

Quiz 1

NAME:

SID:

1. The quiz lasts 45mins.
2. The quiz is not graded, we provide a grading scheme for indicative purposes only.
3. Notes are *not* allowed, except for a one-page, two sided cheat sheet.
4. Do not open the exam until you are told to do so.

The breakdown of points is as follows.

Part	a	b	c		total
1	2	2	2		6
2	2	2	2	4	10

1. *permutation matrix.* A matrix $P \in \mathbb{R}^{n \times n}$ is a permutation matrix if it is a permutation of the columns of the $n \times n$ identity matrix.
 - (a) For a $n \times n$ matrix A , we consider the products PA and AP . Describe in simple terms what these matrices look like with respect to the original matrix A .
 - (b) Show that P is an orthogonal matrix.
 - (c) Show that P is invertible, and that $P^{-1} = P^T$.

2. *Rank.* As seen in class, the rank of a $m \times n$ matrix A is the dimension of its range (also called span) $\mathcal{R}(A) := \{Ax : x \in \mathbb{R}^n\}$.



Figure 1: A black and white image of a painting by Piet Mondrian (1872-1944).

- (a) Assume that A takes the form $A = uv^T$, with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$. Describe the range of A in simple geometrical terms and find its rank.
- (b) Show that the rank of the sum of two matrices of same size A, B is less than the sum of the ranks of A and B . *Hint:* show that $\mathcal{R}(A + B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$, meaning that any vector in the range of $A + B$ can be expressed as the sum of two vectors, each in the range of A and B respectively. You may also use (without proof) that for any two subspaces S_1, S_2 , $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.
- (c) Assume that a $m \times n$ matrix A takes the form $A = UV^T$, with $U \in \mathbb{R}^{m \times k}$, $V \in \mathbb{R}^{n \times k}$. Show that the rank of A is less or equal than k . *Hint:* use part (2b).
- (d) Consider the image in Fig 1, a gray-scale rendering of a painting by Mondrian (1872-1944). We build a 256×256 matrix A of pixels based on this image by ignoring grey zones (i.e. assigning zero to both light/dark grey zones), assigning +1 to horizontal or vertical black lines, +2 at the intersections, and zero elsewhere. The horizontal lines occur at row indices 100, 200 and 230, and the vertical ones, at columns indices 50, 230. What is the rank of the matrix?

