Modern Methods of Data Analysis
Homework 2

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Let $(\Omega, \mathcal{A}, P)$ be a probability space. Recall that, for $1 \leq p \leq +\infty$, $\mathcal{L}^p(\Omega, A, P)$ denotes the space of all $\mathbb{R}$-valued random variables $X : \Omega \to \mathbb{R}$ such that $\mathbb{E}[|X|^p] < +\infty$. For $X \in \mathcal{L}^p(\Omega, A, P)$, we denote

$$\|X\|_p := E[|X|^p]^{1/p}$$

Recall that $\mathcal{L}^\infty(\Omega, A, P)$ denotes the space of all $\mathbb{R}$-valued random variables $X : \Omega \to \mathbb{R}$ for which there exists $C > 0$ such that $P(|X| > C) = 0$. For $X \in \mathcal{L}^\infty(\Omega, A, P)$, we denote

$$\|X\|_\infty := \inf\{C > 0 : P(|X| > C) = 0\}.$$

Task 2

Let $X \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$ be such that $P(X \neq 0) > 0$. Consider the function $m : [1, +\infty) \to [0, +\infty]$ defined by $m(p) := E[|X|^p]$. Show that the set $\mathcal{J} := \{p \in [1, +\infty) : m(p) < +\infty\}$ is an interval.

Solution

Proof via contradiction:
$m(1) < +\infty$ via definition of $\mathcal{L}^1$. Now suppose that there exist a and b such that $1 \leq a \leq b < +\infty$ and

$$\begin{cases} m(1) < +\infty \\ m(a) = +\infty \\ m(b) < +\infty \end{cases} \Leftrightarrow \begin{cases} X \in \mathcal{L}^1 \\ X \notin \mathcal{L}^a \\ X \in \mathcal{L}^b \end{cases}$$

Then $a \notin \mathcal{J}$ and $\mathcal{J}$ is not an interval. From the previous task we know that if $a \leq b$ then $\mathcal{L}^b \subset \mathcal{L}^a$. So if $X \in \mathcal{L}^b$ then $X \in \mathcal{L}^a$, but this is contradiction. Therefore $a \in \mathcal{J}$ and $\mathcal{J}$ is an interval.

Task 4

Suppose that $X \in L^\infty(\Omega, \mathcal{A}, P)$. Show that $\lim_{p \to +\infty} \|X\|_p = \|X\|_\infty$.

Solution

On the one side, $\|X\|_\infty = \inf\{C > 0 : P(|X| > C) = 0\}$ via defenition and then $|X(\omega)| \leq \|X\|_\infty$, P-a.e., so

$$\left(\int_\Omega |X|^p dP\right)^{1/p} \leq \left(\int_\Omega \|X\|_\infty^p dP\right)^{1/p}$$
$$\Rightarrow \|X\|_p \leq \left(\|X\|_\infty^p P(\Omega)\right)^{1/p}$$
$$\Rightarrow \|X\|_p \leq \|X\|_\infty P(\Omega)^{1/p}$$

Since $P(\Omega)^{1/p} = 1$ by definition of a probability space, we have

$$\limsup_{p\to+\infty} \|X\|_p \leq \|X\|_\infty$$

On the other side, by defenition of $\mathcal{L}^\infty$, for all $\varepsilon > 0$ there exists a probability value a such that

$$P(\{\omega : |X(\omega)| \geq \|X\|_\infty - \varepsilon\}) \geq a$$

And then

$$\int_\Omega |X|^p dP \geq a(\|X\|_\infty - \varepsilon)^p$$

Therefore

$$\liminf_{p\to+\infty} \|X\|_p \geq \|X\|_\infty - \varepsilon$$

Since ε is arbitrary, consequently we proved that

$$\lim_{p\to+\infty} \|X\|_p = \|X\|_\infty$$

Task 6

Consider $X \sim \mathcal{N}(m, \sigma^2)$, for $m \in \mathbb{R}$ and $\sigma^2 > 0$. Show that $X \in \mathcal{L}^p(\Omega, \mathcal{A}, P)$, for all $p \in [1, +\infty)$, but that $X \notin \mathcal{L}^\infty(\Omega, \mathcal{A}, P)$.

Solution

Consider the moment-generating function for the random variable $|X| : \Omega \to \mathbb{R}_+$

$$M_{|X|}(t) = E[e^{t|X|}] = \int_{\mathbb{R}} e^{t|x|}\rho d\lambda_1 = \int_{-\infty}^{+\infty} e^{t|x|}\rho(x)dx$$

where

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

If $X \in \mathcal{L}^p(\Omega, \mathcal{A}, P)$ then $E[|X|^p] < +\infty$ via definition, and one of the property of the moment-generating function is $E[|X|^p] = \frac{d^p}{dt^p} M_{|X|}(t)$ where $t = 0$. So, the proof is complete provided we can show that

$$\frac{d^p}{dt^p} M_{|X|}(t) < +\infty$$

for all $p \in [1, +\infty)$ where $t = 0$.
Consider the case where $m = 0$ and $\sigma^2 = 1$. Denote $Y \sim \mathcal{N}(0, 1)$

$$M_{|Y|}(t) = \int_{-\infty}^{+\infty} e^{t|x|} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \exp\left(\frac{t^2}{2}\right)\left(\operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + 1\right)$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ (the error function).
Consider only integers p, denoted as n. And then

$$E[|Y|^n] = \frac{d^n}{dt^n}\left[\exp\left(\frac{t^2}{2}\right)\left(\operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + 1\right)\right]$$

where $t = 0$.
We can observe that n-derivation of types of functions e^x, $\frac{x^m}{C}$, $\operatorname{erf}(x)$ and their compositions produces functions of the same types (and constants) for all n. So the values of these functions are less than $+\infty$ for all n with $t = 0$ and consequenly $E[|Y|^n] < +\infty$. Next, $E[|X|^n] < +\infty$ too because we can make the linear substitution $X = m + \sigma Y$ and get the same inequality. Moreover, for all $p \notin \mathbb{N}$ we can find n such that $p < n$, and from the Task 1 we know that if $X \in \mathcal{L}^n$ then $X \in \mathcal{L}^p$. Hence, $X \in \mathcal{L}^p(\Omega, \mathcal{A}, P)$, for all $p \in [1, +\infty)$.
Finally, for $Y \sim \mathcal{N}(0, 1)$ we can see that

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} > 0$$

for all $x \in (-\infty, +\infty)$ and thereby there is no $C > 0$ such that $P(|Y| > C) = 0$ and consequenly $Y \notin \mathcal{L}^\infty$. We can obtain the same result for X via linear substitution.

Task 8

Let X be a $(0, +\infty)$-valued random variables such that $X \in \mathcal{L}^p(\Omega, \mathcal{A}, P)$ for some $p > 1$. For $t \in [1, p]$, we define $\phi(t) := E[X^t] - E[X]^t$. Show that $\operatorname{Ent}[X] = \phi'(1+) := \lim_{p \downarrow 1} \frac{\phi(p) - \phi(1)}{p - 1}$.

Solution

$$\phi'(1+) := \lim_{p \downarrow 1} \frac{\phi(p) - \phi(1)}{p - 1} = \lim_{\Delta p \downarrow 0} \frac{\phi(1 + \Delta p) - \phi(1)}{\Delta p}$$
$$= \lim_{\Delta p \downarrow 0} \frac{E[X X^{\Delta p}] - E[X] E[X]^{\Delta p} - E[X] + E[X]}{\Delta p}$$

$$
\begin{aligned}
&= \lim_{\Delta p \downarrow 0} \frac{E[XX^{\Delta p}] - E[X]E[X]^{\Delta p}}{\Delta p} \\
&= \left[\frac{E[XX^0] - E[X]E[X]^0}{0} = \frac{0}{0} \right]
\end{aligned}
$$

Apply L'Hopital's rule

$$
\begin{aligned}
&= \lim_{\Delta p \downarrow 0} \frac{\left(E[XX^{\Delta p}] - E[X]E[X]^{\Delta p}\right)'}{\Delta p'} \\
&= \lim_{\Delta p \downarrow 0} \left(E'[XX^{\Delta p}] - (E[X]E[X]^{\Delta p})'\right) \\
&= \lim_{\Delta p \downarrow 0} \left(E'[XX^{\Delta p}] - E[X] \ln E[X]\right)
\end{aligned}
$$

Apply the property of the expectation $E'[X] = E[X']$

$$
\begin{aligned}
&= \lim_{\Delta p \downarrow 0} \left(E\left[(XX^{\Delta p})'\right] - E[X] \ln E[X]\right) \\
&= \lim_{\Delta p \downarrow 0} \left(E[X \ln X] - E[X] \ln E[X]\right) \\
&= E[X \ln X] - E[X] \ln E[X] := \mathrm{Ent}[X]
\end{aligned}
$$