Calculation of sensitivities for the frequency-domain electromagnetic problem

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SUMMARY

A simple derivation is presented for the computation of sensitivities needed to solve parametric inverse problems in electromagnetic induction. It is shown that sensitivities for any component of an electromagnetic field can be obtained by solving two boundary-value problems which are identical except for the specification of the source terms and (possibly) prescribed boundary conditions. The electric fields from these primal and auxiliary problems are multiplied and integrated to produce a numerical value for the sensitivity. Although the final formulae derived here are equivalent to those developed through the use of formal adjoint or Green's functions approaches, our work does not require explicit derivation of the adjoint operator and boundary conditions and does not formally invoke reciprocity.

Key words: electromagnetic induction, Fréchet derivative, inverse problem, sensitivities.

INTRODUCTION

Electromagnetic methods are routinely used in geophysical surveys. The goal of these experiments is to determine the distribution of electrical conductivity in the earth and inverse-theory techniques are required to extract this information from the observations. Although electrical conductivity is a function with infinitely many degrees of freedom we are invariably forced to parametrize the conductivity so that both forward and inverse modelling can be carried out. A major computational difficulty in solving the parametric inverse problem is to compute the sensitivities, or partial derivatives of the data with respect to model parameters. These sensitivities, which establish a linear relationship between changes in the conductivity model and changes in the forward-modelled responses, are used to refine an initial model so that an improved fit to the observed data can be obtained.

General procedures exist for computing the sensitivities. McGillivray & Oldenburg (1990) present a review of commonly used methods and offer an introduction to relevant literature. Numerical procedures generally adopt one of the two following routes: the sensitivity-equation approach or the adjoint-equation approach. In the sensitivity-equation approach the initial operator is differentiated with respect to a model parameter and the subsequent boundary-value problem is solved. If there are M parameters, then the solutions of M new boundary-value problems are required. For the electromagnetic induction problem this approach has been used by Rodi (1976), Jupp & Vozoff (1976) and others. In the adjoint-equation approach the adjoint equation is solved and the sensitivities are obtained by a subsequent integration of electric fields. If there are N observation sites then solutions of N adjoint problems are required. Because the number of unknown parameters in inverse problems usually exceeds the number of data, the adjoint approach is generally computationally more efficient. Our derivation below is guided by this philosophy.

Formal derivations for the adjoint-equation approach exist (e.g. Lanczos 1961; Morse & Feschback 1963; Roach 1982) and its application to electromagnetic induction problems have been presented by Weidelt (1975), Park (1987), Madden & Mackie (1989), Madden (1990), Oldenburg (1990), Ellis & Oldenburg (1993) as well as others. The usual procedure for implementation requires the development of an adjoint operator, i.e. an adjoint-differential operator and appropriate boundary conditions. This is done through the use of the bilinear identity. That identity also establishes the reciprocity condition for EM data which is needed so that the sensitivity computations can be carried out efficiently. The procedure for carrying out the adjoint solution is elegant and general in its conclusions, but it does involve mathematically sophisticated steps. In the work presented here we derive the expression for calculating numerical sensitivities without formally introducing the adjoint operator or appealing to reciprocity. Ultimately this means that our work is less general then formal Green's

function approaches. Nevertheless, the simplicity of the following derivation will be appreciated by those who are faced with solving inverse problems in electromagnetics.

MATHEMATICAL DERIVATION

Consider a finite or infinite spatial domain D which is characterized by constant electrical permittivity ϵ, constant magnetic susceptibility μ, and by a variable conductivity $\sigma(\mathbf{x})$. With the usual constitutive relationships, and assuming a harmonic time dependence $e^{i\omega t}$, Maxwell's equations become

$$\begin{aligned} \nabla \times \mathbf{E} &= -i\omega\mu\mathbf{H} + \mathbf{M}_s \\ \nabla \times \mathbf{H} &= (\sigma + i\omega\epsilon)\mathbf{E} + \mathbf{J}_s \end{aligned} \qquad (1)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths due to imposed electric and magnetic current densities $\mathbf{J}_s$ and $\mathbf{M}_s$. The primal electromagnetic problem is solved by satisfying (1) subject to a boundary condition applied on ∂D which is the boundary of D. The general form of the boundary condition is

$$\alpha_E(\hat{\mathbf{n}} \times \mathbf{E}) + \beta_E(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \nabla \times \mathbf{E}) = \mathbf{S}_E$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2)$

$$\alpha_H(\hat{\mathbf{n}} \times \mathbf{H}) + \beta_H(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \nabla \times \mathbf{H}) = \mathbf{S}_H$$

where $(\alpha_E, \beta_E, \alpha_H, \beta_H)$ are constants and $\mathbf{S}_E$, $\mathbf{S}_H$ are respectively, surface magnetic and electric currents. The form of eq. (2) is general in that it allows electromagnetic problems to be solved by specifying the tangential components of the $\mathbf{E}$ or $\mathbf{H}$ fields, or by specifying an impedance boundary condition. With the understanding that the conditions in eq. (2) can be applied to different portions of ∂D it is convenient to write the boundary conditions as

$$\alpha(\hat{\mathbf{n}} \times \mathbf{U}) + \beta(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \nabla \times \mathbf{U}) = \mathbf{S} \qquad (3)$$

where $\mathbf{U}$ can be either $\mathbf{E}$ or $\mathbf{H}$.

Numerical solutions of the primal problem require that $\sigma(\mathbf{x})$ be represented as

$$\sigma(\mathbf{x}) = \sum_{j=1}^{M} \sigma_j \psi_j(\mathbf{x}) \qquad (4)$$

where σ_j are real constants and $\psi_j(\mathbf{x})$ are chosen basis functions. With this parametrization σ is completely specified by the M-vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_M)$.

In the inverse problem we attempt to find a parameter vector $\boldsymbol{\sigma} \in R^M$ such that the forward-modelled data adequately fit the observations. This necessitates computing the sensitivities $G_{ik} = \partial d_i / \partial \sigma_k$ where d_i is the ith datum. d_i may take many forms; it can be an admittance, an impedance, a component of $\mathbf{E}$ or $\mathbf{H}$, an amplitude or a phase. Irrespective of the choice of d_i, the basic building block for computing G_{ik} is the ability to compute $\partial\mathbf{E}/\partial\sigma_k$ and $\partial\mathbf{H}/\partial\sigma_k$. We now turn attention to this aspect.

Substituting (4) into (1) and differentiating with respect to σ_k produces the sensitivity equations,

$$\nabla \times \frac{\partial \mathbf{E}}{\partial \sigma_k} = -i\omega\mu \frac{\partial \mathbf{H}}{\partial \sigma_k}$$

$$\nabla \times \frac{\partial \mathbf{H}}{\partial \sigma_k} = (\sigma + i\omega\epsilon)\frac{\partial \mathbf{E}}{\partial \sigma_k} + \psi_k(\mathbf{x})\mathbf{E} \qquad (5)$$

and homogeneous boundary conditions

$$\alpha\left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{U}}{\partial \sigma_k}\right) + \beta\left(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \nabla \times \frac{\partial \mathbf{U}}{\partial \sigma_k}\right) = 0 \quad \text{on } \partial D. \qquad (6)$$

Our goal is to derive equations from which $\partial\mathbf{E}/\partial\sigma_k$ and $\partial\mathbf{H}/\partial\sigma_k$ can be evaluated, and in which we need to solve only an equation of the form (1). To this end consider an auxiliary Maxwell problem,

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= -i\omega\mu\tilde{\mathbf{H}} + \tilde{\mathbf{M}}_s \\ \nabla \times \tilde{\mathbf{H}} &= (\sigma + i\omega\epsilon)\tilde{\mathbf{E}} + \tilde{\mathbf{J}}_s \end{aligned} \qquad (7)$$

where the electric and magnetic sources $\tilde{\mathbf{J}}_s$ and $\tilde{\mathbf{M}}_s$ have yet to be defined. The boundary value problem can be solved once the conditions appropriate to $\tilde{\mathbf{J}}_s$ and $\tilde{\mathbf{M}}_s$ are specified on ∂D. These have the form

$$\tilde{\alpha}(\hat{\mathbf{n}} \times \tilde{\mathbf{U}}) + \tilde{\beta}(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \nabla \times \tilde{\mathbf{U}}) = 0. \qquad (8)$$

We note that these boundary conditions may differ from those used to solve the primal problem. For instance, the primal problem might be a magnetotelluric problem whose source is a sheet of current at height, but the source for the auxiliary problem is likely an electric or magnetic dipole inside ∂D. With the exception of the changes in the specifics of the boundary conditions and practical details regarding meshing of the domain, the solution of the forward and auxiliary problems can likely be obtained by using the same computing algorithm.

Use of the vector identity,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \qquad (9)$$

allows (5) and (7) to be combined, yielding,

$$\begin{aligned} \nabla \cdot &\left(\tilde{\mathbf{E}} \times \frac{\partial \mathbf{H}}{\partial \sigma_k} - \frac{\partial \mathbf{E}}{\partial \sigma_k} \times \tilde{\mathbf{H}}\right) \\ &= \tilde{\mathbf{M}}_s \cdot \frac{\partial \mathbf{H}}{\partial \sigma_k} + \tilde{\mathbf{J}}_s \cdot \frac{\partial \mathbf{E}}{\partial \sigma_k} - \tilde{\mathbf{E}} \cdot \mathbf{E}\psi_k(\mathbf{x}). \end{aligned} \qquad (10)$$

Integrating (10) over D and using the divergence theorem yields

$$\begin{aligned} \int_{\partial D} &\left(\tilde{\mathbf{E}} \times \frac{\partial \mathbf{H}}{\partial \sigma_k} - \frac{\partial \mathbf{E}}{\partial \sigma_k} \times \tilde{\mathbf{H}}\right) \cdot \mathbf{n} \, ds \\ &= \int_D \left[\tilde{\mathbf{M}}_s \cdot \frac{\partial \mathbf{H}}{\partial \sigma_k} + \tilde{\mathbf{J}}_s \cdot \frac{\partial \mathbf{E}}{\partial \sigma_k} - \tilde{\mathbf{E}} \cdot \mathbf{E}\psi_k(\mathbf{x})\right] dv. \end{aligned} \qquad (11)$$

Our desired result for computing sensitivities arises when the left-hand side of eq. (11) can be shown to be equal to zero. The circumstances in which this occurs depends upon whether or not the boundary ∂D extends to infinity. The fields $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, $\partial\mathbf{H}/\partial\sigma_k$, $\partial\mathbf{E}/\partial\sigma_k$, all originate from finite sources, and hence the left-hand side of (11) approaches zero as the boundary ∂D extends to infinity. The proof of this is given in Harrington, 1961 p. 117. For a finite domain D the integral is also identically zero if the auxiliary and primal problems have the same boundary conditions. This is often the case in controlled-source experiments and the proof that the left-hand side of (11) vanishes under this condition is given in Appendix 1.

Setting the left-hand side of eq. (11) equal to zero allows

us to write

$$\int_D \left(\tilde{\mathbf{M}}_s \cdot \frac{\partial \mathbf{H}}{\partial \sigma_k} + \tilde{\mathbf{J}}_s \cdot \frac{\partial \mathbf{E}}{\partial \sigma_k} \right) dv = \int_D \tilde{\mathbf{E}} \cdot \mathbf{E} \psi_k(\mathbf{x}) \, dv. \tag{12}$$

This is the main result and shows that the sensitivity for **E** or **H** can be obtained by appropriately specifying the sources for the auxiliary fields and by integrating the dot product of the primal and auxiliary electric fields over the region on which ψ_k is non-zero. For example, to obtain the sensitivities for H_z at an observation location $\mathbf{x}_0$, let $\tilde{\mathbf{M}}_s = \delta(\mathbf{x} - \mathbf{x}_0)\hat{z}$ and $\tilde{\mathbf{J}}_s = 0$. Then (12) becomes

$$\frac{\partial H_z(\mathbf{x}_0)}{\partial \sigma_k} = \int_D \tilde{\mathbf{E}} \cdot \mathbf{E} \psi_k(\mathbf{x}) \, dv. \tag{13}$$

The primal problem is solved for the electric field **E** throughout the domain. The auxiliary problem, with a unit vertical magnetic dipole source placed at $\mathbf{x}_0$, is solved for the auxiliary electric field $\tilde{\mathbf{E}}$ throughout the domain. The quantity $\tilde{\mathbf{E}} \cdot \mathbf{E}$ is then integrated to generate $\partial H_z/\partial \sigma_k$.

To compute the sensitivities for the magnetic field in any other direction, the source for the auxiliary problem must be a unit magnetic dipole in the same direction placed at the observation location. To compute the sensitivities of the electric field, the source must be a unit electric dipole. In all cases the auxiliary electric field is computed using (7), and (12) is then evaluated to generate the desired sensitivity.

EXAMPLE

As a demonstration of the ease with which sensitivities can be derived with the above method, we present a simple example. Let us assume we have a whole space of constant conductivity σ_0 with a plane $\hat{y}$-polarized electromagnetic field propagating in the $\hat{z}$ direction with angular frequency ω. The primal electric field is given by

$$\mathbf{E}(x, y, z) = \hat{y} e^{-ikz}. \tag{14}$$

Let us assume that we wish to compute the sensitivity of the magnetic field strength component H_z located at the origin with respect to variations in conductivity of a $2\,\Delta x \times 2\,\Delta y \times 2\,\Delta z\ m^3$ cell centred at $(x, y, z) = (x_1, y_1, z_1)$. Then, we need to compute the auxiliary electric field resulting from a unit amplitude harmonic vertical magnetic dipole located at the origin. The auxiliary electric field (Ward & Hohmann 1988, eq. 2.56) is

$$\tilde{\mathbf{E}}(x, y, z) = \frac{i\omega\mu}{4\pi r^2}(ikr + 1)e^{-ikr}\left(\frac{z}{r}\hat{y} - \frac{y}{r}\hat{z}\right) \tag{15}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $k = \sqrt{\mu\epsilon\omega^2 - i\mu\sigma_0\omega}$.

Substituting eqs (14) and (15) into (13) yields

$$\begin{aligned}
\frac{\partial H_z}{\partial \sigma} &= \int \mathbf{E} \cdot \tilde{\mathbf{E}} \psi(\mathbf{x}) \, dv \\
&= \int_{z_1-\Delta z}^{z_1+\Delta z} \int_{y_1-\Delta y}^{y_1+\Delta y} \int_{x_1-\Delta x}^{x_1+\Delta x} \frac{i\omega\mu z}{4\pi r^3} \\
&\quad \times (ikr + 1)e^{-ik(z+r)} \, dx \, dy \, dz.
\end{aligned} \tag{16}$$

REFERENCES

Ellis, R. G. & Oldenburg, D. W., 1993. Magnetotelluric inversion using Green's functions and Conjugate Gradients, *Geophys. J. Int.*, submitted.

Jupp, D. L. B. & Vozoff, K., 1976. Two-dimensional magnetotelluric inversion, *Geophys. J. R. astr. Soc.*, **50**, 333–352.

Lanczos, C. 1961. *Linear Differential Operators*, Van Nostrand Company Ltd, Princeton, NJ.

Harrington, R. F., 1961. *Time–Harmonic Electromagnetic Fields*, McGraw-Hill, New York, NY.

Madden, T. R., 1990. Inversion of low frequency electromagnetic data, in *Oceanographic and Geophysical Tomography*, pp. 379–408, eds Desaubies, Y., Tarantola, A. & Vinn-Justin, J., North Holland, New Amsterdam.

Madden, T. R. & Mackie, R. L., 1989. Three-dimensional magnetotelluric modeling and inversion, *Proc. Inst. Electron. Electric. Eng.*, **77**, 318–333.

McGillivray, P. R. & Oldenburg, D. W. 1990. Methods for calculating Fréchet derivatives and sensitivities for the nonlinear inverse problem: a comparative study, *Geophys. Prosp.*, **38**, 499–524.

Morse, P. W. & Feshbach, H., 1953. *Methods of Theoretical Physics*, McGraw-Hill, New York, NY.

Oldenburg, D. W., 1990. Inversion of electromagnetic data: an overview of new techniques, *Surv. Geophys.*, **11**, 231–270.

Park, S. K., 1988. Inversion of magnetotelluric data for multi-dimensional structure, *Institute of Geophysics and Planetary Physics Report* 87/6, University of California, San Diego.

Roach, G. F., 1982. *Greens' Functions*, Cambridge University Press, Cambridge.

Rodi, W. L., 1976. A technique for improving the accuracy of finite element solutions for MT data, *Geophys. J. R. astr. Soc.*, **44**, 483–506.

Ward, S. H. & Hohmann, G. W., 1988. Electromagnetic theory for Geophysical applications, in *Electromagnetic Methods in Applied Geophysics, Vol. 1. Theory*, ed. Nabighian, M. N., SEG investigations in Geophysics, No. 3, Society of Exploration Geophysicists, Tulsa, Oklahoma.

Weidelt, P., 1975. Inversion of two-dimensional conductivity structures, *Phys. Earth planet. Inter.*, **10**, 282–291.

APPENDIX 1

The quantity

$$\int_{\partial D} \left(\tilde{\mathbf{E}} \times \frac{\partial \mathbf{H}}{\partial \sigma_k} - \frac{\partial \mathbf{E}}{\partial \sigma_k} \times \tilde{\mathbf{H}} \right) \cdot \mathbf{n} \, ds \tag{17}$$

on the left-hand side of eq. (11) can be shown to be identically zero under the condition that the domain D is finite and that the auxiliary and primal problems satisfy the same boundary conditions. Use of vector identities $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$ permits the left-hand side of eq. (17) to be written as

$$\int_{\partial D} \left[\frac{\partial \mathbf{H}}{\partial \sigma_k} \cdot (\hat{\mathbf{n}} \times \tilde{\mathbf{E}}) - \tilde{\mathbf{H}} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \sigma_k} \right) \right] ds$$

or as $\tag{18}$

$$-\int_{\partial D} \left[\tilde{\mathbf{E}} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{H}}{\partial \sigma_k} \right) - \frac{\partial \mathbf{E}}{\partial \sigma_k} \cdot (\hat{\mathbf{n}} \times \tilde{\mathbf{H}}) \right] ds.$$

Using eq. (1) with the assumption that the sources $\mathbf{J}_s$ and $\mathbf{M}_s$ are finite and confined inside the domain D, the boundary

conditions in eq. (2) can be written as

$$\alpha(\hat{\mathbf{n}} \times \mathbf{E}) - \beta i\omega\mu(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H}) = \mathbf{S}$$

or   (19)

$$\alpha(\hat{\mathbf{n}} \times \mathbf{H}) + \beta(\sigma + i\omega\epsilon)(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H}) = \mathbf{S}.$$

Differentiating these boundary conditions with respect to σ_k yields

$$\alpha\left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \sigma_k}\right) - \beta i\omega\mu\left(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \frac{\partial \mathbf{H}}{\partial \sigma_k}\right) = \mathbf{0}$$

or   (20)

$$\alpha\left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{H}}{\partial \sigma_k}\right) + \beta(\sigma + i\omega\epsilon)\left(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \sigma_k}\right) = \mathbf{0}$$

where we assume that the probed medium does not extend to the boundary ∂D. If the auxiliary problem satisfies the same boundary conditions as the primal problem (so the $\alpha's$ and $\beta's$ are the same as those in eq. 20) then the boundary conditions for the auxiliary problem are

$$\alpha(\hat{\mathbf{n}} \times \tilde{\mathbf{E}}) - \beta i\omega\mu(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \tilde{\mathbf{H}}) = \tilde{\mathbf{S}}$$

or   (21)

$$\alpha(\hat{\mathbf{n}} \times \tilde{\mathbf{H}}) + \beta(\sigma + i\omega\epsilon)(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \tilde{\mathbf{E}}) = \tilde{\mathbf{S}}$$

where $\tilde{\mathbf{S}}$ is an arbitrary boundary-source term. Substitution of eq. (20) and eq. (21) into eq. (18) shows that both terms in eq. (18) are equal to zero if the applied boundary source $\tilde{\mathbf{S}} = \mathbf{0}$.