Optimization Models

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LECTURE 23

Applications to Control Systems

Control: *to exercise authoritative or dominating influence over; to adjust to a requirement; regulate . . .* From Middle English controllen, from Anglo-Norman contreroller, from Medieval Latin contra-rotulare, to check by duplicate register, from contra-rotulus, duplicate register.

Merriam–Webster Dictionary

Outline

1. Overview
2. Models of control systems
3. Model predictive control
4. Examples
   - Control of micro-satellite constellations
   - Control of a cart
5. Further topics

Some applications of control

Very wide range of applications:

- robotics: autonomous vehicles, androids, etc;
- process & manufacturing: supply chain, assembly;
- health: control of epidemics, radio-therapy, non-invasive surgery, etc;
- power and energy: HVAC systems, electricity production, etc;
- finance: asset management, index tracking, etc;
- environment: flood & fire control, evacuation, etc;
- and many more!

Recent advances in optimization and progress in computing power now allow for optimization to be used in real-time control.

Caveats:

- computing effort still a limit in some applications;
- reliability of optimization algorithm becomes crucial;
- robustness to various perturbations and model mismatch is also critical.

Control systems, discrete-time

Model of a (discrete-time) dynamical system:

$$x(t+1) = f(x(t), u(t), t), \quad y(t) = g(x(t), u(t), t)$$

where

- $x(t) \in \mathbb{R}^n$ is the "state" at time t;
- $u(t) \in \mathbb{R}^p$ is the "control input";
- $y(t) \in \mathbb{R}^q$ is the "measured output";
- f, g are given maps.

If f, g do not depend explicitely on time, the system is said to be time-invariant.

"Continuous-time" version involves differential equations; can be approximated by the above via "discretization".

Linear models

It is often convenient to approximate the model via linearization:

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)$$

where $A(t), B(t), C(t), D(t)$ are matrices obtained by first-order Taylor expansion of the maps f, g.

Time-invariant systems correspond to the case when A, B, C, D are constant.

Example: population dynamics

Yearly population dynamics modeled via

$$x(t+1) \approx Ax(t)$$

where

$$A = \begin{pmatrix} b_1 & b_2 & \cdots & b_{99} \\ 1-d_1 & 0 & \cdots & 0 \\ 0 & 1-d_2 & \cdots & 0 \\ \vdots & \ddots & & \\ 0 & 0 & \cdots & 1-d_{99} \end{pmatrix},$$

and

- state vector $x(t) \in \mathbb{R}^{100}$, with $x_i(t)$ the population with age $i-1$ at time t;
- b_i is is the average number of births per person with age $i-1$;
- d_i is the portion of those aged $i-1$ who will die this year;
- the model assumes a maximum life span of 100 years.

Time-invariance means that we assume the birth and death rates to be constant over time; the model also assumes that there is no immigration or other "inputs".

Control systems, continous-time

Model of a (continuous-time) dynamical system:

$$\frac{d}{dt}x(t) = f(x(t), u(t), t), \quad y(t) = g(x(t), u(t), t)$$

where

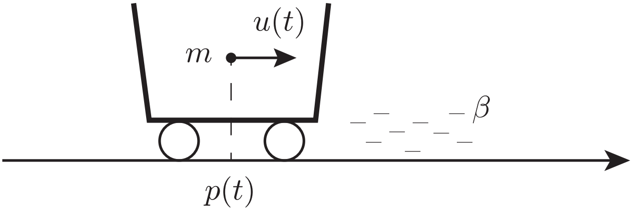
- $x(t) \in \mathbb{R}^n$ is the "state" at time t;
- $u(t) \in \mathbb{R}^p$ is the "control input";
- $y(t) \in \mathbb{R}^q$ is the "measured output";
- f, g are given maps.

The continuous-time aspect often originates from physical laws, *e.g.* equations of motion ($f = ma$).

Example

Cart on rail

- Consider a cart of mass m, moving along a horizontal rail, where it is subject to viscous damping (a damping which is proportional to the velocity) of coefficient β.
- We let $p(t)$ denote the position of the center of mass of the cart, and $u(t)$ the force applied to the center of mass, where t is the (continuous) time.



Cart on rail

- The Newton dynamic equilibrium law then prescribes that

$$u(t) - \beta \dot{p}(t) = m\ddot{p}(t),$$

  which is the second-order differential equation governing the dynamics of this system.

- If we introduce variables (states) $x_1(t) = p(t)$, $x_2(t) = \dot{p}(t)$, we may rewrite the Newton equation in the form of a system of two coupled differential equations of first order:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \alpha x_2(t) + bu(t), \end{aligned}$$

  where we defined

$$\alpha \doteq -\frac{\beta}{m}, \quad b \doteq \frac{1}{m}.$$

Cart on rail

- The system can then be recast in compact matrix form as

$$\dot{x}(t) = A_c x(t) + B_c u(t),$$

where

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

- With this system we can also associate an output equation, representing a signal y that is of some particular interest, e.g., the position of the cart itself:

$$y(t) = Cx(t)., \quad C = [1 \ \ 0].$$

The model is a so-called continuous-time linear time-invariant (LTI) system, of the form

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad y(t) = Cx(t). \tag{1}$$

Lagrange formula

Given the value of the state at some instant t_0, and given the input $u(t)$ for $t \geq t_0$, there exists an explicit formula (usually known as Lagrange's formula) for expressing the evolution in time of the state of the system:

$$x(t) = \mathrm{e}^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau, \quad t \geq t_0, \tag{2}$$

where, for a square matrix M, we define the matrix exponential as

$$e^M \doteq \sum_{k=1}^{\infty} \frac{1}{k!}M^k \approx I + M.$$

Discretization

- Often, a continuous-time systems must be analyzed and controlled by means of digital devices, such DSP and computers. It is therefore common to "convert" a continuous-time system into its discrete-time version, by "taking snapshots" of the system at time instants $t = k\Delta$, where Δ is referred to as the *sampling interval*, assuming the the input signal $u(t)$ remains constant between two successive sampling instants, that is

$$u(t) = u(k\Delta), \quad \forall t \in [k\Delta, (k+1)\Delta).$$

- This discrete-time conversion can be done as follows: given the state of the continuous system (1) at time $t = k\Delta$ (which we denote by $x(k) = x(k\Delta)$), we can use equation (2) to compute the value of the state at instant $(k+1)\Delta$:

$$x(k+1) \quad = \quad \mathrm{e}^{A\Delta} x(k) + \int_0^{\Delta} \mathrm{e}^{A\tau} B \mathrm{d}\tau \, u(k).$$

Discretization

- The sampled version of the continuous-time system (1) evolves according to a discrete-time recursion of the form

$$\begin{aligned} x(k+1) &= A_\Delta x(k) + B_\Delta u(k), \\ y(k) &= Cx(k), \end{aligned}$$

$$A_\Delta = \mathrm{e}^{A\Delta}, \quad B_\Delta = \int_0^\Delta \mathrm{e}^{A\tau} B \mathrm{d}\tau. \tag{3}$$

- For the cart example, with $m = 1$ Kg, $\beta = 0.1$ Ns/m, and $\Delta = 0.1$ s, we obtain a discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned}$$

$$\begin{aligned} A &= \mathrm{e}^{A_c\Delta} = \begin{bmatrix} 1 & \frac{1}{\alpha}(\mathrm{e}^{\alpha\Delta} - 1) \\ 0 & \mathrm{e}^{\alpha\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0.0995017 \\ 0 & 0.9900498 \end{bmatrix} \\ B &= \frac{b}{\alpha} \begin{bmatrix} \frac{1}{\alpha}(\mathrm{e}^{\alpha\Delta} - 1) - \Delta \\ \mathrm{e}^{\alpha\Delta} - 1 \end{bmatrix} = \begin{bmatrix} 0.0049834 \\ 0.0995016 \end{bmatrix}. \end{aligned}$$

Optimal control problems

Finite-horizon optimal control: for a given "horizon" time T

$$\min_{x(\cdot),\, u(\cdot)} L(x(T)) \ : \quad \begin{array}{l} x(t+1) = f(x(t), u(t)), \ \ t = 1, \ldots, T-1, \\ x(t) \in \mathcal{X}, \ \ u(t) \in \mathcal{U}, \ \ t = 1, \ldots, T, \end{array}$$

where sets $\mathcal{X}$, $\mathcal{U}$ encode state and input constraints, and cost function L encodes desired objective, such as closeness to a target state.

- non-convex optimization problem in general;
- assumes that the control input has access to the whole state; problems where some states are not measured are more difficult.

Model predictive control (MPC)

Basic idea

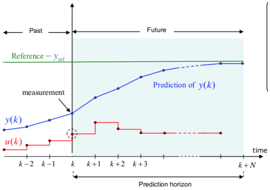


Figure: MPC.

Fix a time horizon, and for each time t:

- determine a model of the system over future time interval $[t, t+T]$;
- solve an optimal control problem, posed over the same interval;
- apply the first control input $u(t+1)$, and iterate.

Linear MPC

With affine dynamics and convex state and input constraints the optimal control problem is convex. Assuming time-invariant dynamics (constant matrices A, B):

$$\begin{array}{ll} \min_{x(\cdot),\, u(\cdot)} & \sum_{t=1}^{T} \left(\|x(t) - x_{\text{des}}(t)\|_2^2 + \|u(t)\|_2^2 \right) \\ \text{s.t.} & x(t+1) = Ax(t) + Bu(t), \;\; t = 1, \ldots, T-1, \\ & \|x(t)\|_\infty \le x_{\max}, \;\; \|u(t)\|_\infty \le u_{\max}, \;\; t = 1, \ldots, T. \end{array}$$

- The above is a QP;
- in practice, we often add a "terminal cost", of the form $\|x(T+N)\|_2^2$, in order to improve the stability of the controlled system.

Linear-Quadratic Regulator (LQR)

Consider the case where state and input constraints are removed:

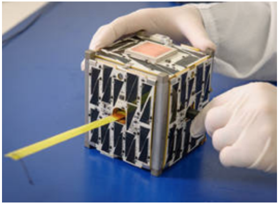
$$\begin{array}{ll} \min\limits_{x(\cdot),\, u(\cdot)} & \sum\limits_{t=1}^{T} \left(\|x(t) - x_{\rm des}(t)\|_2^2 + \|u(t)\|_2^2 \right) \\ \text{s.t.} & x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1. \end{array}$$

The problem reduces to linearly constrained LS.

Setting the horizon to $T = +\infty$, it can be shown that the optimal control has the form of a constant state-feedback (or, "regulator"): $u(t) = Kx(t)$ for some suitable K that can be pre-computed. Due to its low online computational requirements, the LQR approach is widely applied.

Example

Control of micro-satellite constellations[1]



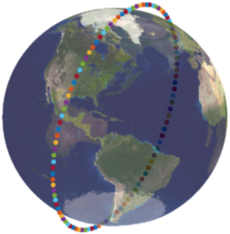


Figure: A micro-satellite, also known as a "cubesat" (left), and a constellation of such (right). The control problem is to make sure the constellation is as close as possible to an equi-angular configuration on a given reference orbital plane.

The satellites' positions can be partially controlled via the atmospheric drag, which is still present at altitudes ≈ 400 km.

[1] From A. Packard and E. Sin.

Linear dynamical model

Linearized, discrete-time dynamics:

$$x(t+1) \approx Ax(t) + Bu(t)$$

where

- $x(t) = (r_i(t), \theta_i(t), \omega_i(t))_{i=1}^n \in \mathbb{R}^{3n}$ contains the satellites' polar coordinates at time t;
- $u(t) \in \mathbb{R}^n$ controls the drags at time t;
- A, B are estimated from nonlinear equations of motion.

Goals: for a given control horizon T,

- keep angular distances of satellites within some error bound ϵ;
- keep each control variable within a magnitude bound $u_{\max}$;
- maximize the smallest altitude across the satellites.

LP model

$$
\max_{x,u} \min_{1\le i\le n} r_i(T) \ : \quad \begin{array}{l} x(t) = (r_i(t), \theta_i(t), \omega_i(t))_{i=1}^n, \ \ t = 1, \ldots, T, \\ x(t+1) = Ax(t) + Bu(t), \ \ t = 1, \ldots, T-1, \\ |\theta_{i+1}(t) - \theta_i(t)| \le \epsilon, \ \ i = 1, \ldots, n-1, \ \ t = 1, \ldots, T-1, \\ |\omega_{i+1}(t) - \omega_i(t)| \le \epsilon, \ \ i = 1, \ldots, n-1, \ \ t = 1, \ldots, T-1, \\ \|u(t)\|_\infty \le u_{\max}, \ \ t = 1, \ldots, T-1. \end{array}
$$

- Solve the above at each step, and apply the first control input;
- Many variants of the formulation are possible.

Simulation

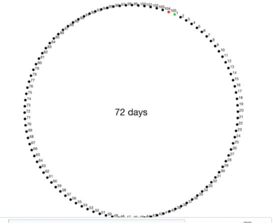


Figure: Coordinated control of a constellation.

Minimum-energy control

Return to the cart example, described by the discrete-time, LTI model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad k = 0, 1, 2, \ldots \\ y(k) &= Cx(k). \end{aligned} \tag{4}$$

The energy of the input signal over the finite time interval $\{0, \ldots, T-1\}$ is defined as

$$\text{energy of } u(\cdot) = \sum_{k=0}^{T-1} u(k)^2.$$

Finding the minimum-energy command sequence that reaches a desired target state x^{des} amounts therefore to solving the linearly constrained least-squares problem:

$$\begin{aligned} \min_{x(\cdot),\, u(\cdot)} \quad & \sum_{k=0}^{T-1} u(k)^2 \\ \text{s.t.} \quad & (4), \; x(T) = x^{\mathrm{des}}. \end{aligned}$$

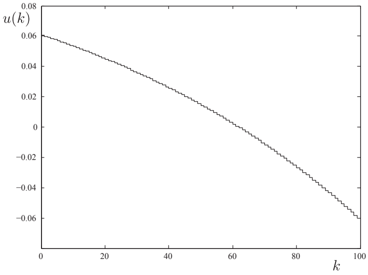
Minimum-fuel control

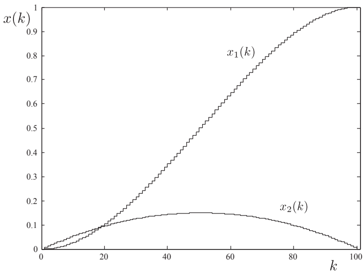
- An alternative approach for determining the command sequence would be to minimize the ℓ_1 norm of the input sequence, instead of the ℓ_2 norm.
- The ℓ_1 norm is proportional to the consumption of "fuel" needed to produce the input commands. For example, in aerospace applications, control inputs are typically forces produced by thrusters that spray compressed gas, hence $\|u(\cdot)\|_1$ would be proportional to the total quantity of gas necessary for control actuation.
- The problem now becomes an LP:

$$\begin{aligned} \min_{\mu_T, s} \quad & \sum_{k=0}^{T-1} s_k \\ \text{s.t.} \quad & |u(k)| \leq s_k, \quad k = 0, \ldots, T-1, \\ & (4), \;\; x(T) = x^{\text{des}}. \end{aligned}$$

Cart example

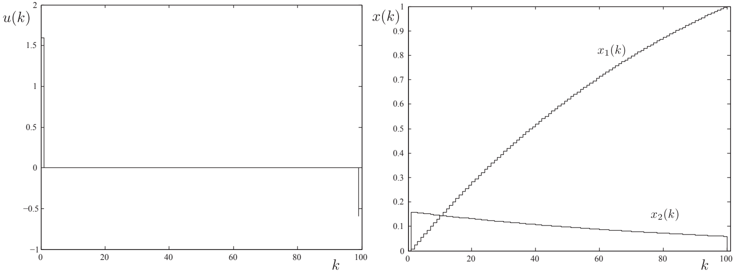
- Given initial conditions $x(0) = [0\ 0]^\top$, we seek minimum-energy and minimum-fuel input sequences that bring the cart into position $p(T_c) = 1$, with zero velocity, at time $T_c = 10$ sec.
- Since the sampling time is $\Delta = 0.1$, we have that the final integer target time is $T = 100$, hence our input sequence is composed of 100 unknown values: $[u(0), \ldots, u(T-1)]^\top$.
- The minimum-energy control sequence is:





Cart example

- The minimum-fuel control sequence is found by solving the LP.
- Observe that the shape of input signals obtained from the minimum-energy approach and from the minimum-fuel one are very different qualitatively. In particular, the minimum-fuel solution is *sparse*, meaning in this case that the control action is zero everywhere except for the initial and the final instants (in the control lingo, this is called a *bang-bang* control sequence).



Trajectory tracking

We next consider the problem of finding a control sequence such that the output $y(k)$ of a discrete-time LTI system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) \tag{5}$$

tracks as closely as possible an assigned reference trajectory $y_{\mathrm{ref}}(k)$, over a given finite time horizon $k \in \{1, \ldots, T\}$. We assume scalar inputs and outputs for simplicity.

We can formulate the problem as

$$\min_{x(\cdot),\, u(\cdot)} \quad \sum_{k=0}^{T} \|y(k) - y_{\mathrm{ref}}(k)\|_2^2 \ : \ (5). \tag{6}$$

The above is a linearly constrained least-squares problem.

Trajectory tracking variants

Many variations on the above theme are possible.

- For instance, one may add explicit constraints on the amplitude of the command signal: $\|u(t)\|_\infty \leq u_{\max}$, where $u_{\max}$ is a given bound.
- We may also consider adding a regularization term to the objective, of the form

$$\gamma \cdot \sum_{k=0}^{T} \|u(t)\|_2^2,$$

  where $\gamma > 0$ is a regularization parameter.
- Also, constraints on the instantaneous rate of variation of the signal (slew rate) can be handled easily, for example via a constraint of the form

$$|u(t+1) - u(t)| \leq s_{\max}, \quad t = 0, \ldots, T-1,$$

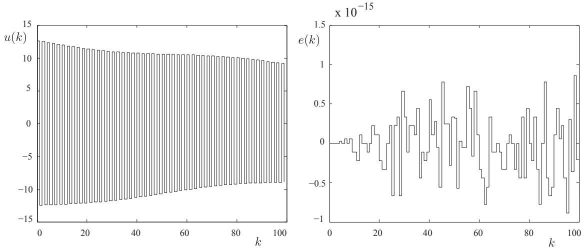
  where $s_{\max}$ is the limit on the input slew rate.

Optimal inputs for trajectory tracking

- Consider again the discretized model of a cart on a rail. Given initial conditions $x(0) = [0\ 0]^\top$ and the following reference output trajectory:

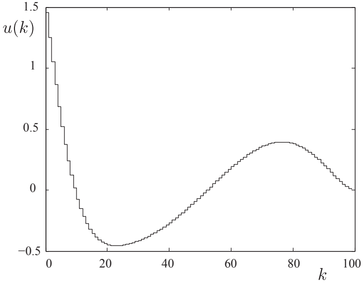
$$y_{\mathrm{ref}}(k) = \sin(\omega k \Delta), \quad \omega = \frac{2\pi}{10}, \quad k = 1, \ldots, 100,$$

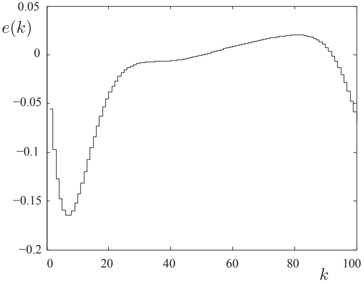
  we seek an input signal such that the output tracks y_{ref}.
- We consider the LS formulation in (6). In this case, since $CB = 0.005 \neq 0$, exact output tracking can (in principle) be achieved.
- The tracking error is numerically zero. However, this is achieved at the expense of an input signal $u(k)$ that is oscillating with high amplitude.



Smoother inputs for trajectory tracking

- Allowing for a nonzero penalization γ of the input energy decreases the condition number of the LS coefficient matrix, and greatly improves the behavior of the input signal.
- For example, assuming $\gamma = 0.1$, we observe that the tracking error is now higher than before (but perhaps still acceptable, depending on the specific application), while the input signal has smaller amplitude and it is substantially smoother than in the case with $\gamma = 0$.





What we have not covered

There is a lot more to control theory and practice:

- frequency-domain analysis, transfer functions;
- asymptotic analysis, stability, controllability;
- observability, observer design, output feedback control synthesis;
- robustness to perturbations and noise;
- semidefinite programming approach;
- reinforcement learning;
- etc ...

Most of these extensions involve optimization problems. We'll describe one topic: semidefinite programming (SDP) for control system synthesis. (This is not on the final.)

Stability analysis

Consider the autonomous, linear, time-invariant discrete-time system:

$$x(t+1) = Ax(t), \quad t = 0, 1, 2, \ldots$$

Asymptotic stability is the property that for *any* initial condition, we have $x(t) \to 0$ as $t \to +\infty$. The classical answer is based on the eigenvalues of A. Here we develop an alternative approach based on so-called "Lyapunov" functions.

Assume that there exist a positive-definite matrix $P = P^\top \succ 0$ and scalar $\alpha \in [0, 1)$ such that the function $V_P : x \to x^\top Px$ decreases at rate α along any trajectory:

$$\forall\, t \geq 0 \;:\; x(t+1)^\top Px(t+1) \leq \alpha^2 x(t)^\top Px(t).$$

Then, the system is asymptotically stable, and $x(t)$ decays exponentially. We refer to the function V_P as a Lyapunov function.

Searching for a Lyapunov function

The previous condition is equivalent to

$$\forall\, x \in \mathbb{R}^n \ : \ (Ax)^\top P(Ax) \leq \alpha^2 x^\top Px,$$

which is the same as $\alpha^2 P \succeq A^\top PA$. Thus, the conditions

$$P \succ 0, \ \ \alpha^2 P - A^\top PA \succeq 0.$$

guarantee that the system is asymptotically stable. The above conditions are convex in P, since the set of PSD matrices is convex.
The problem

$$\max_{t,P} \ t \ \ : \ \ tI \preceq P, \ \ \mathrm{trace}\, P = 1, \ \ \alpha^2 P - A^\top PA \succeq 0$$

is a convex problem, which belongs to the category of semidefinite programs (SDPs).

Stabilizability

Consider the system with inputs:

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots$$

We search for a state-feedback control law, of the form $u(t) = Kx(t)$, with K a fixed matrix, such that the "closed-loop" system

$$x(t+1) = (A + BK)x(t), \quad t = 0, 1, 2, \ldots$$

is asymptotically stable.

The condition becomes

$$P \succ 0, \;\; \alpha^2 P - (A + BK)^\top P(A + BK) \succeq 0.$$

It is not jointly convex in (P, K) ...

Convex stabilizability condition

Using $X \doteq P^{-1}$, $U \doteq KX$ the above becomes:

$$X \succ 0, \;\; \alpha^2 X - (AX + BU)^\top X^{-1}(AX + BU) \succeq 0.$$

Using the Schur complement theorem (lecture 5), in turn, it is equivalent to

$$X \succ 0, \;\; \left(\begin{array}{cc} \alpha X & AX + BU \\ (AX + BU)^\top & \alpha X \end{array} \right) \succeq 0.$$

The above is convex in (X, U).

Theorem 1 (Schur complements)

Let $F \in \mathbb{S}^n$, $G \in \mathbb{R}^{n,m}$, $H \in \mathbb{S}^m$, with $H \succ 0$. The condition $F \succeq G^\top H^{-1} G$ is equivalent to

$$M = \left(\begin{array}{cc} F & G \\ G^\top & H \end{array} \right) \succeq 0.$$

Extension: robust stability

We can extend this SDP approach in many ways. Assume for example that A is only known to be in the set $\{A_1, \dots, A_K\}$, which correspond to the linear system with arbitrary "jumps":

$$x(t+1) = A(t)x(t), \;\; A(t) \in \{A_1, \dots, A_K\}.$$

Analyzing the stability of this system is not trivial ...

The convex condition

$$P \succ 0, \;\; \alpha^2 P - A_i^\top P A_i \succeq 0, \;\; i = 1, \dots, K$$

guarantees stability of the jump linear system. Again, finding such a matrix P is a semidefinite program.

Extension: robust stabilizability

Consider a jump linear system with inputs:

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad [A(t), B(t)] \in \{[A_1, B_1], \ldots, [A_K, B_K]\}.$$

Analyzing the stabilizability of this system is not trivial ...

The convex condition

$$X \succ 0, \quad \alpha^2 X - (A_i X + B_i U)^\top X^{-1}(A_i X + B_i U) \succeq 0, \quad i = 1, \ldots, K$$

guarantees stabilizability of the jump linear system. Again, finding such matrices X, U is a semidefinite program.

Cart example

In the cart example, assume that m, β can change over time, anywhere in the intervals

$$m(t) \in [.8, 1.2], \quad \beta(t) \in [0.07, 0.12].$$

This corresponds to four possible configuration for the parameter $\alpha = \beta/m$.

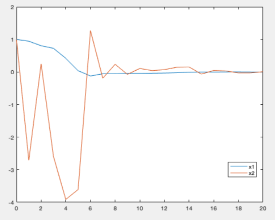


Figure: Quadratic stabilization of the cart with uncertain mass and viscosity parameters.