Predicate_Logic$_{Proof}$

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1 Predicate Logic

Predicate logic is teh pointer where you have to deal with real life application , such that there is a need to have a poroposition when ever something occours ,w hich would ofcourse make a statement for something exist or something not to exist , and those values would add up , when you ahve those statements , an example of this statement could be similar to this $\exists n \in N \implies n = 36$

To deal with real life stuff like this we turn these values into a predicate values such that we give some sort of quantifier to express how many objects are involved and give a domain of the discourse to define the range of objects.

A standard property of *Predicate* is defined via $P(x)$ this notion is there to define a given statement . and allows us to call the given domain of the functin .

For example what you can some thing like this **everyone likes football** and **Some people dislike cats** the domain of this set of infomation is set for all people . suh that this implies that this would go for everyone , The first sentence means all members of teh set like football , teh second is saying atleast one person does not like cats , within that given set .

Another example of this is shown below:

$$P(n) = ((\frac{n}{4})) \implies (n \in \frac{\mathbf{N}}{2})$$

In this example we are stating that for this predicate to exist there must be a value of n such that it is dividable by all natural numbers for n $>$ 0 if that makes sense .

1.1 Type of quantifier

There are many cases where you have some quantifier in which you have some universal pointer to dictate if something would exist o nto , in both cases it would be shown with the following notation :

$$\forall \& \exists \& \exists !$$

For example we could say if that domain of the some section is set for all people you could say this :

$$\forall x P(x) \implies \forall x, P(x) = True$$

Such that you could say that $Everyonelikesfootball$ in teh sense of

$$\forall x \in likes(x, football)$$

Something towards this injuction would proove this is to be true if the statement would hold , which ofcourse it does not . But it is a theory in this repsect where you have to state if something has to be ture or not .

Other examples

$$\exists x P(x) isTrue$$
$$\exists x P(x) isFalse$$
$$\forall x P(x) isTrue$$
$$\forall x P(x) isFalse$$

In this For the first one, we say its true if atleast one person in x is linked within $P(x)$ For teh second one it is the opposite of the first one , the third one states that all x has to be true for $P(x)$ to hold.

1.2 Proof by existence and counter example

To prove that $\forall x P(x)$ is true you have to prove that $P(x)$ is true for all of x in teh domain of teh course that you have . To Prove that $\exists x P(x)$ is true you have to prove that $P(x)$ is true for some value of x in P(x)

This is a way of proof that allows you to prove certian variables via ocunter example and exampel when ever you have a look at a certain style of such given proof .

1.3 Multiple Parameters

Have a look at this statement, such that you have more than one variable that you would have to consider. *"Any Dog on the escalator must be carried by somebody ... "* This can be represented with the following represenation :

$$\forall x \exists y ((on\_escaltor(x) \wedge dog(x)) \implies (Person(y) \wedge (Carried\_by(x, y)))$$

Though this statement is rather long , it dfoes hold in respect to the given value that hve been shown above .

When you look at this at a more mathematical level , consider you have a statement like $y = x$ we can then say the following

$$\forall x \exists y \in P(x, y) \implies \forall x \exists y (y = x) \iff \forall x \exists y \in int(x) \implies (int(y) \wedge (y = x))$$

There are many proposition that you can use to define these varaibles , in each given existence there is osme value that points towards that to be true or false , what you want to state is the first one as that is the easier one to point out .

Quick Look at Negation If you want to negate a given value , what you can do is the following :

$$\neg \forall x P(x) = \exists x \neg P(x) \neg \exists x \neg P(x) = \forall x \neg P(x)$$

you push forward a negated value to prove and show that those values equate towards each other.

2 Well formed Fromulae (WFF)

We can use logical symbols of this to build logical expressions from predicates - in a sense those variables are wirtten using x, y, z and are within some constant of the first few letters of the alphebet .

Here is an example of how this may work .

$$\frac{\S\ \forall x (P(x) \implies \exists y Q(x, y))}{\S\ \ \exists x (P(x) \wedge Q(x, c))}$$

Suppose that the domain is all people, and thatL(x, y) means "xlikesy",F(x) means"xcan speak French", andJ(x) means "xknows Java".

3 Logical Equivalences

WFF has large manipulation forms the basis of predicate calculus , is teh study of equality of logical expressions , we would then consider only first order predicate calculus in which quantifiers can only range over varaibles and not over functions . Have a look at the following :

1. Re-Naming :

$$\forall x A(x) \iff \forall y A(y) \quad \exists x A(x) \iff \exists y A(y)$$

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   in this example you see that you can merely rename variables and it owuld be fine either way This is a well known and is used within the return values when ever we have to change some value that we had .

2. Negation

$$\neg \forall x A(x) \iff \exists x (\neg A(x)) \quad \neg \exists x A(x) \iff \forall x (\neg A(x)))$$

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3. Distributed law works in this as well .

Theorem 3.1.

$$\forall \; and \; \exists \; have \; a \; higher \; precednece \; than \; \wedge \; and \; \vee \; and \; \implies \; and \; \iff$$

Example : Show that $\neg\forall x(P(x) \wedge Q(x) \iff (\exists x \neg P(x) \vee \exists y \neg Q(x)))$

- you know that not fora ll of x within p of x and q f x is teh same as for some value of x in not of p(x) and for some value of y in not of q of y , holds because they are equivelent formulas .
- $\iff \exists x \neg(P(x) \wedge Q(x))$
- $\iff \exists x(\neg(P(x) \vee \neg Q(x)))$
- $\iff \exists x(\neg P(x)) \vee \exists y(\neg Q(x))$

As an example here you would see that , exists statement corrosponds to another value for x in Q , what you want to do is prove that the first statement holds for some value of x , and that the second values hold for some value of x, if this is true then you have proved it correctly . Most cases you would have numbers that are given for you to prove such statment , but in this case when you know these hold , you can look at the rooles within the previous documents on Propositional calculus .

4 Interpretations

A formula in propositional logic has a truth value associated with each possible truthvalue, T or F, of each of its propositional variables.

For example $P \implies Q \; true \; for \; P = T, Q = T \; false \; for \; P = T, Q = F$ A mapping from each of the variables to T or F is called and this is an interpretations and gives meaning to teh formula whether something is true or not . This would normally come in sets of data where you would have to interprete different compliex logic.

5 Semantic Entailment

Definition 5.1. *An interpretation which makes a formulaAtrue is amodelforA. A formulawhich has at least one model is said to beconsistentorsatisfiable. A formula whichhas no models is said to beinconsistentorunsatisfiable. A formula which is true for allinterpretations is said to bevalid. A formula which is neither inconsistent nor valid is saidto becontingent.*

A formula, is sematic formula b if and only if every model of a is also a model for B , that is where you would write

$$A \models B$$

Where A models B Another ways of saying this is that A logically implies B or B is a logical Consequence of A Here are a few examples of how this could occour

$$P = Likes \; Domain = \{Sue, ann, john, bill\} \; Model \; for \; \exists x \forall y(((x! = y) \implies P(x, y)) \wedge \neg P(x, x))$$

We can then say that

$$\forall x P(x) \vee \exists y \neg P(y) \; True$$

This statement above would hold true, as $\forall x \exists y \in Domain \iff x! = y \implies P(x, y)$

When we say $A \models B$ What this means is that any interpreation which makes formula A true will also make formula B true , such that you have $A \models B$

Here is an example o fhow logical consequences would work

$$S \models U$$

we can construct the truth table , and hence write the truth table with a colomn for each of the given proposition in S and the column for U and you want to see if they refer towards each other

Use this : Example -< $\{P, Q, Q \implies (R \vee U) \neg R\} \models U$ Here is a truth table for that

6 Modus Ponens

The Symbols $\vdash$ is infers , where you infer to something and $\models$ means A has consequences to B where $\vdash$ means A infers B

If we know that a given porperty of P implies a property in Q and we know that P is true then we can deduce that Q is true This can be shown with teh following examples

- If it is true that i am on teh seaside then i am happy
- If it is true i am at teh seaside

You can then deduce that you are happy , via that . using Modus Ponens with the inference rule .

$$from\ if\ P\ then\ Deduce\ P$$

this becomes into

$$P \implies Q\ and\ P \implies Q$$

This can be further written with the following given formulas .

$$\frac{P, P \implies Q}{Q}\ or\ \{P, P \implies Q\} \vdash Q$$

In addition to single proposition P and Q qw can have a formula Via A and B The two Values We state that to be true where you have the first P , P value

$$\frac{A, A \implies B}{B}\ or\ \{A, A \implies B\} \vdash B$$

Here is another example of how this could work .

Consider the proposition

P - Interest rates increase

Q - Mortgage rate increase

R - House prices Fall you then hypothesise

$$\{P \implies Q, Q \implies R, P\}$$

Such that Interest rates increase woudl imply that mortgage rates increase and hence that implies that house prices will fall . We can then write this down , as teh following -

$$\{P, P \implies Q\} \models Q\ and\ \{Q, Q \implies R\} \vdash R$$

For P if P is true it implies Q such that for Q if Q is true , it implies R , where $P \vdash Q \vdash R$ Where if any is true, you know that it will hold.

most times when you look a Modu Pones if you have multiple implications, then you can state that P could imply onto R and so on .

7 Soundness

We only want to use this inference rule if things proved using it are correct in teh same sense that if A is proved from S then $S \models A$ if any formula A which can be derived from a set of s , using the given formula that have been provided

Definition 7.1. *Notice how they are lined together with Consequences and You look at Soundness A sound inference rule R is one for which :*

$$S \models A\ (Using R)\ Then\ S \vdash A$$

If you look at the formula $P \implies Q$ then we can see that if P = T then $P \implies Q = T$ then only onw row would follow true to this , such that Q = T We can then show:

$$\{P \implies Q, Q \implies R, P\} \vdash R$$
$$\{P \implies Q, Q \implies R, P\} \vdash R$$

Looking at rules

$$\frac{A}{A \vee B} \quad \frac{\neg(A \wedge B)}{(\neg A) \vee (\neg B)}$$

$$\frac{A \wedge B}{A} \quad \frac{\neg(A \vee B)}{\neg(A) \wedge \neg(B)}$$

$$\frac{A, B}{A \wedge B} \quad \frac{A \vee (B \wedge C)}{A \vee B}$$

$$\frac{A, A \implies B}{B} \quad \frac{A \wedge (B \vee C)}{A \vee B}$$

$$\frac{A \implies B,\ where\ B \implies C}{A \implies C}$$

$$\frac{A \vee B, \neg A}{B}$$

With this we can use direct proofs with inferences . We can derive a result using any combination sound of inference rules.

Say that you have : That is Direct proofs with inferences Using $\vdash$

$$\{P, P \implies Q, Q \implies (R \vee U), \neg R\} \vdash U$$

$$\frac{P, PQ}{Q} \frac{Q \implies (R \wedge U) -- > \neg R}{U}$$

The first thing that you would do is where you have P implies Q , we then state forward Q implies (R or U) and that would have to imply neg R -> we can then state this infers to U.

8 Questions

1. $$\forall N((\frac{N}{3}) \implies (\frac{N+1}{2}\ is\ True \implies N \in odd))$$

   If N is divisable by three then its odd , this is false , where you can use a counter example for how this may not work , in this example we can use , $6 \in \mathbb{Z}\ and\ \frac{6}{2} \implies True$

2. $$\exists n((n+1)^2 = 36)$$

   We know that this is true , because you would have some value of N = 5 where $(n+1)^2 = 36$

3. $$\exists n((n^2 + 3n + 2 = 0) \wedge (n > 1))$$

   If N such that $n^2 + 3n + 2 = 0$, this can get factored down , into a quadratic equation where you would have $(n+1)(n+2)$ such that this cannot hold , as you know that it is false , where there is no value of n for this to be true

4. $$\forall x \exists y \; where \; (x - y = 0)$$

   This is true, where for any value of x there will always be some value of y where you will equal zero , $x \in \mathbb{Z} \& y \in \mathbb{Z} \iff (xy)$

5. $$\forall x \forall y (x \cdot y = 7)$$

   This is true , as you can prove this via existence , such that $if\ x = 7\ and\ y = 1 \implies x \cdot y = 7$

6. $$\forall x \forall y (y > x)$$

   This ofcourse is false , as if x = 5 and y = 4 this is false , and hence does not hold .

7. $$\forall x \forall y ((y > 0) \implies (x + y > 0))$$

   No This is false as $let\ x = -15\ \&\ y = 5 \implies False$

now that you have had a look at those examples have a look at these examples below and see if they hold or not :

- True, by existence take x=3
- True, by existence take x=4,y=3
- False, by contradiction take x=3
- False, by contradiction take x=8
- False, by contradiction take y=1
- True, take x= 15y
- True, take y=x1
- False, take x=y+1
- True, take $x = 2yy^2 10$

Questions

1. $$\forall y \exists x (((y-1)^2 + x) < 0)$$

   This statement would be True as if you take the following : $x = 2y - y^2 - 10 \implies \forall x, y(((y-1)^2) + 2y - y^2 - 10) = 2y^2 - 9$ This would then mean that for all values of y , you would get a value that is less than zero , which in this case this is not true , and hence does not hold .
   We now know that this is false , what we can then use is a logical based pointer to assume that if x = y + 1 then this statement is false as it is not less than 0 have a look : $x = y + 1 \implies (((y-1)^2) + y + 1) < 0 \implies y^2 - y + 2\ such\ that\ \forall y \in \mathbb{Z} \exists\ number > 0$

2.

$$\forall x \exists y (x = y + 1)$$

This is true the reason for why this is true would be if x = y + 1 then this would hold , where you would have the following $Let\ x = 4\ and\ y = 3 \implies 4 = 3 + 1 \iff True$

3.

$$\exists x \forall y ((x + y = 7) \wedge (x - y = 1))$$

For some x , there is every y where you would get x + y = 7 and x - y to give you 1 this statement from my view stands false, because say you have , x = 5 and y = 2 then you would have the land part to give you y = 5 , and hence you are double renaming the given variables that you have .