INFORMS Journal on Computing

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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INFORMS Journal on Computing
Vol. 19, No. 1, Winter 2007, pp. 121–126
ISSN 1091-9856 | EISSN 1526-5528 | 07 | 1901 | 0121

DOI 10.1287/ijoc.1050.0156

Algorithmic Analysis of the Maximum Queue Length in a Busy Period for the $M/M/c$ Retrial Queue

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This paper deals with the maximum number of customers in orbit (and in the system) during a busy period for the $M/M/c$ retrial queue. Determining the distribution for the maximum number of customers in orbit is reduced to computation of certain absorption probabilities. By reducing to the single-server case we arrive at a closed analytic formula. For the multi-server case we develop an efficient algorithmic procedure for computation of this distribution by exploiting the special block-tridiagonal structure of the system. Numerical results illustrate the efficiency of the method and reveal interesting facts concerning the behavior of the $M/M/c$ retrial queue.

Key words: $M/M/c$ retrial queue; maximum orbit size; busy period; continuous-time Markov chain; tridiagonal linear system
History: Accepted by Edward P. C. Kao, Area Editor for Computational Probability and Analysis; received July 2004; revised August 2004; accepted July 2005.

1. Introduction

The $M/M/c$ retrial queue has been extensively studied. Because of the retrial feature, the underlying bivariate Markov chain that represents the state of the system is not space-homogeneous. This creates severe complications and allows for explicit formulae of the stationary distribution only in the single-server case. For the two-server case, the stationary distribution of the system state can be expressed in terms of hypergeometric functions. For models with more than two servers, the stationary distribution cannot be expressed in closed form and the main characteristics are computed through approximations (truncated models, limit theorems, stochastic bounds etc.). Most retrial queues operate under the random-order discipline, so waiting-time analysis is also intricate. In fact, this is a common feature of queueing systems with service in random-order (Cooper 1981, §5.15). Falin and Templeton (1997, Chapter 2) summarize the main findings for the $M/M/c$ retrial queue, while Artalejo and Falin (2002) compare the $M/M/c$ retrial queue with the standard $M/M/c$ queue. For more references see the bibliographic overviews in Artalejo (1999a, b).

Study of the busy period for the $M/M/c$ retrial queue remains an open problem since neither its density function nor a transform solution is known. The only available result is a recursive procedure for computing the moments of the busy period for the single-server model (Choo and Conolly 1979). The related problems of the number of customers served in a busy period and the maximum number of customers observed during a busy period are also unsolved. We begin here a study in this direction. More specifically, we consider the problem of determining the distribution of the maximum number of customers in orbit during a busy period. We reduce this problem to computation of certain absorption probabilities. This idea has been employed for determining maximum length distributions in birth-and-death queues (e.g., Chung 1967, §I.12). Our contribution is the extension of this methodology for the study of a bidimensional space-non-homogeneous queueing model. Recently, Gomez-Corral (2001) and Lopez-Herrero and Neuts (2002) studied the same problem for $M/G/1$ retrial queues, using a completely different approach. Lopez-Herrero (2002) obtains the number of customers served in the $M/G/1$ retrial queue by a recursive scheme.

In §2, we describe the mathematical model. In §3, we reduce the problem to computation of certain absorption probabilities that constitute the unique solution of a certain linear system obtained by employing first-step analysis. We observe the block-tridiagonal form of this system and briefly discuss its algorithmic implications. Then, we exploit further the special form of the system to derive a numerically stable and efficient algorithm for computing the distribution of interest. Analysis of the single-server case that leads to explicit results is presented in §4. Numerical results are given in §5.

2. Model Description

Primary customers arrive in a Poisson process at rate λ. Service is provided by c identical servers and the service times are exponentially distributed with rate ν. The system does not have waiting space. Upon arrival, a customer who finds at least one server free immediately occupies a server and leaves the system after service. However, an arriving customer who finds all servers busy is forced to leave the service area and becomes a retrial customer. This means that he retries for service with exponential interretrial times with parameter μ until he finds a free server. Between trials we said that a customer is in *orbit*. Moreover, we assume that the primary arrival process and the service and retrial times are mutually independent.

We represent the state of the system by a pair $(C(t), N(t))$, where $C(t)$ is the number of busy servers and $N(t)$ denotes the total number of retrial customers in orbit. The process $\{(C(t), N(t));\ t \geq 0\}$ is an irreducible regular continuous-time Markov chain with state space $S = \{(i, j) \mid 0 \leq i \leq c,\ j \geq 0\}$.

3. Analysis of the Maximum Number of Customers in Orbit

The busy period L of the $M/M/c$ retrial queue is defined as the period that starts at the epoch when an arriving customer finds an empty system and ends at the service-completion epoch at which the system is empty again. We now study the distribution of $N_{\max}$, the maximum number of retrial customers during a busy period. Observe that $\{N_{\max} < m\}$ is equivalent to the event that starting from state $(1, 0)$ the process $\{(C(t), N(t));\ t \geq 0\}$ will hit state $(0, 0)$ before hitting (c, m), for $m \geq 1$. Hence, for every arbitrary but fixed m we consider the modification of the process $\{(C(t), N(t));\ t \geq 0\}$ where the states $(0, 0)$ and (c, m) become absorbing. Then the absorption probability at $(0, 0)$ is the probability of interest, $\Pr\{N_{\max} < m\}$.

For a fixed m let $\tau_{(i,j)}$ be the probability that starting from (i, j) the modified process will be absorbed at $(0, 0)$, $0 \leq i \leq c$, $0 \leq j \leq m-1$. By conditioning on the next state visited by the process we obtain the system

$$\tau_{(0,0)} = 1, \tag{1}$$

$$\tau_{(i,j)} = \frac{\lambda}{\lambda + i\nu + j\mu}\tau_{(i+1,j)} + \frac{i\nu}{\lambda + i\nu + j\mu}\tau_{(i-1,j)} + \frac{j\mu}{\lambda + i\nu + j\mu}\tau_{(i+1,j-1)},$$
$$0 \leq i \leq c-1,\ 0 \leq j \leq m-1,\ (i, j) \neq (0, 0) \tag{2}$$

$$\tau_{(c,j)} = \frac{\lambda}{\lambda + c\nu}\tau_{(c,j+1)} + \frac{c\nu}{\lambda + c\nu}\tau_{(c-1,j)}, \quad 0 \leq j \leq m-1, \tag{3}$$

$$\tau_{(c,m)} = 0. \tag{4}$$

We next show the matrix structure of (1)–(4) for the simple case $c = 2$, $m = 3$. The symbol $*$ indicates the negative of the sum of the other elements in the row.

$$\left(\begin{array}{cc|ccc|ccc}
-(\lambda+\nu) & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
2\nu & * & 0 & 0 & \lambda & 0 & 0 & 0 \\
\hline
\mu & 0 & * & \lambda & 0 & 0 & 0 & 0 \\
0 & \mu & \nu & * & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 2\nu & * & 0 & 0 & \lambda \\
\hline
0 & 0 & 0 & 2\mu & 0 & * & \lambda & 0 \\
0 & 0 & 0 & 0 & 2\mu & \nu & * & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 2\nu & -(\lambda+2\nu)
\end{array}\right)
\cdot \left(\begin{array}{c}
\tau_{(1,0)} \\ \tau_{(2,0)} \\ \hline \tau_{(0,1)} \\ \tau_{(1,1)} \\ \tau_{(2,1)} \\ \hline \tau_{(0,2)} \\ \tau_{(1,2)} \\ \tau_{(2,2)}
\end{array}\right)
= \left(\begin{array}{c}
-\nu \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0
\end{array}\right).$$

System (1)–(4) can be expressed in block-tridiagonal matrix form where each block contains the unknowns $\tau_{(i,j)}$, $0 \leq i \leq c$, corresponding to the level j of the orbit. We can solve the system by using any all-purpose algorithm for solving block-tridiagonal linear systems (e.g., Ciarlet 1989, §4.3). Although this procedure is quite efficient, it does not exploit the special sparse nature of the involved matrices. Moreover, numerical errors can be accumulated by subtractions and matrix inversions. We next propose a refined algorithm that exploits both the block tridiagonal form of the overall system and the special structure of the blocks. This algorithm was inspired by Falin and Templeton (1997, §2.4.5) to obtain the stationary probabilities of a truncated $M/M/c$ retrial system.

To develop an algorithm for solving the system (1)–(4) efficiently, we concentrate on (2) for every fixed j and exploit its tridiagonal structure. This enables us to express the unknowns $\tau_{(i,j)}$, $0 \leq i \leq c-1$, in terms of $\tau_{(c,j)}$ and elements of the previously computed orbit levels $0, \ldots, j-1$. Then we use (3) to eliminate several unknowns and relate the probabilities $\tau_{(c,j)}$ for the various indices j. Using the boundary conditions (1) and (4) we compute all $\tau_{(c,j)}$ backwards from $j=m-1$ to $j=0$ and finally compute all $\tau_{(i,j)}$. Based on this scheme we next describe the algorithm of interest.

Algorithm 1. Computation of $\Pr\{N_{\max}=k\}$ for $k=0,\ldots,m-1$ where m is the first integer such that $\Pr\{N_{\max}<m\}>0.99$.

1. Set $m=1$.
2. Define the coefficients b_{i0}, $\bar{D}_{i0}$, $\hat{F}_{(i,0)}$, and $\hat{G}_{(i,0)}$ $(i=1,\ldots,c-1)$ recursively by the relations

$$b_{10}=\nu; \quad b_{i0}=i\nu b_{i-1,0}/(b_{i-1,0}+\lambda), \quad i=2,\ldots,c-1, \tag{5}$$

$$\bar{D}_{i0}=b_{i0}, \quad i=1,\ldots,c-1, \tag{6}$$

$$\hat{F}_{(i,0)}=\frac{1}{\lambda}\sum_{k=i}^{c-1}\bar{D}_{k0}\prod_{n=i}^{k}\frac{\lambda}{b_{n0}+\lambda}, \quad i=1,\ldots,c-1, \tag{7}$$

$$\hat{G}_{(i,0)}=\prod_{k=i}^{c-1}\frac{\lambda}{b_{k0}+\lambda}, \quad i=1,\ldots,c-1. \tag{8}$$

3. Calculate

$$\tau_{(c,0)}=\frac{c\nu\hat{F}_{(c-1,0)}}{\lambda+c\nu(1-\hat{G}_{(c-1,0)})} \tag{9}$$

and then $\tau_{(i,0)}$ by

$$\tau_{(i,0)}=\hat{F}_{(i,0)}+\hat{G}_{(i,0)}\tau_{(c,0)}, \quad 1\le i\le c-1. \tag{10}$$

Set $\Pr\{N_{\max}=0\}=\tau_{(1,0)}=g_0$. If $g_0>0.99$ then go to Step 7; otherwise go to Step 4.

4. Set $m:=m+1$.
5. For $j=m-1$ calculate sequentially the coefficients b_{ij}, $G_{(i,j)}$, $H_{(i,j)}^{(k)}$, $\bar{D}_{ij}$, $\tilde{D}_{ij}$, $\hat{F}_{(i,j)}$, and $\hat{G}_{(i,j)}$ $(i=0,\ldots,c-1$ and $k=1,\ldots,i+1)$ given by

$$b_{0j}=0; \quad b_{ij}=i\nu(b_{i-1,j}+j\mu)/(b_{i-1,j}+\lambda+j\mu), \quad i=1,\ldots,c-1, \tag{11}$$

$$G_{(i,j)}=\prod_{k=i}^{c-1}\frac{\lambda}{b_{kj}+\lambda+j\mu}, \quad i=0,\ldots,c-1, \tag{12}$$

$$H_{(i,j)}^{(k)}=\nu^{i-k+1}\prod_{n=k-1}^{i-1}\frac{n+1}{b_{nj}+\lambda+j\mu}, \quad i=0,\ldots,c-1,\ k=1,\ldots,i+1, \tag{13}$$

$$\bar{D}_{ij}=j\mu\sum_{k=1}^{i+1}\left(\hat{F}_{(k,j-1)}+\hat{G}_{(k,j-1)}\frac{c\nu\hat{F}_{(c-1,j-1)}}{\lambda+c\nu(1-\hat{G}_{(c-1,j-1)})}\right)H_{(i,j)}^{(k)}, \quad i=0,\ldots,c-1, \tag{14}$$

$$\tilde{D}_{ij}=\frac{\lambda j\mu}{\lambda+c\nu(1-\hat{G}_{(c-1,j-1)})}\sum_{k=1}^{i+1}\hat{G}_{(k,j-1)}H_{(i,j)}^{(k)}, \quad i=0,\ldots,c-1, \tag{15}$$

$$\hat{F}_{(i,j)}=\frac{1}{\lambda}\sum_{k=i}^{c-1}\bar{D}_{kj}\prod_{n=i}^{k}\frac{\lambda}{b_{nj}+\lambda+j\mu}, \quad i=0,\ldots,c-1, \tag{16}$$

$$\hat{G}_{(i,j)}=G_{(i,j)}+\frac{1}{\lambda}\sum_{k=i}^{c-1}\tilde{D}_{kj}\prod_{n=i}^{k}\frac{\lambda}{b_{nj}+\lambda+j\mu}, \quad i=0,\ldots,c-1. \tag{17}$$

6. Calculate

$$\tau_{(c,m-1)}=\frac{c\nu\hat{F}_{(c-1,m-1)}}{\lambda+c\nu(1-\hat{G}_{(c-1,m-1)})} \tag{18}$$

and then compute in reverse order $\tau_{(c,m-2)},\ldots,\tau_{(c,0)}$ by

$$\tau_{(c,j-1)}=\frac{c\nu\hat{F}_{(c-1,j-1)}+\lambda\tau_{(c,j)}}{\lambda+c\nu(1-\hat{G}_{(c-1,j-1)})}, \quad j=m-1,\ldots,1. \tag{19}$$

Calculate $\tau_{(i,0)}$ by

$$\tau_{(i,0)}=\hat{F}_{(i,0)}+\hat{G}_{(i,0)}\tau_{(c,0)}, \quad 1\le i\le c-1. \tag{20}$$

Set $\Pr\{N_{\max}\le m-1\}=\tau_{(1,0)}=g_{m-1}$. If $g_{m-1}>0.99$ then go to Step 7; otherwise go to Step 4.

7. Calculate the probability mass function of $N_{\max}$: $\Pr\{N_{\max}=0\}=g_0$, $\Pr\{N_{\max}=k\}=g_k-g_{k-1}$, $k=1,\ldots,m-1$.

Proof. First, consider (2) for $j=0$, which has the tridiagonal form

$$\alpha_{i0}\tau_{(i-1,0)}+\beta_{i0}\tau_{(i,0)}+\gamma_{i0}\tau_{(i+1,0)}=\delta_{i0}, \quad 1\le i\le c-1, \tag{21}$$

where $\alpha_{10}=0$, $\alpha_{i0}=-i\nu$ $(2\le i\le c-1)$, $\beta_{i0}=\lambda+i\nu$ $(1\le i\le c-1)$, $\gamma_{i0}=-\lambda$ $(1\le i\le c-1)$, $\delta_{10}=\nu$, and $\delta_{i0}=0$ $(2\le i\le c-1)$. System (21) can be solved by combining a forward-elimination, backward-substitution argument and the consideration of new appropriate coefficients to avoid subtractions. We omit here these algebraic details (see Falin and Templeton 1997, §2.4.5). Then, we find that

$$\tau_{(i,0)}=\frac{D_{i0}+\lambda\tau_{(i+1,0)}}{b_{i0}+\lambda}, \quad 1\le i\le c-1. \tag{22}$$

Using (22) recursively, we easily prove (10). By defining $\hat{F}_{(c,0)}=0$ and $\hat{G}_{(c,0)}=1$, (10) also remains valid for $i=c$.

For $j=1,\ldots,m-1$, (2) has the same tridiagonal form as (21) but now i varies from $i=0$ to $i=c-1$ and $\alpha_{ij}=-i\nu$, $\beta_{ij}=\lambda+i\nu+j\mu$, $\gamma_{ij}=-\lambda$ and $\delta_{ij}=$

$j\mu\tau_{(i+1,\,j-1)}$. Hence, similarly to the derivation of the case $j=0$ we can prove that

$$\tau_{(i,\,j)} = F_{(i,\,j)} + G_{(i,\,j)}\tau_{(c,\,j)}, \quad 0 \le i \le c-1, \tag{23}$$

where $G_{(i,\,j)}$ was given in (12) and $F_{(i,\,j)} = (1/\lambda)\cdot\sum_{k=i}^{c-1} D_{kj}\prod_{n=i}^{k}\lambda/(b_{nj}+\lambda+j\mu)$ $(i=0,\ldots,c-1)$, $D_{0j} = j\mu\tau_{(1,\,j-1)}$, $D_{ij} = j\mu\tau_{(i+1,\,j-1)} + i\nu D_{i-1,\,j}/(b_{i-1,\,j}+\lambda+j\mu)$ $(i=1,\ldots,c-1)$.

Formula (23) allows us to express $\tau_{(i,\,j)}$ for a given (i,j) in terms of $\tau_{(c,\,j)}$ and $\tau_{(i',\,j')}$ with $j'<j$. The dependence on $\tau_{(i',\,j')}$ with $j'<j$ is implicit through the definition of D_{kj} in $F_{(i,\,j)}$. We now aim to remove this dependence and express $\tau_{(i,\,j)}$ in terms of only $\tau_{(c,\,j)}$ and the system parameters.

By iterating the definition of D_{i1} for $i=1,\ldots,c-1$, we get

$$D_{i1} = \mu\sum_{k=1}^{i+1}\tau_{(k,\,0)}H_{(i,\,1)}^{(k)}, \quad 0 \le i \le c-1. \tag{24}$$

On the other hand, by combining (3) for $j=0$ and (10), we get

$$\tau_{(c,\,0)} = \frac{c\nu\widehat{F}_{(c-1,\,0)} + \lambda\tau_{(c,\,1)}}{\lambda + c\nu(1-\widehat{G}_{(c-1,\,0)})}. \tag{25}$$

Hence, by substituting (10) and (25) into (24) we get $D_{i1} = \overline{D}_{i1} + \widetilde{D}_{i1}\tau_{(c,\,1)}$, where $\overline{D}_{i1}$ and $\widetilde{D}_{i1}$ are given by (14) and (15), respectively. Consequently, coming back to the definition of $F_{(i,\,1)}$ we get $\tau_{(i,\,1)} = \widehat{F}_{(i,\,1)} + \widehat{G}_{(i,\,1)}\tau_{(c,\,1)}$, $0\le i\le c-1$, with $\widehat{F}_{(i,\,1)}$ and $\widehat{G}_{(i,\,1)}$ given by (16) and (17).

The proof can be completed by induction. Suppose that for all orbit levels up to $j-1$ the following expressions are valid

$$\tau_{(i,\,k)} = \widehat{F}_{(i,\,k)} + \widehat{G}_{(i,\,k)}\tau_{(c,\,k)}, \quad 0\le i\le c-1,\ 1\le k\le j-1, \tag{26}$$

$$\tau_{(c,\,k-1)} = \frac{c\nu\widehat{F}_{(c-1,\,k-1)} + \lambda\tau_{(c,\,k)}}{\lambda + c\nu(1-\widehat{G}_{(c-1,\,k-1)})}, \quad 1\le k\le j. \tag{27}$$

Then, by following the steps employed for the case $j=1$ we can prove that (26)–(27) are also true for the orbit level j. Finally, for each fixed m we know from (4) that $\tau_{(c,\,m)}=0$. Thus, (27) for $k=m$ automatically yields (18). This completes the proof. □

We close this section by studying the maximum number of customers in the system $S_{\max}$ during a busy period (i.e., customers in service plus retrial customers). Since the number of retrial customers increases only through states where all servers are busy, we observe that $\{N_{\max}=0\}=\{S_{\max}\le c\}$ and $\{N_{\max}>0\}\subset\{S_{\max}=c+N_{\max}\}$. Thus, we can prove that $\{S_{\max}\le m\}\subset\{N_{\max}\le m-c\}$ for $m\ge c+1$. Furthermore, $S_{\max}\le c+N_{\max}$, so $\{N_{\max}\le m-c\}\subset\{S_{\max}\le m\}$. Turning to probabilities we have $\Pr\{S_{\max}\le m\}=\Pr\{N_{\max}\le m-c\}$, for $m\ge c$.

Now in the case where $S_{\max}\le c-1$ during a busy period, there are no retrial customers during that busy period. In other words, seeing the problem from the viewpoint of the absorption probabilities, we have a one-dimensional process (a birth-death process corresponding to the standard $M/M/c$ queue) modified to evolve in the set $\{(0,0),\ldots,(m+1,0)\}$ by making the states $(0,0)$ and $(m+1,0)$ absorbing. Then the probability $\Pr\{S_{\max}\le m\}$ is exactly the absorption probability at $(0,0)$ starting from $(1,0)$. Therefore we can use Serfozo (1988, equation 2.1) (see also Chung 1967, §I.12) and get $\tau_{(1,\,0)}=\Pr\{S_{\max}\le m\}=1-(\sum_{k=0}^{m}k!/\rho^k)^{-1}$, where $\rho=\lambda/\nu$ and $m=1,\ldots,c-1$. Combining, we obtain

$$\Pr\{S_{\max}\le m\} = \begin{cases} 1-\left(\displaystyle\sum_{k=0}^{m}\frac{k!}{\rho^k}\right)^{-1}, & \text{for } m=1,\ldots,c-1\\[2ex] \Pr\{N_{\max}\le m-c\}, & \text{for } m=c,\ c+1,\ldots\end{cases} \tag{28}$$

A very similar algorithmic analysis can be carried out for the corresponding model under the so-called constant-retrial policy by simply replacing in all places the retrial rates $j\mu$ by the rate μ, for $j>0$. This policy models situations where the server facility conducts the retrial mechanism (e.g., systems where the servers seek customers in orbit).

4. The Single-Server Case

In the single-server case we can get an explicit expression for the probabilities $\Pr\{N_{\max}=k\}$, $k=0,1,\ldots$. Indeed, combining (2) for $i=0$ and (3) results, after simplifications, in

$$\tau_{(1,\,j)}\left(\lambda+\frac{j\mu\nu}{\lambda+j\mu}\right) = \frac{j\mu\nu}{\lambda+j\mu}\tau_{(1,\,j-1)} + \lambda\tau_{(1,\,j+1)}, \quad j=1,\ldots,m-1. \tag{29}$$

Setting $z_j=\tau_{(1,\,j)}-\tau_{(1,\,j-1)}$ $(j=1,\ldots,m)$ and $\rho=\lambda/\nu$ we obtain

$$z_{j+1}\rho = \frac{j}{(\lambda/\mu)+j}z_j, \quad j=1,\ldots,m-1. \tag{30}$$

By iterating (30) we have

$$\tau_{(1,\,j+1)}-\tau_{(1,\,j)} = \frac{j!}{\rho^j((\lambda/\mu)+1)_j}(\tau_{(1,\,1)}-\tau_{(1,\,0)}), \quad j=1,\ldots,m-1, \tag{31}$$

where $(x)_j = x(x+1)\cdots(x+j-1)$ is the well-known Pochhammer's symbol (ascending factorial). Summing (31) for $j=1,\ldots,m-1$ results in

$$\tau_{(1,m)}-\tau_{(1,1)}=\sum_{j=1}^{m-1}\frac{j!}{\rho^j((\lambda/\mu)+1)_j}(\tau_{(1,1)}-\tau_{(1,0)}). \quad (32)$$

But $\tau_{(1,m)}=0$ and $\tau_{(1,0)}=(\lambda/(\lambda+\nu))\tau_{(1,1)}+\nu/(\lambda+\nu)$ (see (3)), so by eliminating $\tau_{(1,1)}$ and $\tau_{(1,m)}$ we obtain the explicit formula

$$\begin{aligned}\tau_{(1,0)}&=\Pr\{N_{\max}\le m-1\}\\&=\frac{\sum_{k=0}^{m-1}\dfrac{k!}{\rho^k((\lambda/\mu)+1)_k}}{\rho+\sum_{k=0}^{m-1}\dfrac{k!}{\rho^k((\lambda/\mu)+1)_k}},\quad m=1,2,\ldots. \quad (33)\end{aligned}$$

5. Numerical Results

We apply the algorithm in §3 to shed light on the behavior of the maximum orbit length in a busy period for the $M/M/c$ retrial queue. More specifically, we present numerical results for various scenarios and comment on the influence of parameter changes.

In our first study we vary the arrival rate λ, keeping the other parameters constant. We choose the parameters $c=5$, $\nu=1$, $\mu=2.5$ to be fixed while $\rho=\lambda/5\in\{0.2,0.4,0.6,0.8\}$. Table 1 contains the probabilities needed to reach the 99th percentile.

The behavior of the model for larger values of ρ indicates that the 99th percentile occurs at larger values of k. Moreover, in any case the mass function is decreasing with k.

In our second example we vary the retrial rate μ while the other parameters are fixed. We also examine the limit behavior for $\mu\to\infty$. More concretely we want to check convergence to the corresponding values for the standard $M/M/c$ queue without retrials. From Serfozo (1988), the probability of having at most $c+m$ customers in the standard $M/M/c$ queue is given by

$$P\{S_{\max}^{\infty}\le c+m\}=1-\frac{1}{\sum_{k=0}^{c-1}k!(\lambda/\nu)^{-k}+\sum_{k=c}^{c+m}c!c^{-c}(\lambda/(c\nu))^{-k}}. \quad (34)$$

In Table 2 we choose the parameters $c=5$, $\lambda=2.5$, and $\nu=1$ to be fixed while $\mu\in\{0.08,0.25,0.5,1,10,1{,}000,\infty\}$. The last column corresponding to $\mu=\infty$ was computed using (34). For $\mu=0.08$ the distribution has two peaks at $k=0$ and $k=8$. Moreover, the probabilities of the retrial model converge to the corresponding probabilities for the classical $M/M/c$ queue with the same parameters as $\mu\to\infty$. As μ increases, the number of customers in orbit decreases, so we observe smaller 99th percentiles.

In Table 3 we vary c and choose λ so that $\lambda/c=0.5$. We also assume that $\nu=1$ and $\mu=0.5$. For $c=12$

Table 1 $P\{N_{\max}=k\}$ as λ Varies

k	$\rho=0.2$	$\rho=0.4$	$\rho=0.6$	$\rho=0.8$
0	0.993506	0.875000	0.610576	0.404651
1		0.064795	0.084468	0.039128
2		0.033159	0.075635	0.037326
3		0.015396	0.063976	0.036805
4		0.006749	0.050648	0.036722
5			0.037685	0.036681
6			0.026620	0.036460
7			0.018055	0.035922
8			0.011881	0.034995
9			0.007649	0.033654
10			0.004848	0.031919
11				0.029845
12				0.027513
⋮				⋮
27				0.002535
28				0.002076

Table 2 $P\{N_{\max}=k\}$ as μ Varies

k	$\mu=0.08$	$\mu=0.25$	$\mu=0.5$	$\mu=1$	$\mu=10$	$\mu=1{,}000$	$\mu=\infty$
0	0.746655	0.746655	0.746655	0.746655	0.746655	0.746655	0.746655
1	0.036021	0.062197	0.075074	0.083846	0.094900	0.097181	0.097211
2	0.022377	0.046763	0.056327	0.061595	0.066991	0.067785	0.067793
3	0.018106	0.038106	0.041965	0.042546	0.041548	0.041065	0.041058
4	0.016745	0.030923	0.029748	0.027399	0.023609	0.022787	0.022777
5	0.016628	0.024093	0.019880	0.016614	0.012720	0.012032	0.012023
6	0.017063	0.017769	0.012586	0.009622	0.006647	0.006186	0.006181
7	0.017599	0.012387	0.007622	0.005392			
8	0.017848	0.008205	0.004460				
9	0.017483	0.005208					
10	0.016320						
11	0.014396						
12	0.011961						
13	0.009377						
14	0.006973						
15	0.004955						

Table 3 $P\{N_{\max}=k\}$ as c Varies with Constant λ/c

k	$c=1$	$c=3$	$c=6$	$c=12$	$c=24$
0	0.666666	0.769230	0.718750	0.434225	0.106855
1	0.133333	0.075596	0.079609	0.098468	0.011572
2	0.069565	0.050181	0.062865	0.109981	0.021147
3	0.044720	0.035662	0.047884	0.106694	0.037617
4	0.030363	0.025075	0.034133	0.088683	0.063483
5	0.020547	0.016970	0.022728	0.064104	0.097860
6	0.013534	0.010981	0.014274	0.041553	0.131308
7	0.008612	0.006814	0.008563	0.024957	0.146607
8	0.005297		0.004965	0.014261	0.133416
9				0.007897	0.100526
10					0.065434
11					0.038601
12					0.021448
13					0.011515
14					0.006065

and $\lambda = 6$, the distribution has two peaks at $k = 0$ and $k = 2$. Similarly, for $c = 24$ and $\lambda = 12$, the distribution has two peaks at $k = 0$ and $k = 7$.

Acknowledgments

The authors are grateful to the referees for their helpful comments on an earlier version of the paper. J. R. Artalejo and M. J. Lopez-Herrero are grateful for the support received from Research Project MTM2005-01248. A. Economou was supported by University of Athens Grant ELKE/70/4/6415 and PYTHAGORAS/2004.

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