Kullback Leibler Divergence

A measure of difference between two probability distribution functions p and q over the same random variable x.

$$D_{KL}(p\|q) = \mathbb{E}_{x\sim p}\left[\log \frac{p(x)}{q(x)}\right]$$

$D_{KL}(p\|q)$ Minimization

$$\nabla_\theta D_{KL}(p\|q) = \nabla_\theta \mathbb{E}_{x\sim p}\left[\log \frac{p(x;\theta)}{q(x)}\right]$$

$$= \int \nabla_\theta \left(p_\theta \log p_\theta - p_\theta \log q\right) dx =$$

$$= \int \underbrace{\nabla_\theta p_\theta}_{p_\theta \nabla \log p_\theta} \log p_\theta + p_\theta \nabla \log p_\theta - \underbrace{\nabla p_\theta}_{p_\theta \nabla \log p_\theta} \log q \; dx =$$

$$= \int p_\theta \left(\nabla \log p_\theta \left(1 + \log p_\theta - \log q\right)\right) dx$$

$$= \mathbb{E}_{x\sim p_\theta}\left[\nabla_\theta \log p(x;\theta)\left(1 + \log p(x;\theta) - \log q(x)\right)\right]$$

Optimization Algorithm

1. sample x from p_θ
2. calculate $\log p_\theta(x)$ and $\log q(x)$
3. backpropagate to get $\nabla_\theta \log p_\theta(x)$
4. update p_θ $\Delta\theta = \nabla_\theta \log p_\theta(x)\left(1 + \log p_\theta(x) - \log q(x)\right)$
5. goto 1.

This is trickier because it resembles reinforcement learning because the sampling happens with the optimized distribution.

$D_{KL}(q\|p)$ Minimization

$$\nabla_\theta D_{KL}(q\|p) = \nabla_\theta \mathbb{E}_{x\sim q}\left[\log \frac{q(x)}{p(x;\theta)}\right] =$$

$$= \int \nabla_\theta \left(q(x) \log \frac{q(x)}{p(x;\theta)} \right) dx = \int q\cancel{(x)} \frac{\cancel{p(x;\theta)}}{\cancel{q(x)}} \cdot \frac{-q(x)}{p(x;\theta)^2} \cdot \nabla_\theta p \, dx$$

$$= \int - \frac{q(x)}{p(x;\theta)} \nabla_\theta p(x;\theta) dx = - \mathbb{E}_{x\sim q}\left[\nabla_\theta \log p(x;\theta)\right]$$

Optimization Algorithm

1. sample x from q
2. calculate $\log p_\theta(x)$
3. backpropagate to get $\nabla_\theta \log p_\theta(x)$
4. update p_θ $\Delta\theta = \nabla_\theta \log p_\theta(x)$
5. goto 1.