Unit II: Linear Algebra

Question Bank

1 Check whether the following are linear transformations/operators where v denotes the vector (v_1, v_2, v_3, v_4) belonging to R^4.

(a) $T(v) = (v_1+v_2, v_3+2, v_4)$

Ans. NO. The observation that T(0) = (0,2,0) rules out the possibility that T is a linear transformation. Thus T is not a linear transformation.

(b) $T(v) = v \cdot v_0$ (inner product) where v_0 is a fixed vector in R^4.

Ans. YES. This is inner product of two (column) vectors. Thus we can verify $T(v + w) = (v + w) \cdot v_0 = v \cdot v_0 + w \cdot v_0 = T(v) + T(w)$, because inner product operation distributes over vector addition.
$T(cv) = (cv) \cdot v_0 = c(v \cdot v_0) = c\,T(v)$ because constant multiplication commutes with the inner product operation. This T is a linear transformation.

2 Let $T(x_1, x_2, x_3, x_4) = (2x_1+x_3+x_4, x_1+2x_2+x_4, x_1+x_2+2x_3, x_2+x_3+2x_4)$ w.r.t. the basis $B_1 = \{(1,0,0,0), (2,1,0,0), (1,2,1,0), (1,1,2,1)\}$ for domain and $B_2 = \{(0,0,0,1),(0,0,1,2),(0,1,2,1),(1,2,1,1)\}$ for co-domain.

Ans. As given,
T(1,0,0,0) = (2,1,1,0),
T(2,1,0,0) = (4,4,3,1),
T(1,2,1,0) = (3,5,5,3),
T(1,1,2,1) = (5,4,6,5).

Thus we have

$$N=\begin{pmatrix} T(v_1) & T(v_2) & T(v_3) & T(v_4)\end{pmatrix}=\begin{pmatrix} 2 & 4 & 3 & 5\\ 1 & 4 & 5 & 4\\ 1 & 3 & 5 & 6\\ 0 & 1 & 3 & 5\end{pmatrix} \text{ ; and}$$

$$B=\begin{pmatrix} w_1 & w_2 & w_3 & w_4\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 1 & 2 & 1\\ 1 & 2 & 1 & 1\end{pmatrix} .$$

We carry out the usual process starting with [B:N] and row-reducing:

$$[B:N]=\begin{pmatrix} 0 & 0 & 0 & 1 & : & 2 & 4 & 3 & 5\\ 0 & 0 & 1 & 2 & : & 1 & 4 & 5 & 4\\ 0 & 1 & 2 & 1 & : & 1 & 3 & 5 & 6\\ 1 & 2 & 1 & 1 & : & 0 & 1 & 3 & 5\end{pmatrix}$$

R_1 <--> R_4 and R_2 <--> R_3 give

$$\begin{pmatrix} 1 & 2 & 1 & 1 & : & 0 & 1 & 3 & 5 \\ 0 & 1 & 2 & 1 & : & 1 & 3 & 5 & 6 \\ 0 & 0 & 1 & 2 & : & 1 & 4 & 5 & 4 \\ 0 & 0 & 0 & 1 & : & 2 & 4 & 3 & 5 \end{pmatrix}$$

Now, R_1 <-- $R_1 - 2R_2$ gives

$$\begin{pmatrix} 1 & 0 & -3 & -1 & : & -2 & -5 & -7 & -7 \\ 0 & 1 & 2 & 1 & : & 1 & 3 & 5 & 6 \\ 0 & 0 & 1 & 2 & : & 1 & 4 & 5 & 4 \\ 0 & 0 & 0 & 1 & : & 2 & 4 & 3 & 5 \end{pmatrix}$$

R_1 <-- $R_1 + 3R_3$ and R_2 <-- $R_2 - 2R_3$ give

$$\begin{pmatrix} 1 & 0 & 0 & 5 & : & 1 & 7 & 8 & 5 \\ 0 & 1 & 0 & -3 & : & -1 & -5 & -5 & -2 \\ 0 & 0 & 1 & 2 & : & 1 & 4 & 5 & 4 \\ 0 & 0 & 0 & 1 & : & 2 & 4 & 3 & 5 \end{pmatrix}$$

R_1 <-- $R_1 - 5R_4$, R_2 <-- $R_2 + 3R_4$ and R_3 <-- $R_3 - 2R_4$ give

$$\begin{pmatrix} 1 & 0 & 0 & 0 & : & -9 & -13 & -7 & -20 \\ 0 & 1 & 0 & 0 & : & 5 & 7 & 4 & 13 \\ 0 & 0 & 1 & 0 & : & -3 & -4 & -1 & -6 \\ 0 & 0 & 0 & 1 & : & 2 & 4 & 3 & 5 \end{pmatrix} .$$

Thus the matrix of T w.r.t, the given bases is:

$$M_T = \begin{pmatrix} -9 & -13 & -7 & -20 \\ 5 & 7 & 4 & 13 \\ -3 & -4 & -1 & -6 \\ 2 & 4 & 3 & 5 \end{pmatrix} .$$

3 Let S be rotation by 90 degrees and T the reflection across the line y = x. For v = (2,5) , calculate TS(v) and ST(v).

Ans. TS(2,5) = T(-5,2) = (2,-5) and ST(2,5) = S(5,2) = (-2,5).

4 Find the range, rank, kernel, nullity of T where $T(x_1, x_2, x_3, x_4) = (x_1+x_4, x_1+x_2, x_2+x_3, x_3+x_4)$

Ans. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. The range of T = C(A). On row reducing, we get by the appropriate sequence of row operations that

$$\begin{pmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\sim\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&1&1&0\\0&0&1&1\end{pmatrix}\sim\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&1&1\end{pmatrix}\sim\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}$$

which leads to pivots in first three columns, showing that the range of T is the span of {(1,1,0,0),(0,1,1,0),(0,0,1,1)} and the rank is consequently 3.

The kernel of T = N(A). Here the above is already in RREF, therefore the kernel of T is spanned by {(-1,1,-1,1)} and consequently the nullity is 1.

5 Find a set of vectors which span $S^{\perp}$ for S = {(3,1,4), (1,2,3), (5,2,3)}

Ans. If $A=\begin{pmatrix}3&1&5\\1&2&2\\4&3&3\end{pmatrix}$, then $S^{\perp} = C(A)^{\perp} = N(A^T)$ which is the left-null space of A. We now start with A^T and row-reduce:

$$A^T=\begin{pmatrix}3&1&4\\1&2&3\\5&2&3\end{pmatrix}\sim\begin{pmatrix}1&2&3\\3&1&4\\5&2&3\end{pmatrix}\sim\begin{pmatrix}1&2&3\\0&-5&-8\\0&-8&-12\end{pmatrix}\sim\begin{pmatrix}1&2&3\\0&-5&-8\\0&0&0.8\end{pmatrix}$$

where the first operation swaps R_1 and R_2, the second step is obtained by clears the first pivot column using two operations, and the third step by clearing the entries below the second pivot.

This implies all columns have pivots. Thus there are no free variables and consequently the $N(A^T)$ = {0}. Thus $S^{\perp} = 0$.

6 Find a spanning set for the 2-d plane through the origin in R^4 that is orthogonal to the plane given by x – y + z + t = 3x + 5z – t = 0.

Ans. The plane is clearly given by N(A) where

$A=\begin{pmatrix}1&-1&1&1\\3&0&5&-1\end{pmatrix}$. We need the space orthogonal to it, which is simply $C(A^T)$ which is the row-space of A. Thus the required plane is the plane spanned by (1,-1,1,1) and (3,0,5,-1).

7 Find the matrix for the projection onto the subspace spanned by (5,2,7) and (6,4,6).

Ans. We are looking for the projection matrix for projection onto C(A) where $A=\begin{pmatrix}5&6\\2&4\\7&6\end{pmatrix}$. The projection matrix is given by $A(A^TA)^{-1}A^T$ which is

$$A=\begin{pmatrix}5 & 6\\2 & 4\\7 & 6\end{pmatrix}$$

8 Find the projection matrix:

a. If B is the projection matrix for projection onto a subspace V of R^n, what is the matrix for the projection onto $V^\perp$?

Ans. We know that given v, Bv is the projection of v onto the subspace V. By definition, v – Bv is a vector in $V^\perp$ and actually v is the sum of its projections onto V and $V^\perp$. Thus v – Bv = (I – B)v is the projection of v onto $V^\perp$. thus the matrix for projection onto $V^\perp$ is I – B.

b. Calculate the projection matrix for projection onto the left-null space of a matrix A, using the projection matrix for the same onto C(A).

Ans. The matrix for projection onto C(A) is $A(A^TA)^{-1}A^T$, and for the left null space $N(A^T)$ it is $I – A(A^TA)^{-1}A^T$.

c. What are the matrices for the projections onto the row space and the null space of A?

Ans. $A^T(AA^T)^{-1}A$ for projection onto row space $C(A^T)$ (obtained by replacing A with A^T in the projection matrix onto C(A); and for $N(A^T)$ it is $I – A^T(AA^T)^{-1}A$ since the null space is the perp of the row space.

9 Given the data set:

x	0	3	5	8	10
y	2	4	7	12	17

a. Can you find the quadratic $y = ax^2 + bx + c$ that best fits the above data? (HINT: Write matrix A with all 1's in the first column, x values in the second and x^2 values in the third).

Ans. We here are looking for a relation of the form
$y = a_0 + a_1x + a_2x^2$.

Thus the equations are:

$2 = a_0 + a_1(0) + a_2(0)^2$
$4 = a_0 + a_1(3) + a_2(3)^2$

...

and so on.

In matrix form this becomes Ax = b where

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 3 & 3^2\\ 1 & 5 & 5^2\\ 1 & 8 & 8^2\\ 1 & 10 & 10^2\end{pmatrix}, \quad x=\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix} \text{ and } b=\begin{pmatrix}2\\ 4\\ 7\\ 12\\ 17\end{pmatrix}$$

We now have that the optimal x is given by $x=(AA^T)^{-1}A^T b$ which is

$$\begin{pmatrix}5 & 26 & 198\\ 26 & 198 & 1664\\ 198 & 1664 & 14802\end{pmatrix}^{-1}\begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 0 & 3 & 5 & 8 & 10\\ 0 & 9 & 25 & 64 & 100\end{pmatrix}\begin{pmatrix}2\\ 4\\ 7\\ 12\\ 17\end{pmatrix}$$

$$=\begin{pmatrix}1.9746\\ 0.4065\\ 0.1089\end{pmatrix}.$$

Thus the quadratic curve of best fit is given by
y = 1.9746 + 0.4065x + 0.1089x^2.

b What size is the matrix A^TA where A is as in the above hint for question b if we look for a model of the form y = p(x) where p is a polynomial of degree k?

Ans. There are columns for each term 1,x,...,x^k in A. Thus A^TA has as many rows as A^T and as many columns as A. Thus the number of rows and columns in A^TA are equal to the number of columns of A. Thus A^TA is a k+1 by k+1 matrix.