$

$$= \oint_c (A_1 i + A_2 j + A_3 k)\cdot(dx\overline{i} + dy\overline{j} + dz\overline{k})$$

$$= \oint_c A_1 dx + A_2 dy + A_3 dz \quad (2)$$

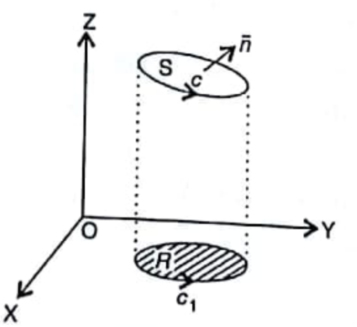

Fig. 16.21

*Sir George Gabriel Stokes (1819–1903) Irish mathematician.

To show that

$$\iint_S (\nabla\times A_1 i)\cdot\overline{n}dS = \oint_c A_1 dx \quad (3)$$

Consider

$$\nabla\times A_1 i = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\overline{j} - \frac{\partial A_1}{\partial y}\overline{k}$$

so that

$$(\nabla\times A_1 i)\cdot\overline{n}dS = \left(\frac{\partial A_1}{\partial z}\overline{n}\cdot j - \frac{\partial A_1}{\partial y}\overline{n}\cdot\overline{k}\right)dS \quad (4)$$

Take the equation of S as $z = f(x, y)$. Then the position vector $\overline{r}$ to any point of S is

$$\overline{r} = x\overline{i} + y\overline{j} + z\overline{k} = x\overline{i} + y\overline{j} + f(x, y)\overline{k}$$

so $$\frac{\partial\overline{r}}{\partial y} = 0 + \overline{j} + \frac{\partial f}{\partial y}\overline{k} \quad (5)$$

Now $$\overline{n}\cdot\frac{\partial\overline{r}}{\partial y} = 0$$

since normal $\overline{n}$ to S is perpendicular to the tangent $\frac{\partial\overline{r}}{\partial y}$ to S.

Thus taking dot product of (5) with $\overline{n}$, we have

$$0 = \overline{n}\cdot\frac{\partial\overline{r}}{\partial y} = \overline{n}\cdot\overline{j} + \frac{\partial f}{\partial y}\overline{n}\cdot\overline{k}$$

or $$\overline{n}\cdot j = -\frac{\partial f}{\partial y}\overline{n}\cdot\overline{k} = -\frac{\partial z}{\partial y}\overline{n}\cdot\overline{k} \quad (6)$$

Substituting (6) in (4), we get

$$(\nabla\times A_1\overline{i})\cdot\overline{n}dS = -\left(\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y}\right)\overline{n}\cdot\overline{k}dS \quad (7)$$

Now on S,

$$A_1(x, y, z) = A_1(x, y, f(x, y)) = F(x, y)$$

so that $$\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y} \quad (8)$$

Using (8) in (7), we get

$$\iint_S (\nabla\times A_1\overline{i})\cdot\overline{n}dS = \iint_S -\frac{\partial F}{\partial y}\overline{n}\cdot\overline{k}dS$$
$$= -\iint_R \frac{\partial F}{\partial y}dxdy \quad (9)$$

where R is the projection of S on xy-plane and

$$\overline{n}\cdot\overline{k}dS = dxdy$$

Applying Green's theorem in plane

$$-\iint_R \frac{\partial F}{\partial y} dx dy = \oint_{c_1} F dx = \oint_c A_1 dx \qquad (10)$$

since at each point (x, y) of c_1 the value of F is the same as the value of A_1 at each point (x, y, z) of c and since dx is same for both the curves c and c_1. Thus from (9) of (10), we arrive at

$$\iint_S (\nabla \times A_1 i) \cdot \bar{n} dS = \oint_c A_1 dx \qquad (3)$$

Similarly by projecting S on to other coordinate planes, we get

$$\iint_S (\nabla \times A_2 \bar{j}) \cdot \bar{n} dS = \oint_c A_2 dy \qquad (11)$$

and

$$\iint_S (\nabla \times A_3 \bar{k}) \cdot \bar{n} dS = \oint_c A_3 dz \qquad (12)$$

Adding (3), (11) and (12), we get (1) the result of Stokes' theorem.

Note 1: Stokes' theorem in rectangular form is

$$\iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \cos\gamma \right] dS$$

$$= \oint_c A_1 dx + A_2 dy + A_3 dz$$

where $\bar{n} = \cos\alpha i + \cos\beta j + \cos\gamma k$, and α, β, γ are angles made by normal $\bar{n}$ with $\bar{i}, \bar{j}, \bar{k}$.

Note 2: Green's theorem in plane is a special case of Stoke's theorem.

Note 3: The *circulation* of $\bar{A}$ around a closed curve c is given by the line integral

$$\oint_c \bar{A} \cdot d\bar{r}$$

where $\bar{A}$ represents the velocity of a fluid circulation has applications in fluid mechanics and aerodynamics.

Worked Out Examples

Example 1: Prove that $\oint_c f d\bar{r} = \iint_S d\bar{S} \times \nabla f$.

Solution: Choose $\bar{A} = f\bar{c}$, where $\bar{c}$ is a constant vector, in the Stoke's theorem. Then

$$\text{L.H.S.} = \oint_c \bar{A} \cdot d\bar{r} = \oint_c f\bar{c} \cdot d\bar{r}$$

$$= \oint_c \bar{c} \cdot (f d\bar{r}) = \bar{c} \cdot \oint_c f d\bar{r}$$

Now $\nabla \times \bar{A} = \nabla \times (f\bar{c}) = (\nabla f) \times \bar{c} + f(\nabla \times \bar{c}) = \nabla f \times \bar{c}$ since $\nabla \times \bar{c} = 0$

So $\qquad (\nabla \times \bar{A}) \cdot \bar{n} = (\nabla f \times \bar{c}) \cdot \bar{n} = \bar{c} \cdot (\bar{n} \times \nabla f)$

$$\text{R.H.S.} = \iint_S (\nabla \times \bar{A}) \cdot \bar{n} ds = \iint_S \bar{c} \cdot (n \times \nabla f) dS$$

$$= \bar{c} \cdot \iint_S (\bar{n} \times \nabla f) dS$$

Thus

$$\bar{c} \cdot \oint_c f d\bar{r} = \bar{c} \cdot \iint_S (\bar{n} \times \nabla f) dS$$

Since this is true for any arbitrary constant $\bar{c}$, hence, we get the result.

$$\oint_c f d\bar{r} = \iint_S (\bar{n} \times \nabla f) dS = \iint_S \bar{n} dS \times \nabla f$$

$$= \iint_S d\bar{S} \times \nabla f.$$

Example 2: Evaluate $\iint_S (\nabla \times \bar{A}) \cdot \bar{n} dS$ over the surface of intersection of the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ which is included in the first octant, given that $\bar{A} = 2yzi - (x + 3y - 2)j + (x^2 + z)k$ (refer Fig. 16.22).

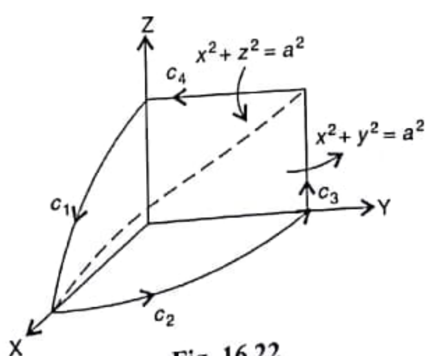

Fig. 16.22

Solution: By Stokes' theorem the given surface integral can be converted to a line integral i.e.,

$$SI = \iint_S (\nabla \times \bar{A}) \cdot \bar{n} dS = \oint_c \bar{A} \cdot d\bar{r} = LI$$

Here c is the curve consisting of the four curves c_1 : $x^2 + z^2 = a^2$, $y = 0$; $c_2 : x^2 + y^2 = a^2$, $z = 0$, $c_3 : x = 0, y = a, 0 \le z \le a$; $c_4 : x = 0, z = a, 0 \le y \le a$

$$LI = \oint_c \bar{A} \cdot d\bar{r} = \int_{c_1+c_2+c_3+c_4} = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4}$$
$$= LI_1 + LI_2 + LI_3 + LI_4$$

On the curve c_1: $y = 0;\ x^2 + z^2 = a^2$

$$LI_1 = \int_{c_1} \bar{A} \cdot d\bar{r} = \int_{c_1} (x^2 + z) dz$$
$$= \int_a^0 \left[(a^2 - z^2) + z\right] dz = -\frac{2}{3}a^3 - \frac{a^2}{2}$$

On the curve c_2: $z = 0, x^2 + y^2 = a^2$

$$LI_2 = \int_{c_2} \bar{A} \cdot d\bar{r} = \int_{c_2} -(x + 3y - 2) dy$$
$$= -\int_0^a \left(\sqrt{a^2 - y^2} + 3y - 2\right) dy$$
$$= -\frac{\pi a^2}{4} - \frac{3}{2}a^2 + 2a$$

On the curve c_3: $x = 0, y = a, 0 \le z \le a$

$$LI_3 = \int_{c_3} \bar{A} \cdot d\bar{r} = \int_0^a z dz = \frac{a^2}{2}$$

On c_4: $x = 0, z = a, 0 \le y \le a$

$$LI_4 = \int \bar{A} \cdot d\bar{r} = \int_a^0 (2 - 3y) dy = -2a + \frac{3a^2}{2}$$

$$SI = \iint_S (\nabla \times \bar{A}) \cdot \bar{n} dS = LI = \left(\frac{-2a^3}{3} - \frac{a^2}{2}\right)$$
$$+ \left(-\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a\right) + \frac{a^2}{2} + \left(-2a + \frac{3a^2}{2}\right)$$

$$SI = \frac{-a^2}{12}(3\pi + 8a).$$

Example 3: Verify Stokes' theorem for $\bar{A} = xzi - yj + x^2yk$ where S is the surface of the region bounded by $x = 0, y = 0, z = 0, 2x + y + 2z = 8$ which is not included in the xz-plane (Fig. 16.23).

Solution: Stokes' theorem states that

$$\oint_c \bar{A} \cdot d\bar{r} = \iint_S (\nabla \times \bar{A}) \cdot \bar{n} dS$$

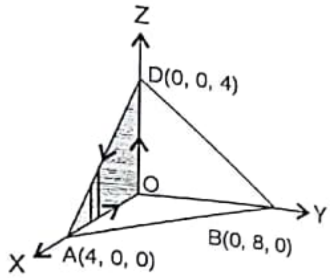

Fig. 16.23

Here c is curve consisting of the straight lines AO, OD and DA.

$$\text{L.H.S.} = \oint_c \bar{A} \cdot d\bar{r} = \int_{AO+OD+DA}$$
$$= \int_{AO} + \int_{OD} + \int_{DA} = LI_1 + LI_2 + LI_3$$

On the straight line AO: $y = 0, z = 0, \bar{A} = 0$ so

$$LI_1 = \int_{AO} \bar{A} \cdot d\bar{r} = 0$$

On the straight line OD: $x = 0, y = 0, \bar{A} = 0$ so

$$LI_2 = \int_{OD} \bar{A} \cdot d\bar{r} = 0$$

On the straight line DA: $x + z = 4$ and $y = 0$ so

$$\bar{A} = xzi = x(4 - x)i$$

$$LI_3 = \int_{DA} \bar{A} \cdot d\bar{r} = \int_0^4 x(4-x)i \cdot dxi = \int_0^4 x(4-x)dx = \frac{32}{3}$$

$$LI = 0 + 0 + \frac{32}{3} = \frac{32}{3}$$

Here the surface S consists of 3 surfaces (planes) $S_1 : OAB$, $S_2 : OBD$, $S_3 : ABD$, so that

$$\text{R.H.S.} = \iint_S (\nabla \times \bar{A}) \cdot \bar{n} dS = \iint_{S_1+S_2+S_3}$$
$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = SI_1 + SI_2 + SI_3$$

$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y & x^2y \end{vmatrix} = x^2 i + x(1 - 2y) j$$

On the surface S_1: **plane** OAB: $z=0, \overline{n}=-\overline{k}$, so

$$(\nabla\times\overline{A})\cdot\overline{n}=\left[x^2 i+x(1-2y)j\right]\cdot(-k)=0$$

$$SI_1=\iint_{S_1}(\nabla\times\overline{A})\cdot\overline{n}\,dS=0$$

On surface S_2: **plane** OBD: plane $x=0, \overline{n}=-i$

so

$$\nabla\times\overline{A}=0$$

$$SI_2=\iint_{S_2}(\nabla\times\overline{A})\cdot\overline{n}\,dS=0$$

On surface S_3: **plane** ABD: $2x+y+2z=8$.

Unit normal $\hat{n}$ to the surface $S_3=\frac{\nabla(2x+y+2z)}{|\nabla(2x+y+2z)|}$

$$\hat{n}=\frac{2i+j+2k}{\sqrt{4+1+4}}=\frac{2i+j+2k}{3}$$

$$(\nabla\times\overline{A})\cdot\overline{n}=\frac{2}{3}x^2+\frac{1}{3}x(1-2y)$$

To evaluate the surface integral on the surface S_3, project S_3 on to say xz-plane i.e., projection of ABD on xz-plane is AOD

$$dS=\frac{dxdz}{n\cdot j}=\frac{dxdz}{\frac{1}{3}}=3dxdz$$

Thus

$$SI_3=\iint_{S_3}(\nabla\times\overline{A})\cdot\overline{n}\,dS$$

$$=\iint_{AOD}\left[\frac{2}{3}x^2+\frac{x}{3}(1-2y)\right]3dxdz$$

$$=\int_{x=0}^{4}\int_{z=0}^{4-x}\left[2x^2+x(1-2y)\right]dzdx$$

since the region AOD is covered by varying z from 0 to $4-x$, while x varies from 0 to 4. Using the equation of the surface S_3, $2x+y+2z=8$, eliminate y, then

$$SI_3=\int_0^4\int_0^{4-x}\left\{2x^2+x[1-2(8-2x-2z)]\right\}dzdx$$

$$=\int_0^4\int_0^{4-x}(6x^2-15x+4xz)dzdx$$

$$=\int_0^4 6x^2z-15xz+\frac{4xz^2}{2}\Big|_0^{4-x}dx$$

$$=\int_0^4(23x^2-4x^3-28x)dx=\frac{32}{3}$$

Thus L.H.S $=$ L.I. $=$ R.H.S. $=$ S.I.

Hence Stokes' theorem is verified.

Example 4: Verify Stokes' theorem for $\overline{A}=y^2i+xyj-xzk$ where S is the hemisphere

$$x^2+y^2+z^2=a^2, z\geq 0.$$

Solution: The curve c which is the boundary of the given hemisphere is the base circle (see Fig. 16.24)

$$x^2+y^2=a^2$$

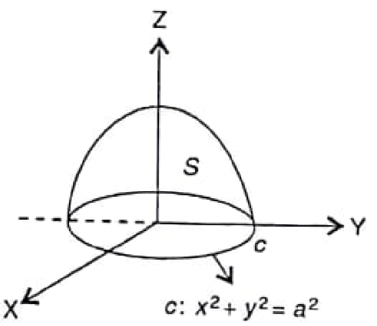

Fig. 16.24

On curve c: $z=0, x^2+y^2=a^2$

$$\text{L.H.S.}=LI=\oint_c\overline{A}\cdot d\overline{r}=\int y^2dx+xydy-xzdz$$

$$=\int y^2dx+xydy$$

Introducing polar coordinates $x=a\cos t$, $y=a\sin t$, with t varying from 0 to 2π

$$LI=\int_0^{2\pi}a^2\sin^2 t\,d(a\cos t)+a\cos t\cdot a\sin t\cdot d(a\sin t)$$

$$=a^3\int_0^{2\pi}(-\sin^3 t+\cos^2 t\sin t)dt=0$$

Now

$$\nabla\times\overline{A}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & xy & -xz\end{vmatrix}=zj-yk$$

Unit normal $\hat{n}$ to the sphere is

$$\hat{n}=\frac{\nabla(x^2+y^2+z^2)}{|\nabla(x^2+y^2+z^2)|}=\frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}}$$

$$=\frac{xi+yj+zk}{a}$$

$$(\nabla\times\overline{A})\cdot\overline{n}=(zj-yk)\cdot\left(\frac{xi+yj+zk}{a}\right)$$

$$=\frac{1}{a}(zy-zy)=0$$

so

$$\text{R.H.S.} = SI = \iint_S (\nabla \times \bar{A}) \cdot \bar{n}\, dS = 0$$

Thus

$$\text{L.H.S.} = \text{L.I.} = 0 = \text{S.I.} = \text{R.H.S.}$$

Hence the Stokes' theorem is verified.

EXERCISE

Stokes' theorem

1. If $\nabla \times \bar{A} = 0$, then prove that $\oint_c \bar{A} \cdot d\bar{r} = 0$ for every closed curve c.
2. Prove that $\iint_S \nabla \times \bar{A} \cdot \bar{n}\, dS = 0$ for any closed surface S.
3. Prove that $\oint_c d\bar{r} \times \bar{B} = \iint_S (\bar{n} \times \nabla) \times \bar{B}\, ds$.

   Hint: Choose $\bar{A} = \bar{B} \times \bar{c}$, where $\bar{c}$ is a constant vector, and apply Stokes' theorem. Note that

   $$(\bar{n} \times \nabla) \times \bar{B} = \nabla(\bar{B} \cdot \bar{n}) - \bar{n}(\nabla \cdot \bar{B}).$$
4. Prove that $\oint_c f \nabla g \cdot d\bar{r} = \iint_S (\nabla f \times \nabla g) \cdot \bar{n}\, dS$ and deduce that $\oint_c f \nabla f \cdot d\bar{r} = 0$.

   Hint: Take $\bar{A} = f \nabla g$ in Stokes' theorem. Note that $\nabla \times \nabla g = 0$.

   For deduction, take $f = g$ and note that $\nabla f \times \nabla f = 0$.
5. Evaluate $\iint_S (\nabla \times \bar{A}) \cdot \bar{n}\, dS$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane and $\bar{A} = (x^2 + y - 4)i + 3xyj + (2xz + z^2)k$.

Ans. -16π

6. If $\bar{A} = (y^2 + z^2 + x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z^2)k$ evaluate $\iint_S (\nabla \times \bar{A}) \cdot \bar{n}\, dS$ taken over the surface $S = x^2 + y^2 - 2ax + az = 0, z \ge 0$.

Ans. $2\pi a^3$

7. Evaluate $\iint_S \nabla \times (yi + zj + xk) \cdot \bar{n}\, dS$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \ge 0$.

Ans. π

8. Evaluate $\iint_S \nabla \times (yi + 2xj + zk) \cdot \bar{n}\, dS$ where S is the paraboloid $z = 1 - x^2 - y^2, z \ge 0$.

Ans. π

9. What is the surface integral of the normal component of the curl of the vector function $(x + y)i + (y - x)j + z^3k$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Ans. -2π

10. Evaluate $\int_c y\,dx + z\,dy + x\,dz$ where c is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$, beginning at the point $(2a, 0, 0)$ and going at first below the z-plane.

Ans. $-2\sqrt{2}\pi a^2$

11. Evaluate $\oint_c \sin z\,dx - \cos x\,dy + \sin y\,dz$ where c: rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

Ans. 2

12. Evaluate $\oint_c y\,dx + z\,dy + x\,dz$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + z = a$.

Ans. $-\pi a^2/\sqrt{2}$.

Verification of Stokes' theorem

13. Evaluate $\oint_c y\,dx + xz^3dy - zy^2dz$ (a) directly (b) using Stokes' theorem, given that c is the circle: $x^2 + y^2 = 4, z = -3$.

Ans. -28.4π

14. Evaluate (a) directly (b) using Stokes' theorem $\oint_c 4z\,dx - 2x\,dy + 2x\,dz$ where c is the ellipse $x^2 + y^2 = 1, z = y + 1$.

Ans. -4π

Verify Stokes' theorem in the following examples for:

15. $\bar{A} = (2x - y)i - yz^2j - y^2zk$ where S: upper half surface of the sphere $x^2 + y^2 + z^2 = 1$

    Hint: Here c: $x^2 + y^2 = 1, z = 0$.

Ans. π

16. $\bar{A} = x^2\bar{i} + xy\bar{j}$ where S is square $0 \le x \le a, 0 \le y \le a$ in the xy-plane

    Hint: c: square $0 \le x \le a, 0 \le y \le a, z = 0$

Ans. $a^3/2$

17. $\bar{A} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ taken around the rectangle bounded by $x = a$, $x = -a$, $y = 0$, $y = b$

Ans. $-4ab^2$

18. $\bar{A} = e^z(i + \sin yj + \cos yk)$ where S: $z = y^2$, $0 \le x \le 4, 0 \le y \le 2$

Ans. $\pm 4(1 - e^4)$ from (0, 0, 0) to (4, 0, 0)
$\mp 4e^4$ from (4, 2, 4) to (0, 2, 4)
The integrals over the parabolas cancel each other.

19. $\bar{A} = y^2 i + z^2 j + x^2 k$ where S: portion of paraboloid $x^2 + y^2 = z$, $y \ge 0$, $z \le 1$

Ans. $\pm\frac{4}{3}$.

Work done around closed curve by Stokes' theorem

Find the work done by the force F in the displacement around the closed curve c where:

20. $\bar{F} = 2xy^3 \sin z\bar{i} + 3x^2y^2 \sin zj + x^2y^3 \cos z\bar{k}$ and c: intersection of paraboloid $z = x^2 + y^2$ and cylinder $(x - 1)^2 + y^2 = 1$

Hint: $\nabla \times \bar{F} = 0$.

Ans. 0

21. $\bar{F} = x^3 i + e^{3y} j + e^{-3z}\bar{k}$, c: $x^2 + 9y^2 = 9$, $z = x^2$

Hint: $\nabla \times \bar{F} = 0$.

Ans. 0.

16.7 GAUSS* DIVERGENCE THEOREM

Transformation between surface integral and volume integral. Let $\bar{A}$ be a vector function of position, having continuous derivatives, in a volume V bounded by a closed surface S then

$$\oiint_S \bar{A} \cdot \bar{n}\, dS = \iiint_V \nabla \cdot \bar{A}\, dV \qquad (1)$$

where $\bar{n}$ is the outward drawn (positive) normal to S.

Proof: Assume that S is such that any line parallel to coordinate axes meets S in at most two points. Let S_1 and S_2 be the lower (below) and upper (top), portions of S having equations $z = f_1(x, y)$ and $z = f_2(x, y)$ and having $\bar{n}_1$ and n_2 as normals respectively (See Fig. 16.25). Let R be the projection of the surface S on the xy-plane. If $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$, then the result of Gauss divergence theorem (1) in component form is

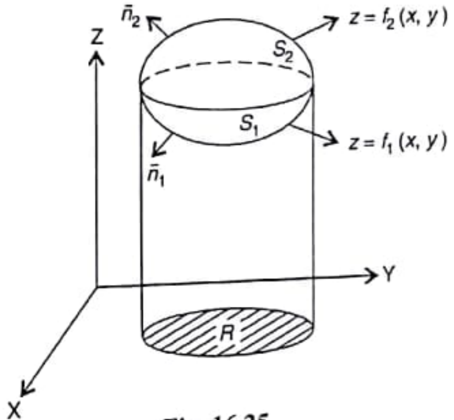

Fig. 16.25

$$\iint_S (A_1 i + A_2 j + A_3 k) \cdot \bar{n}\, dS$$
$$= \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV \qquad (2)$$

Consider

$$\iiint_V \frac{\partial A_3}{\partial z} dV = \iiint_V \frac{\partial A_3}{\partial z} dz dy dx$$
$$= \iint_R \left[\int_{z=f_1}^{z=f_2} \frac{\partial A_3}{\partial z} dz \right] dy dx$$
$$= \iint_R [A_3(x, y, f_2) - A_3(x, y, f_1)]\, dy dx$$
$$= \iint_R A_3\bar{k} \cdot \bar{n}_2 dS_2 - \iint_R A_3\bar{k} \cdot (-\bar{n}_1) dS_1$$

since for upper surface S_2, $k \cdot \bar{n}_2 dS_2 = dydx$ while for lower surface S_1, $\bar{k} \cdot (-\bar{n}_1) dS_1 = dydx$. Thus

$$\iiint_V \frac{\partial A_3}{\partial z} dV = \iint_S A_3\bar{k} \cdot \bar{n} dS \qquad (3)$$

Similarly, projecting S on to yz-plane and xz-planes we have

$$\iiint_V \frac{\partial A_1}{\partial x} dV = \iint_S A_1\bar{i} \cdot \bar{n} dS \qquad (4)$$

$$\iiint_V \frac{\partial A_2}{\partial y} dV = \iint_S A_2\bar{j} \cdot \bar{n} dS \qquad (5)$$

*Karl Frierich Gauss (1777–1855), German mathematician.

Adding (3), (4), (5), we get the required result (2).

Note 1: Gauss divergence theorem (G.D.T.) transforms volume integrals to surface integrals and vice versa.

Note 2: G.D.T in rectangular form

$$\iiint_V \left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right) dxdydz$$

$$= \iint_S (A_1\bar{i}+A_2\bar{j}+A_3\bar{k})\cdot(n_1\bar{i}+n_2\bar{j}+n_3\bar{k})ds$$

$$= \iint_S (A_1n_1+A_2n_2+A_3n_3)dS$$

$$= \iint_S (A_1\cos\alpha+A_2\cos\beta+A_3\cos\gamma)dS$$

where $n_1 = n\cdot i = \cos\alpha, n_2 = n\cdot j = \cos\beta, n_3 = n\cdot k = \cos\gamma$. Here α, β, γ are the angles which $\bar{n}$ makes with the positive x, y, z axes.

Note 3: Apart from (1), G.D.T. can also be written in the following forms:

$$\iint_S \bar{n}\times\bar{A}\, dS = \iiint_V \nabla\times\bar{A}dV$$ (see Example 14 on Page 16.33)

$$\iint_S \bar{n}\phi dS = \iiint_V \nabla\phi dV$$ (see W.O.E. 7 on Page 16.31)

Note 4: G.D.T. is also known as "Green's theorem in space" because G.D.T. generalizes the "Green's theorem in plane" by replacing the (plane) region R and its closed boundary (curve) c by a (space) region V and its closed boundary (surface) S.

Note 5: When $\bar{A} = \bar{V}$ = velocity of a fluid then G.D.T. has the following physical interpretation:

$$\left.\begin{array}{l}\text{Volume of fluid emerging}\\ \text{(diverging) from a closed}\\ \text{surfaces in unit time}\end{array}\right\} = \left\{\begin{array}{l}\text{Volume of fluid}\\ \text{supplied from within}\\ \text{volume } V \text{ in unit time}\end{array}\right.$$

Worked Out Examples

Surface to volume integral using divergence theorem

Example 1: Find the volume V of a region bounded by a surface S.

Solution: By Gauss' divergence theorem

$$\iiint_V \nabla\cdot\bar{A}\,dV = \iint_S \bar{A}\cdot\bar{n}dS \quad (1)$$

Choose $\bar{A} = xi$, so that $\nabla\cdot\bar{A} = 1$, with this (1) reduces to

$$V = \text{Volume} = \iiint 1\cdot dV$$

$$= \iint_S x(\bar{i}\cdot\bar{n})dS = \iint_S xdydz$$

Similarly by taking $\bar{A} = yj$ and $\bar{A} = z\bar{k}$, we get

$$V = \iint_S ydzdx \quad \text{and} \quad V = \iint_S zdxdy$$

or $$V = \frac{1}{3}\iint_S (xdydz + ydzdx + zdxdy).$$

Example 2: Evaluate $\iint_S e^x dydz - ye^x dzdx + 3zdxdy$ where S is the surface of the cylinder $x^2+y^2=c^2, 0\le z\le h$ (Fig. 16.26).

Solution: Here $A_1 = e^x, A_2 = -ye^x, A_3 = 3z$, so that $\nabla\cdot A = \frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z} = e^x - e^x + 3 = 3$ using divergence theorem, given surface integral

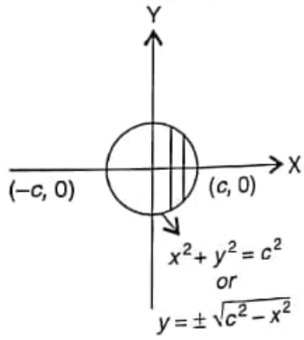

Fig. 16.26

$$= \iiint \nabla\cdot\bar{A}dV = 3\iiint dxdydz$$

$$= 3\int_{z=0}^{h}\int_{-c}^{c}\int_{-\sqrt{c^2-x^2}}^{+\sqrt{c^2-x^2}} dydxdz$$

$$= 3\cdot 2\cdot 2\cdot \int_0^h\int_0^c\int_0^{\sqrt{c^2-x^2}} dydxdz$$

$$= 12h\int_0^c \sqrt{c^2-x^2}dx \quad a = 3\pi hc^2.$$

Example 3: Evaluate $\iint_S \bar{A}\cdot\bar{n}dS$ where $\bar{A} = 2xyi + yz^2j + xzk$, and S is the surface of the region bounded by $x = 0, y = 0, z = 0, y = 3$ and $x + 2z = 6$ (refer Fig. 16.27).

Solution: $\nabla \cdot \bar{A} = 2y + z^2 + x.$

By Gauss' divergence theorem

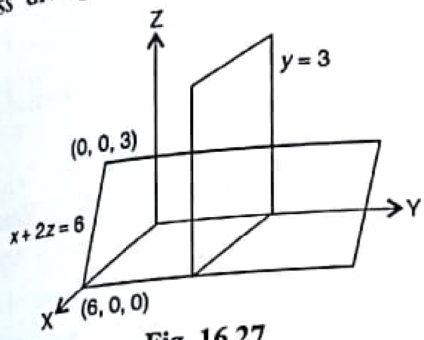

Fig. 16.27

$$SI = \iint_S \bar{A} \cdot \bar{n} dS = \iiint_V \nabla \cdot \bar{A} dV$$

$$= \iiint_V (2y + z^2 + x) dV$$

$$= \int_{y=0}^{3} \int_{x=0}^{6} \int_{z=0}^{\frac{6-x}{2}} (2y + z^2 + x) dz dx dy$$

$$= \int_0^3 \int_0^6 (2y + x)z + \frac{z^3}{3} \Big|_0^{\frac{6-x}{2}}$$

$$= \int_0^3 \int_0^6 \left[y(6 - x) + \frac{6x - x^2}{2} + \frac{1}{24}(6 - x)^3 \right] dx$$

$$= \int_0^3 y \left(6x - \frac{x^2}{2} \right) + \frac{1}{2} \left(\frac{6x^2}{2} - \frac{x^3}{3} \right) - \frac{1}{24} \left(\frac{6 - x}{4} \right)^4 \Big|_0^6$$

$$= \int_0^3 \left[18y + 216 \left(\frac{1}{12} + \frac{1}{16} \right) \right] dy = \frac{351}{2}.$$

Example 4: Evaluate $\iint_S \frac{\bar{r}}{r^2} \cdot \bar{n} dS$.

Solution: Take $\bar{A} = \frac{\bar{r}}{r^2} = \frac{xi + yj + zk}{x^2 + y^2 + z^2}$ so that

$$\nabla \cdot \bar{A} = \nabla \cdot \left(\frac{\bar{r}}{r^2} \right)$$

$$= \frac{(r^2 - 2x^2)}{r^4} + \frac{(r^2 - 2y^2)}{r^4} + \frac{(r^2 - 2z^2)}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4} = \frac{1}{r^2}$$

Applying Gauss divergence theorem, we get

$$\iint \frac{\bar{r}}{r^2} \cdot \bar{n} \, dS = \iint_S \bar{A} \cdot \bar{n} dS = \iiint_V \nabla \cdot \bar{A} dV$$

$$= \iiint_V \frac{1}{r^2} dV$$

Example 5: Evaluate $\iint_S r^5 \bar{n} \, dS$

Solution: Put $f = r^5$ so that $\nabla f = \nabla r^5 = 5r^3 \bar{r}$

Applying Gauss divergence theorem

$$\iint_S r^5 \bar{n} dS = \iint f \bar{n} dS = \iiint_V \nabla f dV$$

$$= \iiint_V 5r^3 \bar{r} dV$$

Example 6: Evaluate $\iint_S \bar{B} \cdot \bar{n} dS$ when $\bar{B} = \nabla \times \bar{A}$ and S is any closed surface.

Solution: By Gauss divergence theorem

$$\iint_S \bar{B} \cdot \bar{n} dS = \iiint_V \nabla \cdot \bar{B} dV$$

$$= \iiint_V \nabla \cdot (\nabla \times \bar{A}) dV = 0$$

since $\nabla \cdot (\nabla \times \bar{A}) = 0$ for any $\bar{A}$.

Example 7: Prove that $\iiint \nabla f dV = \iint_S f \bar{n} \, dS$.

Solution: Choose $\bar{A} = f\bar{c}$ where $\bar{c}$ is a constant vector

so that $\quad \nabla \cdot \bar{A} = \nabla \cdot (f\bar{c}) = \bar{c} \cdot \nabla f + f \nabla \cdot \bar{c} = \bar{c} \cdot \nabla f$

since $\quad \nabla \cdot \bar{c} = 0$

Also $\quad \bar{A} \cdot \bar{n} = (f\bar{c}) \cdot \bar{n} = (f\bar{n}) \cdot \bar{c} = \bar{c} \cdot (f\bar{n})$

Applying Gauss divergence theorem

$$\iiint_V \nabla \cdot \bar{A} \, dV = \iiint_V \bar{c} \cdot \nabla f \, dV$$

$$= \iint_S \bar{A} \cdot \bar{n} dS$$

$$= \iint_S \bar{c} \cdot (f\bar{n}) dS$$

or $\quad \bar{c} \cdot \iiint_V \nabla f dV = \bar{c} \cdot \iint_S f\bar{n} \, dS$

Since $\bar{c}$ is arbitrary constant vector, the result follows.

Example 8: Prove that

$\iint_S \bar{r} \times d\bar{S} = \bar{0}$ for any closed surface S.

Solution: We know that

$$\iint_S \bar{n} \times \bar{B} dS = \iiint_V \nabla \times \bar{B} dV \qquad (1)$$

Consider

$$\iint_S \bar{r} \times d\bar{S} = \iint_S \bar{r} \times \bar{n} dS = -\iint \bar{n} \times \bar{r} dS$$

Choose $\bar{B} = -\bar{r}$ in the above result (1). Note that

$$\nabla \times \bar{B} = \nabla \times (-\bar{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & -z \end{vmatrix} = 0$$

Thus

$$\iint \bar{r} \times d\bar{S} = -\iint \bar{n} \times \bar{r} dS$$

$$= -\iiint_V \nabla \times \bar{r} dV = 0$$

Green's formulas:

Green's first formula (identity)

Example 9: Prove that

$$\iint_S f \frac{\partial g}{\partial n} dS = \iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV$$

Solution: Choose $\bar{A} = f\nabla g$ in the divergence theorem then

$$\nabla \cdot \bar{A} = \nabla \cdot (f\nabla g) = f\nabla \cdot \nabla g + \nabla f \cdot \nabla g$$

$$= f\nabla^2 g + \nabla f \cdot \nabla g$$

$$\bar{A} \cdot \bar{n} = \bar{n} \cdot f\nabla g = f\bar{n} \cdot \nabla g = f\nabla g \cdot \bar{n} = f\frac{\partial g}{\partial n}$$

From divergence theorem

$$\iint_S \bar{A} \cdot \bar{n} dS = \iint_S f\frac{\partial g}{\partial n} dS = \iiint_V \nabla \cdot \bar{A} dV$$

$$= \iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV$$

since $\frac{\partial g}{\partial n} dS = \nabla g \cdot \bar{n} dS = \nabla g \cdot d\bar{S}$

Green's first identity can also be written as

$$\iint_S f\nabla g \cdot d\bar{S} = \iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV$$

Green's second formula (identity) or symmetrical theorem

Example 10: Show that

$$\iiint (f\nabla^2 g - g\nabla^2 f) dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS$$

Solution: From Green's first formula (above Example 9) we have

$$\iint_S f\frac{\partial g}{\partial n} dS = \iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV \qquad (1)$$

Interchanging f and g, we obtain

$$\iint_S g\frac{\partial f}{\partial n} dS = \iiint_V (g\nabla^2 f + \nabla g \cdot \nabla f) dV \qquad (2)$$

Subtracting (2) from (1), we get

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS = \iiint_V (f\nabla^2 g - g\nabla^2 f) dV$$

Note: Since

$$\iint_S [f(\nabla g \cdot \bar{n}) - g\nabla f \cdot \bar{n}] dS$$

$$= \iint_S (f\nabla g - g\nabla f) \cdot \bar{n} dS = \iint_S (f\nabla g - g\nabla f) \cdot d\bar{S}$$

Green's second identity can also be written as

$$\iint_S (f\nabla g - g\nabla f) \cdot d\bar{S} = \iiint_V (f\nabla^2 g - g\nabla^2 f) dV.$$

EXERCISE

Surface to volume integral, using divergence theorem

Using divergence theorem, evaluate the surface integral:

1. $\iint_S yz dy dz + zx dz dx + xy dx dy$ where S: $x^2 + y^2 + z^2 = 4$

Ans. 0

2. $\iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy$ where S: closed surface consisting of the circular cylinder $x^2 + y^2 = a^2$, $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$, $(x^2 + y^2 \le a^2)$.

Ans. $5\pi a^4 b/4$

3. $\iint_S \sin x dydz + (2 - \cos x)ydzdx$ where S: parallelopiped $0 \le x \le 3, 0 \le y \le 2, 0 \le z \le 1$

Ans. 12

4. $\iint_S (ax^2 + by^2 + cz^2)dS$ where S: sphere of unit radius centered at origin.

Ans. $4\pi(a + b + c)/3$

5. $\iint_S (x^2 - yz)dzdy - 2x^2ydzdx + zdxdy$ where S: cube of side b and three of whose edges are along the axes.

Ans. $b^3(b^2 + 3)/3$

6. $\iint_S 9xdydz + y\cosh^2 xdzdx - z\sinh^2 xdxdy$ where S: ellipsoid $4x^2 + y^2 + 9z^2 = 36$

Ans. 480π

7. $\iint_S \sin xdydz + ydzdx + zdxdy$ where S: surface of $0 \le x \le \pi/2, x \le y \le z, 0 \le z \le 1$

Ans. $\frac{3}{2} - \frac{\pi^2}{4}$

8. $\iint_S \overline{r} \cdot \overline{n} dS$ where S: sphere of radius 2 with centre at origin

Ans. 32π

9. $\iint_S \overline{r} \cdot \overline{n} dS$ where S: surface of cube bounded by the planes $x = -1, y = -1, z = -1, x = 1, y = 1, z = 1$

Hint: For examples 6 and 7 use result of worked example 4

Ans. 24

10. $\iint_S \overline{F} \cdot \overline{n} dS$ where $\overline{F} = 2xyi + yz^2j + xzk$ and S : surface of parallelopiped $0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3$

Ans. 30

11. If S is any closed surface enclosing a volume V and $\overline{A} = axi + byj + cz\overline{k}$, then evaluate $\iint_S \overline{A} \cdot \overline{n} dS$.

Ans. $(a + b + c)V$

12. If $\overline{n}$ is the unit outward drawn normal to any closed surface of area S, then evaluate $\iiint \nabla \cdot \overline{n} dV$.

Ans. S

13. Prove that $\iint_S \overline{n} dS = \overline{0}$ for any closed surface S.

Hint: Choose $f = 1$.

14. Prove that $\iint_S \overline{n} \times \overline{B} dS = \iiint_V \nabla \times \overline{B} dV$.

Hint: Take $\overline{A} = \overline{B} \times \overline{C}$ in divergence theorem, with $\overline{C}$ any arbitrary constant vector.

15. Evaluate $\iiint_V \nabla \times \overline{B} dV$ where V is the region bounded by a closed surface S and $\overline{B}$ is always normal to S.

Hint: Normal $\overline{n}$ to S and $\overline{B}$ are parallel, so $\overline{n} \times \overline{B} = 0$. Use result of Exercise Example 14

Ans. 0

16. Prove that $\iint_S \frac{\partial f}{\partial n} dS = \iiint_V \nabla^2 f dV$. Further if f is harmonic (solution of Laplace's equation) in a domain D, then evaluate $\iint \frac{\partial f}{\partial n} dS$.

Hint: Take $\overline{A} = \nabla f$ in divergence theorem and note that $\overline{A} \cdot \overline{n} = \nabla f \cdot \overline{n} = \frac{\partial f}{\partial n}$.

Ans. 0

17. If f and g are harmonic in V then evaluate $\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS$.

Hint: Use Green's second identity (formula) and note that $\nabla^2 f = 0$ and $\nabla^2 g = 0$

Ans. 0

Gauss' theorem

18. Let S be a closed surface and let $\overline{r}$ be the position vector of any point (x, y, z) measured from an origin 0. Then prove that

$$\iint_S \frac{\overline{n} \cdot \overline{r}}{r^3} dS = 0 \quad \text{if o lies outside } S$$
$$= 4\pi \quad \text{if o lies inside } S$$

Hint: If 0 lies outside S, note that $r \ne 0$ and $\nabla \cdot \frac{\overline{r}}{r^3} = 0$. Now use divergence theorem If 0 lies inside S, enclose 0 by a small sphere S^* of radius a then from the above result

$$\iint_{S+S^*} \frac{\overline{n} \cdot \overline{r}}{r^3} dS = 0$$

Note that $\iint_{S^*} \frac{\overline{n} \cdot \overline{r}}{r^3} dS = -4\pi$

19. Prove that $\iint_S \nabla(x^2 + y^2 + z^2) \cdot \overline{n} dS = 6V$ where S is any closed surface enclosing a volume V.

20. If $\overline{A} = (x^2 + y - 4)i + 3xyj + (2xz + z^2)k$ and S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane, evaluate $\iint_S (\nabla \times \overline{A}) \cdot \overline{n} dS$

Ans. -4π.

Verification of Gauss Divergence Theorem

WORKED OUT EXAMPLES

Example 1: Verify the divergence theorem for $\overline{A} = 2x^2y\overline{i} - y^2\overline{j} + 4xz^2\overline{k}$ taken over the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the plane $x = 2$ (refer Fig. 16.28).

Solution: Here $\nabla \cdot \overline{A} = 4xy - 2y + 8xz$

$$\text{R.H.S.} = \iiint_V \nabla \cdot \overline{A}\, dV$$

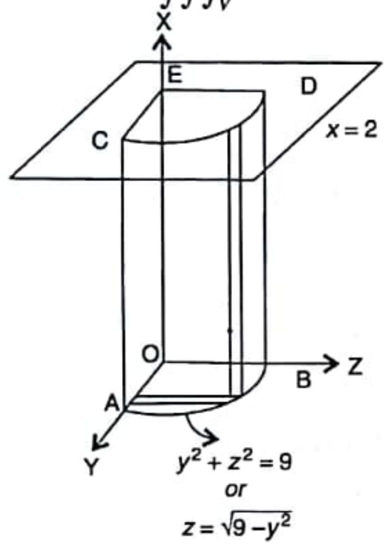

Fig. 16.28

The solid region is covered when z varies from 0 to $\sqrt{9 - y^2}$, y varies from 0 to 3 and x varies from 0 to 2 (height of the cylinder) so

$$\text{R.H.S.} = \int_0^2 \int_0^3 \int_0^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^3 \left[(4xy - 2y)\left(\sqrt{9 - y^2}\right) + 4x(9 - y^2) \right] dy\, dx$$

$$= \int_0^2 \frac{(2 - 4x)}{2} \frac{(9 - y^2)^{\frac{3}{2}}}{3/2} \Bigg|_0^3 dx + \int_0^2 36yx - 4x\frac{y^3}{3} \Bigg|_0^3 dx$$

$$= \int_0^2 -18(1 - 2x)\, dx + \int_0^2 (108x - 36x)\, dx$$

$$\text{R.H.S.} = 180$$

The entire surface S consists of five surfaces S_1, S_2, S_3, S_4, S_5. So

$$\text{L.H.S.} = \iint_S \overline{A} \cdot \overline{n}\, dS = \iint_{S_1} + \iint_{S_2} + \cdots + \iint_{S_5} = SI_1 + SI_2 + \cdots + SI_5$$

On S_1: OAB: $x = 0$, $\hat{n} = -\overline{i}$, $\overline{A} \cdot \overline{n} = 0$ so

$$SI_1 = \iint_{S_1} \overline{A} \cdot \overline{n}\, dS = 0$$

On S_2: CED: $x = 2$, $\hat{n} = \overline{i}$, $\overline{A} \cdot \overline{n} = 8y$ so

$$SI_2 = \iint_{S_2} \overline{A} \cdot \overline{n}\, dS = \iint_{S_2} 8y\, dy\, dz$$

$$= \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\, dy\, dz = \int_0^3 4(9 - z^2)\, dz = 72$$

On S_3: plane $OBDE$: $y = 0$, $\overline{n} = -j$, $\overline{A} \cdot \overline{n} = 0$ so

$$SI_3 = \iint_{S_3} \overline{A} \cdot \overline{n}\, dS = 0$$

On S_4: plane: $OACE$: $z = 0$, $\overline{n} = -\overline{k}$, $\overline{A} \cdot \overline{n} = 0$, so

$$SI_4 = \iint_{S_4} \overline{A} \cdot \overline{n}\, dS = 0$$

On S_5: the curved surface $ABDC$ of the cylinder.

$$y^2 + z^2 = 9$$

unit normal $\hat{n}$ to S_5: $\dfrac{\nabla(y^2+z^2)}{|\nabla(y^2+z^2)|} = \dfrac{2yj + 2zk}{\sqrt{4y^2 + 4z^2}}$

$$\hat{n} = \frac{yj + zk}{3}$$

so that
$$\overline{A} \cdot \hat{n} = \frac{-y^3 + 4xz^3}{3}$$

and
$$\overline{n} \cdot \overline{k} = \frac{yj + zk}{3} \cdot k = \frac{z}{3} = \frac{\sqrt{9 - y^2}}{3}$$

Projecting the surface S_5 on the yx-plane

$$SI_5 = \iint_{S_5} \overline{A} \cdot \overline{n}\, dS = \iint \frac{(4xz^3 - y^3)}{3} \cdot \frac{dx\, dy}{n \cdot k}$$

$$= \iint_R \frac{(4xz^3 - y^3)}{3\frac{\sqrt{9-y^2}}{3}} dx\, dy$$

$$= \int_{x=0}^2 \int_{y=0}^3 \left[4x(9 - y^2) - y^3(9 - y^2)^{-\frac{1}{2}} \right] dy\, dx$$

$$= \int_0^2 72x\, dx + 18 \int_0^2 dx = 144 - 36 = 108$$

So

$$L.H.S. = 0 + 72 + 0 + 0 + 108 = 180$$

Hence the divergence theorem is verified.

Example 2: Compute the flux of the vector field

$$\bar{A} = \left(\frac{x^2 y}{1+y^2} + 6yz^2\right)\bar{i} + 2x \arctan y\, \bar{j} - \frac{2xz(1+y)+1+y^2}{1+y^2}\bar{k}$$

through the outer side of that part of the surface of the paraboloid of revolution $z = 1 - x^2 - y^2$ located above the xy-plane.

Solution: The flux of $\bar{A}$ through a surface S is given by the surface integral,

$$\text{flux} = \iint \bar{A} \cdot \bar{n}\, dS \qquad (1)$$

Since the given surface of S_1 is the surface of the paraboloid of revolution $z = 1 - x^2 - y^2$, which is not a closed surface, so we close this surface from below with the circular portion S_2 of the xy-plane that is bounded by the circle $x^2 + y^2 = 1$, $z = 0$ (see Fig. 16.29).

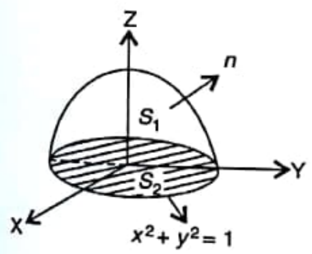

Fig. 16.29

Let V be the volume of the resulting solid bounded above by S_1 and below by S_2.
Now the flux (1) is calculated, using divergence theorem for the closed region V. Thus

$$\text{Flux} = \iint_S \bar{A} \cdot \bar{n}\, dS = \iiint_V (\nabla \cdot \bar{A})\, dV = 0$$

Since

$$\nabla \cdot \bar{A} = \frac{2xy}{1+y^2} + \frac{2x}{1+y^2} \frac{-2x(1+y)}{1+y^2} = 0$$

Flux across $S = S_1 + S_2$ is additive. So

$$\text{Flux} = \iint_S = \iint_{S_1+S_2} = \iint_{S_1} + \iint_{S_2} = 0$$

Thus

$$\iint_{S_1} \bar{A} \cdot \bar{n}\, dS = -\iint_{S_2} \bar{A} \cdot \bar{n}\, dS$$

i.e., flux across the required surface $S_1 = -$flux across the circular region S_2

On S_2: $z = 0$, $x^2 + y^2 \le 1$, $\bar{n} = -\bar{k}$ so that

$$\bar{A} \cdot \bar{n} = \left(\frac{x^2 y}{1+y^2} i + 2x \arctan y\, j - k\right) \cdot k = 1$$

$$\iint_{S_2} \bar{A} \cdot \bar{n}\, dS = \iint_{S_2} dS = S_2 = \text{area of the circular region}$$

$$= \pi r^2 = \pi \cdot 1^2 = \pi$$

Thus the required flux of $\bar{A}$ across the outer side of that part of the surface S_1 of the paraboloid of revolution $z = 1 - x^2 - y^2$ is $-\pi$

Exercise

Verify Gauss divergence theorem for:

1. $\bar{A} = 4xi - 2y^2 j + z^2 k$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

Ans. common value: 84π

2. $\bar{A} = (x^3 - yz)i - 2x^2 yj + z\bar{k}$ taken over the entire surface of the cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Ans. common value: $\frac{a^5}{3} + a^3$

3. $\bar{A} = axi + byj + czk$, theorem taken over the entire surface of the sphere of radius d and centered at origin.

Ans. common value: $\frac{4\pi}{3} d^3(a + b + c)$

4. $\bar{A} = 2xyi + yz^2 j + xzk$ and S is the total surface of the rectangular parallelopiped bounded by the coordinate planes and $x = 1$, $y = 2$, $z = 3$.

Ans. common value: 33

5. $\bar{A} = 2xzi + yzj + z^2 k$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$

Ans. common value: $5\pi a^4/4$

6. $\overline{A} = (x^2 - yz)\overline{i} + (y^2 - zx)j + (z^2 - xy)k$ taken over the rectangular parallelopiped bounded by the coordinate planes and $x = a$, $y = b$ and $z = c$

Ans. common value: $abc(a + b + c)$

7. $\overline{A} = x^2\overline{i} + y^2 j + z^2\overline{k}$ taken over the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Ans. common value: 0

8. $\overline{A} = xi + yj$ taken over the upper half of the unit sphere $x^2 + y^2 + z^2 = 1$

Ans. common value: $4\pi/3$

9. $\overline{A} = x^3 i + x^2 yj + x^2 zk$ taken over the closed region of the cylinder $x^2 + y^2 = a^2$, bounded by the planes $z = 0$ and $z = b$

Ans. common value; $5\pi ba^4/4$

10. Compute the flux of the vector $\overline{A} = 4xi - yj + zk$ through the surface of a torus.

Hint: Volume of a torus with R_1 and R_2 as the inner and outer radii of the torus is $\frac{\pi^2}{4}(R_2 - R_1)^2(R_2 + R_1)$.

Ans. flux $= \pi^2(R_2 - R_1)^2(R_2 + R_1)$.