Modern Methods of Data Analysis

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Task 1.1

(1) Show that U_A is indeed a positive measure

- Check that $\mu(\varnothing) = 0$

$$U_A(\varnothing) = \frac{\lambda_d(A \cap \varnothing)}{\lambda_d(A)} = \frac{\lambda_d(\varnothing)}{\lambda_d(A)} = \frac{0}{\lambda_d(A)} = 0$$

- Check the countable additivity $\mu(A_1 \cup A_2 \cup \ldots) = \mu(A_1) + \mu(A_2) + \ldots$

$$U_A(B_1 \cup B_2 \cup \ldots) = \frac{\lambda_d(A \cap [B_1 \cup B_2 \cup \ldots])}{\lambda_d(A)} = \frac{\lambda_d([A \cap B_1] \cup [A \cap B_2] \cup \ldots)}{\lambda_d(A)}$$
$$= \frac{\lambda_d(A \cap B_1)}{\lambda_d(A)} + \frac{\lambda_d(A \cap B_2)}{\lambda_d(A)} + \cdots = U_A(B_1) + U_A(B_2) + \ldots$$

- Check the monotonicity $\mu(B) \leq \mu(C)$ where $B, C \in \mathcal{S}$ and $B \subset C$

Denote $C = B \cup \bar{B}$, then $U_A(C) = U_A(B \cup \bar{B}) = U_A(B) + U_A(\bar{B})$

$U_A \geq 0$, thereby $U_A(B) \leq U_A(C)$

(2) Show that $U_A \ll \lambda_d$

Chose B such that $\lambda_d(B) = 0$

Obviously, $(A \cap B) \subseteq B$, then via monotonicity $\lambda_d(A \cap B) \leq \lambda_d(B)$

Thereby

$$U_A(B) = \frac{\lambda_d(A \cap B)}{\lambda_d(A)} = \frac{0}{\lambda_d(A)} = 0$$

(3) Show that both U_A and λ_d are σ-finite measures

Def. The measure μ is called a σ-finite if there exists subsets $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{S}, S_n \subset S_{n+1}, \cup_n S_n = S$ and $\mu(S_n) < +\infty$.

- Consider λ_d

Let $S_n = (-n, n)^d$ where $0 \leq n < +\infty$.

Then $S_n \in \mathcal{B}(R^d)$, $S_n \subset S_{n+1}$, $\cup_n S_n = R^d$ and $\lambda_d(S_n) = (2n)^d < +\infty$

- Consider U_A

Let $A = (-m, m)^d$ and $B_n = (-n, n)^d$ where $0 \leq n \leq m < +\infty$.

Then $B_n \in \mathcal{B}(R^d)$, $B_n \subset B_{n+1}$, $\cup_n B_n = (-m, m)^d$ and

$$U_A(B_0) = \frac{\lambda_d(A \cap B_0)}{\lambda_d(A)} = \frac{0}{\lambda_d(A)} = 0$$
$$U_A(B_m) = \frac{\lambda_d(A \cap B_m)}{\lambda_d(A)} = \frac{\lambda_d(A)}{\lambda_d(A)} = 1$$
$$0 \leq U_A(B_n) \leq 1 < +\infty$$

(4) Give the expression of the density of U_A with respect to λ_d

$$\frac{dU_A}{d\lambda_d} := f \Rightarrow U_A(B) = \int_B f d\lambda_d$$

Consider $\lambda_d(A \cap B)$

$$\lambda_d(A \cap B) = \int_{A\cap B} f d\lambda_d = \int_B \mathbb{1}_A d\lambda_d$$

Then

$$U_A(B) = \frac{\lambda_d(A \cap B)}{\lambda_d(A)} = \frac{\int_B \mathbb{1}_A d\lambda_d}{\lambda_d(A)} = \int_B \frac{\mathbb{1}_A}{\lambda_d(A)} d\lambda_d$$

Thereby

$$f = \frac{\mathbb{1}_A}{\lambda_d(A)}$$

Check:

$$U_A(B) = \int_B \frac{\mathbb{1}_A}{\lambda_d(A)} d\lambda_d = \int_S \frac{\mathbb{1}_A \mathbb{1}_B}{\lambda_d(A)} d\lambda_d = \frac{\lambda_d(A \cap B)}{\lambda_d(A)}$$

(5) Suppose that $d = 2$ and that set A is of the form $A = T([0,1]^2) = \{T(x) : x \in [0,1]^2\}$ where $T : R^2 \to R^2$ is defined, for all $x = (x_1, x_2) \in R^2$, by $T(x) = (\alpha(x_1 \cos\theta + x_2 \sin\theta), \alpha(x_2 \cos\theta - x_1 \sin\theta))$, for some $\theta \in [0, 2\pi)$ and $\alpha > 0$. Give the explicit form of the density of U_A with respect to λ_d.

Consider the transformation T

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha\cos\theta & \alpha\sin\theta \\ -\alpha\sin\theta & \alpha\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = MX$$

$$\det M = \alpha^2(\cos^2\theta + \sin^2\theta) = \alpha^2$$

Then

$$\lambda_d(T([0,1]^2)) = |\det M|\lambda_d([0,1]^2) = \alpha^2(1-0)\cdot(1-0) = \alpha^2$$

And consequently

$$\frac{dU_A}{d\lambda_d} = \frac{\mathbb{1}_A}{\lambda_d(A)} = \frac{\mathbb{1}_A}{\alpha^2}$$

where $A = T([0,1]^2)$

(6) Recall that the counting measure η on $(R^d, \mathcal{B}(R^d))$ is defined for any $A \in \mathcal{B}(R^d)$ by $\eta(A) = card(A)$ (i.e. $\eta(A)$ is the number of elements in A). Show that η is not σ-finite.

Consider the sequence $S_1, S_2, \ldots$ where $S_n \subset S_{n+1}$. If $S_1 \cup S_2 \cup \cdots = R^d$, then at least one S_i is open or closed segment and $\eta(S_i) = +\infty$. Thereby $\eta(A)$ is not σ-finite.

Show that $\lambda_d \ll \eta$.

$\eta(B) = 0$ iff $B = \varnothing$ and $\lambda_d(\varnothing) = 0$, then $\lambda_d \ll \eta$

Show that λ_d does not have a density with respect to η: there exists no measurable and positive function $f : R^d \to R^+$ such that

$$\lambda_d(A) = \int_A f d\eta$$

Suppose that λ_d have a density with respect to η. Consider arbitrary point $x \in R^d$

$$\lambda_d(\{x\}) = 0 \Rightarrow \lambda_d(\{x\}) = \int_{\{x\}} f d\eta = \int_{R^d} \mathbb{1}_{\{x\}} f d\eta = 0 \Rightarrow f = 0$$

But it means that for all $A \in \mathcal{B}(R^d)$

$$\lambda_d(A) = 0$$

and it is contradition.

Thereby λ_d does not have a density with respect to η.