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RESEARCH NOTE

Calculating the two-dimensional magnetotelluric Jacobian in finite elements using reciprocity

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SUMMARY

To speed up the calculation of the field Jacobian for 2-D magnetotelluric inversion using finite elements, the principle of electromagnetic reciprocity is applied. The governing relationship for the Jacobian of the field along strike is obtained by differentiating the Helmholtz equation with respect to the resistivity of each region in the finite-element mesh. The result is a similar Helmholtz equation for the Jacobian, with new sources distributed over all nodes within the parameter medium. However, according to the principle of electromagnetic reciprocity, the roles of sources and receivers are interchangeable. Utilizing reciprocity, the field values obtained from the original forward problem and for new unit sources imposed at the receivers are then utilized in the calculation of the Jacobian by simple multiplication and summation with finite-element terms at each rectangle in the mesh. For the auxiliary (across-strike) fields, the Jacobian terms are obtained by solving source vectors loaded with parabola coefficients used in the approximation to Maxwell's equations. Jacobian terms for the apparent resistivity (ρ_a), the impedance phase (ϕ) and the vertical magnetic field (K_{zy}) are then calculated utilizing the parallel- and auxiliary-field Jacobians. Comparison of Jacobian values obtained from reciprocity calculations and by differencing two forward solutions show that the reciprocity method is accurate and can be used to decrease the number of calculations required to obtain sensitivities by one to two orders of magnitude.

Key words: finite-element methods, inversion, magnetotellurics.

INTRODUCTION

The calculation of the Jacobian matrix, or the matrix of the sensitivity of the geophysical response with respect to the model parameters, is an essential component of an inversion algorithm (Hohmann & Raiche 1988). For some previous 2-D differential equation solutions in electromagnetic (EM) geophysics where body geometries were fixed and only a few model resistivities sought (Jupp & Vozoff 1977; Oristaglio & Worthington 1980), it was efficient, with a direct matrix solver, to define each desired sensitivity as an equivalent source distributed over the parameter region. The source could then be evaluated quickly following the original matrix factorization of the forward problem. In our experience with this approach, typically about 10 Jacobian sources can be solved in the same computed time required to factorize the matrix utilizing LU decomposition (no pivoting, specialized for symmetric, banded matrices). In minimum structure inversion, however, thousands of parameters may be encountered, which would require the computer time of tens of forward problems via this method.

Calculation of the Jacobian terms can be sped up enormously by the use of the reciprocity theorem in electromagnetics (Rodi 1976; Parasnis 1988; Mackie & Madden 1993; McGillivray *et al.* 1994; Zhang, Mackie & Madden 1995), which manifests itself as symmetry in the matrix equation (Rodi 1976). The requirements of the many-parameter Jacobian problem are expressed equivalently in terms of sources only at the original receiver locations, which in 2-D again exploits the prior efficiencies of a direct solution. The purpose of this Research Note is to specify Jacobian calculation for the 2-D magnetotelluric (MT) problem in the framework of the finite-element code of Wannamaker, Stodt & Rijo (1987), and to present example calculations to verify its accuracy. We have distributed literally hundreds of copies of this code and it serves as the computational engine for the OCCAM-2 smooth inversion algorithm (deGroot-Hedlin & Constable 1990; deGroot-Hedlin 1995). Up to now, however, the OCCAM-2 program has used the parametrized approach similar to that of Oristaglio & Worthington (1980) to obtain Jacobians, and this is the most time-consuming step in its calculations.

CALCULATION OF SENSITIVITIES FOR THE FIELD PARALLEL TO STRIKE

We consider a finite-element representation of an earth cross-section made up of an arbitrary number of polygonal regions (Fig. 1). Each of these regions has a constant value of conductivity σ (inverse of resistivity ρ) and can be composed of one or many triangular finite elements. A rectilinear mesh is defined by rectangles, each made up of four triangular elements. The x-axis is assigned to the strike.

For the transverse electric (TE) mode in MT, the governing equation for the total electric field parallel to strike E_x is

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{z}}\frac{\partial E_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\hat{z}}\frac{\partial E_x}{\partial z}\right)-\hat{y}E_x=0, \tag{1}$$

where we ignore displacement currents ($\hat{y}=\sigma$), and $\hat{z}=i\omega\mu_0$. Differentiating (1) with respect to conductivity of any region i yields the sensitivity E'_x according to

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{z}}\frac{\partial E'_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\hat{z}}\frac{\partial E'_x}{\partial z}\right)-\hat{y}E'_x=\begin{cases}E_x & \text{in region } i,\\ 0 & \text{elsewhere},\end{cases} \tag{2}$$

where the prime superscript (′) denotes a derivative.

The transverse magnetic (TM) mode forward problem is governed by:

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{y}}\frac{\partial H_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\hat{y}}\frac{\partial H_x}{\partial z}\right)-\hat{z}H_x=0. \tag{3}$$

Similarly, by differentiating (3) with respect to the conductivity of the finite-element region i, the sensitivity H'_x obeys:

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{y}}\frac{\partial H'_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\hat{y}}\frac{\partial H'_x}{\partial z}\right)-\hat{z}H'_x$$
$$=\begin{cases}\dfrac{\partial}{\partial y}\left(\dfrac{1}{\hat{y}_i^2}\dfrac{\partial H_x}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(\dfrac{1}{\hat{y}_i^2}\dfrac{\partial H_x}{\partial z}\right) & \text{in region } i,\\ 0 & \text{elsewhere}.\end{cases} \tag{4}$$

The left-hand operators of (1) and (2) and of (3) and (4) are dual so that a direct matrix factorization for the forward problem serves also for the evaluation of the source vectors defined for the Jacobians. We choose a direct solution (LU decomposition) for the 2-D problem to avoid uncertainties in pre-conditioning and convergence for iterative solutions.

Determining the entries of the finite-element source vector for the TE-mode sensitivities is simple as they are just the integration of the field along strike over the triangular element shape functions within region i (e.g. Wannamaker *et al.* 1987). For the TM mode, the source of (4) is similar to the Helmholtz operator itself, although the term with $\hat{z}$ is absent. Hence, a subset of the finite-element integrals already defined for the forward problem is readily adapted for the Jacobian source term. Solving the total source vector resulting from all elements contained in region i gives the parallel-field sensitivities at all nodes in the mesh. This essentially is the method of Oristaglio & Worthington (1980).

To understand the role of reciprocity, we focus on a subset of a finite-element mesh where a parameter is made up of only one rectangular element i composed of four triangular elements (Fig. 2a). The discrete sources from the foregoing procedure placed at each node around i from (2) or (4) are denoted s_1, s_2, s_3 and s_4. If solved individually by forward and backward substitution, these sources will yield sensitivity terms $F'_{s_j}(j=1,\ldots,4)$ at the receiver nodes, where F denotes either an electric or a magnetic field. According to reciprocity, if a source s_1 was placed at the receiver node it would yield a solution F'_{s_1} at node i_1 of rectangular element i. Therefore the response at node i_1 due to a unit source placed at this receiver node would be:

$$G_1=F'_{s_1}/s_1, \tag{5}$$

and similarly for the other three nodes around the rectangular element i. The sensitivity term for the perturbation of rectangular element i at a receiver node is the summation

$$F'=\sum_{j=1}^{4}G_j s_j. \tag{6}$$

Several rectangular sensitivities may be summed to form the sensitivity of a larger region.

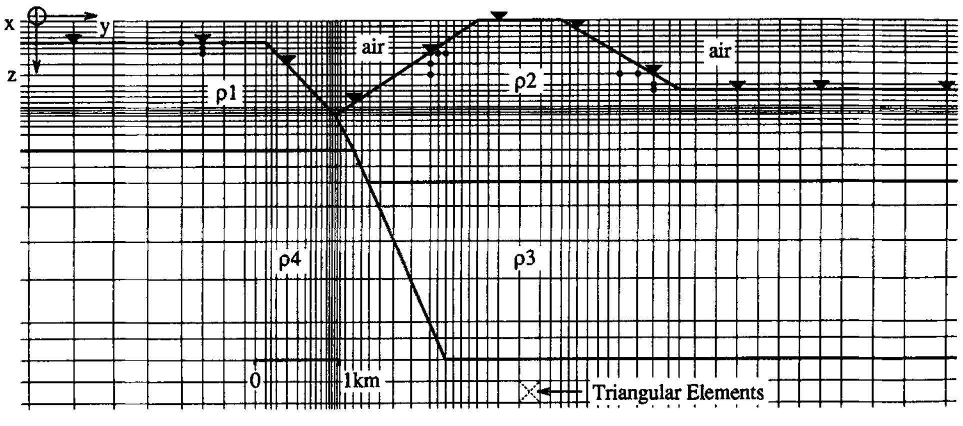

Figure 1. Finite-element mesh for the model of four parameters and the topography used to compare the calculations of Jacobian terms. The example nodal locations used in computing auxiliary fields are shown with dots and are selected according to the character of the topography. Triangles show the stations used in the calculations.

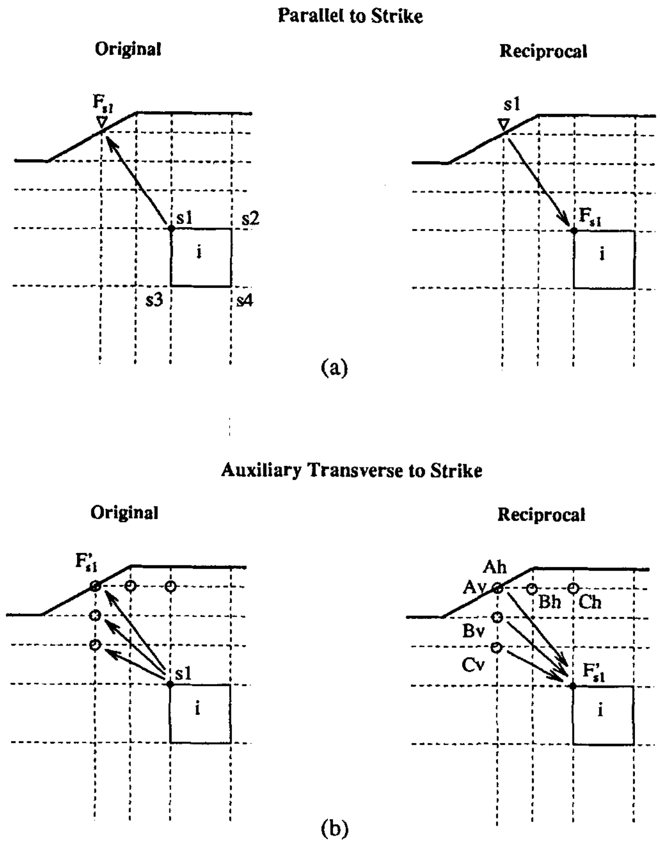

Figure 2. Schematic subregions of the finite-element mesh showing the interchange of the source and the receiver. (a) Field parallel to strike; (b) horizontal auxiliary-field Jacobian obtained from parabola coefficients *Av, Bv* and *Cv* and vertical auxiliary Jacobian obtained from *Ah, Bh* and *Ch.*

Reducing a source at a receiver node in effect provides the sensitivity of that receiver to all triangular elements in the mesh individually. These reciprocal sources constitute some added computational expense, in contrast to the dipole–dipole DC problem where original sources and receivers are coincident and all necessary source reductions were accomplished during the forward calculation (Oldenburg 1994).

We note here that since, on the surface of the earth, the magnetic field parallel to strike sought in the TM mode is constant, its derivative with respect to conductivity is zero. Sensitivities for this component need not be computed.

AUXILIARY FIELD SENSITIVITIES USING RECIPROCITY

In magnetotellurics, inversion is usually performed utilizing the apparent resistivity, the impedance phase and vertical magnetic field data. Therefore, in order to obtain Jacobians for these functions, Jacobian terms for the auxiliary fields (vertical and transverse to strike) are also required.

In our MT forward problem the auxiliary fields are calculated through a numerical approximation to Maxwell's equations (Wannamaker *et al.* 1987). Using the chain rule, the auxiliary Jacobians are defined similarly as

$$H'_y = -\frac{1}{\hat{z}}\frac{\partial E'_x}{\partial z} \tag{7}$$

and

$$H'_z = \frac{1}{\hat{z}}\frac{\partial E'_x}{\partial y} \tag{8}$$

in the TE mode, and

$$E'_y = \frac{1}{\hat{y}}\left(\frac{\partial H'_x}{\partial z} - \delta\hat{y}E_y\right) \tag{9}$$

and

$$E'_z = -\frac{1}{\hat{y}}\left(\frac{\partial H'_x}{\partial y} + \delta\hat{y}E_z\right) \tag{10}$$

in the TM mode, where $\delta\hat{y}$ is the admittivity difference between the 2-D inhomogeneity and its layered host.

The field sensitivity curl terms above can be approximated numerically in the original parametrized approach by fitting piecewise parabolas to the parallel-field Jacobian values in the mesh and then differentiating these analytically, as in the forward problem. For obtaining the auxiliary Jacobians using reciprocity, however, each component of curl first is written as a linear combination of the three nodal Jacobians along strike, i.e.

$$\partial F' = AF'_{n_1} + BF'_{n_2} + CF'_{n_3}, \tag{11}$$

where A, B and C are the coefficients of the differentiated parabola, and F'_{n_k} (at nodes $k = 1, \ldots, 3$) would be the adjacent field Jacobian values used in the approximation (as yet unknown). Subsequently, all three parabola coefficients are loaded into a source vector. The vector solution is then multiplied by mesh parameters and summed as in (6). Fig. 2(b) shows the location of the auxiliary sources in the original and reciprocal models.

For the TE mode, the vertical and horizontal auxiliary magnetic fields are calculated separately with an additional source vector reduction each per receiver (thus, a total of three for this mode). For the TM mode, only the transverse electric field along the topographic slope needs to be calculated. This is accomplished by loading the coefficients for both horizontal and vertical components, weighted according to the local slope. Since H'_x is zero at the receiver, only one source reduction per receiver gives the complete Jacobian for this mode.

For seafloor E'_z, auxiliary nodes in the conductive sea water are used instead of nodes in the resistive rocks below. This is necessary since, for the TM mode, this component and its numerical error can be very large and unstable on the resistive side. When projected onto the sloping seafloor, it can result in large errors in the Jacobian for the electric field along slope.

VERIFICATION AND CONCLUSIONS

Here we compare Jacobian calculations obtained by our procedure using reciprocity with those by differencing two nearly equal forward solutions. We use the finite-element mesh of Fig. 1, which contains four parameters and includes topography. The locations of the neighbouring nodes used for the auxiliary fields and Jacobians are marked by dots. The four parameters are labelled ρ_1, ρ_2, ρ_3 and ρ_4 and have values of 100, 300, 30 and 300 Ωm, respectively. Jacobian terms were calculated at 10 Hz for ρ_1 and ρ_2, and at 1 Hz for ρ_3 and ρ_4 for the 11 stations shown in Fig. 1. Jacobians of log ρ_a with respect to ρ_i are dimensionless, while those of the impedance phase and $K_{zy} = H_z/H_{yt}$ are in radians/log(Ωm) and 1/log(Ωm), respectively.

Figs 3 to 5 show the calculations for the TE mode ρ_a and ϕ, the TE-mode function K_{zy}, and the TM-mode ρ_a and ϕ. For all four parameters and at all 11 stations, the agreement between results obtained with reciprocity and from differenced forward problems is within 1 per cent. Values obtained with the original procedure of eqs (2) and (4) also agreed with the reciprocal results within machine precision at all data points.

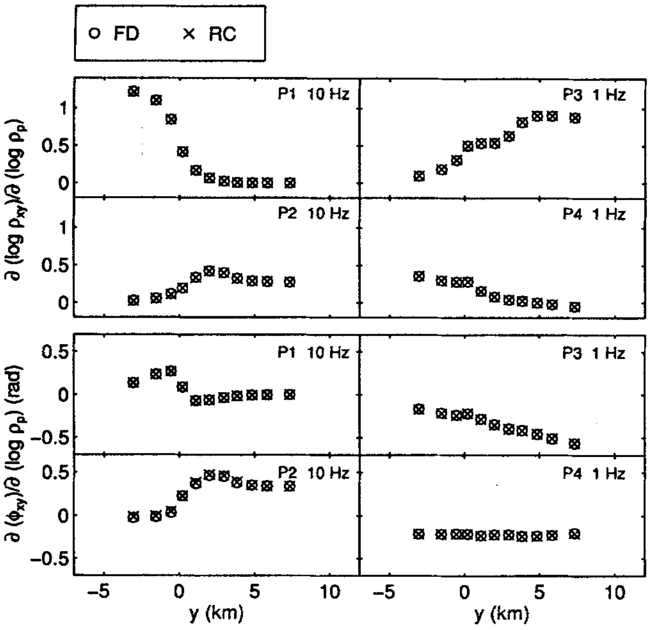

Figure 3. Results of the calculation of the log apparent resistivity and the phase Jacobian (TE mode) using reciprocity (RC) and differencing two forward calculations.

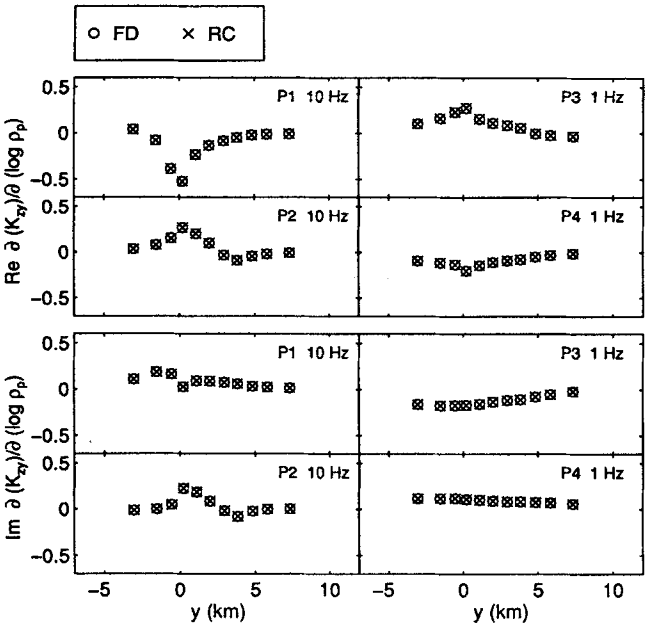

Figure 4. Results of the calculation of the real and imaginary normalized vertical-magnetic-field Jacobian (TE mode) using reciprocity (RC) and differencing two forward calculations (FD).

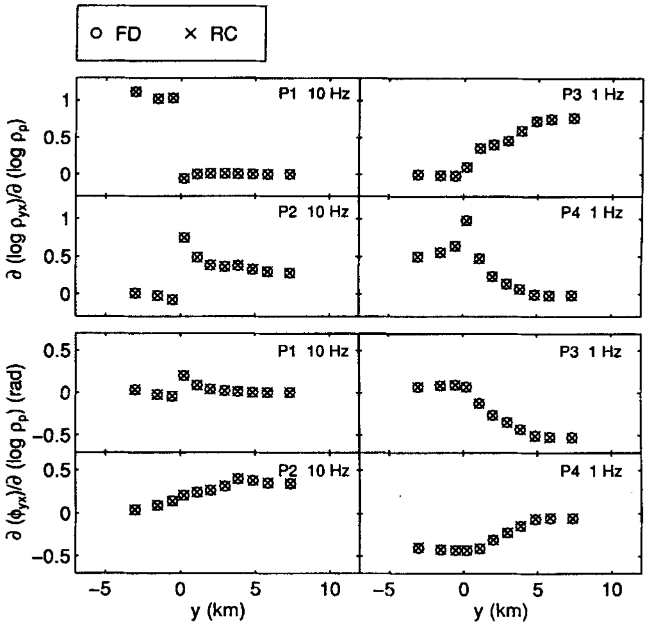

Figure 5. Results of the calculation of the log apparent resistivity and the phase Jacobian (TM mode) using reciprocity (RC) and differencing two forward calculations (FD).

In this particular example, only four parameters were considered to verify computations so that a saving in time using the reciprocal approach is not exemplified. For this model cross-section, Jacobian terms were calculated for 11 stations; therefore, only 11 source reductions for the TM mode (for the auxiliary electric field along the slope) and 33 for the TE mode (11 for the field along strike, 11 each for vertical and horizontal auxiliary magnetic fields) were required under reciprocity.

However, even if more than four distinct media were of interest, only 44 source reductions with TE and TM forward problems would be performed for this example. In a smooth structure estimation, the number of inversion parameters is usually of the order of thousands, while the number of data stations recorded in an average magnetotelluric profile rarely reaches 100. Therefore, reciprocity decreases the computational expense of Jacobians to the order of the number of stations of interest, regardless of the number of parameters used in the inversion.

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