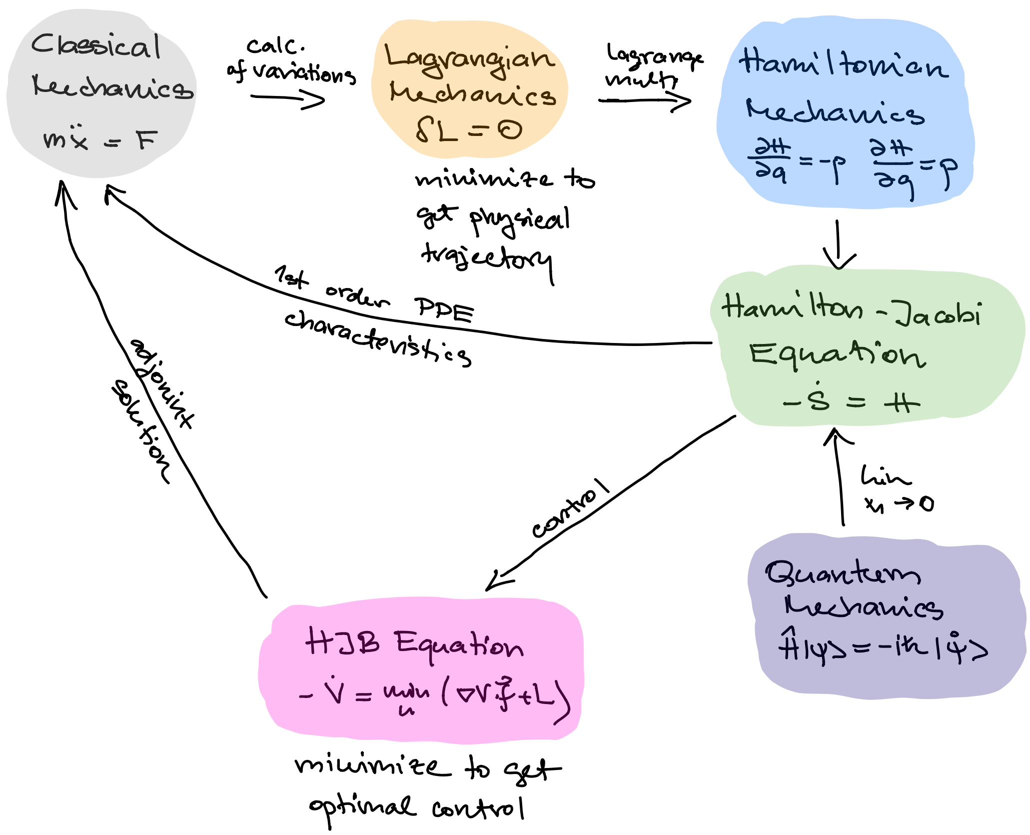


Classical Mechanics
$m\ddot{x} = F$
calc. of variations
Lagrangian Mechanics
$\delta L = 0$
minimize to get physical trajectory
Lagrange multi
Hamiltonian Mechanics
$\frac{\partial H}{\partial q} = -\dot{p}$ $\frac{\partial H}{\partial q} = \dot{p}$
Hamilton-Jacobi Equation
$-\dot{S} = H$
1st order PDE characteristics
Lim $\hbar \to 0$
Quantum Mechanics
$\hat{H}|\psi\rangle = -i\hbar|\dot{\psi}\rangle$
control
adjoint solution
HJB Equation
$-\dot{V} = \min_u (\nabla V^T f + L)$
minimize to get optimal control

- Single 1^{st} order PDE for function S of N generalized coordinates and time t. Generalized momenta don't appear except as derivatives of S.

$$-\frac{S(\vec{q},t)}{\partial t} = \underset{u}{\overset{\text{max if controlled}}{\ }} H\left(\vec{q}, \frac{\partial S(\vec{q},t)}{\partial \vec{q}}, t\right)$$

$$S(\vec{q},t) = \int_0^{\vec{q},t} \mathcal{L}\, dt \quad \text{... action taken along the minimal trajectory}$$

- Trajectory of the system satisfies Euler--Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial q} = 0$$

$\vec{q}$ ... generalized coordinates

$\vec{p}$ ... generalized momentum

H ... Hamiltonian of a mechanical system

S ... Hamilton's principal function

$\mathcal{L}$ ... Lagrangian of the mechanical system

- Eikonal approximation & Schrödinger equation limit

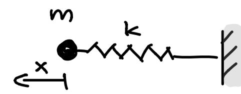
Iso-surfaces of $S(\vec{q},t)$ can be determined at any time t. The motion of the iso-surface can be thought of as a wave in $\vec{q}$ space.

$\psi = \psi_0 \exp(iS/\hbar)$ ... phase wave of S in quantum mechanics

$\hat{H}|\psi\rangle = -i\hbar|\dot{\psi}\rangle$ Schrödinger equation $\quad \frac{\hat{p}^2}{2m} + U = H$

$$L = \frac{1}{2} m v^2 - \frac{1}{2} k x^2$$

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2 \qquad m = k = 1$$



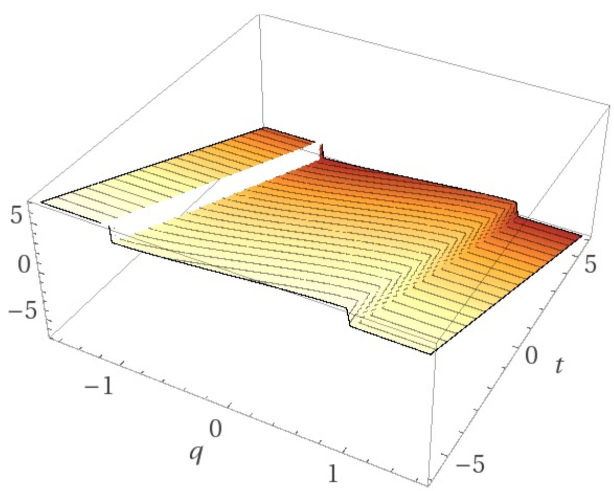
$$-2 \frac{\partial S}{\partial t} = \frac{\partial S}{\partial q}^2 + q^2 \Rightarrow S = \bar{S}(q) - 2Et \,, \; E = \text{const.}$$

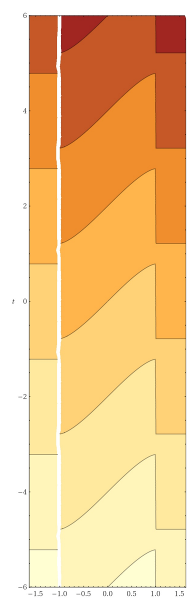
$$E = \left(\frac{\partial \bar{S}}{\partial q}\right)^2 + q^2 \Rightarrow \left|\frac{\partial \bar{S}}{\partial q}\right| = \sqrt{E - q^2}$$

$$\bar{S} = \frac{1}{2}\left(\sqrt{E-q^2}\, q + E \operatorname{atan}\left(\frac{q}{\sqrt{E-q^2}}\right)\right) + C$$

$$S = \frac{1}{2}\left(\sqrt{E-q^2}\, q + E \operatorname{atan}\left(\frac{q}{\sqrt{E-q^2}}\right)\right) - 2Et + C$$

S for $E = 1$, $C = 1$

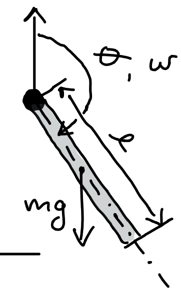




- what are the initial conditions (i.e. $S(p, q, t=0) = ?$)
- what are the boundary conditions for bounded problems ?
- can be known only in retrospect

$$L = \frac{m\ell^2}{2}\omega^2 - \frac{mg\ell}{2}\cos\theta$$

$$H = \frac{p^2}{2m} + \frac{mg\ell}{2}\cos q \qquad m = \ell = g = 1$$



$$-\frac{\partial S}{\partial t} = \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}\cos q$$

$-\dot{S} = H(\vec{q}, \nabla S, t)$

$S = S(\vec{q}, t)$ a PDE $\qquad \nabla S = \vec{p}$

$$S = -Et + \bar{S}(p, q)$$

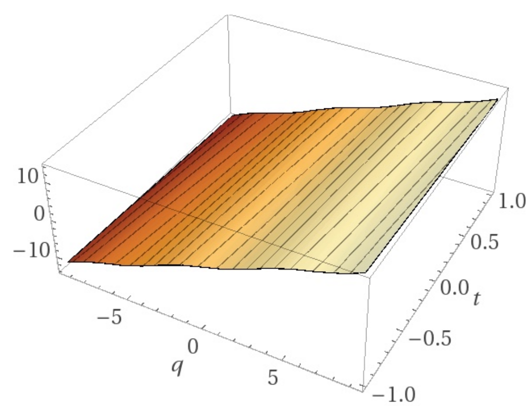
$\mathcal{E}$ ...Ugly eliptic integral function :(

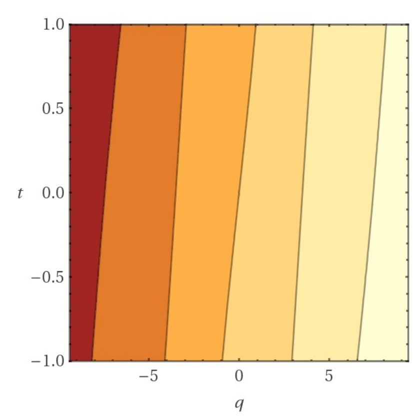
$$\sqrt{2E - \cos q} = \left|\frac{\partial \bar{S}}{\partial q}\right|$$

$$\bar{S} = 2\sqrt{1-2E}\;\mathcal{E}\left(\frac{q}{2}\,\middle|\,\frac{2}{1-2E}\right) \Big/ \sqrt{\frac{2E-\cos q}{2E-1}} + C$$

$$S = 2\sqrt{1-2E}\;\mathcal{E}\left(\frac{q}{2}\,\middle|\,\frac{2}{1-2E}\right) \Big/ \sqrt{\frac{2E-\cos q}{2E-1}} - Et + C$$

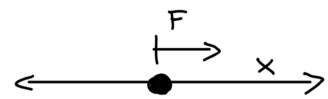
S for $E = 1/2$, $C = 0$





- <u>Slider</u>

$$\begin{bmatrix} \dot{v} \\ x \end{bmatrix} = \begin{bmatrix} F/m \\ v \end{bmatrix}$$



L ... cost function

V ... value function

F ... force ($u \in \langle -1, +1 \rangle$)

x ... ball pos. (x_0)

v ... ball speed (x_1)

m ... ball mass ($= 1$)

$$\begin{bmatrix} \dot{x}_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} u \\ x_0 \end{bmatrix}$$

$$L = -\frac{x_0^2 + x_1^2 + u^2}{2}$$

$$\frac{\partial V}{\partial t} = \min_u \left(\nabla V \cdot \vec{f} + L \right)$$

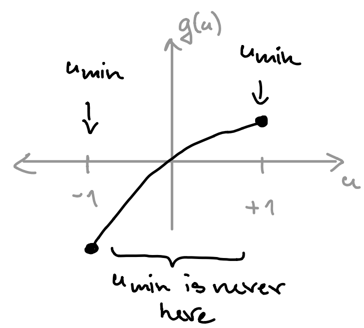
$$\frac{\partial V}{\partial t} = \min_u \left(\underbrace{\frac{\partial V}{\partial x_0} u + \frac{\partial V}{\partial x_1} x_0 - \frac{x_0^2 + x_1^2 + u^2}{2}}_{g(u)} \right)$$

$$\frac{dg}{du}(u_{min}) = 0 = \frac{\partial V}{\partial x_0} - u_{min} \quad \text{or} \quad u_{min} = \pm 1$$

$$\frac{1}{2}\left(\frac{\partial V}{\partial x_0}\right)^2 + \frac{\partial V}{\partial x_1} x_0 - \frac{x_0^2 + x_1^2}{2} \geq \pm 1 \frac{\partial V}{\partial x_0} + \frac{\partial V}{\partial x_1} x_0 - \frac{x_1^2 + x_0^2 + 1^2}{2}$$

$$\frac{1}{2}\left(\left(\frac{\partial V}{\partial x_0}\right)^2 \mp 2 \frac{\partial V}{\partial x_0} + 1\right) = \frac{1}{2}\left(1 \mp \frac{\partial V}{\partial x_0}\right)^2 \geq 0$$

$$u_{min} = -\operatorname{sign} \frac{\partial V}{\partial x_0}$$

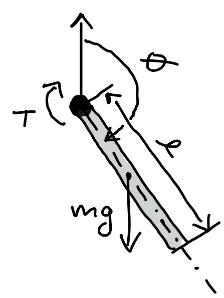


$$\frac{\partial V}{\partial t} = -\left| \frac{\partial V}{\partial x_0} \right| + \frac{\partial V}{\partial x_1} x_0 - \frac{x_0^2 + x_1^2 + 1}{2}$$

Non-linear partial differential HJB equation for optimal value function $V(x_0, x_1, t)$

- Pendulum

$$\begin{bmatrix} \dot{\omega} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{g}{2l} \sin\theta + \frac{u}{ml^2} \\ \omega \end{bmatrix}$$



L ... cost function

V ... value function

T ... torque ($u \in \langle -1, +1 \rangle$)

θ ... pole angle (x_0)

ω ... pole speed (x_1)

m ... pole mass (=1)

l ... pole mass (=1)

g ... gravity acc. (=1)

$$\begin{bmatrix} \dot{x}_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \sin x_1 + u \\ x_0 \end{bmatrix}$$

$$L = -\frac{x_0^2 + x_1^2 + u^2}{2}$$

$$\frac{\partial V}{\partial t} = \min_u \left(\nabla V \cdot \vec{f} + L \right)$$

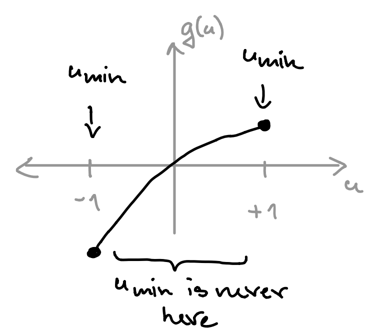
$$\frac{\partial V}{\partial t} = \min_u \left(\underbrace{\frac{\partial V}{\partial x_0} (\sin x_1 + u) + \frac{\partial V}{\partial x_1} x_0 - \frac{x_0^2 + x_1^2 + u^2}{2}}_{g(u)} \right)$$

$$\frac{dg}{du}(u_{min}) = 0 = \frac{\partial V}{\partial x_0} - u_{min} \text{ or } \quad u_{min} = \pm 1$$

$$\frac{1}{2}\left(\frac{\partial V}{\partial x_0}\right)^2 + \dots - \frac{x_0^2 + x_1^2}{2} \geq \pm 1 \frac{\partial V}{\partial x_0} + \dots - \frac{x_1^2 + x_0^2 + 1}{2}$$

$$\frac{1}{2}\left(\left(\frac{\partial V}{\partial x_0}\right)^2 \mp 2 \frac{\partial V}{\partial x_0} + 1 \right) = \frac{1}{2}\left(1 \mp \frac{\partial V}{\partial x_0}\right)^2 \geq 0$$

$$u_{min} = -\operatorname{sign} \frac{\partial V}{\partial x_0}$$



$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x_0} \sin x_1 - \left| \frac{\partial V}{\partial x_0} \right| + \frac{\partial V}{\partial x_1} x_0 - \frac{x_0^2 + x_1^2 + 1}{2}$$

Non-linear partial differential HJB equation with periodic boundary condition (i.e. $V(x_0, x_1, t) = V(x_0, x_1 + 2\pi, t)$) for optimal value function $V(x_0, x_1, t)$