EE 127/227AT 11/7/2017

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# Quiz 2

## NAME:
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The quiz lasts 1h30. Notes are *not* allowed, except for a one-page, two sided cheat sheet.

1. In this problem we examine the convexity of various functions.

   (a) For a given $k \in \{1, \ldots, n\}$ we define the function $s_k : \mathbb{R}^n \to \mathbb{R}$ with values given by
   $$s_k(x) = \sum_{i=1}^{k} x_{[i]},$$
   where $x_{[i]}$ is the $i$-th largest component of $x$. Show that $s_k$ is a convex function. *Hint:* express $s_k$ as the maximum of linear functions. You can try with $n = 3$, $k = 2$ first.

   (b) Assume $n = 2k - 1$ is odd. Consider the average absolute deviation from the median of the components of $x$, which is the function $\phi : \mathbb{R}^n \to \mathbb{R}$ with values given by
   $$\phi(x) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \mathrm{med}(x)|,$$
   where $\mathrm{med}(x) = x_{[k]}$ denotes the *median* of the components of $x$, i.e., a number that leaves half of the entries of $x$ on its left and half on its right. Show that $\phi$ is convex. *Hint:* express $\phi$ in terms of $s_k, s_{k-1}$ and the sum of the components of $x$.

   **Solution:**

   (a) For $n = 3$, $k = 2$, we have
   $$s_2(x) = \max(x_1 + x_2, x_2 + x_3, x_3 + x_1).$$
   More generally:
   $$s_k(x) = \max_{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k} x_{i_1} + \ldots + x_{i_k}.$$
   This shows that $s_k$ is the point-wise maximum of linear function, hence it is convex.

(b) We have, for $n = 2k-1$ odd:

$$\phi(x) = \sum_{i=1}^{k-1}(x_{[i]} - x_{[k]}) + \sum_{i=k+1}^{n}(x_{[k]} - x_{[i]}) = \sum_{i=1}^{k-1} x_{[i]} - \sum_{i=k+1}^{n} x_{[i]} = s_{k-1}(x) + s_k(x) - \sum_{i=1}^{n} x_i,$$

which shows that $\phi$ is convex.

2. A version of the so-called (convex) *trust-region* problem amounts to finding the minimum of a convex quadratic function over an Euclidean ball, that is

$$\begin{array}{ll} \min\limits_x & \frac{1}{2}x^\top Hx + c^\top x + d \\ \text{s.t.} & x^\top x \leq r^2, \end{array}$$

where $H \succ 0$, and $r > 0$ is the given radius of the ball. Prove that the optimal solution to this problem is unique and it is given by

$$x(\lambda^*) = -(H + \lambda^* I)^{-1}c,$$

where $\lambda^* = 0$ if $\|H^{-1}c\|_2 \leq r$, or otherwise $\lambda^*$ is the unique value such that

$$\|(H + \lambda^* I)^{-1}c\|_2 = r.$$

**Solution 1 (A trust-region problem)** The Lagrangian of this problem can be written as

$$\begin{aligned} \mathcal{L}(x, \lambda) &= \frac{1}{2}x^\top Hx + c^\top x + d + \frac{\lambda}{2}(x^\top x - r^2) \\ &= \frac{1}{2}x^\top (H + \lambda I)x + c^\top x + d - \frac{\lambda}{2}r^2. \end{aligned}$$

The Lagrangian is strongly convex, hence it has a unique minimizer

$$x^*(\lambda) = -(H + \lambda I)^{-1}c.$$

The optimal solution of the problem is $x^*(\lambda^*)$, where $\lambda^*$ is the optimal solution of the dual problem

$$\max_{\lambda \geq 0} \ g(\lambda),$$

with

$$g(\lambda) \doteq -\frac{1}{2}c^\top (H + \lambda I)^{-1}c + d - \frac{\lambda}{2}r^2.$$

Let $H = U\Lambda U^\top$ be a spectral factorization of $H$, where $U$ is orthogonal, and $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) \succ 0$. We can write $(H + \lambda I) = U(\Lambda + \lambda I)U^\top$, hence

$$g(\lambda) = d - \frac{\lambda}{2}r^2 - \frac{1}{2}\sum_{i=1}^{n} \frac{\tilde{c}_i^2}{\lambda_i + \lambda},$$

where $\tilde{c}_i$ are the entries of $\tilde{c} \doteq U^\top c$, and it holds that

$$g'(\lambda) \doteq \frac{\mathrm{d}g}{\mathrm{d}\lambda} = -\frac{1}{2}r^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{\tilde{c}_i^2}{(\lambda_i + \lambda)^2}$$
$$= -\frac{1}{2}r^2 + \frac{1}{2}\|(H + \lambda I)^{-1}c\|_2^2.$$

Notice that the term $\|(H + \lambda I)^{-1}c\|_2$ is strictly decreasing over $\lambda \geq 0$. Hence, if $g'(0) \leq 0$ then $g(\lambda) < 0$ for all $\lambda > 0$, which means that $g(\lambda)$ is decreasing for positive $\lambda$, hence the maximum is at the boundary point where $\lambda = 0$. If otherwise $g'(0) > 0$, then the concave function $g$ has a maximum over $\lambda > 0$, at the point where the derivative $g'$ is zero, that is where

$$\|(H + \lambda I)^{-1}c\|_2^2 = r^2,$$

which is what we needed to prove. The value of $\lambda$ satisfying this equation can be found numerically via any univariate search technique. For instance, one may use Newton method, starting with some initial $\lambda > 0$, and iteratively updating

$$\lambda \leftarrow \lambda - \frac{g'(\lambda)}{g''(\lambda)},$$

where $g''(\lambda)$ is the second derivative

$$g''(\lambda) \doteq \frac{\mathrm{d}^2 g}{\mathrm{d}\lambda^2} = -\sum_{i=1}^{n} \frac{\tilde{c}_i^2}{(\lambda_i + \lambda)^3}.$$

3. Let $B_i$, $i = 1, \ldots, m$, be $m$ given Euclidean balls in $\mathbb{R}^n$, with centers $x_i$, and radii $\rho_i \geq 0$. We wish to find a ball $B$ of minimum radius that contains all the $B_i$, $i = 1, \ldots, m$. Explain how to cast this problem into a known convex optimization format.

   **Solution 2 (Robust sphere enclosure)** Let $c \in \mathbb{R}^n$ and $r \geq 0$ denote the center and radius of the enclosing ball $B$, respectively. We express the given balls $B_i$ as

   $$B_i = \{x : x = x_i + \delta_i, \ \|\delta_i\|_2 \leq \rho_i\}, \quad i = 1, \ldots, m.$$

   We have that $B_i \subseteq B$ if and only if

   $$\max_{x \in B_i} \|x - c\|_2 \leq r.$$

   But

   $$\max_{x \in B_i} \|x - c\|_2 = \max_{\|\delta_i\|_2 \leq \rho_i} \|x_i - c + \delta_i\|_2 = \|x_i - c\|_2 + \rho_i.$$

   The problem is then cast as the following SOCP

   $$\begin{array}{ll} \min\limits_{c,r} & r \\ \text{s.t.} & \|x_i - c\|_2 + \rho_i \leq r, \quad i = 1, \ldots, m. \end{array}$$