EECS 127/227AT 3/8/2018
L. El Ghaoui

Midterm

NAME:

SID:

Instruction:

1. The exam lasts 1h20.
2. The maximum score is 30.
3. Notes are *not* allowed, except for a one-page, two sided cheat sheet.
4. Do not open the exam until you are told to do so.

The breakdown of points is as follows.

Part	a	b	c	d	e	total
1	2	4	2	2		10
2	1	2	2	2	2	9
3	2	2	2	2	3	11

1. Consider the 2×2 matrix

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{2}{\sqrt{10}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

   (a) What is an SVD of A? Express it as $A = USV^\top$, with S the diagonal matrix of singular values ordered in decreasing fashion. Make sure to check all the properties required for U, S, V.

   (b) Find the semi-axis lengths and principal axes of the ellipsoid

$$\mathcal{E}(A) = \left\{ Ax \ : x \in \mathbb{R}^2, \ \ \|x\|_2 \leq 1 \right\}.$$

   Hint: Use the SVD of A to show that every element of $\mathcal{E}(A)$ is of the form $y = U\bar{y}$ for some element $\bar{y}$ in $\mathcal{E}(S)$. That is, $\mathcal{E}(A) = \{U\bar{y} \ : \ \bar{y} \in \mathcal{E}(S)\}$. (In other words the matrix U maps $\mathcal{E}(S)$ into the set $\mathcal{E}(A)$.) Then analyze the geometry of the simpler set $\mathcal{E}(S)$.

   (c) What is the set $\mathcal{E}(A)$ when we append a zero vector after the last column of A, that is A is replaced with $\tilde{A} = [A, 0] \in \mathbb{R}^{2\times 3}$?

   (d) Same question when we append a row after the last row of A, that is, A is replaced with $\tilde{A} = [A^\top, 0]^\top \in \mathbb{R}^{3\times 2}$. Interpret geometrically your result.

2. *Student scores and duality.* We are given a $n \times m$ matrix M that contains the scores of n sudents on an exam with m parts, so that $M_{i,j}$ is the score on student i on part j.

   (a) Someone hypothesizes that the score M_{ij} is simply the product of two variables, the i-th student overall academic ability a_i, and the difficulty level of part j, d_j. If that is the case, what is the rank and an SVD of M, in terms of the vectors a, d?

   (b) How would you test the above hypothesis, on real-world data, and estimate vectors a, d? Describe precisely the steps you would take.

   (c) We define an n-vector b, with b_i the largest score, across the whole exam, of student i. Similarly we define an m-vector s, with s_j the smallest score, across students, for part j. Show that for every i, j, $s_j \leq b_i$. *Hint:* show that M_{ij} is between the two quantities.

   (d) Show that

   $$d^* := \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} M_{ij} \leq \min_{1 \leq i \leq n} \max_{1 \leq j \leq m} M_{ij} := p^*.$$

   How does this relate to weak duality? Discuss.

   (e) Assume that M satisfies the hypothesis of part 2a, with positive vectors a, d. Does strong duality hold (that is, $p^* = d^*$) in that case? Justify your answer.

3. *Optimization over a dome.* We are given two n-vectors a, y, and a scalar c, and consider the following optimization problem:

$$p^* := \max_x \; a^\top x \;\; : \;\; \|x\|_2 \le 1, \;\; y^\top x \ge c. \tag{1}$$

(a) Is the problem convex, as stated? Justify your answer.

(b) Show that problem (1) is feasible if and only if $c \le \|y\|_2$, which we henceforth assume.

(c) Interpret problem (1) geometrically; in particular, explain why $r^* = \|a\|_2$ when $c \le -\|y\|_2$ and $r^* = +\infty$ (that is, the problem is infeasible) when $c > \|y\|_2$.

(d) Show that

$$p^* = \max_x \; \min_{\lambda \ge 0} \; a^\top x + \lambda(y^\top x - c) \;\; : \;\; \|x\|_2 \le 1.$$

Hint: fix x and compute the minimum in the above.

(e) Assuming that strong duality holds, show that the problem can be reduced to a one-dimensional search:

$$p^* = \min_{\lambda \ge 0} \; \|a + \lambda y\|_2 - c\lambda.$$