1.3. Fourier Series

Many excitation forces are not harmonic, but instead are a combination of harmonic excitations. In fact, over a finite period of time any excitation can be represented by a combination of harmonic excitations. This fact will be used when we perform signal processing. Applying Fourier Series analysis to a periodic signal demonstrates that a random signal can indeed be represented by a series of harmonic functions. Consider a function $f(t)$ with a periodicity T. Assume that an approximation for that function can be represented by

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)\right) \tag{1.55}$$

where n is an integer. Integrating both sides over one period gives

$$a_0 = \frac{1}{T} \int_0^T f(t)\, dt \tag{1.56}$$

Multiplying both sides by $\cos\left(\frac{2\pi mt}{T}\right)$ where m is also an integer gives

$$\begin{aligned} f(t) \cos\left(\frac{2\pi mt}{T}\right) &= a_0 \cos\left(\frac{2\pi mt}{T}\right) \\ &+ \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi mt}{T}\right)\right) \\ &+ \left(b_n \sin\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi mt}{T}\right)\right) \end{aligned} \tag{1.57}$$

evaluating equation (1.57) integrated over $0 < t < T$ for $m \neq 0$ yields

$$a_n = \frac{2}{T}\int_0^T f\left(t\right)\cos\left(\frac{2\pi nt}{T}\right)dt, n = 0, \ldots, \infty \tag{1.58}$$

Likewise

$$b_n = \frac{2}{T}\int_0^T f\left(t\right)\sin\left(\frac{2\pi nt}{T}\right)dt, n = 0, \ldots, \infty \tag{1.59}$$

Consider another form of the Fourier series where instead of using trigonometric functions we take advantage of Euler's relations and write

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j2\pi nt/T} \tag{1.60}$$

Note that the limits have been changed in the summation. Following the development of the sine based Fourier series, we will multiply both sides of equation (1.60) by $e^{-j2\pi mt/T}$ and integrate from $0 < t < T$ yielding

$$\int_0^T f(t)e^{-j2\pi mt/T}dt = \int_0^T \sum_{-\infty}^{\infty} F_n e^{j2\pi nt/T}e^{-j2\pi mt/T}dt = \int_0^T \sum_{-\infty}^{\infty} F_n e^{j2\pi(n-m)t/T}dt \tag{1.61}$$

Applying the Euler equation (1.2)

$$\int_0^T f(t)e^{-j2\pi mt/T}dt = \int_0^T \sum_{-\infty}^{\infty} F_n\left(\cos(2\pi(n-m)t/T) + j\sin(2\pi(n-m)t/T)\right)dt \tag{1.62}$$

After integration, all terms on the right hand side are zero except that of $n = m$ resulting in

$$\int_0^T f(t)e^{-j2\pi nt/T}dt = \int_0^T F_n 1dt = F_n T \tag{1.63}$$

therefore

$$F_n = \frac{1}{T}\int_0^T f(t)e^{-j2\pi nt/T}dt \tag{1.64}$$

Applying Euler's equation to this expression yields

$$F_n^{'} + jF_n^{''} = \frac{1}{T}\left(\int_0^T f(t)\cos(2\pi nt/T)dt - j\int_0^T f(t)\sin(2\pi nt/T)dt\right) \tag{1.65}$$

where $F^{'}$ is the real part of F, and $F^{''}$ is the imaginary part. It is clear then that

$$F_n = F_n^{'} + jF_n^{''} = \frac{a_n}{2} - j\frac{b_n}{2} \tag{1.66}$$

Also, noting equation (1.65),

$$F_{-n} = \frac{a_n}{2} + j\frac{b_n}{2} = \bar{F}_n \tag{1.67}$$

where $\bar{F}_n$ represents the complex conjugate of F_n.

Example 1.2 *Find the Fourier series of the function $x(t)$ with a period of 4 seconds where*

$$x(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & 2 < t < 4 \\ & \vdots \end{cases}$$

Solution:
Substituting $x(t)$ into equation (1.64) yields

$$
\begin{aligned}
X_n &= \frac{1}{4}\int_0^2 1e^{-j2\pi nt/4}dt + \frac{1}{4}\int_2^4 0e^{-j2\pi nt/4}dt \\
&= \frac{1}{4}\frac{-4}{2\pi nj}e^{-jn\pi t/2}|_0^2 \\
&= \frac{j}{2n\pi}\left(e^{-jn\pi}-1\right) \\
&= \begin{cases} \frac{1}{2} & n=0 \\ \frac{-j}{\pi n} & n \ odd \\ 0 & n \ even \end{cases}
\end{aligned}
$$

Note that in obtaining X_n for $n=0$, $n=0$ must be substituted prior to integrating to avoid a divide by zero error. Substituting X_n into equation (1.60), applying Euler's equation, and simplifying gives

$$
\begin{aligned}
x(t) &= \frac{1}{2} + \sum_{n=-\infty}^{\infty}\frac{-j}{n\pi}e^{j2\pi nt/4}, \qquad n = 1,3,5,\ldots \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin\left(\frac{n\pi t}{2}\right), \qquad n = 1,3,5,\ldots
\end{aligned}
$$

A short cut could have been taken by noting equation (1.66) and directly writing the Fourier expansion in real form. Figure 1.4 illustrates the Gibbs effect*. The Gibbs effect is the observation that Fourier series do not perform well at sharp corners*

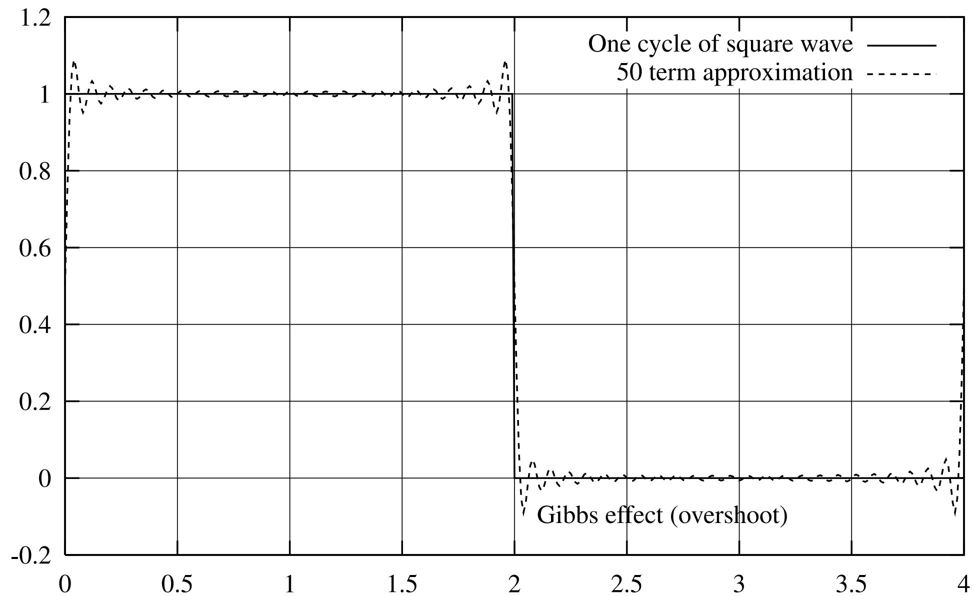

Figure 1.4: 50-term Fourier series representation of a square wave.

due to inevitable truncation errors. Certainly, additional terms will improve the approximation, but later in the text when we discuss the practical aspects of signal processing real limitations on the maximum frequency available exist.

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Example 1.3 *Find the Fourier series of*[2]

$$f(t) = \sum_{m=-\infty}^{\infty} \delta(t-m)$$

where

$$\delta(t) = 0, \qquad t \neq m \tag{1.68}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{1.69}$$

The period of the repeated function is $T = 1$*. Using equation (1.64) yields*

$$F_n = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(t) e^{-j2\pi nt} dt = e^0 = 1$$

Then, substituting into equation (1.60)

$$f(t) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt}$$

For the term $n = 0$, $e^{j2\pi 0t} = 1$ *and for terms* $\pm n$, $e^{j2\pi nt} + e^{j2\pi(-n)t} = 2\cos 2\pi nt$

$$\therefore f(t) = 1 + \sum_{n=1}^{\infty} 2\cos 2\pi nt$$

---

[2] This represents an impulse occuring once per second. Periodic impulses are characteristic of certain types of failures in bearings.

The advantages of this form of the Fourier series are twofold: first, the integrations tend to be much easier for those who don't like "difficult" integrations, and second, it is directly related to the *discrete Fourier transform* (DFT) that is the mainstay of signal processing. Without the DFT and its more expedient algorithm, the *fast Fourier transform* (FFT), modern vibration testing as we know it wouldn't exist.

1.4. Fourier Transform

Consider substituting equation (1.64) into equation (1.60) yielding

$$f(t) = \sum_{-\infty}^{\infty} \left(\frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi mt/T} dt \right) e^{j2\pi nt/T} \tag{1.70}$$

Now $1/T = \Delta f = \Delta\omega/2\pi$ is the change in frequency from one term to the next in the Fourier series, and likewise $n/T = f$ from equation (1.55). Be careful to note that f is used here for frequency, while $f(t)$ refers to a time signal. If we consider applying this relation to a function that doesn't repeat, we must then consider $T \to \infty$. As $T \to \infty$, $\Delta\omega \to d\omega$ and $n/T \to \omega/2\pi$. In performing the integral, we also can't forget the earlier non-repeating part of the function prior to $t = 0$ (there is no well-defined time $t = 0$). We then change the limits of the integral to be from $-\infty$ to ∞, and replace the summation by an integral over $d\omega$ giving

$$f(t) = \int_{-\infty}^{\infty} \left(\frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) e^{j\omega t} \tag{1.71}$$

or

$$f(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) e^{j\omega t} d\omega \tag{1.72}$$

There are multiple ways to parse this expression. In mechanical engineering, the term $1/2\pi$ is traditionally factored to the front of the expression leaving

$$f(t) = \mathcal{F}^{-1}\left\{F(j\omega)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \tag{1.73}$$

where

$$F(j\omega) = \mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{1.74}$$

Here equation (1.74) is commonly called the *forwards Fourier transform* (commonly abbreviated to *Fourier Transform*), and equation (1.73) is commonly referred to as the *backward* or *inverse Fourier transform*. Alternatively, consider the cycle per second form that alleviates the need for the 2π

$$f(t) = \int_{-\infty}^{\infty} F(f)e^{j2\pi ft}df \tag{1.75}$$

and

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt \tag{1.76}$$

simply by recognizing that $f = \omega/2\pi$. Thus in using equations (1.76) and (1.75) there is no extraneous 2π to cause confusion.

The Fourier transform of a function exists for most, but not all, functions. A sufficient, but not necessary, condition is that the function satisfy the *Dirichlet* conditions. To satisfy the Dirichlet conditions a function must:

- be absolutely integrable: i.e. $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$, and
- have a finite number of discontinuities and a finite number of minima and maxima in any arbitrary time interval.

The end result is that almost all practical engineering functions satisfy these conditions. Two important exceptions are the sine and cosine because they are not absolutely integrable. However, the Dirichlet conditions are sufficient, not necessary, and we will see how these too are Fourier transformable.

Example 1.4 *Find the Fourier Transform of the* Dirac delta function $\delta(t)$.

Solution:

Recall that by definition

$$\delta(t) = 0, \qquad t \neq 0 \tag{1.77}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{1.78}$$

The Fourier Transform of $\delta(t)$ is obtained using (1.74).

$$\Delta(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt \tag{1.79}$$

Noting that $\delta(t) = 0$ for $t \neq 0$, the limits of the integral can equivalently be changed to arbitrary non-zero values, say $t = \pm b$.

$$\Delta(\omega) = \int_{-b}^{b} \delta(t)e^{-j\omega t}dt \tag{1.80}$$

Integrating by parts yields

$$\Delta(\omega) = u(t)e^{-j\omega t}\,|_{-b}^{b} - \int_{-b}^{b} u(t)(-j\omega)e^{-j\omega t}dt \tag{1.81}$$

where $u(t)$ is the unit step function (more formally referred to as the Heaviside *function). Evaluating the first term and breaking the integral into two parts,*

$$\Delta(\omega) = e^{-j\omega b} + \int_{-b}^{0} u(t)(j\omega)e^{-j\omega t}dt + \int_{0}^{b} u(t)(j\omega)e^{-j\omega t}dt \tag{1.82}$$

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The first integral is zero, since $u(t) = 0$ for $t < 0$. Over the interval $0 < t < b$, $u(t) = 1$. Therefore $\Delta(\omega)$ becomes

$$\Delta(\omega) = e^{-j\omega b} - e^{-j\omega t}\,|_0^b = e^{-j\omega b} - \left(e^{j\omega b} - e^{0}\right) = 1 \tag{1.83}$$

Since the value of $\Delta(\omega)$ cannot depend on b, $\Delta(\omega) = 1$.
In the continuous case, there is a simple solution. Considering the definition of the Dirac delta function given by equations (1.77) and (1.78), since $e^{-j\omega t}$ is relatively slow varying as compared to $\delta(t)$, we can consider it to be constant over the entire period of time that $\delta(t) \neq 0$. This allows us to factor it outside of the integral and evaluate it at $t = 0$. Thus equation (1.79) becomes

$$\Delta(\omega) = e^{-j\omega 0}\int_{-\infty}^{\infty}\delta(t)dt = 1\int_{-\infty}^{\infty}\delta(t)dt = 1 \tag{1.84}$$

Thus we get the same result with much less effort. This approach can be applied in general when a Dirac delta function is inside the integrand so that

$$\int_{\tau-b}^{\tau+b}\delta(t-\tau)f(t)dt = f(\tau) \tag{1.85}$$

as long at $\dot{f}(t)$ is bounded and is often referred to as the sifting *property of the impulse because it effectively sifts out the value of $f(t)$ at $t = \tau$.*

Example 1.5 *Find the Fourier transform of $f(t) = \sin(\Omega t)$.*
Solution:
Following the steps of the previous problem, the Fourier Transform of $\sin(\Omega t)$ is obtained using (1.74).

$$F(\omega) = \int_{-\infty}^{\infty}\sin(\Omega t)e^{-j\omega t}dt \tag{1.86}$$

It's easier to apply the inverse Fourier transform than the forward Fourier transform. Let's guess $\tilde{F}(\omega) = \delta(\omega - \Omega)$. *If this is correct, we will get* $\tilde{f}(t) = \sin(\Omega t)$

$$\tilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \Omega) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\Omega t} = \frac{1}{2\pi}\left(\cos(\Omega t) + j\sin(\Omega t)\right) \tag{1.87}$$

This doesn't give us the correct answer. However, since $f(t) = \frac{2\pi}{2j}\left(\tilde{f}(t) - \tilde{f}(-t)\right)$

$$\begin{aligned} f(t) &= \frac{2\pi}{2j}\left(\tilde{f}(t) - \tilde{f}(-t)\right) \\ &= \frac{2\pi}{2j}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \Omega) e^{j\omega t} d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \Omega) e^{-j\omega t} d\omega\right) \end{aligned} \tag{1.88}$$

Noting that

$$\int_{-\infty}^{\infty} \delta(\omega - \Omega) e^{-j\omega t} d\omega = \int_{-\infty}^{\infty} \delta(\omega + \Omega) e^{j\omega t} d\omega \tag{1.89}$$

then the second integral can be replaced and $f(t)$ *can be simplified to*

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\left(\frac{\pi}{j}\int_{-\infty}^{\infty} \delta(\omega - \Omega) e^{j\omega t} d\omega - \frac{\pi}{j}\int_{-\infty}^{\infty} \delta(\omega + \Omega) e^{j\omega t} d\omega\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\pi}{j}\left(\delta(\omega - \Omega) - \delta(\omega + \Omega)\right)}_{F(j\omega) = \mathcal{F}\{f(t)\}} e^{j\omega t} d\omega \end{aligned} \tag{1.90}$$

Thus

$$\mathcal{F}\{f(t)\} = \frac{\pi}{j}\left(\delta(\omega - \Omega) - \delta(\omega + \Omega)\right) \tag{1.91}$$

is the Fourier transform of $\sin(\Omega t)$.

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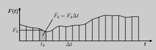
3.3. The Discrete and Fast Fourier Transforms

- The practical application of the DFT is the Fast Fourier Transform, or FFT.
- FFT algorithms result in precisely the same result as a DFT, but in a much more efficient manner, with exceptional computational savings.
- Since FFTs are available in numerous software packages and available subroutines, FFT algorithms will not be covered here.
- Instead, the more illustrative, and simpler, DFT will be considered, with the understanding that the mathematical operations here are illustrative in nature, but do not represent how real calculations are performed.

Fourier's theorem states that any periodic function can be represented by its harmonic components (see section 1.3). The frequency spacing between each harmonic component is $\Delta f = 1/T$ where T is the period of the repeating function. As the period of the repeating function increases, the spacing between adjacent harmonics of the Fourier series diminishes. As the period continues to increase, the functions become non-repeating and the distribution of the harmonics becomes continuous. Such functions can no longer be represented by a discrete Fourier series but must instead be represented by a function of ω as was derived in section 1.4. Here we return to a finite



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period. Discretizing equation (1.75) yields

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(\omega) e^{j2\pi f t} df \\ &= \lim_{N\to\infty} \Delta f \sum_{n=-N/2+1}^{N/2} X_n e^{j\frac{2\pi n}{T}t} \\ &= \lim_{N\to\infty} \Delta f \sum_{n=0}^{N-1} X_n e^{j\frac{2\pi n}{T}t} \end{aligned} \tag{3.70}$$

where we will prove the equivalence of changing the limits later in this section. Consider the case where we have a set of discrete data points, x_m where m is an integer $0 \le m \le N-1$. Substituting

$$t = t_m = \frac{mT}{N} \tag{3.71}$$

into equation (3.70) gives the *inverse discrete Fourier transform* (IDFT)

$$x_m = \Delta f \sum_{n=0}^{N-1} X_n e^{j\frac{2\pi n}{N}m}, \qquad m = 0, 1, 2, \ldots, N-1 \tag{3.72}$$

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The Fourier coefficients, X_n, can be obtained from equation (1.76)

$$X_n = \Delta t \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}n}, \qquad n = 0, 1, 2, \ldots, N-1 \tag{3.73}$$

by noting that $\frac{\Delta t}{T} = \frac{1}{N}$ if $x(t)$ is represented as the discrete set of data points x_m. This is called the forward *discrete Fourier transform*, or DFT. Noting equation (3.71),

$$x_m = x(t_m) = x\left(\frac{mT}{N}\right) = \Delta f \sum_{n=0}^{N-1} X_n e^{j\frac{2\pi n}{N}m} \tag{3.74}$$

Also, comparing the discrete summation of equation (3.70) to its continuous counterpart (also equation (3.70)), the change in frequency from one term to the next (due to incrementing n) is

$$\Delta f = \frac{1}{T} \tag{3.75}$$

and thus we can write equation (3.73) as

$$\begin{aligned} X_n = X(\omega_n) = X\left(n\Delta f\right) &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}n} \\ &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j2\pi m\frac{\Delta t}{T}n} \\ &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j2\pi(m\Delta t)(n\Delta f)}, \qquad n = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{3.76}$$

This yields some very important observations. From equation (3.75), in order to have a very fine frequency resolution (small Δf), the temporal length of the signal, T, should be long. Next consider evaluating $X_{N/2-l} + X_{N/2+l}$. Applying equation (3.76)

$$\begin{aligned}
X_{N/2-l} + X_{N/2+l} &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}(N/2-l)} + \Delta t \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}(N/2+l)} \\
&= \Delta t \sum_{m=0}^{N-1} x_m \left(e^{-j\frac{2\pi m}{N}(N/2-l)} + e^{-j\frac{2\pi m}{N}(N/2+l)} \right) \\
&= \Delta t \sum_{m=0}^{N-1} x_m \left(e^{-j\left(\pi m - \frac{2\pi m l}{N}\right)} + e^{-j\left(\pi m + \frac{2\pi m l}{N}\right)} \right) \\
&= \Delta t \sum_{m=0}^{N-1} x_m \left(e^{-j\pi m} \left(e^{j\frac{2\pi m l}{N}} + e^{-j\frac{2\pi m l}{N}} \right) \right)
\end{aligned} \tag{3.77}$$

Applying Euler's equation (1.2) and equation (1.4), this becomes

$$\begin{aligned}
X_{N/2-l} + X_{N/2+l} &= \Delta t \sum_{m=0}^{N-1} x_m \left(\left(\cos(\pi m) + j\sin(\pi m)\right) 2\cos\left(\frac{2\pi m l}{N}\right) \right) \\
&= \Delta t \sum_{m=0}^{N-1} x_m \left(\cos(\pi m) 2\cos\left(\frac{2\pi m l}{N}\right) \right)
\end{aligned} \tag{3.78}$$

Since every term in the summation is real, we have proven that the imaginary parts of $X_{N/2-l}$ and $X_{N/2+l}$ are complex conjugates. Similarly, $X_{N/2-l} - X_{N/2+l}$ can be shown to be imaginary, proving that the quantities $X_{N/2-l}$ and $X_{N/2+l}$ are indeed complex conjugates. No restrictions were placed on N or l, except that $N/2$ is an integer (N is even, we'll cover N odd). With this in mind, consider the case where $l = N/2 - p$

where $p < N/2$. Evaluating $X_{N/2+l}$ yields

$$
\begin{aligned}
X_{N/2+l} &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}(N/2+l)} \\
X_{N/2+(N/2-p)} &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}(N/2+N/2-p)} \\
X_{N-p} &= \Delta t \sum_{m=0}^{N-1} x_m e^{-j2\pi m} e^{-j\frac{2\pi m}{N}(-p)} \\
X_{N-p} &= \Delta t \sum_{m=0}^{N-1} x_m 1 e^{-j\frac{2\pi m}{N}(-p)} \\
&= X_{-p}
\end{aligned}
\tag{3.79}
$$

- The end result is that the coefficients X_n repeat every N indices, and thus we only need to calculate N of them. However, as was illustrated in equation (3.78), only $N/2+1$ of them are unique (The $n = 0$ and $n = N/2$ terms are real and unique).

- Further, because $X_{N/2-l} = \bar{X}_{N/2+l}$, and $X_{-p} = X_{N-p}$, the terms X_n for $N/2+1 < n < N-1$ correspond to $X(-\omega_n)$ where $1 > n > N/2-1$. These are the same complex conjugate terms we were required to include in Fourier series analysis in equation (1.60). Without them, the summation yielding x_n will not be real. What this means is that we have terms corresponding to positive and negative frequency values, with the maximum magnitude observed frequency magnitude being $f_{N/2} = N/2T$.

- Noting that the sampling rate frequency is $f_s = N/T$, we have derived what is called the Nyquist frequency, $f_n = f_s/2$. This is Shannon's sampling theorem: *the maximum frequency about which information can be obtained using a discrete Fourier transform is* $1/2$ *the sampling frequency.*

- Also note that this allows the change of indices that we made in equation (3.70). Secondly, because the maximum frequency is at $X_{N/2} = X\left(\frac{N}{2T}\right)$, in order to obtain values for higher frequencies, T should be short, and N should be high.

The practical implementation of the DFT and IDFT in code usually results in the ratio $\frac{1}{N}$ in front of the forwards Fourier transform instead of Δf. This is done by multiplying the DFT, equation (3.73), by $\Delta t = \frac{T}{N}$, and dividing the inverse DFT, equation (3.72), by Δt. This allows the DFT and IDFT to be performed with no information other than the length of the data sequence and the sequence itself. Users of software packages such as MATLAB, Octave and other software libraries should be aware of how this factor is included and how it must be compensated. When this is done, the backward and forwards DFTs take the form

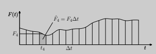
$$x_m = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{X}_n e^{j\frac{2\pi n}{N}m}, \qquad m = 0, 1, 2, \ldots, N-1 \tag{3.80}$$

and

$$\tilde{X}_n = \sum_{m=0}^{N-1} x_m e^{-j\frac{2\pi m}{N}n}, \qquad n = 0, 1, 2, \ldots, N-1 \tag{3.81}$$

respectively where

$$\tilde{X}_n = \frac{1}{\Delta t} X_n \tag{3.82}$$



Example 3.1 *Consider a data sequence of four points ($N = 4$). The Fourier representation of a discrete function x_m may be written as*

$$
\begin{aligned}
x_m &= \frac{1}{4}\left(\tilde{X}_0 e^{j\frac{2\pi m}{4}0} + \tilde{X}_1 e^{j\frac{2\pi m}{4}1} + \tilde{X}_2 e^{j\frac{2\pi m}{4}2} + \tilde{X}_3 e^{j\frac{2\pi m}{4}3}\right) \\
&= \frac{1}{4}\left(\tilde{X}_0 + \tilde{X}_1 e^{j\frac{\pi m}{2}} + \tilde{X}_2 e^{j\pi m} + \tilde{X}_3 e^{j\frac{3\pi m}{2}}\right)
\end{aligned}
\tag{3.83}
$$

If we write this equation for each value of m, and put the resulting equations in matrix form, the result is

$$
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{6\pi}{4}} \\ 1 & e^{j\frac{4\pi}{4}} & e^{j\frac{8\pi}{4}} & e^{j\frac{12\pi}{4}} \\ 1 & e^{j\frac{6\pi}{4}} & e^{j\frac{12\pi}{4}} & e^{j\frac{18\pi}{4}} \end{bmatrix} \begin{bmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{bmatrix}
\tag{3.84}
$$

Solving for the Fourier coefficients, $\tilde{X}_n$, gives the DFT

$$
\begin{bmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{-2\pi}{4}} & e^{j\frac{-4\pi}{4}} & e^{j\frac{-6\pi}{4}} \\ 1 & e^{j\frac{-4\pi}{4}} & e^{j\frac{-8\pi}{4}} & e^{j\frac{-12\pi}{4}} \\ 1 & e^{j\frac{-6\pi}{4}} & e^{j\frac{-12\pi}{4}} & e^{j\frac{-18\pi}{4}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}
\tag{3.85}
$$

One advantage that DFTs have over their continuous counterparts is that for any repeating sequence of N data points, an exact fit can be found using N Fourier coefficients. However, a significant drawback is that, like the Fourier Series, the DFT is defined only for repeating functions. In practice, it is rare that a set of data points can be taken that represent exactly one complete cycle. This can lead to errors due to a phenomenon called leakage.