CSER 2207: Numerical Analysis

Lecture-15
Numerical Integration

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Numerical Integration

The general problem of numerical integration may be stated as follows. Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b y\,dx. \tag{5.28}$$

As in the case of numerical differentiation, one replaces $f(x)$ by an interpolating polynomial $\phi(x)$ and obtains, on integration, an approximate value of the definite integral. Thus, different integration formulae can be obtained depending upon the type of the interpolation formula used. We derive in this section a general formula for numerical integration using Newton's forward difference formula.

Let the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence the integral becomes

$$I = \int_{x_0}^{x_n} y\,dx.$$

Approximating y by Newton's forward difference formula, we obtain

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \cdots \right] dx.$$

Since $x = x_0 + ph$, $dx = h\,dp$ and hence the above integral becomes

$$I = h \int_{0}^{n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \cdots \right] dp,$$

which gives on simplification

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right]. \quad (5.29)$$

From this *general formula*, we can obtain different integration formulae by putting $n = 1, 2, 3, \ldots$, etc. We derive here a few of these formulae but it should be remarked that the trapezoidal and Simpson's 1/3 rules are found to give sufficient accuracy for use in practical problems.

Trapezoidal Rule

Setting $n=1$ in the general formula (5.29), all differences higher than the first will become zero and we obtain

$$\int_{x_0}^{x_1} y\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1). \tag{5.30}$$

For the next interval $[x_1, x_2]$, we deduce similarly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}(y_1 + y_2) \tag{5.31}$$

and so on. For the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}(y_{n-1} + y_n). \tag{5.32}$$

Combining all these expressions, we obtain the rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n], \tag{5.33}$$

which is known as the *trapezoidal rule*.

5.4.2 Simpson's 1/3-Rule

This rule is obtained by putting $n = 2$ in Eq. (5.29), i.e. by replacing the curve by $n/2$ arcs of second-degree polynomials or parabolas. We have then

$$\int_{x_0}^{x_2} y\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly,

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\vdots$$

and finally

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n).$$

Summing up, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + y_n], \tag{5.39}$$

which is known as *Simpson's 1/3-rule*, or simply Simpson's rule. It should be noted that this rule requires the division of the whole range into an even number of subintervals of width h.

5.4.3 Simpson's 3/8-Rule

Setting $n = 3$ in (5.29) we observe that all the differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} y\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3).$$

Similarly

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

and so on. Summing up all these, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots$$
$$+ (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$
$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots$$
$$+ 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n). \tag{5.41}$$

This rule, called Simpson's (3/8)-rule, is not so accurate as Simpson's rule,

5.4.4 Boole's and Weddle's Rules

If we wish to retain differences up to those of the fourth order, we should integrate between x_0 and x_4 and obtain Boole's formula

$$\int_{x_0}^{x_4} y\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4). \qquad (5.42)$$

If, on the other hand, we integrate between x_0 and x_6 retaining differences up to those of the sixth order, we obtain Weddle's rule

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6), \qquad (5.43)$$

Integration Formulas

- **Trapezoidal Rule:** $$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n],$$
- **Simpson's 1/3-Rule:** $$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + y_n],$$
- **Simpson's 3/8-Rule:** $$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n).$$
- **Weddle's Rule:** $$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6),$$

Example

Example 5.7 Find, from the following table, the area bounded by the curve and the x-axis from $x = 7.47$ to $x = 7.52$

x	$f(x)$	x	$f(x)$
7.47	1.93	7.50	2.01
7.48	1.95	7.51	2.03
7.49	1.98	7.52	2.06

We know that

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,dx$$

with $h = 0.01$, the trapezoidal rule (5.32) gives

$$\text{Area} = \frac{0.01}{2}[1.93 + 2(1.95 + 1.98 + 2.01 + 2.03) + 2.06] = 0.0996.$$

Example

Example 5.8 A solid of revolution is formed by rotating about the x-axis the area between the x-axis, the lines $x=0$ and $x=1$, and a curve through the points with the following coordinates:

x	y
0.00	1.0000
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415

Estimate the volume of the solid formed, giving the answer to three decimal places.

If V is the volume of the solid formed, then we know that

$$V=\pi\int_0^1 y^2 dx$$

Cont…

Hence we need the values of y^2 and these are tabulated below, correct to four decimal places

x	y^2
0.00	1.0000
0.25	0.9793
0.50	0.9195
0.75	0.8261
1.00	0.7081

With $h = 0.25$, Simpson's rule gives

$$V = \frac{\pi(0.25)}{3}[1.0000 + 4(0.9793 + 0.8261) + 2(0.9195) + 0.7081]$$

$$= 2.8192.$$

Example

Example 5.11 Apply trapezoidal and Simpson's rules to the integral

$$I = \int_0^1 \sqrt{1-x^2}\,dx$$

continually halving the interval h for better accuracy.

Using 10, 20, 30, 40 and 50 subintervals successively, an electronic computer, with a nine decimal precision, produced the results given in Table below. The true value of the integral is $\pi/4 = 0.785\,398\,163$.

No. of subintervals	*Trapezoidal rule*	*Simpson's's rule*
10	0.776 129 582	0.781 752 040
20	0.782 116 220	0.784 111 766
30	0.783 610 789	0.784 698 434
40	0.784 236 934	0.784 943 838
50	0.784 567 128	0.785 073 144

Thank You