Modern Methods of Data Analysis
Homework 1

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Now, recall from the lecture notes that, for any set $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\lambda_d(A) \in (0, +\infty)$, the uniform measure $\mathcal{U}_A$ on A is defined, for any $B \in \mathcal{B}(\mathbb{R}^d)$, by

$$\mathcal{U}_A(B) := \frac{\lambda_d(A \cap B)}{\lambda_d(A)}$$

Task 1

Show that U_A is indeed a positive measure.

Solution

Check that $\mu(\emptyset) = 0$

$$U_A(\emptyset) = \frac{\lambda_d(A \cap \emptyset)}{\lambda_d(A)} = \frac{\lambda_d(\emptyset)}{\lambda_d(A)} = \frac{0}{\lambda_d(A)} = 0$$

Check the countable additivity $\mu(A_1 \cup A_2 \cup \dots) = \mu(A_1) + \mu(A_2) + \dots$

$$\begin{aligned} U_A(B_1 \cup B_2 \cup \dots) =& \frac{\lambda_d(A \cap [B_1 \cup B_2 \cup \dots])}{\lambda_d(A)} \\ =& \frac{\lambda_d([A \cap B_1] \cup [A \cap B_2] \cup \dots)}{\lambda_d(A)} \\ =& \frac{\lambda_d(A \cap B_1)}{\lambda_d(A)} + \frac{\lambda_d(A \cap B_2)}{\lambda_d(A)} + \dots \\ =& U_A(B_1) + U_A(B_2) + \dots \end{aligned}$$

Check the monotonicity $\mu(B) \leq \mu(C)$ where $B, C \in \mathcal{S}$ and $B \subset C$. Denote $C = B \cup \bar{B}$, then $U_A(C) = U_A(B \cup \bar{B}) = U_A(B) + U_A(\bar{B})$. $U_A \geq 0$, thereby $U_A(B) \leq U_A(C)$

Task 2

Show that $U_A \ll \lambda_d$.

Solution

Chose B such that $\lambda_d(B) = 0$. Obviously, $(A \cap B) \subseteq B$, then via monotonicity $\lambda_d(A \cap B) \leq \lambda_d(B)$. Thereby

$$U_A(B) = \frac{\lambda_d(A \cap B)}{\lambda_d(A)} = \frac{0}{\lambda_d(A)} = 0$$

Task 3

Show that both U_A and λ_d are σ-finite measures.

Solution

- *Def. The measure μ is called a σ-finite if there exists subsets $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{S}, S_n \subset S_{n+1}, \cup_n S_n = S$ and $\mu(S_n) < +\infty$.*

First, consider λ_d. Let $S_n = (-n, n)^d$ where $0 \leq n < +\infty$. Then $S_n \in \mathcal{B}(R^d)$, $S_n \subset S_{n+1}$, $\cup_n S_n = R^d$ and $\lambda_d(S_n) = (2n)^d < +\infty$.
Next, Consider U_A. Let $A = (-m, m)^d$ and $B_n = (-n, n)^d$ where $0 \leq n \leq m < +\infty$. Then $B_n \in \mathcal{B}(R^d)$, $B_n \subset B_{n+1}$, $\cup_n B_n = (-m, m)^d$ and

$$U_A(B_0) = \frac{\lambda_d(A \cap B_0)}{\lambda_d(A)} = \frac{0}{\lambda_d(A)} = 0$$

$$U_A(B_m) = \frac{\lambda_d(A \cap B_m)}{\lambda_d(A)} = \frac{\lambda_d(A)}{\lambda_d(A)} = 1$$

$$0 \leq U_A(B_n) \leq 1 < +\infty$$

Task 4

Give the expression of the density of U_A with respect to λ_d.

Solution

$$\frac{dU_A}{d\lambda_d} := f \Rightarrow U_A(B) = \int_B f d\lambda_d$$

Consider $\lambda_d(A \cap B)$

$$\lambda_d(A \cap B) = \int_{A \cap B} f d\lambda_d = \int_B \mathbf{1}_A d\lambda_d$$

Then

$$U_A(B) = \frac{\lambda_d(A \cap B)}{\lambda_d(A)} = \frac{\int_B \mathbf{1}_A d\lambda_d}{\lambda_d(A)} = \int_B \frac{\mathbf{1}_A}{\lambda_d(A)} d\lambda_d$$

Thereby

$$f = \frac{\mathbf{1}_A}{\lambda_d(A)}$$

Check:

$$U_A(B) = \int_B \frac{\mathbf{1}_A}{\lambda_d(A)} d\lambda_d = \int_S \frac{\mathbf{1}_A \mathbf{1}_B}{\lambda_d(A)} d\lambda_d = \frac{\lambda_d(A \cap B)}{\lambda_d(A)}$$

Task 5

Suppose that $d = 2$ and that set A is of the form $A = T([0,1]^2) = \{T(x) : x \in [0,1]^2\}$ where $T : R^2 \to R^2$ is defined, for all $x = (x_1, x_2) \in R^2$, by $T(x) = (\alpha(x_1 \cos\theta + x_2 \sin\theta), \alpha(x_2 \cos\theta - x_1 \sin\theta))$, for some $\theta \in [0, 2\pi)$ and $\alpha > 0$. Give the explicit form of the density of U_A with respect to λ_d.

Solution

Consider the transformation T

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \cos\theta & \alpha \sin\theta \\ -\alpha \sin\theta & \alpha \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = MX$$

$$\det M = \alpha^2 (\cos^2\theta + \sin^2\theta) = \alpha^2$$

Then

$$\lambda_d(T([0,1]^2)) = |\det M| \lambda_d([0,1]^2) = \alpha^2 (1-0) \cdot (1-0) = \alpha^2$$

And consequently

$$\frac{dU_A}{d\lambda_d} = \frac{\nVdash_A}{\lambda_d(A)} = \frac{\nVdash_A}{\alpha^2}$$

where $A = T([0,1]^2)$

Task 6

Recall that the counting measure η on $(R^d, \mathcal{B}(R^d))$ is defined for any $A \in \mathcal{B}(R^d)$ by $\eta(A) = card(A)$ (i.e. $\eta(A)$ is the number of elements in A).
(1) Show that η is not σ-finite.
(2) Show that $\lambda_d \ll \eta$.
(3) Show that λ_d does not have a density with respect to η: there exists no measurable and positive function $f : R^d \to R^+$ such that

$$\lambda_d(A) = \int_A f d\eta.$$

Solution (1)

Consider the sequence $S_1, S_2, \dots$ where $S_n \subset S_{n+1}$. If $S_1 \cup S_2 \cup \dots = R^d$, then at least one S_i is open or closed segment and $\eta(S_i) = +\infty$. Thereby $\eta(A)$ is not σ-finite.

Solution (2)

$\eta(B) = 0$ iff $B = \emptyset$ and $\lambda_d(\emptyset) = 0$, then $\lambda_d \ll \eta$

Solution (3)

Suppose that λ_d have a density with respect to η. Consider arbitrary point $x \in R^d$

$$\lambda_d(\{x\}) = 0 \Rightarrow \lambda_d(\{x\}) = \int_{\{x\}} f d\eta = \int_{R^d} \mathbb{1}_{\{x\}} f d\eta = 0 \Rightarrow f = 0$$

But it means that for all $A \in \mathcal{B}(R^d) : \lambda_d(A) = 0$ and it is contradition. Thereby λ_d does not have a density with respect to η.