GRAPH THEORY

www.ankurgupta.net

These notes have been prepared from the following book:-

Graph Theory with Applications to Engineering and Computer Science by Narsingh Deo.

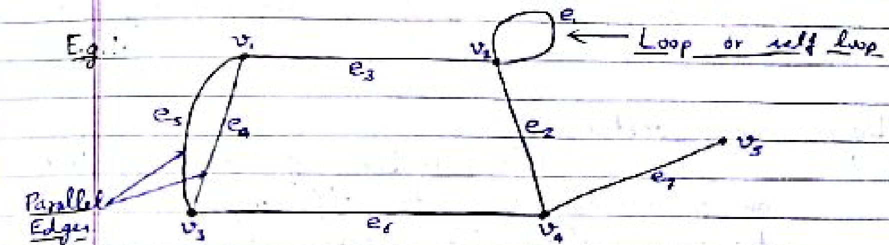
Kindly refer the above book for more details

Graph Theory

Graph :-

A Graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \dots v_n\}$ called vertices, and another set $E = \{e_1, e_2, \dots\}$, whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

E.g. :-



A graph that has neither self-loops nor parallel edges is called a simple graph.

Seating Problem :-

Nine members of a club meet each day at a round table. They decide to sit such that every member has different neighbours at each lunch. How many days can this arrangement last :-

$\left(\frac{n-1}{2}\right)$ if n is odd.

$\left(\frac{n-2}{2}\right)$ if n is even.

Adjacent Edges :-

Two nonparallel edges are said to be adjacent if they are incident on a common vertex.

Eg :- In previous figure, e_2 & e_6 are adjacent, while e_4 & e_8 are not adjacent.

Degree of a vertex :-

The number of edges incident on a vertex v_i, with self-loops counted twice, is called the degree $d(v_i)$ of vertex v_i.

The sum of degrees of all vertices in G is twice the number of edges in G. That is

$$\sum_{i=1}^{n} d(v_i) = 2e$$

The number of vertices of odd degree in a graph is always even.

A graph in which all vertices are of equal degree is called a regular graph.

Q:- Can a simple graph exist with 15 vertices each of degree five?

Ans:- No.

Isolated Vertex :-

A vertex having no incident edge is called an isolated vertex.

Pendant Vertex :-

A vertex of degree one is called a pendant vertex or an end vertex.

Two adjacent edges are said to be in series if their common vertex is of degree two.

Null Graph :-

A graph, without any edges, is called a null graph.

Every vertex in a null graph is an isolated vertex.

A graph must have at least one vertex.

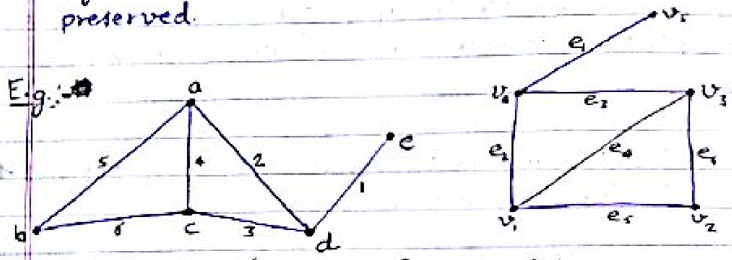
In a simple graph with at least two vertices, there must be two vertices that have the same degree.

→ In a group, there must be two people who know the same number of other people in the group.

Isomorphism :-

Two graphs G and G' are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

E.g. :-



(Isomorphic Graphs)

\# Two isomorphic graphs must have :-

1. The same number of vertices
2. The same number of edges.
3. An equal number of vertices with a given degree.

→ However, these conditions are by no means sufficient.

www.ankurgupta.net

SubGraphs :-

A graph g is said to be a subgraph of a graph G, if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G.

\# Every graph is its own subgraph.

Edge-Disjoint Subgraphs :-

Two (or more) subgraphs g_1 and g_2 of a graph G are said to be edge disjoint if g_1 and g_2 do not have any edges in common.

\# They may have vertices in common.

Vertex Disjoint Subgraphs :-

Sub-graphs that do not even have vertices in common are said to be vertex disjoint.

~~Walks :-~~

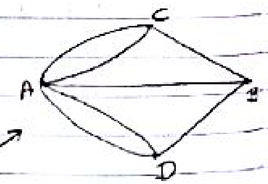
~~A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices~~

Walk :-

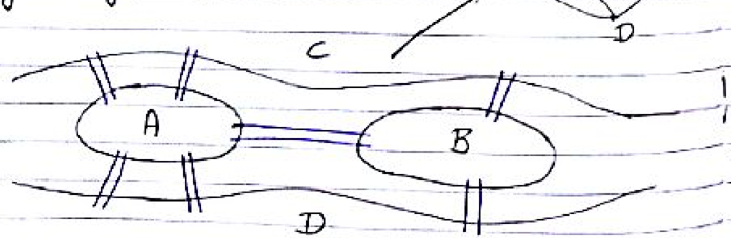
A walk is defined as a finite alternating sequence of vertices and edges, begining and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A vertex, however, may appear more than once.

A walk that begins and ends at the same vertex is called a closed walk.

A walk that is not closed is called an open walk.



Konigsberg Bridge Problem :-



Begining anywhere and ending anywhere, traverse all seven bridges exactly once.

Path :-

An open walk in which no vertex appears more than once is called a path.

A path does not intersect itself.

The number of edges in a path is called the length of a path.

A self loop can be included in a walk but not in a path.

The terminal vertices of a path are of degree one, and the rest of the vertices are of degree two.

Circuit :-

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit.

Connected Graphs, Disconnected Graphs, and Components :-

A graph G is said to be **connected** if there is at least one path between every pair of vertices in G.

Otherwise, G is disconnected.

A null graph of more than one vertex is disconnected.

Each of the connected subgraphs of a disconnected graph G is called a component.

A graph G is disconnected if and only if its vertex set V can be partitioned into two non-empty, disjoint subsets V_1 & V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

A simple graph with n vertices and k components can have at most :-

$$(n-k)(n-k+1)/2 \text{ edges.}$$

A simple graph with n vertices must be connected if it has more than $[(n-1)(n-2)/2]$ edges.

Euler Graphs :-

If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler Line and the graph an Euler Graph.

A given connected graph is an Euler Graph if and only if all vertices of G are of even degree.

Unicursal Graph :-

An open walk in a graph, that includes all edges of the graph, is called a unicursal line or an open Euler Line.

A graph that has a unicursal line is called a unicursal graph.

A connected graph is unicursal if and only if it has exactly two vertices of odd degree.

In a connected graph G, with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

www.ankurgupta.net

classmate

Date ____
Page ____

Operations on Graphs :-

$G_1 \cup G_2 \Rightarrow V_1 \cup V_2$ & $E_1 \cup E_2$

$G_1 \cap G_2 \Rightarrow V_1 \cap V_2$ & $E_1 \cap E_2$

$G_1 \oplus G_2 \Rightarrow V_1 \cup V_2$ & $E_1 \oplus E_2$

$G \cup G = G \cap G = G$

$G \oplus G$ = Null Graph

$G \oplus g = G - g$
where, g is a subgraph of G.

Deletion of a vertex from a graph G, always implies the deletion of all edges incident on that vertex.

Deletion of an edge from a graph G does not imply deletion of its end vertices. Therefore :-

$$G - e_j = G \oplus e_j$$

→ If the simple graph G has n vertices and e edges, then the no of edges in $\overline{G}$

$$= \frac{n(n-1)}{2} - e$$

→ If G is a bipartite graph with n vertices and e edges then :-

$$e \leq n^2/4$$

classmate

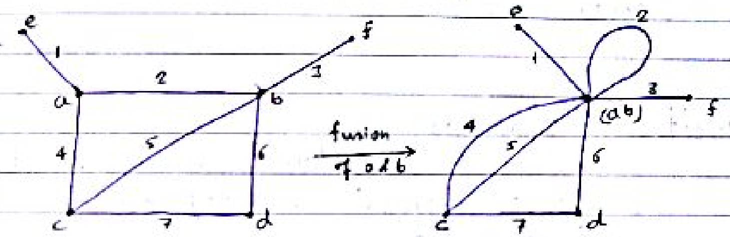
Date ____
Page ____

Fusion :-

A pair of vertices a, b in a graph are said to be fused (merged) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex.

Thus fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one.

Eg :-



Connectedness in Directed Graphs :-

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

More on Euler Graphs :-

A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

Arbitrarily Traceable Graphs :-

An Euler graph G is arbitrarily traceable from vertex v in G if and only if every circuit in G contains v.

Hamiltonian Circuits :-

A Hamiltonian circuit in a connected graph is defined as a closed walk, that traverses every vertex of G exactly once, except of course the starting vertex, at which the walk also terminates.

A circuit in a connected graph G is said to be Hamiltonian, if it includes every vertex of G.

A Hamiltonian circuit in a graph of n-vertices consists of exactly n edges.

If G is a simple graph with n-vertices with $n \geq 3$, such that $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamiltonian Circuit.

Hamiltonian Path :-

If we remove any one edge from a Hamiltonian circuit, we are left with a path, called a Hamiltonian path.

A Hamiltonian path in a graph G traverses every vertex of G.

The length of a Hamiltonian path in a connected graph of n vertices is n-1.

Complete Graph :-

A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

A complete graph with n vertices has $(n-1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3 and $(n-2)/2$ edge-disjoint Hamiltonian circuits, if n is an even number ≥ 4

A sufficient (but by no means necessary) condition for a simple graph G to have a Hamiltonian circuit is that the degree of every vertex in G be at least $n/2$, where n is the number of vertices in G.

→ Konigsberg Bridge Problem :- Euler Graph.
→ Traveling Salesman Problem :- Hamiltonian Graph

Tree :-

A tree is a connected graph without any circuits.

Some Properties of Trees :-

(i) There is one and only one path between every pair of vertices in a tree, T.

(ii) If in a graph G, there is one and only one path between every pair of vertices, G is a tree.

(iii) A tree with n vertices has (n-1) edges.

(iv) Any connected graph with n vertices and (n-1) edges is a tree

(v) A graph is a tree if and only if it is minimally connected.

(vi) A graph G with n vertices, n-1 edges, and no circuits is connected.

A connected graph is said to be minimally connected if removal of any one edge from it disconnects the graph.

In any tree (with two or more vertices), there are at least two pendant vertices.

Distance and Centers in a tree :-

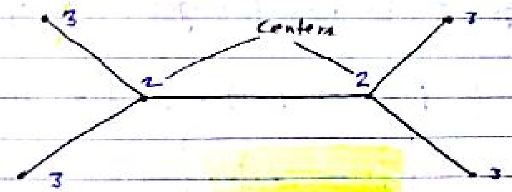
In a connected graph G, the distance $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path between them

The eccentricity $E(v)$ of a vertex v in a graph G is ~~called~~ the distance from v to the vertex farthest from v in G, that is:-

$$E(v) = \max_{v_i \in G} d(v, v_i)$$

A vertex with minimum eccentricity in graph G is called a center of G

Every tree has either one or two centers.



Eccentricities of the vertices of a tree

The eccentricity of a center in a tree is defined as the radius of the tree.

Rooted and Binary Trees :-

A tree in which one vertex is distinguished from all the others is called a rooted tree.

A binary tree is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three.

The number of vertices n in a binary tree is always odd.

Let p be the number of pendant vertices in a binary tree T. Then $n-p-1$ is the number of vertices of degree three. Therefore, the number of edges in T equals :-

$$\frac{1}{2}\left[p\times 1 + 3(n-p-1) + 2\times 1\right] = n-1$$

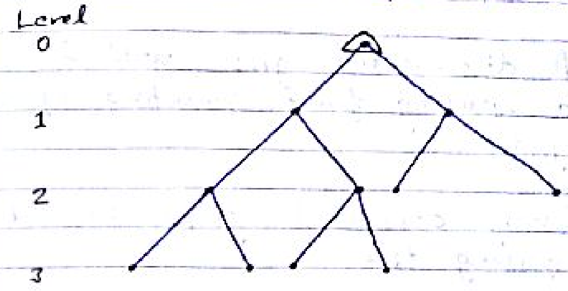
$$\Rightarrow \boxed{p = \frac{n+1}{2}}$$

A non-pendant vertex in a tree is called an internal vertex.

www.ankurgupta.net

The maximum level, l_{max}, of any vertex in a binary tree is called the height of the tree.

$\min l_{max} = \lceil \log_2 (n+1) - 1 \rceil$



$\max l_{max} = \frac{n-1}{2}$

The number of labeled trees with n vertices $(n \geq 2)$ is :-

$$n^{n-2}$$

Spanning Trees :-

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

A disconnected graph with k components has a spanning forest consisting of k-spanning trees.

Every connected graph has at least one spanning tree.

An edge in a spanning tree T is called a branch of T.

An edge of G, that is not in a given spanning tree T is called a chord.

With respect to any of the spanning trees, a connected graph of n vertices and e edges has (n-1) tree branches and (e-n+1) chords.

A pendant edge in a graph G is contained in every spanning tree of G.

www.ankurgupta.net

Rank and Nullity of a Graph :-

Rank $= r = n - k$

Nullity $= \mu = e - n + k$

where, n = No. of vertices in G.
e = No. of edges in G.
k = No. of connected components in G.

Rank of G = Number of branches in any spanning tree (or forest) of G.

Nullity of G = Number of chords in G.

Rank + Nullity = Number of edges in G.

The nullity of a graph is also referred to as its cyclomatic number.

Fundamental Circuits :-

A circuit, formed by adding a chord to a spanning tree is called a fundamental circuit.

The number of fundamental circuits in a graph is equal to the number of chords, μ $(= e - n + k)$

A circuit is a fundamental circuit only with respect to a given spanning tree.

Distance between two spanning trees :-

The distance between two spanning trees T_i and T_j of a graph G is defined as the number of edges of G present in one tree but not in the other.

$$\boxed{d(T_i, T_j) = d(T_j, T_i) = \frac{1}{2} N(T_i \oplus T_j)}$$

where, $T_i \oplus T_j$ = Ring sum of T_i & T_j
& $N(g)$ = No. of edges in graph g.

Starting from any spanning tree of a graph G, we can obtain every spanning tree of G by successive cyclic exchanges.

The maximum distance between any two spanning trees in G is :-

$$\boxed{\text{Max } d(T_i, T_j) \leq \min(\mu, r)}$$

→ A Hamiltonian path is a spanning tree, but a spanning tree ~~is~~ can be a Hamiltonian path or not.

→ The nullity of a graph does not change when we either insert a vertex in the middle of an edge, or remove a vertex of degree two by merging two edges incident on it.

Cut-Sets :-

In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.

Removal of cutset reduces the rank of the graph by one.

Every edge of a tree is a cutset.

Every cutset in a connected graph G must contain atleast one branch of every spanning tree of G, the converse is also true.

Every circuit has an even number of edges in common with any cut-set.

Fundamental Cut-Sets :-

A cut-set S containing exactly one branch of a tree T is called a fundamental cut-set with respect to T.

→ In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.

www.ankurgupta.net

Edge Connectivity :- Minimum no. of edges whose removal disconnects the graph.

Vertex Connectivity :- Minimum no. of vertices whose removal disconnects the graph.

Seperable Graph :- A connected graph with vertex connectivity equal to one. The vertex whose removal disconnects the graph is called a cut vertex or articulation point.

The edge connectivity of a graph G can not exceed the degree of the vertex with the smallest degree in G.

The vertex connectivity of a graph G can never exceed the edge connectivity of G.

The maximum vertex connectivity one can achieve with a graph G of n vertices and e edges ($e \geq n-1$) is :-

$$\lfloor 2e/n \rfloor$$

$$\text{vertex connectivity} \leq \text{edge connectivity} \leq \left\lfloor \frac{2e}{n} \right\rfloor$$

A simple graph with at least two vertices has at least two vertices that are not cut vertices.

A vertex v in a connected graph G is a cut-vertex if and only if there exist two vertices x and y in G such that every path between x and y passes through v.

Planner Graph :- A graph G is said to be planner if there exists some geometric representation of G, which can be drawn on a plane such that no two of its edges intersect.

The complete graph of five vertices is non-planer (K_5).

The regular graph of six vertices and degree ~~three~~ is non-planner ($K_{3,3}$)

K_5 is the non-planer graph with the smallest no. of vertices.

$K_{3,3}$ is the non-planer graph with the smallest no. of edges.

A connected planer graph with n vertices and e edges has :-

$(e - n + 2)$ regions.

In any simple connected planer graph with f regions, n vertices, and e edges ($e > 2$), the following inequalities must hold :-

$$e \geq \frac{3}{2} f \quad [2e \geq 3f]$$

$$e \leq 3n - 6$$

The above 3 - formulas are used to find whether a graph is planner or not.

Although every simple planer graph must satisfy above inequality the mere satisfaction of this inequality does not guarantee the planarity of the graph.

Proper Coloring :-

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the proper coloring or coloring of a graph.

Chromatic Number :-

A graph G that requires K different colors for its proper coloring, and no less, is called a K-chromatic graph, and the number K is called the chromatic number of G.

A complete graph of n vertices is n-chromatic.

A graph containing a complete graph of r-vertices is at least r-chromatic.

A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

Every tree with two or more vertices is 2-chromatic

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

Bipartite Graph :-

A graph G is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2

Every 2-chromatic graph is bipartite. ~~because~~

A simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

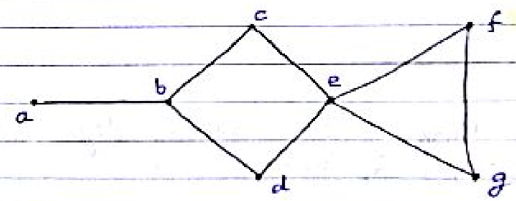
If d_{max} is the maximum degree of the vertices in a graph G, then :-

Chromatic No. of G $\leq 1 + d_{max}$.

Chromatic Partitioning :-

A set of vertices in a graph is said to be an independent set of vertices or simply an independent set if no two vertices in the set are adjacent.

A maximal independent set is an independent set to which no other vertex can be added without destroying its independence property.



The sets {a, c, d, f}, {b, f} etc are maximal independent sets.

A graph may have many maximal independent sets of different sizes.

The number of vertices in the largest independent set of a graph G is called the independence number ($\beta(G)$).

For a K-chromatic graph G of n vertices properly colored with K different colors :-

$$\beta(G) \geq \frac{n}{K}$$

Every planar graph with n vertices has an independent set of size atleast $n/3$

For complete graph of n vertices, the size of any independent set is one.

Chromatic Partitioning :-

Given a simple connected graph G, partition all vertices of G into the smallest possible number of disjoint, independent sets. This is known as the chromatic partitioning of graphs.

Uniquely Colorable Graphs :-

A graph that has only one chromatic partition is called a uniquely colorable graph.

Matchings :-

A matching in a graph is a subset of edges in which no two edges are adjacent.

A maximal matching is a matching to which no edges in the graph can be added.

The maximal matchings with the largest no. of edges are called the largest maximal matchings.

The number of edges in a largest maximal matching is called the matching number of the graph.

For complete graph of n-vertices; no. of perfect matching = $n(n-1)/2$

Coverings :-

In a graph G, a set g of edges is said to cover G if every vertex in G is incident on atleast one edge in g.

A set of edges that covers a graph G is said to be the covering of G.

The number of edges in a minimal covering of the smallest size is called the covering number of the graph.

A covering exists for a graph if and only if the graph has no isolated vertex.

Every pendant edge in a graph is included in every covering of the graph.

A covering of an n-vertex graph will have at least $\lceil n/2 \rceil$ edges.

No minimal covering can contain a circuit.

A minimal covering of an n-vertex graph can contain no more than (n-1) edges.

A covering g of a graph is minimal if and only if g contains no paths of length three or more

www.ankurgupta.net

Some Important Points in Graph Theory

(1) A directed graph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

(2) A directed multigraph having no isolated vertices has an Euler path but not an Euler Circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree one larger than its out-degree and the other that has out-degree one larger than its in-degree.

(3) Detection of Planarity of a Graph :-

(a) Check for the inequalities :-

$$e \geq \frac{3}{2} f$$

$$\& \quad e \leq 3n - 6$$

(b) If this inequality does not hold, it means the graph is not planar.
If this inequality holds it means the graph can be planar or not.

(c) If the inequality holds, then do the Elementary Reduction of the graph :-

(i) If the graph is disconnected, consider only one component at a time.
(ii) Remove all self loops.
(iii) Merge all parallel edges in a single edge.
(iv) Merge all edges in series in a single edge.
(v) Repeat step 3 & 4.

(d) Now check whether the reduced graph is planar or not.

(4) Homeomorphic Graphs :- Two graphs are said to be homeomorphic if one graph can be obtained from the other by the creation of edges in series or by the merger of edges in series.

(5) Condition of planarity :- A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them.

(6) A graph has a dual if and only if it is planar.