UNIT-5 HIGHER ORDER DIFFERENTIAL EQUATIONS

Solve the following differential equations

1. $y''' - 6y'' + 12y' - 8y = 0$

↳ Given differential eqn : $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0$

$\Rightarrow (D^3 - 6D^2 + 12D - 8)y = 0$ where $D = \frac{d}{dx}$

auxiliary equation : $D^3 - 6D^2 + 12D - 8 = 0$

by trial & error, if $D = 2$

$2^3 - 6(2)^2 + 12(2) - 8 = 0$

$\therefore$ $D-2$ is a root of this equation.

$$\begin{array}{c|cccc}
2 & 1 & -6 & 12 & -8 \\
 & & 2 & -8 & 8 \\
\hline
 & 1 & -4 & 4 & \boxed{0} \\
 & \downarrow & \downarrow & \downarrow & \\
 & D^2 & D & \text{constt} &
\end{array}$$

[By synthetic division]

$\therefore (D^2 - 4D + 4)$ is also a factor

$D^2 - 4D + 4 = 0$

$\Rightarrow (D-2)^2 = 0$

$\therefore D = 2, 2, 2$

$\therefore$ CF $= (c_1 x^2 + c_2 x + c_3)e^{2x}$

[Note: This is the same as $(c_1 + c_2 x + c_3 x^2)e^{2x}$]

since $X = 0$

$, PI = 0$

$\therefore$ general solution[complete solution] :

$$y = CF + PI$$

i.e, $\boxed{y = (c_1 x^2 + c_2 x + c_3)e^{2x}}$

2. $y''''' - 3y''' + 2y' = 0.$ $y(0) = 2$, $y'(0) = 0$, $y''(0) = 2$, $y'''(0) = 2$

↳ given $(D^5 - 3D^3 + 2D)y = 0$ where $D = \frac{d}{dx}$

auxiliary eqn:

i.e, $D(D^4 - 3D^2 + 2) = 0$

$\therefore \boxed{D = 0}$

~~Att~~ (or)

$D^4 - 3D^2 + 2 = 0$

By inspection, $\boxed{D = 1}$ & $\boxed{D = -1}$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -3 & 0 & 2 \\ & & 1 & 1 & -2 & -2 \\ \hline & 1 & 1 & -2 & -2 & \boxed{0} \end{array}$$

$D^3 + D^2 - 2D - 2 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -2 & -2 \\ & & -1 & 0 & 2 \\ \hline & 1 & 0 & -2 & \boxed{0} \end{array}$$

$D^2 - 2 = 0$

$\boxed{D = \pm\sqrt{2}}$

$\therefore$ CF, i.e, $\boxed{y = c_1 + c_2 e^{x} + c_3 e^{-x} + c_4 e^{\sqrt{2}x} + c_5 e^{-\sqrt{2}x}}$

There are 5 unknowns & only 4 equations are given - wrong.

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3. $y'''' - 4y''' + 14y'' - 20y' + 25y = 0$

$\hookrightarrow (D^4 - 4D^3 + 14D^2 - 20D + 25)y = 0 \qquad D = \frac{d}{dx}$

auxillary equation

$$D^4 - 4D^3 + 14D^2 - 20D + 25 = 0$$

$\Rightarrow (D^4 + 10D^2 + 25) - 4D^3 - 20D + 4D^2 = 0 \qquad (14D^2 = 10D^2 + 4D^2)$

$\Rightarrow (D^2+5)^2 - 4D(D^2+5) + 4D^2 = 0$

$\Rightarrow (D^2+5)^2 - 2(2D)(D^2+5) + (2D)^2 = 0$

$\Rightarrow (D^2 + 5 - 2D)^2 = 0$

$D^2 + 5 - 2D = 0$

$\Rightarrow D = \dfrac{2 \pm \sqrt{4-20}}{2} \Rightarrow D = 1 \pm 2i$

$\Rightarrow D = 1 \pm 2i, 1 \pm 2i$ (repeated complex roots)

$\therefore$ CF $\Rightarrow \boxed{y = \left[(c_1x + c_2)\cos 2x + (c_3x + c_4)\sin 2x\right]e^x}$

4. $y''''' - 2y'''' + y''' = 0$

↳ $\Rightarrow (D^5 - 2D^4 + D^3)y = 0 \qquad D = \frac{d}{dx}$

auxiliary equation : $D^3(D^2 - 2D + 1) = 0$

$\Rightarrow D = 0, 0, 0$

& $D^2 - 2D + 1 = 0$

$\Rightarrow (D-1)^2 = 0 \Rightarrow D = 1, 1$

$\therefore D = 0, 0, 0, 1, 1.$

$\therefore$ CF $= C_1 + C_2 x + C_3 x^2 + (C_4 + C_5 x)e^x$

PI $= 0$

$\Rightarrow \boxed{y = C_1 + C_2 x + C_3 x^2 + (C_4 + C_5 x)e^x}$

5. $(D^3 - 2D^2 - 5D + 6)y = 2e^x + 4e^{3x} + 7e^{-2x} + 8e^{2x} + 15$

↳ auxiliary equation : $D^3 - 2D^2 - 5D + 6 = 0$

$(D-1)$ is a root

$D = 1$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$D^2 - D - 6 = 0$

$-6D^2$

$-3D + 2D$

$D^2 - 3D + 2D - 6 = 0$

$D(D-3) + 2(D-3) = 0 \Rightarrow D = -2, 3$

$\therefore D = 1, -2, 3$

$\Rightarrow$ $\boxed{CF = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}}$

$$PI = \frac{X}{f(D)}$$

$$= \frac{2e^{x}}{D^3-2D^2-5D+6} + \frac{4e^{3x}}{D^3-2D^2-5D+6} + \frac{7e^{-2x}}{D^3-2D^2-5D+6} + \frac{8e^{2x}}{D^3-2D^2-5D+6} + \frac{15e^{0x}}{D^3-2D^2-5D+6}$$

In the first 3 terms, on replacing D by a, the denominator will become zero. So, we multiply by x and differentiate the denominator

$$2x\frac{e^{x}}{3D^2-4D-5} + 4x\frac{e^{3x}}{3D^2-4D-5} + 7x\frac{e^{-2x}}{3D^2-4D-5} + \frac{8e^{2x}}{D^3-2D^2-5D+6} + \frac{15e^{0x}}{D^3-2D^2-5D+6}$$

replacing D by a:

$$PI = \frac{2xe^{x}}{-6} + \frac{4xe^{3x}}{10} + \frac{7xe^{-2x}}{15} - 2e^{2x} + \frac{5}{2}$$

$$\therefore \boxed{PI = -\frac{x}{3}e^{x} + \frac{2}{5}xe^{3x} + \frac{7}{15}xe^{-2x} - 2e^{2x} + \frac{15}{6}}$$

Complete solution, $\boxed{y = CF + PI}$

6. $(D^2 - p^2)y = \sinh px$

$\hookrightarrow$ given $(D^2-p^2)y = \dfrac{e^{px}-e^{-px}}{2}$

auxiliary eqn: $D^2 - p^2 = 0 \Rightarrow D^2 = p^2 \Rightarrow D = \pm p$

$\therefore$ $\boxed{CF = c_1 e^{px} + c_2 e^{-px}}$

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$$PI = \frac{1}{2}\left[\frac{e^{px}}{D^2-p^2} - \frac{e^{-px}}{D^2-p^2}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{xe^{px}}{2D} - \frac{xe^{-px}}{2D}\right]$$

$$= \frac{1}{2}\left[\frac{xe^{px}}{2p} + \frac{xe^{-px}}{2p}\right]$$

$$= \frac{x}{2p}\left[\frac{e^{px}+e^{-px}}{2}\right]$$

$$\Rightarrow \boxed{PI = \frac{x}{2p}\cosh px}$$

complete solution $\boxed{y = CF + PI}$

7. $(2D^2 - 2D + 1)y = \sin 3x . \cos 2x$

↳ auxiliary equation : $2D^2 - 2D + 1 = 0$

$$D = \frac{2 \pm \sqrt{4-8}}{4} = \frac{2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$$

$$\Rightarrow \boxed{CF = e^{x/2}\left(c_1 \cos\frac{x}{2} + c_2 \cos\frac{x}{2}\right)}$$

$$X = \frac{\sin 5x + \sin x}{2}$$

$$\Rightarrow PI = \frac{1}{2}\left[\frac{\sin 5x}{2D^2-2D+1} + \frac{\sin x}{2D^2-2D+1}\right]$$

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replace D^2 by $-a^2$:

$$PI = \frac{1}{2}\left[\frac{\sin 5x}{-49-2D} + \frac{\sin x}{-1-2D}\right]$$

$$= -\frac{1}{2}\left[\frac{(49-2D)\sin 5x}{2401-4D^2} + \frac{(1-2D)\sin x}{1-4D^2}\right]$$

replace D^2 by $-a^2$

$$= -\frac{1}{2}\left[\frac{49\sin 5x - 10\cos 5x}{2501} + \frac{\sin x - 2\cos x}{5}\right]$$

$$\Rightarrow \boxed{PI = \frac{10\cos 5x - 49\sin 5x}{5002} + \frac{2\cos x - \sin x}{10}}$$

Complete solution, $\boxed{y = CF + PI}$

8. $(D^2+4D+4)y = x^2+2x$ with $y(0)=0, y'(0)=0$.

&> auxiliary equation:

$$D^2+4D+4=0$$
$$\Rightarrow (D+2)^2=0$$
$$\Rightarrow D=-2,-2$$
$$\therefore \boxed{CF = (c_1x+c_2)e^{-2x}.}$$

$$PI = \frac{x^2+2x}{(D+2)^2} = \frac{1}{4}\frac{x^2+2x}{(1+\frac{D}{2})^2}$$

$$= \frac{1}{4}\left(1+\frac{D}{2}\right)^{-2}(x^2+2x)$$

$$= \frac{1}{4}\left[1-D+\frac{3D^2}{4}\right](x^2+2x)$$

$$\left[(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots\right]$$

$$= \frac{1}{4}\left[x^2+2x-2x-2+\frac{3}{2}\right] = \frac{1}{4}\left[x^2-\frac{1}{2}\right]$$

$$\therefore PI = \frac{1}{4}\left[x^2-\frac{1}{2}\right]$$

$$y = CF+PI \Rightarrow y = (c_1x+c_2)e^{-2x} + \frac{1}{4}\left[x^2-\frac{1}{2}\right]$$

given $y(0)=0$:

$$\Rightarrow 0 = c_2 - \frac{1}{8} \Rightarrow \boxed{c_2 = \frac{1}{8}}$$

$$y' = -2(c_1x+c_2)e^{-2x} + c_1e^{-2x} + \frac{x}{2}$$

$y'(0)=0$

$$\Rightarrow 0 = -2c_2 + c_1 \Rightarrow c_1 = 2c_2 \Rightarrow \boxed{c_1 = \frac{2}{8}}$$

$$y = \left(\frac{2}{8}x + \frac{1}{8}\right)e^{-2x} + \frac{1}{8}\left[2x^2 - 1\right]$$

$$\Rightarrow \boxed{y = \frac{1}{8}\left[(2x+1)e^{-2x} + (2x^2-1)\right]}$$

9. $(D^4 - 1)y = \cos x . \cosh x$

auxiliary equation : $D^4 - 1 = 0$

$\Rightarrow D^4 = 1$

$\Rightarrow D^2 = \pm 1 \Rightarrow D^2 = 1,\ D^2 = -1$

$\Rightarrow D = 1, -1, \pm i$

$$\therefore \boxed{CF = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x}$$

$$PI = \frac{\cos x . \cosh x}{D^4 - 1} = \frac{\cos x \left[\frac{e^{x} + e^{-x}}{2}\right]}{D^4 - 1}$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{D^4 - 1} + \frac{e^{-x}\cos x}{D^4 - 1}\right]$$

$\downarrow$ $D \to D+1$ $\qquad$ $\downarrow$ $D \to D-1$

$$= \frac{1}{2}\left[e^{x}\frac{\cos x}{(D+1)^4 - 1} + e^{-x}\frac{\cos x}{(D-1)^4 - 1}\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{\cos x}{(D^2+2D+1)(D^2+2D+1) - 1} + e^{-x}\frac{\cos x}{(D^2-2D+1)(D^2-2D+1) - 1}\right]$$

$\downarrow$ D^2 by -1 $\qquad$ $\downarrow$ D^2 by -1

$$= \frac{1}{2}\left[e^{x}\frac{\cos x}{4D^2 - 1} + e^{-x}\frac{\cos x}{4D^2 - 1}\right]$$

again replacing D^2 by -1:

$$= \frac{1}{2}\left[e^x \frac{\cos x}{-5} + e^{-x}\frac{\cos x}{-5}\right]$$

$$= \frac{\cos x}{-5}\left[\frac{e^x + e^{-x}}{2}\right]$$

$$\Rightarrow \boxed{PI = -\frac{1}{5}\cos x . \cosh x}$$

$\therefore$ complete solution, $\boxed{y = CF + PI}$

10. $(D^2-1)y = x\sin x + x^2 e^x$

$\hookrightarrow$ $D^2 - 1 = 0$ (auxiliary equation)

$$\Rightarrow D = \pm 1$$

$$\therefore \boxed{CF = c_1 e^x + c_2 e^{-x}}$$

$$PI = \frac{x\sin x}{D^2-1} + \frac{x^2 e^x}{D^2-1} = PI_1 + PI_2$$

$$PI_1 = \frac{x\sin x}{D^2-1} = IP \text{ of } \frac{xe^{ix}}{D^2-1}$$

$$= IP \text{ of } e^{ix}\frac{x}{(D+i)^2-1} = IP \text{ of } e^{ix}\frac{x}{D^2+2Di-2}$$

$$= IP \text{ of } \frac{e^{ix}}{-2}\frac{x}{\left(1-\frac{D^2-2Di}{2}\right)}$$

$$= IP \text{ of } -\frac{e^{ix}}{2}\left[1-\left(\frac{D^2}{2}+Di\right)\right]^{-1} x$$

$= IP$ of $\frac{e^{ix}}{-2}(1+Di)x$

$= IP$ of $\frac{e^{ix}}{-2}(x+i) = IP$ of $-\frac{1}{2}\left[(\cos x + i\sin x)(x+i)\right]$

$$\boxed{PI_1 = -\frac{1}{2}(x\sin x + \cos x)}$$

$PI_2 = \frac{x^2e^x}{D^2-1} = e^x \frac{x^2}{(D+1)^2-1}$

$(D+1)^2 - 1 = D^2+2D+\cancel{1}-\cancel{1}$

$= e^x \frac{x^2}{D^2+2D} = \frac{e^x}{2D}\frac{x^2}{(1+\frac{D}{2})}$

$= \frac{e^x}{2D}(1+\frac{D}{2})^{-1}x^2 = \frac{e^x}{2D}\left(1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}\right)(x^2)$

$= \frac{e^x}{2}\left(\frac{1}{D}-\frac{1}{2}+\frac{D}{4}-\frac{D^2}{8}\right)(x^2)$

$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{2}-\frac{1}{4}\right]$

$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{2}\right]-\frac{e^x}{8}$

$\hookrightarrow$ In CF, there is $c_1e^x \rightarrow$ so this term can be included in the CF

$\Rightarrow PI_2 = \frac{e^x x}{2}\left[\frac{2x^2-3x+3}{6}\right]$

$$\therefore \boxed{PI_2 = \frac{xe^x}{12}[2x^2-3x+3]}$$

Complete solution, $\boxed{y = CF + PI_1 + PI_2}$

11. $(D^2-6D+9)y = 6e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$

↳ auxiliary equation : $D^2-6D+9=0$

$\Rightarrow (D-3)^2 = 0$

$\Rightarrow D = 3, 3$

$\Rightarrow \boxed{CF = (C_1 + C_2x)e^{3x}}$

$$PI = \frac{6e^{3x}}{(D-3)^2} + \frac{3x^2e^{3x}}{(D-3)^2} + \frac{7}{25}\frac{e^{-2x}}{(D-3)^2} - \frac{1}{9}\log 2\frac{e^{0x}}{(D-3)^2}$$

$$= 6x\frac{e^{3x}}{2(D-3)} + 3e^{3x}\frac{x^2}{(D+3-3)^2} + \frac{7}{25}\frac{e^{-2x}}{(-2-3)^2} - \frac{1}{9}\frac{\log 2}{(0-3)^2}$$

$$= 6x^2\frac{e^{3x}}{2} + 3e^{3x}\frac{x^2}{D^2} + \frac{7}{25}\frac{e^{-2x}}{25} - \frac{1}{9}\frac{\log 2}{9}$$

$$= 3x^2e^{3x} + 3e^{3x}\iint x^2\,dx + \frac{7}{625}e^{-2x} - \frac{1}{81}\log 2$$

$$\Rightarrow \boxed{PI = 3x^2e^{3x} + \frac{x^4e^{3x}}{4} + \frac{7}{625}e^{-2x} - \frac{1}{81}\log 2}$$

complete solution, $\boxed{y = CF + PI}$

12. Exact same question as 9.

13. $(D^4+D^2+1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$

↳ auxiliary equation : $D^4+D^2+1=0$

$\Rightarrow D^4+D^2+1+D^2-D^2=0$

$\Rightarrow (D^4+2D^2+1)-D^2=0$

$\Rightarrow (D^2+1)^2-D^2=0$

$\Rightarrow (D^2+1+D)(D^2+1-D)=0$

$\Rightarrow D = \frac{-1 \pm i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}$

$$\Rightarrow \boxed{CF = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + e^{x/2}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right]}$$

$$PI = \frac{e^{-x/2}\cos\frac{\sqrt{3}}{2}x}{(D^2+1+D)(D^2+1-D)}$$

$D \to D - \frac{1}{2}$ $\qquad (D-\frac{1}{2})^2 = D^2 - D + \frac{1}{4}$

$$\Rightarrow PI = e^{-x/2}\frac{\cos\frac{\sqrt{3}}{2}x}{(D^2+\frac{5}{4})(D^2-2D+\frac{5}{4})}$$

replace D^2 by $-\frac{3}{4}$

$$\Rightarrow PI = e^{-x/2}\frac{\cos\frac{\sqrt{3}}{2}x}{\frac{1}{2}\cdot(\frac{1}{2}-2D)}$$

$$= 2e^{-x/2}\frac{(\frac{1}{2}+2D)}{(\frac{1}{2}+2D)(\frac{1}{2}-2D)}\cos\frac{\sqrt{3}}{2}x = \frac{2e^{-x/2}(\frac{1}{2}+2D)\cos\frac{\sqrt{3}}{2}x}{\frac{1}{4}-4D^2}$$

$$= 2e^{-x/2}\frac{(\frac{1}{2}+2D)\cos\frac{\sqrt{3}}{2}x}{\frac{1}{4}-4(-\frac{3}{4})} = \frac{2e^{-x/2}\left(\frac{1}{2}\cos\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}x\right)}{\frac{1}{4}+3}$$

$$= \frac{4e^{-x/2}}{13}\left(\cos\frac{\sqrt{3}}{2}x - 2\sqrt{3}\sin\frac{\sqrt{3}}{2}x\right) \qquad \left(\frac{1}{4}+3 = \frac{1+12}{4} = \frac{13}{4}\right)$$

$$= e^{-x/2}\left(\frac{4}{13}\cos\frac{\sqrt{3}}{2}x - \frac{8\sqrt{3}}{13}\sin\frac{\sqrt{3}}{2}x\right)$$

$$\therefore \boxed{y = e^{-x/2}\left[\left(c_1+\frac{4}{13}\right)\cos\frac{\sqrt{3}}{2}x + \left(c_2 - \frac{8\sqrt{3}}{13}\right)\sin\frac{\sqrt{3}}{2}x\right] + e^{x/2}\left[c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right]}$$

14. $(D^3+2D^2+D)y = x^2e^{2x} + \sin^2 x$

$\hookrightarrow$ auxiliary equation: $D^3+2D^2+D=0$

$\Rightarrow \underline{D=0}$

$D^2+2D+1=0$

$\underline{\underline{D=-1,-1,0}}$

$\therefore \boxed{CF = c_1 + (c_2+c_3x)e^{-x}}$

$$PI = \underbrace{\frac{x^2e^{2x}}{D^3+2D^2+D}}_{PI_1} + \underbrace{\frac{1}{2}\frac{1}{D^3+2D^2+D}}_{PI_2} - \underbrace{\frac{1}{2}\frac{\cos 2x}{D^3+2D^2+D}}_{PI_3}$$

$$PI_1 = \frac{x^2e^{2x}}{D^3+2D^2+D} = e^{2x}\frac{x^2}{(D+2)(D+3)^2}$$

$\hookrightarrow D(D^2+2D+1)$

$$= \frac{e^{2x}}{18}\frac{x^2}{(1+\frac{D}{2})(1+\frac{D}{3})^2}$$

$$= \frac{e^{2x}}{18}\left(1+\frac{D}{2}\right)^{-1}\left(1+\frac{D}{3}\right)^{-2}x^2$$

$$= \frac{e^{2x}}{18}\left(1-\frac{D}{2}+\frac{D^2}{4}\right)\left(1-\frac{2D}{3}+\frac{D^2}{3}\right)(x^2)$$

$$= \frac{e^{2x}}{18}\left(1-\frac{2D}{3}+\frac{D^2}{3}-\frac{D}{2}+\frac{2D^2}{6}+\frac{D^2}{4}\right)(x^2)$$

$$= \frac{e^{2x}}{18}\left(1-\frac{7D}{6}+\frac{11}{12}D^2\right)(x^2)$$

$$\Rightarrow \boxed{PI_1 = \frac{e^{2x}}{18}\left(1-\frac{7x}{3}+\frac{11}{6}\right)}$$

$$PI_2 = \frac{1}{2}\frac{e^{0x}}{D^3+2D^2+D} = \frac{x}{2}\frac{e^{0x}}{3D^2+4D+1}$$

$$\Rightarrow \boxed{PI_2 = \frac{x}{2}}$$

$$PI_3 = -\frac{1}{2}\frac{\cos 2x}{D^3+2D^2+D}$$

$$= -\frac{1}{2}\frac{\cos 2x}{-4D-8+D}$$

$$= \frac{1}{2}\frac{(8-3D)}{(8-3D)}\frac{\cos 2x}{(8+3D)}$$

$$= \frac{1}{2}\cdot\frac{1}{100}(8\cos 2x + 6\sin 2x)$$

$$\Rightarrow \boxed{PI_3 = \frac{1}{100}(3\sin 2x + 4\cos 2x)}$$

$\therefore$ complete solution, $\boxed{y = CF + PI_1 + PI_2 + PI_3}$

15. $(D^2+2)y = x^2e^{3x} + e^x\cos 2x$

↳ auxiliary equation: $D^2+2=0$

$$\Rightarrow D = \pm\sqrt{2}i$$

$$\boxed{CF = C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x}$$

$$PI = \underset{(D\to D+3)}{\frac{x^2e^{3x}}{D^2+2}} + \underset{(D\to D+1)}{\frac{e^x\cos 2x}{D^2+2}}$$

$$= e^{3x}\frac{x^2}{D^2+6D+11} + e^x\frac{\cos 2x}{D^2+2D+3}$$

$$= \frac{e^{3x}}{11} \frac{x^2}{\left(1 + \frac{D^2+6D}{11}\right)} + e^x \frac{(2D+1)\cos 2x}{(2D+1)(2D-1)}$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{(D^2+6D)}{11} + \frac{36D^2}{121}\right](x^2) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$$

$$\therefore \boxed{PI = \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] + \frac{e^x}{17}(4\sin 2x - \cos 2x)}$$

$$\boxed{y = CF + PI}$$

16. $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

↳ auxiliary equation : $(D-2)^2 = 0 \quad D = 2, 2$

$$\boxed{CF = (C_1 x + C_2)e^{2x}}$$

$$PI = 8\left[\frac{e^{2x}}{(D-2)^2} + \frac{\sin 2x}{D^2 - 4D + 4} + \frac{x^2}{(D-2)^2}\right]$$

$$= 8\left[\frac{x^2 e^{2x}}{2} + \frac{\sin 2x}{-4D} + \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2\right]$$

$$= 8\left[\frac{x^2 e^{2x}}{2} - \frac{1}{4}\int \sin 2x \, dx + \frac{1}{4}\left[1 + D + \frac{3D^2}{4}\right](x^2)\right]$$

$$= 4x^2 e^{2x} + \cos 2x + 2\left(x^2 + 2x + \frac{3}{2}\right)$$

$$\Rightarrow \boxed{PI = 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3}$$

$$\boxed{y = CF + PI}$$

17. $(D^2 - 2D + 1)y = xe^x \sin x$

↳ auxiliary equation : $(D-1)^2 = 0 \Rightarrow D = 1, 1$

$$\boxed{CF = (C_1 x + C_2)e^x}$$

$$PI = \frac{xe^x \sin x}{(D-1)^2} \qquad D \to D+1$$

$$\Rightarrow PI = e^x \frac{x \sin x}{D^2} \qquad = e^x \int \int \underset{I}{x} \underset{II}{\sin x} \, dx$$

$$= e^x \int (-x\cos x + \sin x)\, dx$$

$$= -e^x [x \sin x + \cos x] - e^x \cos x$$

$$\Rightarrow \boxed{PI = -e^x [x \sin x + 2\cos x]}$$

$$\boxed{y = CF + PI}$$

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VARIATION OF PARAMETERS

Solve the following differential equations by the method of variation of parameters.

1) $y'' - 2y' + y = e^{x} \log x$

Let $\frac{d}{dx} = D$.

The given equations become.

$(D^2 - 2D + 1)\, y = e^{x} \log x$

Consider the auxillary equation.

$D^2 - 2D + 1 = 0$.

$D = 1, 1$

$\therefore$ Complimentary Function $C.F = (c_1 x + c_2) e^{x}$,

$C.F = c_1 x e^{x} + c_2 e^{x}$

Let $y_1 = x e^{x}$

$y_2 = e^{x}$

$$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} x e^{x} & e^{x}(x+1) \\ e^{x} & e^{x} \end{vmatrix}$$

$= -e^{2x}$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -x e^{x} \int \frac{e^{2x} \log x}{-e^{2x}} dx + e^{x} \int \frac{x e^{2x} \log x}{-e^{2x}} dx$$

$$= x e^{x} \int \log x \, dx - e^{x} \int x \log x \, dx.$$

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Let $I_1 = \int \log x \, dx$; $I_2 = \int x \log x \, dx$

$x = e^t$; $x = e^t$

$dx = e^t dt$; $dx = e^t dt$

$I_1 = \int t e^t dt$; $I_2 = \int t e^{2t} dt$

$I_1 = e^t(t-1)$; $I_2 = \frac{e^{2t}}{2}\left(t - \frac{1}{2}\right)$

$I_1 = x(\log x - 1)$; $I_2 = \frac{x^2}{4}(2\log x - 1)$

$P.I = xe^x(x(\log x - 1)) - e^x\left(\frac{x^2}{4}(2\log x - 1)\right)$

$PI = \frac{x^2 e^x \log x}{2} - 3\frac{x^2 e^x}{4} = \frac{x^2 e^x}{4}(2\log x - 3)$

$y = C.F + P.I = (c_1 x + c_2)e^x + \frac{x^2 e^x}{4}(2\log x - 3)$

2) $y'' + a^2 y = \mathrm{Sec}(ax)$

Let $\frac{d}{dx} = D$

The given equation becomes.

$(D^2 + a^2)y = \mathrm{Sec}(ax)$

Consider the Auxillary Equation

$D^2 + a^2 = 0.$

$D = \pm ia$

$\therefore$ Complimentary Function $C.F = c_1 \mathrm{Cos}(ax) + c_2 \mathrm{Sin}(ax)$

Let $y_1 = \mathrm{Cos}(ax)$

$y_2 = \mathrm{Sin}(ax)$

$$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} \cos(ax) & -a\sin(ax) \\ \sin(ax) & a\cos(ax) \end{vmatrix}$$

$$= a$$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -\cos(ax) \int \frac{\sin(ax)\sec(ax)}{a} dx + \sin(ax) \int \frac{\cos(ax)\sec(ax)}{a} dx$$

$$= -\cos(ax)\left(\frac{\log(\sec(ax))}{a^2}\right) + \sin(ax)\left(\frac{x}{a}\right)$$

$$= \frac{1}{a^2}\cos(ax)\log(\cos(ax)) + \frac{x}{a}\sin(ax)$$

$$y = C.F + P.I.$$

3) $y'' + y = \frac{1}{1+\sin(x)}$

Let $\frac{d}{dx} = D$

The given equation becomes.

$$(D^2+1)y = \frac{1}{1+\sin(x)}$$

Consider the Auxillary Equation

$$D^2+1=0$$

$$D = \pm i$$

$\therefore$ Complimentary Function $C.F = c_1 \cos(x) + c_2 \sin(x)$

Let $y_1 = \cos(x)$

$y_2 = \sin(x)$

$$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{vmatrix}$$

$= 1$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -\cos(x) \int \frac{\sin(x)}{1+\sin(x)} dx + \sin(x) \int \frac{\cos(x)}{1+\sin(x)} dx$$

Let $I_1 = \int \frac{\sin x}{1+\sin x} dx$; $I_2 = \int \frac{\cos x}{1+\sin x} dx$

$I_1 = \int 1 - \frac{1}{1+\sin x} dx$; $\sin x = t$

$\cos x \, dx = dt$

$I_1 = \int 1 - \frac{1-\sin x}{1-\sin^2 x} dx$; $I_2 = \int \frac{dt}{1+t}$

$I_1 = \int 1 - \frac{1-\sin x}{\cos^2 x} dx$; $I_2 = \log(1+t)$

$I_1 = \int 1 - \sec^2 x + \sec x \tan x \, dx$;

$I_1 = x - \tan x + \sec x$; $I_2 = \log(1+\sin x)$.

$P.I = -\cos(x)(x - \tan x + \sec x) + \sin(x)(\log(1+\sin(x)))$;

$P.I = \sin(x) \log(1+\sin(x)) - x\cos(x) - 1 + \sin(x)$

Can be neglected.

$y = C.F + P.I$

4). $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$

Let $\frac{d}{dx} = D$.

The given equation becomes

$$(D^2 - 3D + 2)y = \frac{e^x}{1+e^x}$$

Consider the Auxillary Equation

$$D^2 - 3D + 2 = 0.$$

$$D = 1, 2.$$

$\therefore$ Complimentary Function $C.F = C_1 e^x + C_2 e^{2x}$

Let $y_1 = e^x$

$y_2 = e^{2x}$

$$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^x \\ e^{2x} & 2e^{2x} \end{vmatrix}$$

$$= e^{3x}$$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -e^x \int \frac{e^{2x}}{e^{3x}} \left(\frac{e^x}{1+e^x}\right) dx + e^{2x} \int \frac{e^x}{e^{3x}} \left(\frac{e^x}{1+e^x}\right) dx$$

Let $I_1 = \int \frac{1}{1+e^x} dx$; $I_2 = \int \frac{e^{-x}}{1+e^x} dx$

$I_1 = \int \frac{e^{-x}}{1+e^{-x}} dx$; I_2 $e^{-x} = t$;

$-e^{-x} dx = dt$

$I_1 = -\log(1+e^{-x})$;

$; I_2 = \int \frac{-dt}{1+\frac{1}{t}}$

$; I_2 = -\int \frac{t}{1+t}$

$; I_2 = -\int 1 - \frac{1}{1+t}$

$; I_2 = \log(1+t) - t$

$I_1 = -\log(1+e^{-x})$ $\quad ; I_2 = \log(1+e^{-x}) - e^{-x}$

$P.I = -e^{x}(-\log(1+e^{-x})) + e^{2x}(\log(1+e^{-x}) - e^{-x})$

$P.I = e^{x}(\log(1+e^{x}) - x) + e^{x} + e^{2x}(\log(1+e^{x}) - x)$

$P.I = \log(1+e^{x})x(e^{x}+e^{2x}) \cancel{-e^{2x}} - xe^{x} + e^{x} - xe^{2x}$

5. $(D^2+2D+1)y = e^{-x}\log(x)$

The Auxillary Equation is

$D^2+2D+1=0$

$D = -1, -1.$

$\therefore$ Complimentary Function $C.F = (c_1 x + c_2)e^{-x}$

$C.F = c_1 x e^{-x} + c_2 e^{-x}$

$y_1 = x e^{-x}$

$y_2 = e^{-x}$

$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} xe^{-x} & e^{-x}(1-x) \\ e^{-x} & -e^{-x} \end{vmatrix}$

$= -e^{-2x}$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -(xe^{-x}) \int \frac{e^{-x}(e^{-x}\log(x))}{-e^{-2x}} dx + e^{-x} \int \frac{xe^{-x}(e^{-x}\log(x))}{-e^{-2x}} dx$$

$$P.I = xe^{-x} \int \log(x)\,dx - e^{-x} \int x\log(x)\,dx$$

$$= xe^{-x}(x(\log x - 1)) - e^{-x}\left(\frac{x^2}{4}(2\log x - 1)\right)$$

$$P.I = \frac{x^2 e^{-x}}{4}(2\log(x) - 3)$$

$$y = C.F + P.I$$

6) $(D^2 - 3D + 2)y = \frac{1}{1+e^{-x}}$

The Auxillary Equation is

$$D^2 - 3D + 2 = 0$$

$$D = 1, 2.$$

$\therefore$ Complimentary function $C.F = c_1 e^x + c_2 e^{2x}$

$$y_1 = e^x$$

$$y_2 = e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^x \\ e^{2x} & 2e^{2x} \end{vmatrix} = e^{3x}$$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -e^{x} \int \frac{e^{2x}}{e^{3x}} \left(\frac{1}{1+e^{-x}}\right) dx + e^{2x} \int \frac{e^{x}}{e^{3x}} \left(\frac{1}{1+e^{-x}}\right) dx$$

$$P.I = -e^{x} \int \frac{1}{1+e^{x}} dx + e^{2x} \int \frac{e^{-x}}{1+e^{x}} dx$$

$$P.I = -e^{x}(-\log(1+e^{-x})) + e^{2x}(\log(1+e^{-x}) - e^{-x})$$

$$= +e^{x}(\log(1+e^{x}) - x) + e^{2x}(\log(1+e^{x}) - x - e^{-x})$$

$$= (e^{x} + e^{2x}) \log(1+e^{x}) - xe^{x} - xe^{2x} - e^{x}$$

$$y = CF + P.I$$

7). $(D^2 - 1) y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

Auxillary Equation is

$$D^2 - 1 = 0$$

$$D = \pm 1$$

$\therefore$ Complimentary Function $C.F = c_1 e^{x} + c_2 e^{-x}$

$$y_1 = e^{x}$$

$$y_2 = e^{-x}$$

$$W = \begin{vmatrix} e^{x} & e^{x} \\ e^{-x} & -e^{-x} \end{vmatrix} = -2.$$

The particular integral $P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$P.I = -e^{x}\int \frac{e^{-x}\left(e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right)}{-2}\,dx$$

$$+\, e^{-x}\int \frac{e^{x}\left(e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right)}{-2}\,dx$$

$$P.I = \frac{e^{x}}{2}\int e^{-2x}\sin(e^{-x}) + e^{-x}\cos(e^{-x})$$

$$-\frac{e^{-x}}{2}\int \sin(e^{-x}) + e^{x}\cos(e^{-x})$$

Let $I_1 = \int e^{-2x}\sin(e^{-x}) + e^{-x}\cos(e^{-x})\,dx$

$I_2 = \int \sin(e^{-x}) + e^{x}\cos(e^{-x})\,dx$

$$I_1 = \int e^{-2x}\sin(e^{-x}) + e^{-x}\cos(e^{-x})\,dx$$

Let $e^{-x} = t$

$-e^{-x}\,dx = dt$

$$I_1 = -\int t\sin t + \cos t\,dt$$

$$= -(\sin t - t\cos t + \sin t)$$

$$= t\cos t - 2\sin t = e^{-x}\cos e^{-x} - 2\sin e^{-x}$$

$$I_2 = \int \sin(e^{-x}) + e^{x}\cos(e^{-x})\,dx$$

Let $e^{-x} = t$

$-e^{-x}\,dx = dt$

$dx = \frac{-1}{t}\,dt$

$$I_2 = \int -\frac{1}{t}\left(\sin t + \frac{1}{t}\cos t\right)dt$$

$$= -\int \frac{t\sin t + \cos t}{t^2}\,dt$$

$$I_2 = \int \sin(e^{-x}) + e^{x}\cos(e^{-x})\, dx$$

$$\frac{d}{dx}\left(e^{x}\cos(e^{-x})\right) = e^{x}\cos(e^{-x}) + \sin(e^{-x})$$

$$I_2 = e^{x}\cos(e^{-x})$$

$$PI = \frac{e^{x}}{2}\left(e^{-x}\cos e^{-x} - 2\sin e^{-x}\right) + \frac{e^{-x}}{2}\left(e^{x}\cos e^{-x}\right)$$

$$= \frac{\cos e^{-x}}{2} - e^{x}\sin e^{-x} - \frac{\cos e^{-x}}{2}$$

$$= -e^{x}\sin e^{-x}$$

$$y = C.F + P.I$$

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Solve the following Cauchy's and Legendre's differential Equation.

1. $x^2 y'' - 3xy' + 5y = x^2 \sin(\log x)$

Put $x = e^t$

$\log x = t$.

then

$$[D(D-1) - 3D + 5]y = e^{2t} \sin t.$$

where $D = \frac{d}{dt}$.

$$(D^2 - 4D + 5)y = e^{2t} \sin t.$$

Auxillary Equation

$D^2 - 4D + 5 = 0$.

$D = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$

$D = \frac{4 \pm \sqrt{16 - 20}}{2}$

$D = 2 \pm i$

Hence

$$C.F = e^{2t}(c_1 \cos t + c_2 \sin t)$$

$$P.I = e^{2t} \frac{\sin t}{(D+2)^2 - 4(D+2) + 5}.$$

$$= e^{2t} \frac{\sin t}{D^2 + 4 - 8 + 5}$$

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$$P.I = e^{2t} \frac{\sin t}{D^2+1}$$

$$= e^{2t} \frac{t \sin t}{2D}.$$

$$= \frac{t e^{2t}}{2} (-\cos t)$$

$$P.I = \frac{-t e^{2t} \cos t}{2}.$$

$t = \log x$

$$P.I = -\frac{(\log x)(x^2)\cos(\log x)}{2}$$

$$P.I = -\frac{x^2 \cos(\log x) \log x}{2}$$

$$C.F = x^2(c_1\cos(\log x) + c_2\sin(\log x))$$

$y = C.F + P.I$

$$\boxed{y = x^2\sin(\log x) - \frac{x^2}{2}(\log x)\cos(\log x) + x^2\cos(\log x)}$$

2) $x^2y'' - xy' + 4y = \cos(\log x) + x\sin(\log x)$

Put

$x = e^t$

$t = \log x$

then where $D = \frac{d}{dt}$.

$[D(D-1) - D + 4]\, y = \cos t + e^t \sin t.$

$= [D^2 - 2D + 4]\, y = \cos t + e^t \sin t.$

Auxillary Eq^n

$D^2 - 2D + 4 = 0.$

$$D = \frac{2 \pm \sqrt{4-4(1)(4)}}{2} = \frac{2 \pm \sqrt{-12}}{2}$$

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$D = 1 \pm \sqrt{3}\, i$

$D = 1 \pm \sqrt{3}\, i$

$C.F = e^{t}\left(c_1 \cos\sqrt{3}\, t + c_2 \sin\sqrt{3}\, t\right)$

$$P.I = \frac{\cos t + e^{t}\sin t}{D^2 - 2D + 4}$$

$$= \frac{\cos t}{D^2 - 2D + 4} + e^{t}\,\frac{\sin t}{(D+1)^2 - 2(D+1) + 4}$$

$$= \frac{\cos t}{-2D + 3} + e^{t}\,\frac{\sin t}{D^2 + 3}$$

$$= \frac{(-2D-3)\cos t}{4D^2 - 9} + e^{t}\,\frac{\sin t}{2}$$

$$= \frac{-2\sin t - 3\cos t}{4D^2 - 9} + \frac{e^{t}}{2}\sin t$$

$$= \frac{2}{13}\sin t + \frac{3}{13}\cos t + \frac{e^{t}}{2}\sin t.$$

$t = \log x$

$$P.I = \frac{2}{13}\sin(\log x) + \frac{3}{13}\cos(\log x) + \frac{x}{2}\sin(\log x)$$

Complete $sol^n = C.F + P.I$

$$y = x\left(c_1\cos\sqrt{3}\log x + c_2 \sin\sqrt{3}\log x\right) + \frac{2}{13}\sin(\log x) + \frac{3}{13}\cos(\log x) + \frac{x}{2}\sin(\log x)$$

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3.) $x^2 y'' - 2xy' + 2y = (\log x)^2 - \log x^2$

Put $x = e^t$ $(t = \log x)$

$(D(D-1) - 2D + 2)y = t^2 - \log e^{2t}$

or $\quad$ where $D = \frac{d}{dt}$.

$(D^2 - 3D + 2)y = t^2 - 2t.$

Auxillary Equation is

$D^2 - 3D + 2 = 0$

$D = \frac{3 \pm \sqrt{9 - 4(1)(2)}}{2}$

$= \frac{3 \pm \sqrt{9-8}}{2}$

$D = \frac{3 \pm 1}{2} \Rightarrow D = 2$ and $D = 1$

$C.F = c_1 e^{2t} + c_2 e^{t}$

$P.I = \frac{t^2 - 2t}{(D^2 - 3D + 2)}$

$= \frac{t^2 - 2t.}{2\left(1 + \frac{D^2 - 3D}{2}\right)}$

$= \frac{1}{2}\left(1 - \left(\frac{D^2 - 3D}{2}\right)\right)(t^2 - 2t)$

$= \frac{1}{2}\left[t^2 - 2t - \frac{1}{2}\left((D^2(t^2 - 2t)) - 3D(t^2 - 2t)\right)\right]$

$= \frac{1}{2}\left[t^2 - 2t - 1 + \frac{3}{2}(2t - 2)\right]$

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$$P.I = \frac{1}{2}\left[t^2 - 2t - 1 + 3t - 3\right]$$

$$P.I = \frac{1}{2}\left[t^2 + t - 4\right]$$

$$t = \log x$$

$$P.I = \frac{1}{2}\left[(\log x)^2 + \log x - 4\right]$$

$$y = C.F + P.I$$

$$y = \cancel{C.F}\ c_1 x^2 + c_2 x + \frac{1}{2}\left[(\log x)^2 + \log x - 4\right]$$

4.) $(x+1)^3 y'' + 3(x+1)^2 y' + (x+1)y = 6\log(x+1)$

$$(x+1)^2 y'' + 3(x+1)y' + y = \frac{6}{(x+1)}\log(x+1)$$

Put $x+1 = e^t$

$\log(x+1) = t.$

then

$$\left[D(D-1) + 3D + 1\right]y = \frac{6}{e^t}t$$

$$\left[D^2 + 2D + 1\right]y = 6te^{-t} \quad \text{where } D = \frac{d}{dt}$$

Auxillary Eq^n is

$$D^2 + 2D + 1 = 0,$$

$$(D+1)^2 = 0.$$

$$D = -1, -1$$

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$$C.F = e^{-t}(c_1 \cancel{t} + c_2)$$

$$= e^{-t}(c_1 t + c_2)$$

$$P.I = \frac{6 e^{-t}\, t}{(D+1)^2}$$

$D \to D-1$

$$= 6 e^{-t} \frac{t}{D^2}$$

$$= 6 e^{-t} \iint t .$$

$$= 6 e^{-t} \int \frac{t^2}{2}$$

$$3 e^{-t} \int t^2$$

$$\boxed{P.I = e^{-t} t^3}$$

$\cancel{t = e}$

$x + 1 = e^t$

$$C.S = C.F + P.I$$

$$y = \frac{1}{(x+1)}\left(c_1 \log(x+1) + c_2\right) + \frac{1}{(x+1)}\left(\log(x+1)\right)^3$$

5) $(2x+3)^2 y'' + (2x+3) y' - 2y = 24x^2$

Put $2x + 3 = e^t$. $2x = e^t - 3$

$x = \frac{e^t - 3}{2}$

$$\left[4D(D-1) + 2D - 2\right] y = 24\left(\frac{e^t - 3}{2}\right)^2$$

$$= \left[4D^2 - 2D - 2\right] y = 6\,(e^t - 3)^2$$

Auxillary Eq^n

$$2D^2 - D - 1 = 0.$$

$$D = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{4} = \frac{1 \pm \sqrt{9}}{4}$$

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$D = 1, \frac{-1}{2}$

$C.F = c_1 e^{t} + c_2 e^{-t/2}$

$P.I = 6 \left[\frac{e^{2t} - 6e^{t} + 9}{4D^2 - 2D - 2} \right]$

$2 - 1 - 1$

$2 - 2$

$= 3 \left[\frac{e^{2t}}{2D^2 - D - 1} - \frac{6e^{t}}{2D^2 - D - 1} + \frac{9e^{0t}}{2D^2 - D - 1} \right]$

$2(4) - 2 - 1$

$2 - 0$

$2(1) - 2$

$8 - 3$

$= 5$

$= 3 \left[\frac{e^{2t}}{5} - \frac{6te^{t}}{4D - 1} - 9 \right]$

$= 3 \left[\frac{e^{2t}}{5} - 2te^{t} - 9 \right]$

$y = C.F + P.I$

$y = c_1 (2x+3) + \frac{c_2}{\sqrt{2x+3}} - 27 + \frac{3}{5}(2x+3)^{2}$

$- 6 \log(2x+3) \times (2x+3)$

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6.) $(4x+1)^2 y'' + (4x+1) y' + 4y = 8(4x+1)^2$

Put $4x+1 = e^t$

$\log(4x+1) = t.$

$\left[16 D(D-1) + 4D + 4\right] y = 8 e^{2t}$

Auxillary Eq^n

$16D^2 - 12D + 4 = 0.$

$4D^2 - 3D + 1 = 0.$

$D = \dfrac{3 \pm \sqrt{9 - 4(4)(1)}}{2 \times 4}$

$= \dfrac{3 \pm \sqrt{9 - 16}}{8}$

$D = \dfrac{3 \pm \sqrt{7} i}{8} = \dfrac{3}{8} \pm \dfrac{\sqrt{7}}{8} i.$

$C.F = e^{\frac{3}{8}t}\left(c_1 \cos \dfrac{\sqrt{7}}{8} t + c_2 \sin \dfrac{\sqrt{7}}{8} t\right)$

$P.I = \dfrac{8 \quad e^{2t}}{16D^2 - 12D + 4}.$

$= \dfrac{8 \quad e^{2t}}{16(4) - 12(2) + 4}$

$= \dfrac{8 \quad e^{2t}}{44}.$

$4D^2 \; \frac{-4}{-1}$

$-3D$

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$$y = C.F + P.I$$

$$t = \log(4x+1)$$

$$y = (4x+1)^{3/8}\left(c_1 \cos\frac{\sqrt{7}}{8}\log(4x+1) + c_2 \sin\frac{\sqrt{7}}{8}\log(4x+1)\right) + \frac{2}{11}(4x+1)^2$$

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2. Solve the following problem using power series $(1-x^2)y' - y = 0$.

soln:- the coefficient of y' is $1 \neq 0$ at $x=0$

$\therefore x=0$ is an ordinary point of the differential equation.

Assume,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n + \ldots \infty \quad \text{(i)}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n a_n x^{n-1} \ldots \quad \text{(ii)}$$

Substituting for y and y' in the given DE,

$$(1-x^2)\left[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n a_n x^{n-1} \ldots\right] - \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \infty\right] = 0 \quad \text{(iii)}$$

Equating to zero the coefficients of various powers of x,

$[\text{coeff of } x^0]$

$$a_1 - a_0 = 0$$

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$a_1 = a_0$

$2a_2 - a_1 = 0$ [coeff of x^1]

$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$3a_3 - a_1 - a_2 = 0$ [coeff of x^2]

$3a_3 - a_1 - \frac{a_1}{2} = 0$

$3a_3 = \frac{3a_1}{2}$

$a_3 = \frac{a_1}{2} = \frac{a_0}{2}$

$\cancel{4a_4 - 3a_3 = a}$

$-2a_2 + 4a_4 - a_3 = 0$

$-\cancel{2}\cdot\frac{a_1}{\cancel{2}} + 4a_4 - \frac{a_1}{2} = 0$

$4a_4 = \frac{3a_1}{2}$

$a_4 = \frac{3}{8}a_1 = \frac{3}{8}a_0$

Substituting these value in (i)

$y = a_0 + a_0 x + \frac{a_0}{2}x^2 + \frac{a_0}{2}x^3 + \frac{3}{8}a^0 \ldots$

Scanned by CamScanner

$$y = a_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3}{8}x^4 + \cdots\right)$$

Scanned by CamScanner

3. $y' - 2xy = 0 \longrightarrow (1)$

$$\frac{dy}{dx} - 2x \cdot y = 0$$

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$\Rightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

Subst in (1) :

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots 6a_6 x^5$$
$$- 2x\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right] = 0$$

Const :

$a_1 = 0$

x :

$2a_2 - 2a_0 = 0$

$a_2 = a_0$

x^2 :

$3a_3 - 2a_1 = 0$

$a_3 = 0$

$\Rightarrow a_5, a_7, a_9, \ldots = 0.$

x^3 : $4a_4 - 2a_2 = 0$

$a_4 = \frac{a_0}{2}$

x^5 : $6a_6 - 2a_4 = 0$

$a_6 = \frac{a_4}{3} = \frac{a_0}{3 \cdot 2} = \frac{a_0}{3!}$

$\Rightarrow a_8 = \frac{a_0}{4!}$ and so on....

$$\Rightarrow y = a_0\left[1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right]$$

$$= a_0\left[1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots\right]$$

$$\Rightarrow \boxed{y = a_0 e^{x^2}}$$