The general class of (sparse) random graphs arising from exchangeable point processes

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Big Picture

Motivating Problem

Need families of random graphs for modelling network structures

Example

- network: friendships among n users of a social network
- model: family of random graphs $(G_n)_{n \in \mathbb{N}}$

Sparse Networks

Real world networks are sparsely connected

Random graph models should be *sparse*: $o(n^2)$ edges as number of observed nodes n becomes large.

Problem

No general framework for the statistical analysis of sparsely connected networks.

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Sparse Graphs

We have no satisfactory answers to some fundamental questions:

1. how should we parameterize the space of distributions on sparse graphs?
2. what can we learn about a large graph if we observe only a small subgraph? and what do we mean when we say "observe"?

Results

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We derive and study a general class of random graphs suitable for modelling network structures.

Special cases

- All dense (graphon) models
- Caron & Fox models
- Sparse graphs with e.g. small world and power law behaviour

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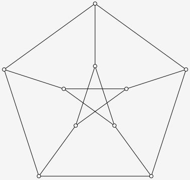
Graphs, Adjacency, and Pixel Pictures



$$
\begin{matrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
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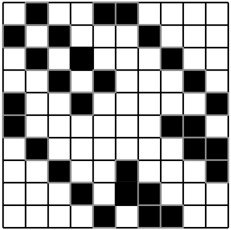
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{matrix}
$$



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Dense Graphs

Graph models

- Basic object: infinite random binary matrix (X_{ij})
- Random graphs: $(G_n)_n$ defined by adjacency matrix as $n \times n$ upper left submatrix

Joint exchangeability of infinite random matrices

$(X_{ij}) \stackrel{d}{=} (X_{\sigma(i)\sigma(j)})$ for all permutations $\sigma \in S_\infty$ of the positive integers

Aldous-Hoover-Kallenberg (translated)

- Primitive of inference: $W : [0,1]^2 \to [0,1]$ (a *graphon*)
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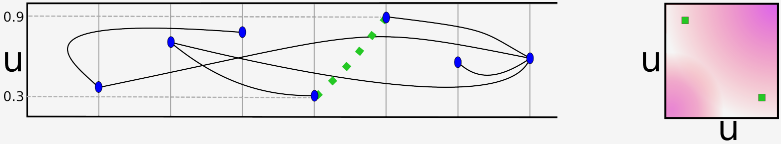
Graphon Generative Model

Given a graphon $W : [0,1]^2 \rightarrow [0,1]$, sample a random graph by:

1. Assign each vertex i an iid $U[0,1]$ latent random variable
2. Include each edge (i,j) independently with probability $W(U_i, U_j)$



Edge $(4,5)$ is included with probability $W(0.3, 0.9) = W(0.9, 0.3)$.
The graphon W is shown as a heatmap on the right.

Representation Theorem

Recipe for constructing statistical models

- Assume a probabilistic symmetry on some infinite random structure
- Associated representation theorem picks out privileged family of distributions

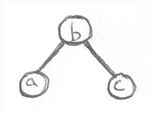
Examples

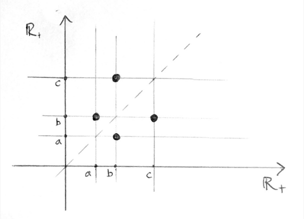
- de Finetti's representation theorem for exchangeable sequences
- Aldous-Hoover-Kallenberg theorem for exchangeable arrays

Caron & Fox 2014: (Sparse) Exchangeable Graphs

Key insights

- adjacency matrix $\rightarrow$ point process on $\mathbb{R}_+^2$
- array joint exchangeability $\rightarrow$ point process joint exchangeability





Graph edges correspond to points on $\mathbb{R}_+^2$

(Sparse) Graph Representation Theorem

Setup

- Random structure: point process on $\mathbb{R}_+^2$
- Finite graph G_s: truncate to $s \times s$ box
- Symmetry: joint exchangeability of point process

Representation theorem*

Distribution characterized by a *graphex*: a symmetric, integrable function $W : \mathbb{R}_+^2 \to [0, 1]$

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Graphex Model

Generative Model

Given W an infinite random graph is sampled by:

1. Sample a (latent) unit rate Poisson process Π on $\theta \times \vartheta$.
2. For each pair of points (θ_i, ϑ_i), $(\theta_j, \vartheta_j) \in \Pi$ include edge (θ_i, θ_j) with probability $W(\vartheta_i, \vartheta_j)$.
3. Include θ_i as a vertex whenever θ_i participates in at least one edge.

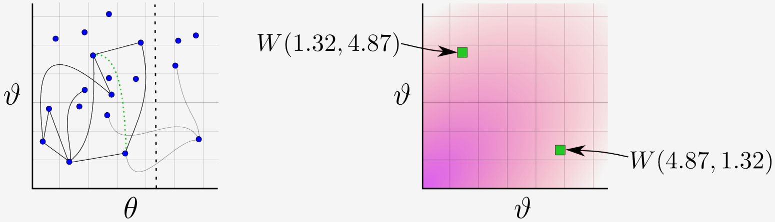

Figure : Graphex Model. **Graphex W is magenta heatmap.**

Sampling Distribution Results

Given graphex W we know:

1. the expected number of vertices and edges as a function of the size s
2. the asymptotic degree distribution
3. the asymptotic connectivity structure for certain families of graphexes.

Punchline

These models include a wide range of interesting graphs.

Estimation

Problem

How can we estimate a graphex?

Setup

- Observation: G_s (and s)
- Want Estimator: $\hat{W}_{G_s} : \mathbb{R}_+^2 \to [0, 1]$

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Dense Graph Estimation

Empirical Graphon

Let $\widetilde{W}_n : [0,1]^2 \to \{0,1\}$ be the step function corresponding to the (arbitrarily permuted) adjacency matrix. $\widetilde{W}_n$ is a general non-parametric estimator for graphons.

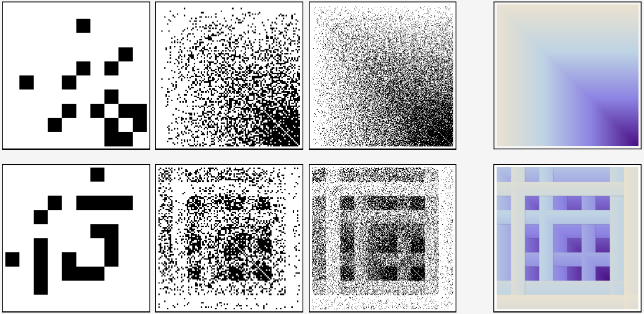

Figure : Empirical Graphon (from Orbanz Roy 2015)

Sparse Graph Estimation

Empirical Graphex

Let $\widetilde{W}^{G_s}$ be the empirical graphex of G_s, and define the empirical graphex $\widehat{W}^{G_s} : [0, \frac{v_s}{s}] \to \{0, 1\}$ by

$$\widehat{W}^{G_s}(x, y) = \widetilde{W}^{G_s}(\frac{v_s}{s}x, \frac{v_s}{s}y)$$

Theorem

Let $\widehat{G}_r^{(s)}$ be generated by $\widehat{W}^{G_s}(x, y)$ and let $\eta_r^{G_s}$ be the (random, G_s measurable) law of η^{G_s}. Then, almost surely,

$$\lim_{s \to \infty} \|\eta_r^{G_s} - \mathscr{L}(G_r)\|_{\mathrm{TV}} \to 0,$$

where $\mathscr{L}(G_r)$ is the law of a size r graph generated by W and $\|\cdot\|_{\mathrm{TV}}$ is the total variation distance.

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Sparse Graph Estimation

Estimation is also possible even when the size s is unknown
(but it's a little bit too long to state here)

Summary

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- Representation theorem for sparse random graphs
- Extends dense (graphon) theory to sparse graphs
- Formulas for sampling distribution properties in terms of graphex
- General non-parametric estimator

Arxiv

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