CSER 2207_8: Numerical Analysis-I

Lecture-4
Solution of equation in single variable

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2.2 Fixed-Point Iteration

A *fixed point* for a function is a number at which the value of the function does not change when the function is applied.

Definition 2.2 The number p is a **fixed point** for a given function g if $g(p) = p$. ■

Fixed-point results occur in many areas of mathematics, and are a major tool of economists for proving results concerning equilibria. Although the idea behind the technique is old, the terminology was first used by the Dutch mathematician L. E. J. Brouwer (1882–1962) in the early 1900s.

In this section we consider the problem of finding solutions to fixed-point problems and the connection between the fixed-point problems and the root-finding problems we wish to solve. Root-finding problems and fixed-point problems are equivalent classes in the following sense:

- Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as

$$g(x) = x - f(x) \quad \text{or as} \quad g(x) = x + 3f(x).$$

- Conversely, if the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

has a zero at p.

Although the problems we wish to solve are in the root-finding form, the fixed-point form is easier to analyze, and certain fixed-point choices lead to very powerful root-finding techniques.

We first need to become comfortable with this new type of problem, and to decide when a function has a fixed point and how the fixed points can be approximated to within a specified accuracy.

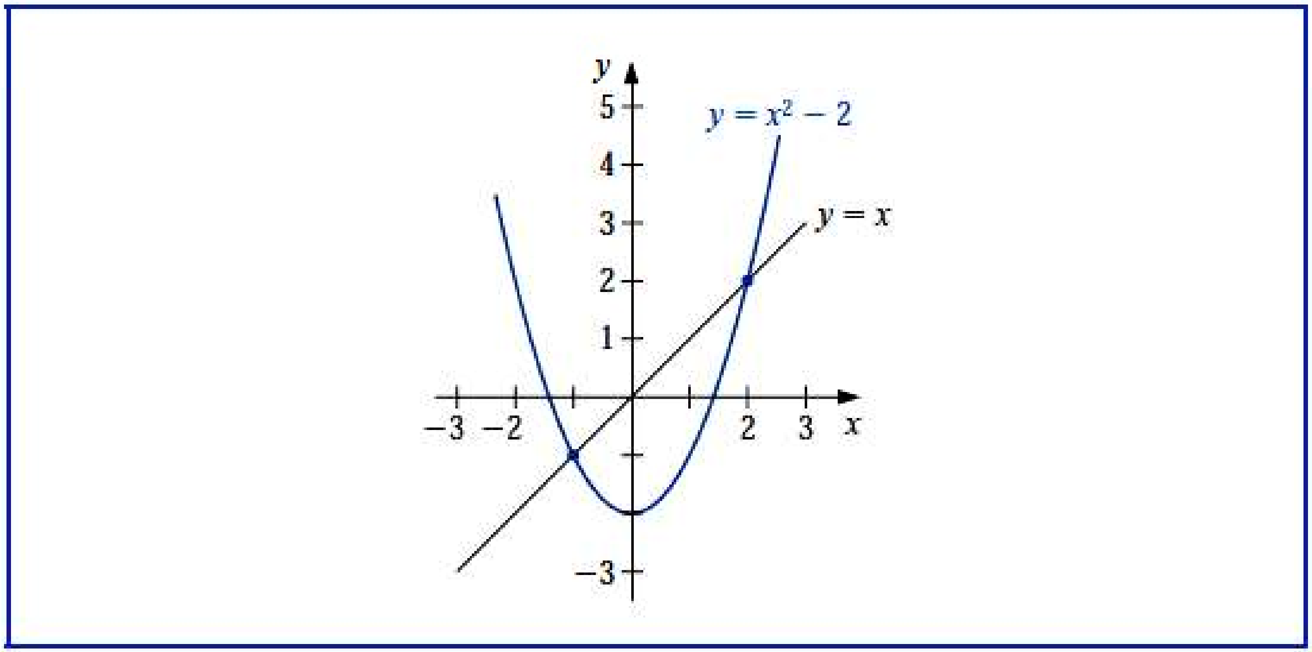
Example 1 Determine any fixed points of the function $g(x) = x^2 - 2$.

Solution A fixed point p for g has the property that

$$p = g(p) = p^2 - 2 \quad \text{which implies that} \quad 0 = p^2 - p - 2 = (p+1)(p-2).$$

A fixed point for g occurs precisely when the graph of $y = g(x)$ intersects the graph of $y = x$, so g has two fixed points, one at $p = -1$ and the other at $p = 2$. These are shown in Figure 2.3. ■

Figure 2.3



Theorem

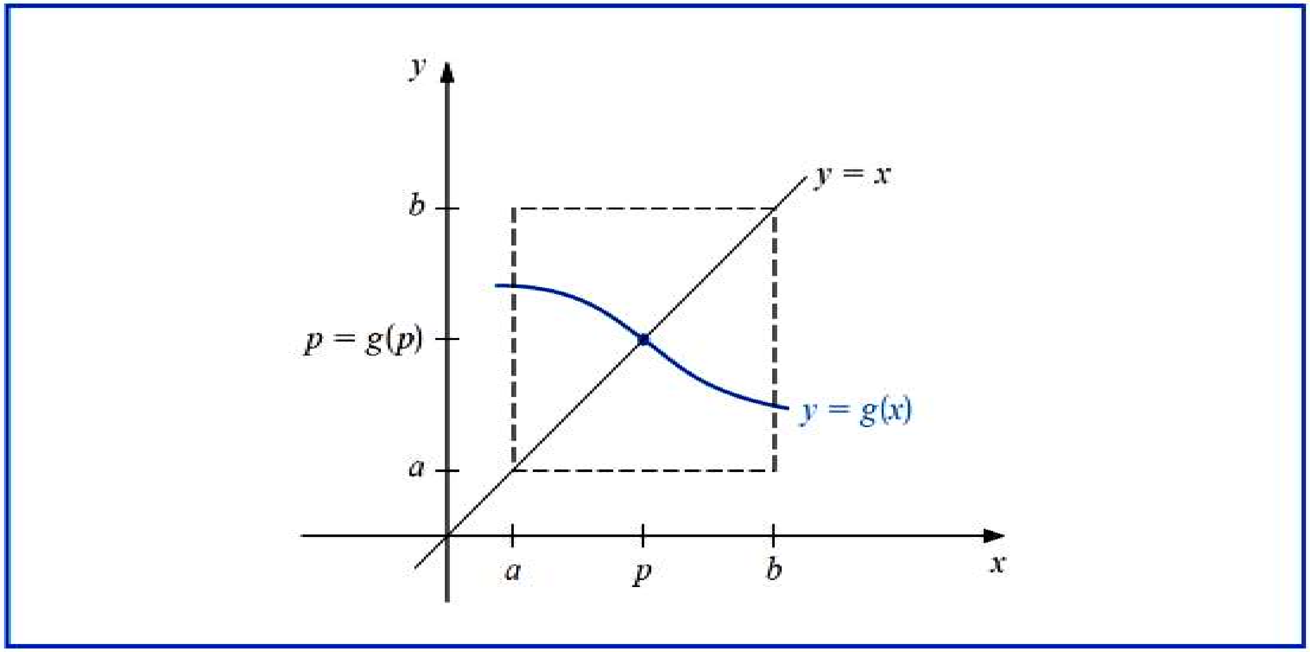
Theorem 2.3 **(i)** If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.

(ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$. (See Figure 2.4.) ∎

Figure 2.4



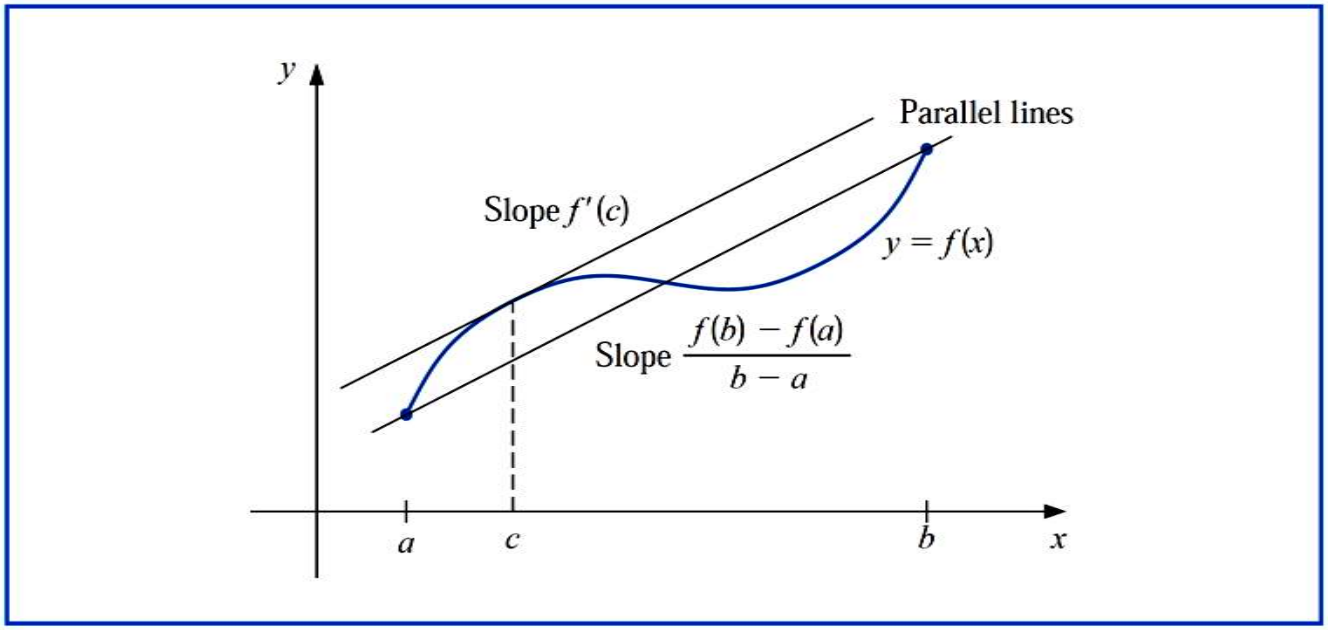
Mean Value Theorem

Theorem 1.8 **(Mean Value Theorem)**

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with (See Figure 1.4.)

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Figure 1.4



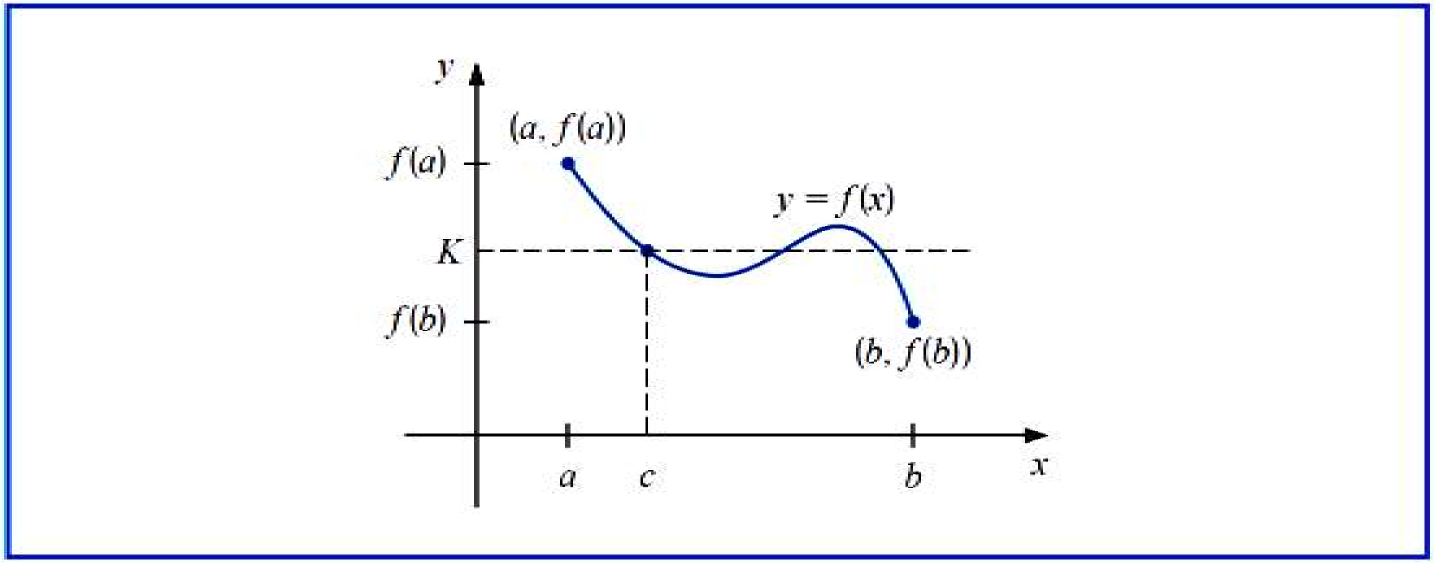
Intermediate Value Theorem

Theorem 1.11 **(Intermediate Value Theorem)**

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$. ■

Figure 1.7 shows one choice for the number that is guaranteed by the Intermediate Value Theorem. In this example there are two other possibilities.

Figure 1.7



Existence of Fixed-Point

Proof

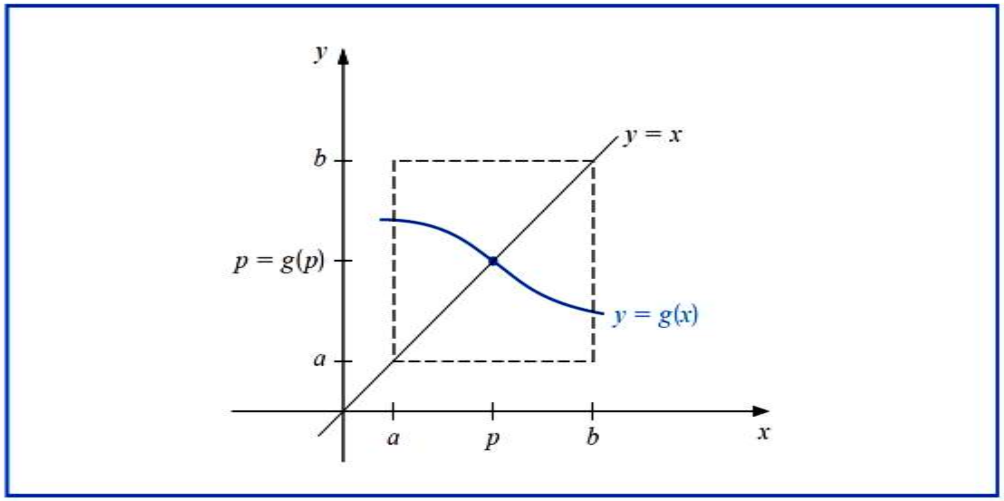
(i) If $g(a) = a$ or $g(b) = b$, then g has a fixed point at an endpoint. If not, then $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a, b]$, with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

The Intermediate Value Theorem implies that there exists $p \in (a, b)$ for which $h(p) = 0$. This number p is a fixed point for g because

$$0 = h(p) = g(p) - p \quad \text{implies that} \quad g(p) = p.$$

Figure 2.4



Uniqueness of Fixed-Point

Proof

(ii) Suppose, in addition, that $|g'(x)| \leq k < 1$ and that p and q are both fixed points in $[a, b]$. If $p \neq q$, then the Mean Value Theorem implies that a number ξ exists between p and q, and hence in $[a, b]$, with

$$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

Thus

$$|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \leq k|p - q| < |p - q|,$$

which is a contradiction. This contradiction must come from the only supposition, $p \neq q$. Hence, $p = q$ and the fixed point in $[a, b]$ is unique. ■ ■ ■

Convergence Property

Theorem 2.4 (Fixed-Point Theorem)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$. ■

Proof Theorem 2.3 implies that a unique point p exists in $[a,b]$ with $g(p) = p$. Since g maps $[a,b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$, and $p_n \in [a,b]$ for all n. Using the fact that $|g'(x)| \leq k$ and the Mean Value Theorem 1.8, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)||p_{n-1} - p| \leq k|p_{n-1} - p|,$$

where $\xi_n \in (a,b)$. Applying this inequality inductively gives

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots \leq k^n|p_0 - p|. \quad (2.4)$$

Since $0 < k < 1$, we have $\lim_{n\to\infty} k^n = 0$ and

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p| = 0.$$

Hence $\{p_n\}_{n=0}^{\infty}$ converges to p. ■ ■ ■

Algorithm

Fixed-Point Iteration

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. *(Compute p_i.)*

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. *(Update p_0.)*

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(The procedure was unsuccessful.)
STOP.

Illustration

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation. For example, to obtain the function g described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm\frac{1}{2}(10 - x^3)^{1/2}.$$

To obtain a positive solution, $g_3(x)$ is chosen. It is not important for you to derive the functions shown here, but you should verify that the fixed point of each is actually a solution to the original equation, $x^3 + 4x^2 - 10 = 0$.

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\dfrac{10}{4 + x}\right)^{1/2}$

(e) $x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

Cont…

With $p_0 = 1.5$, Table 2.2 lists the results of the fixed-point iteration for all five choices of g.

Table 2.2

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

The actual root is 1.365230013, as was noted in Example 1 of Section 2.1. Comparing the results to the Bisection Algorithm given in that example, it can be seen that excellent results have been obtained for choices (c), (d), and (e) (the Bisection method requires 27 iterations for this accuracy). It is interesting to note that choice (a) was divergent and that (b) became undefined because it involved the square root of a negative number. □

Example

Example 2.7 Find the root of the equation $2x = \cos x + 3$ correct to three decimal places.

We rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3) \qquad \text{(i)}$$

so that

$$\phi(x) = \frac{1}{2}(\cos x + 3),$$

and

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1.$$

Hence the iteration method can be applied to the eq. (i) and we start with $x_0 = \pi/2$. The successive iterates are

$$x_1 = 1.5, \quad x_2 = 1.535, \quad x_3 = 1.518,$$
$$x_4 = 1.526, \quad x_5 = 1.522, \quad x_6 = 1.524,$$
$$x_7 = 1.523, \quad x_8 = 1.524.$$

Hence we take the solution as 1.524 correct to three decimal places.

Thank You