EE 127/227AT
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11/7/2017

Quiz 2

Your Name:

Your SID:

Student sitting on your left:

Student sitting on your right:

Instruction:

1. The quiz lasts 1h20.
2. Notes are *not* allowed, except for two one-page, two sided cheat sheets.
3. Do not open the exam until you are told to do so.

1. In this problem we examine the convexity of various functions.

   (a) For a given $k \in \{1, \ldots, n\}$ we define the function $s_k : \mathbb{R}^n \to \mathbb{R}$ with values given by
   $$s_k(x) = \sum_{i=1}^{k} x_{[i]},$$
   where $x_{[i]}$ is the i-th largest component of x. Show that s_k is a convex function. *Hint:* express s_k as the maximum of linear functions. You can try with $n = 3$, $k = 2$ first.

   (b) Assume $n = 2k - 1$ is odd. Consider the average absolute deviation from the median of the components of x, which is the function $\phi : \mathbb{R}^n \to \mathbb{R}$ with values given by
   $$\phi(x) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \operatorname{med}(x)|,$$
   where $\operatorname{med}(x) = x_{[k]}$ denotes the *median* of the components of x, i.e., a number that leaves half of the entries of x on its left and half on its right. Show that ϕ is convex. *Hint:* express ϕ in terms of s_k, s_{k-1} and the sum of the components of x.

2. A version of the so-called (convex) *trust-region* problem amounts to finding the minimum of a convex quadratic function over an Euclidean ball, that is

$$\begin{array}{ll} \min\limits_{x} & \frac{1}{2}x^\top Hx + c^\top x + d \\ \text{s.t.} & x^\top x \le r^2, \end{array}$$

where $H \succ 0$, and $r > 0$ is the given radius of the ball. Prove that the optimal solution to this problem is unique and it is given by

$$x(\lambda^*) = -(H + \lambda^* I)^{-1}c,$$

where $\lambda^* = 0$ if $\|H^{-1}c\|_2 \le r$, or otherwise λ^* is the unique value such that

$$\|(H + \lambda^* I)^{-1}c\|_2 = r.$$

3. Let B_i, $i = 1, \ldots, m$, be m given Euclidean balls in $\mathbb{R}^n$, with centers x_i, and radii $\rho_i \geq 0$. We wish to find a ball B of minimum radius that contains all the B_i, $i = 1, \ldots, m$. Explain how to cast this problem into a known convex optimization format.

Extra Space