

Towards a provably resilient scheme for graph-based watermarking

Lucila Maria Souza Bento^{1,2}
Davidson Boccardo²
Raphael Carlos Santos Machado^{1,2}
Vinícius Gusmão Pereira de Sá¹
Jayme Luiz Szwarcfiter^{1,2,3}

¹ Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brasil,
lucilabento@ppgi.ufrj.br, vigusmao@dcc.ufrj.br

² Instituto Nacional de Metrologia, Qualidade e Tecnologia, Rio de Janeiro, Brasil,
drboccardo@inmetro.gov.br, rcmachado@inmetro.gov.br

³ COPPE-Sistemas, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brasil,
jayme@nce.ufrj.br

Abstract

Techniques of watermarking/fingerprinting concern the embedding of identification data into a digital object, allowing for later claims of authorship/ownership and therefore discouraging piracy. Graph-based watermarking schemes comprise an encoding algorithm, which translates a given number (the identifier, usually a positive integer) onto some appropriately tailored graph (the watermark), and a decoding algorithm, which extracts the original identifier from a given watermark. Collberg, Kobourov, Carter and Thomborson (*Error-correcting graphs for software watermarking*, WG'03) introduced one such scheme, meant for software watermarking, in which an integer key was encoded onto a reducible permutation graph. A number of interesting ideas have further improved the original scheme, including the formulation of a particularly promising linear-time codec by Chroni and Nikolopoulos. We extend the work of these authors in various aspects. First, we characterize the class of graphs constituting the image of Chroni and Nikolopoulos's encoding function. Furthermore, we formulate a novel, linear-time watermark-to-key decoding algorithm which detects and recovers from ill-intentioned removals of $k \leq 2$ edges. Finally, our results also include the detection of $k \leq 5$ edge modifications (insertions/deletions) in polynomial time and a proof that such bound is tight, so the resilience of the considered watermarking scheme is fully determined. Our proof that graphs of a well characterized class can detect—and recover from—bounded-magnitude distortive attacks reinforces the interest in regarding those graphs as possible watermarking solutions to numerous applications.

Keywords. algorithms; reducible permutation graphs; graph-based watermarking; software security

1 Introduction

Preventing the theft of intellectual property has been an everlasting, highly sought after goal. For ages have books been photocopied, pictures counterfeited, recordings unscrupulously shared. More recently, the criminal reproduction of software known as software piracy has become a big concern, and the struggle between the opposing forces of pirates and security engineers has been increasingly fierce.

Watermarking/fingerprinting an object is the act of embedding an identifier of authorship/ownership within that object, so to discourage illegal copying. Different approaches to software watermarking have been devised to date, still none of them was ever proved to be sufficiently resilient, let alone immune, to the numerous forms of program transformation attacks. Naturally, a lot of research towards strengthening such methods has been endeavored. The pioneering graph-based watermarking algorithm formulated by Davidson and Myrhvold [10] inspired the first watermarking algorithm where a positive integer *key* meant for identification was encoded as a special digraph to be disguised into a method's flow graph, by Venkatesan, Vazirani and Sinha [14]. Other graph-based software watermarking schemes include [6, 8, 9, 13].

In this paper, we consider the ingenious graph-based watermarking scheme introduced by Collberg, Kobourov, Carter and Thomborson [7], and afterwards developed and improved upon by Chroni and Nikolopoulos [1]. These latter authors proposed a linear-time codec where the watermarks are instances of the reducible permutation graphs introduced in [7]. (A series of papers on watermarking by the same authors include, but is not limited to, [2–5].) Though the mechanics of encoding-decoding the watermark is well described in [1], the class of such special graphs has not been fully characterized. Moreover, not much was known thus far about the resilience of Chroni and Nikolopoulos's graphs to malicious attacks, even though their ability to withstand attacks in the form of a single edge modification has been suggested without proofs. A thorough scrutiny of the structural properties of the aforementioned graph class allowed us to give it a formal characterization, as well as to introduce a linear-time decoding algorithm that retrieves the correct, untampered with encoded key even when $k \leq 2$ edges of the watermark are missing. Such algorithm allows for the determination of the exact resilience level of such graphs against distortive attacks.

The paper is organized as follows. In Section 2, we recall the codec from [1], formulating and proving a number of properties of the employed structures. In Section 3, we characterize the class of canonical reducible permutation graphs. In Section 4, we tackle the edge-removal attack model, proving that, for keys of size $n > 2$, it is always possible to identify and recover from attacks that remove $k \leq 2$ edges. Finally, in Section 5, we formulate a linear-time algorithm that reconstructs the original digraph, in case $k \leq 2$ edges are missing, recovering the encoded key thereafter. As a corollary of the results hitherto presented, we fully determine the resilience of the considered watermarking scheme. Section 6 concludes the paper with our final remarks and future directions.

Throughout the paper, many proofs were omitted due to space constraints, and shall be included in a future extended version of the text.

2 The watermark by Chroni and Nikolopoulos

We start by briefly recalling the watermarking codec described in [1].

Let ω be a positive integer key, with n the size of the binary representation B of ω . Let

also n_0 and n_1 be the number of 0's and 1's, respectively, in B , and let f_0 be the index¹ of the leftmost 0 in B . The extended binary B^* is obtained by concatenating n digits 1, followed by the one's complement² of B and by a single digit 0. We let $n^* = 2n + 1$ denote the size of B^* , and we define $Z_0 = (z_i^0)$, $i = 1, \dots, n_1 + 1$, as the ascending sequence of indexes of 0's in B^* , and $Z_1 = (z_i^1)$, $i = 1, \dots, n + n_0$, as the ascending sequence of indexes of 1's in B^* .

Let S be a sequence. We denote by S^R the sequence formed by the elements of S in backward order. If $S = (s_i)$, $i = 1, \dots, t$, is a sequence of size t , and there is an integer $k \leq t$ such that the subsequence consisting of the elements of S with indexes less than or equal to k is ascending, and the subsequence consisting of the elements of S with indexes greater than or equal to k is descending, then we say S is *bitonic*. If all t elements of a sequence S are distinct and belong to $\{1, \dots, t\}$, then S is a *permutation*. If S is a permutation of size t , and, for all $1 \leq i \leq t$, the equality $i = s_{s_i}$ holds, then we say S is *self-inverting*. In this case, the unordered pair (i, s_i) is called a *2-cycle* of S , if $i \neq s_i$, and a *1-cycle* of S , if $i = s_i$. If S_1, S_2 are sequences, we denote by $S_1 || S_2$ the sequence formed by the elements of S_1 followed by the elements of S_2 .

Back to Chroni and Nikolopoulos's codec, the bitonic permutation $P_b = Z_0 || Z_1^R = (b_i)$, $i = 1, \dots, n^*$, is obtained by appending to Z_0 the elements of Z_1 in backward order, and, finally, the self-inverting permutation P_s is obtained from P_b as follows: for $i = 1, \dots, n^*$, index b_i in P_s is assigned to element $s_{b_i} = b_{n^*-i+1}$, and index b_{n^*-i+1} in P_s is assigned to element $s_{b_{n^*-i+1}} = b_i$. In other words, the 2-cycles of P_s correspond to the n unordered pairs of distinct elements of P_b that are equidistant from the extremes of P_b , namely the pairs $(p, q) = (b_i, b_{n^*-i+1})$, for $i = 1, \dots, n$. Since the central index $i = n + 1$ of P_b is the solution of equation $n^* - i + 1 = i$, element b_{n+1} —and no other—will constitute a 1-cycle in P_s . We refer to such element of P_s as its *fixed element*, and we let f denote it.

The watermark generated by the codec from [1] belongs to the class of reducible permutation graphs first defined in [7]. It is a directed graph G whose vertex set is $\{0, 1, \dots, 2n + 2\}$, and whose edge set contains $4n + 3$ edges, to wit: a *path edge* $(u, u - 1)$ for $u = 1, \dots, 2n + 2$, constituting a Hamiltonian path that will be unique in G , and a *tree edge* from u to $q(u)$, for $u = 1, \dots, n^*$, where $q(u)$ is defined as the vertex $v > u$ with the greatest index in P_s to the left of u , if such v exists, or $2n + 2$ otherwise³. A graph so obtained is called a *canonical reducible permutation graph*.

Let us glance at an example. For $\omega = 43$, we have $B = 101011$, $n = 6$, $n_0 = 2$, $n_1 = 4$, $f_0 = 2$, $B^* = 1111110101000$, $n^* = 13$, $Z_0 = (7, 9, 11, 12, 13)$, $Z_1 = (1, 2, 3, 4, 5, 6, 8, 10)$, $P_b = (7, 9, 11, 12, 13, 10, 8, 6, 5, 4, 3, 2, 1)$, $P_s = (7, 9, 11, 12, 13, 10, 1, 8, 2, 6, 3, 4, 5)$ and $f = 8$. The generated watermark G will have, along with the path edges in Hamiltonian path $14 \rightarrow 13 \rightarrow \dots \rightarrow 0$, also the tree edges $(1, 10)$, $(2, 8)$, $(3, 6)$, $(4, 6)$, $(5, 6)$, $(6, 8)$, $(7, 14)$, $(8, 10)$, $(9, 14)$, $(10, 13)$, $(11, 14)$, $(12, 14)$ and $(13, 14)$, as illustrated in Figure 1.

We now state a number of properties of canonical reducible permutation graphs and the special permutations they are associated to. Let G still be the canonical reducible permutation graph associated to a key ω of size n , and P_b and P_s , respectively, the bitonic and the self-inverting permutations dealt with during the construction of G .

Property 1 For $1 \leq i \leq n$, element b_{n+i+1} in P_b is equal $n - i + 1$, that is, the n rightmost elements in P_b are $1, 2, \dots, n$ when read from right to left.

Property 2 The elements whose indexes are $1, 2, \dots, n$ in P_s are all greater than n .

¹The index of the first element in all sequences considered in the text is 1.

²The *one's complement* of a binary R is obtained by swapping all 0's in R for 1's and vice-versa.

³The rationale behind the name *tree edge* is the fact that such edges induce a spanning tree of $G \setminus \{0\}$.

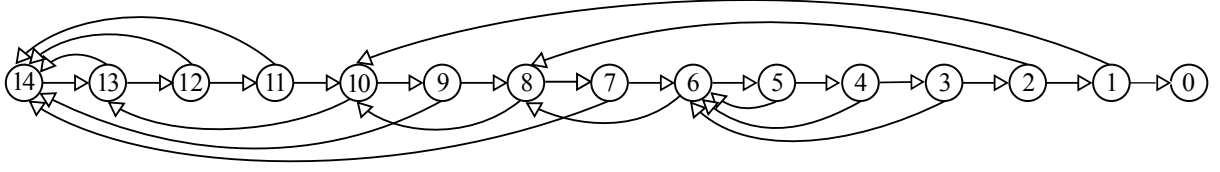


Figure 1: Watermark for key $\omega = 43$.

Property 3 *The fixed element f satisfies $f = n + f_0$, unless the key ω is equal to $2^k - 1$ for some integer k , whereupon $f = n^* = 2n + 1$.*

Property 4 *In self-inverting permutation P_s , elements indexed $1, 2, \dots, f - n - 1$ are respectively equal to $n + 1, n + 2, \dots, f - 1$, and elements indexed $n + 1, n + 2, \dots, f - 1$ are respectively equal to $1, 2, \dots, f - n - 1$.*

Property 5 *The first element in P_s is $s_1 = n + 1$, and the central element in P_s is $s_{n+1} = 1$.*

Property 6 *If $f \neq n^*$, then the index of element n^* in P_s is equal to $n_1 + 1$, and vice-versa. If $f = n^*$, then the index of element n^* in P_s is also n^* .*

Property 7 *The subsequence of P_s consisting of elements indexed $1, 2, \dots, n + 1$ is bitonic.*

Property 8 *For $u \neq 2n + 1$, $(u, 2n + 2)$ is a tree edge of watermark G if, and only if, $u - n$ is the index of a digit 1 in the binary representation B of the key ω represented by G .*

Property 9 *If (u, k) is a tree edge of watermark G , with $k \neq 2n + 2$, then (i) element k precedes u in P_s ; and (ii) if v is located somewhere between k and u in P_s , then $v < u$.*

3 Canonical reducible permutation graphs

This section is devoted to the characterization of the class of canonical reducible permutation graphs. After describing some terminology, we define the class using purely graph-theoretical predicates, then we prove it corresponds exactly to the set of all watermark instances possibly produced by Chroni and Nikolopoulos's encoding algorithm [1]. Finally, we characterize it in a way that suits the design of a new, resilient, linear-time decoding algorithm.

Given a graph G , we let $V(G)$ and $E(G)$ denote, as usual, the vertex set and edge set of G , respectively. Also, we let $N_G^+(v)$ and $N_G^-(v)$ respectively denote the set of out-neighbors and in-neighbors of vertex $v \in V(G)$. A *reducible flow graph* [11, 12] is a directed graph G with source $s \in V(G)$, such that, for each cycle C of G , every path from s to C reaches C at a same vertex. It is well known that a reducible flow graph has at most one Hamiltonian cycle.

Definition 10 *A self-labeling reducible flow graph relative to $n > 1$ is a directed graph G s.t.*

- (i) $|V(G)| = 2n + 3$;
- (ii) G presents exactly one directed Hamiltonian path, hence there is a unique labeling function $\sigma : V(G) \rightarrow \{0, 1, \dots, 2n + 2\}$ of the vertices of G such that the order of the labels along the Hamiltonian path is precisely $2n + 2, 2n + 1, \dots, 0$;
- (iii) considering the labeling σ as in the previous item, $N_G^+(0) = \emptyset$, $N_G^-(0) = \{1\}$, $N_G^+(2n + 2) = \{2n + 1\}$, $|N_G^-(2n + 2)| \geq 2$, and, for all $v \in V(G) \setminus \{0, 2n + 2\}$, $N_G^+(v) = \{v - 1, w\}$, for some $w > v$.

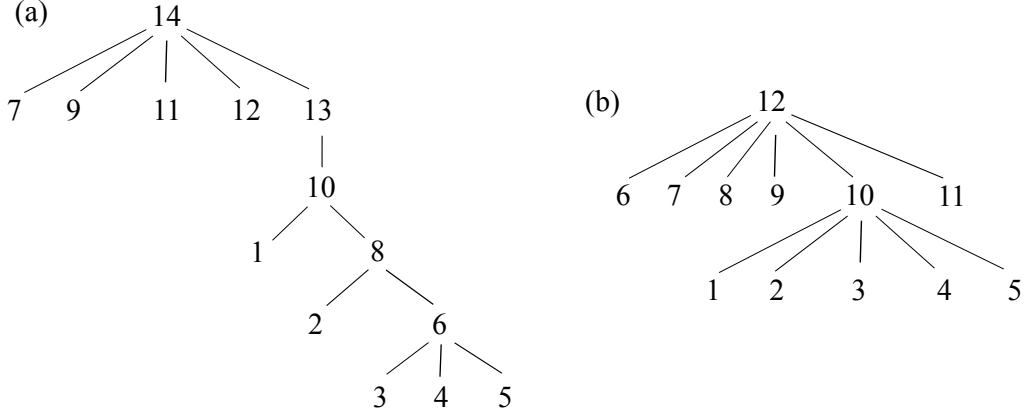


Figure 2: Representative trees of the watermarks for keys (a) $\omega = 43$ and (b) $\omega = 31$.

From now on, without loss of generality, we shall take σ for granted and assume the vertex set of any self-labeling reducible flow graph G is the very set $V(G) = \{0, 1, \dots, 2n+2\}$, yielding the unique Hamiltonian path $2n+2, 2n+1, \dots, 0$ in G .⁴

Let G be a self-labeling reducible flow graph and H its unique Hamiltonian path. We define a tree T with vertex set $V(T) = V(G) \setminus \{0\}$, and edge set $E(T)$ comprising all edges in $E(G) \setminus E(H)$ deprived of their orientation. We call T the *representative tree* of G , and we regard it as a *rooted* tree whose root is $2n+2$, and where the children of each $v \in V(T)$, denoted $N_T(v)$, are exactly the in-neighbors of v in $G \setminus E(H)$. In addition, we regard T as an *ordered* tree, that is, for each $v \in V(T)$, the children of v are always considered according to an ascending order of their labels. Finally, for $v \in T$, we denote by $N_T^*(v)$ the set of descendants of v in T . Figure 2 depicts two representative trees.

Observation 11 *The representative tree T of a self-labeling reducible flow graph G satisfies the max-heap property, that is, if vertex u is a child of vertex v in T , then $v > u$.*

We convey the idea that a representative tree T satisfies the max-heap property by saying that T is a *descending, ordered, rooted* tree.

Definition 12 *A self-inverting permutation S of size $2n+1$ is said to be canonical if:*

- (i) *there is exactly one 1-cycle in S ;*
- (ii) *each 2-cycle (s_i, s_j) of S satisfies $1 \leq i \leq n$, for $s_i > s_j$;*
- (iii) *s_1, \dots, s_{n+1} is a bitonic subsequence of S starting at $s_1 = n+1$ and ending at $s_{n+1} = 1$.*

Lemma 13 *In any canonical self-inverting permutation of $\{1, \dots, 2n+1\}$, the fixed element f satisfies $f \in [n+2, 2n+1]$.*

Let T be some representative tree, therefore a descending, ordered, rooted tree. The *preorder traversal* P of T is a sequence of its vertices that is recursively defined as follows. If T is empty, P is also empty. Otherwise, P starts at the root r of T , followed by the preorder traversal of the subtree whose root is the first (i.e. smallest) child of r , followed by the preorder traversal of the subtree whose root is the second smallest child of r , and so on. The last (rightmost) element of P is referred to as the rightmost element of T as well.

⁴By doing so, we may simply compare two vertices, e.g. $v > u$ (or v greater than u , in full writing), whereas we would otherwise need to compare their images under σ , e.g. $\sigma(v) > \sigma(u)$.

Lemma 14 *The preorder traversal of a representative tree T is unique. Conversely, a representative tree T is uniquely determined by its preorder traversal.*

If we remove the first element of P , the remaining sequence is said to be the *root-free* preorder traversal of T .

Definition 15 *A canonical reducible permutation graph G is a self-labeling reducible graph such that the root-free preorder traversal of the representative tree of G is a canonical self-inverting permutation.*

Theorem 16 *A digraph G is a watermark instance produced by Chroni and Nikolopoulos's encoding algorithm [1] if, and only if, G is a canonical reducible permutation graph.*

Let T be the representative tree of some canonical reducible permutation graph G , and P a canonical self-inverting permutation corresponding to the root-free preorder traversal of T . We refer to the fixed element f of P also as the fixed element (or vertex) of both G and T . Similarly, the 2-cyclic elements of P correspond to *cyclic* elements (or vertices) of both G and T . The concepts we describe next will be employed in the characterization of canonical reducible permutation graphs.

A vertex $v \in V(T) \setminus \{2n+2\}$ is considered *large* when $n < v \leq 2n+1$; otherwise, $v \leq n$ and v is dubbed as *small*. Denote by X, Y , respectively, the subsets of large and small vertices in T , so $|X| = n+1$ and $|Y| = n$. By Lemma 13, $f \in X$. We then define $X_c = X \setminus \{f\} = \{x_1, \dots, x_n\}$ as the set of large cyclic vertices in T .

We say that T is a *type-1* tree—please see Figure 3(a)—when

- (i) $n+1, n+2, \dots, 2n+1$ are children of the root $2n+2$ in T ; and
- (ii) $1, 2, \dots, n$ are children of $2n$.

Elseways, we say that T is a *type-2* tree relative to f —please see Figure 3(b)—when

- (i) $n+1 = x_1 < x_2 < \dots < x_\ell = 2n+1$ are the children of $2n+2$, for some $\ell \in [2, n-1]$;
- (ii) $x_i > x_{i+1}$ and x_i is the parent of x_{i+1} , for all $i \in [\ell, n-1]$;
- (iii) $1, 2, \dots, f-n-1$ are children of x_n ;
- (iv) $x_i = n+i$, for $1 \leq i \leq f-n-1$;

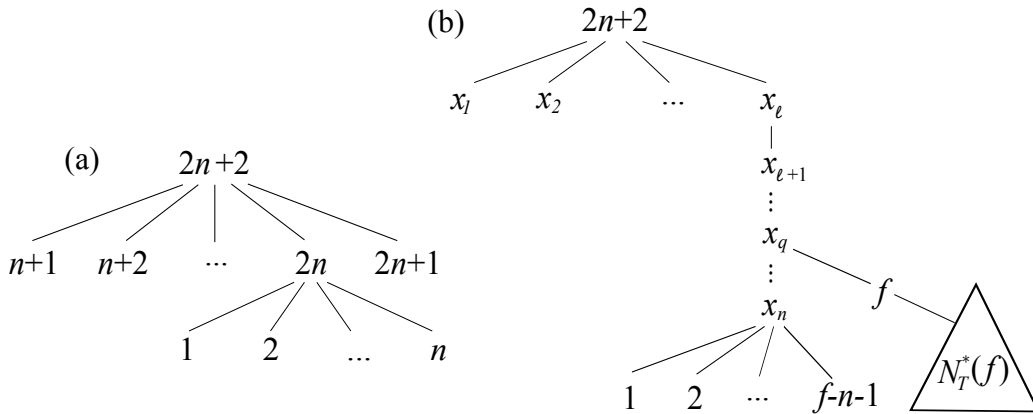


Figure 3: (a) A type-1 representative tree. (b) A type-2 representative tree.

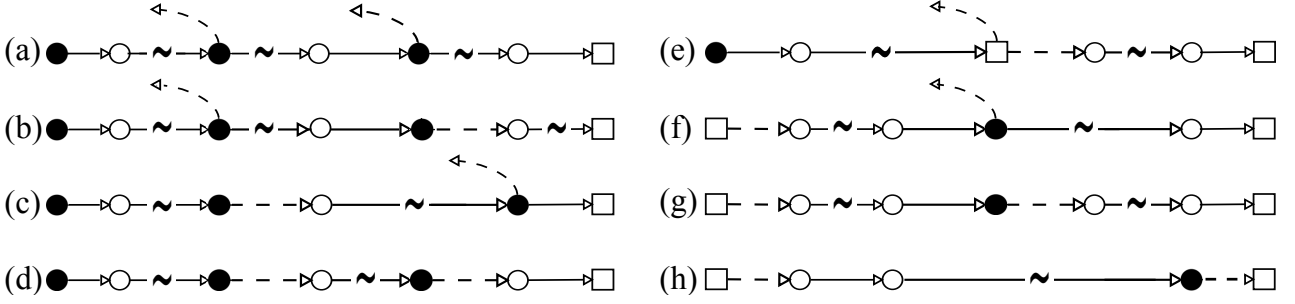


Figure 4: Possible scenarios for the Hamiltonian path of a damaged watermark G . Dashed arrows indicate missing edges. Broken arrows (with a tilde in the middle) indicate paths of arbitrary size. Squares, solid circles and hollow circles represent vertices whose out-degrees in G' are, respectively, 0, 1 and 2. The tree edges of G' are not being shown (unless those removed by the attacker, in dashed curved lines).

- (v) f is a child of x_q , for some $q \in [\ell, n]$ satisfying $x_{q+1} < f$ whenever $q < n$; and
- (vi) $N_T^*(f) = \{f - n, f - n + 1, \dots, n\}$ and $y_i \in N_T^*(f)$ has index $x_{y_i} - f + 1$ in the preorder traversal of $N_T^*[f]$.

Lemma 17 *If y_r is the rightmost vertex of a type-2 representative tree T relative to some $f \neq 2n + 1$, then $y_r = \ell$.*

The following theorem characterizes canonical reducible permutation graphs in terms of the above defined trees.

Theorem 18 *A digraph G is a canonical reducible permutation graph if, and only if, G is a self-labeling reducible graph and*

- (i) *the fixed element of G is $2n + 1$ and G has a type-1 representative tree; or*
- (ii) *the fixed element of G belongs to $[n + 2, 2n]$ and G has a type-2 representative tree.*

4 Restoring a damaged watermark

In this section, we analyze the effects of a distortive attack against a canonical reducible permutation graph G whereby two edges $e_1, e_2 \in E(G)$ were removed. Let $G' = G \setminus \{e_1, e_2\}$.

4.1 Reconstructing the Hamiltonian path

The algorithm given in pseudocode as Procedure 1 reconstructs the Hamiltonian path H of G , in case e_1 or e_2 belonged to H , and classifies each missing edge as either a path edge or a tree edge. The input is the damaged watermark graph G' . If $v \in V(G')$, we denote by $H(v)$ the subsequence of the Hamiltonian path of G that ends at v and starts as far as possible in G' . Also, we denote by $first(H(v))$ the first vertex of the subsequence $H(v)$.

The mechanics of Procedure 1 is that of simply sewing together the $k' \leq 3$ maximal directed paths resulting from deletion of $k \leq 2$ edges from the original, unique hamiltonian path of G . Its correctness can be inferred by simple code inspection, where the different possible scenarios are illustrated in Figure 4.

Procedure 1: Reconstructing the Hamiltonian path

```

 $V_0 \leftarrow \{v \in V(G') \text{ s.t. } |N_{G'}^+| = 0\}$ 
 $V_1 \leftarrow \{v \in V(G') \text{ s.t. } |N_{G'}^+| = 1\}$ 
if  $|V_0| = 1$  then
  let  $v_0$  be the unique element in  $V_0$ 
  if  $|H(v_0)| = 2n + 3$  then  $H \leftarrow H(v_0)$ , return  $H$  and edge types as in Figure 4(a)
  else if  $\exists v_1 \in V_1$  such that  $|H(v_0)| + |H(v_1)| = 2n + 3$  then
     $H \leftarrow H(v_0) || H(v_1)$ , return  $H$  and edge types as in Figures 4(b,c)
  else
    let  $v_1, v'_1$  be such that
       $|H(v_0)| + |H(v_1)| + |H(v'_1)| = 2n + 3$  and  $N_{G'}^+(first(H(v_1) \cap H(v'_1))) \neq \emptyset$ 
     $H \leftarrow H(v_0) || H(v_1) || H(v'_1)$ , return  $H$  and edge types as in Figure 4(d)
else
  let  $v_0, v'_0$  be the elements in  $V_0$ 
  if  $|H(v_0)| + |H(v'_0)| = 2n + 3$  then
    let  $v_0$  be such that  $N_{G'}^+(first(H(v_0))) \cap H(v'_0) \neq \emptyset$ 
     $H \leftarrow H(v_0) || H(v'_0)$ , return  $H$  and edge types as in Figures 4(e,f)
  else
    let  $v'_0 \in V_0$  and  $v_1 \in V_1$  be such that  $v'_0 \in N_{G'}^+(first(H(v_1)))$ 
     $H \leftarrow H(v_0) || H(v_1) || H(v'_0)$ , return  $H$  and edge types as in Figures 4(g,h)

```

As for the time complexity of the algorithm, note that, in general, $1 \leq |V_0| \leq 2$ and $1 \leq |V_1| \leq 3$. The latter follows from the definition of a self-labeling reducible graph and from the fact that two edges were removed from G . Moreover, observe that each path $H(v_i)$ can be computed in time $|H(v_i)|$ easily. Consequently, the entire algorithm has complexity $O(n)$.

4.2 Determining the fixed vertex

Suppose the watermark G has been attacked, which resulted in a damaged watermark G' , where two unknown edges are missing. Now we shall recognize the fixed vertex of the original watermark, given the damaged one. Getting to know the fixed vertex of G will play a crucial role in retrieving the missing tree edges and consequently restoring the original key w .

We describe some characterizations that lead to an efficient computation of the fixed vertex f of G . Let T be the representative tree of the original watermark G . We consider the case where the two edges that have been removed belong to T . Denote by F the forest obtained from T by the removal of two edges. First, we consider the case $f = 2n + 1$.

Theorem 19 : *Let F be a forest obtained from the representative tree T by removing two edges, where $n > 2$. Then $f = 2n + 1$ if, and only if,*

- (i) *vertex $2n + 1$ is a leaf of F ; and*
- (ii) *the n small vertices of G' are children of $2n$ in F , with the possible exception of at most two of them, in which case they must be isolated vertices.*

Next, we characterize the case $f < 2n + 1$. Figure 5 helps to visualize the three conditions of the theorem.

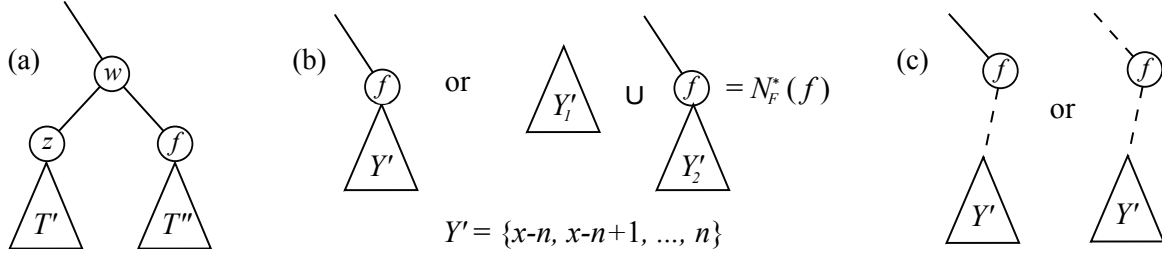


Figure 5: (a-c) Conditions (i), (ii) and (iii) of Theorem 20, respectively.

Procedure 2: Finding $f \neq 2n + 1$

1. If F contains a large vertex x having a sibling z
then let $f \leftarrow \max\{x, z\}$ and terminate the algorithm. Otherwise,
2. For each large vertex x of F satisfying $N_F(x) \neq \emptyset$ and each small $y \in N_F(x)$,
let $Y' = \{x - n, x - n + 1, \dots, n\}$. If $N_F^*(x) = Y'$ or $N_F^*(x) \subset Y'$,
and $Y' \setminus N_F^+(x)$ is the vertex set of one of the trees of F ,
then let $f \leftarrow x$ and terminate the algorithm. Otherwise,
3. Find the preorder traversals of the three trees of F , and
then let f be the unique vertex that is both large and the rightmost element
of the preorder traversal of some tree of F .

Theorem 20 : *Let F be a forest obtained from the representative tree T of watermark G by removing two of its edges, and let $x \leq 2n$ be a large vertex of T which is not a child of $2n + 2$. Then x is the fixed vertex f of G if, and only if,*

- (i) *the large vertex x has a sibling z in F , and $x > z$; or*
- (ii) *the subset of small vertices $Y' \subset Y$, $Y' = \{x - n, x - n + 1, \dots, n\}$ can be partitioned into at most two subsets Y'_1, Y'_2 , such that $\emptyset \neq Y'_1 = N_F^+(x)$ and Y'_2 is the vertex set of one of the trees which form F ; or, whenever the previous conditions do not hold,*
- (iii) *the large vertex x is the rightmost vertex of one of the trees of F , while the rightmost vertices of the remaining trees are all small vertices.*

The above theorems lead to an algorithm that finds the fixed vertex of G in linear time. The input is the forest F , obtained from the representative tree T of G by the removal of two edges. The algorithm is as follows: first, check whether $f = 2n + 1$. This is a direct task, applying Theorem 19: just verify if $2n + 1$ is a leaf of F and whether all small vertices are children of $2n$, except possibly two, which must be isolated vertices. If both conditions are satisfied then set $f = 2n + 1$ and terminate the algorithm. Otherwise, proceed to determining $f < 2n + 1$ by checking conditions (i), (ii) and (iii) as described in Theorem 20.

4.3 Determining the root's children

After having identified the fixed vertex of the watermark, we are almost in a position to determine the tree edges that have been removed.

Observe that, when $f = 2n + 1$, the task is trivial, since, in this case, by Theorem 18, there can be only one canonical reducible permutation graph G relative to n . Such graph is precisely

Procedure 3: Constructing the set of ascending large vertices

1. If $F[X_c] \cup 2n + 2$ is connected then $A \leftarrow N_F(2n + 2)$ and terminate the algorithm. Otherwise,
2. If $F[X_c] \cup 2n + 2$ contains no isolated vertices then $A \leftarrow N_F(2n + 2) \cup 2n + 1$ and terminate the algorithm. Otherwise,
3. If $F[X_c] \cup 2n + 2$ contains two isolated vertices x, x' then $A \leftarrow N_F(2n + 2) \cup \{x, x'\}$ and terminate the algorithm. Otherwise,
4. If $F[X_c] \cup 2n + 2$ contains a unique isolated vertex x then
 - if $|N_F^*(f)| = 2n - f + 1$ then
 - let y_r be the rightmost vertex of $N_F^*(f)$
 - if $|N_F(2n + 2)| < y_r$ then $A \leftarrow N_F(2n + 2) \cup \{x, 2n + 1\}$
 - else $A \leftarrow N_F(2n + 2)$
 - else $A \leftarrow N_F(2n + 2) \cup \{x\}$

the one with a type-1 representative tree T , which is unique for each $n > 2$ (cf. Property 3 of canonical reducible permutation graphs, in Section 2). By definition, the root-free preorder traversal of a type-1 representative tree, when $f = 2n + 1$, is $n + 1, n + 2, \dots, 2n, 1, 2, \dots, n, 2n + 1$.

We therefore want to determine the children of $2n + 2$ restricted to the case where $f < 2n + 1$. Let G be a watermark, T its representative tree and F the forest obtained from T by the removal of two edges. As usual, f stands for the fixed vertex of T , X is the set of large vertices other than $2n + 2$, and $X_c = X \setminus \{f\}$. Finally, denote by $A \subseteq X_c$ the subsets of ascending large cyclic vertices of T , which we shall refer to simply as the *ascending* vertices, and denote by D the set $D = X_c \setminus A$ of descending large cyclic vertices of T , or simply the *descending* vertices. Given the forest F and its fixed vertex f , the algorithm below computes the set A , which, due to the representative tree properties of T , corresponds precisely to the children of its root $2n + 2$.

Theorem 21 *Procedure 3 correctly computes the set of ascending vertices of T in linear time.*

Once we manage to recover the set A of children of the root $2n + 2$ in T , the decoding algorithm proposed in the next section can be run, retrieving the encoded key ω . This suffices to prove that the original, undamaged watermark can be fully restored: a simple possibility is to run the (linear) encoding algorithm from scratch, with ω as input. We have, however, an even simpler algorithm that restores the watermark based on the reconstitution of its preorder traversal, but we shall not present it in the present paper due to space constraints.

5 A new decoding algorithm

We can now formulate our new decoding algorithm. If the input watermark presents $k \leq 2$ missing edges, the algorithm is able to fix it, prior to running the decoding step. The decoding step itself is absolutely straightforward, and relies on the following theorem.

Theorem 22 *Let ω be a given key and G the watermark corresponding to ω . Let $A' = x_1, \dots, x_{\ell-1}$ be the ascending sequence of children of $2n + 2$, in the representative tree T of G , that are different from $2n + 1$. Then $\omega = \sum_{x_i \in A'} 2^{2n-x_i}$.*

As a consequence of the above theorem, whenever the input watermark has *not* been tampered with, the proposed algorithm simply retrieves the encoded key in a faster, less complicated fashion than the original decoding algorithm from [1].⁵

Algorithm 4: Obtaining the key from a possibly damaged watermark

1. Let $k \leftarrow |E(G)| - (4n + 3)$.
2. If $k > 2$, report the occurrence of k edge removals and halt.
3. If $0 < k \leq 2$, proceed to the reconstitution (Procedures 1–3).
4. Calculate and return the key ω as indicated by Theorem 22.

Theorem 23 *Algorithm 4 retrieves the correct key $\omega \geq 4$, encoded in a watermark with up to two missing edges, in linear time.*

Corollary 24 *Distortive attacks in the form of k edge modifications (insertions/deletions) against canonical reducible permutation graphs G , with $|V(G)| = 2n + 3$, $n > 2$, can be detected in polynomial time, if $k \leq 5$, and also recovered from, if $k \leq 2$. Such bounds are tight.*

6 Final considerations

After characterizing the class of canonical reducible permutation graphs, we formulated a linear-time algorithm that restores a member of that class presenting two missing edges. The case where a single edge is missing is less demanding and can be solved by simpler algorithms. Our results therefore have proved that canonical reducible permutation graphs are always able to detect and recover from malicious attacks in the form of $k \leq 2$ edge removals⁶ in linear time. Moreover, we have shown that attacks in the form of $k \leq 5$ edge modifications (insertions/deletions) can be detected in polynomial time. Such level of resilience, we remark, is a very important feature of the original watermarking scheme.

Future directions. A necessary condition for a watermark G_1 to recover from the removal of a subset of edges $E'_1 \subset E(G_1)$, with $|E'_1| = k$, is that $G'_1 = G_1 \setminus E'_1$ is not isomorphic to some graph G'_2 obtained from watermark $G_2 \neq G_1$ by the removal of k edges. For $k \leq 2$, we have shown this condition is always satisfied, provided $n > 2$, and we have proved this is not always true for $k \geq 3$. An interesting open problem is therefore to characterize the set of keys $\Omega(k)$ whose corresponding watermarks can always recover from the removal of $k \geq 3$ edges.

Future research focusing on the development of watermarking schemes resilient to attacks of greater magnitude may consider extending the concept of canonical reducible permutation graphs by allowing permutations with h -cycles, with $h > 2$, as well as multiple fixed elements.

⁵Note that, in this case, it is not even necessary to obtain the representative tree of the watermark, since the set A can be determined as $A = N_G^-(2n + 2)$.

⁶The sole exceptions are two very small instances G, G' corresponding to keys of size $n = 2$, namely keys $\omega = 2$ (binary $B = 10$) and $\omega' = 3$ (binary $B = 11$), respectively, which become isomorphic when edges $(1, 5), (4, 5)$ are removed from G_1 and edges $(1, 4), (4, 6)$ are removed from G_2 .

References

- [1] M. Chroni and S.D. Nikolopoulos, *Efficient encoding of watermark numbers as reducible permutation graphs*, arXiv:1110.1194v1 [cs.DS], 2011.
- [2] M. Chroni and S.D. Nikolopoulos, Encoding watermark numbers as cographs using self-inverting permutations, *12th Int'l Conference on Computer Systems and Technologies, CompSysTech'11*, ACM ICPS **578** (2011), 142–148 (Best Paper Award).
- [3] M. Chroni and S.D. Nikolopoulos, An efficient graph codec system for software watermarking, *36th IEEE Conference on Computers, Software, and Applications (COMPSAC'12)*, IEEE Proceedings (2012), 595–600.
- [4] M. Chroni and S.D. Nikolopoulos, Multiple encoding of a watermark number into reducible permutation graphs using cotrees, *13th Int'l Conference on Computer Systems and Technologies (CompSysTech'12)*, ACM ICPS Proceedings (2012), 118–125.
- [5] M. Chroni and S.D. Nikolopoulos, An embedding graph-based model for software watermarking, *8th Int'l Conference on Intelligent Information Hiding and Multimedia Signal Processing, IIH-MSP' 12*, IEEE Proceedings (2012), 261–264.
- [6] C. Collberg, A. Huntwork, E. Carter, G. Townsend and M. Stepp, More on graph theoretic software watermarks: implementation, analysis and attacks, *Information and Software Technology* **51** (2009), 56–67.
- [7] C. Collberg, S. Kobourov, E. Carter and C. Thomborson, Error-correcting graphs for software watermarking, *29th Workshop on Graph-Theoretic Concepts in Computer Science, WG'03*, LNCS **2880** (2003), 156–167.
- [8] C. Collberg and C. Thomborson, Software watermarking models and dynamic embeddings, *Proc. 26th ACM SIGPLAN-SIGACT on Principles of Programming Languages, POPL'99* (1999), 311–324.
- [9] C. Collberg, C. Thomborson and G. M. Townsend, Dynamic graph-based software fingerprinting, *ACM Transactions on Programming Languages and Systems* **29** (2007), 1–67.
- [10] R. L. Davidson and N. Myhrvold, Method and system for generating and auditing a signature for a computer program, US Patent 5.559.884, Microsoft Corporation (1996).
- [11] M. S. Hecht and J. D. Ullman, Flow graph reducibility, *SIAM J. Computing* **1** (1972), 188–202.
- [12] M. S. Hecht and J. D. Ullman, Characterizations of reducible flow graphs, *Journal of the ACM* **21** (1974), 367–375.
- [13] J. Zhu, Y. Liu and K. Yin, A novel dynamic graph software watermark scheme, *1st Int'l Workshop on Education Technology and Computer Science* **3** (2009), 775–780.
- [14] R. Venkatesan, V. Vazirani, S. Sinha, A graph theoretic approach to software watermarking, *4th International Information Hiding Workshop* (2001), 157–168.

Appendix: Full proofs

Proofs of the properties given in Section 2

Proof of Property 1. When read from right to left, the n rightmost elements in P_b correspond to the n first elements in Z_1 , i.e. the n first indexes, in B^* , where a digit 1 is located. Since B^* starts with a sequence of n contiguous 1's, the property ensues. \square

Proof of Property 2. In B^* , digits with indexes $1, 2, \dots, n$ are all 1, by construction. Since the n rightmost elements in P_b (i.e. elements indexed $n+2 \leq i \leq n^*$ in P_b) correspond to the first n elements in Y , and therefore to the first n indexes of 1's in B^* , those will always be precisely the elements of set $S = \{1, 2, \dots, n\}$. In other words, if $s \in S$, then s will have index $n^* - s + 1 > n + 1$ in P_b . By the time the elements of P_b are gathered together in pairs with views to defining their placement in P_s , element s will be paired with element q whose index is $n^* - (n^* - s + 1) + 1 = s$. Because $s \leq n$, such q clearly does not belong to S , hence $q > n$. Now, because s will be assigned index q in P_s , the element with index s in P_s will be its pair $q > n$, concluding the proof. \square

Proof of Property 3. The bitonic permutation P_b is assembled in such a way that its $(n+1)$ -th element $f = b_{n+1}$ is either:

- (i) the $(n+1)$ -th element of Z_0 , in case B^* has at least $n+1$ digits 0; or
- (ii) the $(n+1)$ -th element of Z_1 , otherwise.

By construction, the number of 0's in B^* is one unit greater than the number of 1's in B .

If (i) holds, then B corresponds to a key w that is the predecessor of a power of 2, implying all n digits of B are 1's. If that is the case, then the desired property follows immediately, once the $(n+1)$ -th element of Z_0 will be the index of the $(n+1)$ -th—i.e. the last—digit 0 in B^* . Such index is, by construction, n^* .

If (ii) holds, then f is the index of the $(n+1)$ -th digit 1 in B^* . By construction, the n first digits 1 in B^* occupy positions with indexes $1, \dots, n$, and the $(n+1)$ -th digit 1 in B^* corresponds to the first digit 1 in the one's complement of B . Since that digit has index f_0 in the one's complement of B , and there are in B^* exactly n digits to the left of the one's complement of B , the property follows. \square

Proof of Property 4. From the construction of P_s and Property 1, it follows that the elements that occupy positions with indexes $1, 2, \dots, n$ in P_s are the first n elements in P_b . It just occurs that the first $n_1 + 1$ numbers in P_b are the elements of Z_0 , i.e. the indexes of 0's in B^* . Now, the last digit in B^* —the one indexed n^* —is always a 0. Besides that 0, the other digits 0 in B^* have indexes $z = n + d$, where each d is the index of a digit 1 in B (the original binary representation of key w). While the first digit in B is always 1, it is also true that:

- (i) the $f_0 - 1$ first digits in B constitute a seamless sequence of 1's, in case there is at least one 0 in B ; or
- (ii) all n digits of B are 1's, in which case ω is the predecessor of a power of 2.

Whichever the case, Property 3 allows us to state that there is a sequence of $f - n - 1$ digits 1 in B starting at the first digit of B . Such sequence will show up, in B^* , starting at index $n+1$, in such a way that the $f - n - 1$ first elements of Z_0 will be $n+1, n+2, \dots, n+(f-n-1) = f-1$.

Those elements, as we have seen, will be precisely the first numbers in P_b . Because there are no more than n such elements, they will be paired against elements $1, 2, \dots, f - n - 1 \leq n$ (located from the right end of P_b leftwards) in order to determine their placement in P_s , and the property follows. \square

Proof of Property 5. If the key ω is not the predecessor of a power of 2, then its binary representation B , whose first digit is always a 1, contains some digit 0. In light of this, Property 3 implies $f \geq n + 2$ for all integers ω , and the first equality now follows from Property 4. The second equality is granted by the self-invertibility of P_s , whereby $s_j = u \iff s_u = j$. \square

Proof of Property 6. First, note that $f \neq n^*$ corresponds to the case where the key ω is not the predecessor of a power of 2, i.e. $n_1 < n$. Because the sequence Z_0 has exactly $n_1 + 1$ elements, the last of which being the index n^* of the rightmost digit in B^* , element n^* will always be assigned index $n_1 + 1$ in P_b . As we have seen in the proof of Property 4, for $i \leq n$, the i -th element in P_b will also be the i -th element in P_s , for it will be paired against element i , indexed $n^* - i + 1$ in P_b (due to the starting sequence of n digits 1 in B^*). That being said, element n^* , indexed $n_1 + 1 \leq n$ in P_b , will have index $n_1 + 1$ in P_s as well. If $f = n^*$, then the definition of f verifies the property trivially. \square

Proof of Property 7. We employ again the fact, noted for the first time in the proof of Property 4, that the subsequence consisting of the first n elements in P_s and the subsequence consisting of the first n elements in P_b are one and the same. Since P_b is bitonic, whatever subsequence of P_b is bitonic too, particularly the one containing its first n elements. By Property 5, the central element s_{n+1} of P_s is always equal to 1, therefore the bitonic property of the subsequence consisting of the leftmost elements of P_s will not be broken after its length has grown from n to $n + 1$, that is, after element $s_{n+1} = 1$ has been appended to it. \square

Proof of Property 8. The first $n_1 + 1$ elements of the bitonic permutation P_b are the elements of Z_0 , corresponding to the indexes of 0's in the extended binary B^* (which consists, we recall, of n digits 1, followed by the one's complement of the binary representation B of the key ω encoded by G , followed by a single digit 0). Those elements constitute the ascending prefix $A = n + z_1, n + z_2, \dots, n + z_{n_1}, 2n + 1$, where, for $i \in \{1, \dots, n_1\}$, z_i is the index of a digit 1 in B . From the proof of Property 4, we know that, for $i \leq n$, the i -th element in P_b will also be the i -th element in the self-inverting permutation P_s . Since $n_1 \leq n$, we have that the n_1 first elements of P_s are precisely the n_1 first elements of A , hence the tree edge tailed at each of those elements must point, by construction, to $2n + 2$. It remains to show that no element $u \notin A \cup \{2n + 1\}$ is the tail of a tree edge pointing to $2n + 2$. But this comes easily from the fact that, by Property 6, the $(n_1 + 1)$ -th element in P_s is $2n + 1$. Since all vertices u with indexes $i > n_1 + 1$ in P_s are certainly smaller than $2n + 1$, they can only be the tail of tree edges pointing to vertices $q(u) \leq 2n + 1$, and the proof is complete. \square

Proof of Property 9. Both items are trivially verified, since, by construction, every tree edge $(u, k) \in G$ is such that either $k > u$ is the element that is closest to u and to the left of u in P_s , or $k = 2n + 2$. \square

Proofs of results from Section 3

Proof of Observation 11. Direct from the way T is rooted and from property (iii) in the definition of self-labeling reducible flow graphs, whereby the in-neighbors of w in $G \setminus E(H)$ comprise only vertices $v < w$. \square

Proof of Lemma 13. By property (ii) of canonical self-inverting permutations, each 2-cycle of S must contain at least one element whose index i satisfies $1 \leq i \leq n$. From property (i), and given the size of S , it follows that the number of 2-cycles in S is n , hence, by the pigeonhole principle, each and every 2-cycle in S contains *exactly* one such element s_i with $1 \leq i \leq n$. But this means the other element in each 2-cycle, namely s_j , satisfies $s_j \in [n+1, 2n+1]$. Since there are $n+1$ values in that range and only n such elements s_j , there must be exactly one element $s_k \in [n+1, 2n+1]$ which is not part of a 2-cycle, and therefore $s_k = f$. Now, by property (iii), $n+1 = s_1$, hence $f \neq n+1$, and the lemma follows. \square

Proof of Lemma 14. We use induction on $|V(T)|$. If $|V(T)| \leq 1$, the lemma holds trivially. Let $|V(T)| > 1$, and let v_k be the uniquely defined leaf of T for which the path v_1, \dots, v_k from the root v_1 of T to v_k has the property that each v_i , for $1 < i \leq k$, is the greatest vertex among the children of v_{i-1} . By the induction hypothesis, the preorder traversal P' of $T \setminus \{v_k\}$ is unique. Because v_k is necessarily the rightmost vertex of T , the preorder traversal P of T is uniquely determined as $P' || v_k$.

Conversely, let P be a preorder traversal of some representative tree T . If $|P| \leq 1$, there is nothing to prove. Otherwise, suppose the lemma holds for preorder traversals of size $\leq k$, and consider $|P| = k$. Let v_k be the rightmost element of P . Clearly, v_k must be a leaf of T , and the rightmost (i.e. greatest) vertex among the children of its parent. Now define $P' = P - v_k$ (i.e. P without its rightmost element v_k). By induction, there is a unique tree T' whose preorder traversal is P' . Let v_{k-1} be the rightmost element of P' . We obtain T from T' , by making v_k the rightmost child of the smallest ancestor v_j of v_{k-1} satisfying $v_j > v_k$, so P is clearly the preorder traversal of T . Since no other parent for v_k would be possible without breaking the ascending order of siblings in a representative tree, T is uniquely defined by P . \square

Proof of Theorem 16. Recall, from Section 2, that the watermark (i.e. the canonical reducible permutation graph) G associated to key ω , whose binary representation B has size n , is constructed with vertex set $V(G) = \{0, \dots, 2n+2\}$ and an edge set $E(G)$ which can be partitioned into path edges and tree edges in such a way that all conditions in the definition of self-labeling reducible graphs are satisfied, as can be easily checked. Now, by Property 9 of canonical reducible permutation graphs, the tree edges of G constitute a representative tree T of G whose root-free preorder traversal is precisely the self-inverting permutation P_s determined by the encoding algorithm from [1] as a function of B . Consequently, what is left to prove is that $P_s = P$ is canonical. The first condition to P being canonical (the uniqueness of the 1-cycle in P) is met by construction, also asserted by Property 3 of reducible permutation graphs; the second condition is given by Property 2; and, finally, Properties 5 and 7 guarantee the third and final condition, so P is indeed canonical.

Conversely, let G be a canonical reducible permutation graph, and T its representative tree. By Lemma 14, T is uniquely defined by its preorder traversal P . We show that P corresponds to the self-inverting permutation P_s generated by the encoding algorithm of [1] (please refer to

Section 2 for details) when computing the watermark for some integer key ω . By definition, $P = (s_i), i = 1, \dots, 2n + 1$, is a canonical self-inverting permutation presenting a single 1-cycle f and a number n of 2-cycles (p, q) . Those 2-cycles (p, q) define exactly one bitonic permutation $P_b = (b_j), j = 1, \dots, 2n + 1$ satisfying

- (i) $b_{n+1} = f$;
- (ii) for all $j \in \{1, \dots, n\}$, $b_j = p$ if, and only if, $b_{2n+1-j} = q$; and
- (iii) Property 1 of canonical reducible permutation graphs, concerning the bitonic permutation employed by the encoding algorithm.

Such bitonic permutation P_b can be regarded as $Z_0 || Z_1^R$ by assigning to Z_0 the prefix of P_b comprising its maximal ascending subsequence, and now the indexes of 0's and 1's in the extended binary B^* are totally determined. We proceed by extracting the binary B that is the one's complement of the subsequence of B^* with digits from the $(n + 1)$ -th to the $(2n)$ -th position in B^* . Since B is the binary representation of a well determined positive integer, such is the positive integer ω whose image under the encoding function of Chroni and Nikolopoulos is precisely G . \square

Proof of Lemma 17. *Proof:* By the definition of a type-2 representative tree, the only non-leaf child of the root $2n + 2$ of T is its rightmost child x_ℓ , therefore each child x_i of $2n + 2$, $1 \leq i \leq \ell$, appears precisely at the i -th position in the root-free preorder traversal P of T . Since, by definition, P is self-inverting, and y_r is the last, $(2n + 1)$ -th element of P , it follows that y_r must be equal to the index of $2n + 1 = x_\ell$ in P , that is, $y_r = \ell$. \square

Proof of Theorem 18. Let G be a canonical reducible permutation graph and T its representative tree. By definition, G is a self-labeling reducible graph. Let $P = s_1, \dots, s_{2n+1}$ be the root-free preorder traversal of T , hence a canonical self-inverting permutation, also by definition. This means, among other things, that P has a unique fixed element f , and that $P' = s_1, \dots, s_{n+1}$ is a bitonic subsequence of P . Since T is descending, it follows that the prefix A of P' constituting its maximal ascending subsequence must comprise solely vertices that are children of the root $2n + 2$, the rightmost of which certainly being $2n + 1$.

First, let $f = 2n + 1$. Since f constitutes a 1-cycle of P , f must occupy the rightmost, $(2n + 1)$ -th position in P , hence f is a leaf of T . Furthermore, by Property 4 of canonical reducible permutation graphs, it follows that P' consists of elements $n + 1, n + 2, \dots, 2n, 1$, hence $A = n + 1, n + 2, \dots, 2n$, and these vertices are therefore children of $2n + 2$ in T . Now, again by Property 4, elements $1, \dots, n$ appear, in this order, to the right of A in P . Considering that P is a preorder traversal and a representative tree satisfies the max-heap property, we conclude that vertices $1, \dots, n$ can only be children of $2n$, hence T is a type-1 tree, as required.

Next, suppose $f < 2n + 1$. By Lemma 13, it follows that $f \in [n + 2, 2n]$. We already know that the children of $2n + 2$ are the vertices of A . Let D be the subset formed by the remaining vertices of P' . Clearly, the vertices of D must appear in descending order. Since T satisfies the max-heap property and P is a preorder traversal of T , it follows that the largest vertex of D is a child of $2n + 1$, and subsequently each vertex in D is the parent in T of the vertex placed to its left along the sequence D . Again, because T satisfies the max-heap property, $f \in [2n + 2, 2n]$ must be the child of the smallest vertex $x_q \in D \cup \{2n + 1\}$ satisfying $x_q > f$. Let us again examine the ascending subsequence A . We know that the first vertex of A is $n + 1$. Suppose the leading k vertices of A are $n + 1, n + 2, \dots, n_k$, for some k . Because P is self-inverting, it

follows that the vertices $1, 2, \dots, k$ must be the children of the last (i.e. smallest) vertex of D , and $k = f - n - 1$ by Property 4. It remains solely to describe how the remaining small vertices, namely $f - n, f - n + 1, \dots, n$, are placed in T . Since they appear after f in P , it can only be that this subset comprises exactly the descendants $N_T^*(f)$ of f in T . Each of the vertices $y \in N_T^*(f)$ constitute a 2-cycle with some vertex x belonging to the bitonic subsequence P' , hence the index of y in P is exactly x , and all the conditions for a type-2 tree have thus been verified.

Conversely, let G be a self-labeling reducible graph. First, suppose that (i) applies and let T be the corresponding type-1 representative tree. Then the root-free preorder traversal P of T is

$$n + 1, n + 2, \dots, 2n, 1, 2, \dots, n, 2n + 1.$$

Regarding P as a permutation of $\{1, \dots, 2n+1\}$, we observe that $2n + 1$ is the only fixed vertex on it; for $1 \leq i \leq n$, each element $n + i$ of P has index i , while i has index $n + i$, and $n + 1, n + 2, \dots, 2n, 1$ form a bitonic subsequence of P . Consequently, P is a canonical self-inverting permutation, and G is a canonical reducible permutation graph.

Finally, suppose (ii) applies. Let T be the corresponding type-2 representative tree relative to some $f \in [n + 2, 2n]$. The root-free preorder traversal P of T consists of

$$x_1, \dots, x_\ell, x_{\ell+1}, \dots, x_q, x_{q+1}, \dots, x_n, 1, 2, \dots, f - n - 1, f, P(N_T^*(f)),$$

where $x_1 = n + 1; x_\ell = 2n + 1; x_i = n + i$ for $1 \leq i \leq f - n - 1$; $x_1, x_2, \dots, x_n, 1$ is a bitonic subsequence of P ; and $P(N_T^*(f))$ denotes the preorder traversal of the vertices of $N_T^*(f)$, in which each $y_i \in N_T^*(f)$ has index $x_{y_i} - f + 1$. Observe that, for $1 \leq i \leq f - n - 1$, $(n + i, i)$ constitutes a 2-cycle in P . Moreover, for $f - n \leq i \leq n$, vertex x_i forms a 2-cycle with an element $y_j \in N_T^*(f)$. All conditions have been met, thus P is a canonical self-inverting permutation and G is a canonical reducible permutation graph. \square

Proofs of results from Section 4

Proof of Theorem 19. From Theorem 18, we know that, when $f = 2n + 1$, f is the rightmost vertex of T , hence a leaf of F , implying the necessity of (i). Again by Theorem 18, the small vertices of T must immediately follow the rightmost cyclic vertex of T , namely $2n$. Since two edges have been deleted from T , it follows that all small vertices are children of $2n$ in F , with the possible exception of at most two of them, which then became isolated vertices, so (ii) is also necessary.

Conversely, suppose conditions (i) and (ii) hold, and assume $f \neq 2n + 1$. Then the second case covered by Theorem 18 applies for T . If $f = 2n$, we know from $n > 2$ that vertex $2n + 1$ has at least 3 children in T , making it impossible for $2n + 1$ to become a leaf of F by the removal of only two edges, therefore contradicting (i). If, on the other hand, $f < 2n$, then, again by Theorem 18, vertex $2n + 1$ cannot have any small children, contradicting (ii). Therefore $f > 2n$, implying $f = 2n + 1$. \square

Proof of Theorem 20. For the sufficiency of (i), let x be a large vertex of F , z a sibling of x in F and x_q their parent. By Theorem 18, the only large vertex of T which is not a child of $2n + 2$ and has some sibling z is precisely the fixed vertex f . Clearly, the removal of edges of T cannot create new vertices having this property. Furthermore, $x_q \notin \{x_n, 2n + 1\}$ implies

that f has a unique sibling x_{q+1} , hence $f > x_{q+1}$ according to the ascending order of siblings in F , whereas $x_q \in \{x_n, 2n+1\}$ implies every sibling y of f is a small vertex, hence $f > y$. Consequently, $x = f$.

Now suppose condition (ii) holds. First, assume that $Y'_2 = \emptyset$. In this case, $Y' = Y'_1 = \{x - n, x - n + 1, \dots, n\} = N_F^*(x)$. Again, according to Theorem 18, we can locate a unique vertex f fulfilling this property, implying $x = f$. In addition, when $Y'_2 \neq \emptyset$, we can again select a unique vertex f , where $N_F^*(f) \cup Y'_2 = Y'$. Thus, $x = f$ indeed.

Finally, assume neither condition (i) nor condition (ii) hold. Because (i) is not satisfied, we have that either $x_q \notin \{x_n, 2n+1\}$, and the edge from x_q to one of its children has been deleted; or $x_q \in \{x_n, 2n+1\}$, and the edge (x_q, f) has been deleted. Additionally, since condition (ii) is not satisfied, f must have a unique child y , and the edge (f, y) has also been removed. Next, assume that, in such a context, condition (iii) is verified. For the sake of contradiction, suppose the theorem is false, so that $x \neq f$. Since $x \neq 2n+1$ and x is not a child of $2n+2$, it follows that it must be a descending vertex, whereupon the fact that x is the rightmost vertex of the tree of F containing it implies that x is a leaf of F . Now the latter implies that the edge (x_q, x) of T has been removed, where x_q is the parent of x in T . Because (i) is not satisfied, at least one edge has been removed from T , and because (ii) is not satisfied, at least one more edge has been deleted from T . Since no more than two edges overall have been removed, we conclude that the assumption is false, and therefore, here again, $x = f$.

Conversely, assume that x is the large vertex of F satisfying $x = f$. We prove that conditions (i) or (ii) hold, otherwise condition (iii) is satisfied.

Let x_q be the parent of $x = f$ in T . If x_q has at least two children in F , then f is larger than its siblings, by Theorem 18, and condition (i) holds. Alternatively, if f is not a leaf of F , then the set $Y' = \{x - n, x - n + 1, \dots, n\}$ either satisfies $Y' = N_F^*(f)$ or it can be split into two subsets $Y'_1 \cup Y'_2 = Y'$, where $Y'_1 = N_F^*(f)$ and Y'_2 is the vertex set of one of the trees of F . In this situation, condition (ii) holds. Assume, next, that neither (i) nor (ii) hold. Then the parent x_q of f in T has at most one child in F , whereas f has no children. The latter implies that f is the rightmost vertex of the tree of F containing it. Since $x_q \neq f$, we know that no more than two edges have been deleted from T , hence no large vertex other than f can be a leaf of F . Consequently, condition (iii) holds, completing the proof. \square

Proof of Theorem 21. We follow the different conditions that are checked by the algorithm. Assume $F[X_c] \cup \{2n+2\}$ is connected. Then $N_T(2n+2) = N_F(2n+2)$, implying $A = N_F(2n+2)$. The algorithm is therefore correct if it terminates at Step 1.

Assume $F[X_c] \cup \{2n+2\}$ is disconnected, but has no isolated vertices. Then either $N_F(2n+2) = N_F(2n+2)$ or the edge $(2n+2, 2n+1)$ was one of those that might have been removed from T . In any of these situations, we can write $A = N_F(2n+2) \cup 2n+1$, implying that the algorithm is also correct if it terminates at Step 2.

Assume $F[X_c] \cup \{2n+2\}$ contains two distinct isolated vertices x, x' . The only possibility is $x, x' \in N_T(2n+2)$. So, the action of constructing A as the union of x, x' and $N_F(2n+2)$ assures correctness, whenever the algorithm terminates at Step 3.

The last situation is $F[X_c] \cup \{2n+2\}$ containing a unique isolated vertex x . We consider the following alternatives. If $|N_F^+(f)| = 2n - f + 1$, it implies that $N_T^*(f) = N_F^*(f)$, because the set of descendants of f in T comprises exactly $y_{f_0}, y_{f_0+1}, \dots, y_n$. The number of such descendants of f is therefore $n - f_0 + 1$, which, by Property 3 of canonical reducible permutation graphs, is

equal to $2n - f + 1$. Now, by Theorem 18, $|N_T(2n + 2)| = y_r$, where y_r is the rightmost vertex of $N_F^*(f)$. In this situation, $|N_F(2n + 2)| < y_r$ implies that x necessarily belongs to $N_F(2n + 2)$. In addition, edge $(2n + 2, 2n + 1)$ might also have been deleted from T , since a single edge deletion suffices to turn x into an isolated vertex. Observe, on the other hand, that isolating a large vertex which is not a child of $2n + 2$ requires the removal of at least two edges, provided $n > 2$. Thus, $A = N_F(2n + 2) \cup \{x, 2n + 1\}$, and the algorithm is correct. In case $|N_F(2n + 2)| = y_r$, we know that $N_T(2n + 2) = N_F(2n + 2)$, hence $A = N_F(2n + 2)$, ensuring the correctness of the algorithm. Finally, when $|N_F^*(f)| \neq 2n - f + 1$, it means some edge inside the subtree rooted at f has been deleted from T . In this case, the isolated vertex x is necessarily a child of $2n + 2$ in T , implying $A = N_F(2n + 2) \cup \{x\}$, and the algorithm is correct.

Checking that the algorithm runs in linear time is an easy task. \square

Proofs of results from Section 5

Proof of Theorem 23. Since the final step of the algorithm clearly runs in linear time, its overall time complexity relies on the fact that Procedures 1–3 run in linear time themselves. The correctness of the algorithm follows from Theorem 22, and from the fact that those procedures guarantee the reconstruction of the original watermark G encoding a key ω , with $|V(G)| = 2n + 3, n > 2$, whenever $k \leq 2$ edges have been removed from G . Note that the condition $n > 2$ is equivalent to $\omega \geq 4$. \square

Proof of Corollary 24. From Theorem 23, we know that, for $n > 2$, there are no two watermarks G_1, G_2 , with $|V(G_1)| = |V(G_2)| = 2n + 3$, such that $|E(G_1) \setminus E(G_2)| \leq 2$, otherwise it would not always be possible to recover from the removal of up to two edges. Thus, for $n > 2$, any two canonical permutation graphs G_1, G_2 satisfy

$$|E(G_1) \setminus E(G_2)| = |E(G_2) \setminus E(G_1)| \geq 3, \quad (1)$$

hence G_1 cannot be transformed into G_2 by less than 6 edge modifications. Since the class of canonical permutation graphs can be recognized in polynomial-time in light of the characterization given in Theorem 18, and since any number $k \leq 5$ of edge modifications made to a graph G of the class produces a graph G' that does not belong to the class, all distortive attacks of such magnitude ($k \leq 5$) can be detected. Now, for $k = 2$, we have three possibilities:

- (i) two edges were removed;
- (ii) two edges were inserted;
- (iii) one edge was removed and one edge was inserted.

If case (i) applies, Theorem 23 grants the original graph can be successfully restored. If case (ii) or case (iii) apply, then a simple algorithm in which all possible sets of two edge modifications are attempted against the damaged graph G' suffices to prove that the original graph G can be restored in polynomial time, since, as we already know, exactly one such set shall turn G' into a canonical reducible permutation graph. The case $k = 1$ is simpler and can be tackled in analogous manner.

It remains to show that such bounds are tight. We present a pair—of plenty there are—of canonical permutation graphs G_1, G_2 , with $|V(G_1)| = |V(G_2)| = 2n + 3, n > 2$, such that inequation (1) holds with equality. Let G_1, G_2 be the watermarks relative to keys $\omega_1 = 8, \omega_2 = 9$, respectively. Their edge sets are such that $E(G_1) \setminus \{(2, 3), (7, 8), (8, 9)\} = E(G_2) \setminus \{(2, 4), (7, 9), (8, 10)\}$, completing the proof. \square