UNIT - 1

MATRICES AND GAUSSIAN ELIMINATION

Introduction

One of the most fundamental and important problems in linear algebra is finding a solution x to the equation Ax = b.

Definition :

A system of m equations in n unknowns can be represented in matrix form as

$$Ax = b$$

where A is called the ***coefficient matrix***, x is called the ***matrix of unknowns*** and b is the ***column vector*** consisting of the right side constants.

Definition :

A ***solution*** of the system Ax = b is a set of values (x_1, x_2, …., x_n) that satisfies all the equations present in the system. Any system of equations Ax = b has either

1. a unique solution or
2. infinity of solutions or
3. no solution

Definition:

The system $Ax = b$ is said to be *consistent* if it possesses a solution. Otherwise it is called *inconsistent*.

The Geometry Of Linear Equations

Consider a system of two equations in two unknowns

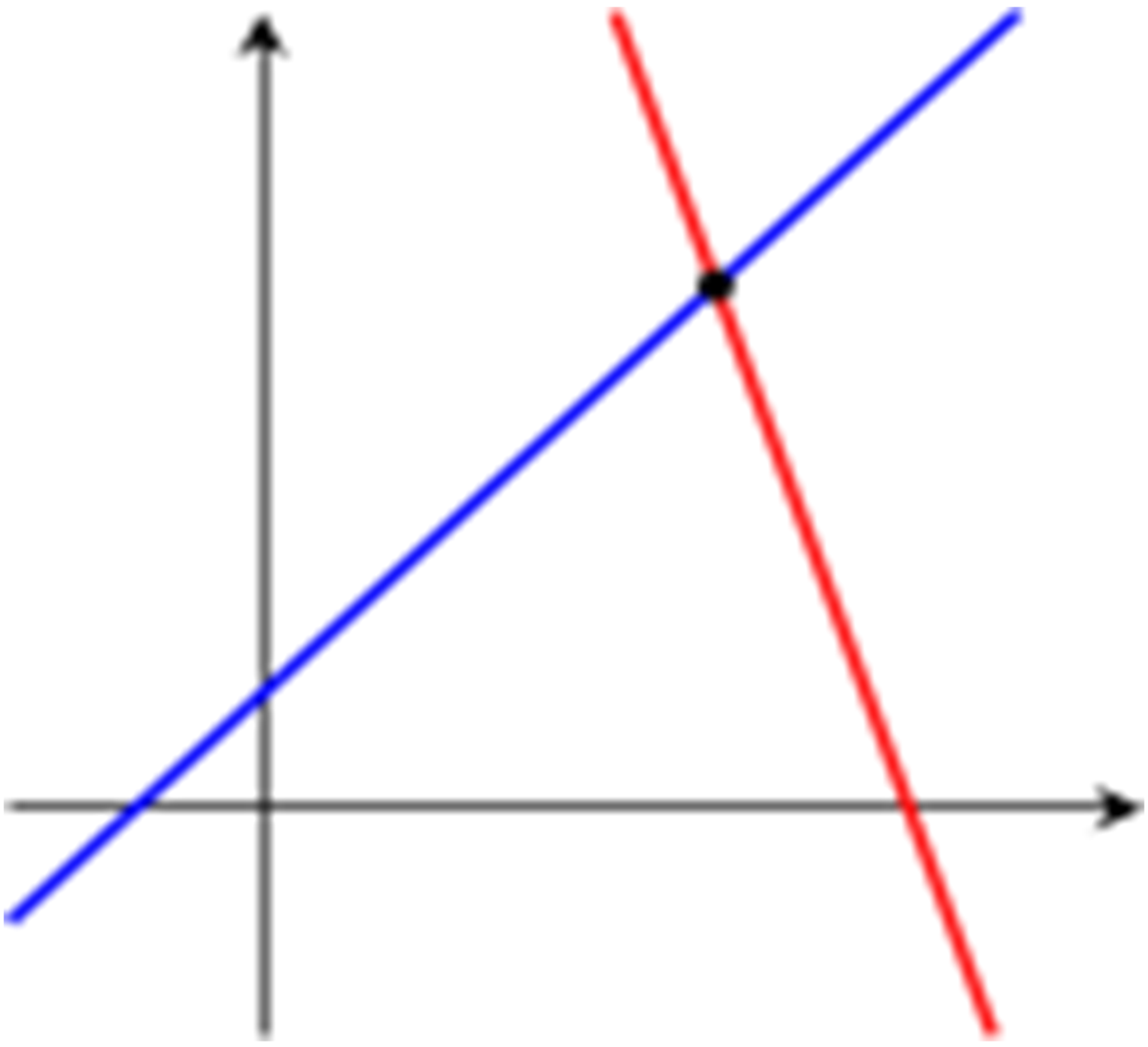
say x and y.

We can look at the system by rows or by columns.

ROW PICTURE

The first approach concentrates on the separate equations which produce two straight lines in two dimensions. The point of intersection lies on both the lines and it is the *only solution* to both equations.

TWO INTERSECTING LINES



example

$$2x - y = 0$$
$$-x + 2y = 3$$

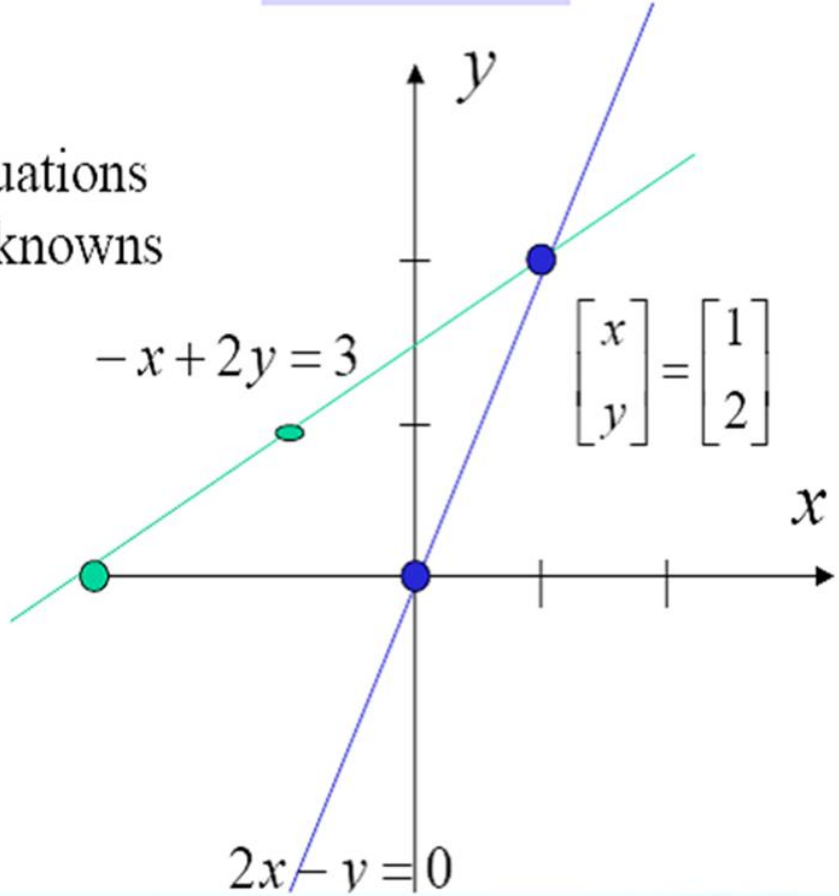
Two equations
Two unknowns

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

matrix

$$AX = b$$

Row picture



COLUMN PICTURE

The second approach looks at the columns of the linear system.

The problem here is to find the ***combination of the column vectors*** on the left side that produces the vector on the right side.

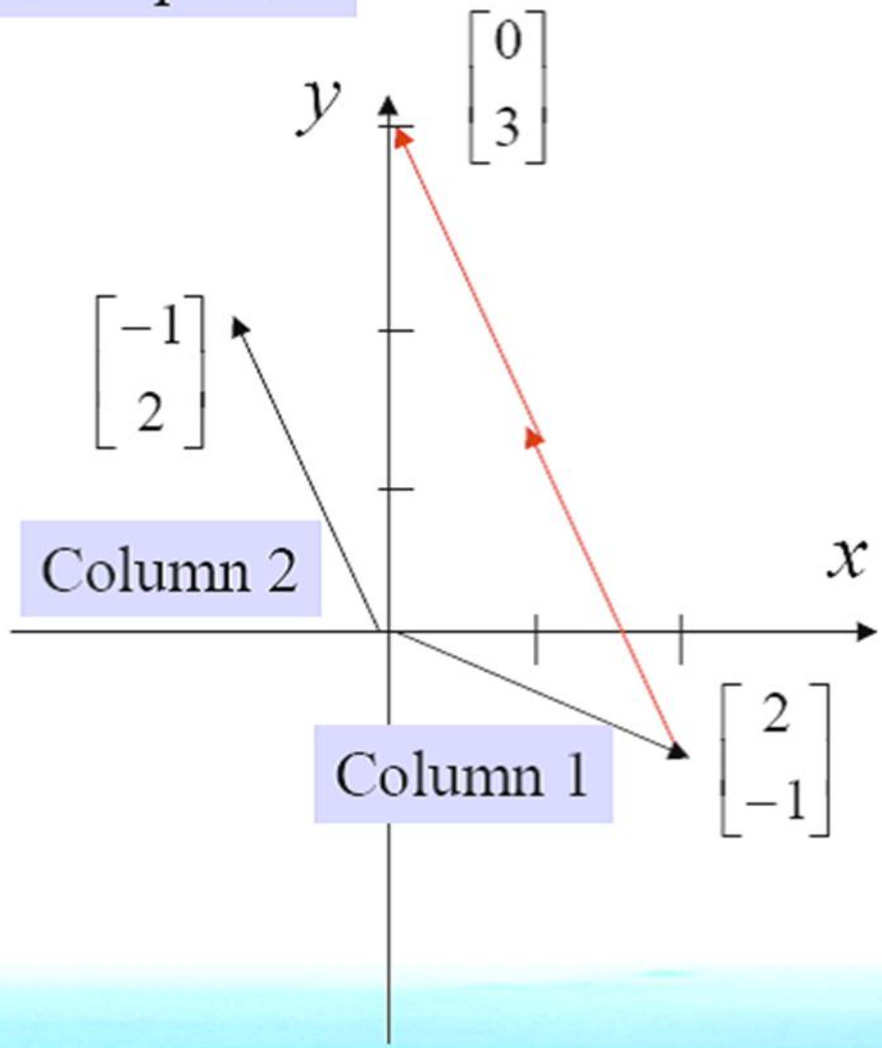
column picture

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Linear combination of columns

$$1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Right numbers to produce $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$



What are all the combination ?

Consider a system of 3 equations in 3 unknowns

$$2x - y = 0$$

$$-x + 2y - z = -1$$

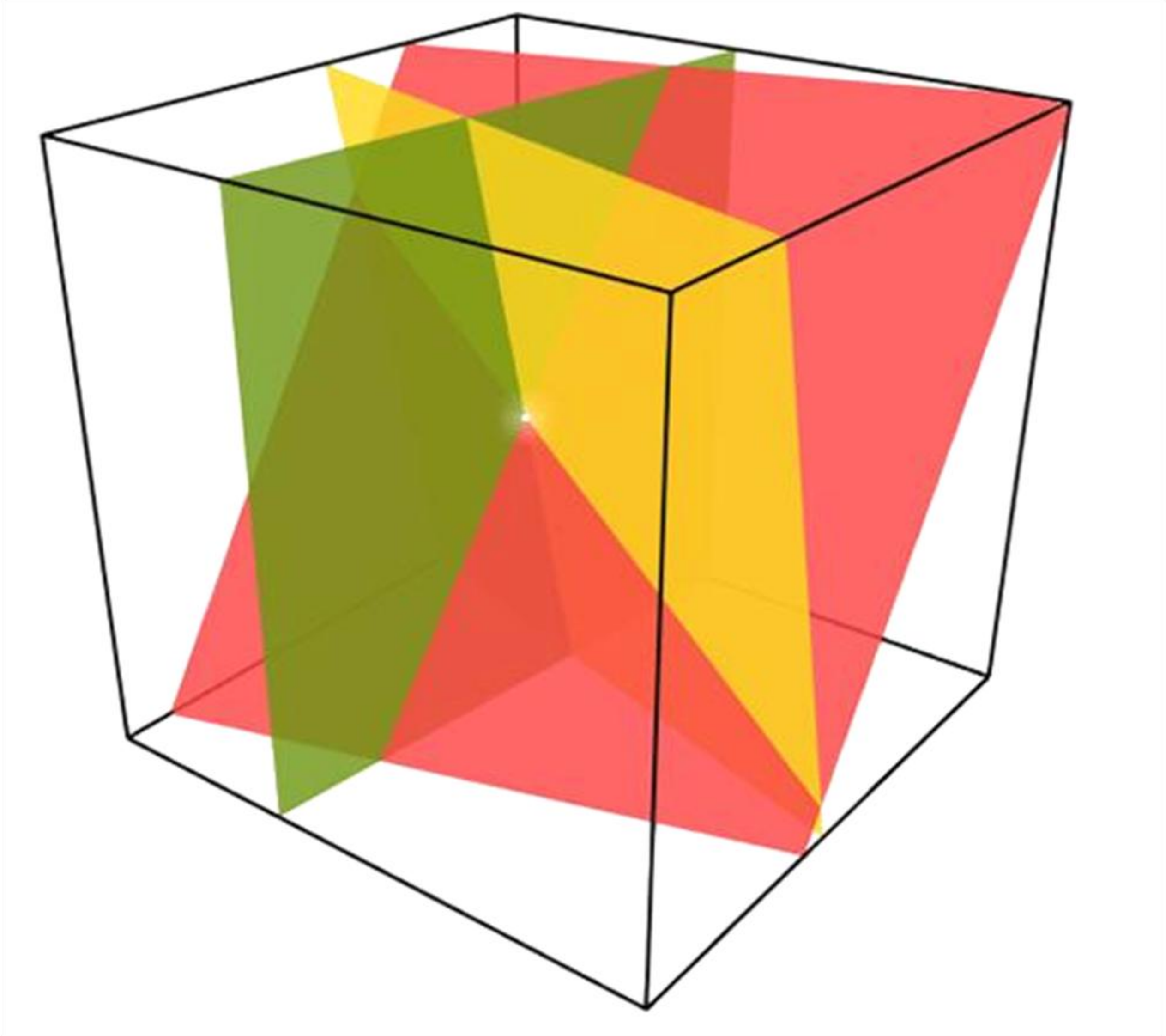
$$-3y + 4z = 4$$

In matrix form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Each equation describes a plane in three dimensions. The intersection of the first two planes is a straight line. The third plane intersects this line at a point. This point of intersection solves the linear system.

3D Row Picture



3-D COLUMN PICTURE

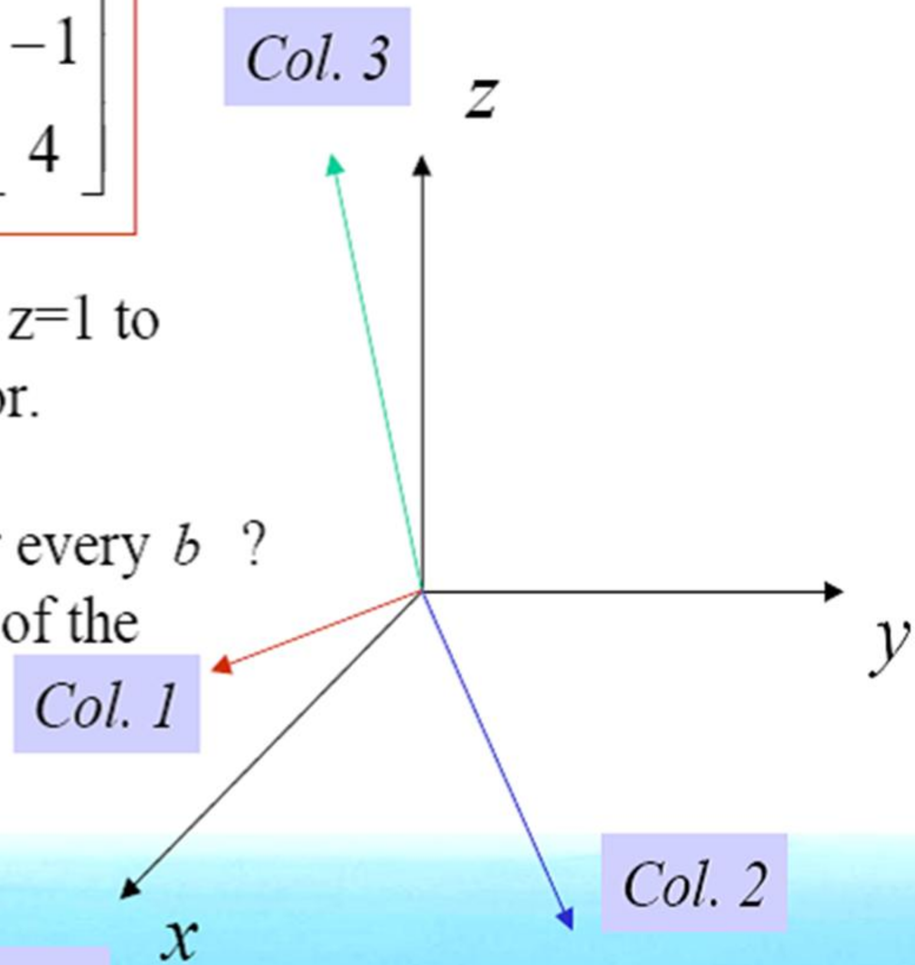
The three columns of the matrix A are three dimensional column vectors. The vector b is identified with the point whose coordinates are (0 , -1, 4).

Change b to $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \rightarrow$ x=1,y=1,z=0

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}$$

Right combination x=0, y=0, z=1 to produce right hand side vector.

1. Can I solve $Ax = b$ for every b ?
2. Do the linear combination of the columns fill 3-D space?



For this matrix A, the answer is yes.

Row Picture:

Intersection of lines / planes.

Column Picture :

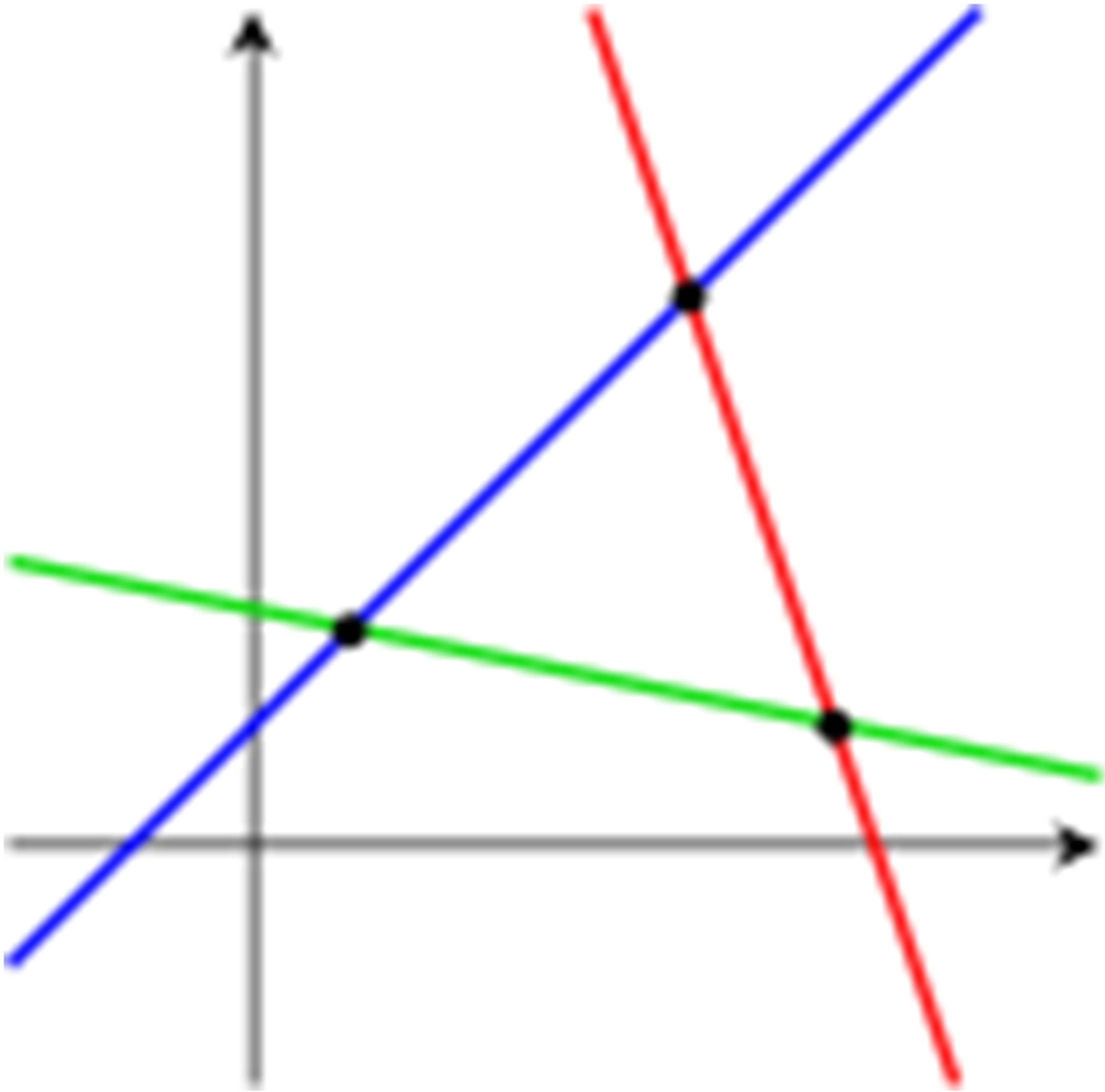
Combination of columns.

The Singular Case

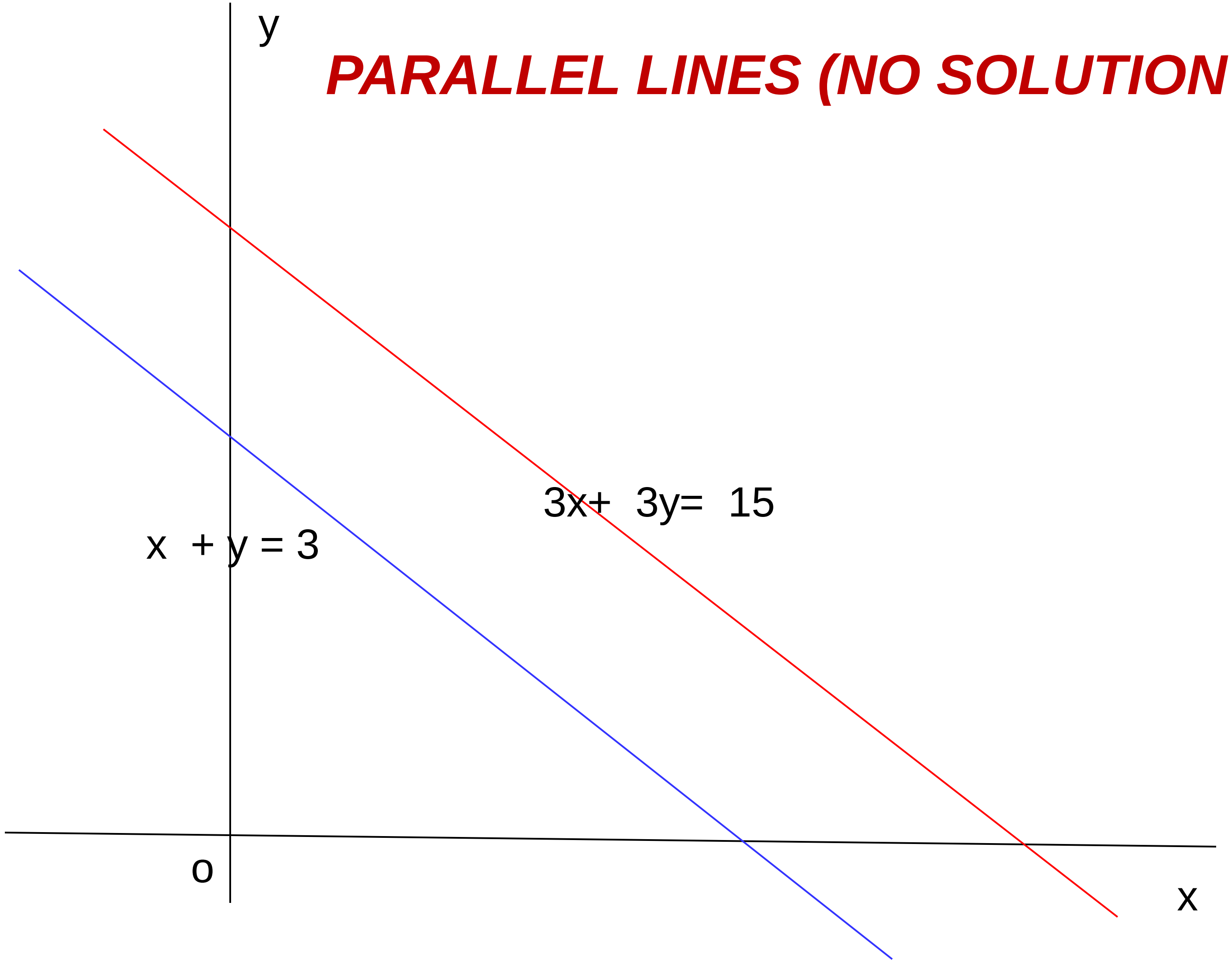
In two dimensions, if the lines are parallel or if they coincide then the number of solutions is none or too many. In this case, the matrix A will have dependent column (s) / row (s) and it is called a *singular matrix*.

For such a matrix, det A = 0.

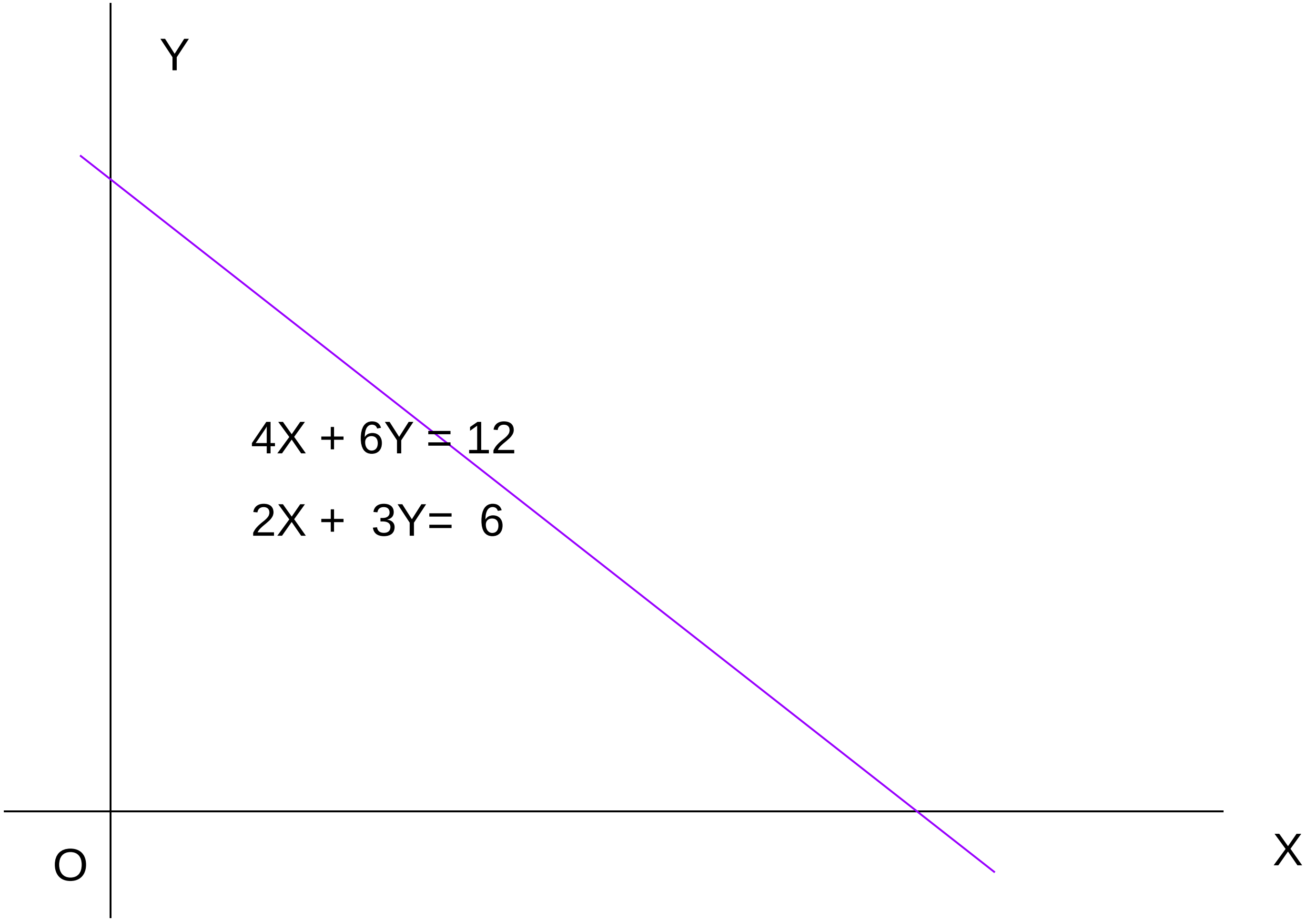
LINES INTERSECTING IN PAIRS



PARALLEL LINES (NO SOLUTION)



COINCIDENT LINES (INFINITY OF SOLUTIONS)



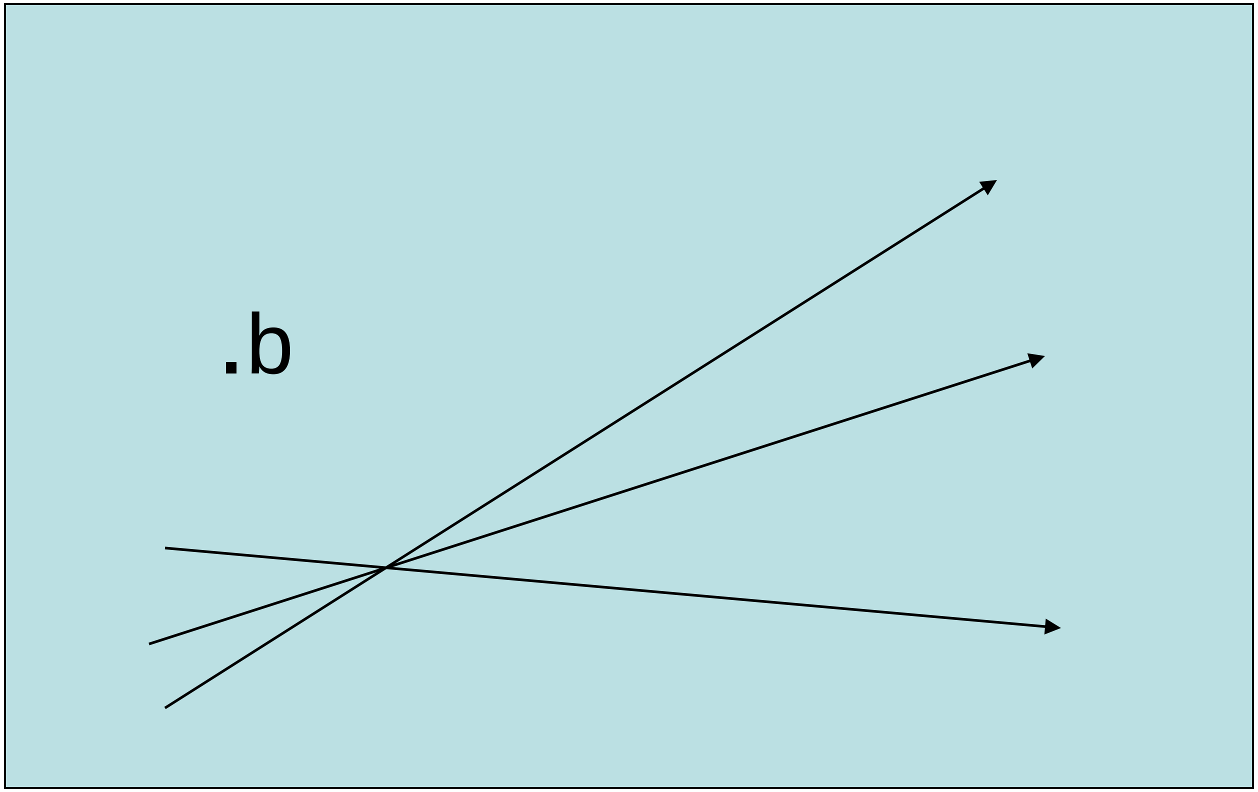
THE SINGULAR CASES IN 3-D

In three dimensions, if the three planes do not intersect then either

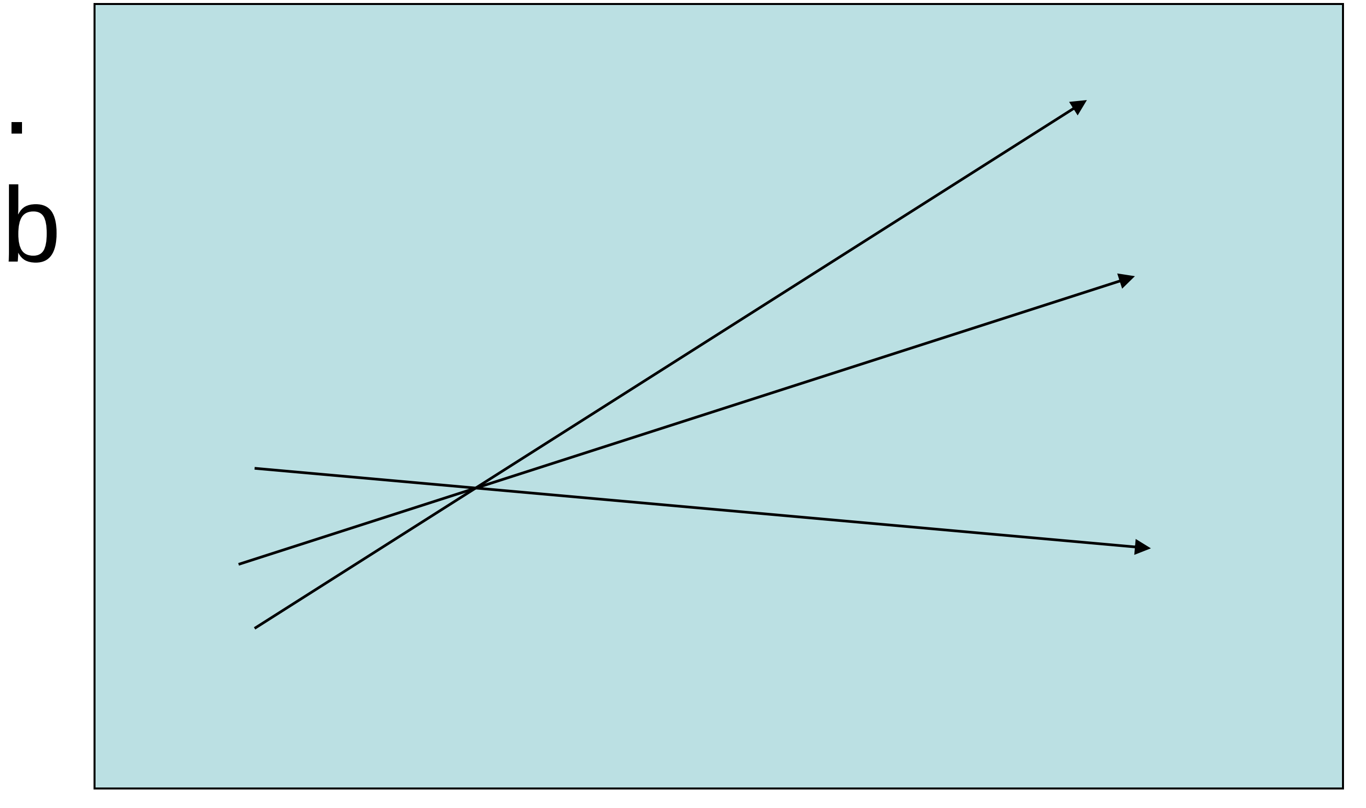
1. **Every pair of planes will intersect in a line and those lines will be parallel or**
2. **The three planes will have a whole line in common or**
3. **All the three planes will be parallel or**
4. **Two planes may be parallel or**
5. **Two planes will overlap and the third intersects them or**
6. **Two planes will overlap and the third will be parallel to them or**
7. **All the three planes will overlap.**

In the column picture, we still have three columns on the left side of the equation but these column vectors lie in a plane. Then, every combination is also in the plane. If the vector b is not in that plane then no solution is possible. If b lies in the plane then there are too many solutions; the three columns can be combined in infinitely many ways to produce b.

b in the plane - infinity of solutions



b not in the plane - no solution



Gaussian Elimination: Solving Systems of Equations

Gaussian elimination is the method which is commonly used to solve large systems of equations.

Consider the following system of 3 equations in 3 unknowns

$$2x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 + 4x_3 = 2$$
$$-4x_1 + 2x_2 + 2x_3 = -6$$

This system can be written using matrix multiplication as follows:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$$

This gives us pivot at the first row i.e 2

$$R_2 = R_2 - (2)R_1$$

$$R_3 = R_3 - (-2)R_1$$

The result after these first two operations is:

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

We continue this to get more zeros.

So

$$R_3 = R_3 - (4)R_2$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

This has created an *upper triangular* matrix, one where all elements
below the main diagonal are zero. The system of equations in matrix form now is

$$Ux = c$$

This system can now be easily solved by ***back-substitution***.
The solution is

$$X_3 = 1$$
$$x_2 = -2.$$
$$X_1 = 1.$$

Gaussian Elimination has produced three pivots 2, 1, -4.

With a full set of 3 pivots there is only one solution. The system is nonsingular and it is solved by forward elimination and back substitution.

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Breakdown of Elimination

If a zero appears in a pivot position, elimination has to stop – either temporarily or permanently. The system might or might not be singular. **I**n many cases this problem can be cured and elim**i**nation can proceed. Such a system **i**s still singular; **I**n other cases a breakdown **i**s unavoidable. Those systems have no solution or infinitely many and a full set of pivots can not be found.

Elimination Matrices

Consider the system of equations

$$2x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 + 4x_3 = 2$$
$$4x_1 + 2x_2 + 2x_3 = -6$$

Gaussian elimination applied on this system gives us three multipliers 2, -2 and 4. These are obtained from the three row transformations that carry A to an upper triangular form U.

The effect of these row transformations is the same as multiplying A by what are called the *elementary matrices* or *elimination matrices*. There is an elimination matrix associated with each row transformation of Gaussian elimination. In our example above, there are three elimination matrices that take A to an upper triangular structure.

They are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = F$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = G$$

It can be seen that GFEAx = GFEb or Ux = c where the product GFEA = U.

Triangular factors and row exchange

The triangular factorization can be written A =LDU, where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & -2 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} = LDU$$

Row exchange and permutation matrices

We need to exchange rows whenever we meet a zero at pivot position

Zero in the pivot position $\begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Exchange rows

$$\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

We can exchange rows by multiplying with matrices called permutation matrices eg:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } PA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$$

In case of a 3 x 3 matrix

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3/2 & 3/2 \\ 0 & 1 & 7/5 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix P has the same effect on b exchanging b_1 and b_2. The new system is

$$PAX = Pb.$$

The unknowns u and v are not reversed in the row exchange. A *permutation matrix* P has the same rows as the identity, in some order.The product of two permutation matrices is another permutation.

There are n! = (n)(n-1).....(1) permutation of size n.

So for a 3 by 3 square matrices there can be 3!= (3)(2)(1)=6.

$$I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad P = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \quad P = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

$$P = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \quad P = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \quad P = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

The lower triangular matrix L in A = LDU has the following characteristics:

1. It has 1's along the principal diagonal
2. The multipliers obtained from Gaussian elimination in their respective places, below the diagonal.

The diagonal matrix D has the pivots on its diagonal.

The upper triangular matrix U has 1's on its diagonal.

INVERSE OF A MATRIX

Definition:

The ***inverse*** of a square matrix A is the matrix B such that AB = I and BA = I**.** We write $B = A^{-1}$.

Note, however, that A^{-1} might not exist. If it does, A is said to be *invertible*, and its inverse A^{-1} will be unique. It can also be shown that the product of invertible matrices has an inverse, which is the product of the inverses in reverse order: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

This can be extended to three or more matrices.

Gauss-Jordan method of finding A^{-1}

We now find the inverse of a matrix by a set of row operations that transform **A** to **I and then I** to **A**$^{-1}$.

Suppose that we want to find the inverse of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & 2 & 2 \end{bmatrix}$$

We will start by augmenting **A** with the identity matrix, then we will perform row operations to reduce **A** to upper triangular form:

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ -4 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 1 & 0 \\ 0 & 4 & 4 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & -2 & 1 & 0 \\ 0 & 0 & -4 & : & 10 & -4 & 1 \end{bmatrix}$$

We thus have arrived at [A I] ~ [U L^{-1}]

We now continue the procedure by making the coefficients above each of the pivots zeros. Thus we get

$$\begin{bmatrix} 2 & 0 & -1 & : & 3 & -1 & 0 \\ 0 & 1 & 2 & : & -2 & 1 & 0 \\ 0 & 0 & -4 & : & 10 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & : & 1/2 & 0 & -1/4 \\ 0 & 1 & 0 & : & 3 & -1 & 1/2 \\ 0 & 0 & -4 & : & 10 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 1/4 & 0 & -1/8 \\ 0 & 1 & 0 & : & 3 & -1 & 1/2 \\ 0 & 0 & 1 & : & -5/2 & 1 & -1/4 \end{bmatrix}$$

This transforms [A I] to [I A^{-1}]

The matrix on the right is $\mathbf{A}^{-1}$, the inverse of **A.**

Note that our algorithm would have failed if we had encountered an incurable zero pivot, *i.e.* if **A** was singular.

So the inverse matrix A is :

$$\begin{bmatrix} 1/4 & 0 & -1/8 \\ 3 & -1 & 1/2 \\ -5/2 & 1 & -1/4 \end{bmatrix}$$

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Note :

1. The inverse of a matrix A exists if and only **i**f elimination produces n pivots (row exchanges allowed).
2. **If** A is invertible then $Ax = 0$ can have only the zero solution $x = 0$.
3. **If** A is invertible then the determinant of A is the product of the pivots.

The Transpose of a Matrix

Definition :

The ***transpose*** of a matrix A of order m x n is the matrix A^T obtained from A by interchanging its rows and columns.

Note :

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(AB)^T = B^T A^T$
4. $(A^{-1})^T = (A^T)^{-1}$ provided A^{-1} exists

Symmetric Matrices

Definition :

A ***symmetric matrix*** is a matrix that equals its own transpose. i.e $A = A^T$.

Note :

1. Symmetric matrices are necessarily square.
2. A symmetric matrix may or may not be invertible. If it is invertible, its inverse is also symmetric.
3. If A is symmetric then $A = LDU$ where $L = U^T$.
4. For any A, both AA^T and A^TA are symmetric.