The double pendulum

Is it really that chaotic ?

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LINMA2361 - Nonlinear dynamical systems
EPL UCLouvain

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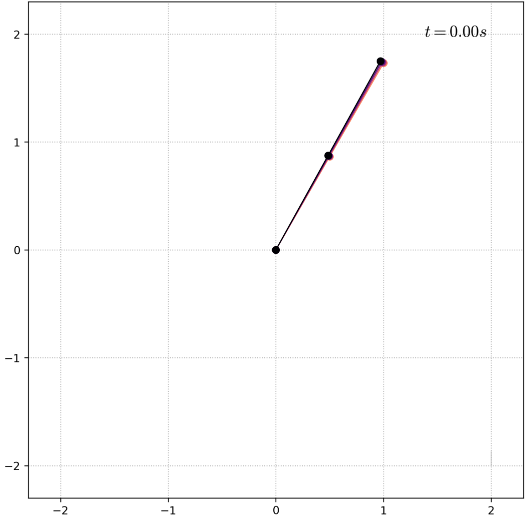
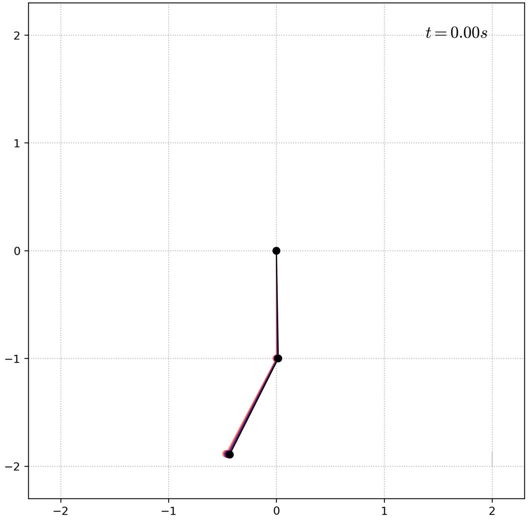
Introduction: A chaotic trajectory



Introduction: A non-chaotic trajectory



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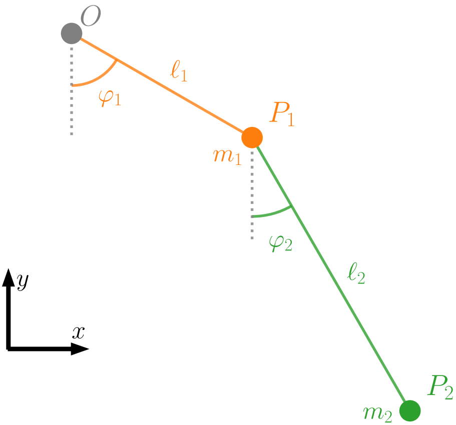
Description



Equations of motion

The Lagrangian of the system:

$$\mathcal{L} = \mathcal{K} - \mathcal{U}$$

$$\mathcal{K} = \frac{m_1}{2}\ell_1^2\dot{\varphi}_1^2 + \frac{m_2}{2}\left[\ell_1^2\,\dot{\varphi}_1^2 + \ell_2^2\,\dot{\varphi}_2^2 + 2\ell_1\ell_2\,\dot{\varphi}_1\dot{\varphi}_2\cos\overbrace{(\varphi_1 - \varphi_2)}^{:=\theta}\right]$$

$$\mathcal{U} = \left[(m_1 + m_2)\,g\ell_1\,(1 - \cos\varphi_1) + m_2 g\ell_2\,(1 - \cos\varphi_2)\right]$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_i}\right) = \frac{\partial\mathcal{L}}{\partial\varphi_i}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_i}\right) = \frac{\partial\mathcal{L}}{\partial\varphi_i}$$

$$(m_1 + m_2)\ell_1\ddot{\varphi}_1 + m_2\ell_2\ddot{\varphi}_2\cos\theta = -m_2\ell_2\dot{\varphi}_2^2\sin\theta - (m_1 + m_2)g\sin\varphi_1$$

$$\ell_1\ddot{\varphi}_1\cos\theta + \ell_2\ddot{\varphi}_2 = \ell_1\dot{\varphi}_1^2\sin\theta - g\sin\varphi_2$$

State-space formulation

A choice of nondimensionalization ?

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$$\lambda = \frac{\ell_2}{\ell_1} \qquad \mu = \frac{m_2}{m_1 + m_2} \qquad \tau = \sqrt{\frac{g}{\ell_1}}\, t$$

State-space formulation

A choice of nondimensionalization ?

$$\lambda = \frac{\ell_2}{\ell_1} \qquad \mu = \frac{m_2}{m_1 + m_2} \qquad \tau = \sqrt{\frac{g}{\ell_1}}\, t$$

We obtain the following system

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\varphi_1 \\ \varphi_2 \\ \omega_1 \\ \omega_2\end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \left[\mu \cos\theta(\sin\varphi_2 - \omega_1^2 \sin\theta) - \lambda\mu\omega_2^2 \sin\theta - \sin\varphi_1\right] \frac{1}{1-\mu\cos^2\theta} \\ \frac{1}{\lambda}\sin\theta\left[\cos\varphi_1 + \omega_1^2 + \lambda\mu\omega_2^2\cos\theta\right] \frac{1}{1-\mu\cos^2\theta} \end{bmatrix}$$

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What about the equilibria

- How many ?
- What coordinates ?
- Which one(s) is/are stable ?

Eigenvalues of the 4 equilibria

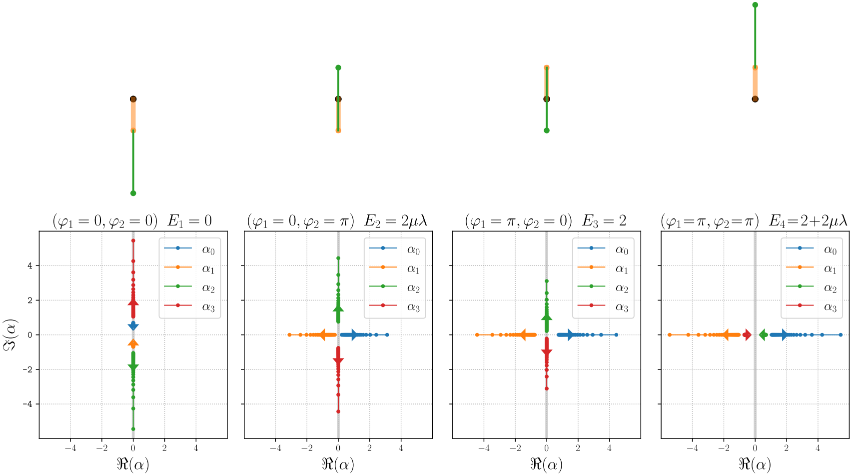

Figure: Eigenvalues for $\lambda = 2$ and $\mu \in [0.05, 0.95]$.

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Simple system

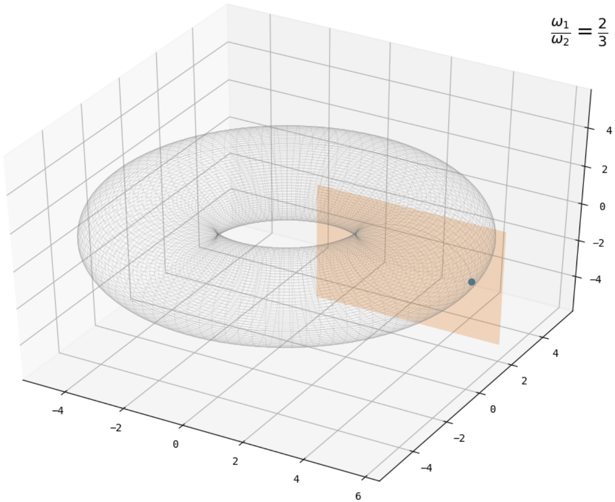
Example

$$\dot{\theta}_1 = \omega_1$$
$$\dot{\theta}_2 = \omega_2$$

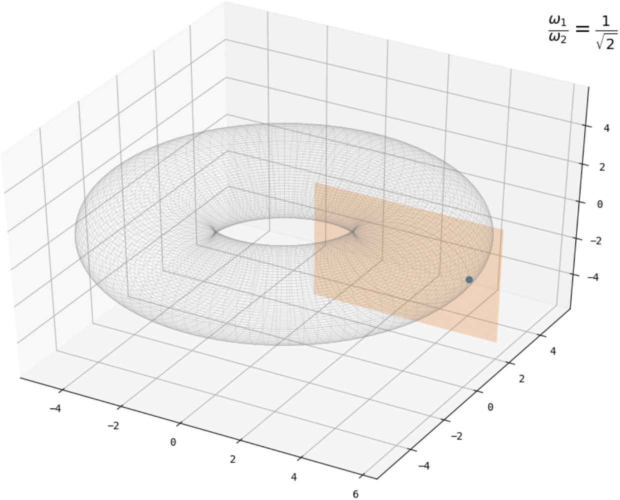
Two possibilities

- $\omega_1/\omega_2 = p/q$ rational
- ω_1/ω_2 irrational

Commensurable frequencies



Incommensurable frequencies



Recap

- Periodic orbits: finite number of points
- Quasiperiodic orbits: curve-filling points
- Chaotic orbits: area-filling points

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What is a Poincaré section

- Cannot visualize 4-dimensional state-space
- Pick a lower-dimensional slice of the state-space and look for intersection points with the trajectory

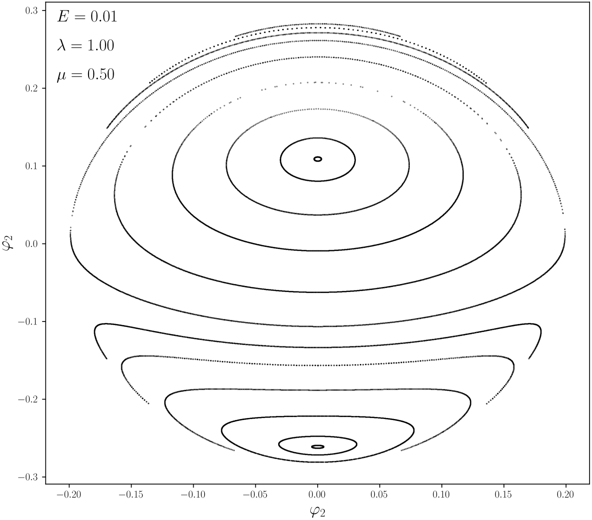
What is a Poincaré section

- Cannot visualize 4-dimensional state-space
- Pick a lower-dimensional slice of the state-space and look for intersection points with the trajectory

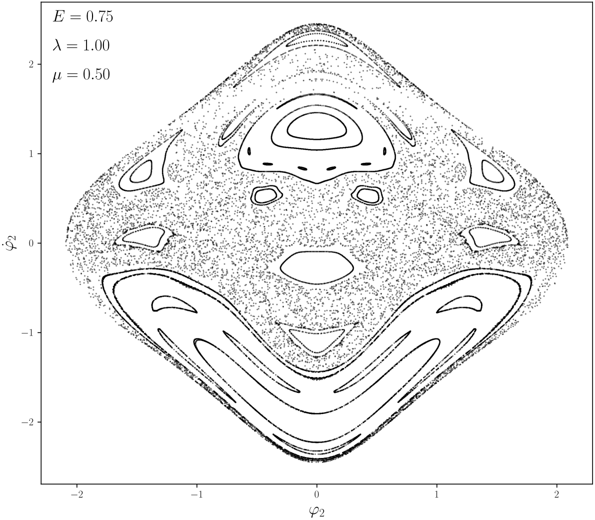
In the case of the double pendulum, section can be defined:

$$\begin{array}{lccc} & \varphi_1 = 0 & \text{and} & \dot{\varphi}_1 + \mu\lambda\dot{\varphi}_2 \cos\varphi_2 > 0 \\ \text{or} & \varphi_2 = 0 & \text{and} & \dot{\varphi}_2 + \dfrac{1}{\lambda}\dot{\varphi}_1 \cos\varphi_1 > 0 \end{array}$$

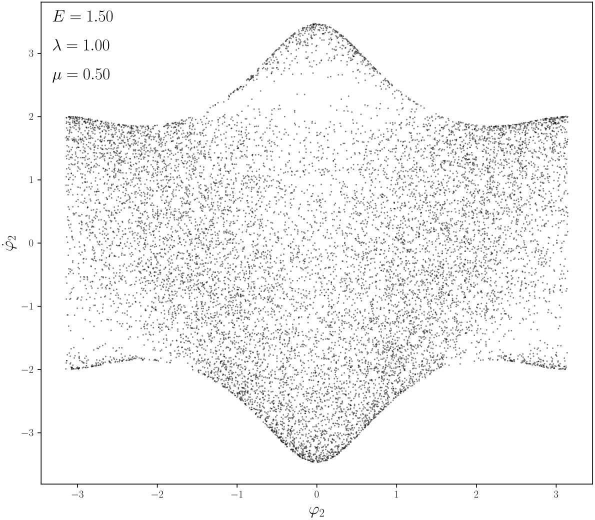
Numerical experiment - low energy



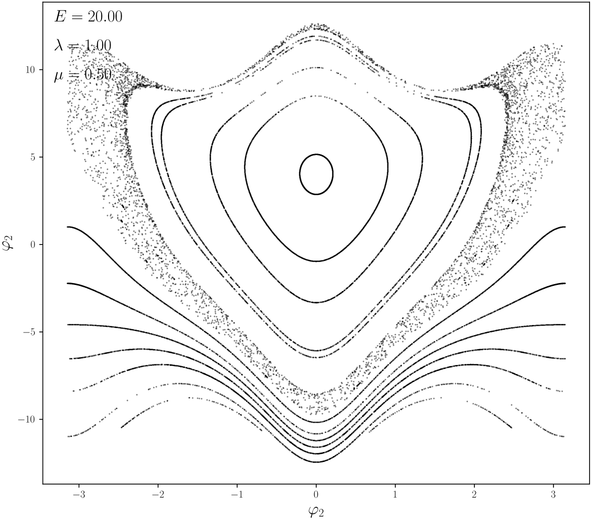
Numerical experiment - mid-low energy



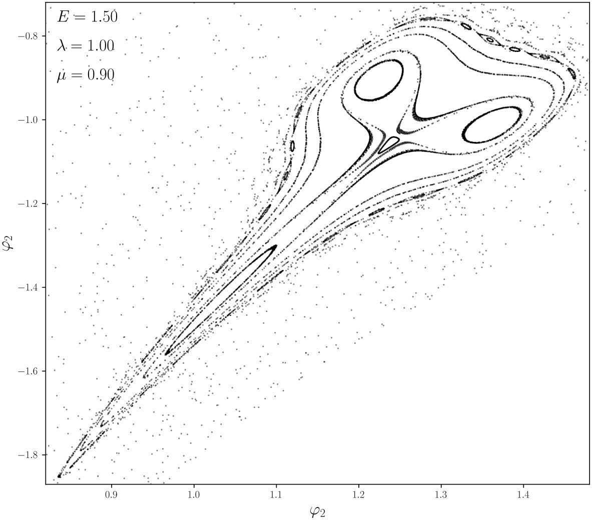
Numerical experiment - mid-high energy



Numerical experiment - high energy



Numerical experiment - Fractal



Recap

- Low energy: 2 normal modes of the linear model which are periodic and their associated quasiperiodic orbits
- Mid-range energy: new types of periodic orbits being progressively replaced by chaotic regions
- High energies: chaotic regions replaced by (quasi)periodic orbits similar to a simple pendulum

Extra - Energy influence - 1

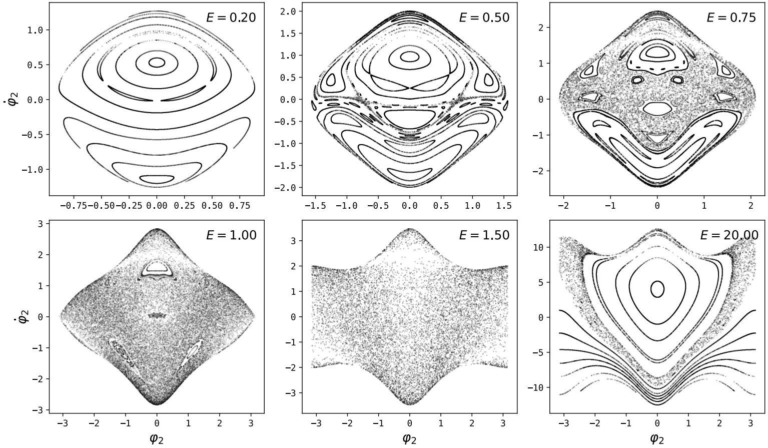


Figure: Poincaré sections for $\lambda = 1$ and $\mu = 0.5$, with condition $\varphi_1 = 0$.

Extra - Energy influence - 2

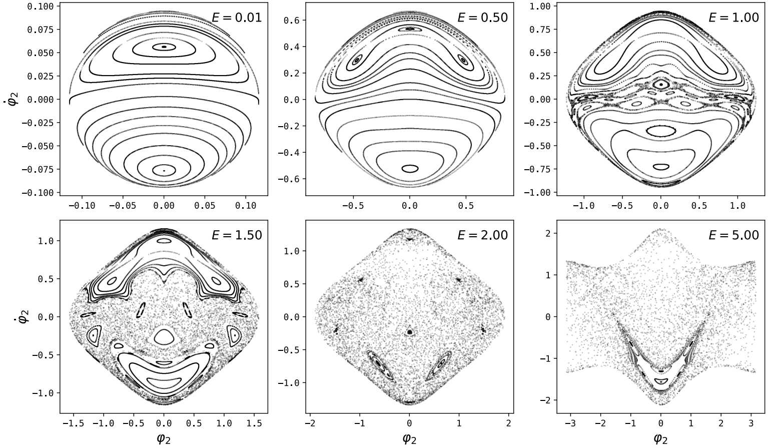


Figure: Poincaré sections for $\lambda = 3$ and $\mu = 0.5$, with condition $\varphi_1 = 0$.

Extra - Energy influence - 3

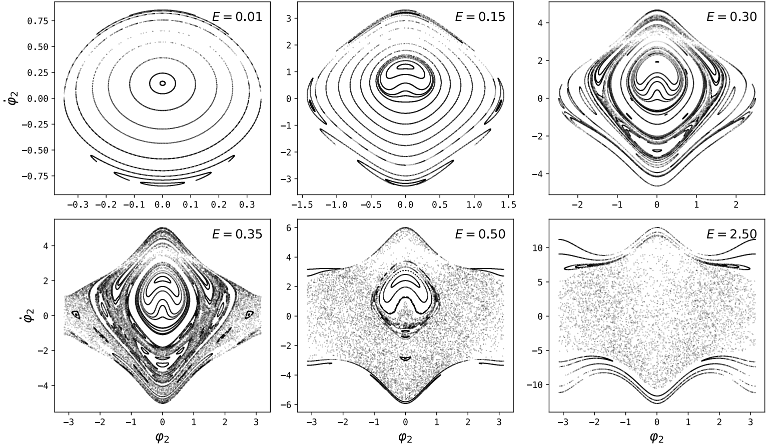


Figure: Poincaré sections for $\lambda = 1/3$ and $\mu = 0.5$, with condition $\varphi_1 = 0$.

Extra - Energy influence - 4

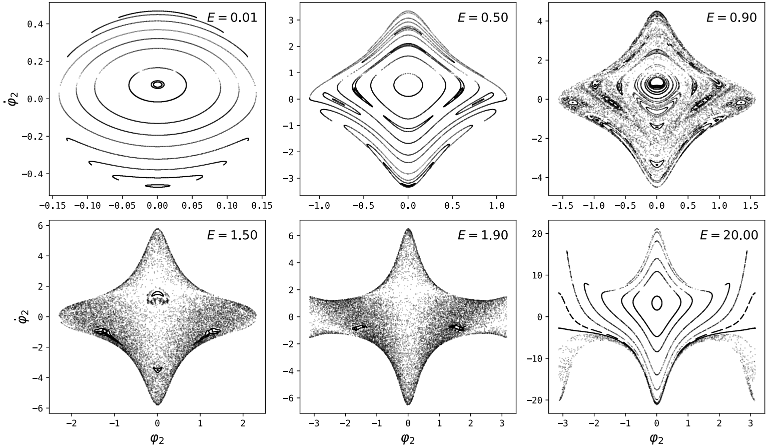


Figure: Poincaré sections for $\lambda = 1$ and $\mu = 0.9$, with condition $\varphi_1 = 0$.

Extra - Energy influence - 5

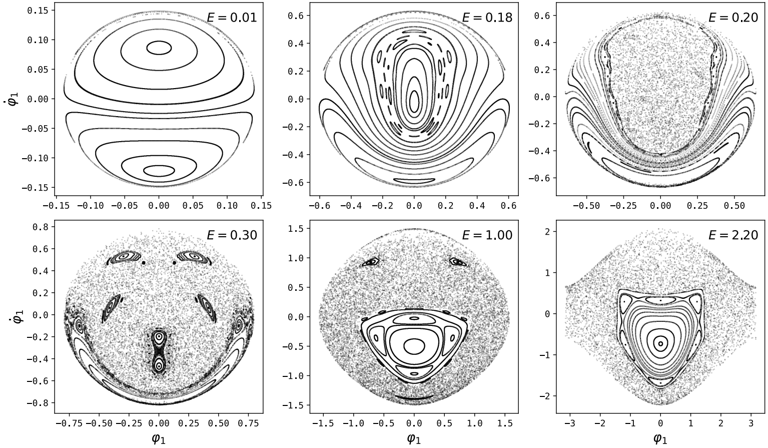


Figure: Poincaré sections for $\lambda = 1$ and $\mu = 0.1$, with condition $\varphi_2 = 0$.