Proofs of Proposition 2.2 and 2.5

Proposition 2.2 *Let B_1, B_2 be ℓ_∞-balls with radii r centered at $x, y \in \mathbb{R}^d$ respectively; then*

$$\frac{Vol(B_1 \cap B_2)}{Vol(B_1)} \geq 1 - \frac{||x-y||_1}{2r}. \tag{1}$$

Proof: We first note that $Vol(B_1) = (2r)^d$ and

$$Vol(B_1 \cap B_2) = \prod_{i=1}^{d} \left(2r - |x_i - y_i|\right) \quad \text{and thus:} \quad \frac{Vol(B_1 \cap B_2)}{Vol(B_1)} = \prod_{i=1}^{d} \left(1 - \frac{|x_i - y_i|}{2r}\right)$$

Our goal is to solve the optimization problem:

$$\min_{z \in \mathbb{R}^d,\, \|z-x\|_1 \leq c} \prod_{i=1}^{d} \left(1 - \frac{|z_i - x_i|}{2r}\right) \tag{2}$$

The minimum is attained as the objective is continuous and the constraint set is closed and bounded and it has to hold $||z-x||_1 = c$ for the optimum; thus, we replace the constraint $\|z-x\|_1 \leq c$ by $\|z-x\|_1 = c$. It holds that the value of the Problem 2 lower bounds the value of the left-hand-side of Equation 1 when $\|x-y\|_1 = c$. After the replacement, the constraint set stays closed and bounded.

We note that if x and z agree up to a coordinate j and $|z_j - x_j| = \|z-x\|_1 = c \leq 2r$ (otherwise the intersection is empty) then

$$\frac{Vol(B_1 \cap B_2)}{Vol(B_1)} = 1 - \frac{\|z-x\|_1}{2r} = 1 - \frac{c}{2r}.$$

We claim that such a z achieves the minimal value of optimization Problem 2. In the following we assume without loss of generality that $z_i - x_i \geq 0$ for all $i = 1, \ldots, d$ (the problem is translation invariant, so we can identify x with the origin and the volume of the intersection does not depend on the orthant in which z lies).

For the sake of contradiction assume there exists a minimizer z' that differs from x for at least two coordinates i, j (again we can assume without loss of generality that $z'_i - x_i > 0$. Let $u_r(t) = z'_r$ for $r \neq i, j$ and set $u_i(t) = z'_i + t$ and $u_j(t) = z'_j - t$. There exists a sufficiently small ϵ and $t \in [-\epsilon, \epsilon]$ so it holds $0 \leq u_j(t) - x_j \leq 2r$ and $0 \leq u_i(t) - x_i \leq 2r$. We note that for $t \in [-\epsilon, \epsilon]$, it holds $\|u(t) - x\|_1 = \sum_{r=1}^{d} |u_r(t) - x_r| = \sum_{i=1}^{d} u_r(t) - x_r = \|z' - x\|_1 = c$. We denote by $V(t)$ the ratio $\frac{Vol(B_1 \cap B_2)}{Vol(B_1)}$ for the given pair x and $u(t)$. Then for $t \in [-\epsilon, \epsilon]$, it holds

$$V(t) = \prod_{i=1}^{d} \left(1 - \frac{|u_i(t) - x_i|}{2r}\right) = \left(\prod_{r \neq i,j} \left(1 - \frac{z'_r - x_r}{2r}\right)\right) \left(1 - \frac{z'_i + t - x_i}{2r}\right) \left(1 - \frac{z'_j - t - x_j}{2r}\right).$$

We note that $V(t)$ is strictly concave in t due to the negative quadratic term. Thus, the minimum of V is attained at the interval borders $t = -\epsilon$ or $t = \epsilon$ and $t = 0$ which corresponds to $u(0) = z'$ cannot be a minimizer, which contradicts the assumption that z' is a minimizer of the full problem. We conclude that a minimizer of the optimization problem can differ in at most one coordinate from x. Thus the minimal value needs to be attained for x, z differing in one coordinate and $\|x - z\|_1 = c$ and thus as derived before it holds

$$\min_{z \in \mathbb{R}^d,\, ||x-z||_1 \leq c} \prod_{i=1}^{d} \left(1 - \frac{|x_i - z_i|}{2r}\right) = 1 - \frac{c}{2r}.$$

□

Proposition 2.5 *Let B_1, B_2 be ℓ_∞-balls with radii r centered at $x, y \in [0,1]^d$ respectively; then*

$$\frac{Vol(B_1 \cap B_2)}{Vol(B_1)} \geq \left(1 - \frac{1}{2r}\right)^{\lfloor \|x-y\|_1 \rfloor} \left(1 - \frac{\|x-y\|_1 - \lfloor \|x-y\|_1 \rfloor}{2r}\right), \tag{3}$$

where $\lfloor u \rfloor = \max\{m \in \mathbb{Z} \,|\, m \leq u\}$.

Proof: The proof strategy is very similar to the previous proposition except that now we optimize both variables as the problem is no longer translation invariant as before. We have to solve the optimization problem:

$$\min_{w,z \in [0,1]^d,\, \|z-w\|_1 \leq c} \prod_{i=1}^{d} \left(1 - \frac{|z_i - w_i|}{2r}\right) \tag{4}$$

First we relax the box constraint to

$$\min_{||z-w||_\infty \leq 1,\, ||z-w||_1 \leq c} \prod_{i=1}^{d} \left(1 - \frac{|z_i - w_i|}{2r}\right)$$

We note that this problem is translation invariant and thus without loss of generality we can introduce $u = w - z$ and solve:

$$\min_{||u||_\infty \leq 1,\, ||u||_1 \leq c} \prod_{i=1}^{d} \left(1 - \frac{|u_i|}{2r}\right) \tag{5}$$

The value of Problem 5 is thus a lower bound for the original optimization Problem 4 which is a lower bound for the left-hand-side of Equation 3. Due to the compact constraint set, the minimizer is attained and again it holds for the optimal u^* that $\|u^*\|_1 = c$ (unless $c > d$, but such case cannot be attained for $\|x-y\|_1$). Thus, we replace the constraint $\|u\|_1 \leq c$ by $\|u\|_1 = c$ while keeping the constraint set to be closed and bounded. We claim that $\hat{u}$ is optimal when $|\hat{u}_i| = 1$ for $\lfloor c \rfloor$ coordinates and $\hat{u}_i = 0$ for $d - \lfloor c \rfloor - 1$ coordinates and for the coordinate which is not fixed yet we require $|\hat{u}_i| = c - \lfloor c \rfloor$. One can check that $\|\hat{u}\|_1 = c$ and $\|\hat{u}\|_\infty = 1$ and it holds

$$\frac{Vol(B_1 \cap B_2)}{Vol(B_1)} \geq \left(1 - \frac{1}{2r}\right)^{\lfloor \|\hat{u}\|_1 \rfloor} \left(1 - \frac{\|\hat{u}\|_1 - \lfloor \|\hat{u}\|_1 \rfloor}{2r}\right) = \left(1 - \frac{1}{2r}\right)^{\lfloor c \rfloor} \left(1 - \frac{c - \lfloor c \rfloor}{2r}\right)$$

As noted above $u - w - z$. There exists even $\hat{w}, \hat{z} \in [0,1]^d$ which would realize such a $\hat{u}$, namely set $\hat{w}_i = 0$ and $\hat{z}_i = 1$ for $\lfloor c \rfloor$ coordinates and $\hat{w}_i = \hat{z}_i$ for $d - \lfloor c \rfloor - 1$ coordinates and finally fix the remaining coordinate such that $|\hat{w} - \hat{z}| = c - \lfloor c \rfloor$. Suppose now that there exists a u which is optimal but does not fulfill the conditions listed above. Then there need to exist at least two coordinates i, j where $0 < |\hat{u}_i| < 1$ and $0 < |\hat{u}_j| < 1$. Without loss of generality we assume that $\hat{u}_i \geq 0$ and introduce $v(t) \in \mathbb{R}^d$ such that $v_r(t) = \hat{u}_r$ for $r \neq i, j$ and $v_i(t) = \hat{u}_i + t$ and $v_j(t) = \hat{u}_j - t$. For sufficiently small $t \in [-\epsilon, \epsilon]$ it holds $0 \leq v_i(t) \leq 1$ for all $i = 1, \ldots, d$ and thus $\|v(t)\|_\infty \leq 1$ as well as $\|v(t)\|_1 = c$. The rest of the proof then can be done exactly as in the one of Proposition 2.2 and we get the final result:

$$\frac{Vol(B_1 \cap B_2)}{Vol(B_1)} \geq \left(1 - \frac{1}{2r}\right)^{\lfloor ||x-y||_1 \rfloor} \left(1 - \frac{||x-y||_1 - \lfloor ||x-y||_1 \rfloor}{2r}\right)$$

We note that due to convexity of $e^{\alpha x}$ it holds for $0 \leq \lambda \leq 1$ and any $\alpha \in \mathbb{R}$,

$$e^{\alpha\lambda} = e^{\alpha(\lambda 1 + (1-\lambda)0)} \leq (1-\lambda) + e^{\alpha}\lambda = 1 - \lambda(1 - e^{\alpha})$$

Now we set for $a \geq 1$, $\alpha := \log\left(1 - \frac{1}{a}\right)$ and get for $0 \leq x \leq 1$

$$\left(1 - \frac{1}{a}\right)^x \leq 1 - x\left(1 - (1 - \frac{1}{a})\right) = 1 - \frac{x}{a}.$$

Using this inequality we get the final result:

$$\left(1 - \frac{1}{2r}\right)^{\lfloor ||x-y||_1 \rfloor} \left(1 - \frac{||x-y||_1 - \lfloor ||x-y||_1 \rfloor}{2r}\right) \geq \left(1 - \frac{1}{2r}\right)^{||x-y||_1},$$

when $r \geq \frac{1}{2}$. $\square$