AUTOMATA, FORMAL LANGUAGES AND LOGIC

MODULE 5

Propositional Logic

Dr Pooja Agarwal

Department of Computer Science & Engineering

Outline

- Proof by resolution
  - Unit resolution
  - Conjunctive normal form
  - Resolution algorithm

Proof by Resolution - Conjunctive Normal Form (CNF)

- **Conjunctive Normal Form (CNF)**

every sentence of propositional logic is logically equivalent to a conjunction of clauses.

A sentence expressed as a **conjunction** of clauses **(disjunction of literals)** is said to be in **conjunctive normal form or CNF**.

Existing Knowledge Base

R_1: $\neg P_{1,1}$

R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

R4: $\neg B_{1,1}$

R5: $B_{2,1}$

$R_6 : (B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

$R_7 : ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

$R_8 : (\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1}))$

$R_9 : \neg(P_{1,2} \vee P_{2,1})$

R10 : $\neg$ P1,2 $\wedge$ $\neg$P2,1

R11 : $\neg$B1,2

R12 : B1,2 $\Leftrightarrow$ (P1,1 $\vee$ P2,2 $\vee$ P1,3)

R13 : $\neg$P2,2

R14 : $\neg$P1,3

R15: P1,1 $\vee$ P2,2 $\vee$ P3,1

R16: P1,1 $\vee$ P3,1

R17: P3,1

Proof by Resolution - Conjunctive Normal Form (CNF)

R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

converting the sentence B1,1 ⇔ (P1,2 ∨ P2,1) into CNF. The steps are as follows:

1. Eliminate ⇔, replacing α ⇔ β with (α ⇒ β) ∧ (β ⇒ α). Biconditional Elimination

(B1,1 ⇒ (P1,2 ∨ P2,1)) ∧ ((P1,2 ∨ P2,1) ⇒ B1,1) .

2. Eliminate ⇒, replacing α ⇒ β with ¬α ∨ β:

(¬B1,1 ∨ P1,2 ∨ P2,1) ∧ (¬(P1,2 ∨ P2,1) ∨ B1,1) .

3. CNF requires ¬ to appear only in literals

¬(¬α) ≡ α (double-negation elimination)

¬(α ∧ β) ≡ (¬α ∨ ¬β) (De Morgan)

¬(α ∨ β) ≡ (¬α ∧ ¬β) (De Morgan)

In the example, we require just one application of the last rule:

(¬B1,1 ∨ P1,2 ∨ P2,1) ∧ **((¬P1,2 ∧ ¬P2,1)** ∨ B1,1) .

4. Now we have a sentence containing nested ∧ and ∨ operators applied to literals. We apply the distributivity law

(¬B1,1 ∨ P1,2 ∨ P2,1) ∧ (¬P1,2 ∨ B1,1) ∧ (¬P2,1 ∨ B1,1) .

Proof by Resolution - Conjunctive Normal Form (CNF)

Empty Clause

A disjunction of no disjuncts – is equivalent to False because a disjunction is true only if at least one of its disjuncts is true.

Another way to see empty clause represents a contradiction is to observe that it arises only from resolving two complementary unit clauses such as P and ¬P.

Resolving two singleton clauses leads to the *empty clause*; i.e. the clause consisting of no literals at all, as shown below. The derivation of the empty clause means that the database contains a contradiction.

$$\frac{\begin{array}{c}\{p\}\\ \{\neg p\}\end{array}}{\{\}}$$

Proof by Resolution

A Resolution Algorithm

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  $clauses \leftarrow$ the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  $new \leftarrow \{\}$
  loop do
    for each C_i, C_j **in** *clauses* **do**
      $resolvents \leftarrow$ PL-RESOLVE(C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      $new \leftarrow new \cup resolvents$
    if $new \subseteq clauses$ **then return** *false*
    $clauses \leftarrow clauses \cup new$

Proof by Resolution

Example :-

- Given

$KB = (B_{11} \Leftrightarrow (P_{12} \vee P_{21})) \wedge \neg B_{11}$, prove $\neg P_{12}$

Resolution:

Convert to CNF and $\alpha = \neg P_{12}$

show $KB \wedge \neg\alpha$ is unsatisfiable

$\boxed{\neg P_{21} \vee B_{11}}$ $\boxed{\neg B_{11} \vee P_{12} \vee P_{21}}$ $\boxed{\neg P_{12} \vee B_{11}}$ $\boxed{\neg B_{11}}$ $\boxed{P_{12}}$

Proof by Resolution

Example :- ❑Given

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Resolution: Convert to CNF and $\alpha = \neg P_{12}$

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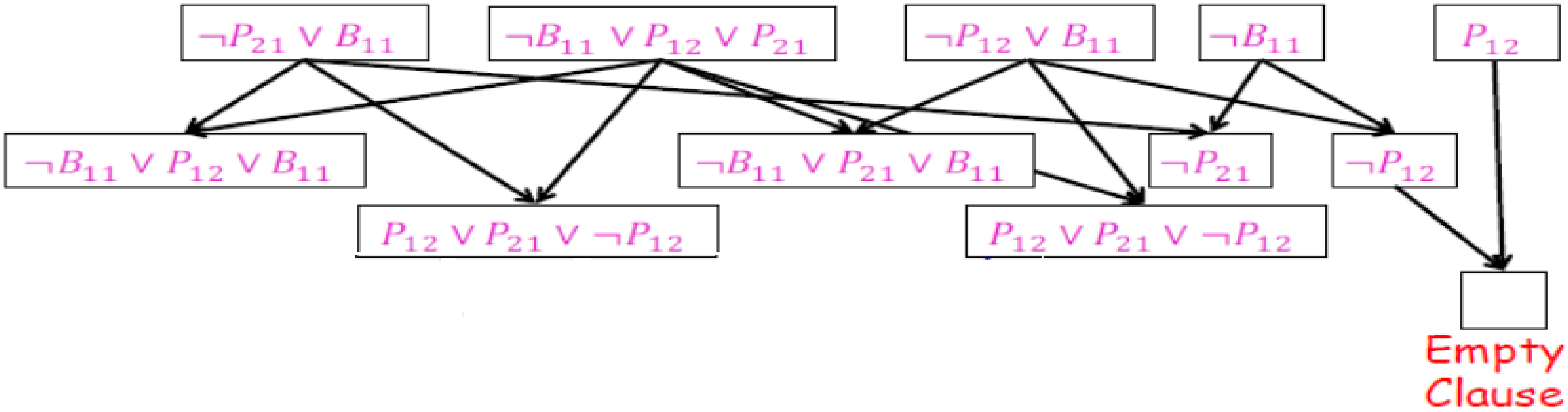

Exercise

- Given the premises ($p \Rightarrow q$) and ($r \Rightarrow s$), use Propositional Resolution to prove the conclusion ($p \vee r \Rightarrow q \vee s$).
- Ans:

1.	{~p,q}	Premise
2.	{~r,s}	Premise
3.	{p,r}	Goal
4.	{~q}	Goal
5.	{~s}	Goal
6.	{~p}	Resolution: 1, 4
7.	{~r}	Resolution: 2, 5
8.	{r}	Resolution: 3, 6
9.	{}	Resolution: 8, 7

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Exercise

In each of the following questions, say which of the answers best characterizes the result of applying resolution to the clauses shown.

1. $\{p, q, \neg r\}$ and $\{r, s\}$
   - (selected) $\{p, q, s\}$
   - $\{p, q, r, s\}$
   - $\{p, q, \neg r, s\}$
   - There are no resolvents.

2. $\{p, q, r\}$ and $\{r, \neg s, \neg t\}$
   - $\{p, q, r, \neg s, \neg t\}$
   - $\{p, q, \neg s, \neg t\}$
   - (selected) There are no resolvents.

Exercise

In each of the following questions, say which of the answers best characterizes the result of applying resolution to the clauses shown.

3. $\{q, \neg q\}$ and $\{q, \neg q\}$
   - (•) $\{q, \neg q\}$
   - ( ) $\{q\}$
   - ( ) $\{\neg q\}$
   - ( ) $\{\}$
   - ( ) There are no resolvents.

Exercise

A Propositional 2-CNF expression is a conjunction of clauses, where each clause contains exactly 2-literals.

$(A \lor B) \land (\sim A \lor C) \land (\sim B \lor D) \land (\sim C \lor G) \land (\sim D \lor G)$

Prove using Resolution that the above sentence entails G.

Ans:

Given "KB $\land$ ~ Alpha" is

R1: $(A \lor B)$, R2: $(\sim A \lor C)$, R3: $(\sim B \lor D)$, R4: $(\sim C \lor G)$,

R5: $(\sim D \lor G)$ and R6: $\sim G$

Resolving R1 and R2: R7 = $B \lor C$

Resolving R3 and R7: R8 = $C \lor D$

Resolving R4 and R8: R9 = $D \lor G$

Resolving R5 and R9: R10 = G

Resolving R6 and R10: R11 = {}

Hence Contradiction, So Alpha is TRUE and the given Statement entails G.

THANK YOU

Dr Pooja Agarwal
Department of Computer Science and Engineering
poojaagarwal@pes.edu