(Sparse) exchangeable random graphs

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based on joint work with:

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Big Picture

Motivating Problem

Need families of random graphs for modelling network structures

Example

- network: friendships among n users of a social network
- model: family of random graphs $(G_n)_{n \in \mathbb{N}}$

Sparse Networks

Real world networks are sparsely connected

Random graph models should be *sparse*: $o(n^2)$ edges as number of observed nodes n becomes large.

Problem

No general framework for the statistical analysis of sparsely connected networks.

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Sparse Graphs

We have no satisfactory answers to some fundamental questions:

1. how should we parameterize the space of distributions on sparse graphs?
2. what can we learn about a large graph if we observe only a small subgraph? and what do we mean when we say "observe"?

Results (VR 2015)

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We derive and study a general class of random graphs suitable for modelling network structures.

Special cases

- All dense (graphon) models
- Sparse graphs with e.g. small world and power law behaviour
- Caron & Fox models

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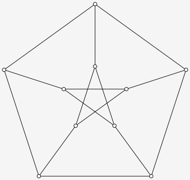
Graphs, Adjacency, and Pixel Pictures



$$
\begin{matrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
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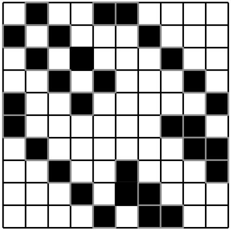
$$



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Dense Graphs

Graph models

- Basic object: infinite random binary matrix (X_{ij})
- Random graphs: $(G_n)_{n \in \mathbb{N}}$ defined by taking adjacency matrix to be upper left $n \times n$ submatrix

Joint exchangeability of infinite random matrices

$(X_{ij}) \stackrel{d}{=} (X_{\sigma(i)\sigma(j)})$ for all permutations $\sigma \in S_\infty$ of the positive integers

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Graphon Generative Model

Aldous-Hoover (translated)

Every infinite exchangeable array has a representation in terms of a *graphon* $W : [0,1]^2 \to [0,1]$

Given a graphon $W : [0,1]^2 \to [0,1]$, sample a random graph by:

1. Assign each vertex i an iid $U[0,1]$ latent random variable U_i
2. Include each edge (i,j) independently with probability $W(U_i, U_j)$

Representation Theorem

Recipe for constructing statistical models

- Assume a probabilistic symmetry on some infinite random structure
- Associated representation theorem picks out privileged family of distributions

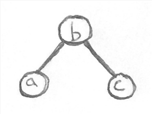
Examples

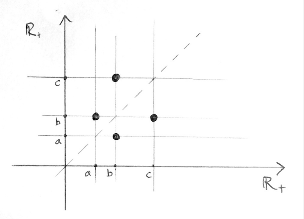
- de Finetti's representation theorem for exchangeable sequences
- Aldous-Hoover-Kallenberg theorem for exchangeable arrays

Caron & Fox 2014: (Sparse) Exchangeable Graphs

Key insights

- adjacency matrix $\rightarrow$ point process on $\mathbb{R}_+^2$
- array joint exchangeability $\rightarrow$ point process joint exchangeability





Graph edges correspond to points on $\mathbb{R}_+^2$

(Sparse) Graph Representation Theorem

Setup

- Random structure: point process on $\mathbb{R}_+^2$
- Finite graph Γ_s: truncate to $[0, s]^2$
- Symmetry: joint exchangeability of point process

Representation theorem*

Distribution characterized by a *graphex*: (I, S, W) where $I \in \mathbb{R}_+$, $S : \mathbb{R}_+ \to \mathbb{R}_+$, and the *graphon* $W : \mathbb{R}_+^2 \to [0, 1]$.
W is symmetric, and S and W satisfy certain weak integrability conditions (integrability suffices).

For this talk, restrict to $(0, 0, W)$ with W integrable.

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Graphex Model

Generative Model

Given W, an infinite random graph is sampled by:

1. Sample a (latent) unit rate Poisson process Π on $\theta \times \vartheta$.
2. For each pair of points (θ_i, ϑ_i), $(\theta_j, \vartheta_j) \in \Pi$ include edge (θ_i, θ_j) with probability $W(\vartheta_i, \vartheta_j)$.
3. Include θ_i as a vertex whenever θ_i participates in at least one edge.

Finite graph Γ_s is given by restricting to $[0, s]^2$

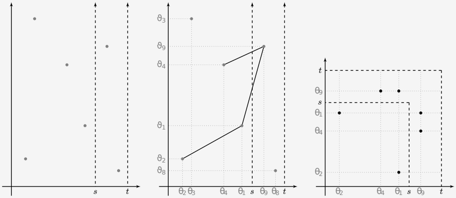

Figure: Graphex Model

Sampling Distribution Results

Given graphon W we know:

1. the expected number of vertices and edges as a function of the size s
2. the asymptotic degree distribution for certain families of graphexes
3. the asymptotic connectivity structure for certain families of graphexes.

Punchline

These models include a wide range of interesting graphs.

Example - Dense Exchangeable Graphs

Let $\widetilde{W} : [0,1]^2 \to [0,1]$ and let

$$W(x,y) = \begin{cases} \widetilde{W}(x,y) & x \leq 1, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Generative model for Γ_s

$$N_s \sim \mathrm{Poi}(s) \tag{1}$$

$$\{\theta_i\} \mid N_s \overset{iid}{\sim} \mathrm{Uni}[0,s] \tag{2}$$

$$\{\vartheta_i\} \mid N_s \overset{iid}{\sim} \mathrm{Uni}[0,1] \tag{3}$$

$$(\theta_i, \theta_j) \mid \widetilde{W}, \vartheta_i, \vartheta_j \overset{ind}{\sim} \mathrm{Bern}(\widetilde{W}(\vartheta_i, \vartheta_j)). \tag{4}$$

Example - Slow Decay

$$W(x,y) = \begin{cases} 0 & x = y \\ (x+1)^{-2}(y+1)^{-2} & \text{otherwise} \end{cases}$$

Properties

$$\mathbb{E}[e_s] = \frac{1}{2}s^2$$

$$\mathbb{E}[v_s] \sim \sqrt{\frac{\pi}{3}s^{3/2}}, s \to \infty$$

$$\mathrm{P}(D_s = k \mid \Gamma_s) \xrightarrow{\mathrm{p}} \frac{\Gamma\left(-\frac{1}{2}+k\right)}{2\sqrt{\pi}k!}, s \to \infty$$

$$\sim k^{-3/2}, k \to \infty$$

Example - Fast Decay

$$W(x,y) = \begin{cases} 0 & x = y \\ e^{-x}e^{-y} & \text{otherwise} \end{cases}$$

Properties

$$\mathbb{E}\left[e_s\right] = \frac{1}{2}s^2$$

$$\mathbb{E}\left[v_s\right] \sim s\log s \ , s \to \infty$$

$$\mathrm{P}\left(D_s \leq s^\beta \mid \Gamma_s\right) \xrightarrow{\mathrm{p}} \beta \ , s \to \infty$$

VR16: Sampling and Estimation

Sampling

Let $\mathscr{G}(\Gamma_s)$ be the unlabeled graph associated with Γ_s

Definition (p-sampling)

A p-sampling of a graph G is a random subgraph given by selecting each vertex of G independently with probability p, and then returning the induced edge set.

Theorem

Let $(\Gamma_s)_{s\in\mathbb{R}_+}$ be generated by W. If G_r is an r/s-sampling of $\mathscr{G}(\Gamma_s)$, then

$$G_r \overset{d}{=} \mathscr{G}(\Gamma_s)$$

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Estimation

Problem

How can we estimate a graphon?

Setup

- Observation: $G_s = \mathscr{G}(\Gamma_s)$ (and s)
- Estimator: $\widehat{W}_{(G_s, s)} : \mathbb{R}_+^2 \to [0, 1]$

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Dense Graph Estimation

Empirical Graphon

Let $\widetilde{W}_G : [0,1]^2 \to \{0,1\}$ be the step function corresponding to the (arbitrarily permuted) adjacency matrix of G.

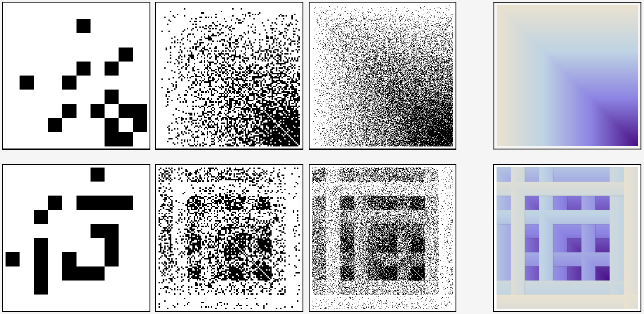

Figure: Convergence of Empirical Graphon (from Orbanz Roy 2015)

Formalization

Definition

Write $\mathrm{uKEG}(W, s) = \mathrm{P}(\mathscr{G}(\Gamma_s) \in \cdot)$ for the distribution of the unlabeled size-s graph generated by W.

Definition

Write $W_k \to_{\mathrm{GP}} W$ as $k \to \infty$, when $\mathrm{uKEG}(W_k, s) \to \mathrm{uKEG}(W, s)$ weakly as $k \to \infty$, for all $s \in \mathbb{R}_+$.

Theorem (Kallenberg 1999)

Let $(\Gamma_s)_{s \in \mathbb{R}_+}$ be generated by compactly supported W, let $s_1, s_2, \ldots \uparrow \infty$ and let $G_k = \mathscr{G}(\Gamma_{s_k})$ for all $k \in \mathbb{N}$. Then $\widetilde{W}_{G_k} \to_{\mathrm{GP}} W$ a.s., as $k \to \infty$.

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Estimation with Known Sizes

Definition (Dilated empirical graphon)

The s-dilated empirical graphon of G, $\widehat{W}_{(G,s)} : [0, \frac{v_s}{s}] \to \{0,1\}$, is defined by

$$\widehat{W}_{(G,s)}(x,y) = \widetilde{W}_G(\frac{v_s}{s}x, \frac{v_s}{s}y)$$

Observations

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Convergence of Dilated Empirical Graphon

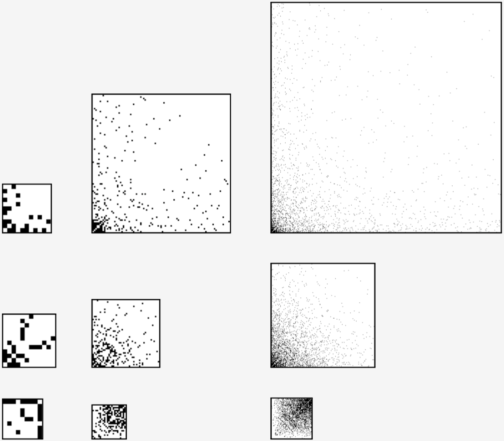

Figure: Convergence of the dilated empirical graphons. Generating graphon in rightmost column. Each square shows a realization of the s_k-dilated empirical graphon of G_{s_k}.

Estimation with Unknown Sizes

What if the observations are just $G_1, G_2, \ldots$? i.e., what if the sizes $(s_k)_{k \in \mathbb{N}}$ aren't included?

Fact

If $W^c(\cdot,\cdot) = W(\cdot/c, \cdot/c)$ then $\mathrm{uKEG}(W, s) = \mathrm{uKEG}(W^c, s/c)$.

Corollary

The dilation of W is not identifiable.

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We need a notion of estimation that is 'dilation invariant':

Definition

The graph sequence $\mathbb{G}(\Gamma)$ of Γ is the ordered sequence of all distinct unlabeled graph structures of $(\mathscr{G}(\Gamma_s))_{s\in\mathbb{R}_+}$

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Write $W_k \to_{\mathrm{GS}} W$ as $k \to \infty$ when $\mathbb{G}(\Gamma^k) \xrightarrow{d} \mathbb{G}(\Gamma)$ as $k \to \infty$, for Γ^k generated by W_k and Γ generated by W

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Estimation with Unknown Sizes

Intuitively, our estimator is the dilated empirical graphon up to dilation:

Theorem (Estimation with unknown sizes)

$\widetilde{W}_{G_k} \to_{\mathrm{GS}} W$ *in probability as* $k \to \infty$

BCCV16+: Sampling Perspectives on Exchangeability and Graph Limits

Background

In the dense graph setting, graphons were independently discovered two different ways:

- The representation theorem work of Aldous–Hover–Kallenberg, and
- The graph limits of Lovasz, Sós, Szegedy, Borgs, Chayes, Vesztergombi, Schrijver, Freedman

These perspectives were unified by Diaconis and Janson (2008)

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Questions

1. Is there an analogous unification in this setting?
2. There are several sparse graph limit theories — why this one?
3. Why do we represent random graphs as point processes?
4. What does exchangeability mean in our setting?

Contribution

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Sampling Convergence

Definition

A sequence of graphs $G_1, G_2, \ldots$ is sampling convergent if the $r/\sqrt{2e_j}$-samplings of G_j converge in distribution as $j \to \infty$, for all $r \in \mathbb{R}_+$

Theorem

If $G_1, G_2, \ldots$ is a sampling convergent sequence then there is some graphex $\mathscr{W}$ such that the limiting distribution of the $r/\sqrt{2e_j}$ is $\mathrm{uKEG}(\mathscr{W}, r)$

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Exchangeability and Graph Limits

- Map graphs $G_1, G_2, \ldots$ to point processes $\mathrm{Lbl}(G_1), \mathrm{Lbl}(G_2), \ldots$ by independently labeling each vertex of G_j by a $\mathrm{Uni}[0, \sqrt{2e_j}]$ random variable
    - These are obviously (finitely) exchangeable
- Notice that each vertex has a label in $[0, r]$ independently with probability $r/\sqrt{2e_j}$
    - Sampling convergence implies that $\mathrm{Lbl}(G_j)(\cdot \cap [0, r]^2)$ converges in distribution for all r
    - This is enough for convergence in distribution of $\mathrm{Lbl}(G_j)$
- Limiting point process is exchangeable, with distribution characterized by same graphex $\mathscr{W}$ that is the sampling convergent limit

Graphexes from p-sampling

- We can get graphex models from graph limits, without any direct appeal to point processes or the associated exchangeability
- Can we get these models directly from p-sampling, without an appeal to graph limits?

Graphexes from p-sampling

Definition

Call $(G_s)_{s\in\mathbb{R}_+}$ an unlabeled random graph process indexed by $\mathbb{R}_+$ if, for all s, G_s is a finite unlabeled graph, and, for all $s \leq t$, it holds that $G_s \subseteq G_t$ in the sense that there is some subgraph of G_t that is isomorphic to G_s.

Theorem

Let $(G_s)_{s\in\mathbb{R}_+}$ be an unlabeled random graph process such that $e_s \uparrow \infty$ a.s. as $s \to \infty$. For each $s \in \mathbb{R}_+$ and $p \in (0,1)$, let $\mathrm{Smpl}_p(G_s)$ be a p-sampling of G_s. If for all $s \in \mathbb{R}_+$ and $p \in (0,1)$ it holds that

$$\mathrm{Smpl}_p(G_s) \overset{d}{=} G_{ps},$$

then there is some (possibly random, possibly non-integrable) almost surely non-zero graphex $\mathscr{W}$ such that, for all $s \in \mathbb{R}_+$,
$G_s \mid \mathscr{W} \sim \mathrm{uKEG}(\mathscr{W}, s)$.

Summary

VR15

- Representation theorem for sparse random graphs
- Extends dense (graphon) theory to sparse graphs
- Formulas for sampling distribution properties in terms of graphex

VR16

- *p*-sampling
- Estimation via the (dilated) empirical graphon

BCCV16+

- Sampling convergence (graph limits)
- Graphex models from *p*-sampling
- Point process construction can be purged from definitions