On the Geodetic Radon Number of Grids*

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Abstract

It is NP-hard to determine the Radon number of graphs in the geodetic convexity. However, for certain classes of graphs, this well-known convexity parameter can be determined efficiently. In this paper, we focus on geodetic convexity spaces built upon d-dimensional grids, which are the Cartesian products of d paths. After revisiting a result of Eckhoff concerning the Radon number of $\mathbb{R}^d$ in the convexity defined by Manhattan distance, we present a series of theoretical findings that disclose some very nice combinatorial aspects of the problem for grids. We also give closed expressions for the Radon number of the product of P_2's and the product of P_3's, as well as computer-aided results covering the Radon number of all possible Cartesian products of d paths for $d \leq 9$.

Keywords: Radon partition; Radon number; geodetic convexity; Manhattan distance; grid graph; Cartesian product

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1 Introduction

Radon's famous theorem [7] states that every set of at least $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect. This result has been generalized in many ways and considered for various convexity spaces ever since. Formally, a *convexity space* is a pair $(V, \mathcal{C})$ where V is a set and $\mathcal{C}$ is a collection of subsets of V — the *$\mathcal{C}$-convex sets* — such that

- $\emptyset, V \in \mathcal{C}$ and
- $\mathcal{C}$ is closed under arbitrary intersections.

The *$\mathcal{C}$-convex hull* $H_{\mathcal{C}}(R)$ of some subset R of V is the intersection of all $C \in \mathcal{C}$ with $R \subseteq C$, that is, $H_{\mathcal{C}}(R)$ is the smallest $\mathcal{C}$-convex set containing R. A *$\mathcal{C}$-Radon partition* of R is a partition $R = R_1 \cup R_2$ such that $H_{\mathcal{C}}(R_1)$ and $H_{\mathcal{C}}(R_2)$ intersect. The set R is a *$\mathcal{C}$-anti-Radon set* if it does not have a $\mathcal{C}$-Radon partition. The *$\mathcal{C}$-Radon number* $r(\mathcal{C})$ is the smallest r such that every subset R of V with at least r elements has a *$\mathcal{C}$-Radon partition*. Equivalently, $r(\mathcal{C}) - 1$ is the maximum cardinality of a $\mathcal{C}$-anti-Radon set.

While $r(\mathcal{C})$ is trivially well-defined whenever V is finite, it does not need to exist in general.

Natural examples of convexity spaces are induced by metrics. If d is a metric on V, then the set $\mathcal{C}$ that consists of all subsets C of V with

$$\forall u, v \in C : \forall w \in V : d(u,v) = d(u,w) + d(w,v) \Rightarrow w \in C$$

defines a convexity space $(V, \mathcal{C})$. If $\left(\mathbb{R}^d, \mathcal{C}\left(\mathbb{R}^d_2\right)\right)$ denotes the convexity space defined on $\mathbb{R}^d$ by the Euclidean metric $(u,v) \mapsto ||u-v||_2$, then Radon's theorem can be restated as

$$r(\mathcal{C}(\mathbb{R}^d_2)) = d + 2.$$

In [3], Eckhoff determined the Radon number of the convexity space $\left(\mathbb{R}^d, \mathcal{C}\left(\mathbb{R}^d_1\right)\right)$ defined on $\mathbb{R}^d$ by the Manhattan metric $(u,v) \mapsto ||u-v||_1$ as

$$r(\mathcal{C}(\mathbb{R}^d_1)) \quad = \quad r(d) := \min\left\{ r \in \mathbb{N} : \binom{r}{\lfloor \frac{r}{2} \rfloor} > 2d \right\}. \tag{1}$$

While a simple counting argument implies $r(\mathcal{C}(\mathbb{R}^d_1)) \leq r(d)$, Eckhoff needed an ingenious combinatorial construction of an anti-Radon set of order $r(d) - 1$ in order to prove $r(\mathcal{C}(\mathbb{R}^d_1)) \geq r(d)$. Jamison-Waldner [6] generalized Eckhoff's simple upper bound argument to discrete convexity spaces defined by products of trees and observed that Eckhoff's construction is possible whenever all tree factors have sufficiently large diameter. For trees of small diameter, Jamison-Waldner stated that improvements were possible without presenting any.

Our starting point in the present paper is exactly this last observation. More specifically, we study the Radon number $r(\mathrm{Grid}(n_1, n_2, \ldots, n_d))$ of the convexity space induced by the shortest path metric on the d-dimensional Cartesian product

$$\mathrm{Grid}(n_1, n_2, \ldots, n_d) := P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$$

of paths $P_{n_1}, P_{n_2}, \ldots, P_{n_d}$ where P_n denotes the path with n vertices.

Note that the shortest path metric on the vertex set of $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ coincides with the Manhattan metric restricted to the integral points in $[1, n_1] \times [1, n_2] \times \cdots \times [1, n_d]$. As

Jamison-Waldner observed, $r(n_1, n_2, \ldots, n_d)$ equals $r(d)$ as defined in (1) provided that all n_i are sufficiently large. We show how to exploit Eckhoff's method also when some n_i are small and estimate the Radon number of grids in many cases.

The outline of our paper is as follows. After collecting some useful notation and terminology in Subsection 1.1, we show how to exploit Eckhoff's construction in the context of grids in Section 2, which contains our main theoretical contributions. In Section 3 we apply the results from Section 2 and give closed formulas for some Radon numbers of grids. In Section 4 we report the results of our substantial computational effort, which allows to determine the Radon number of all grids up to dimension 9. Finally, in Section 5 we conclude with some open problems.

1.1 Notation and terminology

Throughout the paper we only consider finite, simple, undirected, and connected graphs G, and the convexity space $\mathcal{C}(G)$ defined on their vertex set $V(G)$ by the shortest path metric $(u, v) \mapsto \text{dist}_G(u, v)$, that is, some set C of vertices of G belongs to $\mathcal{C}(G)$ exactly if

$$\forall u, w \in C : \forall v \in V(G) : \text{dist}_G(u, w) = \text{dist}_G(u, v) + \text{dist}_G(v, w) \Rightarrow v \in C.$$

Since we consider only one type of convexity $\mathcal{C}(G)$ defined on $V(G)$, we will simply speak of the *convex hull* $H(R)$ of a set R of vertices of G, *Radon partitions*, *anti-Radon sets*, and the *Radon number* $r(G)$ of G instead of the $\mathcal{C}(G)$-convex hull $H_{\mathcal{C}(G)}(R)$ of R, $\mathcal{C}(G)$-Radon partitions, $\mathcal{C}(G)$-anti-Radon sets, and the $\mathcal{C}(G)$-Radon number, respectively.

For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$.

If A is a set and $k \in \mathbb{N}_0$, then $\binom{A}{k} = \{B \subseteq A \mid |B| = k\}$. Let $\mathcal{P}(A)$ denote the power set of A.

We denote the vertices of the graph $\text{Grid}(n_1, n_2, \ldots, n_d)$ in the obvious way by elements of $[n_1] \times [n_2] \times \cdots \times [n_d]$, that is, two vertices $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ are adjacent in $\text{Grid}(n_1, n_2, \ldots, n_d)$ exactly if their Manhattan distance is 1, that is, $||u - v||_1 = |u_1 - v_1| + \cdots + |u_d - v_d| = 1$.

If $R = \{u^1, \ldots, u^r\}$ is a set of vertices of $\text{Grid}(n_1, n_2, \ldots, n_d)$ with $u^j = (u^j_1, \ldots, u^j_d)$ for $j \in [r]$, then the convex hull $H(R)$ of R is the set of integral points in

$$\left[\min_{j\in[r]} u^j_1, \max_{j\in[r]} u^j_1\right] \times \left[\min_{j\in[r]} u^j_2, \max_{j\in[r]} u^j_2\right] \times \cdots \times \left[\min_{j\in[r]} u^j_d, \max_{j\in[r]} u^j_d\right]. \tag{2}$$

In fact, $H(R)$ equals the Cartesian product of the 1-dimensional convex hulls of the d projections of R onto the different coordinates.

An *ordered partition* of a set V is a tuple $(V^1, \ldots, V^n)$ where $V = V^1 \cup \cdots \cup V^n$ is a partition of V.

2 Leveraging Eckhoff's construction

In this section we leverage Eckhoff's construction for d-dimensional grids. We start with a characterization of anti-Radon sets in our context.

Theorem 1 *Let $d, n_1, \ldots, n_d, r \in \mathbb{N}$. The graph* $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ *has an anti-Radon set R of order r if and only if there are d ordered partitions of $[r]$*

$$(V_1^1, \ldots, V_1^{n_1}), (V_2^1, \ldots, V_2^{n_2}), \ldots, (V_d^1, \ldots, V_d^{n_d})$$

such that, for every subset S of $[r]$, there are indices $i \in [d]$ and $j \in [n_i]$ with either

- $S = V_i^1 \cup \cdots \cup V_i^j$ *or*
- $S = [r] \setminus \left(V_i^1 \cup \cdots \cup V_i^j\right)$.

Proof: Let $R = \{u^1, \ldots, u^r\}$ be a set of vertices of $G = \mathrm{Grid}(n_1, n_2, \ldots, n_d)$ with $u^k = (u_1^k, \ldots, u_d^k)$ for $k \in [r]$. For $i \in [d]$, let ρ_i denote the i-th dimension of G, and π_i the projection of R onto ρ_i, that is, $\pi_i(R) = \{u_i^1, \ldots, u_i^r\} \subseteq [n_i]$.[1] By (2), the partition $R = R_1 \cup R_2$ is a Radon partition of R if and only if $\pi_i(R) = \pi_i(R_1) \cup \pi_i(R_2)$ is a Radon partition of $\pi_i(R)$ for every $i \in [d]$. Therefore, R is an anti-Radon set of $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ exactly if, for every subset R_1 of R, there is some $i \in [d]$ such that $\pi_i(R) = \pi_i(R_1) \cup \pi_i(R \setminus R_1)$ is not a Radon partition of $\pi_i(R)$. This happens exactly if, in that i-th dimension, the projections of the elements of R_1 are either all strictly to the left or all strictly to the right of the projections of the elements of $R \setminus R_1$.[2] If, for $i \in [d]$ and $j \in [n_i]$, the set V_i^j consists of all indices $k \in [r]$ such that $u_i^k = j$, then $(V_i^1, \ldots, V_i^{n_i})$ is an ordered partition of $[r]$. Furthermore, for a subset R_1 of R, the projections in $\pi_i(R_1)$ are all strictly to the left (resp. all strictly to the right) of the projections in $\pi_i(R \setminus R_1)$ if and only if there is some $j \in [n_i]$ with $S = V_i^1 \cup \cdots \cup V_i^j$ (resp. $S = [r] \setminus \left(V_i^1 \cup \cdots \cup V_i^j\right)$), where $S = \{k \in [r] \mid u^k \in R_1\}$. Altogether, the necessity follows.

In order to prove the sufficiency, we assume that the d ordered partitions $\left(V_i^j\right)_{j \in [n_i]}$ of $[r]$, for $i \in [d]$, exist as specified. For $k \in [r]$, let $u^k = (u_1^k, \ldots, u_d^k)$ be such that for $i \in [d]$, we have $u_i^k = j$ exactly if $k \in V_i^j$. In view of the above, it follows easily that $R = \{u^1, \ldots, u^r\}$ is an anti-Radon set of $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ of order r. This completes the proof. $\square$

As an example, consider the set $R = \{u^1, u^2, u^3, u^4\}$ where

$$u^1 = (2, 4, 3), u^2 = (1, 3, 1), u^3 = (1, 2, 4), \text{ and } u^4 = (2, 1, 2).$$

We shall see that R is an anti-Radon set of $\mathrm{Grid}(2, 4, 4)$. Figure 1 illustrates the three ordered partitions

$$\begin{aligned}
\rho_1 &= (V_1^1, V_1^2) = (\{u_1^2, u_1^3\}, \{u_1^1, u_1^4\}), \\
\rho_2 &= (V_2^1, V_2^2, V_2^3, V_2^4) = (\{u_2^4\}, \{u_2^3\}, \{u_2^2\}, \{u_2^1\}), \text{ and} \\
\rho_3 &= (V_3^1, V_3^2, V_3^3, V_3^4) = (\{u_3^2\}, \{u_3^4\}, \{u_3^1\}, \{u_3^3\})
\end{aligned}$$

derived from R as explained in the proof of Theorem 1.[3] It is easy to check that the conditions given in Theorem 1 are satisfied. More specifically, let $R_1 \cup R_2$ be a partition of R. If R_1 is

[1] Note that $u_i^{k_1} = u_i^{k_2}$ might occur for $k_1 \neq k_2$, that is, several of the elements of R might be projected onto the same point. Therefore, we consider $\pi_i(R)$ as a multiset.

[2] More formally, the elements of $\pi_i(R_1)$ are all strictly to the left (respectively, all strictly to the right) of the elements of $\pi_i(R \setminus R_1)$ exactly if $\max\{u_i^k : u^k \in R_1\} < \min\{u_i^k : u^k \in R \setminus R_1\}$ (resp. $\min\{u_i^k : u^k \in R_1\} > \max\{u_i^k : u^k \in R \setminus R_1\}$).

[3] In Figure 1, partite sets R_1 of order 3 are not indicated under braces, yet their complements are.

either empty or R, then $R_1 \cup R_2$ is clearly no Radon partition. In view of ρ_1, $R_1 \cup R_2$ is no Radon partition if

$$R_1 \in \{\{u^2, u^3\}, \{u^1, u^4\}\};$$

in view of ρ_2, $R_1 \cup R_2$ is no Radon partition if

$$R_1 \in \{\{u^4\}, \{u^4, u^3\}, \{u^4, u^3, u^2\}, \{u^3, u^2, u^1\}, \{u^2, u^1\}, \{u^1\}\};$$

and, finally, in view of ρ_3, $R_1 \cup R_2$ is no Radon partition if

$$R_1 \in \{\{u^2\}, \{u^2, u^4\}, \{u^2, u^4, u^1\}, \{u^4, u^1, u^3\}, \{u^1, u^3\}, \{u^3\}\}.$$

Since this covers all $2^4 = 16$ cases, R is an anti-Radon set of Grid$(2, 4, 4)$.

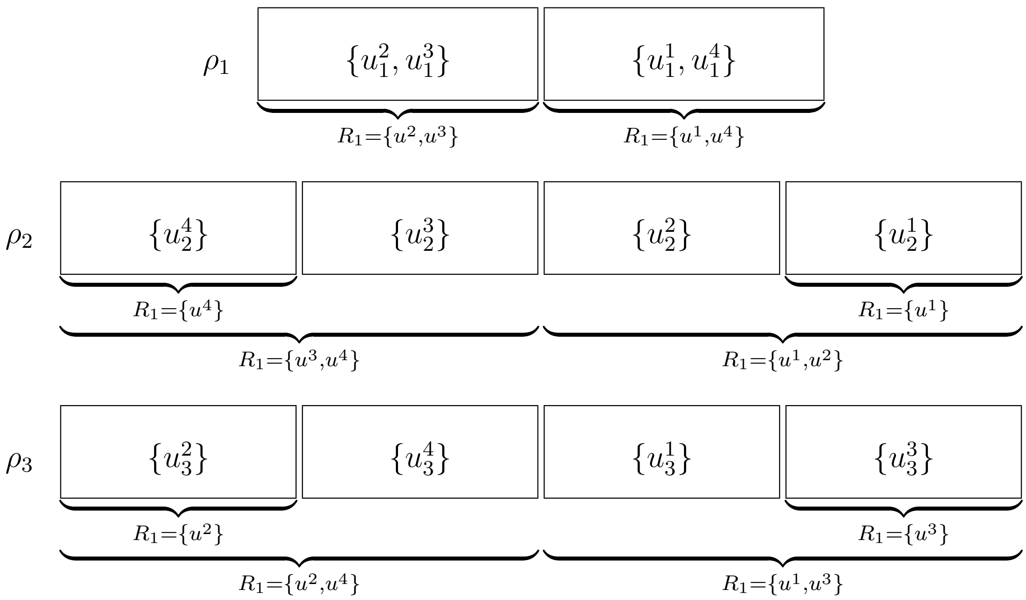

Figure 1: Example of anti-Radon set R of Grid$(2, 4, 4)$ with $r = 4$ elements.

If $n_i \geq r$ for every $i \in [d]$ in Theorem 1, then we may assume without loss of generality that each of the sets V_i^j contains at most one element, which easily implies the following observation that is implicit in Eckhoff's work.

Corollary 2 (Eckhoff [3]) *Let $d, r \in \mathbb{N}$. The convexity space $\left(\mathbb{R}^d, \mathcal{C}\left(\mathbb{R}_1^d\right)\right)$ has an anti-Radon set of order r if and only if there are d permutations $\rho_1, \rho_2, \ldots, \rho_d$ of $[r]$ such that for every subset S of $[r]$, there is some $i \in [d]$ with either $S = \{\rho_i(1), \rho_i(2), \ldots, \rho_i(|S|)\}$ or $S = \{\rho_i(r - |S| + 1), \rho_i(r - |S| + 2), \ldots, \rho_i(r)\}$.*

The Radon number of d-dimensional grids is always bounded from above by the Radon number of $\mathbb{R}^d$ with Manhattan metric.

Corollary 3 *If $d, n_1, \ldots, n_d \in \mathbb{N}$, then $r(\text{Grid}(n_1, n_2, \ldots, n_d)) \leq r(d)$ where $r(d)$ is as in (1).*

Proof: This follows immediately from the simple observation that every anti-Radon set of $\text{Grid}(n_1, n_2, \dots, n_d)$ of order k is an anti-Radon set of $\left(\mathbb{R}^d, \mathcal{C}\left(\mathbb{R}^d_1\right)\right)$ of order k. $\Box$

As observed by Jamison-Waldner, $r(\text{Grid}(n_1, n_2, \dots, n_d))$ will be smaller than $r(d)$ whenever the n_i are not large enough. The next corollary of Theorem 1 quantifies this observation to some extent.

Corollary 4 *Let $d, n_1, \dots, n_d, r \in \mathbb{N}$. The Radon number of* $\text{Grid}(n_1, n_2, \dots, n_d)$ *is at most the smallest integer r with*

$$2 + \sum_{i=1}^{d} 2(\min\{n_i, r\} - 1) < 2^r.$$

Proof: If $\text{Grid}(n_1, n_2, \dots, n_d)$ has an anti-Radon set of order r, then Theorem 1 implies the existence of d ordered partitions $\left(V_i^j\right)_{j \in [n_i]}$ of $[r]$ for $i \in [d]$ as specified. Since $[r]$ has exactly 2^r subsets and for $i \in [d]$, the set

$$\left\{V_i^1 \cup \dots \cup V_i^j \mid j \in [n_i]\right\} \cup \left\{[r] \setminus \left(V_i^1 \cup \dots \cup V_i^j\right) \mid j \in [n_i]\right\}$$

contains $\emptyset$, $[r]$, and exactly $2(\min\{n_i, r\} - 1)$ further subsets of $[r]$, the conditions stated in Theorem 1 imply

$$2 + \sum_{i=1}^{d} 2(\min\{n_i, r\} - 1) \geq 2^r,$$

which immediately implies the desired statement. $\Box$

If the n_i are too small, then Corollary 4 implies that no set of $r(d) - 1$ vertices of $\text{Grid}(n_1, n_2, \dots, n_d)$ is an anti-Radon set. Relying on Eckhoff's construction of large anti-Radon sets in $(\mathbb{R}^d, \mathcal{C}(\mathbb{R}^d_1))$, we can at least guarantee — see Theorem 6 below — the existence of a set R of $r(d) - 1$ vertices of $\text{Grid}(n_1, n_2, \dots, n_d)$ that does not allow Radon partitions $R = R_1 \cup R_2$ where the two partite sets R_1 and R_2 are of similar cardinalities, that is, the set R has no *balanced* Radon partition.

The following lemma summarizes some combinatorial observations by Eckhoff that are important for his construction of large anti-Radon sets.

Lemma 5 (Eckhoff [3]) *Let $r \in \mathbb{N}$.*

(i) For every $k \in \left\lfloor \frac{r}{2} \right\rfloor - 1$, there is an injective function $f : \binom{[r]}{k} \to \binom{[r]}{k+1}$ such that $A \subseteq f(A)$ for every $A \in \binom{[r]}{k}$.

(ii) If G is the graph with vertex set $\binom{[r]}{\lfloor \frac{r}{2} \rfloor}$ where two vertices A and B are adjacent exactly if they are disjoint, then G has a matching of order $\left\lfloor \frac{n(G)}{2} \right\rfloor$.

Proof: (i) This follows easily by applying Hall's theorem [5] to the bipartite graph with partite sets $V_1 = \binom{[r]}{k}$ and $V_2 = \binom{[r]}{k+1}$ where $u \in V_1$ is adjacent to $v \in V_2$ exactly if $u \subseteq v$. Hall's condition for the existence of a matching M covering all of V_1 follows from the simple observation that the degrees of the vertices in V_1 are all larger than the degrees of the vertices in V_2. The matching M defines the desired injective function in an obvious way.

(ii) If $n(G)$ is even, the trivial involution of $\binom{[r]}{\lfloor \frac{r}{2} \rfloor}$ defined by $A \mapsto [r] \setminus A$ easily implies the existence of the desired perfect matching. If $n(G)$ is odd, then the existence of the desired almost-perfect matching follows from Baranyai's theorem [1] (see Theorem 8 below and the corresponding comments on page 72 of [4]). $\Box$

Theorem 6 *Let $d, n_1, \ldots, n_d \in \mathbb{N}$. Let $n^* = \min\limits_{i \in [d]} n_i$.*

There is a set R of vertices of $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ *of order $r(d) - 1$ such that R has no Radon-partition $R = R_1 \cup R_2$ with*

$$\left\lfloor \frac{r(d)-1}{2} \right\rfloor - \left\lfloor \frac{n^*}{2} \right\rfloor + 1 \leq |R_1| \leq |R_2|.$$

Proof: Our proof strategy is to construct suitable ordered partitions and to argue similarly as in the proof of Theorem 1. The construction of the ordered partitions is split into two steps. In a first stept we establish the existence of certain pairs of disjoint sets $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ and $\mathcal{R}$ both contain about half the vertices of the grid. In a second step each such pair $(\mathcal{L}, \mathcal{R})$ will be refined into an ordered partition considering chains of subsets of $\mathcal{L}$ and $\mathcal{R}$, respectively. The subsets of $\mathcal{L}$ will make up the 'left' part of the ordered partition and the subsets of $\mathcal{R}$ will make up the 'right' part of the ordered partition.

Let $r = r(d) - 1$ and let $p = \lfloor \frac{r}{2} \rfloor$. By the definition of $r(d)$, we have $\binom{r}{p} \leq 2d$. By Lemma 5 (ii), there are d pairs of subsets of $[r]$

$$(\mathcal{L}_1(p), \mathcal{R}_1(p)), (\mathcal{L}_2(p), \mathcal{R}_2(p)), \ldots, (\mathcal{L}_d(p), \mathcal{R}_d(p))$$

such that

- $\mathcal{L}_i(p)$ and $\mathcal{R}_i(p)$ are disjoint for every $i \in [d]$ and
- $\binom{[r]}{p} = \{\mathcal{L}_i(p) \mid i \in [d]\} \cup \{\mathcal{R}_i(p) \mid i \in [d]\}$.

These pairs $(\mathcal{L}_i(p), \mathcal{R}_i(p))$ correspond to the edges $\mathcal{L}_i(p)\mathcal{R}_i(p)$ of the matching with possibly one additional pair for the unmatched vertex.

By Lemma 5 (i), there are sets $\mathcal{L}_i(k)$ and $\mathcal{R}_i(k)$ for every $i \in [d]$ and $k \in [p-1]$ such that

- $|\mathcal{L}_i(k)| = |\mathcal{R}_i(k)| = k$,
- $\mathcal{L}_i(1) \subseteq \mathcal{L}_i(2) \subseteq \ldots \subseteq \mathcal{L}_i(p-1) \subseteq \mathcal{L}_i(p)$,
- $\mathcal{R}_i(1) \subseteq \mathcal{R}_i(2) \subseteq \ldots \subseteq \mathcal{R}_i(p-1) \subseteq \mathcal{R}_i(p)$, and
- $\bigcup\limits_{k \in [p]} \binom{[r]}{k} = \bigcup\limits_{i \in [d]} \bigcup\limits_{k \in [p]} \{\mathcal{L}_i(k), \mathcal{R}_i(k)\}$.

For $i \in [d]$, let the ordered partition $(V_i^1, \ldots, V_i^{n_i})$ of $[r]$ be such that (cf. Figure 2 for an illustration)

$$\begin{array}{rcl}
V_i^1 & = & \mathcal{L}_i\left(\left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{n^*}{2} \right\rfloor + 1\right) \\
V_i^1 \cup V_i^2 & = & \mathcal{L}_i\left(\left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{n^*}{2} \right\rfloor + 2\right) \\
\cdots & & \cdots \\
V_i^1 \cup \cdots \cup V_i^{\lfloor \frac{n^*}{2} \rfloor} & = & \mathcal{L}_i\left(\left\lfloor \frac{r}{2} \right\rfloor\right)
\end{array}$$

and

$$
\begin{aligned}
V_i^{n_i} &= \mathcal{R}_i\left(\left\lfloor\frac{r}{2}\right\rfloor - \left\lfloor\frac{n^*}{2}\right\rfloor + 1\right)\\
V_i^{n_i-1} \cup V_i^{n_i} &= \mathcal{R}_i\left(\left\lfloor\frac{r}{2}\right\rfloor - \left\lfloor\frac{n^*}{2}\right\rfloor + 2\right)\\
\cdots \quad & \quad \cdots\\
V_i^{n_i-\lfloor\frac{n^*}{2}\rfloor+1} \cup \cdots \cup V_i^{n_i} &= \mathcal{R}_i\left(\left\lfloor\frac{r}{2}\right\rfloor\right).
\end{aligned}
$$

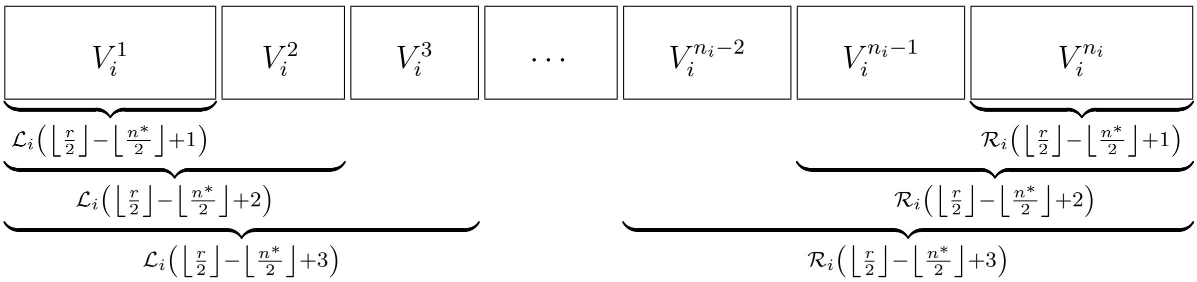

Figure 2: Illustration of a construction.

In view of the above conditions, the existence of ordered partitions $(V_i^1, \ldots, V_i^{n_i})$ of $[r]$ with these properties is straightforward.

Furthermore, since

$$\bigcup_{i\in[d]} \quad \bigcup_{k=\lfloor\frac{r}{2}\rfloor-\lfloor\frac{n^*}{2}\rfloor+1}^{\lfloor\frac{r}{2}\rfloor} \{\mathcal{L}_i(k), \mathcal{R}_i(k)\}$$

contains all subsets of $[r]$ of orders between $\lfloor\frac{r}{2}\rfloor - \lfloor\frac{n^*}{2}\rfloor + 1$ and $\lfloor\frac{r}{2}\rfloor$, the construction at the end of the proof of Theorem 1 implies the existence of a set R of vertices of $\mathrm{Grid}(n_1, n_2, \ldots, n_d)$ such that for every partition $R = R_1 \cup R_2$ with

$$\left\lfloor\frac{r}{2}\right\rfloor - \left\lfloor\frac{n^*}{2}\right\rfloor + 1 \le |R_1| \le \left\lfloor\frac{r}{2}\right\rfloor,$$

we have $H(R_1) \cap H(R_2) = \emptyset$, that is, $R = R_1 \cup R_2$ is not a Radon partition.[4] $\square$

Theorem 6 contains Eckhoff's construction as a special case. The next result essentially captures Eckhoff's formula (1).

Corollary 7 *Let $d, n_1, \ldots, n_d \in \mathbb{N}$. If $\min\limits_{i\in[d]} n_i \ge r(d) - 1$, then $r(\mathrm{Grid}(n_1, n_2, \ldots, n_d)) = r(d)$.*

Proof: This follows immediately from Theorem 6, since $\left\lfloor\frac{r(d)-1}{2}\right\rfloor - \lfloor\frac{n^*}{2}\rfloor \le 0$. $\square$

We believe that Theorem 6 can be improved in several ways. In fact, since the binomial coefficients $\binom{r}{\lfloor\frac{r}{2}\rfloor}, \binom{r}{\lfloor\frac{r}{2}\rfloor-1}, \ldots, \binom{r}{1}$ are strictly and very quickly decreasing, many of the smaller

[4] Since we want to consider $|R_1| \le |R_2|$, without loss of generality, the $\lfloor\frac{r}{2}\rfloor$ upper bound for $|R_1|$ is immediate.

subsets of $[r]$ occur several times in $\bigcup\limits_{i\in[d]}\bigcup\limits_{k\in[p]}\{\mathcal{L}_i(k),\mathcal{R}_i(k)\}$ considered as a multiset. For the construction of the anti-Radon set, each such subset would have to occur in fact only once. We believe that this effect can be used to prove that $\mathrm{Grid}(n_1,n_2,\ldots,n_d)$ has an anti-Radon set of order $r(d)-1$ as soon as

- all n_i are at least 2,
- among the latter, a sufficiently large portion less than 1 are at least 3,
- among the latter, a sufficiently large portion less than 1 are at least 4,
- and so forth.

Unfortunately, we are not able to make the term "sufficiently large portion" more precise.

We have shown in Theorem 6 how to exclude Radon partitions where the two partite sets are of similar order. With our next result, we show how to exclude Radon partitions where one of the two partite sets is small and the other is large.

Again we rely on Baranyai's theorem [1].

Theorem 8 (Baranyai [1]) *If $a_1,\ldots,a_t,r,k\in\mathbb{N}$ are such that $a_1+\cdots+a_t=\binom{r}{k}$, then $\binom{[r]}{k}$ can be partitioned into sets $S_1,\ldots,S_t$ such that for every $i\in[t]$*

- *$|S_i|=a_i$ and*
- *every element of $[r]$ occurs in either $\left\lfloor\frac{a_ik}{r}\right\rfloor$ or $\left\lceil\frac{a_ik}{r}\right\rceil$ many of the elements of S_i.*

Theorem 9 *Let $d,n_1,\ldots,n_d,r,p\in\mathbb{N}$ be such that $2p\le r$ and $2p\le n_i$ for every $i\in[d]$.*

If $\binom{r}{p}\le 2d$, then there is a set R of vertices of $\mathrm{Grid}(n_1,n_2,\ldots,n_d)$ of order r such that R has no Radon-partition $R=R_1\cup R_2$ with $|R_1|\le p$.

Proof: By Theorem 8 for $a_1=\ldots=a_{t-1}=2$ and $a_t\in\{1,2\}$, if G is a graph with vertex set $\binom{[r]}{p}$ where two vertices A and B are adjacent exactly if they are disjoint, then G has a matching of order $\left\lfloor\frac{n(G)}{2}\right\rfloor$. Using this perfect or almost-perfect matching together with Lemma 5 (i) as in the proof of Theorem 6, we obtain sets $\mathcal{L}_i(k)$ and $\mathcal{R}_i(k)$ such that, for every $i\in[d]$ and $k\in[p]$,

- $\mathcal{L}_i(p)$ and $\mathcal{R}_i(p)$ are disjoint,
- $|\mathcal{L}_i(k)|=|\mathcal{R}_i(k)|=k$,
- $\mathcal{L}_i(1)\subseteq\mathcal{L}_i(2)\subseteq\ldots\subseteq\mathcal{L}_i(p-1)\subseteq\mathcal{L}_i(p)$,
- $\mathcal{R}_i(1)\subseteq\mathcal{R}_i(2)\subseteq\ldots\subseteq\mathcal{R}_i(p-1)\subseteq\mathcal{R}_i(p)$, and
- $\bigcup\limits_{k\in[p]}\binom{[r]}{k}=\bigcup\limits_{i\in[d]}\bigcup\limits_{k\in[p]}\{\mathcal{L}_i(k),\mathcal{R}_i(k)\}$.

Now, for $i \in [d]$, since $n_i \geq 2p$, there are ordered partitions $(V_i^1, \ldots, V_i^{n_i})$ of $[r]$ such that

$$\begin{array}{rcl} V_i^1 & = & \mathcal{L}_i(1) \\ V_i^1 \cup V_i^2 & = & \mathcal{L}_i(2) \\ \ldots & & \ldots \\ V_i^1 \cup \cdots \cup V_i^p & = & \mathcal{L}_i(p) \end{array}$$

and

$$\begin{array}{rcl} V_i^{n_i} & = & \mathcal{R}_i(1) \\ V_i^{n_i-1} \cup V_i^{n_i} & = & \mathcal{R}_i(2) \\ \ldots & & \ldots \\ V_i^{n_i-p+1} \cup \cdots \cup V_i^{n_i} & = & \mathcal{R}_i(p)\,. \end{array}$$

Note that each of the sets V_i^j that we specified contains exactly one element. At this point the proof can be completed as for Theorem 6. $\square$

3 Some special cases

In this section we consider some special cases related to small values of n_i.

Proposition 10 *Let $d, n_1, \ldots, n_d \in \mathbb{N}$. If $n_i = 1$ for some $i \in [d]$, then* $r(\mathrm{Grid}(n_1, \ldots, n_d)) = r(\mathrm{Grid}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d))$

Proof: This follows easily from Theorem 1. The intuitive reason is that all vertices agree in the i-th coordinate, which is therefore not helpful for the construction of an anti-Radon set. $\square$

Proposition 11 *If $d \in \mathbb{N}$, then* $r(P_2^d) = r(\mathrm{Grid}(\underbrace{2, \ldots, 2}_{d \text{ factors}})) = \lfloor \log_2 (d+1) \rfloor + 2.$

Proof: By Theorem 1, P_2^d contains an anti-Radon set of order r if and only if there are d pairs $(\mathcal{L}_1, \mathcal{R}_1), \ldots, (\mathcal{L}_d, \mathcal{R}_d)$ of subsets $\mathcal{L}_i$ and $\mathcal{R}_i$ of $[r]$ such that

- $\mathcal{L}_i$ and $\mathcal{R}_i$ are disjoint for $i \in [d]$ and
- $\{\mathcal{L}_1, \ldots, \mathcal{L}_d\} \cup \{\mathcal{R}_1, \ldots, \mathcal{R}_d\}$ contains all subsets of $[r]$ that are distinct from $\emptyset$ and $[r]$.

If such pairs exist, then the second property implies $2^r - 2 \leq 2d$. Conversely, if $2^r - 2 \leq 2d$ and we let $h = 2^{r-1} - 1$ denote the number of proper subsets of $[r]$ that contain the element r, then we have $h \leq d$, and we can choose $\mathcal{L}_1, \ldots, \mathcal{L}_h$ to be exactly those proper subsets, choosing $\mathcal{L}_j$ arbitrarily among the subsets of $[r]$ for $h < j \leq d$. Now, letting $\mathcal{R}_i = [r] \setminus \mathcal{L}_i$ for $i \in [d]$ yields the d pairs $(\mathcal{L}_i, \mathcal{R}_i)$ with the two desired properties. Altogether, we have that $r(P_2^d) = \min \{r \in \mathbb{N} \mid 2^r - 2 > 2d\}$, which directly implies the desired result. $\square$

Proposition 12 *If $d \in \mathbb{N}$, then* $r(P_3^d) = r(\mathrm{Grid}(\underbrace{3, \ldots, 3}_{d \text{ factors}})) = \lfloor \log_2 d \rfloor + 3.$

Proof: By Corollary 4, $r(P_3^d)$ is at most the smallest integer r with $4d+2 < 2^r$. This integer is precisely $r = \lfloor \log_2 (4d+2) \rfloor + 1$, implying $r(P_3^d) \leq \lfloor \log_2 (8d+4) \rfloor$.

Now we constructively prove that $r(P_3^d) \geq \lfloor \log_2 (8d) \rfloor$ by presenting an anti-Radon set of order $r = \lfloor \log_2 (8d) \rfloor - 1$ for P_3^d. By Theorem 1, it suffices to show d tuples $(\mathcal{L}_1, \mathcal{M}_1, \mathcal{R}_1), \dots, (\mathcal{L}_d, \mathcal{M}_d, \mathcal{R}_d)$ of subsets $\mathcal{L}_i, \mathcal{M}_i, \mathcal{R}_i$ of $[r]$, such that

- $\mathcal{L}_i, \mathcal{M}_i$, and $\mathcal{R}_i$ are pairwise disjoint for $i \in [d]$ and
- $\{\mathcal{L}_1, \dots, \mathcal{L}_d\} \cup \{\mathcal{R}_1, \dots, \mathcal{R}_d\} \cup \{\mathcal{L}_1 \cup \mathcal{M}_1, \dots, \mathcal{L}_d \cup \mathcal{M}_d\} \cup \{\mathcal{R}_1 \cup \mathcal{M}_1, \dots, \mathcal{R}_d \cup \mathcal{M}_d\}$ contains all subsets of $[r]$ that are distinct from $\emptyset$ and $[r]$.

First, let $\mathcal{L}_d = [r-1], \mathcal{M}_d = \{r\}, \mathcal{R}_d = \emptyset$. Now, since $r = \lfloor \log_2 (8d) \rfloor - 1$, the number h of proper subsets of $[r-1]$ that contain the element $r-1$ satisfies

$$h = 2^{r-2} - 1 = 2^{\lfloor \log_2 (8d) \rfloor - 3} - 1 = 2^{\lfloor \log_2 (d) \rfloor} - 1 \leq d - 1.$$

We can therefore proceed in a fashion similar to that employed in the proof of Proposition 11 and choose $\mathcal{L}_1, \dots, \mathcal{L}_h$ to be exactly those proper subsets of $[r-1]$ that contain $\{r-1\}$, and choose $\mathcal{L}_j$ arbitrarily among the subsets of $[r-1]$ for $h < j \leq d-1$. Now we let $\mathcal{R}_i = [r-1] \setminus \mathcal{L}_i$ and $\mathcal{M}_i = \{r\}$, for $i \in [d-1]$. All we need to show is that the tuples $(\mathcal{L}_1, \mathcal{M}_1, \mathcal{R}_1), \dots, (\mathcal{L}_d, \mathcal{M}_d, \mathcal{R}_d)$ possess the two desired properties.

The first property is guaranteed by construction. As for the second, notice that the proper, non-empty subsets of $[r]$ can be partitioned into four types:

(i) those that are also non-empty subsets of $[r-2]$, therefore not containing $r-1$ or r;

(ii) those that contain $r-1$ but not r;

(iii) those that contain r but not $r-1$; and

(iv) those that contain both $r-1$ and r.

By choosing the tuples $(\mathcal{L}_i, \mathcal{M}_i, \mathcal{R}_i)$ for $i \in [d]$ as described above, all subsets of the first type will belong to $\bigcup_{j \in [d]} \{\mathcal{R}_j\}$, all subsets of the second type on their turn will belong to $\bigcup_{j \in [d]} \{\mathcal{L}_j\}$, all those of the third type will belong to $\bigcup_{j \in [d]} \{\mathcal{R}_j \cup \mathcal{M}_j\}$, and, finally, all those subsets of the fourth type will be elements of $\bigcup_{j \in [d]} \{\mathcal{L}_j \cup \mathcal{M}_j\}$. The existence of an anti-Radon set of order $r = \lfloor \log_2 (8d) \rfloor - 1$ is therefore guaranteed. Thus,

$$\lfloor \log_2 (8d) \rfloor \leq r(P_3^d) \leq \lfloor \log_2 (8d+4) \rfloor.$$

To conclude the proof, let 2^t be the smallest power of 2 that is greater than $8d$. Since d is a positive integer, $t > 3$ and $2^t - 8d$ is a multiple of 8, hence $8d+4$ is still less than 2^{t+1}, and the two logarithms in the expression above always round down to the same integer. Therefore $r(P_3^d) = \lfloor \log_2 (8d) \rfloor = 3 + \lfloor \log_2 d \rfloor$. $\square$

4 Complete results for $d \leq 9$

Unfortunately, the results presented in Sections 2 and 3 do not give exact values for all grids of whatever size or dimension. We therefore determined their Radon numbers up to dimension 9 with the help of a computer. Since an efficient algorithm to perform this computation for a given d-dimensional grid G — if one exists — is yet unknown, we employed a brute force approach of checking, for a decreasing parameter r starting at the best available upper bound on the Radon number of G (cf. Corollaries 3 and 4) minus 1, whether any subset of $V(G)$ of size r happened to be an anti-Radon set of G. In spite of the exponential nature of the task, some practical improvements could be made to what would have been the simplest, naivest, most straightforward implementation of the idea — and the results were far from disappointing.

To begin with, for certain instances we could obtain the Radon number directly (cf. Corollary 7, Propositions 11 and 12) or solve an equivalent problem on smaller grids (cf. Proposition 10). When this was not the case, however, then for a given size r of the anti-Radon set being searched for, we certainly did not have to test all subsets of V, since Theorem 1 allows us to look for ordered partitions of $[r]$ instead, one in each dimension, such that all proper, non-empty subsets of $[r]$ appear strictly to the left or strictly to the right in at least one such ordered partition, as discussed in the proof of that same Theorem 1. Admittedly, we were still facing an exponential task anyway, yet many orders of magnitude could be shaved off of the overall effort — so higher dimensions came within our reach — by observing some further properties of the Radon number of grids:

Observation 13 *If $G = \mathrm{Grid}(n_1, \ldots, n_d)$ has Radon number $r + 1$, then G admits an anti-Radon set R of order r such that, for $i \in [d]$, the projections of the elements of R onto the i-th dimension defines an ordered partition with exactly $\max\{n_i - r, 0\}$ empty partite sets, all of them to the right of the non-empty partite sets.*

Proof: This follows because applying an $\leq$-order preserving function to the elements of an anti-Radon set yields another anti-Radon set. $\square$

Observation 14 *If $G = \mathrm{Grid}(n_1, \ldots, n_d)$ has Radon number $r+1$, then there is an anti-Radon set $R = \{u^1, \ldots, u^r\}$ of G such that, for $i = 1, \ldots, \lceil r/2 \rceil$, the vertex u^i is the unique element of R whose i-th coordinate is 1.*

Proof: Let $R = \{u^1, \ldots, u^r\}$ be an anti-Radon set of G. By Theorem 1, for every $j \in [r]$, we can select some $i^j \in [d]$ such that the i^j-th coordinate of u^j is either the unique smallest or the unique largest i^j-th coordinate among all elements of R. Let B be the bipartite graph with vertex set $R \cup [d]$ and edge set $\{u^j i^j \mid j \in [r]\}$. By construction, the degree of u^j in B is 1 for all $j \in [r]$ and the degree of i in B is at most 2 for all $i \in [d]$. Therefore, B has exactly r edges and maximum degree at most 2. If M is a maximum matching of B, then $|M| \geq \lceil r/2 \rceil$. Now suitably reversing and exchanging some dimensions according to M yields the desired result. $\square$

For an anti-Radon set R of a grid G, let $A_R(i)$ be the set of all subsets of R whose orthogonal projections appear either strictly to the left or strictly to the right for one of the first i dimensions. Let $A_R(0) = \emptyset$.

Observation 15 *For $d \geq 2$, if $G = \mathrm{Grid}(n_1, \ldots, n_d)$ has Radon number $r+1$, then G admits an anti-Radon set R of order r such that, for $j \in [d]$, either $A_R(j)$ strictly includes $A_R(j-1)$ or $A_R(j-1) = \mathcal{P}(R)$.*

Proof: Let $R = \{u^1, \ldots, u^r\}$ be an anti-Radon set of G. By construction, $\emptyset = A_R(0) \subseteq A_R(1) \subseteq \ldots \subseteq A_R(d) = \mathcal{P}(R)$. If $j^* \in [d]$ is a smallest index such that $A_R(j^*) = A_R(j^*-1)$ but $A_R(j^*-1) \neq \mathcal{P}(R)$, then there is some $j' > j^*$ such that $A_R(j')$ strictly includes $A_R(j^*-1)$. Exchanging dimensions j^* and j' results in a new anti-Radon set R' that satisfies the desired properties for all $j \in [j^*]$. Iteratively repeating this operation results in an anti-Radon set with the desired property. $\square$

Observation 16 *Let $G = \mathrm{Grid}(n_1 \ldots, n_d)$ has Radon number $r+1$, then G has an anti-Radon set R of order r such that, for every $i \in [d]$, there is a labeling $u^1, \ldots, u^r$ of the vertices in R such that the projection of u^j onto the i-th dimension has coordinate $u_i^j = j$ for all $j \in [\min\{n_i, r\}]$.*

Proof: By Observation 13, G admits an anti-Radon set R of order r such that, out of the n_i available coordinates of the i-th coordinate, the leftmost $q = \min\{n_i, r\}$ coordinates will occur. By relabeling the vertices of R in such a way that, for all $j \in [q]$, one arbitrarily chosen vertex whose i-th coordinate is j is labeled u^j, the desired property is easily obtained. $\square$

Tables 1, 2 and 3 encode the Radon number of all grids $\mathrm{Grid}(n_1, \ldots, n_d)$ up to dimension 9. Notice that the lengths $(n_1, \ldots, n_d)$ of the path factors are ordered, and that an "∞" symbol indicates that the corresponding length is at least $(r(d)-1)-1$. Clearly, if $n_{\pi(i)} \geq n'_i$ for every $i \in [d]$ and some permutation π of $[d]$, then $r\left(\mathrm{Grid}(n_1, \ldots, n_d)\right) \geq r\left(\mathrm{Grid}(n'_1, \ldots, n'_d)\right)$. Therefore, the Radon number for every grid $\mathrm{Grid}(n_1, \ldots, n_d)$ for $1 \leq d \leq 9$ can be determined using Tables 1, 2 and 3.

d	dimension sizes	Cl 3	Cl 4	r(G)	max anti-Radon set
1	1	3	2	2	(1)
	∞		3	3	(1),(2)
2	2,2	4	3	3	(1,1),(1,2)
	2,∞		4	4	(1,3),(1,1),(2,2)
3	2,2,2	5	4	4	(1,2,2),(2,2,1),(1,1,1)
	2,3,∞		4	4	(1,1,1),(2,1,3),(1,3,3)
	2,∞,∞		5	5	(2,4,3),(1,3,1),(1,2,4),(2,1,2)
	3,3,3		4	4	(1,1,3),(1,1,1),(3,3,2)
	3,3,∞		5	5	(2,1,4),(1,1,1),(3,2,2),(1,3,3)
4	2,2,2,2	5	4	4	(1,1,2,2),(2,2,2,1),(1,1,1,1)
	2,2,2,∞		4	4	(1,1,1,1),(2,2,2,2),(1,1,1,3)
	2,2,3,3		4	4	(1,1,1,1),(2,2,1,3),(1,1,3,3)
	2,2,3,∞		5	5	(1,2,1,3),(1,1,2,1),(2,2,3,2),(1,1,3,4)
	2,3,3,3		5	5	(1,2,1,3),(1,1,2,1),(2,1,3,3),(1,3,3,2)
5	2,2,2,2,2	6	4	4	(1,2,2,1,1),(1,1,1,1,1),(2,2,1,2,2)
	2,2,2,2,3		4	4	(1,2,2,1,1),(2,2,1,2,3),(1,1,1,1,2)
	2,2,2,2,4		5	5	(1,2,2,1,3),(2,1,2,2,2),(1,1,1,1,1),(1,1,1,2,4)
	2,2,2,2,∞		5	5	(1,2,2,1,3),(2,1,2,2,2),(1,1,1,1,1),(1,1,1,2,4)
	2,2,2,3,3		5	5	(1,1,1,3,3),(1,1,1,1,1),(2,1,2,3,1),(1,2,2,2,2)
	2,2,2,3,∞		5	5	(1,2,2,2,2),(2,1,2,3,4),(1,1,1,3,1),(1,1,1,1,3)
	2,2,3,3,3		5	5	(1,1,1,3,3),(1,1,1,1,1),(2,1,2,3,1),(1,2,2,2,2)
	2,2,∞,∞,∞		5	5	(2,1,4,4,3),(1,2,3,1,4),(2,1,2,2,1),(1,1,1,3,2)
	2,3,3,3,3		5	5	(1,1,1,3,3),(1,1,1,1,1),(2,1,2,3,1),(1,2,2,2,2)
	2,∞,∞,∞,∞		6	5	(1,4,4,2,4),(2,3,2,4,2),(1,2,1,1,1),(1,1,3,3,3)
	3,3,3,3,3		5	5	(3,3,3,2,1),(3,2,2,1,3),(1,1,2,1,1),(2,1,1,3,2)
	3,3,4,4,∞		5	5	(2,1,1,4,3),(1,1,3,2,1),(3,3,4,3,4),(3,2,2,1,2)
	3,3,4,∞,∞		6	6	(2,1,3,4,1),(3,3,1,5,3),(3,2,4,2,4),(1,1,1,3,5),(1,3,2,1,2)
	3,4,4,4,4		5	5	(2,1,4,1,3),(3,4,3,4,4),(3,2,1,2,2),(1,3,2,3,1)
	3,4,4,4,∞		6	6	(1,1,4,2,3),(3,3,2,1,2),(2,4,4,4,1),(1,4,1,3,4),(3,2,3,4,5)
	4,4,4,4,4		6	6	(4,4,3,4,1),(2,4,4,1,3),(3,2,2,3,4),(1,3,1,2,1),(1,1,4,4,2)

Table 1: The Radon number for all d-dimensional grids for $1 \leq d \leq 5$. The "Cl 3" and "Cl 4" columns show, respectively, the upper bounds given by Corollary 3 (fixed for each d, regardless of the orders) and Corollary 4 (taking the orders into account). The "max anti-Radon set" column presents an anti-Radon set of order $r(G) - 1$.

d	dimension sizes	Cl 3	Cl 4	r(G)	max anti-Radon set
6	2,2,2,2,2,2	6	4	4	(1,1,2,1,1,1),(2,2,2,2,2,2),(1,2,1,1,1,1)
	2,2,2,2,2,3		5	5	(1,1,2,1,2,3),(1,2,1,2,1,3),(2,1,1,2,2,2),(1,1,1,1,1,1)
	2,2,2,∞,∞,∞		6	5	(2,1,2,2,1,1),(2,2,1,3,2,2),(1,1,2,4,4,4),(1,1,1,1,3,3)
	2,2,3,3,∞,∞		5	5	(2,1,3,1,4,4),(1,1,1,1,1,3),(1,2,2,2,3,1),(1,1,3,3,2,2)
	2,2,3,4,4,∞		5	5	(1,1,3,4,3,4),(2,1,3,2,1,2),(1,1,1,1,2,3),(2,2,2,3,4,1)
	2,2,3,4,∞,∞		6	6	(2,2,2,3,3,1),(1,1,1,1,1,2),(2,1,1,2,4,5),(1,2,3,1,5,3),(1,1,3,4,2,4)
	2,2,4,4,4,4		5	5	(2,1,2,2,1,2),(1,1,1,3,2,3),(2,1,4,4,3,4),(1,2,3,1,4,1)
	2,2,4,4,4,∞		6	6	(2,1,1,2,4,4),(2,1,3,4,2,2),(1,2,4,1,3,3),(1,2,2,3,1,5),(1,1,1,1,1,1)
	2,3,3,3,4,∞		5	5	(1,1,2,1,1,3),(1,3,2,3,2,2),(2,3,3,1,4,4),(1,2,1,2,3,1)
	2,3,3,4,4,4		5	5	(2,2,1,3,4,1),(2,3,2,2,1,2),(1,3,3,4,3,4),(1,1,2,1,2,3)
	2,3,3,4,4,∞		6	6	(1,1,2,1,1,1),(2,2,3,2,1,5),(2,3,1,3,2,2),(1,1,3,4,3,3),(2,1,1,1,4,4)
	2,3,4,4,4,4		6	6	(1,3,1,2,1,1),(2,3,4,1,2,3),(1,2,2,3,3,4),(2,3,3,4,4,1),(1,1,4,4,1,2)
	3,3,3,3,3,∞		5	5	(3,3,3,1,3,4),(2,2,1,2,2,3),(1,1,2,1 1,1),(3,3,2,3,3,2)
	3,3,3,3,4,4		5	5	(1,1,2,1,1,3),(3,3,2,3,2,2),(2,2,1,2,3,1),(3,3,3,1,4,4)
	3,3,3,3,4,∞		6	6	(1,2,1,2,2,5),(3,1,3,3,1,4),(1,1,2,1,1,1),(3,1,1,1,4,3),(2,3,2,3,3,2)
	3,3,3,4,4,4		6	6	(3,3,2,3,4,3),(1,3,1,4,2,1),(2,2,3,4,1,4),(1,1,3,2,3,2),(3,3,3,1,1,1)
7	2,2,2,2,2,2,2	6	5	5	(1,1,2,1,1,1,1),(1,2,1,1,1,2,2),(2,2,2,1,2,2,1),(1,2,2,2,2,1,2)
	2,2,2,3,4,4,4		5	5	(2,1,2,3,4,3,4),(1,2,2,2,1,4,1),(1,1,1,1,3,2,3),(1,1,2,3,2,1,2)
	2,2,2,3,4,4,∞		6	6	(2,2,1,2,4,3,3),(1,2,1,3,2,1,5),(1,1,1,1,1,4,4),(2,1,2,3,1,2,2),(1,1,1,1,3,1,1)
	2,2,2,4,4,4,4		6	6	(1,1,1,1,1,1,1),(1,2,2,3,2,4,1),(2,1,2,4,3,1,2),(2,1,2,2,1,3,4),(1,2,1,2,4,2,3)
	2,2,3,3,3,3,3		5	5	(1,1,2,1,1,1,1),(1,2,1,1,1,2,2),(2,2,2,1,2,2,1),(1,2,2,2,2,1,2)
	2,2,3,3,∞,∞,∞		6	6	(2,2,2,2,5,5,3),(1,1,1,1,1,4,4),(1,1,1,3,4,2,5),(2,1,1,3,2,3,1),(1,1,3,1,3,1,2)
	2,2,3,4,4,4,4		6	6	(2,2,2,3,4,2,1),(2,2,3,1,2,4,3),(1,1,3,4,1,3,1),(1,2,3,4,3,1,4),(1,2,1,2,1,1,2)
	2,3,3,3,3,3,4		5	5	(2,3,2,3,3,2,4),(1,3,1,3,3,1,2),(1,1,3,1,1,1,3),(1,2,3,2,2,3,1)
	2,3,3,3,3,4,4		6	6	(2,3,2,3,3,2,4),(1,1,3,3,1,1,3),(2,3,3,1,2,1,1),(1,3,1,3,1,3,1),(1,2,3,2,3,4,2)
	3,3,3,3,3,3,3		5	5	(3,3,1,3,3,1,1),(3,3,2,3,3,2,3),(2,2,3,2,2,3,1),(1,1,3,1,1,1,2)
	3,3,3,3,3,3,4		6	6	(3,1,3,2,1,3,1),(1,3,3,3,1,1,2),(3,3,1,3,3,2,1),(3,2,3,2,3,1,4),(2,3,2,1,2,3,3)

Table 2: The Radon number for all d-dimensional grids, $6 \leq d \leq 7$.

d	dimension sizes	Cl 3	Cl 4	r(G)	max anti-Radon set
8	2,2,2,2,2,2,2,2	6	5	5	(1,1,2,1,1,1,1,1),(1,2,1,1,1,1,2,2),(2,2,2,2,1,2,2,1),(1,2,2,1,2,2,1,2)
	2,2,2,2,2,4,4,4		5	5	(2,1,2,1,1,4,4,4),(1,2,1,2,1,3,1,1),(1,1,2,2,2,2,2,2),(1,1,1,1,1,1,3,3)
	2,2,2,2,2,4,4,∞		6	6	(1,1,1,1,1,1,2,1),(2,1,1,2,2,1,4,4),(1,2,1,1,2,2,1,5),(1,2,1,2,1,4,3,3),(2,1,2,2,2,3,1,2)
	2,2,2,2,3,3,3,∞		5	5	(1,2,1,2,2,2,2,3),(2,1,2,1,1,3,3,4),(1,1,2,2,3,3,3,2),(1,1,1,1,1,1,1,1)
	2,2,2,2,3,3,4,4		5	5	(2,1,2,1,1,3,4,4),(1,1,2,2,3,3,2,2),(1,1,1,1,1,1,1,3),(1,2,1,2,2,2,3,1)
	2,2,2,2,3,3,4,∞		6	6	(2,2,1,2,3,3,1,3),(1,2,1,1,2,1,2,5),(1,1,1,1,1,2,1,1),(1,1,1,2,1,3,4,4),(2,1,2,2,2,1,3,2)
	2,2,2,2,3,4,4,4		6	6	(2,2,2,1,3,2,2,4),(1,1,1,2,3,2,1,1),(2,2,1,2,2,3,4,2),(2,1,1,2,1,4,1,4),(1,1,1,1,1,1,3,3)
	2,2,2,3,3,3,3,4		5	5	(1,2,1,2,2,2,2,3),(1,1,2,3,3,3,3,2),(2,1,2,1,3,3,3,4),(1,1,1,1,1,1,1,1)
	2,2,2,3,3,3,3,∞		6	6	(1,2,1,2,1,1,2,5),(2,1,1,1,3,2,3,4),(1,1,1,1,2,1,1,1),(1,1,2,2,1,3,3,2),(2,2,1,3,2,3,1,3)
	2,2,2,3,3,3,4,4		6	6	(2,1,1,1,3,3,4,1),(1,2,1,1,2,2,3,4),(2,2,2,2,1,3,2,2),(2,1,1,3,3,1,2,3),(1,1,1,1,1,1,1,1)
	2,2,3,3,3,3,3,3		5	5	(2,1,3,2,2,2,2,2),(1,1,2,3,3,3,3,3),(1,2,3,3,1,3,3,3),(1,1,1,1,1,1,1,1)
	2,2,3,3,3,3,3,4		6	6	(2,1,1,1,3,2,3,4),(2,1,3,2,3,1,1,2),(1,1,1,1,1,2,1,1),(2,2,2,2,1,3,3,2),(1,2,1,3,2,1,2,3)
	2,3,3,3,3,3,3,3		6	6	(1,2,1,1,3,3,2,3),(2,3,2,2,2,2,1,2),(1,3,1,3,1,3,3,1),(1,2,3,2,1,1,3,3),(1,1,1,1,1,1,1,1)
9	2,2,2,2,2,2,2,2,2	6	5	5	(1,2,2,1,1,2,2,1,2),(2,2,2,2,2,1,2,2,1),(1,1,2,1,1,1,1,1,1),(1,2,1,1,1,1,1,2,2)
	2,2,2,2,2,2,2,4,∞		5	5	(1,1,1,1,1,1,1,1,3),(1,1,2,2,2,1,1,2,2),(1,2,1,2,1,1,1,3,1),(2,1,2,1,1,2,2,4,4)
	2,2,2,2,2,2,3,3,3		5	5	(1,1,1,2,2,2,2,3,3),(2,2,1,2,1,1,3,3,3),(1,1,1,1,1,1,1,1,1),(1,1,2,1,2,1,3,2,2)
	2,2,2,2,2,2,3,3,∞		5	5	(2,1,2,1,1,2,3,3,4),(1,1,1,1,1,1,1,1,1),(1,2,1,2,1,1,2,2,3),(1,1,2,2,2,1,3,3,2)
	2,2,2,2,2,2,3,4,4		6	6	(1,1,1,1,1,1,1,1,3),(1,2,2,1,1,3,3,2,2),(2,2,1,1,2,2,2,3,4),(2,1,1,2,2,3,1,2,1), (1,1,1,1,2,1,3,4,1)
	2,2,2,2,2,3,3,3,4		6	6	(1,1,1,1,1,3,3,2,4),(1,1,1,1,1,1,1,1,1),(1,2,2,1,2,1,3,3,2),(2,2,1,1,3,2,2,1,3), (2,1,1,2,2,3,1,3,2)
	2,2,2,2,3,3,3,3,3		5	5	(2,1,1,2,2,3,3,3,3),(1,1,1,1,1,1,1,1,1),(1,2,1,2,3,1,3,3,3),(2,1,2,1,3,2,2,2,2)
	2,2,2,2,3,3,3,3,4		6	6	(1,1,1,1,1,3,3,2,4),(1,1,1,1,1,1,1,1,1),(1,2,2,1,2,1,3,3,2),(2,2,1,1,3,2,2,1,3), (2,1,1,2,2,3,1,3,2)
	2,2,2,3,3,3,3,3,3		6	6	(2,1,1,3,3,3,2,1,1),(1,2,1,3,2,2,1,2,3),(2,1,2,2,2,1,1,3,1),(2,1,1,2,1,3,3,3,3), (1,1,1,1,1,1,2,1,2)

Table 3: The Radon number for all d-dimensional grids, $8 \leq d \leq 9$.

5 Conclusion

There are numerous obvious challenging problems and research questions motivated by our results.

- As we have already stated, we believe that there are considerable improvements of Theorem 6.
- The results in Section 3 can be considered mere examples and one might try to obtain closed formulas for further cases. In fact, the specific constructions used in order to prove such formulas might indicate general improvements of Theorem 6. A first case to look at might be $P_2^{d_2} \times P_3^{d_3}$.
- In Section 4 we reported computational results. In fact, it is easy to see that computing the (geodetic) Radon number of general graphs is at least as hard as computing their clique number, which even implies very strong non-approximability results. Nevertheless, there might be a more efficient way of calculating the Radon number of grids, which are rather special and simply structured graphs.

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