EE 127/227AT 9/26/2017
G. Calafiore

Quiz 1

NAME:

SID:

Instruction:

1. The quiz lasts 1h20.
2. Notes are *not* allowed, except for a one-page, two sided cheat sheet.
3. Do not open the exam until you are told to do so.

1. (a) Consider the trace function on a square matrix $X \in \mathbb{R}^{n,n}$, defined as the sum of the elements on the diagonal of X: $\text{trace}(X) = \sum_{i=1}^{n} X_{ii}$. Prove that:

    i. $\text{trace}(A) = \text{trace}(A^\top)$, for any square matrix A;

    ii. $\text{trace}(AB) = \text{trace}(BA)$, for any square matrices A, B;

    iii. Prove that if a square matrix A is diagonalizable, then the trace of A equals the sum of all the eigenvalues of A (this fact actually holds for general A, but you are here asked to prove it only for the diagonalizable case).

   (b) Consider the space $\mathbb{R}^{m,n}$ of $m \times n$ matrices as a vector space, endowed with the inner product $\langle A, B\rangle \doteq \text{trace}(A^\top B)$, for any $A, B \in \mathbb{R}^{m,n}$. Prove that $\langle A, B\rangle \doteq \text{trace}(A^\top B)$ is indeed an inner product on $\mathbb{R}^{m,n}$, that is, it satisfies the following three properties:

    i. $\langle A, A\rangle \geq 0$ for all $A \in \mathbb{R}^{m,n}$, and $\langle A, A\rangle = 0$ if and only if $A = 0$;

    ii. $\langle A, B + C\rangle = \langle A, B\rangle + \langle A, C\rangle$;

    iii. $\langle \alpha A, B\rangle = \alpha\langle A, B\rangle$;

    iv. $\langle A, B\rangle = \langle B, A\rangle$.

2. We are given two sets of points in $\mathbb{R}^n$: $\mathcal{A} = \{a_1, \ldots, a_m\}$ and $\mathcal{B} = \{b_1, \ldots, b_m\}$, with $a_i, b_i \in \mathbb{R}^n$, $i = 1, \ldots, m$. Let $A \doteq [a_1 \cdots a_m] \in \mathbb{R}^{n,m}$ and $B \doteq [b_1 \cdots b_m] \in \mathbb{R}^{n,m}$. We know that the points in $\mathcal{B}$ are related to the points in $\mathcal{A}$ via an orthogonal map, plus noise, that is
$$b_i = Qa_i + \epsilon_i, \quad i = 1, \ldots, m,$$
where ϵ_i are unknown noise terms, and $Q \in \mathbb{R}^{n,n}$ is an unknown orthogonal matrix.

We are interested in approximately "recovering" the unknown Q matrix from the data in A and B. More precisely, we seek an orthogonal matrix $\hat{Q}$ that minimizes $\|B - QA\|_F^2$ over all orthogonal matrices Q, i.e., in solving
$$\hat{Q} = \arg \min_{\{Q:\, QQ^\top = I\}} \|B - QA\|_F^2.$$

(a) Prove that $\|YAX\|_F^2 = \|A\|_F^2 =$ for any matrix A and orthogonal matrices X, Y.

(b) Show that the optimal solution $\hat{Q}$ can be expressed in terms of the Singular Value Decomposition (SVD) of the matrix $M = BA^\top$. Find this solution explicitly in terms of the SVD factors of $BA^\top = U\Sigma V^\top$.

3. Given the following three points in $\mathbb{R}^5$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \; x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \; x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

compute:

(a) the projections $\hat{x}_1, \hat{x}_2, \hat{x}_3$ of x_1, x_2, x_3 onto the line $\mathcal{L} \doteq \{x = \bar{x} + \gamma v, \gamma \in \mathbb{R}\}$, where

$$\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix};$$

(b) the distances d_1, d_2, d_3 of the points x_1, x_2, x_3 from the hyperplane

$$\mathcal{H} \doteq \{x : \sum_{i=1}^{5} x_i = 0\}.$$