Chapter 1

The Laplace Transform

DEFINITION OF THE LAPLACE TRANSFORM

Let $F(t)$ be a function of t specified for $t > 0$. Then the *Laplace transform* of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

The Laplace transform of $F(t)$ is said to *exist* if the integral (1) *converges* for some value of s; otherwise it does not exist. For sufficient conditions under which the Laplace transform does exist, see Page 2.

NOTATION

If a function of t is indicated in terms of a capital letter, such as $F(t)$, $G(t)$, $Y(t)$, etc., the Laplace transform of the function is denoted by the corresponding lower case letter, i.e. $f(s)$, $g(s)$, $y(s)$, etc. In other cases, a tilde ($\sim$) can be used to denote the Laplace transform. Thus, for example, the Laplace transform of $u(t)$ is $\tilde{u}(s)$.

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

The adjacent table shows Laplace transforms of various elementary functions. For details of evaluation using definition (1), see Problems 1 and 2. For a more extensive table see Appendix B, Pages 245 to 254.

	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1.	1	$\dfrac{1}{s}$ $\quad s > 0$
2.	t	$\dfrac{1}{s^2}$ $\quad s > 0$
3.	t^n $n = 0, 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$ $\quad s > 0$ *Note.* Factorial $n = n! = 1 \cdot 2 \cdots n$ Also, by definition $0! = 1$.
4.	e^{at}	$\dfrac{1}{s-a}$ $\quad s > a$
5.	$\sin at$	$\dfrac{a}{s^2 + a^2}$ $\quad s > 0$
6.	$\cos at$	$\dfrac{s}{s^2 + a^2}$ $\quad s > 0$
7.	$\sinh at$	$\dfrac{a}{s^2 - a^2}$ $\quad s > \lvert a \rvert$
8.	$\cosh at$	$\dfrac{s}{s^2 - a^2}$ $\quad s > \lvert a \rvert$

SECTIONAL OR PIECEWISE CONTINUITY

A function is called *sectionally continuous* or *piecewise continuous* in an interval $\alpha \leqq t \leqq \beta$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

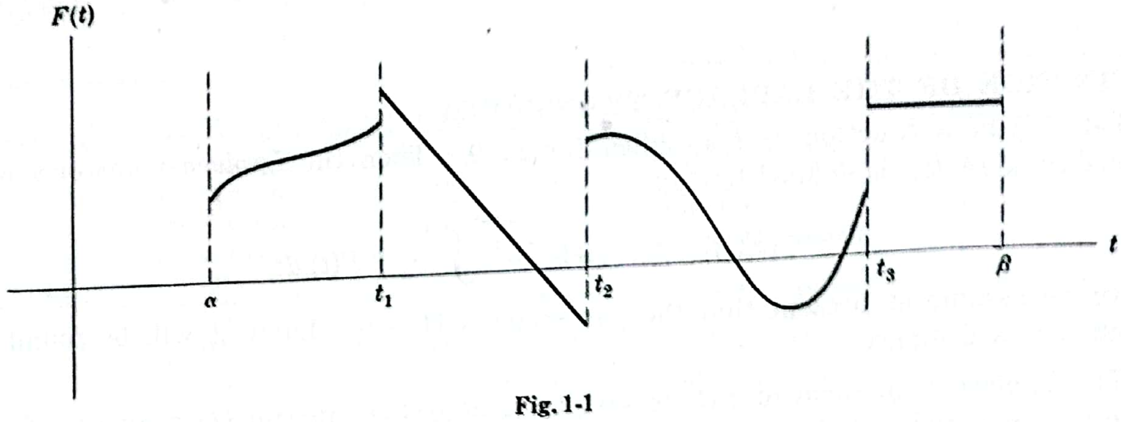

Fig. 1-1

An example of a function which is sectionally continuous is shown graphically in Fig. 1-1 above. This function has discontinuities at t_1, t_2 and t_3. Note that the right and left hand limits at t_2, for example, are represented by $\lim_{\epsilon \to 0} F(t_2 + \epsilon) = F(t_2 + 0) = F(t_2+)$ and $\lim_{\epsilon \to 0} F(t_2 - \epsilon) = F(t_2 - 0) = F(t_2-)$ respectively, where ϵ is positive.

FUNCTIONS OF EXPONENTIAL ORDER

If real constants $M > 0$ and γ exist such that for all $t > N$

$$|e^{-\gamma t} F(t)| < M \qquad \text{or} \qquad |F(t)| < Me^{\gamma t}$$

we say that $F(t)$ is a *function of exponential order* γ *as* $t \to \infty$ or, briefly, is of *exponential order*.

Example 1. $F(t) = t^2$ is of exponential order 3 (for example), since $|t^2| = t^2 < e^{3t}$ for all $t > 0$.

Example 2. $F(t) = e^{t^3}$ is not of exponential order since $|e^{-\gamma t} e^{t^3}| = e^{t^3 - \gamma t}$ can be made larger than any given constant by increasing t.

Intuitively, functions of exponential order cannot "grow" in absolute value more rapidly than $Me^{\gamma t}$ as t increases. In practice, however, this is no restriction since M and γ can be as large as desired.

Bounded functions, such as $\sin at$ or $\cos at$, are of exponential order.

SUFFICIENT CONDITIONS FOR EXISTENCE OF LAPLACE TRANSFORMS

Theorem 1-1. If $F(t)$ is sectionally continuous in every finite interval $0 \leqq t \leqq N$ and of exponential order γ for $t > N$, then its Laplace transform $f(s)$ exists for all $s > \gamma$.

For a proof of this see Problem 47. It must be emphasized that the stated conditions are *sufficient* to guarantee the existence of the Laplace transform. If the conditions are not satisfied, however, the Laplace transform may or may not exist [see Problem 32]. Thus the conditions are not *necessary* for the existence of the Laplace transform.

For other sufficient conditions, see Problem 145.

SOME IMPORTANT PROPERTIES OF LAPLACE TRANSFORMS

In the following list of theorems we assume, unless otherwise stated, that all functions satisfy the conditions of *Theorem 1-1* so that their Laplace transforms exist.

1. Linearity property.

Theorem 1-2. If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} = c_1f_1(s) + c_2f_2(s) \tag{2}$$

The result is easily extended to more than two functions.

Example. $\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\}$

$$= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right)$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

The symbol $\mathcal{L}$, which transforms $F(t)$ into $f(s)$, is often called the *Laplace transformation operator.* Because of the property of $\mathcal{L}$ expressed in this theorem, we say that $\mathcal{L}$ is a *linear operator* or that it has the *linearity property.*

2. First translation or shifting property.

Theorem 1-3. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a) \tag{3}$$

Example. Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Second translation or shifting property.

Theorem 1-4. If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then

$$\mathcal{L}\{G(t)\} = e^{-as}f(s) \tag{4}$$

Example. Since $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$, the Laplace transform of the function

$$G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$$

is $6e^{-2s}/s^4$.

4. Change of scale property.

Theorem 1-5. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right) \tag{5}$$

Example. Since $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, we have

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3}\,\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$$

Theorem 1-6. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F'(t)\} = s\,f(s) - F(0) \tag{6}$$

if $F(t)$ is continuous for $0 \leqq t \leqq N$ and of exponential order for $t > N$ while $F'(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

Example. If $F(t) = \cos 3t$, then $\mathcal{L}\{F(t)\} = \dfrac{s}{s^2+9}$ and we have

$$\mathcal{L}\{F'(t)\} = \mathcal{L}\{-3\sin 3t\} = s\left(\frac{s}{s^2+9}\right) - 1 = \frac{-9}{s^2+9}$$

The method is useful in finding Laplace transforms without integration [see Problem 15].

Theorem 1-7. If in *Theorem 1-6*, $F(t)$ fails to be continuous at $t = 0$ but $\lim_{t\to 0} F(t) = F(0+)$ exists [but is not equal to $F(0)$, which may or may not exist], then

$$\mathcal{L}\{F'(t)\} = s\,f(s) - F(0+) \tag{7}$$

Theorem 1-8. If in *Theorem 1-6*, $F(t)$ fails to be continuous at $t = a$, then

$$\mathcal{L}\{F'(t)\} = s\,f(s) - F(0) - e^{-as}\{F(a+) - F(a-)\} \tag{8}$$

where $F(a+) - F(a-)$ is sometimes called the *jump* at the discontinuity $t = a$. For more than one discontinuity, appropriate modifications can be made.

Theorem 1-9. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - s\,F(0) - F'(0) \tag{9}$$

if $F(t)$ and $F'(t)$ are continuous for $0 \leqq t \leqq N$ and of exponential order for $t > N$ while $F''(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

If $F(t)$ and $F'(t)$ have discontinuities, appropriate modification of (9) can be made as in *Theorems 1-7* and *1-8*.

Theorem 1-10. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - s\,F^{(n-2)}(0) - F^{(n-1)}(0) \tag{10}$$

if $F(t), F'(t), \ldots, F^{(n-1)}(t)$ are continuous for $0 \leqq t \leqq N$ and of exponential order for $t > N$ while $F^{(n)}(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

6. Laplace transform of integrals.

Theorem 1-11. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s} \tag{11}$$

Example. Since $\mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+4}$, we have

$$\mathcal{L}\left\{\int_0^t \sin 2u\,du\right\} = \frac{2}{s(s^2+4)}$$

as can be verified directly.

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if $F(t), F'(t), \ldots, F^{(n-1)}(t)$ are continuous for $0 \leqq t \leqq N$ and of exponential order for $t > N$ while $F^{(n)}(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

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$$\mathcal{L}\left\{\int_0^t \sin 2u\,du\right\} = \frac{2}{s(s^2+4)}$$

as can be verified directly.

7. Multiplication by t^n.

Theorem 1-12. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s) \tag{12}$$

Example. Since $\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$, we have

$$\mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

$$\mathcal{L}\{t^2e^{2t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$$

8. Division by t.

Theorem 1-13. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du \tag{13}$$

provided $\lim_{t \to 0} F(t)/t$ exists.

Example. Since $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$ and $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$, we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{du}{u^2+1} = \tan^{-1}(1/s)$$

9. Periodic functions.

Theorem 1-14. Let $F(t)$ have period $T > 0$ so that $F(t+T) = F(t)$ [see Fig. 1-2].

Then
$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-sT}} \tag{14}$$

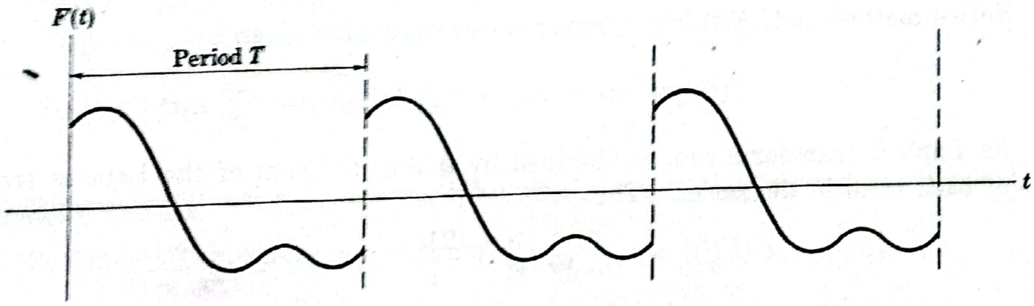

Fig. 1-2

10. Behavior of $f(s)$ as $s \to \infty$.

Theorem 1-15. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\lim_{s\to\infty} f(s) = 0 \tag{15}$$

11. Initial-value theorem.

Theorem 1-16. If the indicated limits exist, then

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} s\,f(s) \tag{16}$$

12. **Final-value theorem.**

Theorem 1-17. If the indicated limits exist, then

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s\,f(s) \tag{17}$$

13. **Generalization of initial-value theorem.**

If $\lim_{t \to 0} F(t)/G(t) = 1$, then we say that for values of t near $t = 0$ [small t], $F(t)$ is *close to* $G(t)$ and we write $F(t) \sim G(t)$ as $t \to 0$.

Similarly if $\lim_{s \to \infty} f(s)/g(s) = 1$, then we say that for large values of s, $f(s)$ is *close to* $g(s)$ and we write $f(s) \sim g(s)$ as $s \to \infty$.

With this notation we have the following generalization of *Theorem 1-16*.

Theorem 1-18. If $F(t) \sim G(t)$ as $t \to 0$, then $f(s) \sim g(s)$ as $s \to \infty$ where $f(s) = \mathcal{L}\{F(t)\}$ and $g(s) = \mathcal{L}\{G(t)\}$.

14. **Generalization of final-value theorem.**

If $\lim_{t \to \infty} F(t)/G(t) = 1$, we write $F(t) \sim G(t)$ as $t \to \infty$. Similarly if $\lim_{s \to 0} f(s)/g(s) = 1$, we write $f(s) \sim g(s)$ as $s \to 0$. Then we have the following generalization of *Theorem 1-17*.

Theorem 1-19. If $F(t) \sim G(t)$ as $t \to \infty$, then $f(s) \sim g(s)$ as $s \to 0$ where $f(s) = \mathcal{L}\{F(t)\}$ and $g(s) = \mathcal{L}\{G(t)\}$.

METHODS OF FINDING LAPLACE TRANSFORMS

Various means are available for determining Laplace transforms as indicated in the following list.

1. **Direct method.** This involves direct use of definition (*1*).

2. **Series method.** If $F(t)$ has a power series expansion given by

$$F(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{n=0}^{\infty} a_n t^n \tag{18}$$

its Laplace transform can be obtained by taking the sum of the Laplace transforms of each term in the series. Thus

$$\mathcal{L}\{F(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2!\,a_2}{s^3} + \cdots = \sum_{n=0}^{\infty} \frac{n!\,a_n}{s^{n+1}} \tag{19}$$

A condition under which the result is valid is that the series (*19*) be convergent for $s > \gamma$. See Problems 34, 36, 39 and 48.

3. **Method of differential equations.** This involves finding a differential equation satisfied by $F(t)$ and then using the above theorems. See Problems 34 and 48.

4. **Differentiation with respect to a parameter.** See Problem 20.

5. **Miscellaneous methods** involving special devices such as indicated in the above theorems, for example *Theorem 1-13*.

6. **Use of Tables** (see Appendix).

EVALUATION OF INTEGRALS

If $f(s) = \mathcal{L}\{F(t)\}$, then

$$\int_0^\infty e^{-st} F(t)\, dt = f(s) \tag{20}$$

Taking the limit as $s \to 0$, we have

$$\int_0^\infty F(t)\, dt = f(0) \tag{21}$$

assuming the integral to be convergent.

The results (20) and (21) are often useful in evaluating various integrals. See Problems 45 and 46.

SOME SPECIAL FUNCTIONS

I. The Gamma function.

If $n > 0$, we define the *gamma function* by

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u}\, du \tag{22}$$

The following are some important properties of the gamma function.

1. $$\Gamma(n+1) = n\,\Gamma(n), \quad n > 0$$

   Thus since $\Gamma(1) = 1$, we have $\Gamma(2) = 1$, $\Gamma(3) = 2! = 2$, $\Gamma(4) = 3!$ and in general $\Gamma(n+1) = n!$, if n is a positive integer. For this reason the function is sometimes called the *factorial function*.

2. $$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

3. $$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1$$

4. For large n,
   $$\Gamma(n+1) \sim \sqrt{2\pi n}\, n^n e^{-n}$$

   [Here $\sim$ means "approximately equal to for large n". More exactly, we write $F(n) \sim G(n)$ if $\lim_{n\to\infty} F(n)/G(n) = 1$.] This is called *Stirling's formula*.

5. For $n < 0$ we can define $\Gamma(n)$ by
   $$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

II. Bessel functions.

We define a *Bessel function of order n* by

$$J_n(t) = \frac{t^n}{2^n\,\Gamma(n+1)}\left\{1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\} \tag{23}$$

Some important properties are

1. $J_{-n}(t) = (-1)^n J_n(t)$ if n is a positive integer

2. $J_{n+1}(t) = \dfrac{2n}{t} J_n(t) - J_{n-1}(t)$

3. $\dfrac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t)$. If $n = 0$, we have $J_0'(t) = -J_1(t)$.

4. $e^{\frac{1}{2}t(u-1/u)} = \displaystyle\sum_{n=-\infty}^{\infty} J_n(t) u^n$

This is called the *generating function* for the Bessel functions.

5. $J_n(t)$ satisfies *Bessel's differential equation*.

$$t^2 Y''(t) + t Y'(t) + (t^2 - n^2) Y(t) = 0$$

It is convenient to define $J_n(it) = i^{-n} I_n(t)$ where $I_n(t)$ is called the *modified Bessel function of order n*.

III. The Error function is defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du \tag{24}$$

IV. The Complementary Error function is defined as

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du \tag{25}$$

V. The Sine and Cosine integrals are defined by

$$\operatorname{Si}(t) = \int_0^t \frac{\sin u}{u}\, du \tag{26}$$

$$\operatorname{Ci}(t) = \int_t^\infty \frac{\cos u}{u}\, du \tag{27}$$

VI. The Exponential integral is defined as

$$\operatorname{Ei}(t) = \int_t^\infty \frac{e^{-u}}{u}\, du \tag{28}$$

VII. The Unit Step function, also called *Heaviside's unit function*, is defined as

$$\mathcal{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases} \tag{29}$$

See Fig. 1-3.

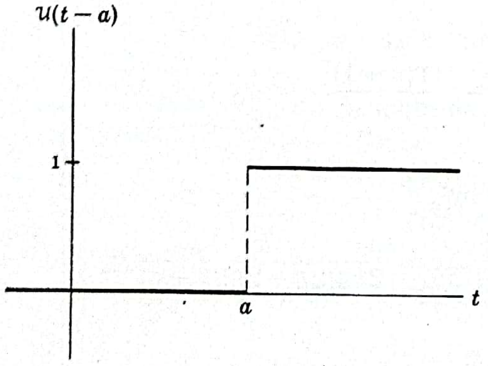

Fig. 1-3

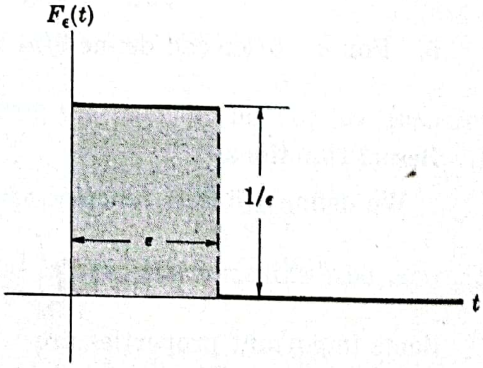

Fig. 1-4

VIII. The Unit Impulse function or Dirac delta function.

Consider the function

$$F_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \leqq t \leqq \epsilon \\ 0 & t > \epsilon \end{cases} \tag{30}$$

where $\epsilon > 0$, whose graph appears in Fig. 1-4.

It is geometrically evident that as $\epsilon \to 0$ the height of the rectangular shaded region increases indefinitely and the width decreases in such a way that the area is always equal to 1, i.e. $\int_0^\infty F_\epsilon(t)\,dt = 1$.

This idea has led some engineers and physicists to think of a limiting function, denoted by $\delta(t)$, approached by $F_\epsilon(t)$ as $\epsilon \to 0$. This limiting function they have called the *unit impulse function* or *Dirac delta function*. Some of its properties are

1. $\int_0^\infty \delta(t)\,dt = 1$

2. $\int_0^\infty \delta(t)\,G(t)\,dt = G(0)$ for any continuous function $G(t)$.

3. $\int_0^\infty \delta(t-a)\,G(t)\,dt = G(a)$ for any continuous function $G(t)$.

Although mathematically speaking such a function does not exist, manipulations or operations using it can be made rigorous.

IX. Null functions. If $\mathcal{N}(t)$ is a function of t such that for all $t > 0$

$$\int_0^t \mathcal{N}(u)\,du = 0 \qquad (31)$$

we call $\mathcal{N}(t)$ a *null function*.

Example. The function $F(t) = \begin{cases} 1 & t = 1/2 \\ -1 & t = 1 \\ 0 & \text{otherwise} \end{cases}$ is a null function.

In general, any function which is zero at all but a countable set of points [i.e. a set of points which can be put into one-to-one correspondence with the natural numbers $1, 2, 3, \ldots$] is a null function.

LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

In the following table we have listed Laplace transforms of various special functions. For a more extensive list see Appendix B, Page 245.

Table of Laplace transforms of special functions

	$F(t)$	$f(s) = \mathcal{L}\{F(t)\}$
1.	t^n	$\dfrac{\Gamma(n+1)}{s^{n+1}}$ Note that if $n = 0, 1, 2, \ldots$ this reduces to entry 3, Page 1.
2.	$J_0(at)$	$\dfrac{1}{\sqrt{s^2+a^2}}$
3.	$J_n(at)$	$\dfrac{(\sqrt{s^2+a^2}-s)^n}{a^n\sqrt{s^2+a^2}}$
4.	$\sin\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4s}$
5.	$\dfrac{\cos\sqrt{t}}{\sqrt{t}}$	$\sqrt{\dfrac{\pi}{s}}\,e^{-1/4s}$

Table of Laplace transforms of special functions (cont.)

	$F(t)$	$f(s) = \mathcal{L}\{F(t)\}$
6.	erf (t)	$\dfrac{e^{s^2/4}}{s}$ erfc $(s/2)$
7.	erf $(\sqrt{t})$	$\dfrac{1}{s\sqrt{s+1}}$
8.	Si (t)	$\dfrac{1}{s}\tan^{-1}\dfrac{1}{s}$
9.	Ci (t)	$\dfrac{\ln(s^2+1)}{2s}$
10.	Ei (t)	$\dfrac{\ln(s+1)}{s}$
11.	$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
12.	$\delta(t)$	1
13.	$\delta(t-a)$	e^{-as}
14.	$\mathcal{N}(t)$	0

Solved Problems

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

1. Prove that: (a) $\mathcal{L}\{1\} = \dfrac{1}{s}$, $s>0$; (b) $\mathcal{L}\{t\} = \dfrac{1}{s^2}$, $s>0$; (c) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s>a$.

(a)
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\,dt$$
$$= \lim_{P\to\infty} \frac{e^{-st}}{-s}\bigg|_0^P = \lim_{P\to\infty}\frac{1-e^{-sP}}{s} = \frac{1}{s} \quad \text{if } s>0$$

(b)
$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t)\,dt = \lim_{P\to\infty}\int_0^P t\,e^{-st}\,dt$$
$$= \lim_{P\to\infty} (t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\bigg|_0^P = \lim_{P\to\infty}\left(\frac{1}{s^2} - \frac{e^{-sP}}{s^2} - \frac{Pe^{-sP}}{s}\right)$$
$$= \frac{1}{s^2} \quad \text{if } s>0$$

where we have used integration by parts.

(c)
$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \lim_{P\to\infty}\int_0^P e^{-(s-a)t}\,dt$$
$$= \lim_{P\to\infty} \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^P = \lim_{P\to\infty}\frac{1-e^{-(s-a)P}}{s-a} = \frac{1}{s-a} \quad \text{if } s>a$$

For methods not employing direct integration, see Problem 15.

2. Prove that (a) $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$, (b) $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$ if $s>0$.

(a)
$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\sin at\,dt$$
$$= \lim_{P\to\infty}\left.\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right|_0^P$$
$$= \lim_{P\to\infty}\left\{\frac{a}{s^2+a^2} - \frac{e^{-sP}(s\sin aP + a\cos aP)}{s^2+a^2}\right\}$$
$$= \frac{a}{s^2+a^2} \quad \text{if } s>0$$

(b)
$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\cos at\,dt$$
$$= \lim_{P\to\infty}\left.\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right|_0^P$$
$$= \lim_{P\to\infty}\left\{\frac{s}{s^2+a^2} - \frac{e^{-sP}(s\cos aP - a\sin aP)}{s^2+a^2}\right\}$$
$$= \frac{s}{s^2+a^2} \quad \text{if } s>0$$

We have used here the results

$$\int e^{\alpha t}\sin\beta t\,dt = \frac{e^{\alpha t}(\alpha\sin\beta t - \beta\cos\beta t)}{\alpha^2+\beta^2} \tag{1}$$

$$\int e^{\alpha t}\cos\beta t\,dt = \frac{e^{\alpha t}(\alpha\cos\beta t + \beta\sin\beta t)}{\alpha^2+\beta^2} \tag{2}$$

Another method. Assuming that the result of Problem 1(c) holds for complex numbers (which can be proved), we have

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \tag{3}$$

But $e^{iat} = \cos at + i\sin at$. Hence

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st}(\cos at + i\sin at) \tag{4}$$
$$= \int_0^\infty e^{-st}\cos at\,dt + i\int_0^\infty e^{-st}\sin at\,dt = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}$$

From (3) and (4) we have on equating real and imaginary parts,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

(c) $$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \lim_{P\to\infty}\int_0^P e^{-(s-a)t}\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^P = \lim_{P\to\infty}\frac{1-e^{-(s-a)P}}{s-a} = \frac{1}{s-a} \quad \text{if } s>a$$

For methods not employing direct integration, see Problem 15.

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(a) $$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\sin at\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\Big|_0^P$$

$$= \lim_{P\to\infty}\left\{\frac{a}{s^2+a^2} - \frac{e^{-sP}(s\sin aP + a\cos aP)}{s^2+a^2}\right\}$$

$$= \frac{a}{s^2+a^2} \quad \text{if } s>0$$

(b) $$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\cos at\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\Big|_0^P$$

$$= \lim_{P\to\infty}\left\{\frac{s}{s^2+a^2} - \frac{e^{-sP}(s\cos aP - a\sin aP)}{s^2+a^2}\right\}$$

$$= \frac{s}{s^2+a^2} \quad \text{if } s>0$$

We have used here the results

$$\int e^{\alpha t}\sin\beta t\,dt = \frac{e^{\alpha t}(\alpha\sin\beta t - \beta\cos\beta t)}{\alpha^2+\beta^2} \tag{1}$$

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$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st}(\cos at + i\sin at) \tag{4}$$

$$= \int_0^\infty e^{-st}\cos at\,dt + i\int_0^\infty e^{-st}\sin at\,dt = \mathcal{L}\{\cos at\} + i\,\mathcal{L}\{\sin at\}$$

From (3) and (4) we have on equating real and imaginary parts,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

3. Prove that (a) $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2 - a^2}$, (b) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2 - a^2}$ if $s > |a|$.

(a)
$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$
$$= \frac{1}{2}\int_0^\infty e^{-st} e^{at}\, dt - \frac{1}{2}\int_0^\infty e^{-st} e^{-at}\, dt$$
$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2} \quad \text{for } s > |a|$$

Another method. Using the linearity property of the Laplace transformation, we have at once

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2} \quad \text{for } s > |a|$$

(b) As in part (a),

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{s}{s^2 - a^2} \quad \text{for } s > |a|$$

4. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} 5 & 0 < t < 3 \\ 0 & t > 3 \end{cases}$.

By definition,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^3 e^{-st}(5)\, dt + \int_3^\infty e^{-st}(0)\, dt$$
$$= 5\int_0^3 e^{-st}\, dt = 5\left.\frac{e^{-st}}{-s}\right|_0^3 = \frac{5(1 - e^{-3s})}{s}$$

THE LINEARITY PROPERTY

5. Prove the *linearity property* [*Theorem 1-2*, Page 3].

Let $\mathcal{L}\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)\, dt$ and $\mathcal{L}\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)\, dt$. Then if c_1 and c_2 are any constants,

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} = \int_0^\infty e^{-st}\{c_1 F_1(t) + c_2 F_2(t)\}\, dt$$
$$= c_1\int_0^\infty e^{-st} F_1(t)\, dt + c_2\int_0^\infty e^{-st} F_2(t)\, dt$$
$$= c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s)$$

The result is easily generalized [see Problem 61].

3. Prove that (a) $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$, (b) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$ if $s > |a|$.

(a)
$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$
$$= \frac{1}{2}\int_0^\infty e^{-st}e^{at}\,dt - \frac{1}{2}\int_0^\infty e^{-st}e^{-at}\,dt$$
$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2} \quad \text{for } s > |a|$$

Another method. Using the linearity property of the Laplace transformation, we have at once

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2} \quad \text{for } s > |a|$$

(b) As in part (a),

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{s}{s^2-a^2} \quad \text{for } s > |a|$$

4. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} 5 & 0 < t < 3 \\ 0 & t > 3 \end{cases}$

By definition,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = \int_0^3 e^{-st}(5)\,dt + \int_3^\infty e^{-st}(0)\,dt$$
$$= 5\int_0^3 e^{-st}\,dt = 5\left.\frac{e^{-st}}{-s}\right|_0^3 = \frac{5(1-e^{-3s})}{s}$$

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$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = \int_0^\infty e^{-st}\{c_1F_1(t) + c_2F_2(t)\}\,dt$$
$$= c_1\int_0^\infty e^{-st}F_1(t)\,dt + c_2\int_0^\infty e^{-st}F_2(t)\,dt$$
$$= c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\}$$
$$= c_1f_1(s) + c_2f_2(s)$$

The result is easily generalized [see Problem 61].

6. Find $\mathcal{L}\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$.

By the linearity property [Problem 5] we have

$$\begin{aligned}\mathcal{L}\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\} &= 4\mathcal{L}\{e^{5t}\} + 6\mathcal{L}\{t^3\} - 3\mathcal{L}\{\sin 4t\} + 2\mathcal{L}\{\cos 2t\} \\ &= 4\left(\frac{1}{s-5}\right) + 6\left(\frac{3!}{s^4}\right) - 3\left(\frac{4}{s^2+16}\right) + 2\left(\frac{s}{s^2+4}\right) \\ &= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}\end{aligned}$$

where $s > 5$.

TRANSLATION AND CHANGE OF SCALE PROPERTIES

7. Prove the *first translation or shifting property*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

We have
$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = f(s)$$

Then
$$\begin{aligned}\mathcal{L}\{e^{at}F(t)\} &= \int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt \\ &= \int_0^\infty e^{-(s-a)t}F(t)\,dt = f(s-a)\end{aligned}$$

8. Find (*a*) $\mathcal{L}\{t^2e^{3t}\}$, (*b*) $\mathcal{L}\{e^{-2t}\sin 4t\}$, (*c*) $\mathcal{L}\{e^{4t}\cosh 5t\}$, (*d*) $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$.

(*a*) $\mathcal{L}\{t^2\} = \dfrac{2!}{s^3} = \dfrac{2}{s^3}$. Then $\mathcal{L}\{t^2e^{3t}\} = \dfrac{2}{(s-3)^3}$.

(*b*) $\mathcal{L}\{\sin 4t\} = \dfrac{4}{s^2+16}$. Then $\mathcal{L}\{e^{-2t}\sin 4t\} = \dfrac{4}{(s+2)^2+16} = \dfrac{4}{s^2+4s+20}$.

(*c*) $\mathcal{L}\{\cosh 5t\} = \dfrac{s}{s^2-25}$. Then $\mathcal{L}\{e^{4t}\cosh 5t\} = \dfrac{s-4}{(s-4)^2-25} = \dfrac{s-4}{s^2-8s-9}$.

Another method.

$$\begin{aligned}\mathcal{L}\{e^{4t}\cosh 5t\} &= \mathcal{L}\left\{e^{4t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\right\} = \frac{1}{2}\mathcal{L}\{e^{9t}+e^{-t}\} \\ &= \frac{1}{2}\left\{\frac{1}{s-9} + \frac{1}{s+1}\right\} = \frac{s-4}{s^2-8s-9}\end{aligned}$$

(*d*)
$$\begin{aligned}\mathcal{L}\{3\cos 6t - 5\sin 6t\} &= 3\mathcal{L}\{\cos 6t\} - 5\mathcal{L}\{\sin 6t\} \\ &= 3\left(\frac{s}{s^2+36}\right) - 5\left(\frac{6}{s^2+36}\right) = \frac{3s-30}{s^2+36}\end{aligned}$$

Then
$$\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = \frac{3(s+2)-30}{(s+2)^2+36} = \frac{3s-24}{s^2+4s+40}$$

9. Prove the *second translation or shifting property:*

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$.

$$\begin{aligned} \mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st} G(t)\,dt = \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt \\ &= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} F(t-a)\,dt \\ &= \int_a^\infty e^{-st} F(t-a)\,dt \\ &= \int_0^\infty e^{-s(u+a)} F(u)\,du \\ &= e^{-as}\int_0^\infty e^{-su} F(u)\,du \\ &= e^{-as} f(s) \end{aligned}$$

where we have used the substitution $t = u + a$.

10. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$.

Method 1.

$$\begin{aligned} \mathcal{L}\{F(t)\} &= \int_0^{2\pi/3} e^{-st}(0)\,dt + \int_{2\pi/3}^\infty e^{-st}\cos(t - 2\pi/3)\,dt \\ &= \int_0^\infty e^{-s(u+2\pi/3)} \cos u\,du \\ &= e^{-2\pi s/3}\int_0^\infty e^{-su}\cos u\,du = \frac{se^{-2\pi s/3}}{s^2+1} \end{aligned}$$

Method 2. Since $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$, it follows from Problem 9, with $a = 2\pi/3$, that

$$\mathcal{L}\{F(t)\} = \frac{se^{-2\pi s/3}}{s^2+1}$$

11. Prove the *change of scale property:* If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$.

$$\begin{aligned} \mathcal{L}\{F(at)\} &= \int_0^\infty e^{-st} F(at)\,dt \\ &= \int_0^\infty e^{-s(u/a)} F(u)\,d(u/a) \\ &= \frac{1}{a}\int_0^\infty e^{-su/a} F(u)\,du \\ &= \frac{1}{a} f\left(\frac{s}{a}\right) \end{aligned}$$

using the transformation $t = u/a$.

12. Given that $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$, find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$.

By Problem 11,

$$\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}\{1/(s/a)\} = \frac{1}{a}\tan^{-1}(a/s)$$

Then $\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}(a/s)$.

LAPLACE TRANSFORM OF DERIVATIVES

13. Prove *Theorem 1-6*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s\,f(s) - F(0)$.

Using integration by parts, we have

$$\begin{aligned}\mathcal{L}\{F'(t)\} &= \int_0^\infty e^{-st}F'(t)\,dt = \lim_{P\to\infty}\int_0^P e^{-st}F'(t)\,dt \\ &= \lim_{P\to\infty}\left\{e^{-st}F(t)\Big|_0^P + s\int_0^P e^{-st}F(t)\,dt\right\} \\ &= \lim_{P\to\infty}\left\{e^{-sP}F(P) - F(0) + s\int_0^P e^{-st}F(t)\,dt\right\} \\ &= s\int_0^\infty e^{-st}F(t)\,dt - F(0) \\ &= s\,f(s) - F(0)\end{aligned}$$

using the fact that $F(t)$ is of exponential order γ as $t\to\infty$, so that $\lim_{P\to\infty} e^{-sP}F(P) = 0$ for $s>\gamma$.

For cases where $F(t)$ is not continuous at $t=0$, see Problem 68.

14. Prove *Theorem 1-9*, Page 4: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$.

By Problem 13,

$$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0) = s\,g(s) - G(0)$$

Let $G(t) = F'(t)$. Then

$$\begin{aligned}\mathcal{L}\{F''(t)\} &= s\,\mathcal{L}\{F'(t)\} - F'(0) \\ &= s\,[s\,\mathcal{L}\{F(t)\} - F(0)] - F'(0) \\ &= s^2\mathcal{L}\{F(t)\} - s\,F(0) - F'(0) \\ &= s^2 f(s) - s\,F(0) - F'(0)\end{aligned}$$

The generalization to higher order derivatives can be proved by using mathematical induction [see Problem 65].

15. Use *Theorem 1-6*, Page 4, to derive each of the following Laplace transforms:

(a) $\mathcal{L}\{1\} = \frac{1}{s}$, (b) $\mathcal{L}\{t\} = \frac{1}{s^2}$, (c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

Theorem 1-6 states, under suitable conditions given on Page 4, that

$$\mathcal{L}\{F'(t)\} = s\,\mathcal{L}\{F(t)\} - F(0) \tag{1}$$

(a) Let $F(t) = 1$. Then $F'(t) = 0$, $F(0) = 1$, and (1) becomes

$$\mathcal{L}\{0\} = 0 = s\,\mathcal{L}\{1\} - 1 \qquad \text{or} \qquad \mathcal{L}\{1\} = 1/s \tag{2}$$

(b) Let $F(t) = t$. Then $F'(t) = 1$, $F(0) = 0$, and (1) becomes using part (a)

$$\mathcal{L}\{1\} = 1/s = s\,\mathcal{L}\{t\} - 0 \qquad \text{or} \qquad \mathcal{L}\{t\} = 1/s^2 \tag{3}$$

By using mathematical induction we can similarly show that $\mathcal{L}\{t^n\} = n!/s^{n+1}$ for any positive integer n.

(c) Let $F(t) = e^{at}$. Then $F'(t) = ae^{at}$, $F(0) = 1$, and (1) becomes

$$\mathcal{L}\{ae^{at}\} = s\,\mathcal{L}\{e^{at}\} - 1, \quad \text{i.e.} \quad a\,\mathcal{L}\{e^{at}\} = s\,\mathcal{L}\{e^{at}\} - 1 \quad \text{or} \quad \mathcal{L}\{e^{at}\} = 1/(s-a)$$

16. Use *Theorem 1-9* to show that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$.

Let $F(t) = \sin at$. Then $F'(t) = a\cos at$, $F''(t) = -a^2 \sin at$, $F(0) = 0$, $F'(0) = a$. Hence from the result

$$\mathcal{L}\{F''(t)\} = s^2\,\mathcal{L}\{F(t)\} - s\,F(0) - F'(0)$$

we have

$$\mathcal{L}\{-a^2 \sin at\} = s^2\,\mathcal{L}\{\sin at\} - s\,(0) - a$$

i.e.

$$-a^2\,\mathcal{L}\{\sin at\} = s^2\,\mathcal{L}\{\sin at\} - a$$

or

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

LAPLACE TRANSFORM OF INTEGRALS

17. Prove *Theorem 1-11*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = f(s)/s$.

Let $G(t) = \int_0^t F(u)\,du$. Then $G'(t) = F(t)$ and $G(0) = 0$. Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0) = s\,\mathcal{L}\{G(t)\} = f(s)$$

Thus

$$\mathcal{L}\{G(t)\} = \frac{f(s)}{s} \qquad \text{or} \qquad \mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$$

18. Find $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\}$.

We have by the Example following *Theorem 1-13* on Page 5,

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$$

Thus by Problem 17,

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

MULTIPLICATION BY POWERS OF t

19. Prove *Theorem 1-12*, Page 5:

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s)$ where $n = 1, 2, 3, \ldots$.

We have
$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

Then by Leibnitz's rule for differentiating under the integral sign,

$$\frac{df}{ds} = f'(s) = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} F(t)\, dt$$
$$= \int_0^\infty -te^{-st} F(t)\, dt$$
$$= -\int_0^\infty e^{-st}\{t F(t)\}\, dt$$
$$= -\mathcal{L}\{t F(t)\}$$

Thus
$$\mathcal{L}\{t F(t)\} = -\frac{df}{ds} = -f'(s) \tag{1}$$

which proves the theorem for $n = 1$.

To establish the theorem in general, we use *mathematical induction*. Assume the theorem true for $n = k$, i.e. assume

$$\int_0^\infty e^{-st}\{t^k F(t)\}\, dt = (-1)^k f^{(k)}(s) \tag{2}$$

Then
$$\frac{d}{ds}\int_0^\infty e^{-st}\{t^k F(t)\}\, dt = (-1)^k f^{(k+1)}(s)$$

or by Leibnitz's rule,
$$-\int_0^\infty e^{-st}\{t^{k+1} F(t)\}\, dt = (-1)^k f^{(k+1)}(s)$$

i.e.
$$\int_0^\infty e^{-st}\{t^{k+1} F(t)\}\, dt = (-1)^{k+1} f^{(k+1)}(s) \tag{3}$$

It follows that *if* (2) is true, i.e. if the theorem holds for $n = k$, then (3) is true, i.e. the theorem holds for $n = k+1$. But by (1) the theorem is true for $n = 1$. Hence it is true for $n = 1+1 = 2$ and $n = 2+1 = 3$, etc., and thus for all positive integer values of n.

To be completely rigorous, it is necessary to prove that Leibnitz's rule can be applied. For this, see Problem 166.

20. Find (*a*) $\mathcal{L}\{t \sin at\}$, (*b*) $\mathcal{L}\{t^2 \cos at\}$.

(*a*) Since $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2 + a^2}$, we have by Problem 19

$$\mathcal{L}\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

(b) By Problem 23, since $T = 2\pi$, we have

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}F(t)\,dt \\ &= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt \\ &= \frac{1}{1-e^{-2\pi s}}\left\{\frac{e^{-st}(-s\sin t - \cos t)}{s^2+1}\right\}\Bigg|_0^{\pi} \\ &= \frac{1}{1-e^{-2\pi s}}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = \frac{1}{(1-e^{-\pi s})(s^2+1)}\end{aligned}$$

using the integral (1) of Problem 2, Page 11.

The graph of the function $F(t)$ is often called a *half wave rectified sine curve.*

INITIAL AND FINAL VALUE THEOREMS

25. Prove the *initial-value theorem:* $\lim\limits_{t\to 0} F(t) = \lim\limits_{s\to\infty} s\,f(s)$.

By Problem 13,

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}F'(t)\,dt = s\,f(s) - F(0) \tag{1}$$

But if $F'(t)$ is sectionally continuous and of exponential order, we have

$$\lim_{s\to\infty}\int_0^\infty e^{-st}F'(t)\,dt = 0 \tag{2}$$

Then taking the limit as $s\to\infty$ in (1), assuming $F(t)$ continuous at $t=0$, we find that

$$0 = \lim_{s\to\infty} s\,f(s) - F(0) \qquad \text{or} \qquad \lim_{s\to\infty} s\,f(s) = F(0) = \lim_{t\to 0} F(t)$$

If $F(t)$ is not continuous at $t=0$, the required result still holds but we must use *Theorem 1-7*, Page 4.

26. Prove the *final-value theorem:* $\lim\limits_{t\to\infty} F(t) = \lim\limits_{s\to 0} s\,f(s)$.

By Problem 13,

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}F'(t)\,dt = s\,f(s) - F(0)$$

The limit of the left hand side as $s\to 0$ is

$$\begin{aligned}\lim_{s\to 0}\int_0^\infty e^{-st}F'(t)\,dt &= \int_0^\infty F'(t)\,dt = \lim_{P\to\infty}\int_0^P F'(t)\,dt \\ &= \lim_{P\to\infty}\{F(P)-F(0)\} = \lim_{t\to\infty} F(t) - F(0)\end{aligned}$$

The limit of the right hand side as $s\to 0$ is

$$\lim_{s\to 0} s\,f(s) - F(0)$$

Thus

$$\lim_{t\to\infty} F(t) - F(0) = \lim_{s\to 0} s\,f(s) - F(0)$$

or, as required,

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s\,f(s)$$

If $F(t)$ is not continuous, the result still holds but we must use *Theorem 1-7*, Page 4.

27. Illustrate Problems 25 and 26 for the function $F(t) = 3e^{-2t}$.

We have $F(t) = 3e^{-2t}$, $f(s) = \mathcal{L}\{F(t)\} = \dfrac{3}{s+2}$.

By the initial-value theorem (Problem 25),

$$\lim_{t\to 0} 3e^{-2t} = \lim_{s\to\infty} \frac{3s}{s+2}$$

or $3 = 3$, which illustrates the theorem.

By the final-value theorem (Problem 26),

$$\lim_{t\to\infty} 3e^{-2t} = \lim_{s\to 0} \frac{3s}{s+2}$$

or $0 = 0$, which illustrates the theorem.

THE GAMMA FUNCTION

28. Prove: (a) $\Gamma(n+1) = n\,\Gamma(n)$, $n > 0$; (b) $\Gamma(n+1) = n!$, $n = 1, 2, 3, \ldots$.

(a)
$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty u^n e^{-u}\,du = \lim_{P\to\infty} \int_0^P u^n e^{-u}\,du \\
&= \lim_{P\to\infty} \left\{ (u^n)(-e^{-u})\Big|_0^P - \int_0^P (-e^{-u})(nu^{n-1})\,du \right\} \\
&= \lim_{P\to\infty} \left\{ -P^n e^{-P} + n\int_0^P u^{n-1} e^{-u}\,du \right\} \\
&= n\int_0^\infty u^{n-1} e^{-u}\,du = n\,\Gamma(n) \quad \text{if } n > 0
\end{aligned}$$

(b)
$$\Gamma(1) = \int_0^\infty e^{-u}\,du = \lim_{P\to\infty}\int_0^P e^{-u}\,du = \lim_{P\to\infty}(1 - e^{-P}) = 1.$$

Put $n = 1, 2, 3, \ldots$ in $\Gamma(n+1) = n\,\Gamma(n)$. Then

$$\Gamma(2) = 1\,\Gamma(1) = 1, \quad \Gamma(3) = 2\,\Gamma(2) = 2\cdot 1 = 2!, \quad \Gamma(4) = 3\,\Gamma(3) = 3\cdot 2! = 3!$$

In general, $\Gamma(n+1) = n!$ if n is a positive integer.

29. Prove: $\displaystyle\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$.

Let $I_P = \displaystyle\int_0^P e^{-x^2}\,dx = \int_0^P e^{-y^2}\,dy$ and let $\displaystyle\lim_{P\to\infty} I_P = I$, the required value of the integral. Then

$$\begin{aligned}
I_P^2 &= \left(\int_0^P e^{-x^2}\,dx\right)\left(\int_0^P e^{-y^2}\,dy\right) \\
&= \int_0^P\int_0^P e^{-(x^2+y^2)}\,dx\,dy \\
&= \iint_{\mathcal{R}_P} e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}$$

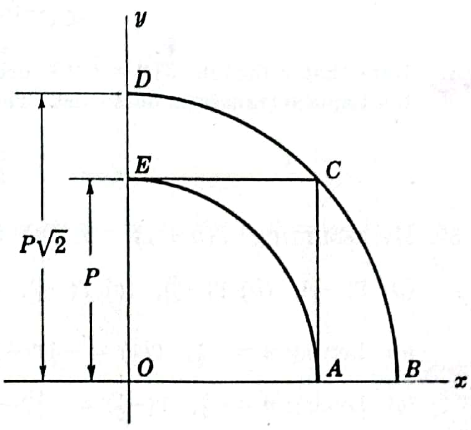

Fig. 1-6

where $\mathcal{R}_P$ is the square $OACE$ of side P [see Fig. 1-6].

Since the integrand is positive, we have

$$\iint_{\mathcal{R}_1} e^{-(x^2+y^2)}\,dx\,dy \quad \leq \quad I_P^2 \quad \leq \quad \iint_{\mathcal{R}_2} e^{-(x^2+y^2)}\,dx\,dy \tag{1}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the regions in the first quadrant bounded by the circles having radii P and $P\sqrt{2}$ respectively.

Using polar coordinates (r,θ) we have from (1),

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{P} e^{-r^2}\,r\,dr\,d\theta \quad \leq \quad I_P^2 \quad \leq \quad \int_{\theta=0}^{\pi/2}\int_{r=0}^{P\sqrt{2}} e^{-r^2}\,r\,dr\,d\theta \tag{2}$$

or

$$\frac{\pi}{4}(1-e^{-P^2}) \quad \leq \quad I_P^2 \quad \leq \quad \frac{\pi}{4}(1-e^{-2P^2}) \tag{3}$$

Then taking the limit as $P \to \infty$ in (3), we find $\lim\limits_{P\to\infty} I_P^2 = I^2 = \pi/4$ and $I = \sqrt{\pi}/2$.

30. Prove: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$\Gamma(\frac{1}{2}) = \int_0^\infty u^{-1/2}\,e^{-u}\,du$. Letting $u = v^2$, this integral becomes on using Problem 29

$$2\int_0^\infty e^{-v^2}\,dv \quad = \quad 2\left(\frac{\sqrt{\pi}}{2}\right) \quad = \quad \sqrt{\pi}$$

31. Prove: $\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$ if $n > -1$, $s > 0$.

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\,t^n\,dt$. Letting $st = u$, assuming $s > 0$, this becomes

$$\mathcal{L}\{t^n\} \quad = \quad \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n d\left(\frac{u}{s}\right) \quad = \quad \frac{1}{s^{n+1}}\int_0^\infty u^n\,e^{-u}\,du \quad = \quad \frac{\Gamma(n+1)}{s^{n+1}}$$

32. Prove: $\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}$, $s > 0$.

Let $n = -1/2$ in Problem 31. Then

$$\mathcal{L}\{t^{-1/2}\} \quad = \quad \frac{\Gamma(\frac{1}{2})}{s^{1/2}} \quad = \quad \frac{\sqrt{\pi}}{s^{1/2}} \quad = \quad \sqrt{\frac{\pi}{s}}$$

Note that although $F(t) = t^{-1/2}$ does not satisfy the sufficient conditions of *Theorem 1-1, Page 2*, the Laplace transform does exist. The function does satisfy the conditions of the theorem in Prob. 145.

33. By assuming $\Gamma(n+1) = n\Gamma(n)$ holds for all n, find:

(a) $\Gamma(-\frac{1}{2})$, (b) $\Gamma(-\frac{3}{2})$, (c) $\Gamma(-\frac{5}{2})$, (d) $\Gamma(0)$, (e) $\Gamma(-1)$, (f) $\Gamma(-2)$.

(a) Letting $n = -\frac{1}{2}$, $\Gamma(\frac{1}{2}) = -\frac{1}{2}\Gamma(-\frac{1}{2})$. Then $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$.

(b) Letting $n = -\frac{3}{2}$, $\Gamma(-\frac{1}{2}) = -\frac{3}{2}\Gamma(-\frac{3}{2})$. Then $\Gamma(-\frac{3}{2}) = -\frac{2}{3}\Gamma(-\frac{1}{2}) = (2)(\frac{2}{3})\sqrt{\pi} = \frac{4}{3}\sqrt{\pi}$ by part (a).

(c) Letting $n = -\frac{5}{2}$, $\Gamma(-\frac{3}{2}) = -\frac{5}{2}\Gamma(-\frac{5}{2})$. Then $\Gamma(-\frac{5}{2}) = -\frac{2}{5}\Gamma(-\frac{3}{2}) = -(2)(\frac{2}{3})(\frac{2}{5})\sqrt{\pi} = -\frac{8}{15}\sqrt{\pi}$ by part (b).

(d) Letting $n = 0$, $\Gamma(1) = 0 \cdot \Gamma(0)$ and it follows that $\Gamma(0)$ must be infinite, since $\Gamma(1) = 1$.

(e) Letting $n = -1$, $\Gamma(0) = -1\,\Gamma(-1)$ and it follows that $\Gamma(-1)$ must be infinite.

(f) Letting $n = -2$, $\Gamma(-1) = -2\,\Gamma(-2)$ and it follows that $\Gamma(-2)$ must be infinite.

In general if p is any positive integer or zero, $\Gamma(-p)$ is infinite and [see Problem 170],

$$\Gamma(-p-\tfrac{1}{2}) \quad = \quad (-1)^{p+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\cdots\left(\frac{2}{2p+1}\right)\sqrt{\pi}$$

BESSEL FUNCTIONS

34. (a) Find $\mathcal{L}\{J_0(t)\}$ where $J_0(t)$ is the Bessel function of order zero.

(b) Use the result of (a) to find $\mathcal{L}\{J_0(at)\}$.

(a) ***Method 1, using series.*** Letting $n=0$ in equation (*23*), Page 7, we find

$$J_0(t) \quad = \quad 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\,4^2} - \frac{t^6}{2^2\,4^2\,6^2} + \cdots$$

Then

$$\begin{aligned} \mathcal{L}\{J_0(t)\} &= \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2\,4^2}\frac{4!}{s^5} - \frac{1}{2^2\,4^2\,6^2}\frac{6!}{s^7} + \cdots \\ &= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^4}\right) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^6}\right) + \cdots\right\} \\ &= \frac{1}{s}\left\{\left(1+\frac{1}{s^2}\right)^{-1/2}\right\} \quad = \quad \frac{1}{\sqrt{s^2+1}} \end{aligned}$$

using the binomial theorem [see Problem 172].

Method 2, using differential equations. The function $J_0(t)$ satisfies the differential equation

$$t\,J_0''(t) \;+\; J_0'(t) \;+\; t\,J_0(t) \quad = \quad 0 \qquad (1)$$

[see Property 5, Page 8, with $n=0$]. Taking the Laplace transform of both sides of (*1*) and using *Theorems 1-6* and *1-9*, Page 4, and *Theorem 1-12*, Page 5, together with $J_0(0) = 1$, $J_0'(0) = 0$, $y = \mathcal{L}\{J_0(t)\}$, we have

$$-\frac{d}{ds}\{s^2y - s(1) - 0\} \;+\; \{sy - 1\} \;-\; \frac{dy}{ds} \quad = \quad 0$$

from which

$$\frac{dy}{ds} \quad = \quad -\frac{sy}{s^2+1}$$

Thus

$$\frac{dy}{y} \quad = \quad -\frac{s\,ds}{s^2+1}$$

and by integration

$$y \quad = \quad \frac{c}{\sqrt{s^2+1}}$$

Now $\lim_{s\to\infty} s\,y(s) = \dfrac{cs}{\sqrt{s^2+1}} = c$ and $\lim_{t\to 0} J_0(t) = 1$. Thus by the initial-value theorem [Page 5], we have $c=1$ and so $\mathcal{L}\{J_0(t)\} = 1/\sqrt{s^2+1}$.

For another method, see Problem 165.

(b) By Problem 11,

$$\mathcal{L}\{J_0(at)\} \quad = \quad \frac{1}{a}\,\frac{1}{\sqrt{(s/a)^2+1}} \quad = \quad \frac{1}{\sqrt{s^2+a^2}}$$

35. Find $\mathcal{L}\{J_1(t)\}$, where $J_1(t)$ is Bessel's function of order one.

From Property 8 for Bessel functions, Page 7, we have $J_0'(t) = -J_1(t)$. Hence

$$\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\} = -[s\,\mathcal{L}\{J_0(t)\} - 1]$$

$$= 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}$$

The methods of infinite series and differential equations can also be used [see Problem 17, Page 41].

THE SINE, COSINE AND EXPONENTIAL INTEGRALS

36. Prove: $\mathcal{L}\{\text{Si}(t)\} = \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$.

Method 1. Let $F(t) = \int_0^t \frac{\sin u}{u}\,du$. Then $F(0) = 0$ and $F'(t) = \frac{\sin t}{t}$ or $t\,F'(t) = \sin t$.

Taking the Laplace transform,

$$\mathcal{L}\{t\,F'(t)\} = \mathcal{L}\{\sin t\} \qquad \text{or} \qquad -\frac{d}{ds}\{s\,f(s) - F(0)\} = \frac{1}{s^2+1}$$

i.e.

$$\frac{d}{ds}\{s\,f(s)\} = \frac{-1}{s^2+1}$$

Integrating,

$$s\,f(s) = -\tan^{-1} s + c$$

By the initial value theorem, $\lim_{s\to\infty} s\,f(s) = \lim_{t\to 0} F(t) = F(0) = 0$ so that $c = \pi/2$. Thus

$$s\,f(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\frac{1}{s} \qquad \text{or} \qquad f(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Method 2. See Problem 18.

Method 3. Using infinite series, we have

$$\int_0^t \frac{\sin u}{u}\,du = \int_0^t \frac{1}{u}\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots\right) du$$

$$= t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!} + \cdots$$

Then

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\right\} = \mathcal{L}\left\{t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!} + \cdots\right\}$$

$$= \frac{1}{s^2} - \frac{1}{3\cdot 3!}\cdot\frac{3!}{s^4} + \frac{1}{5\cdot 5!}\cdot\frac{5!}{s^6} - \frac{1}{7\cdot 7!}\cdot\frac{7!}{s^8} + \cdots$$

$$= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \cdots$$

$$= \frac{1}{s}\left\{\frac{(1/s)}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \cdots\right\}$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}$$

using the series $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \cdots$, $|x| < 1$.

Method 4. Letting $u = tv$,

$$\int_0^t \frac{\sin u}{u}\,du = \int_0^1 \frac{\sin tv}{v}\,dv$$

Then

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} &= \mathcal{L}\left\{\int_0^1 \frac{\sin tv}{v}\,dv\right\} \\
&= \int_0^\infty e^{-st}\left\{\int_0^1 \frac{\sin tv}{v}\,dv\right\}dt \\
&= \int_0^1 \frac{1}{v}\left\{\int_0^\infty e^{-st}\sin tv\,dt\right\}dv \\
&= \int_0^1 \frac{\mathcal{L}\{\sin tv\}}{v}\,dv = \int_0^1 \frac{dv}{s^2+v^2} \\
&= \frac{1}{s}\tan^{-1}\frac{v}{s}\bigg|_0^1 = \frac{1}{s}\tan^{-1}\frac{1}{s}
\end{aligned}$$

where we have assumed permissibility of change of order of integration.

37. Prove: $\mathcal{L}\{\mathrm{Ci}(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\,du\right\} = \frac{\ln(s^2+1)}{2s}$.

We use the principle of Method 1 in Problem 36. Let $F(t) = \int_t^\infty \frac{\cos u}{u}\,du$ so that $F'(t) = -\frac{\cos t}{t}$ and $tF'(t) = -\cos t$. Taking the Laplace transform, we have

$$-\frac{d}{ds}\{s\,f(s) - F(0)\} = \frac{-s}{s^2+1} \qquad \text{or} \qquad \frac{d}{ds}\{s\,f(s)\} = \frac{s}{s^2+1}$$

Then by integration,

$$s\,f(s) = \tfrac{1}{2}\ln(s^2+1) + c$$

By the final-value theorem, $\lim_{s\to 0} s\,f(s) = \lim_{t\to\infty} F(t) = 0$ so that $c = 0$. Thus

$$s\,f(s) = \tfrac{1}{2}\ln(s^2+1) \qquad \text{or} \qquad f(s) = \frac{\ln(s^2+1)}{2s}$$

We can also use Method 4 of Problem 36 [see Problem 153].

38. Prove: $\mathcal{L}\{\mathrm{Ei}(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u}\,du\right\} = \frac{\ln(s+1)}{s}$.

Let $F(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$. Then $tF'(t) = -e^{-t}$. Taking the Laplace transform, we find

$$-\frac{d}{ds}\{s\,f(s) - F(0)\} = \frac{-1}{s+1} \qquad \text{or} \qquad \frac{d}{ds}\{s\,f(s)\} = \frac{1}{s+1}$$

Integrating,

$$s\,f(s) = \ln(s+1) + c$$

Applying the final-value theorem as in Problem 37, we find $c = 0$ and so

$$f(s) = \frac{\ln(s+1)}{s}$$

For another method similar to that of Method 4, Problem 36, see Problem 153.

THE ERROR FUNCTION

39. Prove: $\mathcal{L}\{\text{erf}\sqrt{t}\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\} = \frac{1}{s\sqrt{s+1}}$.

Using infinite series, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\} &= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right)du\right\} \\
&= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\cdot 2!} - \frac{t^{7/2}}{7\cdot 3!} + \cdots\right)\right\} \\
&= \frac{2}{\sqrt{\pi}}\left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5\cdot 2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7\cdot 3!\,s^{9/2}} + \cdots\right\} \\
&= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{s^{7/2}} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{s^{9/2}} + \cdots \\
&= \frac{1}{s^{3/2}}\left\{1 - \frac{1}{2}\frac{1}{s} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{s^2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{s^3} + \cdots\right\} \\
&= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}}
\end{aligned}$$

using the binomial theorem [see Problem 172].

For another method, see Problem 175(*a*).

IMPULSE FUNCTIONS. THE DIRAC DELTA FUNCTION.

40. Prove that $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$ where $\mathcal{U}(t-a)$ is Heaviside's unit step function.

We have $\mathcal{U}(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$. Then

$$\begin{aligned}
\mathcal{L}\{\mathcal{U}(t-a)\} &= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt \\
&= \lim_{P\to\infty}\int_a^P e^{-st}\,dt = \lim_{P\to\infty}\left.\frac{e^{-st}}{-s}\right|_a^P \\
&= \lim_{P\to\infty}\frac{e^{-as} - e^{-sP}}{s} = \frac{e^{-as}}{s}
\end{aligned}$$

Another method.

Since $\mathcal{L}\{1\} = 1/s$, we have by Problem 9, $\mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}/s$.

41. Find $\mathcal{L}\{F_\epsilon(t)\}$ where $F_\epsilon(t)$ is defined by (*30*), Page 8.

We have $F_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \leqq t \leqq \epsilon \\ 0 & t > \epsilon \end{cases}$. Then

$$\begin{aligned}
\mathcal{L}\{F_\epsilon(t)\} &= \int_0^\infty e^{-st}F_\epsilon(t)\,dt \\
&= \int_0^\epsilon e^{-st}(1/\epsilon)\,dt + \int_\epsilon^\infty e^{-st}(0)\,dt \\
&= \frac{1}{\epsilon}\int_0^\epsilon e^{-st}\,dt = \frac{1 - e^{-s\epsilon}}{\epsilon s}
\end{aligned}$$

42. (*a*) Show that $\lim_{\epsilon \to 0} \mathcal{L}\{F_\epsilon(t)\} = 1$ in Problem 41.

(*b*) Is the result in (*a*) the same as $\mathcal{L}\left\{\lim_{\epsilon \to 0} F_\epsilon(t)\right\}$? Explain.

(*a*) This follows at once since

$$\lim_{\epsilon \to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon} = \lim_{\epsilon \to 0} \frac{1 - (1 - s\epsilon + s^2\epsilon^2/2! - \cdots)}{s\epsilon} = \lim_{\epsilon \to 0} \left(1 - \frac{s\epsilon}{2!} + \cdots\right) = 1$$

It also follows by use of L'Hospital's rule.

(*b*) Mathematically speaking, $\lim_{\epsilon \to 0} F_\epsilon(t)$ does not exist, so that $\mathcal{L}\left\{\lim_{\epsilon \to 0} F_\epsilon(t)\right\}$ is not defined. Nevertheless it proves useful to consider $\delta(t) = \lim_{\epsilon \to 0} F_\epsilon(t)$ to be such that $\mathcal{L}\{\delta(t)\} = 1$. We call $\delta(t)$ the *Dirac delta function* or *impulse function*.

43. Show that $\mathcal{L}\{\delta(t-a)\} = e^{-as}$, where $\delta(t)$ is the Dirac delta function.

This follows from Problem 9 and the fact that $\mathcal{L}\{\delta(t)\} = 1$.

44. Indicate which of the following are null functions.

(*a*) $F(t) = \begin{cases} 1 & t = 1 \\ 0 & \text{otherwise} \end{cases}$, (*b*) $F(t) = \begin{cases} 1 & 1 \leqq t \leqq 2 \\ 0 & \text{otherwise} \end{cases}$, (*c*) $F(t) = \delta(t)$.

(*a*) $F(t)$ is a null function, since $\int_0^t F(u)\,du = 0$ for all $t > 0$.

(*b*) If $t < 1$, we have $\int_0^t F(u)\,du = 0$.

If $1 \leqq t \leqq 2$, we have $\int_0^t F(u)\,du = \int_1^t (1)\,du = t - 1$.

If $t > 2$, we have $\int_0^t F(u)\,du = \int_1^2 (1)\,du = 1$.

Since $\int_0^t F(u)\,du \neq 0$ for all $t > 0$, $F(t)$ is not a null function.

(*c*) Since $\int_0^t \delta(u)\,du = 1$ for all $t > 0$, $\delta(t)$ is not a null function.

EVALUATION OF INTEGRALS

45. Evaluate (*a*) $\int_0^\infty t\,e^{-2t}\cos t\,dt$, (*b*) $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt$.

(*a*) By Problem 19,

$$\begin{aligned} \mathcal{L}\{t\cos t\} &= \int_0^\infty t\,e^{-st}\cos t\,dt \\ &= -\frac{d}{ds}\mathcal{L}\{\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2} \end{aligned}$$

Then letting $s = 2$, we find $\int_0^\infty te^{-2t} \cos t \, dt = \frac{3}{25}$.

(b) If $F(t) = e^{-t} - e^{-3t}$, then $f(s) = \mathcal{L}\{F(t)\} = \frac{1}{s+1} - \frac{1}{s+3}$. Thus by Problem 21,

$$\mathcal{L}\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^\infty \left\{\frac{1}{u+1} - \frac{1}{u+3}\right\} du$$

or

$$\int_0^\infty e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \ln\left(\frac{s+3}{s+1}\right)$$

Taking the limit as $s \to 0+$, we find $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \, dt = \ln 3$.

46. Show that (a) $\int_0^\infty J_0(t) \, dt = 1$, (b) $\int_0^\infty e^{-t} \operatorname{erf} \sqrt{t} \, dt = \sqrt{2}/2$.

(a) By Problem 34,

$$\int_0^\infty e^{-st} J_0(t) \, dt = \frac{1}{\sqrt{s^2+1}}$$

Then letting $s \to 0+$ we find $\int_0^\infty J_0(t) \, dt = 1$.

(b) By Problem 39,

$$\int_0^\infty e^{-st} \operatorname{erf} \sqrt{t} \, dt = \frac{1}{s\sqrt{s+1}}$$

Then letting $s \to 1$, we find $\int_0^\infty e^{-t} \operatorname{erf} \sqrt{t} \, dt = \sqrt{2}/2$.

MISCELLANEOUS PROBLEMS

47. Prove *Theorem 1-1*, Page 2.

We have for any positive number N,

$$\int_0^\infty e^{-st} F(t) \, dt = \int_0^N e^{-st} F(t) \, dt + \int_N^\infty e^{-st} F(t) \, dt$$

Since $F(t)$ is sectionally continuous in every finite interval $0 \leqq t \leqq N$, the first integral on t right exists. Also the second integral on the right exists, since $F(t)$ is of exponential order γ f $t > N$. To see this we have only to observe that in such case

$$\left|\int_N^\infty e^{-st} F(t) \, dt\right| \leqq \int_N^\infty |e^{-st} F(t)| \, dt$$

$$\leqq \int_0^\infty e^{-st} |F(t)| \, dt$$

$$\leqq \int_0^\infty e^{-st} M e^{\gamma t} \, dt = \frac{M}{s - \gamma}$$

Thus the Laplace transform exists for $s > \gamma$.

48. Find $\mathcal{L}\{\sin\sqrt{t}\}$.

Method 1, using series.

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

Then the Laplace transform is

$$\begin{aligned}\mathcal{L}\{\sin\sqrt{t}\} &= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\,s^{5/2}} + \frac{\Gamma(7/2)}{5!\,s^{7/2}} - \frac{\Gamma(9/2)}{7!\,s^{9/2}} + \cdots \\ &= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{(1/2^2 s)^2}{2!} - \frac{(1/2^2 s)^3}{3!} + \cdots\right\} \\ &= \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/2^2 s} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}\end{aligned}$$

Method 2, using differential equations.

Let $Y(t) = \sin\sqrt{t}$. Then by differentiating twice we find

$$4tY'' + 2Y' + Y = 0$$

Taking the Laplace transform, we have if $y = \mathcal{L}\{Y(t)\}$

$$-4\frac{d}{ds}\{s^2 y - s\,Y(0) - Y'(0)\} + 2\{s\,y - Y(0)\} + y = 0$$

or

$$4\,s^2 y' + (6\,s - 1)y = 0$$

Solving,

$$y = \frac{c}{s^{3/2}}\,e^{-1/4s}$$

For small values of t, we have $\sin\sqrt{t} \sim \sqrt{t}$ and $\mathcal{L}\{\sqrt{t}\} = \sqrt{\pi}/2s^{3/2}$. For large s, $y \sim c/s^{3/2}$. It follows by comparison that $c = \sqrt{\pi}/2$. Thus

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}$$

49. Find $\mathcal{L}\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Let $F(t) = \sin\sqrt{t}$. Then $F'(t) = \dfrac{\cos\sqrt{t}}{2\sqrt{t}}$, $F(0) = 0$. Hence by Problem 48,

$$\mathcal{L}\{F'(t)\} = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\,f(s) - F(0) = \frac{\sqrt{\pi}}{2\,s^{1/2}}\,e^{-1/4s}$$

from which

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}\,e^{-1/4s}$$

The method of series can also be used [see Problem 175(*b*)].

50. Show that

$$\mathcal{L}\{\ln t\} = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

where $\gamma = .5772156\ldots$ is *Euler's constant.*

We have

$$\Gamma(r) = \int_0^\infty u^{r-1}\,e^{-u}\,du$$

Then differentiating with respect to r, we find

$$\Gamma'(r) = \int_0^\infty u^{r-1} e^{-u} \ln u \, du$$

from which

$$\Gamma'(1) = \int_0^\infty e^{-u} \ln u \, du$$

Letting $u = st$, $s > 0$, this becomes

$$\Gamma'(1) = s \int_0^\infty e^{-st} (\ln s + \ln t) \, dt$$

Hence

$$\mathcal{L}\{\ln t\} = \int_0^\infty e^{-st} \ln t \, dt = \frac{\Gamma'(1)}{s} - \ln s \int_0^\infty e^{-st} \, dt$$

$$= \frac{\Gamma'(1)}{s} - \frac{\ln s}{s} = -\frac{\gamma + \ln s}{s}$$

Another method. We have for $k > -1$,

$$\int_0^\infty e^{-st} t^k \, dt = \frac{\Gamma(k+1)}{s^{k+1}}$$

Then differentiating with respect to k,

$$\int_0^\infty e^{-st} t^k \ln t \, dt = \frac{\Gamma'(k+1) - \Gamma(k+1) \ln s}{s^{k+1}}$$

Letting $k = 0$ we have, as required,

$$\int_0^\infty e^{-st} \ln t \, dt = \mathcal{L}\{\ln t\} = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

Supplementary Problems

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

51. Find the Laplace transforms of each of the following functions. In each case specify the values of s for which the Laplace transform exists.

(*a*) $2e^{4t}$	*Ans.* (*a*) $2/(s-4)$,	$s > 4$
(*b*) $3e^{-2t}$	(*b*) $3/(s+2)$,	$s > -2$
(*c*) $5t - 3$	(*c*) $(5-3s)/s^2$,	$s > 0$
(*d*) $2t^2 - e^{-t}$	(*d*) $(4+4s-s^3)/s^3(s+1)$,	$s > 0$
(*e*) $3 \cos 5t$	(*e*) $3s/(s^2+25)$,	$s > 0$
(*f*) $10 \sin 6t$	(*f*) $60/(s^2+36)$,	$s > 0$
(*g*) $6 \sin 2t - 5 \cos 2t$	(*g*) $(12-5s)/(s^2+4)$,	$s > 0$
(*h*) $(t^2+1)^2$	(*h*) $(s^4+4s^2+24)/s^5$,	$s > 0$
(*i*) $(\sin t - \cos t)^2$	(*i*) $(s^2-2s+4)/s(s^2+4)$,	$s > 0$
(*j*) $3 \cosh 5t - 4 \sinh 5t$	(*j*) $(3s-20)/(s^2-25)$,	$s > 5$

52. Evaluate (*a*) $\mathcal{L}\{(5e^{2t}-3)^2\}$, (*b*) $\mathcal{L}\{4\cos^2 2t\}$.

Ans. (*a*) $\dfrac{25}{s-4} - \dfrac{30}{s-2} + \dfrac{9}{s}$, $s > 4$ (*b*) $\dfrac{2}{s} + \dfrac{2s}{s^2+16}$, $s > 0$

53. Find $\mathcal{L}\{\cosh^2 4t\}$. *Ans.* $\dfrac{s^2-32}{s(s^2-64)}$

54. Find $\mathcal{L}\{F(t)\}$ if (*a*) $F(t) = \begin{cases} 0 & 0 < t < 2 \\ 4 & t > 2 \end{cases}$, (*b*) $F(t) = \begin{cases} 2t & 0 \leqq t \leqq 5 \\ 1 & t > 5 \end{cases}$

Ans. (*a*) $4e^{-2s}/s$ (*b*) $\dfrac{2}{s^2}(1-e^{-5s}) - \dfrac{9}{s}e^{-5s}$

55. Prove that $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $n = 1, 2, 3, \ldots$.

56. Investigate the existence of the Laplace transform of each of the following functions.

(*a*) $1/(t+1)$, (*b*) e^{t^2-t}, (*c*) $\cos t^2$ *Ans.* (*a*) exists, (*b*) does not exist, (*c*) exists

LINEARITY, TRANSLATION AND CHANGE OF SCALE PROPERTIES

57. Find $\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$.

Ans. $\dfrac{72}{s^5} - \dfrac{12}{s^4} + \dfrac{4}{s+3} - \dfrac{10}{s^2+25} + \dfrac{3s}{s^2+4}$

58. Evaluate each of the following.

(*a*) $\mathcal{L}\{t^3 e^{-3t}\}$ *Ans.* (*a*) $6/(s+3)^4$

(*b*) $\mathcal{L}\{e^{-t}\cos 2t\}$ (*b*) $(s+1)/(s^2+2s+5)$

(*c*) $\mathcal{L}\{2e^{3t}\sin 4t\}$ (*c*) $8/(s^2-6s+25)$

(*d*) $\mathcal{L}\{(t+2)^2 e^t\}$ (*d*) $(4s^2-4s+2)/(s-1)^3$

(*e*) $\mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\}$ (*e*) $(20-4s)/(s^2-4s+20)$

(*f*) $\mathcal{L}\{e^{-4t}\cosh 2t\}$ (*f*) $(s+4)/(s^2+8s+12)$

(*g*) $\mathcal{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$ (*g*) $(1-5s)/(s^2+2s-3)$

59. Find (*a*) $\mathcal{L}\{e^{-t}\sin^2 t\}$, (*b*) $\mathcal{L}\{(1+te^{-t})^3\}$.

Ans. (*a*) $\dfrac{2}{(s+1)(s^2+2s+5)}$ (*b*) $\dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$

60. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$. *Ans.* $2e^{-s}/s^3$

61. If $F_1(t), F_2(t), \ldots, F_n(t)$ have Laplace transforms $f_1(s), f_2(s), \ldots, f_n(s)$ respectively and $c_1, c_2, \ldots, c_n$ are any constants, prove that

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t) + \cdots + c_n F_n(t)\} = c_1 f_1(s) + c_2 f_2(s) + \cdots + c_n f_n(s)$$

62. If $\mathcal{L}\{F(t)\} = \dfrac{s^2 - s + 1}{(2s+1)^2(s-1)}$, find $\mathcal{L}\{F(2t)\}$. *Ans.* $(s^2 - 2s + 4)/4(s+1)^2(s-2)$

63. If $\mathcal{L}\{F(t)\} = \dfrac{e^{-1/s}}{s}$, find $\mathcal{L}\{e^{-t}F(3t)\}$. *Ans.* $\dfrac{e^{-3/(s+1)}}{s+1}$

64. If $f(s) = \mathcal{L}\{F(t)\}$, prove that for $r > 0$,

$$\mathcal{L}\{r^t F(at)\} = \frac{1}{s - \ln r} f\left(\frac{s - \ln r}{a}\right)$$

LAPLACE TRANSFORMS OF DERIVATIVES

65. (a) If $\mathcal{L}\{F(t)\} = f(s)$, prove that

$$\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - s F'(0) - F''(0)$$

stating appropriate conditions on $F(t)$.

(b) Generalize the result of (a) and prove by use of mathematical induction.

66. Given $F(t) = \begin{cases} 2t & 0 \leq t \leq 1 \\ t & t > 1 \end{cases}$. (a) Find $\mathcal{L}\{F(t)\}$. (b) Find $\mathcal{L}\{F'(t)\}$. (c) Does the result $\mathcal{L}\{F'(t)\} = s\,\mathcal{L}\{F(t)\} - F(0)$ hold for this case? Explain.

Ans. (a) $\dfrac{2}{s^2} - \dfrac{e^{-s}}{s} - \dfrac{e^{-s}}{s^2}$, (b) $\dfrac{2}{s} - \dfrac{e^{-s}}{s}$

67. (a) If $F(t) = \begin{cases} t^2 & 0 < t \leq 1 \\ 0 & t > 1 \end{cases}$, find $\mathcal{L}\{F''(t)\}$.

(b) Does the result $\mathcal{L}\{F''(t)\} = s^2\mathcal{L}\{F(t)\} - s\,F(0) - F'(0)$ hold in this case? Explain.

Ans. (a) $2(1 - e^{-s})/s$

68. Prove: (a) *Theorem 1-7*, Page 4; (b) *Theorem 1-8*, Page 4.

LAPLACE TRANSFORMS OF INTEGRALS

69. Verify directly that $\mathcal{L}\left\{\displaystyle\int_0^t (u^2 - u + e^{-u})\,du\right\} = \dfrac{1}{s}\mathcal{L}\{t^2 - t + e^{-t}\}$.

70. If $f(s) = \mathcal{L}\{F(t)\}$, show that $\mathcal{L}\left\{\displaystyle\int_0^t dt_1 \int_0^{t_1} F(u)\,du\right\} = \dfrac{f(s)}{s^2}$.

[The double integral is sometimes briefly written as $\displaystyle\int_0^t\int_0^t F(t)\,dt^2$.]

71. Generalize the result of Problem 70.

72. Show that $\mathcal{L}\left\{\displaystyle\int_0^t \frac{1 - e^{-u}}{u}\,du\right\} = \dfrac{1}{s}\ln\left(1 + \dfrac{1}{s}\right)$.

73. Show that $\displaystyle\int_{t=0}^{\infty}\int_{u=0}^{t} \frac{e^{-t}\sin u}{u}\,du\,dt = \frac{\pi}{4}$.

MULTIPLICATION BY POWERS OF t

74. Prove that (a) $\mathcal{L}\{t \cos at\} = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$

(b) $\mathcal{L}\{t \sin at\} = \dfrac{2as}{(s^2 + a^2)^2}$

75. Find $\mathcal{L}\{t(3 \sin 2t - 2 \cos 2t)\}$. *Ans.* $\dfrac{8 + 12s - 2s^2}{(s^2 + 4)^2}$

76. Show that $\mathcal{L}\{t^2 \sin t\} = \dfrac{6s^2 - 2}{(s^2 + 1)^3}$.

77. Evaluate (a) $\mathcal{L}\{t \cosh 3t\}$, (b) $\mathcal{L}\{t \sinh 2t\}$. *Ans.* (a) $(s^2 + 9)/(s^2 - 9)^2$, (b) $4s/(s^2 - 4)^2$

78. Find (a) $\mathcal{L}\{t^2 \cos t\}$, (b) $\mathcal{L}\{(t^2 - 3t + 2) \sin 3t\}$.

Ans. (a) $(2s^3 - 6s)/(s^2 + 1)^3$, (b) $\dfrac{6s^4 - 18s^3 + 126s^2 - 162s + 432}{(s^2 + 9)^3}$

79. Find $\mathcal{L}\{t^3 \cos t\}$. *Ans.* $\dfrac{6s^4 - 36s^2 + 6}{(s^2 + 1)^4}$

80. Show that $\displaystyle\int_0^\infty t\, e^{-3t} \sin t\, dt = \frac{3}{50}$.

DIVISION BY t

81. Show that $\mathcal{L}\left\{\dfrac{e^{-at} - e^{-bt}}{t}\right\} = \ln\left(\dfrac{s + b}{s + a}\right)$.

82. Show that $\mathcal{L}\left\{\dfrac{\cos at - \cos bt}{t}\right\} = \dfrac{1}{2}\ln\left(\dfrac{s^2 + b^2}{s^2 + a^2}\right)$.

83. Find $\mathcal{L}\left\{\dfrac{\sinh t}{t}\right\}$. *Ans.* $\dfrac{1}{2}\ln\left(\dfrac{s + 1}{s - 1}\right)$

84. Show that $\displaystyle\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt = \ln 2$.

[*Hint.* Use Problem 81.]

85. Evaluate $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$. *Ans.* $\ln(3/2)$

86. Show that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$.

PERIODIC FUNCTIONS

87. Find $\mathcal{L}\{F(t)\}$ where $F(t)$ is the periodic function shown graphically in Fig. 1-7 below.

Ans. $\dfrac{1}{s} \tanh \dfrac{s}{2}$

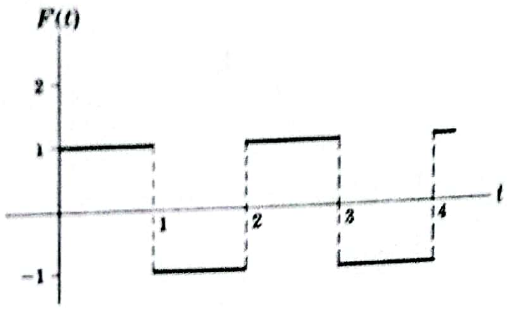

Fig. 1-7

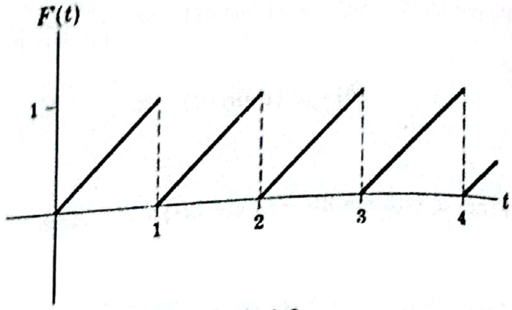

Fig. 1-8

88. Find $\mathcal{L}\{F(t)\}$ where $F(t)$ is the periodic function shown graphically in Fig. 1-8 above.

Ans. $\dfrac{1}{s^2} - \dfrac{e^{-s}}{s(1-e^{-s})}$

89. Let $F(t) = \begin{cases} 3t & 0 < t < 2 \\ 6 & 2 < t < 4 \end{cases}$ where $F(t)$ has period 4. (a) Graph $F(t)$. (b) Find $\mathcal{L}\{F(t)\}$.

Ans. (b) $\dfrac{3 - 3e^{-2s} - 6se^{-4s}}{s^2(1-e^{-4s})}$

90. If $F(t) = t^2$, $0 < t < 2$ and $F(t+2) = F(t)$, find $\mathcal{L}\{F(t)\}$.

Ans. $\dfrac{2 - 2e^{-2s} - 4se^{-2s} - 4s^2e^{-2s}}{s^3(1-e^{-2s})}$

91. Find $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$ and $F(t+2) = F(t)$ for $t > 0$.

Ans. $\dfrac{1 - e^{-s}(s+1)}{s^2(1-e^{-2s})}$

92. (a) Show that the function $F(t)$ whose graph is the triangular wave shown in Fig. 1-9 has the Laplace transform $\dfrac{1}{s^2} \tanh \dfrac{s}{2}$.

(b) How can the result in (a) be obtained from Problem 87? Explain.

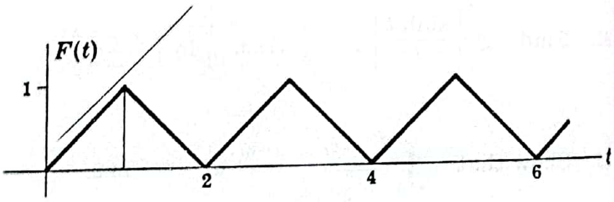

Fig. 1-9

INITIAL AND FINAL-VALUE THEOREMS

93. Verify the initial-value theorem for the functions (a) $3 - 2\cos t$, (b) $(2t+3)^2$, (c) $t + \sin 3t$.

94. Verify the final-value theorem for the functions (a) $1 + e^{-t}(\sin t + \cos t)$, (b) $t^3 e^{-2t}$.

95. Discuss the applicability of the final-value theorem for the function $\cos t$.

96. If $F(t) \sim ct^p$ as $t \to 0$ where $p > -1$, prove that $f(s) \sim c\,\Gamma(p+1)/s^{p+1}$ as $s \to \infty$.

97. If $F(t) \sim ct^p$ as $t \to \infty$ where $p > -1$, prove that $f(s) \sim c\,\Gamma(p+1)/s^{p+1}$ as $s \to \infty$.

THE GAMMA FUNCTION

98. Evaluate (a) $\Gamma(5)$, (b) $\dfrac{\Gamma(3)\,\Gamma(4)}{\Gamma(7)}$, (c) $\Gamma(5/2)$, (d) $\dfrac{\Gamma(3/2)\,\Gamma(4)}{\Gamma(11/2)}$.

    Ans. (a) 24, (b) 1/60, (c) $3\sqrt{\pi}/4$, (d) 32/315

99. Find (a) $\mathcal{L}\{t^{1/2}+t^{-1/2}\}$, (b) $\mathcal{L}\{t^{-1/3}\}$, (c) $\mathcal{L}\{(1+\sqrt{t})^4\}$.

    Ans. (a) $(2s+1)\sqrt{\pi}/2s^{3/2}$, (b) $\Gamma(2/3)/s^{2/3}$, (c) $(s^2+2\sqrt{\pi}\,s^{3/2}+6s+3\sqrt{\pi}\,s^{1/2}+2)/s^3$

100. Find (a) $\mathcal{L}\left\{\dfrac{e^{-2t}}{\sqrt{t}}\right\}$, (b) $\mathcal{L}\{t^{7/2}\,e^{3t}\}$.

    Ans. (a) $\sqrt{\pi/(s+2)}$, (b) $105\sqrt{\pi}/16(s-3)^{9/2}$

BESSEL FUNCTIONS

101. Show that $\mathcal{L}\{e^{-at}J_0(bt)\} = \dfrac{1}{\sqrt{s^2-2as+a^2+b^2}}$.

102. Show that $\mathcal{L}\{t\,J_0(at)\} = \dfrac{s}{(s^2+a^2)^{3/2}}$.

103. Find (a) $\mathcal{L}\{e^{-3t}J_0(4t)\}$, (b) $\mathcal{L}\{t\,J_0(2t)\}$. *Ans.* (a) $\dfrac{1}{\sqrt{s^2+6s+25}}$, (b) $\dfrac{s}{(s^2+4)^{3/2}}$

104. Prove that (a) $J_0'(t) = -J_1(t)$, (b) $\dfrac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t)$.

105. If $I_0(t) = J_0(it)$, show that $\mathcal{L}\{I_0(at)\} = \dfrac{1}{\sqrt{s^2-a^2}}$, $a>0$.

106. Find $\mathcal{L}\{t\,J_0(t)\,e^{-t}\}$. *Ans.* $(s-1)/(s^2-2s+2)^{3/2}$

107. Show that (a) $\displaystyle\int_0^\infty J_0(t)\,dt = 1$, (b) $\displaystyle\int_0^\infty e^{-t}J_0(t)\,dt = \frac{\sqrt{2}}{2}$.

108. Find the Laplace transform of $\dfrac{d^2}{dt^2}\{e^{2t}J_0(2t)\}$. *Ans.* $\dfrac{s^2}{\sqrt{s^2-4s+8}} - s - 2$

109. Show that $\mathcal{L}\{t\,J_1(t)\} = \dfrac{1}{(s^2+1)^{3/2}}$.

110. Prove that $\mathcal{L}\{J_0(a\sqrt{t})\} = \dfrac{e^{-a^2/4s}}{s}$.

111. Evaluate $\displaystyle\int_0^\infty t\,e^{-3t}J_0(4t)\,dt$. *Ans.* 3/125

112. Prove that $\mathcal{L}\{J_n(t)\} = \dfrac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}$ and thus obtain $\mathcal{L}\{J_n(at)\}$.

THE SINE, COSINE AND EXPONENTIAL INTEGRALS

113. Evaluate (a) $\mathcal{L}\{e^{2t}\,\mathrm{Si}(t)\}$, (b) $\mathcal{L}\{t\,\mathrm{Si}(t)\}$.

    Ans. (a) $\tan^{-1}(s-2)/(s-2)$, (b) $\dfrac{\tan^{-1}s}{s^2} - \dfrac{1}{s(s^2+1)}$

114. Show that $\mathcal{L}\{t^2\,\text{Ci}\,(t)\} = \dfrac{\ln(s^2+1)}{s^3} - \dfrac{3s^2+1}{s(s^2+1)^2}$.

115. Find (a) $\mathcal{L}\{e^{-3t}\,\text{Ei}\,(t)\}$, (b) $\mathcal{L}\{t\,\text{Ei}\,(t)\}$.

Ans. (a) $\dfrac{\ln(s+4)}{s+3}$, (b) $\dfrac{\ln(s+1)}{s^2} - \dfrac{1}{s(s+1)}$

116. Find (a) $\mathcal{L}\{e^{-t}\,\text{Si}\,(2t)\}$, (b) $\mathcal{L}\{te^{-2t}\,\text{Ei}\,(3t)\}$.

Ans. (a) $\dfrac{\tan^{-1}(s+1)/2}{s+1}$, (b) $\dfrac{1}{(s+2)^2}\ln\left(\dfrac{s+5}{3}\right) - \dfrac{1}{(s+2)(s+5)}$

THE ERROR FUNCTION

117. Evaluate (a) $\mathcal{L}\{e^{3t}\,\text{erf}\,\sqrt{t}\}$, (b) $\mathcal{L}\{t\,\text{erf}\,(2\sqrt{t})\}$.

Ans. (a) $\dfrac{1}{(s-3)\sqrt{s-2}}$, (b) $\dfrac{3s+8}{s^2(s+4)^{3/2}}$

118. Show that $\mathcal{L}\{\text{erfc}\,\sqrt{t}\} = \dfrac{1}{\sqrt{s+1}\,\{\sqrt{s+1}+1\}}$.

119. Find $\mathcal{L}\left\{\displaystyle\int_0^t \text{erf}\,\sqrt{u}\,du\right\}$. *Ans.* $1/s^2\sqrt{s+1}$

THE UNIT STEP FUNCTION, IMPULSE FUNCTIONS, AND THE DIRAC DELTA FUNCTION

120. (a) Show that in terms of Heaviside's unit step function, the function $F(t) = \begin{cases} e^{-t} & 0 < t < 3 \\ 0 & t > 3 \end{cases}$ can be written as $e^{-t}\{1 - \mathcal{U}(t-3)\}$. (b) Use $\mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}/s$ to find $\mathcal{L}\{F(t)\}$.

Ans. (b) $\dfrac{1-e^{-3(s+1)}}{s+1}$

121. Show that $F(t) = \begin{cases} F_1(t) & 0 < t < a \\ F_2(t) & t > a \end{cases}$ can be written as

$$F(t) = F_1(t) + \{F_2(t) - F_1(t)\}\,\mathcal{U}(t-a)$$

122. If $F(t) = F_1(t)$ for $0 < t < a_1$, $F_2(t)$ for $a_1 < t < a_2$, ..., $F_{n-1}(t)$ for $a_{n-2} < t < a_{n-1}$, and $F_n(t)$ for $t > a_{n-1}$, show that

$$F(t) = F_1(t) + \{F_2(t) - F_1(t)\}\,\mathcal{U}(t-a_1) + \cdots + \{F_n(t) - F_{n-1}(t)\,\mathcal{U}(t-a_{n-1})$$

123. Express in terms of Heaviside's unit step functions.

(a) $F(t) = \begin{cases} t^2 & 0 < t < 2 \\ 4t & t > 2 \end{cases}$ (b) $F(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$

Ans. (a) $t^2 + (4t - t^2)\,\mathcal{U}(t-2)$, (b) $\sin t + (\sin 2t - \sin t)\,\mathcal{U}(t-\pi) + (\sin 3t - \sin 2t)\,\mathcal{U}(t-2\pi)$

124. Show that $\mathcal{L}\{t^2\,\mathcal{U}(t-2)\} = \dfrac{2}{s^3} - \dfrac{2e^{-2s}}{s^3}(1+2s+2s^2)$, $s > 0$.

125. Evaluate (a) $\int_{-\infty}^{\infty} \cos 2t\, \delta(t-\pi/3)\, dt$, (b) $\int_{-\infty}^{\infty} e^{-t}\, \mathcal{U}(t-2)\, dt$. *Ans.* (a) $-1/2$, (b) e^{-2}

126. (a) If $\delta'(t-a)$ denotes the formal derivative of the delta function, show that

$$\int_0^{\infty} F(t)\, \delta'(t-a)\, dt \;=\; -F'(a)$$

(b) Evaluate $\int_0^{\infty} e^{-4t}\, \delta'(t-2)\, dt$.
Ans. (b) $4e^{-8}$

127. Let $G_\epsilon(t) = 1/\epsilon$ for $0 \leqq t < \epsilon$, 0 for $\epsilon \leqq t < 2\epsilon$, $-1/\epsilon$ for $2\epsilon \leqq t < 3\epsilon$, and 0 for $t \geqq 3\epsilon$.
(a) Find $\mathcal{L}\{G_\epsilon(t)\}$. (b) Find $\lim_{\epsilon \to 0} \mathcal{L}\{G_\epsilon(t)\}$. (c) Is $\lim_{\epsilon \to 0} \mathcal{L}\{G_\epsilon(t)\} = \mathcal{L}\left\{\lim_{\epsilon \to 0} G_\epsilon(t)\right\}$? (d) Discuss geometrically the results of (a) and (b).

128. Generalize Problem 127 by defining a function $G_\epsilon(t)$ in terms of ϵ and n so that $\lim_{\epsilon \to 0} G_\epsilon(t) = s^n$ where $n = 2, 3, 4, \ldots$.

EVALUATION OF INTEGRALS

129. Evaluate $\int_0^{\infty} t^3\, e^{-t} \sin t\, dt$. *Ans.* 0

130. Show that $\int_0^{\infty} \frac{e^{-t} \sin t}{t}\, dt = \frac{\pi}{4}$.

131. Prove that (a) $\int_0^{\infty} J_n(t)\, dt = 1$, (b) $\int_0^{\infty} t\, J_n(t)\, dt = 1$.

132. Prove that $\int_0^{\infty} u\, e^{-u^2} J_0(au)\, du = \frac{1}{2} e^{-a^2/4}$.

133. Show that $\int_0^{\infty} t\, e^{-t}\, \mathrm{Ei}\,(t)\, dt = \ln 2 - \frac{1}{2}$.

134. Show that $\int_0^{\infty} u\, e^{-u^2}\, \mathrm{erf}\, u\, du = \frac{\sqrt{2}}{4}$.

MISCELLANEOUS PROBLEMS

135. If $F(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$, show that $\mathcal{L}\{F(t)\} = \frac{1 + e^{-\pi s}}{s^2 + 1}$.

136. If $F(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$, find $\mathcal{L}\{F(t)\}$. *Ans.* $\frac{s + (s-1)e^{-\pi s}}{s^2 + 1}$

137. Show that $\mathcal{L}\{\sin^3 t\} = \frac{6}{(s^2+1)(s^2+9)}$.

138. Establish entries (a) 16, (b) 17, (c) 20, (d) 28 in the Table of Page 246.

139. Find (a) $\mathcal{L}\{\sinh^3 2t\}$, (b) $\mathcal{L}\{t^3 \cos 4t\}$.

Ans. (a) $\dfrac{48}{(s^2-36)(s^2-4)}$, (b) $\dfrac{6s^4 - 576s^2 + 1536}{(s^2+16)^4}$

140. If $F(t) = 5 \sin 3(t - \pi/4)$ for $t > \pi/4$ and 0 for $t < \pi/4$, find $\mathcal{L}\{F(t)\}$. *Ans.* $e^{-\pi s/4}/(s^2+9)$

141. If $\mathcal{L}\{t\,F(t)\} = \dfrac{1}{s(s^2+1)}$, find $\mathcal{L}\{e^{-t}F(2t)\}$.

142. Find (a) $\mathcal{L}\{\sinh 2t \cos 2t\}$, (b) $\mathcal{L}\{\cosh 2t \cos 2t\}$.

Ans. (a) $2(s^2-8)/(s^4+64)$, (b) $s^3/(s^4+64)$

143. Let $F(t) = \begin{cases} t+n & 2n \leqq t < 2n+1 \\ n-t & 2n+1 \leqq t < 2n+2 \end{cases}$, $n = 0, 1, 2, \ldots$. Show that

$$\mathcal{L}\{F(t)\} = \frac{1}{s^2}\sum_{n=0}^{\infty}\{(3ns+1)\,e^{-2ns} - 2[(2n+1)s+1]\,e^{-(2n+1)s} + [(n+2)s+1]\,e^{-(2n+2)s}\}$$

144. (a) Show that $\mathcal{L}\{\sin^5 t\} = \dfrac{120}{(s^2+1)(s^2+9)(s^2+25)}$.

(b) Using the results of part (a) and Problem 137, can you arrive at a corresponding result for $\mathcal{L}\{\sin^{2n-1} t\}$ where n is any positive integer? Justify your conjectures.

145. Suppose that $F(t)$ is unbounded as $t \to 0$. Prove that $\mathcal{L}\{F(t)\}$ exists if the following conditions are satisfied:

(a) $F(t)$ is sectionally continuous in any interval $N_1 \leqq t \leqq N$ where $N_1 > 0$,

(b) $\lim\limits_{t\to 0} t^n F(t) = 0$ for some constant n such that $0 < n < 1$,

(c) $F(t)$ is of exponential order γ for $t > N$.

146. Show that, (a) $\mathcal{L}\{J_0(t)\sin t\} = \dfrac{1}{\sqrt{s}\,\sqrt[4]{s^2+4}} \sin\{\tfrac{1}{2}\tan^{-1}(2/s)\}$

(b) $\mathcal{L}\{J_0(t)\cos t\} = \dfrac{1}{\sqrt{s}\,\sqrt[4]{s^2+4}} \cos\{\tfrac{1}{2}\tan^{-1}(2/s)\}$

147. Let $F(t) = \begin{cases} t\,G(t) & t > 1 \\ 0 & 0 < t < 1 \end{cases}$. Prove that $\mathcal{L}\{F(t)\} = -\dfrac{d}{ds}[e^{-s}\,\mathcal{L}\{G(t+1)\}]$.

148. If $\mathcal{L}\{F''(t)\} = \tan^{-1}(1/s)$, $F(0) = 2$ and $F'(0) = -1$, find $\mathcal{L}\{F(t)\}$.

Ans. $\dfrac{2s - 1 + \tan^{-1} 1/s}{s^2}$

149. Prove that $\mathcal{L}\{e^{\alpha t}F(\beta t)\} = \dfrac{1}{\beta} f\left(\dfrac{s-\alpha}{\beta}\right)$ where α and β are constants and $\mathcal{L}\{F(t)\} = f(s)$.

150. Show that the Laplace transform of e^{e^t} does not exist, while the Laplace transform of e^{-e^t} does exist.

138. Establish entries (a) 16, (b) 17, (c) 20, (d) 28 in the Table of Page 246.

139. Find (a) $\mathcal{L}\{\sinh^3 2t\}$, (b) $\mathcal{L}\{t^3 \cos 4t\}$.

Ans. (a) $\dfrac{48}{(s^2-36)(s^2-4)}$, (b) $\dfrac{6s^4 - 576s^2 + 1536}{(s^2+16)^4}$

140. If $F(t) = 5 \sin 3(t - \pi/4)$ for $t > \pi/4$ and 0 for $t < \pi/4$, find $\mathcal{L}\{F(t)\}$. *Ans.* $e^{-\pi s/4}/(s^2+9)$

141. If $\mathcal{L}\{t\,F(t)\} = \dfrac{1}{s(s^2+1)}$, find $\mathcal{L}\{e^{-t} F(2t)\}$.

142. Find (a) $\mathcal{L}\{\sinh 2t \cos 2t\}$, (b) $\mathcal{L}\{\cosh 2t \cos 2t\}$.
Ans. (a) $2(s^2-8)/(s^4+64)$, (b) $s^3/(s^4+64)$

143. Let $F(t) = \begin{cases} t+n & 2n \leqq t < 2n+1 \\ n-t & 2n+1 \leqq t < 2n+2 \end{cases}$, $n = 0, 1, 2, \ldots$. Show that

$$\mathcal{L}\{F(t)\} = \frac{1}{s^2}\sum_{n=0}^{\infty} \{(3ns+1)\,e^{-2ns} - 2[(2n+1)s+1]\,e^{-(2n+1)s} + [(n+2)s+1]\,e^{-(2n+2)s}\}$$

144. (a) Show that $\mathcal{L}\{\sin^5 t\} = \dfrac{120}{(s^2+1)(s^2+9)(s^2+25)}$.

(b) Using the results of part (a) and Problem 137, can you arrive at a corresponding result for $\mathcal{L}\{\sin^{2n-1} t\}$ where n is any positive integer? Justify your conjectures.

145. Suppose that $F(t)$ is unbounded as $t \to 0$. Prove that $\mathcal{L}\{F(t)\}$ exists if the following conditions are satisfied:

(a) $F(t)$ is sectionally continuous in any interval $N_1 \leqq t \leqq N$ where $N_1 > 0$,

(b) $\lim_{t\to 0} t^n F(t) = 0$ for some constant n such that $0 < n < 1$,

(c) $F(t)$ is of exponential order γ for $t > N$.

146. Show that, (a) $\mathcal{L}\{J_0(t) \sin t\} = \dfrac{1}{\sqrt{s}\,\sqrt[4]{s^2+4}} \sin\{\tfrac{1}{2}\tan^{-1}(2/s)\}$

(b) $\mathcal{L}\{J_0(t) \cos t\} = \dfrac{1}{\sqrt{s}\,\sqrt[4]{s^2+4}} \cos\{\tfrac{1}{2}\tan^{-1}(2/s)\}$

147. Let $F(t) = \begin{cases} t\,G(t) & t > 1 \\ 0 & 0 < t < 1 \end{cases}$. Prove that $\mathcal{L}\{F(t)\} = -\dfrac{d}{ds}[e^{-s}\mathcal{L}\{G(t+1)\}]$.

148. If $\mathcal{L}\{F''(t)\} = \tan^{-1}(1/s)$, $F(0) = 2$ and $F'(0) = -1$, find $\mathcal{L}\{F(t)\}$.

Ans. $\dfrac{2s - 1 + \tan^{-1} 1/s}{s^2}$

149. Prove that $\mathcal{L}\{e^{\alpha t} F(\beta t)\} = \dfrac{1}{\beta} f\left(\dfrac{s-\alpha}{\beta}\right)$ where α and β are constants and $\mathcal{L}\{F(t)\} = f(s)$.

150. Show that the Laplace transform of e^{e^t} does not exist, while the Laplace transform of e^{-e^t} does exist.

151. (a) Show that $\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$.

(b) Evaluate $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt$.

Ans. (b) $\frac{1}{4}\ln 5$

152. (a) Find $\mathcal{L}\left\{\frac{1-J_0(t)}{t}\right\}$. (b) Show that $\int_0^\infty \frac{e^{-t}\{1-J_0(t)\}}{t}\,dt = \ln\left(\frac{\sqrt{2}+1}{2}\right)$.

153. Work Problems 37 and 38 by using Method 4 of Problem 36.

154. Suppose that $\mathcal{L}\{F(t)\}$ exists for $s=a$ where a is real. Prove that it also exists for all $s>a$.

155. Find the Laplace transform of the periodic function $F(t)$ shown graphically in Fig. 1-10.

Ans. $\frac{1-e^{-as}-as\,e^{-as}}{s^2(1-e^{-as})}\tan\theta_0$

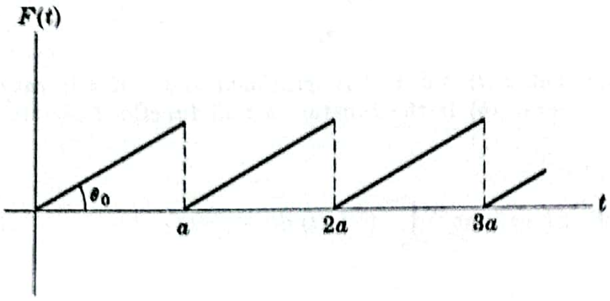

Fig. 1-10

156. Prove that

$$\mathcal{L}\{\sin t^2\} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n-2)!}{(2n-1)!\,s^{2n+1}}$$

157. Show that $\mathcal{L}\{\sin^6 t\} = \frac{6!}{s(s^2+4)(s^2+16)(s^2+36)}$ and generalize [see Prob. 144].

158. Find $\mathcal{L}\{t\,e^{-2t}J_0(t\sqrt{2})\}$. *Ans.* $\frac{s+2}{(s^2+4s+6)^{3/2}}$

159. Find $\mathcal{L}\{t\,\mathcal{U}(t-1) + t^2\,\delta(t-1)\}$. *Ans.* $e^{-s}(s^2+s+1)/s^2$

160. Find $\mathcal{L}\{\cos t \ln t\ \delta(t-\pi)\}$. *Ans.* $-e^{-\pi s}\ln\pi$

161. Let $F(t)$ and $G(t)$ be sectionally continuous in every finite interval and of exponential order as $t\to\infty$. Prove that $\mathcal{L}\{F(t)\,G(t)\}$ exists.

162. The *Laguerre polynomials* $L_n(t)$ are defined by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\{t^n e^{-t}\} \qquad n = 0, 1, 2, \ldots$$

(a) Find $L_0(t), L_1(t), \ldots, L_4(t)$. (b) Find $\mathcal{L}\{L_n(t)\}$.

163. (a) Let a, b, α, β and Λ be constants. Prove that

$$\mathcal{L}\{at^{-\alpha}+bt^{-\beta}\} = \Lambda\{as^{-\alpha}+bs^{-\beta}\}$$

if and only if $\alpha+\beta = 1$ and $\Lambda = \pm\sqrt{\pi \csc \alpha\pi}$.

(b) A function $F(t)$ is said to be its own Laplace transform if $\mathcal{L}\{F(t)\} = F(s)$. Can the function $F(t) = at^{-\alpha}+bt^{-\beta}$ be its own Laplace transform? Explain.

164. If $F(t)$ and $G(t)$ have Laplace transforms, is it true that $F(t)\,G(t)$ also has a Laplace transform? Justify your conclusion.

165. Use the result $J_0(t) = \frac{1}{\pi}\int_0^{\pi} \cos(t \sin\theta)\,d\theta$ to show that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$.

166. Prove that Leibnitz's rule can be applied in Problem 19, stating suitable restrictions on $F(t)$.

167. (a) Prove that $\displaystyle\int_0^\infty e^{-st}\left(\frac{1-\cos t}{t^2}\right)dt = \frac{\pi}{2} - s\ln\left(\frac{s^2}{s^2+1}\right) + 2\tan^{-1}s$.

(b) Prove that $\displaystyle\int_0^\infty \frac{1-\cos t}{t^2}\,dt = \frac{\pi}{2}$.

168. Let $F(t) = 0$ if t is irrational and 1 if t is rational. (a) Prove that $\mathcal{L}\{F(t)\}$ exists and is equal to zero. (b) Is the function a null function? Explain.

169. Show that $\displaystyle\int_0^\infty t^2 J_0(t)\,dt = -1$.

170. Prove that if p is any positive integer,

$$\Gamma(-p-\tfrac{1}{2}) = (-1)^{p+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\cdots\left(\frac{2}{2p+1}\right)\sqrt{\pi}$$

171. Verify the entries (a) 55, (b) 61, (c) 64, (d) 65, (e) 81 in the Table of Appendix B, Pages 248 and 250.

172. Using the binomial theorem show that for $|x| < 1$,

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot3}{2\cdot4}x^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3 + \cdots$$

and thus verify the summation of the infinite series in Problems 34 and 39.

173. Use infinite series to find the Laplace transforms of (a) $\sin t$, (b) $\cos t$, (c) e^{at}, (d) $\cos\sqrt{t}$.

174. Prove that $\mathcal{L}\{\operatorname{erf}(t)\} = \frac{e^{s^2/4}}{s}\operatorname{erfc}(s/2)$ and thus find $\mathcal{L}\{\operatorname{erf}(at)\}$.

175. (a) Find $\mathcal{L}\{\operatorname{erf}\sqrt{t}\}$ by using the method of differential equations.

(b) Find $\mathcal{L}\{\cos\sqrt{t}/\sqrt{t}\}$ using infinite series.

176. Show that (a) $\displaystyle\int_0^\infty J_0(2\sqrt{tu})\cos u\,du = \sin t$,

(b) $\displaystyle\int_0^\infty J_0(2\sqrt{tu})\sin u\,du = \cos t$.

177. Show that $\int_0^\infty J_0(2\sqrt{tu})\, J_0(u)\, du = J_0(t)$.

178. Use (a) infinite series, (b) differential equations to find $\mathcal{L}\{J_1(t)\}$. See Problem 35.

179. If $s > 0$ and $n > 1$, prove that

$$\mathcal{L}\left\{\frac{t^{n-1}}{1-e^{-t}}\right\} = \Gamma(n)\left\{\frac{1}{s^n} + \frac{1}{(s+1)^n} + \frac{1}{(s+2)^n} + \cdots\right\}$$

180. Prove that if $n > 1$,

$$\zeta(n) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{t^{n-1}}{e^t - 1}\, dt = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots$$

The function $\zeta(n)$ is called the *Riemann zeta function.*

181. If $f(s) = \mathcal{L}\{F(t)\}$, show that

$$\mathcal{L}\left\{\int_0^\infty \frac{t^u F(u)}{\Gamma(u+1)}\, du\right\} = \frac{f(\ln s)}{s \ln s}$$

182. If $L_n(t)$, $n = 0, 1, 2, \ldots$, are the Laguerre polynomials [see Problem 162], prove that

$$\sum_{n=0}^{\infty} \frac{L_n(t)}{n!} = e\, J_0(2\sqrt{t})$$

183. Let $J(a,t) = \int_0^\infty e^{-u^2 t} \cos au\, du$. (a) Show that $\frac{\partial J}{\partial a} = -\frac{a}{2t} J$ where $J(0,t) = \sqrt{\pi}/2\sqrt{t}$. (b) By solving the differential equation in (a) show that

$$J(a,t) = \int_0^\infty e^{-u^2 t} \cos au\, du = \frac{\sqrt{\pi}}{2\sqrt{t}}\, e^{-a^2/4t}$$

184. Use Problem 183 to find $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$ [see Problem 49, Page 29].

185. Prove that $\int_0^\infty \frac{e^{-\sqrt{2}t} \sinh t \sin t}{t}\, dt = \frac{\pi}{8}$.

Chapter 2

The Inverse Laplace Transform

DEFINITION OF INVERSE LAPLACE TRANSFORM

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e. if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an *inverse Laplace transform* of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the *inverse Laplace transformation operator*.

Example. Since $\mathcal{L}\{e^{-3t}\} = \dfrac{1}{s+3}$ we can write

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

UNIQUENESS OF INVERSE LAPLACE TRANSFORMS. LERCH'S THEOREM

Since the Laplace transform of a null function $\mathcal{N}(t)$ is zero [see Chapter 1, Page 9], it is clear that if $\mathcal{L}\{F(t)\} = f(s)$ then also $\mathcal{L}\{F(t)+\mathcal{N}(t)\} = f(s)$. From this it follows that we can have two different functions with the same Laplace transform.

Example. The two different functions $F_1(t) = e^{-3t}$ and $F_2(t) = \begin{cases} 0 & t = 1 \\ e^{-3t} & \text{otherwise} \end{cases}$ have the same Laplace transform, i.e. $1/(s+3)$.

If we allow null functions, we see that the inverse Laplace transform is not unique. It is unique, however, if we disallow null functions [which do not in general arise in cases of physical interest]. This result is indicated in

Theorem 2-1. *Lerch's theorem.* If we restrict ourselves to functions $F(t)$ which are sectionally continuous in every finite interval $0 \leqq t \leqq N$ and of exponential order for $t > N$, then the inverse Laplace transform of $f(s)$, i.e. $\mathcal{L}^{-1}\{f(s)\} = F(t)$, is unique. We shall always assume such uniqueness unless otherwise stated.

SOME INVERSE LAPLACE TRANSFORMS

The following results follow at once from corresponding entries on Page 1.

Table of Inverse Laplace Transforms

	$f(s)$	$\mathcal{L}^{-1}\{f(s)\} = F(t)$
1.	$\dfrac{1}{s}$	1
2.	$\dfrac{1}{s^2}$	t
3.	$\dfrac{1}{s^{n+1}} \quad n = 0, 1, 2, \ldots$	$\dfrac{t^n}{n!}$
4.	$\dfrac{1}{s-a}$	e^{at}
5.	$\dfrac{1}{s^2+a^2}$	$\dfrac{\sin at}{a}$
6.	$\dfrac{s}{s^2+a^2}$	$\cos at$
7.	$\dfrac{1}{s^2-a^2}$	$\dfrac{\sinh at}{a}$
8.	$\dfrac{s}{s^2-a^2}$	$\cosh at$

SOME IMPORTANT PROPERTIES OF INVERSE LAPLACE TRANSFORMS

In the following list we have indicated various important properties of inverse Laplace transforms. Note the analogy of Properties 1-8 with the corresponding properties on Pages 3-5.

1. Linearity property.

Theorem 2-2. If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$\begin{aligned}\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} &= c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\} \\ &= c_1 F_1(t) + c_2 F_2(t)\end{aligned} \tag{1}$$

The result is easily extended to more than two functions.

Example.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} &= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} \\ &\quad + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t\end{aligned}$$

Because of this property we can say that $\mathcal{L}^{-1}$ is a *linear operator* or that it has the *linearity property*.

2. First translation or shifting property.

Theorem 2-3. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t) \tag{2}$$

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}e^t \sin 2t$$

3. Second translation or shifting property.

Theorem 2-4. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases} \tag{3}$$

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/3}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi/3) & \text{if } t>\pi/3 \\ 0 & \text{if } t<\pi/3 \end{cases}$$

4. Change of scale property.

Theorem 2-5. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F\left(\frac{t}{k}\right) \tag{4}$$

Example. Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$, we have

$$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\frac{4t}{2} = \frac{1}{2}\cos 2t$$

as is verified directly.

5. Inverse Laplace transform of derivatives.

Theorem 2-6. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f^{(n)}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\} = (-1)^n t^n F(t) \tag{5}$$

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{-2s}{(s^2+1)^2}$, we have

$$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -t\sin t \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$$

6. Inverse Laplace transform of integrals.

Theorem 2-7. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t} \tag{6}$$

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+1}\right\} = 1-e^{-t}$, we have

$$\mathcal{L}^{-1}\left\{\int_s^\infty\left(\frac{1}{u}-\frac{1}{u+1}\right)du\right\} = \mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\} = \frac{1-e^{-t}}{t}$$

7. Multiplication by s^n.

Theorem 2-8. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then

$$\mathcal{L}^{-1}\{s\,f(s)\} = F'(t) \tag{7}$$

Thus multiplication by s has the effect of *differentiating* $F(t)$.

If $F(0) \neq 0$, then

$$\mathcal{L}^{-1}\{s\,f(s) - F(0)\} = F'(t) \tag{8}$$

or

$$\mathcal{L}^{-1}\{s\,f(s)\} = F'(t) + F(0)\,\delta(t) \tag{9}$$

where $\delta(t)$ is the Dirac delta function or unit impulse function [see Page 9].

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\sin 0 = 0$, then

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{d}{dt}(\sin t) = \cos t$$

Generalizations to $\mathcal{L}^{-1}\{s^n f(s)\}$, $n = 2, 3, \ldots$, are possible.

8. Division by s.

Theorem 2-9. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du \tag{10}$$

Thus division by s (or multiplication by $1/s$) has the effect of *integrating* $F(t)$ from 0 to t.

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,du = \frac{1}{4}(1-\cos 2t)$$

Generalizations to $\mathcal{L}^{-1}\{f(s)/s^n\}$, $n = 2, 3, \ldots$, are possible [see Problem 70].

9. The Convolution property.

Theorem 2-10. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du = F * G \tag{11}$$

We call $F*G$ the *convolution* or *faltung* of F and G, and the theorem is called the *convolution theorem* or *property*.

From Problem 21, we see that $F*G = G*F$.

Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \int_0^t e^u\,e^{2(t-u)}\,du = e^{2t} - e^t$$

METHODS OF FINDING INVERSE LAPLACE TRANSFORMS

Various means are available for determining inverse Laplace transforms, as indicated in the following list. Compare with Page 6.

1. **Partial fractions method.** Any rational function $P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials, with the degree of $P(s)$ less than that of $Q(s)$, can be written as the sum of rational functions [called *partial fractions*] having the form $\dfrac{A}{(as+b)^r}$, $\dfrac{As+B}{(as^2+bs+c)^r}$ where $r = 1,2,3,\ldots$. By finding the inverse Laplace transform of each of the partial fractions, we can find $\mathcal{L}^{-1}\{P(s)/Q(s)\}$.

   Example 1. $$\frac{2s-5}{(3s-4)(2s+1)^3} = \frac{A}{3s-4} + \frac{B}{(2s+1)^3} + \frac{C}{(2s+1)^2} + \frac{D}{2s+1}$$

   Example 2. $$\frac{3s^2-4s+2}{(s^2+2s+4)^2(s-5)} = \frac{As+B}{(s^2+2s+4)^2} + \frac{Cs+D}{s^2+2s+4} + \frac{E}{s-5}$$

   The constants A, B, C, etc., can be found by clearing of fractions and equating of like powers of s on both sides of the resulting equation or by using special methods [see Problems 24-28]. A method related to this uses the *Heaviside expansion formula* [see below].

2. **Series methods.** If $f(s)$ has a series expansion in inverse powers of s given by

   $$f(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \cdots \tag{12}$$

   then under suitable conditions we can invert term by term to obtain

   $$F(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \cdots \tag{13}$$

   See Problem 40. Series expansions other than those of the form (12) can sometimes be used. See Problem 41.

3. **Method of differential equations.** See Problem 41.

4. **Differentiation with respect to a parameter.** See Problems 13 and 38.

5. **Miscellaneous methods using the above theorems.**

6. **Use of Tables** (see Appendix B).

7. **The Complex Inversion formula.** This formula, which supplies a powerful direct method for finding inverse Laplace transforms, uses complex variable theory and is considered in Chapter 6.

THE HEAVISIDE EXPANSION FORMULA

Let $P(s)$ and $Q(s)$ be polynomials where $P(s)$ has degree less than that of $Q(s)$. Suppose that $Q(s)$ has n distinct zeros α_k, $k = 1,2,3,\ldots,n$. Then

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t} \tag{14}$$

This is often called *Heaviside's expansion theorem* or *formula*. See Problems 29-31.

The formula can be extended to other cases [see Problems 105 and 111].

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THE BETA FUNCTION

If $m > 0$, $n > 0$, we define the *beta function* as

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\,du \tag{15}$$

We can show the following properties [see Problems 32 and 33]:

1. $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

2. $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$

EVALUATION OF INTEGRALS

The Laplace transformation is often useful in evaluating definite integrals. See, for example, Problems 35-37.

Solved Problems

INVERSE LAPLACE TRANSFORMS

1. Prove that (a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$, $n = 0,1,2,3,\ldots$, where $0! = 1$,

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{\sin at}{a}$, (d) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$, (e) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{\sinh at}{a}$,

(f) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$.

(a) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$. Then $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$.

(b) $\mathcal{L}\left\{\dfrac{t^n}{n!}\right\} = \dfrac{1}{n!}\mathcal{L}\{t^n\} = \dfrac{1}{n!}\left(\dfrac{n!}{s^{n+1}}\right) = \dfrac{1}{s^{n+1}}$. Then $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$ for $n = 0,1,2,3,\ldots$.

(c) $\mathcal{L}\left\{\dfrac{\sin at}{a}\right\} = \dfrac{1}{a}\mathcal{L}\{\sin at\} = \dfrac{1}{a}\cdot\dfrac{a}{s^2+a^2} = \dfrac{1}{s^2+a^2}$. Then $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{\sin at}{a}$.

(d) $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$. Then $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$.

(e) $\mathcal{L}\left\{\dfrac{\sinh at}{a}\right\} = \dfrac{1}{a}\mathcal{L}\{\sinh at\} = \dfrac{1}{a}\cdot\dfrac{a}{s^2-a^2} = \dfrac{1}{s^2-a^2}$. Then $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{\sinh at}{a}$.

(f) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$. Then $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$.

2. Prove that $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{\Gamma(n+1)}$ for $n > -1$.

$$\mathcal{L}\left\{\frac{t^n}{\Gamma(n+1)}\right\} = \frac{1}{\Gamma(n+1)}\cdot\mathcal{L}\{t^n\} = \frac{1}{\Gamma(n+1)}\cdot\frac{\Gamma(n+1)}{s^{n+1}} = \frac{1}{s^{n+1}}, \quad n > -1$$

by Problem 31, Page 32.

Then $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{\Gamma(n+1)}$, $n > -1$. Note that if $n = 0, 1, 2, 3, \ldots$, then $\Gamma(n+1) = n!$ and the result is equivalent to that of Problem 1(b).

3. Find each of the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\}$ (e) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-16}\right\}$ (g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{3/2}}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4}{s-2}\right\}$ (d) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2}\right\}$ (f) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-3}\right\}$

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \dfrac{\sin 3t}{3}$ [Problem 1(c)]

(b) $\mathcal{L}^{-1}\left\{\dfrac{4}{s-2}\right\} = 4e^{2t}$ [Problem 1(a)]

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{t^3}{3!} = \dfrac{t^3}{6}$ [Problems 1(b) or 2]

(d) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2}\right\} = \cos\sqrt{2}\,t$ [Problem 1(d)]

(e) $\mathcal{L}^{-1}\left\{\dfrac{6s}{s^2-16}\right\} = 6\cosh 4t$ [Problem 1(f)]

(f) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-3}\right\} = \dfrac{\sinh\sqrt{3}\,t}{\sqrt{3}}$ [Problem 1(e)]

(g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{3/2}}\right\} = \dfrac{t^{1/2}}{\Gamma(3/2)} = \dfrac{t^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \dfrac{2t^{1/2}}{\sqrt{\pi}} = 2\sqrt{\dfrac{t}{\pi}}$ [Problem 2]

LINEARITY, TRANSLATION AND CHANGE OF SCALE PROPERTIES

4. Prove the *linearity property* for the inverse Laplace transformation [*Theorem* 2-2, Page 43].

By Problem 5, Page 12, we have

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} = c_1f_1(s) + c_2f_2(s)$$

Then
$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 F_1(t) + c_2 F_2(t) = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$

The result is easily generalized [see Problem 52].

5. Find (a) $\mathcal{L}^{-1}\left\{\dfrac{5s+4}{s^3} - \dfrac{2s-18}{s^2+9} + \dfrac{24-30\sqrt{s}}{s^4}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{6}{2s-3} - \dfrac{3+4s}{9s^2-16} + \dfrac{8-6s}{16s^2+9}\right\}$.

(a) $$\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+9} + \frac{18}{s^2+9} + \frac{24}{s^4} - \frac{30}{s^{7/2}}\right\}$$
$$= 5t + 4(t^2/2!) - 2\cos 3t + 18(\tfrac{1}{3}\sin 3t) + 24(t^3/3!) - 30\{t^{5/2}/\Gamma(7/2)\}$$
$$= 5t + 2t^2 - 2\cos 3t + 6\sin 3t + 4t^3 - 16t^{5/2}/\sqrt{\pi}$$

since $\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{15}{8}\sqrt{\pi}$.

(b) $$\mathcal{L}^{-1}\left\{\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{3}{s-3/2} - \frac{1}{3}\left(\frac{1}{s^2-16/9}\right) - \frac{4}{9}\left(\frac{s}{s^2-16/9}\right) + \frac{1}{2}\left(\frac{1}{s^2+9/16}\right) - \frac{3}{8}\left(\frac{s}{s^2+9/16}\right)\right\}$$
$$= 3e^{3t/2} - \tfrac{1}{4}\sinh 4t/3 - \tfrac{4}{9}\cosh 4t/3 + \tfrac{2}{3}\sin 3t/4 - \tfrac{3}{8}\cos 3t/4$$

6. Prove the *first translation or shifting property*: If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then
$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

By Problem 7, Page 13, we have $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$. Then
$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

Another method. Since $f(s) = \int_0^\infty e^{-st}F(t)\,dt$, we have
$$f(s-a) = \int_0^\infty e^{-(s-a)t}F(t)\,dt = \int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt = \mathcal{L}\{e^{at}F(t)\}$$

Then
$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

7. Find each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s+12}{s^2+8s+16}\right\}$ (d) $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{2s+3}}\right\}$

(a) $$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$
$$= 6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$$
$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t = 2e^{2t}(3\cos 4t + \sin 4t)$$

(b)
$$\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$$
$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$
$$= 4\,e^{-4t} - 4t\,e^{-4t} = 4\,e^{-4t}(1-t)$$

(c)
$$\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$
$$= 3\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2-4}\right\}$$
$$= 3\,e^t\cosh 2t + 5\,e^t\sinh 2t = e^t(3\cosh 2t + 5\sinh 2t)$$
$$= 4\,e^{3t} - e^{-t}$$

For another method, see Problem 24.

(d)
$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\} = \frac{1}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{1}{(s+3/2)^{1/2}}\right\}$$
$$= \frac{1}{\sqrt{2}}\,e^{-3t/2}\,\frac{t^{-1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{2\pi}}\,t^{-1/2}\,e^{-3t/2}$$

8. Prove the *second translation or shifting property:*

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{e^{-as}f(s)\} = G(t)$ where
$$G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

Method 1. By Problem 9, Page 14, we have $\mathcal{L}\{G(t)\} = e^{-as}f(s)$. Then
$$\mathcal{L}^{-1}\{e^{-as}f(s)\} = G(t)$$

Method 2. Since $f(s) = \int_0^\infty e^{-st}F(t)\,dt$, we have
$$e^{-as}f(s) = \int_0^\infty e^{-as}e^{-st}F(t)\,dt = \int_0^\infty e^{-s(t+a)}F(t)\,dt$$
$$= \int_a^\infty e^{-su}F(u-a)\,du \qquad [\text{letting } t+a=u]$$
$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}F(t-a)\,dt$$
$$= \int_0^\infty e^{-st}G(t)\,dt$$

from which the required result follows.

It should be noted that we can write $G(t)$ in terms of the Heaviside unit step function as $F(t-a)\,\mathcal{U}(t-a)$.

9. Find each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{e^{-5s}}{(s-2)^4}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{se^{-4\pi s/5}}{s^2+25}\right\}$, (c) $\mathcal{L}^{-1}\left\{\dfrac{(s+1)e^{-\pi s}}{s^2+s+1}\right\}$, (d) $\mathcal{L}^{-1}\left\{\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right\}$.

(a) Since $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{t^3e^{2t}}{3!} = \dfrac{1}{6}t^3e^{2t}$, we have by Problem 8,

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases}\frac{1}{6}(t-5)^3e^{2(t-5)} & t>5\\ 0 & t<5\end{cases}$$

$$= \tfrac{1}{6}(t-5)^3e^{2(t-5)}\,\mathcal{U}(t-5)$$

(b) Since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+25}\right\} = \cos 5t$,

$$\mathcal{L}^{-1}\left\{\frac{se^{-4\pi s/5}}{s^2+25}\right\} = \begin{cases}\cos 5(t-4\pi/5) & t>4\pi/5\\ 0 & t<4\pi/5\end{cases}$$

$$= \begin{cases}\cos 5t & t>4\pi/5\\ 0 & t<4\pi/5\end{cases}$$

$$= \cos 5t\;\mathcal{U}(t-4\pi/5)$$

(c) We have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\} + \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}/2}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}\,t}{2} + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}\,t}{2}$$

$$= \frac{e^{-\frac{1}{2}t}}{\sqrt{3}}\left(\sqrt{3}\cos\frac{\sqrt{3}\,t}{2} + \sin\frac{\sqrt{3}\,t}{2}\right)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+s+1}\right\}$$

$$= \begin{cases}\dfrac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}}{2}(t-\pi) + \sin\dfrac{\sqrt{3}}{2}(t-\pi)\right\} & t>\pi\\ 0 & t<\pi\end{cases}$$

$$= \frac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}}\left\{\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi) + \sin\frac{\sqrt{3}}{2}(t-\pi)\right\}\mathcal{U}(t-\pi)$$

(d) We have $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+4)^{5/2}}\right\} = e^{-4t}\mathcal{L}^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$

$$= e^{-4t}\frac{t^{3/2}}{\Gamma(5/2)} = \frac{4t^{3/2}e^{-4t}}{3\sqrt{\pi}}$$

Thus
$$\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^4\,\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\}$$
$$= \begin{cases} \dfrac{4e^4(t-3)^{3/2}e^{-4(t-3)}}{3\sqrt{\pi}} & t > 3 \\ 0 & t < 3 \end{cases}$$
$$= \begin{cases} \dfrac{4(t-3)^{3/2}e^{-4(t-4)}}{3\sqrt{\pi}} & t > 3 \\ 0 & t < 3 \end{cases}$$
$$= \frac{4(t-3)^{3/2}e^{-4(t-4)}}{3\sqrt{\pi}}\,\mathcal{U}(t-3)$$

10. Prove the *change of scale property*: If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then
$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F(t/k)$$

Method 1. By Problem 11, Page 14, we have on replacing a by $1/k$, $\mathcal{L}\{F(t/k)\} = k\,f(ks)$. Then
$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F(t/k)$$

Method 2. Since $f(s) = \int_0^\infty e^{-st}F(t)\,dt$, we have
$$f(ks) = \int_0^\infty e^{-kst}F(t)\,dt = \int_0^\infty e^{-su}F(u/k)\,d(u/k) \qquad [\text{letting } u = kt]$$
$$= \frac{1}{k}\int_0^\infty e^{-su}F(u/k)\,du = \frac{1}{k}\mathcal{L}\{F(t/k)\}$$

Then $\mathcal{L}^{-1}\{f(ks)\} = \dfrac{1}{k}F(t/k)$.

11. If $\mathcal{L}^{-1}\left\{\dfrac{e^{-1/s}}{s^{1/2}}\right\} = \dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find $\mathcal{L}^{-1}\left\{\dfrac{e^{-a/s}}{s^{1/2}}\right\}$ where $a > 0$.

By Problem 10, replacing s by ks, we have
$$\mathcal{L}^{-1}\left\{\frac{e^{-1/ks}}{(ks)^{1/2}}\right\} = \frac{1}{k}\,\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi(t/k)}} = \frac{1}{\sqrt{k}}\,\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}$$
or
$$\mathcal{L}^{-1}\left\{\frac{e^{-1/ks}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}$$
Then letting $k = 1/a$,
$$\mathcal{L}^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

INVERSE LAPLACE TRANSFORMS OF DERIVATIVES AND INTEGRALS

12. Prove *Theorem 2-6*, Page 44: $\mathcal{L}^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t)$, $n = 1,2,3,\ldots$.

Since $\mathcal{L}\{t^n F(t)\} = (-1)^n \dfrac{d^n}{ds^n}f(s) = (-1)^n f^{(n)}(s)$ [see Problem 19, Page 17], we have
$$\mathcal{L}^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t)$$

13. Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

We have $\dfrac{d}{ds}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{-2s}{(s^2+a^2)^2}$. Thus $\dfrac{s}{(s^2+a^2)^2} = -\dfrac{1}{2}\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right)$.

Then since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{\sin at}{a}$, we have by Problem 12,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{1}{2}t\left(\frac{\sin at}{a}\right) = \frac{t\sin at}{2a}$$

Another method. Differentiating with respect to the parameter a, we find,

$$\frac{d}{da}\left(\frac{s}{s^2+a^2}\right) = \frac{-2as}{(s^2+a^2)^2}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{d}{da}\left(\frac{s}{s^2+a^2}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

or

$$\frac{d}{da}\left\{\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)\right\} = -2a\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

i.e.

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2a}\frac{d}{da}(\cos at) = -\frac{1}{2a}(-t\sin at) = \frac{t\sin at}{2a}$$

14. Find $\mathcal{L}^{-1}\left\{\ln\left(1+\dfrac{1}{s^2}\right)\right\}$.

Let $f(s) = \ln\left(1+\dfrac{1}{s^2}\right) = \mathcal{L}\{F(t)\}$. Then $f'(s) = \dfrac{-2}{s(s^2+1)} = -2\left\{\dfrac{1}{s}-\dfrac{s}{s^2+1}\right\}$.

Thus since $\mathcal{L}^{-1}\{f'(s)\} = -2(1-\cos t) = -t\,F(t)$, $F(t) = \dfrac{2(1-\cos t)}{t}$.

MULTIPLICATION AND DIVISION BY POWERS OF s

15. Prove *Theorem 2-9:* $\mathcal{L}^{-1}\left\{\dfrac{f(s)}{s}\right\} = \displaystyle\int_0^t F(u)\,du$.

Let $G(t) = \displaystyle\int_0^t F(u)\,du$. Then $G'(t) = F(t)$, $G(0) = 0$. Thus

$$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0) = s\,\mathcal{L}\{G(t)\} = f(s)$$

and so

$$\mathcal{L}\{G(t)\} = \frac{f(s)}{s} \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(u)\,du$$

Compare Problem 17, Page 16.

16. Prove that $\mathcal{L}^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = \displaystyle\int_0^t \int_0^v F(u)\,du\,dv.$

Let $G(t) = \displaystyle\int_0^t \int_0^v F(u)\,du\,dv$. Then $G'(t) = \displaystyle\int_0^t F(u)\,du$ and $G''(t) = F(t)$. Since $G(0) = G'(0) = 0$,

$$\mathcal{L}\{G''(t)\} = s^2\mathcal{L}\{G(t)\} - s\,G(0) - G'(0) = s^2\mathcal{L}\{G(t)\} = f(s)$$

Thus $\qquad \mathcal{L}\{G(t)\} = \dfrac{f(s)}{s^2} \qquad$ or $\qquad \mathcal{L}^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = G(t) = \displaystyle\int_0^t \int_0^v F(u)\,du\,dv$

The result can be written $\qquad \mathcal{L}^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = \displaystyle\int_0^t \int_0^t F(t)\,dt^2.$

In general, $$\mathcal{L}^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t F(t)\,dt^n$$

17. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}.$

Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$, we have by repeated application of Problem 15,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = 1 - \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)\,du = t - \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u-\sin u)\,du = \frac{t^2}{2} + \cos t - 1$$

Check: $$\mathcal{L}\left\{\frac{t^2}{2} + \cos t - 1\right\} = \frac{1}{s^3} + \frac{s}{s^2+1} - \frac{1}{s} = \frac{s^2+1+s^4-s^2(s^2+1)}{s^3(s^2+1)} = \frac{1}{s^3(s^2+1)}$$

18. Given that $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{1}{2}t\sin t$, find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}.$

Method 1. By *Theorem 2-9* [Problem 15], we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2+1)^2}\right\} = \int_0^t \tfrac{1}{2}u\sin u\,du \\ &= (\tfrac{1}{2}u)(-\cos u) - (\tfrac{1}{2})(-\sin u)\Big|_0^t \\ &= \tfrac{1}{2}(\sin t - t\cos t)\end{aligned}$$

Method 2. By *Theorem 2-8*, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{s\cdot\frac{s}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s^2+1-1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} \\ &= \frac{d}{dt}\{\tfrac{1}{2}t\sin t\} = \tfrac{1}{2}(t\cos t + \sin t)\end{aligned}$$

Then $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \tfrac{1}{2}(t\cos t + \sin t) = \tfrac{1}{2}(\sin t - t\cos t)$$

16. Prove that $\mathcal{L}^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = \displaystyle\int_0^t \int_0^v F(u)\,du\,dv$.

Let $G(t) = \displaystyle\int_0^t \int_0^v F(u)\,du\,dv$. Then $G'(t) = \displaystyle\int_0^t F(u)\,du$ and $G''(t) = F(t)$. Since $G(0) = G'(0) = 0$,

$$\mathcal{L}\{G''(t)\} = s^2\mathcal{L}\{G(t)\} - s\,G(0) - G'(0) = s^2\mathcal{L}\{G(t)\} = f(s)$$

Thus $\quad \mathcal{L}\{G(t)\} = \dfrac{f(s)}{s^2} \quad$ or $\quad \mathcal{L}^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = G(t) = \displaystyle\int_0^t \int_0^v F(u)\,du\,dv$

The result can be written $\quad \mathcal{L}^{-1}\left\{\dfrac{f(s)}{s^2}\right\} = \displaystyle\int_0^t \int_0^t F(t)\,dt^2$.

In general, $$\mathcal{L}^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t F(t)\,dt^n$$

17. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$.

Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$, we have by repeated application of Problem 15,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = 1 - \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)\,du = t - \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u - \sin u)\,du = \frac{t^2}{2} + \cos t - 1$$

Check: $$\mathcal{L}\left\{\frac{t^2}{2} + \cos t - 1\right\} = \frac{1}{s^3} + \frac{s}{s^2+1} - \frac{1}{s} = \frac{s^2+1+s^4-s^2(s^2+1)}{s^3(s^2+1)} = \frac{1}{s^3(s^2+1)}$$

18. Given that $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{1}{2}t\sin t$, find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$.

Method 1. By *Theorem 2-9* [Problem 15], we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2+1)^2}\right\} = \int_0^t \tfrac{1}{2}u\sin u\,du$$

$$= (\tfrac{1}{2}u)(-\cos u) - (\tfrac{1}{2})(-\sin u)\Big|_0^t$$

$$= \tfrac{1}{2}(\sin t - t\cos t)$$

Method 2. By *Theorem 2-8*, we have

$$\mathcal{L}^{-1}\left\{s\cdot\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+1-1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$= \frac{d}{dt}\{\tfrac{1}{2}t\sin t\} = \tfrac{1}{2}(t\cos t + \sin t)$$

Then $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \tfrac{1}{2}(t\cos t + \sin t) = \tfrac{1}{2}(\sin t - t\cos t)$$

19. Find $\mathcal{L}^{-1}\left\{\frac{1}{s}\ln\left(1+\frac{1}{s^2}\right)\right\}$.

Using Problem 14, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\ln\left(1+\frac{1}{s^2}\right)\right\} = \int_0^t \frac{2(1-\cos u)}{u}\,du = 2\int_0^t \frac{1-\cos u}{u}\,du$$

THE CONVOLUTION THEOREM

20. Prove the *convolution theorem*: If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du = F * G$$

Method 1. The required result follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = f(s)\,g(s) \tag{1}$$

where $f(s) = \mathcal{L}\{F(t)\}$, $g(s) = \mathcal{L}\{G(t)\}$. To show this we note that the left side of (*1*) is

$$\int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} F(u)\,G(t-u)\,du\right\} dt$$

$$= \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}\,F(u)\,G(t-u)\,du\,dt = \lim_{M\to\infty} s_M$$

where
$$s_M = \int_{t=0}^{M}\int_{u=0}^{t} e^{-st}\,F(u)\,G(t-u)\,du\,dt \tag{2}$$

The region in the tu plane over which the integration (*2*) is performed is shown shaded in Fig. 2-1.

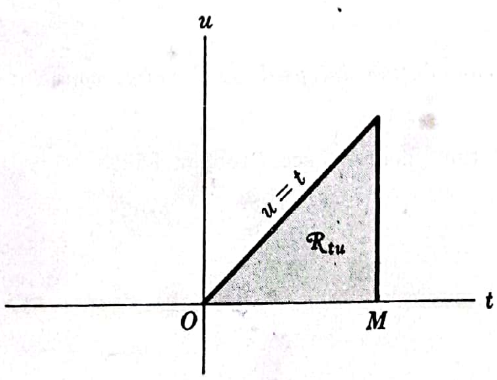

Fig. 2-1

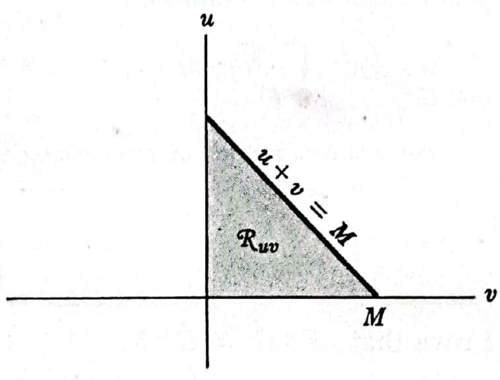

Fig. 2-2

Letting $t-u=v$ or $t=u+v$, the shaded region $\mathcal{R}_{tu}$ of the tu plane is transformed into the shaded region $\mathcal{R}_{uv}$ of the uv plane shown in Fig. 2-2. Then by a theorem on transformation of multiple integrals, we have

$$s_M = \iint_{\mathcal{R}_{tu}} e^{-st}\,F(u)\,G(t-u)\,du\,dt = \iint_{\mathcal{R}_{uv}} e^{-s(u+v)}\,F(u)\,G(v)\left|\frac{\partial(u,t)}{\partial(u,v)}\right| du\,dv \tag{3}$$

where the *Jacobian of the transformation* is

$$J \;=\; \frac{\partial(u,t)}{\partial(u,v)} \;=\; \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \;=\; \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \;=\; 1$$

Thus the right side of (3) is

$$s_M \;=\; \int_{v=0}^{M}\int_{u=0}^{M-v} e^{-s(u+v)}\, F(u)\, G(v)\, du\, dv \qquad (4)$$

Let us define a new function

$$K(u,v) \;=\; \begin{cases} e^{-s(u+v)}\, F(u)\, G(v) & \text{if } u+v \leqq M \\ 0 & \text{if } u+v > M \end{cases} \qquad (5)$$

This function is defined over the square of Fig. 2-3 but, as indicated in (5), is zero over the unshaded portion of the square. In terms of this new function we can write (4) as

$$s_M \;=\; \int_{v=0}^{M}\int_{u=0}^{M} K(u,v)\, du\, dv$$

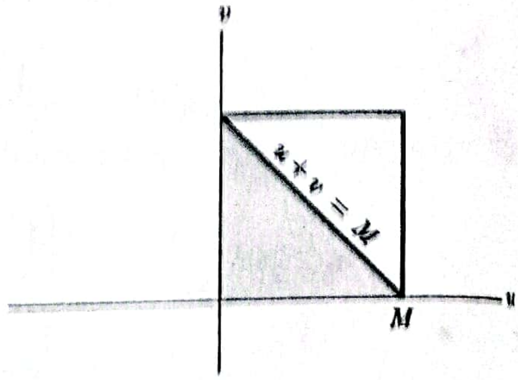

Fig. 2-3

Then

$$\begin{aligned} \lim_{M\to\infty} s_M \;&=\; \int_0^{\infty}\int_0^{\infty} K(u,v)\, du\, dv \\ &=\; \int_0^{\infty}\int_0^{\infty} e^{-s(u+v)}\, F(u)\, G(v)\, du\, dv \\ &=\; \left\{\int_0^{\infty} e^{-su} F(u)\, du\right\}\left\{\int_0^{\infty} e^{-sv} G(v)\, dv\right\} \\ &=\; f(s)\, g(s) \end{aligned}$$

which establishes the theorem.

We call $\int_0^t F(u)\, G(t-u)\, du = F*G$ the *convolution integral* or, briefly, *convolution* of F and G.

For a direct method of establishing the convolution theorem, see Problem 85.

21. Prove that $F*G = G*F$.

Letting $t-u=v$ or $u=t-v$, we have

$$\begin{aligned} F*G \;&=\; \int_0^t F(u)\, G(t-u)\, du \;=\; \int_0^t F(t-v)\, G(v)\, dv \\ &=\; \int_0^t G(v)\, F(t-v)\, dv \;=\; G*F \end{aligned}$$

This shows that the *convolution* of F and G obeys the *commutative law* of algebra. It also obeys the *associative law* and *distributive law* [see Problems 80 and 81].

22. Evaluate each of the following by use of the convolution theorem.

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.

(a) We can write $\dfrac{s}{(s^2+a^2)^2} = \dfrac{s}{s^2+a^2}\cdot\dfrac{1}{s^2+a^2}$. Then since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{\sin at}{a}$, we have by the convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\,du \\
&= \frac{1}{a}\int_0^t (\cos au)(\sin at \cos au - \cos at \sin au)\,du \\
&= \frac{1}{a}\sin at \int_0^t \cos^2 au\,du - \frac{1}{a}\cos at \int_0^t \sin au \cos au\,du \\
&= \frac{1}{a}\sin at \int_0^t \left(\frac{1+\cos 2au}{2}\right)du - \frac{1}{a}\cos at \int_0^t \frac{\sin 2au}{2}\,du \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right) \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right) - \frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right) \\
&= \frac{t\sin at}{2a}
\end{aligned}$$

Compare Problem 13, Page 53.

(b) We have $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$, $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = te^{-t}$. Then by the convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} &= \int_0^t (ue^{-u})(t-u)\,du \\
&= \int_0^t (ut-u^2)\,e^{-u}\,du \\
&= (ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\Big|_0^t \\
&= te^{-t} + 2e^{-t} + t - 2
\end{aligned}$$

Check:

$$\begin{aligned}
\mathcal{L}\{te^{-t}+2e^{-t}+t-2\} &= \frac{1}{(s+1)^2} + \frac{2}{s+1} + \frac{1}{s^2} - \frac{2}{s} \\
&= \frac{s^2+2s^2(s+1)+(s+1)^2-2s(s+1)^2}{s^2(s+1)^2} = \frac{1}{s^2(s+1)^2}
\end{aligned}$$

23. Show that $\displaystyle\int_0^t\int_0^v F(u)\,du\,dv = \int_0^t (t-u)\,F(u)\,du$.

By the convolution theorem, if $f(s) = \mathcal{L}\{F(t)\}$, we have

$$\mathcal{L}\left\{\int_0^t (t-u)\,F(u)\,du\right\} = \mathcal{L}\{t\}\,\mathcal{L}\{F(t)\} = \frac{f(s)}{s^2}$$

Then by Problem 10,

$$\int_0^t (t-u)\,F(u)\,du = \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^v F(u)\,du\,dv$$

The result can be written

$$\int_0^t \int_0^t F(t)\,dt^2 = \int_0^t (t-u)\,F(u)\,du$$

In general, we can prove that [see Problems 83 and 84],

$$\int_0^t \int_0^t \cdots \int_0^t F(t)\,dt^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!}\,F(u)\,du$$

PARTIAL FRACTIONS

24. Find $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$.

Method 1.

$$\frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \qquad (1)$$

Multiplying by $(s-3)(s+1)$, we obtain

$$3s + 7 = A(s+1) + B(s-3) = (A+B)s + A - 3B$$

Equating coefficients, $A+B = 3$ and $A - 3B = 7$; then $A = 4$, $B = -1$,

$$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}$$

and

$$\mathcal{L}^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\} = 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$
$$= 4e^{3t} - e^{-t}$$

Method 2. Multiply both sides of (1) by $s-3$ and let $s \to 3$. Then

$$\lim_{s\to 3} \frac{3s+7}{s+1} = A + \lim_{s\to 3} \frac{B(s-3)}{s+1} \qquad \text{or} \qquad A = 4$$

Similarly multiplying both sides of (1) by $s+1$ and letting $s \to -1$, we have

$$\lim_{s\to -1} \frac{3s+7}{s-3} = \lim_{s\to -1} \frac{A(s+1)}{s-3} + B \qquad \text{or} \qquad B = -1$$

Using these values we obtain the result in Method 1. See also Problem 7(c), Page 50.

25. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

We have

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3} \qquad ($$

Let us use the procedure of Method 2, Problem 24.

Multiply both sides of (*1*) by $s+1$ and let $s \to -1$; then

$$A = \lim_{s \to -1} \frac{2s^2-4}{(s-2)(s-3)} = -\frac{1}{6}$$

Multiply both sides of (*1*) by $s-2$ and let $s \to 2$; then

$$B = \lim_{s \to 2} \frac{2s^2-4}{(s+1)(s-3)} = -\frac{4}{3}$$

Multiply both sides of (*1*) by $s-3$ and let $s \to 3$; then

$$C = \lim_{s \to 3} \frac{2s^2-4}{(s+1)(s-2)} = \frac{7}{2}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/6}{s+1} + \frac{-4/3}{s-2} + \frac{7/2}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

The procedure of Method 1, Problem 24, can also be used. However, it will be noted that the present method is less tedious. It can be used whenever the denominator has *distinct linear factors.*

26. Find $\mathcal{L}^{-1}\left\{\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$.

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2} \tag{1}$$

A procedure analogous to that of Problem 25 can be used to find A and B.

Multiply both sides of (*1*) by $s+1$ and let $s \to -1$; then

$$A = \lim_{s \to -1} \frac{5s^2-15s-11}{(s-2)^3} = \frac{-1}{3}$$

Multiply both sides of (*1*) by $(s-2)^3$ and let $s \to 2$; then

$$B = \lim_{s \to 2} \frac{5s^2-15s-11}{s+1} = -7$$

The method fails to determine C and D. However, since we know A and B, we have from (*1*),

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-1/3}{s+1} + \frac{-7}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2} \tag{2}$$

To determine C and D we can substitute two values for s, say $s=0$ and $s=1$, from which we find respectively,

$$\frac{11}{8} = -\frac{1}{3} + \frac{7}{8} + \frac{C}{4} - \frac{D}{2}, \qquad \frac{21}{2} = -\frac{1}{6} + 7 + C - D$$

i.e. $3C - 6D = 10$ and $3C - 3D = 11$, from which $C = 4$, $D = 1/3$. Thus

$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/3}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{1/3}{s-2}\right\}$$

$$= -\frac{1}{3}e^{-t} - \frac{7}{2}t^2e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}$$

Another method. On multiplying both sides of (2) by s and letting $s \to \infty$, we find $0 = -\frac{1}{3} + D$ which gives $D = \frac{1}{3}$. Then C can be found as above by letting $s = 0$.

This method can be used when we have some *repeated linear factors.*

27. Find $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$.

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \tag{1}$$

Multiply both sides by $s-1$ and let $s \to 1$; then $A = \lim_{s\to 1} \dfrac{3s+1}{s^2+1} = 2$ and

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{Bs+C}{s^2+1} \tag{2}$$

To determine B and C, let $s = 0$ and 2 (for example); then

$$-1 = -2 + C, \qquad \frac{7}{5} = 2 + \frac{2B+C}{5}$$

from which $C = 1$ and $B = -2$. Thus we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{-2s+1}{s^2+1}\right\}\\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= 2e^t - 2\cos t + \sin t\end{aligned}$$

Another method. Multiplying both sides of (2) by s and letting $s \to \infty$, we find at once that $B = -2$.

28. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$.

Method 1.

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \tag{1}$$

Multiplying by $(s^2+2s+2)(s^2+2s+5)$,

$$\begin{aligned}s^2 + 2s + 3 &= (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)\\ &= (A+C)s^3 + (2A+B+2C+D)s^2 + (5A+2B+2C+2D)s + 5B + 2D\end{aligned}$$

Then $A+C = 0$, $2A+B+2C+D = 1$, $5A+2B+2C+2D = 2$, $5B+2D = 3$. Solving, $A = 0$, $B = \frac{1}{3}$, $C = 0$, $D = \frac{2}{3}$. Thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}\right\}\\ &= \tfrac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}\\ &= \tfrac{1}{3}e^{-t}\sin t + \tfrac{2}{3}\cdot\tfrac{1}{2}e^{-t}\sin 2t\\ &= \tfrac{1}{3}e^{-t}(\sin t + \sin 2t)\end{aligned}$$

Method 2. Let $s = 0$ in (*1*): $\quad \dfrac{3}{10} = \dfrac{B}{2} + \dfrac{D}{5}$

Multiply (*1*) by s and let $s \to \infty$: $\quad 0 = A + C$

Let $s = 1$: $\quad \dfrac{3}{20} = \dfrac{A+B}{5} + \dfrac{C+D}{8}$

Let $s = -1$: $\quad \dfrac{1}{2} = -A + B + \dfrac{D-C}{4}$

Solving, $A = 0,\ B = \frac{1}{3},\ C = 0,\ D = \frac{2}{3}$ as in Method 1.

This illustrates the case of *non-repeated quadratic factors.*

Method 3. Since $s^2 + 2s + 2 = 0$ for $s = -1 \pm i$, we can write

$$s^2 + 2s + 2 = (s+1-i)(s+1+i)$$

Similarly
$$s^2 + 2s + 5 = (s+1-2i)(s+1+2i)$$

Then
$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{s^2+2s+3}{(s+1-i)(s+1+i)(s+1-2i)(s+1+2i)}$$
$$= \frac{A}{s+1-i} + \frac{B}{s+1+i} + \frac{C}{s+1-2i} + \frac{D}{s+1+2i}$$

Solving for A, B, C, D, we find $A = 1/6i,\ B = -1/6i,\ C = 1/6i,\ D = -1/6i$. Thus the required inverse Laplace transform is

$$\frac{e^{-(1-i)t}}{6i} - \frac{e^{-(1+i)t}}{6i} + \frac{e^{-(1-2i)t}}{6i} - \frac{e^{-(1+2i)t}}{6i} = \tfrac{1}{3}e^{-t}\left(\frac{e^{it}-e^{-it}}{2i}\right) + \tfrac{1}{3}e^{-t}\left(\frac{e^{2it}-e^{-2it}}{2i}\right)$$
$$= \tfrac{1}{3}e^{-t}\sin t + \tfrac{1}{3}e^{-t}\sin 2t$$
$$= \tfrac{1}{3}e^{-t}(\sin t + \sin 2t)$$

This shows that the case of non-repeated quadratic factors can be reduced to non-repeated linear factors using complex numbers.

HEAVISIDE'S EXPANSION FORMULA

29. Prove Heaviside's expansion formula (*14*), Page 46.

Since $Q(s)$ is a polynomial with n distinct zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$, we can write according to the method of partial fractions,

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \cdots + \frac{A_k}{s-\alpha_k} + \cdots + \frac{A_n}{s-\alpha_n} \tag{1}$$

Multiplying both sides by $s - \alpha_k$ and letting $s \to \alpha_k$, we find using L'Hospital's rule,

$$A_k = \lim_{s\to\alpha_k} \frac{P(s)}{Q(s)}(s-\alpha_k) = \lim_{s\to\alpha_k} P(s)\left\{\frac{s-\alpha_k}{Q(s)}\right\}$$
$$= \lim_{s\to\alpha_k} P(s) \lim_{s\to\alpha_k}\left(\frac{s-\alpha_k}{Q(s)}\right) = P(\alpha_k)\lim_{s\to\alpha_k}\frac{1}{Q'(s)} = \frac{P(\alpha_k)}{Q'(\alpha_k)}$$

Thus (*1*) can be written

$$\frac{P(s)}{Q(s)} = \frac{P(\alpha_1)}{Q'(\alpha_1)}\frac{1}{s-\alpha_1} + \cdots + \frac{P(\alpha_k)}{Q'(\alpha_k)}\frac{1}{s-\alpha_k} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)}\frac{1}{s-\alpha_n}$$

Then taking the inverse Laplace transform, we have as required

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t} + \cdots + \frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)}e^{\alpha_n t} = \sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$$

30. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

We have $P(s) = 2s^2-4$, $Q(s) = (s+1)(s-2)(s-3) = s^3-4s^2+s+6$, $Q'(s) = 3s^2-8s+1$, $\alpha_1 = -1$, $\alpha_2 = 2$, $\alpha_3 = 3$. Then the required inverse is by Problem 29,

$$\frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t} = \frac{-2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t} = -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

Compare with Problem 25.

31. Find $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$.

We have $P(s) = 3s+1$, $Q(s) = (s-1)(s^2+1) = s^3-s^2+s-1$, $Q'(s) = 3s^2-2s+1$, $\alpha_1 = 1$, $\alpha_2 = i$, $\alpha_3 = -i$ since $s^2+1 = (s-i)(s+i)$. Then by the Heaviside expansion formula the required inverse is

$$\frac{P(1)}{Q'(1)}e^{t} + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it} \qquad (1)$$

$$= \frac{4}{2}e^t + \frac{3i+1}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

$$= 2e^t + (-1-\tfrac{1}{2}i)(\cos t + i\sin t) + (-1+\tfrac{1}{2}i)(\cos t - i\sin t)$$

$$= 2e^t - \cos t + \tfrac{1}{2}\sin t - \cos t + \tfrac{1}{2}\sin t$$

$$= 2e^t - 2\cos t + \sin t$$

Compare with Problem 27.

Note that some labor can be saved by observing that the last two terms in *(1)* are complex conjugates of each other.

THE BETA FUNCTION

32. Prove that $B(m,n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ where $m>0$, $n>0$.

Consider

$$G(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx$$

Then by the convolution theorem, we have

$$\mathcal{L}\{G(t)\} = \mathcal{L}\{t^{m-1}\}\,\mathcal{L}\{t^{n-1}\}$$

$$= \frac{\Gamma(m)}{s^m}\cdot\frac{\Gamma(n)}{s^n} = \frac{\Gamma(m)\,\Gamma(n)}{s^{m+n}}$$

Thus $$G(t) = \mathcal{L}^{-1}\left\{\frac{\Gamma(m)\,\Gamma(n)}{s^{m+n}}\right\} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

or $$\int_0^t x^{m-1}(t-x)^{n-1}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

Letting $t=1$, we obtain the required result.

33. Prove that $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta \;=\; \frac{1}{2}B(m,n) \;=\; \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$.

From Problem 32, we have

$$B(m,n) \;=\; \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;=\; \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Letting $x = \sin^2\theta$, this becomes

$$B(m,n) \;=\; 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta \;=\; \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

from which the required result follows.

34. Evaluate (a) $\displaystyle\int_0^{\pi/2} \sin^4\theta\,\cos^6\theta\,d\theta$, (b) $\displaystyle\int_0^{\pi} \cos^4\theta\,d\theta$, (c) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}}$.

(a) Let $2m-1 = 4$, $2n-1 = 6$ in Problem 33. Then $m = 5/2$, $n = 7/2$, and we have

$$\int_0^{\pi/2} \sin^4\theta\,\cos^6\theta\,d\theta \;=\; \frac{\Gamma(5/2)\,\Gamma(7/2)}{2\,\Gamma(6)} \;=\; \frac{(3/2)(1/2)\sqrt{\pi}\cdot(5/2)(3/2)(1/2)\sqrt{\pi}}{2\cdot5\cdot4\cdot3\cdot2\cdot1} \;=\; \frac{3\pi}{512}$$

(b) Since $\cos\theta$ is symmetric about $\theta = \pi/2$, we have

$$\int_0^{\pi} \cos^4\theta\,d\theta \;=\; 2\int_0^{\pi/2} \cos^4\theta\,d\theta$$

Then letting $2m-1 = 0$ and $2n-1 = 4$, i.e. $m = 1/2$ and $n = 5/2$ in Problem 33, we find

$$2\int_0^{\pi/2} \cos^4\theta\,d\theta \;=\; 2\left[\frac{\Gamma(1/2)\,\Gamma(5/2)}{2\,\Gamma(3)}\right]$$

$$=\; 2\left[\frac{\sqrt{\pi}\cdot(3/2)(1/2)\sqrt{\pi}}{2\cdot2\cdot1}\right] \;=\; \frac{3\pi}{8}$$

(c) $$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} \;=\; \int_0^{\pi/2} \sin^{-1/2}\theta\,\cos^{1/2}\theta\,d\theta$$

Letting $2m-1 = -1/2$ and $2n-1 = 1/2$, or $m = 1/4$ and $n = 3/4$ in Problem 33, we find

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} \;=\; \frac{\Gamma(1/4)\,\Gamma(3/4)}{2\,\Gamma(1)} \;=\; \frac{1}{2}\,\frac{\pi}{\sin(\pi/4)} \;=\; \frac{\pi\sqrt{2}}{2}$$

using the result $\Gamma(p)\,\Gamma(1-p) = \pi/(\sin p\pi)$, $0 < p < 1$.

EVALUATION OF INTEGRALS

35. Evaluate $\displaystyle\int_0^t J_0(u)\,J_0(t-u)\,du$.

Let $G(t) = \displaystyle\int_0^t J_0(u)\,J_0(t-u)\,du$. Then by the convolution theorem,

$$\mathcal{L}\{G(t)\} \;=\; \mathcal{L}\{J_0(t)\}\,\mathcal{L}\{J_0(t)\} \;=\; \left(\frac{1}{\sqrt{s^2+1}}\right)\left(\frac{1}{\sqrt{s^2+1}}\right) \;=\; \frac{1}{s^2+1}$$

Hence $$G(t) \;=\; \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \;=\; \sin t$$

and so $$G(t) \;=\; \int_0^t J_0(u)\,J_0(t-u)\,du \;=\; \sin t$$

36. Show that $\displaystyle\int_0^\infty \cos x^2\,dx = \tfrac{1}{2}\sqrt{\pi/2}$.

Let $G(t) = \displaystyle\int_0^\infty \cos tx^2\,dx$. Then taking the Laplace transform, we find

$$\begin{aligned}
\mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st}\,dt \int_0^\infty \cos tx^2\,dx \\
&= \int_0^\infty dx \int_0^\infty e^{-st}\cos tx^2\,dt \\
&= \int_0^\infty \mathcal{L}\{\cos tx^2\}\,dx \quad = \quad \int_0^\infty \frac{s}{s^2+x^4}\,dx
\end{aligned}$$

Letting $x^2 = s\tan\theta$ or $x = \sqrt{s}\sqrt{\tan\theta}$, this integral becomes on using Problem 34(e),

$$\frac{1}{2\sqrt{s}}\int_0^{\pi/2} (\tan\theta)^{-1/2}\,d\theta \quad = \quad \frac{1}{2\sqrt{s}}\left(\frac{\pi\sqrt{2}}{2}\right) \quad = \quad \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

Inverting, we find

$$G(t) \quad = \quad \int_0^\infty \cos tx^2\,dx \quad = \quad \frac{\pi\sqrt{2}}{4}\,\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s}}\right) \quad = \quad \left(\frac{\pi\sqrt{2}}{4}\right)\left(\frac{t^{-1/2}}{\sqrt{\pi}}\right) \quad = \quad \frac{\sqrt{2\pi}}{4}\,t^{-1/2}$$

Letting $t = 1$ we have, as required,

$$\int_0^\infty \cos x^2\,dx \quad = \quad \frac{\sqrt{2\pi}}{4} \quad = \quad \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

MISCELLANEOUS PROBLEMS

37. Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}$.

Consider $G(t) = \displaystyle\int_0^\infty e^{-tx^2}\,dx$. Then taking Laplace transforms,

$$\mathcal{L}\{G(t)\} \quad = \quad \int_0^\infty \frac{dx}{s+x^2} \quad = \quad \frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\Big|_0^\infty \quad = \quad \frac{\pi}{2\sqrt{s}}$$

Thus by inverting,

$$G(t) \quad = \quad \int_0^\infty e^{-tx^2}\,dx \quad = \quad \frac{\pi}{2}\,\frac{t^{-1/2}}{\sqrt{\pi}} \quad = \quad \frac{1}{2}\sqrt{\pi}\,t^{-1/2}$$

and the required result follows on letting $t = 1$.

Another method.

Letting $x^2 = u$ or $x = \sqrt{u}$, the required integral becomes

$$\frac{1}{2}\int_0^\infty u^{-1/2}e^{-u}\,du \quad = \quad \tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

But by Problem 32 with $m = n = \frac{1}{2}$, we have

$$\begin{aligned}
\{\Gamma(\tfrac{1}{2})\}^2 &= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx \quad = \quad \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \\
&= \int_0^1 \frac{dx}{\sqrt{\frac{1}{4}-(\frac{1}{2}-x)^2}} \quad = \quad \sin^{-1}(1-2x)\Big|_0^1 \quad = \quad \pi
\end{aligned}$$

Thus $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and so the required integral has the value $\frac{1}{2}\sqrt{\pi}$. See also Problem 29, Page 22.

38. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+a^2)^{3/2}}\right\}$.

We have [see Problem 34, Page 23], $\mathcal{L}\{J_0(at)\} = \dfrac{1}{\sqrt{s^2+a^2}}$. Then differentiating with respect to a, we find

$$\frac{d}{da}\mathcal{L}\{J_0(at)\} = \frac{d}{da}\left\{\frac{1}{\sqrt{s^2+a^2}}\right\} \quad \text{or} \quad \mathcal{L}\left[\frac{d}{da}\{J_0(at)\}\right] = \frac{-a}{(s^2+a^2)^{3/2}}$$

i.e.

$$\mathcal{L}\{t\,J_0'(at)\} = \frac{-a}{(s^2+a^2)^{3/2}}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^{3/2}}\right\} = -\frac{t}{a}J_0'(at) = \frac{t\,J_1(at)}{a}$$

since $J_0'(u) = -J_1(u)$.

39. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+2s+5)^{3/2}}\right\}$.

The required inverse can be written as

$$\mathcal{L}^{-1}\left\{\frac{1}{[(s+1)^2+4]^{3/2}}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^{3/2}}\right\} = \frac{te^{-t}}{2}J_1(2t)$$

using Problem 38.

40. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-1/s}}{s}\right\}$.

Using infinite series, we find

$$\begin{aligned}\frac{1}{s}e^{-1/s} &= \frac{1}{s}\left\{1 - \frac{1}{s} + \frac{1}{2!\,s^2} - \frac{1}{3!\,s^3} + \cdots\right\}\\ &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{2!\,s^3} - \frac{1}{3!\,s^4} + \cdots\end{aligned}$$

Inverting term by term,

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{s}e^{-1/s}\right\} &= 1 - t + \frac{t^2}{(2!)^2} - \frac{t^3}{(3!)^2} + \cdots\\ &= 1 - t + \frac{t^2}{1^2\,2^2} - \frac{t^3}{1^2\,2^2\,3^2} + \cdots\\ &= 1 - \frac{(2t^{1/2})^2}{2^2} + \frac{(2t^{1/2})^4}{2^2\,4^2} - \frac{(2t^{1/2})^6}{2^2\,4^2\,6^2} + \cdots\\ &= J_0(2\sqrt{t})\end{aligned}$$

41. Find $\mathcal{L}^{-1}\{e^{-\sqrt{s}}\}$.

Let $y = e^{-\sqrt{s}}$; then $y' = -\dfrac{e^{-\sqrt{s}}}{2s^{1/2}}$, $y'' = \dfrac{e^{-\sqrt{s}}}{4s} + \dfrac{e^{-\sqrt{s}}}{4s^{3/2}}$. Thus

$$4s\,y'' + 2y' - y = 0 \qquad (1)$$

Now $y'' = \mathcal{L}\{t^2Y\}$ so that $sy'' = \mathcal{L}\left\{\frac{d}{dt}(t^2Y)\right\} = \mathcal{L}\{t^2Y' + 2tY\}$. Also, $y' = \mathcal{L}\{-tY\}$. Thus (1) can be written

$$4\mathcal{L}\{t^2Y' + 2tY\} - 2\mathcal{L}\{tY\} - \mathcal{L}\{Y\} = 0 \quad \text{or} \quad 4t^2Y' + (6t-1)Y = 0$$

which can be written

$$\frac{dY}{Y} + \left(\frac{6t-1}{4t^2}\right)dt = 0 \quad \text{or} \quad \ln Y + \frac{3}{2}\ln t + \frac{1}{4t} = c_1$$

i.e.

$$Y = \frac{c}{t^{3/2}}e^{-1/4t}$$

Now $tY = \frac{c}{t^{1/2}}e^{-1/4t}$. Thus

$$\mathcal{L}\{tY\} = -\frac{d}{ds}\mathcal{L}\{Y\} = -\frac{d}{ds}(e^{-\sqrt{s}}) = \frac{e^{-\sqrt{s}}}{2\sqrt{s}}$$

For large t, $tY \sim \frac{c}{t^{1/2}}$ and $\mathcal{L}\{tY\} \sim \frac{c\sqrt{\pi}}{s^{1/2}}$. For small s, $\frac{e^{-\sqrt{s}}}{2\sqrt{s}} \sim \frac{1}{2s^{1/2}}$. Hence by the final value theorem, $c\sqrt{\pi} = 1/2$ or $c = 1/2\sqrt{\pi}$. It follows that

$$\mathcal{L}^{-1}\{e^{-\sqrt{s}}\} = \frac{1}{2\sqrt{\pi}\,t^{3/2}}e^{-1/4t}$$

Another method. Using infinite series, we have formally

$$\mathcal{L}^{-1}\{e^{-\sqrt{s}}\} = \mathcal{L}^{-1}\left\{1 - s^{1/2} + \frac{s}{2!} - \frac{s^{3/2}}{3!} + \frac{s^2}{4!} - \frac{s^{5/2}}{5!} + \cdots\right\}$$

$$= \mathcal{L}^{-1}\{1\} - \mathcal{L}\{s^{1/2}\} + \mathcal{L}^{-1}\left\{\frac{s}{2!}\right\} - \mathcal{L}^{-1}\left\{\frac{s^{3/2}}{3!}\right\} + \cdots \qquad (1)$$

Using the results of Problem 170, Page 40 [see also Problem 33, Page 22] we have for p equal to zero or any positive integer,

$$\mathcal{L}^{-1}\{s^{p+1/2}\} = \frac{t^{-p-3/2}}{\Gamma(-p-\frac{1}{2})}$$

$$= \frac{(-1)^{p+1}}{\sqrt{\pi}}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2p+1}{2}\right)t^{-p-3/2} \qquad (2)$$

while $\mathcal{L}^{-1}\{s^p\} = 0$. Then from (1) using (2) we have

$$\mathcal{L}^{-1}\{e^{-\sqrt{s}}\} = \frac{t^{-3/2}}{2\sqrt{\pi}} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{t^{-5/2}}{3!\sqrt{\pi}} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{t^{-7/2}}{5!\sqrt{\pi}} + \cdots$$

$$= \frac{1}{2\sqrt{\pi}\,t^{3/2}}\left\{1 - \left(\frac{1}{2^2t}\right) + \frac{(1/2^2t)^2}{2!} - \frac{(1/2^2t)^3}{3!} + \cdots\right\} = \frac{1}{2\sqrt{\pi}\,t^{3/2}}e^{-1/4t}$$

42. Find $\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}}}{s}\right\}$.

From Problems 41 and 15 we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s}}}{s}\right\} = \int_0^t\left\{\frac{1}{2\sqrt{\pi}\,u^{3/2}}e^{-1/4u}\right\}du = \frac{2}{\sqrt{\pi}}\int_{1/2\sqrt{t}}^{\infty}e^{-v^2}\,dv \qquad (\text{letting } u = 1/4v^2)$$

$$= \operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right)$$

43. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-x\sqrt{s}}}{s}\right\}$.

In Problem 42 use the change of scale property (4), Page 44, with $k = x^2$. Then

$$\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{x^2 s}}}{x^2 s}\right\} = \frac{1}{x^2}\operatorname{erfc}\left(\frac{1}{2\sqrt{t/x^2}}\right)$$

from which

$$\mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

Note that this is entry 87 in the Table on Page 250.

44. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^3+10s^2+8s+40}{s^2(s^2+9)}\right\}$.

Since $\dfrac{1}{s^2(s^2+9)} = \dfrac{1}{9}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+9}\right)$, we have

$$\begin{aligned}\frac{2s^3+10s^2+8s+40}{s^2(s^2+9)} &= \frac{1}{9}\left\{\frac{2s^3+10s^2+8s+40}{s^2} - \frac{2s^3+10s^2+8s+40}{s^2+9}\right\}\\ &= \frac{1}{9}\left\{\left(2s+10+\frac{8}{s}+\frac{40}{s^2}\right) - \left(2s+10+\frac{-10s-50}{s^2+9}\right)\right\}\\ &= \frac{1}{9}\left\{\frac{8}{s}+\frac{40}{s^2}+\frac{10s}{s^2+9}+\frac{50}{s^2+9}\right\}\end{aligned}$$

and so

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s^3+10s^2+8s+40}{s^2(s^2+9)}\right\} &= \frac{1}{9}\left(8 + 40t + 10\cos 3t + \frac{50}{3}\sin 3t\right)\\ &= \frac{1}{27}(24 + 120t + 30\cos 3t + 50\sin 3t)\end{aligned}$$

We can also use the method of partial fractions.

45. Prove that $J_0(t) = \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} e^{itw}(1-w^2)^{-1/2}\,dw$.

We have [see Problem 34, Page 23],

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

Now

$$\frac{1}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s+i}}\cdot\frac{1}{\sqrt{s-i}}$$

Using the fact that $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{s+a}}\right\} = \dfrac{t^{-1/2}e^{-at}}{\sqrt{\pi}}$, we have by the convolution theorem,

$$\begin{aligned}J_0(t) &= \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+i}}\cdot\frac{1}{\sqrt{s-i}}\right\}\\ &= \int_0^t \frac{u^{-1/2}e^{-iu}}{\sqrt{\pi}}\cdot\frac{(t-u)^{-1/2}e^{i(t-u)}}{\sqrt{\pi}}\,du\\ &= \frac{1}{\pi}\int_0^t e^{i(t-2u)}\,u^{-1/2}(t-u)^{-1/2}\,du\end{aligned}$$

Letting $u = tv$ this becomes

$$J_0(t) = \frac{1}{\pi}\int_0^1 e^{it(1-2v)}\,v^{-1/2}(1-v)^{-1/2}\,dv$$

or if $1-2v = w$,

$$J_0(t) = \frac{1}{\pi}\int_{-1}^1 e^{itw}(1-w^2)^{-1/2}\,dw$$

46. Prove that $J_0(t) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(t\cos\theta)\,d\theta$.

Let $w = \cos\theta$ in the result of Problem 45. Then

$$J_0(t) = \frac{1}{\pi}\int_0^\pi e^{it\cos\theta}\,d\theta = \frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)\,d\theta + \frac{i}{\pi}\int_0^\pi \sin(t\cos\theta)\,d\theta$$

Equating real and imaginary parts or by showing directly that the last integral is zero, we have as required

$$J_0(t) = \frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)\,d\theta$$

Another method.

Let $G(t) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(t\cos\theta)\,d\theta = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$. Then taking Laplace transforms,

$$\mathcal{L}\{G(t)\} = \frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+\cos^2\theta}\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{s^2\tan^2\theta + s^2 + 1}\,d\theta$$

$$= \frac{2}{\pi}\frac{1}{\sqrt{s^2+1}}\tan^{-1}\left(\frac{s\tan\theta}{\sqrt{s^2+1}}\right)\bigg|_0^{\pi/2} = \frac{1}{\sqrt{s^2+1}}$$

Thus $G(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{s^2+1}}\right\} = J_0(t)$, as required.

Supplementary Problems

INVERSE LAPLACE TRANSFORMS

47. Determine each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s+4}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{8s}{s^2+16}\right\}$ (e) $\mathcal{L}^{-1}\left\{\dfrac{3s-12}{s^2+8}\right\}$ (g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ (i) $\mathcal{L}^{-1}\left\{\dfrac{12}{4-3s}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{2s-5}\right\}$ (d) $\mathcal{L}^{-1}\left\{\dfrac{6}{s^2+4}\right\}$ (f) $\mathcal{L}^{-1}\left\{\dfrac{2s-5}{s^2-9}\right\}$ (h) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{7/2}}\right\}$ (j) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^{4/3}}\right\}$

Ans. (a) $3e^{-4t}$ (e) $3\cos 2\sqrt{2}\,t - 3\sqrt{2}\sin 2\sqrt{2}\,t$ (i) $-4e^{4t/3}$

(b) $\frac{1}{2}e^{5t/2}$ (f) $2\cosh 3t - \frac{5}{3}\sinh 3t$ (j) $(t^{-2/3} + 3t^{1/3})/\Gamma(\frac{1}{3})$

(c) $8\cos 4t$ (g) $t^4/24$

(d) $3\sin 2t$ (h) $8t^{5/2}/15\sqrt{\pi}$

48. Find (a) $\mathcal{L}^{-1}\left\{\left(\frac{\sqrt{s}-1}{s}\right)^2\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$.

Ans. (a) $1 + t - 4t^{1/2}/\sqrt{\pi}$ (b) $1 + e^{-t}$

49. Find (a) $\mathcal{L}^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}$.

Ans. (a) $\frac{3}{4}\cos 5t/2 - \frac{4}{5}\sin 5t/2$ (b) $\frac{5}{9}\cosh 4t/3 + \frac{5}{6}\sinh 4t/3$

50. (a) Show that the functions $F(t) = \begin{cases} t & t \neq 3 \\ 5 & t = 3 \end{cases}$ and $G(t) = t$ have the same Laplace transforms.

(b) Discuss the significance of the result in (a) as far as uniqueness of inverse Laplace transforms is concerned.

51. Find (a) $\mathcal{L}^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{3s-2}{s^{5/2}} - \frac{7}{3s+2}\right\}$.

Ans. (a) $3\cos 2t - 4\sin 2t - 4\cosh 4t + 6\sinh 4t$
(b) $6t^{1/2}/\sqrt{\pi} - 8t^{3/2}/3\sqrt{\pi} - \frac{7}{3}e^{-2t/3}$

52. (a) If $F_1(t) = \mathcal{L}^{-1}\{f_1(s)\}$, $F_2(t) = \mathcal{L}^{-1}\{f_2(s)\}$, $F_3(t) = \mathcal{L}^{-1}\{f_3(s)\}$, and c_1, c_2, c_3 are any constants, prove that

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s) + c_3 f_3(s)\} = c_1 F_1(t) + c_2 F_2(t) + c_3 F_3(t)$$

stating any restrictions. (b) Generalize the result of part (a) to n functions.

53. Find $\mathcal{L}^{-1}\left\{\frac{3(s^2-1)^2}{2s^5} + \frac{4s-18}{9-s^2} + \frac{(s+1)(2-s^{1/2})}{s^{5/2}}\right\}$.

Ans. $\frac{1}{2} - t - \frac{3}{2}t^2 + \frac{1}{16}t^4 + 4t^{1/2}/\sqrt{\pi} + 8t^{3/2}/3\sqrt{\pi} - 4\cosh 3t + 6\sinh 3t$

54. Find (a) $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^5}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\}$.

Ans. (a) $\frac{e^{-t}}{24}(4t^3 - t^4)$, (b) $\frac{2t^{1/2}(3-2t)}{3\sqrt{\pi}}$

55. Find (a) $\mathcal{L}^{-1}\left\{\frac{3s-14}{s^2-4s+8}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{8s+20}{s^2-12s+32}\right\}$.

Ans. (a) $e^{2t}(3\cos 2t - 4\sin 2t)$, (b) $2e^{6t}(4\cosh 2t + 17\sinh 2t) = 21e^{8t} - 13e^{4t}$

56. Find (a) $\mathcal{L}^{-1}\left\{\frac{3s+2}{4s^2+12s+9}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}$.

Ans. (a) $\frac{3}{4}e^{-3t/2} - \frac{5}{8}te^{-3t/2}$, (b) $\frac{e^{-2t/3}}{15}\{25\cos 2\sqrt{5}\,t/3 - 24\sqrt{5}\sin 2\sqrt{5}\,t/3\}$

57. Find (a) $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt[3]{8s-27}}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2-4s+20}}\right\}$.

Ans. (a) $t^{-2/3}e^{27t/8}/2\Gamma(\frac{1}{3})$, (b) $e^{2t}J_0(4t)$

58. Find (a) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{8e^{-3s}}{s^2+4}\right\}$, (c) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{\sqrt{s+1}}\right\}$.

Ans. (a) $\begin{cases} t-2 & t>2 \\ 0 & t<2 \end{cases}$ or $(t-2)\,\mathcal{U}(t-2)$. (b) $\begin{cases} 4\sin 2(t-3) & t>3 \\ 0 & t<3 \end{cases}$ or $4\sin 2(t-3)\,\mathcal{U}(t-3)$

(c) $\begin{cases} (t-1)^{-1/2}/\sqrt{\pi} & t>1 \\ 0 & t<1 \end{cases}$ or $(t-1)^{-1/2}\,\mathcal{U}(t-1)/\sqrt{\pi}$.

59. Find (a) $\mathcal{L}^{-1}\left\{\dfrac{se^{-2s}}{s^2+3s+2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2-2s+5}\right\}$.

Ans. (a) $\begin{cases} 2e^{-2(t-2)}-e^{-(t-2)} & t>2 \\ 0 & t<2 \end{cases}$ or $\{2e^{-2(t-2)}-e^{-(t-2)}\}\,\mathcal{U}(t-2)$

(b) $\begin{cases} \frac{1}{2}e^{(t-3)}\sin 2(t-3) & t>3 \\ 0 & t<3 \end{cases}$ or $\frac{1}{2}e^{(t-3)}\sin 2(t-3)\,\mathcal{U}(t-3)$

60. If $\int_0^\infty e^{-st}F(t)\,dt = f(s)$ and $\int_0^\infty e^{-st}G(t)\,dt = f(ps+q)$, where p and q are constants, find a relationship between $F(t)$ and $G(t)$. *Ans.* $G(t) = e^{-qt/p}F(t/p)/p$

61. If $\mathcal{L}^{-1}\left\{\dfrac{1}{s\sqrt{s+1}}\right\} = \operatorname{erf}\sqrt{t}$, find $\mathcal{L}^{-1}\left\{\dfrac{1}{s\sqrt{s+a}}\right\}$, $a>0$. *Ans.* $\operatorname{erf}\sqrt{at}/\sqrt{a}$

62. If $\mathcal{L}^{-1}\left\{\dfrac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}\right\} = J_n(t)$, find $\mathcal{L}^{-1}\left\{\dfrac{(\sqrt{s^2+a^2}-s)^n}{\sqrt{s^2+a^2}}\right\}$. *Ans.* $a^nJ_n(at)$

63. Find (a) $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{s}\,(s-1)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{\sqrt{s^2+9}}\right\}$.

Ans. (a) $e^t\operatorname{erf}\sqrt{t}$, (b) $\begin{cases} J_0(3t-6) & t>2 \\ 0 & t<2 \end{cases}$ or $J_0(3t-6)\,\mathcal{U}(t-2)$

INVERSE LAPLACE TRANSFORMS OF DERIVATIVES AND INTEGRALS

64. Use *Theorem 2-6*, Page 44, to find

(a) $\mathcal{L}^{-1}\{1/(s-a)^3\}$ given that $\mathcal{L}^{-1}\{1/(s-a)\} = e^{at}$,

(b) $\mathcal{L}^{-1}\{s/(s^2-a^2)^2\}$ given that $\mathcal{L}^{-1}\{1/(s^2-a^2)\} = (\sinh at)/a$.

65. Use the fact that $\mathcal{L}^{-1}\{1/s\} = 1$ to find $\mathcal{L}^{-1}\{1/s^n\}$ where $n = 2,3,4,\ldots$. Thus find $\mathcal{L}^{-1}\{1/(s-a)^n\}$.

66. Find $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s^2+2s+2)^2}\right\}$. *Ans.* $\frac{1}{2}te^{-t}\sin t$

67. Find (a) $\mathcal{L}^{-1}\left\{\ln\left(\dfrac{s+2}{s+1}\right)\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s}\ln\left(\dfrac{s+2}{s+1}\right)\right\}$.

Ans. (a) $(e^{-t}-e^{-2t})/t$, (b) $\displaystyle\int_0^t \frac{e^{-u}-e^{-2u}}{u}\,du$

68. Find $\mathcal{L}^{-1}\{\tan^{-1}(2/s^2)\}$. *Ans.* $2\sin t\sinh t/t$

69. Find $\mathcal{L}^{-1}\left\{\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$. *Ans.* $\int_0^t \frac{\cos au - \cos bu}{u}\,du$

MULTIPLICATION AND DIVISION BY POWERS OF s

70. Prove that $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^3}\right\} = \int_0^t \int_0^v \int_0^w F(u)\,du\,dv\,dw.$

Can the integral be written as $\int_0^t \int_0^t \int_0^t F(t)\,dt^3$? Explain.

71. Evaluate (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\}$, (c) $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^3}\right\}$.

Ans. (a) $1 - t + \frac{1}{2}t^2 - e^{-t}$, (b) $\frac{2}{3}t + \frac{1}{9} - \frac{1}{9}e^{-3t}$, (c) $1 - e^{-t}(1 + t + \frac{1}{2}t^2)$

72. Find (a) $\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+4}}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s^2+a^2}}\right\}$.

Ans. (a) $\frac{1}{2}\operatorname{erf}(2\sqrt{t})$, (b) $\int_0^t J_0(au)\,du$

73. Find (a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^5(s+2)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^5(s+1)}\right\}$ and discuss the relationship between these inverse transforms.

Ans. (a) $\frac{e^t}{72}\left(t^4 - \frac{4}{3}t^3 + \frac{4}{3}t^2 - \frac{8}{9}t + \frac{8}{27}\right) - \frac{e^{-2t}}{243}$

(b) $e^{2t}\left(\frac{t^4}{36} + \frac{t^3}{54} - \frac{t^2}{54} + \frac{t}{81} - \frac{1}{243}\right) + \frac{e^{-t}}{243}$

74. If $F(t) = \mathcal{L}^{-1}\{f(s)\}$, show that

(a) $\mathcal{L}^{-1}\{s\,f'(s)\} = -t\,F'(t) - F(t)$

(b) $\mathcal{L}^{-1}\{s\,f''(s)\} = t^2 F'(t) + 2t\,F(t)$

(c) $\mathcal{L}^{-1}\{s^2 f''(s)\} = t^2 F''(t) + 4t\,F'(t) + 2\,F(t)$

75. Show that $\mathcal{L}^{-1}\{s^2 f'(s) + F(0)\} = -t\,F''(t) - 2\,F'(t)$.

THE CONVOLUTION THEOREM

76. Use the convolution theorem to find (a) $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$.

Ans. (a) $\frac{1}{4}(e^t - e^{-3t})$, (b) $\frac{1}{16}(e^{2t} - e^{-2t} - 4te^{-2t})$

77. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$. *Ans.* $\frac{1}{2}(\sin t - \cos t + e^{-t})$

78. Find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$. *Ans.* $\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$

79. Find (a) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^3}\right\}$.

Ans. (a) $\frac{1}{8}\{(3-t^2)\sin t - 3t\cos t\}$, (b) $\frac{1}{64}t(\sin 2t - 2t\cos 2t)$

80. Prove that $F*(G*H) = (F*G)*H$, i.e. the *associative law for convolutions.*

81. Prove that (a) $F*(G+H) = F*G + F*H$, (b) $(F+G)*H = F*H + G*H$.

82. Show that $1*1*1*\ldots*1$ (n ones) $= t^{n-1}/(n-1)!$ where $n = 1,2,3,\ldots$.

83. Show that $\int_0^t\int_0^t\int_0^t F(t)\,dt^3 = \int_0^t \frac{(t-u)^2}{2!}F(u)\,du$.

84. Show that $\int_0^t\int_0^t\cdots\int_0^t F(t)\,dt^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!}F(u)\,du$.

85. Prove the convolution theorem directly by showing that

$$\begin{aligned} f(s)\,g(s) &= \left\{\int_0^\infty e^{-su}F(u)\,du\right\}\left\{\int_0^\infty e^{-sv}G(v)\,dv\right\} \\ &= \int_0^\infty\int_0^\infty e^{-s(u+v)}F(u)\,G(v)\,du\,dv \\ &= \int_0^\infty e^{-st}\left\{\int_0^t F(u)\,G(t-u)\,du\right\}dt\,. \end{aligned}$$

86. Using the convolution theorem, verify that

$$\int_0^t \sin u\cos(t-u)\,du = \tfrac{1}{2}t\sin t$$

87. Show that $\frac{1}{\pi}\int_0^t \frac{e^{(a-b)u}}{\sqrt{u(t-u)}}\,du = e^{(a-b)t/2}\,I_0\{\tfrac{1}{2}(a-b)t\}$.

PARTIAL FRACTIONS

88. Use partial fractions to find (a) $\mathcal{L}^{-1}\left\{\frac{3s+16}{s^2-s-6}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^3-s}\right\}$.

Ans. (a) $5e^{3t} - 2e^{-2t}$, (b) $1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{t}$

89. Find (a) $\mathcal{L}^{-1}\left\{\frac{s+1}{6s^2+7s+2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}\right\}$.

Ans. (a) $\frac{1}{2}e^{-t/2} - \frac{1}{3}e^{-2t/3}$, (b) $5e^{2t} - \frac{3}{2}e^{t/2} + 2e^{-t}$

90. Find (a) $\mathcal{L}^{-1}\left\{\frac{27-12s}{(s+4)(s^2+9)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{s^3+16s-24}{s^4+20s^2+64}\right\}$.

Ans. (a) $3e^{-4t} - 3\cos 3t$, (b) $\frac{1}{2}\sin 4t + \cos 2t - \sin 2t$

91. Find $\mathcal{L}^{-1}\left\{\frac{s-1}{(s+3)(s^2+2s+2)}\right\}$. *Ans.* $\frac{1}{5}e^{-t}(4\cos t - 3\sin t) - \frac{4}{5}e^{-3t}$

92. Find (a) $\mathcal{L}^{-1}\left\{\frac{s^2-2s+3}{(s-1)^2(s+1)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{3s^3-3s^2-40s+36}{(s^2-4)^2}\right\}$.

Ans. (a) $\frac{1}{2}(2t-1)e^{t} + \frac{3}{2}e^{-t}$, (b) $(5t+3)e^{-2t} - 2te^{2t}$

93. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right\}$.

Ans. $\frac{3}{50}e^{3t} - \frac{1}{25}e^{-2t} - \frac{1}{50}e^{-t}\cos 2t + \frac{9}{25}e^{-t}\sin 2t$

94. Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}\right\}$. *Ans.* $\frac{1}{2}\sin t \sinh t$

95. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right\}$. *Ans.* $\frac{1}{2}\sin t + \frac{1}{2}t\cos t - te^{-t}$

96. Use partial fractions to work (*a*) Problem 44, (*b*) Problem 71, (*c*) Problem 73, (*d*) Problem 76, (*e*) Problem 77.

97. Can Problems 79(*a*) and 79(*b*) be worked by partial fractions? Explain.

HEAVISIDE'S EXPANSION FORMULA

98. Using Heaviside's expansion formula find (*a*) $\mathcal{L}^{-1}\left\{\dfrac{2s-11}{(s+2)(s-3)}\right\}$, (*b*) $\mathcal{L}^{-1}\left\{\dfrac{19s+37}{(s-2)(s+1)(s+3)}\right\}$.

Ans. (*a*) $3e^{-2t} - e^{3t}$, (*b*) $5e^{2t} - 3e^{-t} - 2e^{-3t}$

99. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$. *Ans.* $\frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$

100. Find $\mathcal{L}^{-1}\left\{\dfrac{s+5}{(s+1)(s^2+1)}\right\}$. *Ans.* $2e^{-t} + 3\sin t - 2\cos t$

101. Use Heaviside's expansion formula to work (*a*) Problem 76(*a*), (*b*) Problem 77, (*c*) Problem 88, (*d*) Problem 89, (*e*) Problem 90.

102. Find $\mathcal{L}^{-1}\left\{\dfrac{s-1}{(s+3)(s^2+2s+2)}\right\}$. Compare with Problem 91.

103. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right\}$. Compare with Problem 93.

104. Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}\right\}$. Compare with Problem 94.

105. Suppose that $f(s) = P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials as in Problem 29 but that $Q(s)=0$ has a repeated root a of multiplicity m while the remaining roots, $b_1, b_2, \ldots, b_n$ do not repeat.

(*a*) Show that

$$f(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{(s-a)^m} + \frac{A_2}{(s-a)^{m-1}} + \cdots + \frac{A_m}{s-a} + \frac{B_1}{s-b_1} + \frac{B_2}{s-b_2} + \cdots + \frac{B_n}{s-b_n}$$

(*b*) Show that $A_k = \lim_{s\to a} \dfrac{1}{(k-1)!}\dfrac{d^k}{ds^k}\{(s-a)^m f(s)\}, \quad k = 1, 2, \ldots, m.$

(*c*) Show that $\mathcal{L}^{-1}\{f(s)\} = e^{at}\left\{\dfrac{A_1 t^{m-1}}{(m-1)!} + \dfrac{A_2 t^{m-2}}{(m-2)!} + \cdots + A_m\right\} + B_1 e^{b_1 t} + \cdots + B_n e^{b_n t}.$

106. Use Problem 105 to find (*a*) $\mathcal{L}^{-1}\left\{\dfrac{2s^2-9s+19}{(s-1)^2(s+3)}\right\}$, (*b*) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{(s+1)^2(s+2)^2}\right\}$.

Ans. (*a*) $(3t-2)e^t + 4e^{-3t}$, (*b*) $t(e^{-t}-e^{-2t})$

107. Find $\mathcal{L}^{-1}\left\{\dfrac{11s^3-47s^2+56s+4}{(s-2)^3(s+2)}\right\}$. *Ans.* $(2t^2-t+5)e^{2t} + 6e^{-2t}$

108. Use Problem 105 to work (*a*) Problem 26, (*b*) Problem 44, (*c*) Problem 71, (*d*) Problem 73, (*e*) Problem 76(*b*).

109. Can the method of Problem 105 be used to work Problems 79(*a*) and 79(*b*)? Explain.

110. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right\}$ using Problem 105. Compare with Problem 95.

111. Develop a Heaviside expansion formula which will work for the case of repeated quadratic factors.

112. Find $\mathcal{L}^{-1}\left\{\dfrac{4s^4+5s^3+6s^2+8s+2}{(s-1)(s^2+2s+2)^2}\right\}$ using the method developed in Problem 111.

Ans. $e^t + e^{-t}\{(3-2t)\cos t - 3\sin t\}$

THE BETA FUNCTION

113. Evaluate each of the following: (*a*) $\displaystyle\int_0^1 x^{3/2}(1-x)^2\,dx$, (*b*) $\displaystyle\int_0^4 x^3(4-x)^{-1/2}\,dx$, (*c*) $\displaystyle\int_0^2 y^4\sqrt{4-y^2}\,dy$

Ans. (*a*) 16/315, (*b*) 4096/35, (*c*) 2π

114. Show that $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx = \pi/4$.

115. Evaluate each of the following: (*a*) $\displaystyle\int_0^{\pi/2}\cos^6\theta\,d\theta$, (*b*) $\displaystyle\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$, (*c*) $\displaystyle\int_0^{\pi}\sin^4\theta\cos^4\theta\,d\theta$.

Ans. (*a*) $5\pi/32$, (*b*) $\pi/32$, (*c*) $3\pi/128$

116. Prove that

$$\int_0^{\pi/2}\sin^p\theta\,d\theta = \int_0^{\pi/2}\cos^p\theta\,d\theta = \begin{cases} (a)\ \dfrac{1\cdot3\cdot5\cdots(p-1)}{2\cdot4\cdot6\cdots p}\,\dfrac{\pi}{2} & \text{if } p \text{ is an even positive integer,} \\ (b)\ \dfrac{2\cdot4\cdot6\cdots(p-1)}{1\cdot3\cdot5\cdots p}\cdot & \text{if } p \text{ is an odd positive integer.}\end{cases}$$

117. Given that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, show that $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$ where $0 < p < 1$.

[*Hint.* Let $x/(1+x) = y$.]

118. Use Problem 117 to show that $\displaystyle\int_0^\infty \frac{y^2\,dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$.

106. Use Problem 105 to find (a) $\mathcal{L}^{-1}\left\{\frac{2s^2-9s+19}{(s-1)^2(s+3)}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{2s+3}{(s+1)^2(s+2)^2}\right\}$.

Ans. (a) $(3t-2)e^t + 4e^{-3t}$, (b) $t(e^{-t}-e^{-2t})$

107. Find $\mathcal{L}^{-1}\left\{\frac{11s^3-47s^2+56s+4}{(s-2)^3(s+2)}\right\}$. *Ans.* $(2t^2-t+5)e^{2t} + 6e^{-2t}$

108. Use Problem 105 to work (a) Problem 26, (b) Problem 44, (c) Problem 71, (d) Problem 72, (e) Problem 76(b).

109. Can the method of Problem 105 be used to work Problems 79(a) and 79(b)? Explain.

110. Find $\mathcal{L}^{-1}\left\{\frac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right\}$ using Problem 105. Compare with Problem 95.

111. Develop a Heaviside expansion formula which will work for the case of repeated quadratic factors.

112. Find $\mathcal{L}^{-1}\left\{\frac{4s^4+5s^3+6s^2+8s+2}{(s-1)(s^2+2s+2)^2}\right\}$ using the method developed in Problem 111.

Ans. $e^t + e^{-t}\{(3-2t)\cos t - 3\sin t\}$

THE BETA FUNCTION

113. Evaluate each of the following: (a) $\int_0^1 x^{3/2}(1-x)^2\,dx$, (b) $\int_0^4 x^3(4-x)^{-1/2}\,dx$, (c) $\int_0^2 y^4\sqrt{4-y^2}\,dy$

Ans. (a) 16/315, (b) 4096/35, (c) 2π

114. Show that $\int_0^1 \sqrt{1-x^2}\,dx = \pi/4$.

115. Evaluate each of the following: (a) $\int_0^{\pi/2}\cos^6\theta\,d\theta$, (b) $\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$, (c) $\int_0^{\pi}\sin^4\theta\cos^4\theta\,d\theta$.

Ans. (a) $5\pi/32$, (b) $\pi/32$, (c) $3\pi/128$

116. Prove that

$$\int_0^{\pi/2}\sin^p\theta\,d\theta = \int_0^{\pi/2}\cos^p\theta\,d\theta = \begin{cases} \text{(a)}\ \dfrac{1\cdot3\cdot5\cdots(p-1)}{2\cdot4\cdot6\cdots p}\dfrac{\pi}{2} & \text{if } p \text{ is an even positive integer,} \\ \text{(b)}\ \dfrac{2\cdot4\cdot6\cdots(p-1)}{1\cdot3\cdot5\cdots p} & \text{if } p \text{ is an odd positive integer.} \end{cases}$$

117. Given that $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, show that $\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$ where $0<p<1$.

[*Hint.* Let $x/(1+x) = y$.]

118. Use Problem 117 to show that $\int_0^\infty \frac{y^2\,dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$.

119. Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \dfrac{\pi\sqrt{2}}{2}$.

EVALUATION OF INTEGRALS

120. Show that $\int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$.

121. Evaluate $\int_0^\infty \dfrac{\sin x}{x}\, dx$. *Ans.* $\pi/2$

122. Show that $\int_0^\infty x \cos x^3\, dx = \dfrac{\pi}{3\sqrt{3}\,\Gamma(1/3)}$.

123. Prove that if $0 < p < 1$, (*a*) $\int_0^\infty \dfrac{\sin x}{x^p}\, dx = \dfrac{\pi}{2\,\Gamma(p)\sin(p\pi/2)}$

(*b*) $\int_0^\infty \dfrac{\cos x}{x^p}\, dx = \dfrac{\pi}{2\,\Gamma(p)\cos(p\pi/2)}$.

124. Use the results in Problem 123 to verify the results of Problems 120, 121 and 122.

125. (*a*) Show that $\int_0^\infty x^2 e^{-x^2}\, dx$ converges.

(*b*) If $t > 0$, is $\mathcal{L}\left\{\int_0^\infty x^2 e^{-tx^2}\, dx\right\} = \int_0^\infty \mathcal{L}\{x^2 e^{-tx^2}\}\, dx$?

(*c*) Can the method of Problem 37 be used to evaluate the integral in (*a*)? Explain.

126. Evaluate $\int_0^t J_0(u)\, J_1(t-u)\, du$. *Ans.* $J_0(t) - \cos t$

MISCELLANEOUS PROBLEMS

127. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3+1}\right\}$. *Ans.* $\dfrac{1}{3}\left\{e^{-t} - e^{t/2}\left(\cos\dfrac{\sqrt{3}}{2}t - \sqrt{3}\sin\dfrac{\sqrt{3}}{2}t\right)\right\}$

128. Prove that $\int_a^b (x-a)^p (b-x)^q\, dx = (b-a)^{p+q+1} B(p+1, q+1)$ where $p > -1$, $q > -1$ and $b > a$.

[*Hint.* Let $x - a = (b-a)y$.]

129. Evaluate (*a*) $\int_2^4 \dfrac{dx}{\sqrt{(x-2)(4-x)}}$, (*b*) $\int_1^5 \sqrt[4]{(5-x)(x-1)}\, dx$. *Ans.* (*a*) π, (*b*) $\dfrac{2\{\Gamma(1/4)\}^2}{3\sqrt{\pi}}$

130. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}(1-e^{-s})}{s(s^2+1)}\right\}$. *Ans.* $\{1-\cos(t-1)\}\,\mathcal{U}(t-1) - \{1-\cos(t-2)\}\,\mathcal{U}(t-2)$

131. Show that $\mathcal{L}^{-1}\left\{\dfrac{e^{-x\sqrt{s}}}{\sqrt{s}}\right\} = \dfrac{e^{-x^2/4t}}{\sqrt{\pi t}}$.

132. Prove that $\int_0^t J_0(u)\sin(t-u)\, du = \frac{1}{2}t\, J_1(t)$.

133. (a) Show that the function $f(s) = \dfrac{1 - e^{-2\pi s}}{s}$ is zero for infinitely many complex values of s. What are these values? (b) Find the inverse Laplace transform of $f(s)$.

Ans. (a) $s = \pm i, \pm 2i, \pm 3i, \ldots$ (b) $F(t) = \begin{cases} 1 & t > 2\pi \\ 0 & 0 < t < 2\pi \end{cases}$ or $F(t) = \mathcal{U}(t - 2\pi)$

134. Find $\mathcal{L}^{-1}\left\{\ln\left(\dfrac{s + \sqrt{s^2+1}}{2s}\right)\right\}$. *Ans.* $\dfrac{1 - J_0(t)}{t}$

135. Show that $\displaystyle\int_0^2 u(8 - u^3)^{1/3}\, du = \frac{16\sqrt{3}\,\pi}{27}$.

136. Let $F(t) = t^2$ at all values of t which are irrational, and $F(t) = t$ at all values of t which are rational. (a) Prove that $\mathcal{L}\{F(t)\} = 2/s^3$, $s > 0$. (b) Discuss the significance of the result in (a) from the viewpoint of the uniqueness of inverse Laplace transforms.

137. Show how series methods can be used to evaluate (a) $\mathcal{L}^{-1}\{1/(s^2+1)\}$, (b) $\mathcal{L}^{-1}\{\ln(1 + 1/s)\}$, (c) $\mathcal{L}^{-1}\{\tan^{-1}(1/s)\}$.

138. Find $\mathcal{L}^{-1}\{e^{-3s - 2\sqrt{s}}\}$. *Ans.* $\dfrac{1}{\sqrt{\pi(t-3)^3}}\, e^{-1/(t-3)}\, \mathcal{U}(t-3)$

139. Show that $\displaystyle\int_0^\infty \frac{u \sin tu}{1+u^2}\, du = \frac{\pi}{2} e^{-t}$, $t > 0$.

140. If $F(t) = t^{-1/2}$, $t > 0$ and $G(t) = \begin{cases} t^{-1/2} & 0 < t < 1 \\ 0 & t > 1 \end{cases}$, show that

$$F(t) * G(t) = \begin{cases} \pi & 0 < t < 1 \\ \pi - 2\tan^{-1}\sqrt{t-1} & t > 1 \end{cases}$$

141. Show that $\mathcal{L}^{-1}\left\{\dfrac{\sqrt{s+1} - \sqrt{s}}{\sqrt{s+1} + \sqrt{s}}\right\} = \dfrac{e^{-t/2}\, I_1(t/2)}{t}$.

142. Find $\mathcal{L}^{-1}\left\{\dfrac{\sqrt{s}}{s-1}\right\}$. *Ans.* $t^{-1/2}/\sqrt{\pi} + e^t \operatorname{erf}\sqrt{t}$

143. Show that (a) $\displaystyle\int_0^{\pi/2} \sin(t \sin^2\theta)\, d\theta = \tfrac{1}{2}\pi \sin(t/2)\, J_0(t/2)$

(b) $\displaystyle\int_0^{\pi/2} \cos(t \cos^2\theta)\, d\theta = \tfrac{1}{2}\pi \cos(t/2)\, J_0(t/2)$.

144. Let $\mathcal{L}^{-1}\{f(s)\} = F(t)$ have period $T > 0$. Prove that

$$\mathcal{L}^{-1}\{f(s)(1 - e^{-sT})\} = F(t) \text{ if } 0 < t < T \text{ and zero if } t > T.$$

145. (a) Show that $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3+1}\right\} = \dfrac{t^2}{2!} - \dfrac{t^5}{5!} + \dfrac{t^8}{8!} - \dfrac{t^{11}}{11!} + \cdots$.

(b) Discuss the relationship of the result in (a) to that of Problem 127.

146. Can Heaviside's expansion formula be applied to the function $f(s) = 1/(s \cosh s)$? Explain.

147. Prove that $\int_0^\infty J_0(x^2)\,dx = 1/4\sqrt{\pi}$.

148. Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \cdots$$

$$= \frac{i}{2}\{J_0(2e^{\pi i/4}\sqrt{t}) - J_0(2e^{-\pi i/4}\sqrt{t})\}$$

149. Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \cdots$$

150. Find $\mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{s}}\right\}$. *Ans.* $t^{-1/2}/\sqrt{\pi} - e^t \operatorname{erfc}(\sqrt{t})$

151. Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s+e^{-s}}\right\} = \sum_{n=0}^{[t]} \frac{(-1)^n (t-n)^n}{n!}$$

where $[t]$ denotes the greatest integer less than or equal to t.

152. Show that $\mathcal{L}^{-1}\left\{\frac{1}{s} J_0\left(\frac{2}{\sqrt{s}}\right)\right\} = 1 - \frac{t}{(1!)^3} + \frac{t^2}{(2!)^3} - \frac{t^3}{(3!)^3} + \cdots$

Solved Problems

ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. Solve $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$.

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$\mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = \mathcal{L}\{t\}, \qquad s^2y - sY(0) - Y'(0) + y = \frac{1}{s^2}$$

$$s^2y - s + 2 + y = \frac{1}{s^2}$$

Then
$$y = \mathcal{L}\{Y\} = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$
$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$
$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

and
$$Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\} = t + \cos t - 3\sin t$$

Check: $Y = t + \cos t - 3\sin t$, $Y' = 1 - \sin t - 3\cos t$, $Y'' = -\cos t + 3\sin t$. Then $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$ and the function obtained is the required solution.

For another method, using the convolution integral, see Problem 7 and let $a = 1$, $F(t) = t$.

2. Solve $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$.

We have
$$\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\{e^{2t}\}$$

$$\{s^2y - sY(0) - Y'(0)\} - 3\{sy - Y(0)\} + 2y = \frac{4}{s-2}$$

$$\{s^2y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2}$$

$$(s^2 - 3s + 2)y + 3s - 14 = \frac{4}{s-2}$$

$$y = \frac{4}{(s^2-3s+2)(s-2)} + \frac{14-3s}{s^2-3s+2}$$
$$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$
$$= \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

Thus
$$Y = \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\} = -7e^t + 4e^{2t} + 4te^{2t}$$

which can be verified as the solution.

3. Solve $Y'' + 2Y' + 5Y = e^{-t}\sin t,\ Y(0) = 0,\ Y'(0) = 1.$

We have
$$\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}\sin t\}$$
$$\{s^2y - sY(0) - Y'(0)\} + 2\{sy - Y(0)\} + 5y = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$
$$\{s^2y - s(0) - 1\} + 2\{sy - 0\} + 5y = \frac{1}{s^2+2s+2}$$
$$(s^2+2s+5)y - 1 = \frac{1}{s^2+2s+2}$$
$$y = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+2)(s^2+2s+5)}$$
$$= \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

Then [see Problem 28, Page 60]
$$Y = \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

4. Solve $Y''' - 3Y'' + 3Y' - Y = t^2e^t,\ Y(0) = 1,\ Y'(0) = 0,\ Y''(0) = -2.$

We have
$$\mathcal{L}\{Y'''\} - 3\mathcal{L}\{Y''\} + 3\mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = \mathcal{L}\{t^2e^t\}$$
$$\{s^3y - s^2Y(0) - sY'(0) - Y''(0)\} - 3\{s^2y - sY(0) - Y'(0)\} + 3\{sy - Y(0)\} - y = \frac{2}{(s-1)^3}$$

Thus
$$(s^3 - 3s^2 + 3s - 1)y - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$
$$y = \frac{s^2-3s+1}{(s-1)^3} + \frac{2}{(s-1)^6}$$
$$= \frac{s^2-2s+1-s}{(s-1)^3} + \frac{2}{(s-1)^6}$$
$$= \frac{(s-1)^2-(s-1)-1}{(s-1)^3} + \frac{2}{(s-1)^6}$$
$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

and
$$Y = e^t - te^t - \frac{t^2e^t}{2} + \frac{t^5e^t}{60}$$

5. Find the general solution of the differential equation in Problem 4.

In this case, the initial conditions are arbitrary. If we assume $Y(0) = A,\ Y'(0) = B,\ Y''(0) = C,$ we find as in Problem 4,
$$(s^3y - As^2 - Bs - C) - 3(s^2y - As - B) + 3(sy - A) - y = \frac{2}{(s-1)^3}$$

or
$$y = \frac{As^2 + (B-3A)s + 3A - 3B + C}{(s-1)^3} + \frac{2}{(s-1)^6}$$

Since A, B and C are arbitrary, so also is the polynomial in the numerator of the first term on the right. We can thus write

$$y = \frac{c_1}{(s-1)^3} + \frac{c_2}{(s-1)^2} + \frac{c_3}{s-1} + \frac{2}{(s-1)^6}$$

and invert to find the required general solution

$$y = \frac{c_1 t^2}{2} e^t + c_2 t e^t + c_3 e^t + \frac{t^5 e^t}{60}$$

$$= c_4 t^2 + c_5 t e^t + c_6 e^t + \frac{t^5 e^t}{60}$$

where the c_k's are arbitrary constants.

It should be noted that finding the general solution is easier than finding the particular solution since we avoid the necessity of determining the constants in the partial fraction expansion.

6. Solve $Y'' + 9Y = \cos 2t$ if $Y(0) = 1$, $Y(\pi/2) = -1$.

Since $Y'(0)$ is not known, let $Y'(0) = c$. Then

$$\mathcal{L}\{Y''\} + 9\mathcal{L}\{Y\} = \mathcal{L}\{\cos 2t\}$$

$$s^2 y - s\,Y(0) - Y'(0) + 9y = \frac{s}{s^2+4}$$

$$(s^2+9)y - s - c = \frac{s}{s^2+4}$$

and

$$y = \frac{s+c}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)}$$

$$= \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

$$= \frac{4}{5}\left(\frac{s}{s^2+9}\right) + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)}$$

Thus

$$Y = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$$

To determine c, note that $Y(\pi/2) = -1$ so that $-1 = -c/3 - 1/5$ or $c = 12/5$. Then

$$Y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$$

7. Solve $Y'' + a^2 Y = F(t)$, $Y(0) = 1$, $Y'(0) = -2$.

We have

$$\mathcal{L}\{Y''\} + a^2\mathcal{L}\{Y\} = \mathcal{L}\{F(t)\} = f(s)$$

$$s^2 y - s\,Y(0) - Y'(0) + a^2 y = f(s)$$

$$s^2 y - s + 2 + a^2 y = f(s)$$

and so

$$y = \frac{s-2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}$$

Then using the convolution theorem,

$$Y = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2+a^2}\right\}$$

$$= \cos at - \frac{2\sin at}{a} + F(t) * \frac{\sin at}{a}$$

$$= \cos at - \frac{2\sin at}{a} + \frac{1}{a}\int_0^t F(u)\sin a(t-u)\,du$$

Note that in this case the actual Laplace transform of $F(t)$ does not enter into the final solution.

8. Find the general solution of $Y'' - a^2Y = F(t)$.

Let $Y(0) = c_1$, $Y'(0) = c_2$. Then taking the Laplace transform, we find

$$s^2y - sc_1 - c_2 - a^2y = f(s)$$

or

$$y = \frac{sc_1 + c_2}{s^2 - a^2} + \frac{f(s)}{s^2 - a^2}$$

Thus

$$Y = c_1\cosh at + \frac{c_2}{a}\sinh at + \frac{1}{a}\int_0^t F(u)\sinh a(t-u)\,du$$

$$= A\cosh at + B\sinh at + \frac{1}{a}\int_0^t F(u)\sinh a(t-u)\,du$$

which is the required general solution.

ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

9. Solve $tY'' + Y' + 4tY = 0$, $Y(0) = 3$, $Y'(0) = 0$.

We have

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + \mathcal{L}\{4tY\} = 0$$

or

$$-\frac{d}{ds}\{s^2y - s\,Y(0) - Y'(0)\} + \{sy - Y(0)\} - 4\frac{dy}{ds} = 0$$

i.e.,

$$(s^2+4)\frac{dy}{ds} + sy = 0$$

Then

$$\frac{dy}{y} + \frac{s\,ds}{s^2+4} = 0$$

and integrating

$$\ln y + \tfrac{1}{2}\ln(s^2+4) = c_1 \quad \text{or} \quad y = \frac{c}{\sqrt{s^2+4}}$$

Inverting, we find

$$Y = c\,J_0(2t)$$

To determine c note that $Y(0) = c\,J_0(0) = c = 3$. Thus

$$Y = 3\,J_0(2t)$$

10. Solve $tY'' + 2Y' + tY = 0, \; Y(0+) = 1, \; Y(\pi) = 0.$

Let $Y'(0+) = c$. Then taking the Laplace transform of each term

$$-\frac{d}{ds}\{s^2y - s\,Y(0+) - Y'(0+)\} + 2\{sy - Y(0+)\} - \frac{d}{ds}y = 0$$

or

$$-s^2y' - 2sy + 1 + 2sy - 2 - y' = 0$$

i.e.,

$$-(s^2+1)y' - 1 = 0 \quad \text{or} \quad y' = \frac{-1}{s^2+1}$$

Integrating,

$$y = -\tan^{-1} s + A$$

Since $y \to 0$ as $s \to \infty$, we must have $A = \pi/2$. Thus

$$y = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\frac{1}{s}$$

Then by the Example following *Theorem 1-13* on Page 5,

$$Y = \mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} = \frac{\sin t}{t}$$

This satisfies $Y(\pi) = 0$ and is the required solution.

11. Solve $Y'' - tY' + Y = 1, \; Y(0) = 1, \; Y'(0) = 2.$

We have

$$\mathcal{L}\{Y''\} - \mathcal{L}\{tY'\} + \mathcal{L}\{Y\} = \mathcal{L}\{1\} = \frac{1}{s}$$

i.e.,

$$s^2y - s\,Y(0) - Y'(0) + \frac{d}{ds}\{sy - Y(0)\} + y = \frac{1}{s}$$

or

$$s^2y - s - 2 + sy' + y + y = \frac{1}{s}$$

Then

$$sy' + (s^2+2)y = s + 2 + \frac{1}{s}$$

or

$$\frac{dy}{ds} + \left(s + \frac{2}{s}\right)y = 1 + \frac{2}{s} + \frac{1}{s^2}$$

An integrating factor is $e^{\int \left(s+\frac{2}{s}\right)ds} = e^{\frac{1}{2}s^2 + 2\ln s} = s^2 e^{\frac{1}{2}s^2}$. Then

$$\frac{d}{ds}\{s^2 e^{\frac{1}{2}s^2} y\} = \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{1}{2}s^2}$$

or integrating,

$$\begin{aligned} y &= \frac{1}{s^2} e^{-\frac{1}{2}s^2} \int \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{1}{2}s^2}\, ds \\ &= \frac{1}{s^2} e^{-\frac{1}{2}s^2} \int (s^2 + 2s + 1)\, e^{\frac{1}{2}s^2}\, ds \\ &= \frac{1}{s^2} e^{-\frac{1}{2}s^2} [s e^{\frac{1}{2}s^2} + 2e^{\frac{1}{2}s^2} + c] \\ &= \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2} e^{-\frac{1}{2}s^2} \end{aligned}$$

To determine c, note that by series expansion,

$$y = \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2}(1 - \tfrac{1}{2}s^2 + \tfrac{1}{8}s^4 - \cdots)$$

$$= \frac{1}{s} + \frac{c+2}{s^2} - c(\tfrac{1}{2} - \tfrac{1}{8}s^2 + \cdots)$$

Then since $\mathcal{L}^{-1}\{s^k\} = 0,\ k = 0, 1, 2, \ldots,$ we obtain on inverting,

$$Y = 1 + (c+2)t$$

But $Y'(0) = 2$, so that $c = 0$ and we have the required solution

$$Y = 1 + 2t$$

SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

12. Solve $\begin{cases} \dfrac{dX}{dt} = 2X - 3Y \\ \dfrac{dY}{dt} = Y - 2X \end{cases}$ subject to $X(0) = 8,\ Y(0) = 3$.

Taking the Laplace transform, we have, if $\mathcal{L}\{X\} = x,\ \mathcal{L}\{Y\} = y$,

$$sx - 8 = 2x - 3y \quad \text{or} \quad (1) \quad (s-2)x + 3y = 8$$

$$sy - 3 = y - 2x \quad \text{or} \quad (2) \quad 2x + (s-1)y = 3$$

Solving (1) and (2) simultaneously,

$$x = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

Then

$$X = \mathcal{L}^{-1}\{x\} = 5e^{-t} + 3e^{4t}$$

$$Y = \mathcal{L}^{-1}\{y\} = 5e^{-t} - 2e^{4t}$$

13. Solve $\begin{cases} X'' + Y' + 3X = 15e^{-t} \\ Y'' - 4X' + 3Y = 15\sin 2t \end{cases}$ subject to $X(0) = 35,\ X'(0) = -48,\ Y(0) = 27,$ $Y'(0) = -55$.

Taking the Laplace transform, we have

$$s^2x - s(35) - (-48) + sy - 27 + 3x = \frac{15}{s+1}$$

$$s^2y - s(27) - (-55) - 4\{sx - 35\} + 3y = \frac{30}{s^2+4}$$

or

$$(s^2+3)x + sy = 35s - 21 + \frac{15}{s+1} \tag{1}$$

$$-4sx + (s^2+3)y = 27s - 195 + \frac{30}{s^2+4} \tag{2}$$

Solving (*1*) and (*2*) simultaneously,

$$x = \frac{\begin{vmatrix} 35s - 21 + \dfrac{15}{s+1} & s \\ 27s - 195 + \dfrac{30}{s^2+4} & s^2+3 \end{vmatrix}}{\begin{vmatrix} s^2+3 & s \\ -4s & s^2+3 \end{vmatrix}}$$

$$= \frac{35s^3 - 48s^2 + 300s - 63}{(s^2+1)(s^2+9)} + \frac{15(s^2+3)}{(s+1)(s^2+1)(s^2+9)} - \frac{30s}{(s^2+1)(s^2+4)(s^2+9)}$$

$$= \frac{30s}{s^2+1} - \frac{45}{s^2+9} + \frac{3}{s+1} + \frac{2s}{s^2+4}$$

$$y = \frac{\begin{vmatrix} s^2+3 & 35s - 21 + \dfrac{15}{s+1} \\ -4s & 27s - 195 + \dfrac{30}{s^2+4} \end{vmatrix}}{\begin{vmatrix} s^2+3 & s \\ -4s & s^2+3 \end{vmatrix}}$$

$$= \frac{27s^3 - 55s^2 - 3s - 585}{(s^2+1)(s^2+9)} + \frac{60s}{(s+1)(s^2+1)(s^2+9)} + \frac{30(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)}$$

$$= \frac{30s}{s^2+9} - \frac{60}{s^2+1} - \frac{3}{s+1} + \frac{2}{s^2+4}$$

Then

$$X = \mathcal{L}^{-1}\{x\} = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$$

$$Y = \mathcal{L}^{-1}\{y\} = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t$$

APPLICATIONS TO MECHANICS

14. A particle P of mass 2 grams moves on the X axis and is attracted toward origin O with a force numerically equal to $8X$. If it is initially at rest at $X = 10$, find its position at any subsequent time assuming (*a*) no other forces act, (*b*) a damping force numerically equal to 8 times the instantaneous velocity acts.

(*a*) Choose the positive direction to the right [see Fig. 3-5]. When $X > 0$, the net force is to the left (i.e. is negative) and must be given by $-8X$. When $X < 0$ the net force is to the right (i.e. is positive) and must be given by $-8X$. Hence in either case the net force is $-8X$. Then by Newton's law,

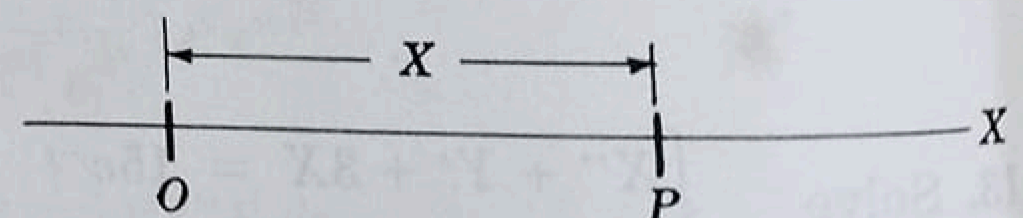

Fig. 3-5

$$(\text{Mass}) \cdot (\text{Acceleration}) = \text{Net force}$$

$$2 \cdot \frac{d^2X}{dt^2} = -8X$$

or

$$\frac{d^2X}{dt^2} + 4X = 0 \tag{1}$$