CSER 2207: Numerical Analysis

Lecture-8
Interpolation and
Polynomial Approximation

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Forward Difference

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *differences* of y. Denoting these differences by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0, \qquad \Delta y_1 = y_2 - y_1, \ldots, \qquad \Delta y_{n-1} = y_n - y_{n-1},$$

where Δ is called the *forward difference operator* and $\Delta y_0, \Delta y_1, \ldots,$ are called *first forward differences*. The differences of the first forward differences

$$\begin{aligned}
\Delta^2 y_0 = \Delta y_1 - \Delta y_0 &= y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0, \\
\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 &= y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0 \\
\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 &= y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0) \\
&= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.
\end{aligned}$$

Newton's Forward Difference Interpolation Formula

Given the set of $(n+1)$ values, viz., $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. let

$$x_i = x_0 + ih, \quad i = 0, 1, 2, \ldots, n.$$

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$\left.\begin{aligned} y_n(x) = a_0 &+ a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots \\ &+ a_n(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}). \end{aligned}\right\} \tag{3.9}$$

Cont…

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

$$a_0 = y_0;\ a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h};\ a_2 = \frac{\Delta^2 y_0}{h^2 2!};\ a_3 = \frac{\Delta^3 y_0}{h^3 3!};\cdots;\ a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots, a_n$, Eq. (3.9) gives

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0, \tag{3.10}$$

which is *Newton's forward difference interpolation formula* and is useful for interpolation *near the beginning* of a set of tabular values.

Newton's Backward Difference Interpolation Formula

Instead of assuming $y_n(x)$ as in (3.9) if we choose it in the form

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1).$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots, x_2, x_1, x_0$, we obtain (after some simplification)

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n y_n, \quad (3.14)$$

where $p = (x - x_n)/h$.

This is *Newton's backward difference interpolation formula* and it uses tabular values to the left of y_n. This formula is therefore useful for interpolation *near the end of* the tabular values.

Example

Example 3.4 Find the cubic polynomial which takes the following values: $y(1) = 24$, $y(3) = 120$, $y(5) = 336$, and $y(7) = 720$. Hence, or otherwise, obtain the value of $y(8)$.

We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1 + 2p$ or $p = (x-1)/2$. Substituting this value of p in Eq. (3.10), we obtain

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$= x^3 + 6x^2 + 11x + 6.$$

To determine $y(8)$, we observe that $p = 7/2$. Hence, formula (3.10) gives:

$$y(8) = 24 + \frac{7}{2}(96) + \frac{(7/2)(7/2-1)}{2}(120) + \frac{(7/2)(7/2-1)(7/2-2)}{6}(48) = 990.$$

Extrapolation

Note: This process of finding the value of y for some value of x *outside* the given range is called extrapolation and this example demonstrates the fact that if a tabulated function is a polynomial, then both interpolation and extrapolation would give exact values.

Example

Example 3.6 Values of x (in degrees) and $\sin x$ are given in the following table:

x (in degrees)	$\sin x$
15	0.2588190
20	0.3420201
25	0.4226183
30	0.5
35	0.5735764
40	0.6427876

Determine the value of $\sin 38°$.

Solution

The difference table is

x	$\sin x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
15	0.2588190					
		0.0832011				
20	0.3420201		−0.0026029			
		0.0805982		−0.0006136		
25	0.4226183		−0.0032165		0.0000248	
		0.0773817		−0.0005888		0.0000041
30	0.5		−0.0038053		0.0000289	
		0.0735764		−0.0005599		
35	0.5735764		−0:0043652			
		0.0692112				
40	0.6427876					

Cont…

To find sin 38°, we use Newton's backward difference formula with $x_n = 40$ and $x = 38$. This gives

$$p = \frac{x - x_n}{h} = \frac{38 - 40}{5} = -\frac{2}{5} = -0.4.$$

Hence, using formula (3.14), we obtain

$$y(38) = 0.6427876 - 0.4\,(0.0692112) + \frac{-0.4(-0.4-1)}{2}(-0.0043652)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(-0.0005599)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{24}(0.0000289)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{120}(0.0000041)$$

$$= 0.6427876 - 0.02768448 + 0.00052382 + 0.00003583 - 0.00000120$$

$$= 0.6156614.$$

Thank You