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Quiz 2 Practice

1. We consider a resource allocation problem of the form

$$\max_{w \in \mathcal{W}} \min_{r \in \mathcal{E}} r^T w$$

where $\mathcal{W} := \{w \in \mathbb{R}^n_+ \ : \ w_1 + \ldots + w_n = 1\}$, and

$$\mathcal{E} := \{\hat{r} + Du \ : \ \|u\|_2 \leq 1\}.$$

Here, $\hat{r} \in \mathbb{R}^n$ and $D = \mathbf{diag}(\sigma_1, \ldots, \sigma_n)$ are given, with $\sigma \in \mathbb{R}^n$ $\sigma > 0$.

The above problem appears when trying to allocate resources to various revenue-generating processes (which could be ads, financial investments, physical sensors, etc). The revenue vector r is unknown but bounded, and the goal is to maximize the worst-case total revenue $\min_{r \in \mathcal{E}} r^T w$.

(a) Describe the shape of $\mathcal{E}$ in simple geometrical terms.

(b) Use the Cauchy-Schwartz inequality to prove that

$$\min_{r \in \mathcal{E}} r^T w = \hat{r}^T w - \|Dw\|_2.$$

(c) Express the problem as an SOCP in standard format, involving the variable w and one extra scalar variable.

2. When a user goes to a website, an advertisement from a set of n ads, labeled $1, \ldots, n$, will be displayed. This is called an *impression* of ad i. We divide a day into T periods, labeled $t = 1, \ldots, T$. Let $N_{it} \geq 0$ denote the number of impressions in period t of ad i. The total number of ad impressions in period t is $I_t > 0$, so we must have $\sum_{i=1}^n N_{it} = I_t$, for $t = 1, \ldots, T$. For simplicity, you can treat all these integer numbers as real. (This is justified since they are typically very large.)

(a) The revenue for displaying ad i in period t is $R_{it} \geq 0$ per impression. What is the total daily revenue for displaying ads $(1, \ldots, n)$? Given I_t (the number of impressions in period t) and R_{it} (the revenue for displaying ad i in period t), how would you choose N_{it} to maximize revenue for a given day?

(b) In reality, we also have in place a set of m contracts that require us to display certain numbers of ads, or mixes of ads (say, associated with the products of one company), over certain periods, with a penalty for any shortfalls. For $j = 1, \ldots, m$, contract j is characterized by a set of ads $\mathcal{A}_j \subseteq \{1, \ldots, n\}$, a set of periods $\mathcal{T}_j \subseteq \{1, \ldots, T\}$, a target number of impressions $q_j \geq 0$. The *shortfall* s_j for contract $j = 1, \ldots, m$ is the difference between the target number of impressions of the contract and the total number of ads (single ad or mix of ads, as specified in the contract) which are displayed; if the number of impressions is larger than the target, we count the shortfall as 0. Show that the shortfall for contract j can be written as

$$s_j = \max(0, q_j - L_j(N)),$$

where L_j is a linear function of matrix N, which you will determine.

(c) Contractually, a unit shortfall in contract j results in a given penalty $p_j > 0$, $j = 1, \ldots, m$. What is the total penalty payment? What is the net profit?

(d) Explain how to find the display numbers N_{it} that maximize net profit via linear programming. Remember that the data in this problem are $R \in \mathbb{R}^{n \times T}$, $I \in \mathbb{R}^T$ (here I is the vector of impressions, not the identity matrix), and the contract data $\mathcal{A}_j$, $\mathcal{T}_j$, q_j and p_j, $j = 1, \ldots, m$. Make sure to state precisely what the objective function, the variables and the constraints are. Explain clearly why your problem is a linear programming problem.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, with domain the entire space $\mathbb{R}^n$.

(a) Show that we can represent f as a pointwise maximum of affine functions, specifically

$$\forall\, x \in \mathbb{R}^n \ : \ f(x) = \max_{(a,b) \in \mathcal{A}} a^T x + b, \tag{1}$$

where $\mathcal{A} \subseteq \mathbb{R}^{n+1}$ is to be determined. *Hint:* use the inequality, valid for any differentiable convex function f, and every $x, y \in \mathbb{R}^n$: $f(x) \geq f(y) + (x-y)^T \nabla f(y)$, and maximize over y for a fixed x.

(b) We define the *perspective* of f as the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with values

$$g(x,t) = \begin{cases} t f(x/t) & \text{if } t > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Prove that g is convex. *Hint:* express g when you rewrite f using (1), leading to a similar expression for g.

(c) Show that the function $h : \mathbb{R}^n \to \mathbb{R}$ with values

$$h(x) = \begin{cases} \dfrac{(a^T x + b)^2}{c^T x + d} & \text{if } c^T x + d > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex. *Hint:* compute the perspective of the function f with values $f(z) = z^2$ on $\mathbb{R}$.

4. The log-sump-exp function is defined as the function $f : \mathbb{R}^n \to \mathbb{R}$ with values

$$f(z) = \log\left(\sum_{i=1}^{n} e^{z_i}\right).$$

Often, the convexity of this function is proven by deriving the Hessian. In this exercise, you will show that f is convex, using another method.

   (a) Show that, for any given $s > 0$, we have

$$\log s = -1 + \min_{v} \; se^v - v.$$

   (b) Show that

$$f(z) = -1 + \min_{v} \; \sum_{i=1}^{n} e^{z_i + v} - v.$$

   (c) Prove convexity of f based on the above result. *Hint:* use the notion of joint convexity.

5. We consider an investment problem of the form

$$p^* = \max_{x \geq 0} \; r^T x - \frac{1}{2} x^T C x$$

where $r \in \mathbb{R}^n$ is a vector of expected returns, $C = C^T \succ 0$ is a covariance matrix. We assume that C is given as a so-called "single factor model", that is

$$C = D + ff^T,$$

where D is diagonal positive-definite, and $f \in \mathbb{R}^n$.

   (a) First show a preliminary result: for every scalars ρ and $\delta > 0$,

$$\max_{\xi \geq 0} \; \rho\xi - \frac{1}{2}\delta\xi^2 = \frac{1}{2\delta} \max(0, \rho)^2.$$

   with unique minimizer $\xi^* = \max(0, \rho)/\delta$.

   (b) Based on the expression

$$p^* = \max_{x \geq 0} \; r^T x - \frac{1}{2} x^T D x - \frac{1}{2} z^2 \; : \; z = f^T x,$$

   show that a dual can be written as

$$\frac{1}{2} \min_{\nu} \; \nu^2 + \sum_{i=1}^{n} \frac{\max(0, r_i - \nu f_i)^2}{D_{ii}}.$$

   Hint: for the minimization of the Lagrangian over x, use part 5a .

(c) Does strong duality hold? Justify your answer.

(d) Explain how to recover an optimal primal dual point from an optimal dual variable ν^*.