Solve the following differential equations by the method of undetermined coefficient.

(i) $(D^2-2D+1)y = x\sin x$

Solution:

Given differential equation is,

$$(D^2-2D+1)y = x\sin x \quad \text{--- (1)}$$

let $y = e^{mx}$ be a trial solution of $(D^2-2D+1)y=0$ --- (2)

then ~~particular integral of (1) is,~~

~~$y_p = Ax^2$~~

$$(2) \Rightarrow (m^2-2m+1)e^{mx} = 0$$

$\therefore$ Auxiliary equation is, $m^2-2m+1 = 0$, since $e^{mx} \neq 0$

$$\text{or, } (m-1)^2 = 0$$

$$\text{or, } m = 1, 1$$

undetermined co-efficient function is, $y_c = (c_1+c_2x)e^x$

$\therefore$ $y_c = (c_1+c_2x)e^x$, where c_1, c_2 are arbitrary constants.

Here undetermined co-efficient function is $R(x) = x\sin x$.

Undetermined co-efficient set of $x\sin x$ is,

$$S = \{x\sin x, x\cos x, \sin x, \cos x\}$$

$\therefore$ particular integral of (1) is,

$$y_p = Ax\sin x + Bx\cos x + C\sin x + E\cos x \quad \text{--- (3)}$$

$\therefore y_p' = A\sin x + Ax\cos x + B\cos x - Bx\sin x$
$+ C\cos x - E\sin x.$

$y_p'' = A\cos x + A\cos x - Ax\sin x - B\sin x - B\sin x$
$- Bx\cos x - C\sin x - E\cos x$

Now in (1) we put y_p, y_p', y_p'' instead of y, Dy, D^2y respectively we get

$2A\cos x - Ax\sin x - 2B\sin x - Bx\cos x - C\cos x$
$- E\cos x - 2A\sin x - 2Ax\cos x - 2B\cos x + 2Bx\sin x - 2C\cos x + 2E\sin x + Ax\sin x + Bx\cos x$
$+ C\sin x + E\cos x = x\sin x$

or, $(2A - 2B - 2C)\cos x + (\cancel{+}2B + 2E - 2A)\sin x$
$+ 2Bx\sin x - 2Ax\cos x = x\sin x$

Equating like terms from both sides we get

$A = 0, \; 2B = 1$ or, $B = \frac{1}{2}$

$2A - 2B - 2C = 0$

or, $0 - 1 - 2C = 0$

or, $C = -\frac{1}{2}$

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$-2B - 2A + 2E = 0$

or, $-1 - 0 + 2E = 0$

or, $E = \frac{1}{2}$

Now putting the values of A, B, C, E in (3) we get,

$$y_p = 0 + \frac{1}{2} x \cos x - \frac{1}{2}\sin x + \frac{1}{2}\cos x$$

$$\Rightarrow y_p = \frac{1}{2}(1+x)\cos x - \frac{1}{2}\sin x .$$

General solution is,

$$y = y_c + y_p$$

$$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{1}{2}(1+x)\cos x - \frac{1}{2}\sin x .$$

(ii) Given differential equation is,

$$\frac{d^2y}{dx^2} - 3 \cdot \frac{dy}{dx} + 2y = 4x^2 \quad \text{--- (1)}$$

let $y = e^{mx}$ be the trial solution of

$$\frac{d^2y}{dx^2} - 3 \cdot \frac{dy}{dx} + 2y = 0 \quad \text{--- (2)}$$

then, $(m^2 - 3m + 2)e^{mx} = 0$

$\therefore$ Auxiliary equation is,

$$m^2 - 3m + 2 = 0, \; e^{mx} \neq 0$$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow m(m-2) - 1(m-2) = 0$$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 1, 2$.

$\therefore y_c = c_1 \cdot e^{x} + c_2 \cdot e^{2x}$, where c_1 and c_2 are arbitrary constants.

Here undetermined coefficient function is,
$R(x) = 4x^2$

Undetermined co-efficient set of $4x^2$ is $S = \{x^2, x, 1\}$

$\therefore$ particular Integral of (1) is.

$$y_p = Ax^2 + Bx + C \quad \ldots (3)$$

$$\therefore y'_p = 2xA + B + 0$$
$$= 2Ax + B$$

$$\therefore y''_p = 2A.$$

Now in (1) we put y_p, y'_p, y''_p instead of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ respectively we get,

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$$

or $$2Ax^2 + (2B - 6A)x + 2A - 3B + 2C = 4x^2$$

Equating like terms from both sides we get,

$2A = 4$,

$\Rightarrow A = 2$

and

$2A - 3B + 2C = 0$

$2B - 6A = 0$

or, $2B = 6 \cdot 2$

or, $B = 6$

or, $2 \cdot 2 - 3 \cdot 6 + 2C = 0$

or, $C = 7$

Now putting the values of A, B, C in (3) we get,

$$y_p = 2x^2 + 6x + 7$$

$\therefore$ General solution is,

$$y = y_c + y_p$$

$$\therefore y = c_1 e^{x} + c_2 e^{2x} + 2x^2 + 6x + 7$$

(iii) Given equation is,

$$(D^2 - 2D + 3) y = x^3 + \sin x \quad \text{--- (1)}$$

let $y = e^{mx}$ be the trial solution of

$(D^2 - 2D + 3) y = 0$ --- (2) then

(2) $\Rightarrow$ $(\cancel{D^2 - 2D + 3})^{m^2 - 2m + 3} e^{mx} = 0$ where $e^{mx} \neq 0$

$\Rightarrow$ D^2 $\cancel{m^2 - 2m + 3 = 0}$

$\therefore$ Auxiliary equation is,

$$m^2 - 2m + 3 = 0$$

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$$\therefore m = \frac{-(-2) \pm \sqrt{4 - 3 \cdot 12}}{2 \cdot 1}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 1}{2} = 1 \pm i\sqrt{2}$$

$\therefore$ Complementary function $y_c = e^x (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$

Here, undetermined co-efficient function $R(x)$ is, $x^3 + \sin x$

$\therefore$ undetermined co-efficient set of x^3 is, $\{x^3, x^2, x, 1\}$

$\therefore$ underdetermined coefficient set of $\sin x$ is $\{\sin x, \cos x\}$

$\therefore$ particular integral of (1) is,

$y_p = Ax^3 + Bx^2 + Cx + D + E\sin x + F\cos x$ ... (3)

$\therefore y_p' = 3x^2A + 2Bx + C + 0 + E\cos x - F\sin x$

$\therefore y_p'' = 6xA + 2B - E\sin x - F\cos x$

Now, in (1) we put y_p, y_p', y_p'' instead of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ respectively we get,

$$(D^2-2D+3)y = x^3+\sin x$$

$$6Ax+2B-E\sin x-F\cos x-6x^2A$$

$$-4Bx-2C-2E\cos x+2F\sin x$$

$$+3(Ax^3+Bx^2+Cx+D+E\sin x+F\cos x) = x^3+\sin x$$

$$\Rightarrow 6Ax+2B-E\sin x-F\cos x-6x^2A-4Bx-2C$$

$$-2E\cos x+2F\sin x+3Ax^3+3Bx^2+3Cx+3D$$

$$+3E\sin x+3F\cos x = x^3+\sin x \quad -(iii)$$

Equating like terms from both sides of the equation (iii) we get,

$$3A=1, \quad \Rightarrow A=\frac{1}{3}$$

$$-6A+3B=0$$

$$\Rightarrow -6\cdot\frac{1}{3}+3B=0$$

$$\Rightarrow 3B=2$$

$$\therefore B=\frac{2}{3}$$

$$6A-4B+3C=0$$

$$\Rightarrow 6\cdot\frac{1}{3}-4\cdot\frac{2}{3}+3C=0$$

$$\Rightarrow 2-\frac{8}{3}+3C=0$$

$$\Rightarrow 3C=\frac{8}{3}-2$$

$$\Rightarrow 3C=\frac{8-6}{3}$$

$$\Rightarrow 3C=\frac{2}{3}$$

$$\Rightarrow C=\frac{2}{3}\times\frac{1}{3}$$

$$\therefore C=\frac{2}{9}$$

$$3D-2C+2B=0$$

$$\Rightarrow 3D-2\times\frac{2}{9}+2\times\frac{2}{3}=0$$

$$\Rightarrow 3D=\frac{4}{9}-\frac{4}{3}$$

$$\Rightarrow 3D=\frac{4-12}{9}$$

$$\Rightarrow 3D=\frac{-8}{9}$$

$$\Rightarrow D=\frac{-8}{27}$$

$\therefore -E + 2F + 3E = 1$

$\Rightarrow 2F + 2E = 1$ — (3)

$-F - 2E + 3F = 0$

$\Rightarrow \cancel{3F}\ 2F - 2E = 0$ — (4)

(3)+(4) করে,

$4F = 1$

$\Rightarrow F = \frac{1}{4}$

Now, if $F = \frac{1}{4}$ then,

$2E = 2 \times \frac{1}{4}$

$\Rightarrow 2E = \frac{1}{2}$

$\Rightarrow E = \frac{1}{4}$

Now, putting the values of A, B, C, D, E and F in (3) we get,

$$y_p = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}\sin x + \frac{1}{4}\cos x.$$

$\therefore$ General solution is,

$$y = y_c + y_p$$

$$= e^x(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + \frac{1}{27}(9x^3 + 18x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)$$

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(Ans.)

Ex-2 $y_2 - 3y_1 = 8e^{3x} + 4\sin x \cdots (1)$

Let $y = e^{mx}$ be the trial solution of $y_2 - 3y_1 = 0$ (2)

$\therefore (m^2 - 3m) e^{mx} = 0 \quad , e^{mx} \neq 0$

$\Rightarrow m^2 - 3m = 0$

$\Rightarrow m(m-3) = 0$

$\Rightarrow m = 0, m = 3$

$\therefore y_c = c_1 \cdot e^{0x} + c_2 \cdot e^{3x}$

$= c_1 + c_2 \cdot e^{3x}$

$\therefore$ Let undetermined co-efficient function $R(x)$ is,

$8e^{3x} + 4\sin x$

$\therefore$ undetermined co-efficient set of $8e^{3x} + 4\sin x$

is $S_1 = \{ e^{3x} \}$

Since, e^{3x} is an element of y_c; Hence multiplying S_1 by x,

$S_1 = \{ xe^{3x} \}$

Now, undetermined co-efficient set of $\sin x$

is $S_2 = \{ \sin x, \cos x \}$

$\therefore$ particular Integral is $y_p =$

$Axe^{3x} + B\sin x + C\cos x$.

$y'' - 3y = 8e^{3x} + 4\sin x$

$\therefore y'_p = A e^{3x} + 3Axe^{3x} + B\cos x - C\sin x$

$\Rightarrow y''_p = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x} - B\sin x - C\cos x$
$= 6Ae^{3x} + 9Axe^{3x} - B\sin x - C\cos x$

Now, putting y_p, y_p' and y_p'' instead of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, in (1) we get,

$6Ae^{3x} + 9Axe^{3x} - B\sin x - C\cos x - 3(Ae^{3x} + 3Axe^{3x} + B\cos x - C\sin x)$
$= 8e^{3x} + 4\sin x$

$\Rightarrow 6Ae^{3x} + 9Axe^{3x} - B\sin x - C\cos x - 3Ae^{3x} - 9Axe^{3x} - 3B\cos x + 3C\sin x$
$= 8e^{3x} + 4\sin x.$

Now equating like terms from both sides of the equations,

$6A - 3A = 8$; $\cancel{9A}$ $-1 - 3B = 0$

$\Rightarrow 3A = 8$; $\Rightarrow 3B = -1$

$\Rightarrow A = \frac{8}{3}$; $\Rightarrow B = \frac{-1}{3}$

$\Rightarrow 3Ae^{3x} - B\sin x + 3C\sin x - \cos x - 3B\cos x = 8e^{3x} + 4\sin x$

$\Rightarrow 3Ae^{3x} + (3C - B)\sin x + (-3B - 1)\cos x = 8e^{3x} + 4\sin x$

$-B + 3C = 4$

$\Rightarrow \cancel{3C = 4 + B}$

$\Rightarrow \cancel{3C = 4 - \frac{1}{3}}$

$\Rightarrow \cancel{3C = \frac{12-1}{3}}$

$\Rightarrow \cancel{3C = \frac{11}{3}}$

$\Rightarrow \cancel{3C = \frac{11}{3}}$

$\Rightarrow \cancel{C = \frac{11}{9}}$

$\therefore \cancel{P.I. = \frac{8}{3} x e^{3x} + -\frac{1}{3} \sin x + \frac{11}{9} \cos x}$

P.I. =

$\therefore y_p = \frac{8}{3} x e^{3x} - \frac{1}{3} \sin x + \frac{13}{9} \cos x$

$\therefore$ The solution is,

$y = y_c + y_p$

$= c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} - \frac{1}{3} \sin x + \frac{13}{9} \cos x$

$3A = 8$	$-3B - 1 = 0$	$3C - B = 4$
$\Rightarrow A = \frac{8}{3}$	$\Rightarrow B = -\frac{1}{3}$	$\Rightarrow 3C = 4 + \frac{1}{3}$
		$C = \frac{13}{3 \times 3} = \frac{13}{9}$

<u>Ex: 03</u> | Solve the following differential equations by the method of undetermined co-efficients.

(1) $\frac{d^3y}{dx^3} + 2 \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + x^2$ —— (i)

let $y = e^{mx}$ be the trial solution of $\frac{d^3y}{dx^3} + 2 \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ - - (ii)

then, $(m^3 + 2m^2 - m - 2) e^{mx} = 0$

$\therefore$ Auxiliary Equation is,

$$m^3+2m^2-m-2=0, \quad e^{mx}\neq 0$$

$$\Rightarrow m^2(m+2)-1(m+2)=0$$

$$\Rightarrow (m+2)(m^2-1)=0$$

$$\Rightarrow m=-2, \quad m=1,-1.$$

$$\therefore y_c = \cancel{c_1+c_2x} \; c_1e^{-2x}+\cancel{(c_2+c_3x)}\,e^{x}+c_3e^{-x}$$

Here, Undetermined function is,

$R(x) = e^x+x^2$

$\therefore$ undetermined set of e^x is, $S_1=\{e^x\}$

Since e^x is an element of y_c. Hence, multiply S_1 by x, $\therefore S_1'=\{xe^x\}$

undetermined set of x^2 is $S_2=\{x^2, x, 1\}$

$\therefore$ particular Integral of (1) is,

$$y_p = Axe^x+Bx^2+Cx+D = \cancel{e^x+x^2} \quad (ii)$$

$$y_p' = Ae^x+Axe^x+2Bx+C+0$$

$$y_p'' = Ae^x+Ae^x+Axe^x+2B+0$$

$$y_p''' = 2Ae^x+Ae^x+0$$

$$= 3Ae^x$$

Now, putting y_p, y_p', y_p'' and y_p''' instead of

$\frac{d^3y}{d\theta}$ y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$

$\therefore$ From (1),

$$3Ae^x + 2Ae^x + 2Ae^x + 2Axe^x + 4B - Ae^x - Axe^x - 2Bx - C - 2(Axe^x + Bx^2 + Cx + D) = e^x + x^2$$

$$\Rightarrow 6Ae^x - Axe^x + 4B - 2Bx - C - 2Bx^2 - 2Cx - 2D = e^x + x^2$$

Equating ~~same~~ like terms from both sides of the equation,

$-2B = 1$

$\Rightarrow B = -\frac{1}{2}$

$6A = 1$

$\Rightarrow A = \frac{1}{6}$

$4B - C - 2D = 0$

$\Rightarrow 4\cdot\left(-\frac{1}{2}\right) - \frac{1}{2} - 2D = 0$

$\Rightarrow -2 - \frac{1}{2} - 2D = 0$

$\Rightarrow 2D = -2 - \frac{1}{2}$

$\Rightarrow 2D = \frac{-4-1}{2}$

$\Rightarrow 2D = \frac{-5}{2}$

$\Rightarrow D = \frac{-5}{4}$

$-2B - 2C = 0$

$\Rightarrow B - C = 0$

$\Rightarrow \cancel{B}\, C = -B$

$\Rightarrow C = \frac{1}{2}$

$\therefore y_p = \frac{1}{6} x e^x - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{5}{4}$

$\therefore$ The General solution is, $y = y_c + y_p$

$= c_1 e^{-2x} + c_2 e^x + c_3 e^{-x} + \frac{1}{6} x e^x - \frac{1}{2} x^2$
$+ \frac{1}{2} x - \frac{5}{4}$

(11) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$ --- (i)

let $y = e^{mx}$ be the trial solution of

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ --- (ii)

Then, $(m^2 - 3m + 2) e^{mx} = 0$, ~~$e^{mx} \neq 0$~~

$\therefore$ ~~Aux~~ Auxiliary equation is,

$m^2 - 3m + 2 = 0, e^{mx} \neq 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\therefore y_c = c_1 e^x + c_2 e^{2x}$ $\quad\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 1, 2.$

$\therefore$ Undetermined function is, $R(x)$

$= 2x^2 + e^x + 2xe^x + 4e^{3x}$

$\therefore$ Undetermined Coefficient set of x^2 is, $S_1 = \{x^2, x, 1\}$

" " " e^x is $S_2 = \{e^x\}$

Here, e^x is an element of y_c. $\therefore S_2 = \{xe^x\}$

undetermined coefficient set of xe^x is $S_3 = \{xe^x, e^x\}$
→ Hence e^x is an element of y_c ∴ $S_3' = \{x^2e^x, xe^x\}$
" " " " e^{3x} is $S_4 = \{e^{3x}\}$

∴ particular integral of ① is,

$$y_p = Ax^2 + Bx + C + \cancel{De^x} + Exe^x + Fe^{3x} + Gx^2e^x$$

$$\Rightarrow y_p' = 2xA + B + 0 + Exe^x + Ee^x + F3e^{3x} + Gx^2e^x + 2Gxe^x$$

$$\Rightarrow y_p'' = 2A + 0 + Ee^x + Exe^x + Ee^x + F9e^{3x} + 2Gxe^x + Gx^2e^x + 2Gxe^x + 2Ge^x$$

$$\Rightarrow y_p'' = 2A + Ee^x + Exe^x + Ee^x + 9Fe^{3x} + 2Gxe^x + Gx^2e^x + 2Gxe^x + 2Ge^x$$

Now putting the value of y_p, y_p', y_p'' instead of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ respectively,

$$2A + Ee^x + Exe^x + Ee^x + 9Fe^{3x} + 2Gxe^x + Gx^2e^x + 2Gxe^x + 2Ge^x - 3(2xA + B + Exe^x + Ee^x + F3e^{3x} + Gx^2e^x + 2Gxe^x) + 2(Ax^2 + Bx + C + Exe^x + Fe^{3x} + Gx^2e^x) = 0$$

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

$$\Rightarrow 2A + 2Ee^x + 2Ge^x + 4Gxe^x + Exe^x + 9Fe^{3x} + Gx^2e^x$$
$$- 6xA - 3B - 3Exe^x - 3Ee^x - 9Fe^{3x} - 3Gx^2e^x$$
$$- 6Gxe^x + 2Ax^2 + 2Bx + 2C + 2Exe^x + 2Fe^{3x} + 2Gx^2e^x = 0$$

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

$$\Rightarrow 2A - 3B + 2C + (2E + 2G - 3E)\, e^x + (9F - 9F + 2F)e^{3x}$$
$$+ (4G + E - 3E - 6G + 2E)\, xe^x + (G - 3G + 2G)\, x^2e^x$$
$$+ (\cancel{4G+E} \cdot (-6A + 2B))x + 2Ax^2 = \cancel{0}$$
$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

Now, equating like terms of both sides of the equations,

$$2A = 2, \qquad -6A + 2B = 0, \qquad 2F = 4$$
$$\Rightarrow A = 1 \qquad \Rightarrow 2B = 6A \qquad \Rightarrow F = 2$$
$$\Rightarrow 2B = 6$$
$$\therefore B = 3 \qquad\qquad 2G - E = 1$$

$$2A - 3B + 2C = 0 \qquad\qquad 2E - G = 1$$
$$\Rightarrow 2\cdot 1 - 3\cdot 3 + 2C = 0 \qquad \Rightarrow 2E = 1 + G$$
$$\Rightarrow 2 - 9 + 2C = 0 \qquad \Rightarrow E = 0 \qquad \Rightarrow E = 2$$
$$= -2$$
$$\Rightarrow 2C = 7$$
$$= -3$$
$$\Rightarrow C = 7/2$$

$-2C_2 = 2$

$\Rightarrow C_2 = -1$

$\therefore y_p = x^2 + 3x + \frac{7}{2} + 0 + 2e^{3x} - x^2 e^x$

$\therefore$ The general solution is, $y = y_c + y_p$

$= c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^x - 3xe^x$ ~~$-3xe^x$~~

(Ans.)

Chapter: 6 Modeling

Modeling in Biology: let at a time t, population is N, P or y.

$\therefore \frac{dy}{dt} \propto y$ or, $\frac{dP}{dt} \propto P$ or, $\frac{dN}{dt} \propto N$

$\Rightarrow \frac{dP}{dt} = kP$

$\Rightarrow \int \frac{dP}{P} = \int k\,dt$

$\Rightarrow \ln P = k \int dt$

$\Rightarrow \ln P = k[t]$

Example: 1

কোন নির্দিষ্ট সময়ে জনসংখ্যার বৃদ্ধির হার ঐ সময়ের জনসমষ্টির সাথে সমানুপাতিক। জনসংখ্যা 50 বছরে দ্বিগুণ হইলে, কত বছরে উহা তিনগুণ হবে।

The population of a community is known to increase at a rate proportional to the number of people present at time t. If the population has doubled in 50 years, how long will it take to triple?

Solution:

let at time t, the number of people is y. then the rate of increase of people is $\frac{dy}{dt}$.

Since, $\frac{dy}{dt} \propto y$

$\Rightarrow \frac{dy}{dt} = ky$ where $k>0$, since population is increased

$\Rightarrow \frac{dy}{y} = k\,dt \quad \cdots (1)$

let when $t=0$, then $y = y_0$

Given that when $t=50$ then $y = 2y_0$

Now integrating (1) by taking limits of t and y.

i.e. $\int_{y_0}^{2y_0} \frac{dy}{y} = k \int_0^{50} dt$

$\Rightarrow [\ln y]_{y_0}^{2y_0} = k\cdot[t]_0^{50}$

$\Rightarrow \ln 2y_0 - \ln y_0 = k(50-0)$

$\Rightarrow \ln \frac{2y_0}{y_0} = k\cdot 50$

$\Rightarrow 50k = \ln 2$

$\Rightarrow k = \frac{\ln 2}{50} \quad \cdots (4)$

let at time $t = t_1$ the population will be $y = 3y_0$

Again when $t=0$ then $y = y_0$

Again integrating (1) by taking limits of t and y.

i.e. $\displaystyle\int_{y_0}^{3y_0} \frac{dy}{y} = k\int_0^{t_1} dt$

$\Rightarrow [\ln y]_{y_0}^{3y_0} = k[t]_0^{t_1}$

$\Rightarrow \ln 3y_0 - \ln y_0 = k(t_1 - 0)$

$\Rightarrow \ln \frac{3y_0}{y_0} = kt_1$

$\Rightarrow kt_1 = \ln 3$

$\Rightarrow t_1 = \frac{\ln 3}{k}$

$\Rightarrow t_1 = \frac{\ln 3}{(\ln 2)/50}$

$\Rightarrow t_1 = \frac{50 \ln 3}{\ln 2}$

$\simeq 79$ year (approximate)

~~79~~ After 79 year, the population will be tripled.

উদাহরণ : 02| একটি জনসংখ্যার N, $\frac{dN}{dt} = kN$ নিয়মে বৃদ্ধি পায় যেখানে k একটি ধনাত্মক ধ্রুবক। সময় t বছরে পরিমাপ করা হইলে, জনসংখ্যার তিনগুণ হওয়ার সময় নির্ণয় কর। $\lim_{t \to \infty} N(t)$ নির্ণয় কর।

A populatation N grows according to the law $\frac{dN}{dt} = kN$, where k is a positive constant. Determine how long it takes the population to triple in size, where the time ~~as~~ t is measured in years. Find $\lim_{t\to\infty} N(t)$

Solution:

Given that, $\frac{dN}{dt} = KN$

$\Rightarrow \frac{dN}{N} = K dt \quad \cdots (1)$

let at time $t=0$, the population was $N = N_0$ and at time $t = T$, the population will be $N = 3N_0$ on

$T = ?$

Integrating (1) we get

$\int_{N_0}^{3N_0} \frac{dN}{N} = K \int_0^T dt$

$\Rightarrow [\ln N]_{N_0}^{3N_0} = K[t]_0^T$

$\Rightarrow \ln 3N_0 - \ln N_0 = K(T-0)$

$\Rightarrow \ln \frac{3N_0}{N_0} = KT$

$\Rightarrow \ln 3 = KT$

$\Rightarrow T = \frac{\ln 3}{K}$

$\therefore$ After $\frac{\ln 3}{K}$, the population will be tripled.

Integrating (1) we get,

$\ln N = Kt + \ln C$

$\Rightarrow \ln N - \ln C = Kt$

$\Rightarrow \ln \frac{N}{C} = Kt$

$\Rightarrow \frac{N}{C} = e^{Kt}$

$\Rightarrow N = C \cdot e^{Kt} \cdots (2)$

let at time $t=0$ the population was $N = N_0$

Now putting $t=0$ and $N = N_0$ in (2) we get

$N_0 = C \cdot e^0 \Rightarrow C = N_0$

$\therefore (2) \Rightarrow N(t) = N_0 \cdot e^{Kt}$

$\Rightarrow \lim_{t\to\infty} N(t) = N_0 \cdot \lim_{t\to\infty} e^{Kt}$

$= N_0 e^{\infty}$

$\Rightarrow \lim_{t \to \infty} N(t) = \infty$

$\therefore$ After infinite time, the population will be infinite.

উদাহরণ:03

কোনো স্থানের জীবাণুর বৃদ্ধি হয় ঐ সময়ে জীবাণুর সংখ্যার সহিত সমানুপাতিক। প্রাথমিক অবস্থায় জীবাণুর সংখ্যা x এবং এক ঘন্টা পরে উহার সংখ্যা $\frac{3x}{2}$ হইলে, কত সময়ে উহা তিনগুণ হবে।

In a certain bacteria culture the rate of increase in the number of bacteria is proportional to the number present. If initial number of bacteria is x and the number is $\frac{3x}{2}$ after one hour, how long will it take to triple.

Solution:

let at time t, the number of bacteria is y then the rate of increase of bacteria is $\frac{dy}{dt}$.

Since, $\frac{dy}{dt} \propto y$

$\Rightarrow \frac{dy}{dt} = ky$, $k > 0$, since the number of bacteria is increased.

$\Rightarrow \frac{dy}{y} = k\,dt$ --- (1)

Given that number of bacteria is x at time $t = 0$ and the number is $\frac{3x}{2}$ at time $t = 1$ hr.

Now integrating (1) by taking limits of t and y.

i.e., $$\int_{x}^{3x/2} \frac{dy}{y} = k \cdot \int_{0}^{1} dt$$

$$\Rightarrow \left[\ln y\right]_{x}^{3x/2} = k \cdot [t]_0^1$$

$$\Rightarrow \ln\frac{3x}{2} - \ln x = k(1-0)$$

$$\Rightarrow \ln\left(\frac{3x/2}{x}\right) = k$$

$$\Rightarrow k = \ln\left(\frac{3}{2}\right) \cdots (2)$$

Again integrating (1) by taking limits of t and y,

$$\int_{x}^{3x} \frac{dy}{y} = k\int_{0}^{T} dt$$

or, $$\left[\ln y\right]_{x}^{3x} = k[t]_0^T$$

$$\Rightarrow \ln 3x - \ln x = k(T-0)$$

let at time $t = T$ the number of bacteria will be $y = 3x$ ∴

$$\Rightarrow \ln\frac{3x}{x} = kT$$

$$\Rightarrow \ln 3 = kT \Rightarrow k = \frac{\ln 3}{T}$$

$$\Rightarrow T = \frac{\ln 3}{k} \Rightarrow T = \frac{\ln 3}{\ln(3/2)}$$

$$= 2.71 \text{ hour.}$$

উদাহরণ-14

কোনো জীবাণু কালচারে জীবাণুর বৃদ্ধির হার ঐ সময়ে জীবাণু সংখ্যার সমানুপাতিক। যদি 3 ঘন্টা পরে উহার সংখ্যা 400 এবং 10 ঘন্টা পরে উহার সংখ্যা 2000 হয়, তাহা হইলে প্রাথমিক অবস্থায় উহার সংখ্যা কত ছিল?

In a certain bacteria culture the rate of increase in the number of bacteria is proportional to the number present. If that number is 400 after 3 hours and 2000 after 10 hours, then find the number initially present.?

Solution: let y be the number of bacteria after t hours, then the rate of increase of bacteria $= \frac{dy}{dt}$

since, $\frac{dy}{dt} \propto y$

$\Rightarrow \frac{dy}{dt} = ky$, when $k =$ Constant

$\Rightarrow \frac{dy}{y} = k\,dt \cdots (1)$

When $t = 3$ then $y = 400$

when $t = 10$ then $y = 2000$

Now integrating (1) by taking limits of t and y i.e., $\int_{400}^{2000} \frac{dy}{y} = k \int_{3}^{10} dt$

$\Rightarrow [\ln y]_{400}^{2000} = k[t]_3^{10}$

$\Rightarrow \ln 2000 - \ln 400 = k(10-3)$

$\Rightarrow \ln \frac{2000}{400} = k \cdot 7$

$\Rightarrow 7k = \ln 5 \cdots (2)$

let when $t = 0$, then number of bacteria $y = x$ and when $t = 3$ then $y = 400$.

Now integrating (1) by taking limits of t and y.

$$\int_{x}^{400} \frac{dy}{y} = k\int_{0}^{3} dt$$

$$\Rightarrow \left[\ln y\right]_{x}^{400} = k\left[t\right]_{0}^{3}$$

$$\Rightarrow \ln 400 - \ln x = k[3-0]$$

$$\Rightarrow \ln \frac{400}{x} = 3k$$

$$\Rightarrow \frac{400}{x} = e^{3k}$$

$$\Rightarrow \frac{x}{400} = e^{-3k}$$

$$\Rightarrow x = 400e^{-3k}$$

$$\Rightarrow x = 400\left(e^{7k}\right)^{-3/7}$$

$$\Rightarrow x = 400\left(e^{\ln 5}\right)^{-3/7} \quad [4(i)]$$

$$\Rightarrow x = 400(5)^{-3/7}$$

$$= \frac{400}{5^{3/7}}$$

$$= \frac{400}{5^{.43}}$$

$$\Rightarrow x = \frac{400}{1.997}$$

$$\approx \frac{400}{2}$$

$$= 200$$

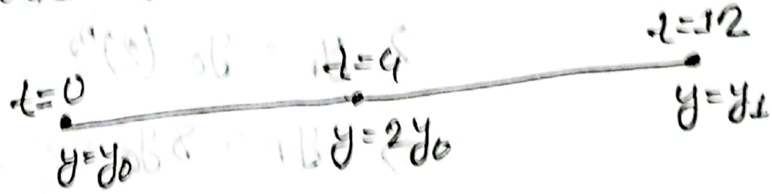
$\therefore$ At primary state, the number of bacteria was $= 200$

<u>উদাহরণ-5:</u> কোন জীবাণু চাষে জীবাণুর বৃদ্ধির হার ঐ সময়ের জীবাণুর সংখ্যার সমানুপাতিক। 4 ঘণ্টায় উহার সংখ্যা দ্বিগুণ হলে 12 ঘণ্টা পর উহার সংখ্যা কত হইবে? যদি 3 ঘণ্টা পরে উহার সংখ্যা 10^4 এবং 5 ঘণ্টা পরে উহার সংখ্যা 4×10^4 হয়, তাহলে প্রাথমিক অবস্থায় উহার সংখ্যা কত ছিল?

In a certain bacteria culture, the rate of increase in the number of bacteria is proportional to the number present. If the number double in 4 hours, how many will be present in 12 hours? If the number 10^4 in 3 hours and 4×10^4 in 5 hours, then find the numbers initially present?

Solution:

Let y be the number of bacteria after t hours, then the rate of increase of bacteria is $\frac{dy}{dt}$.



Since, $\frac{dy}{dt} \propto y$

$\Rightarrow \frac{dy}{dt} = ky$ when $k>0$ and since the number of bacteria is increased

$\Rightarrow \frac{dy}{y} = k\,dt \quad \cdots (1)$

Integrating (1) we get, $\ln y = kt + \ln c$

$\Rightarrow \ln\frac{y}{c} = kt$

$\Rightarrow y = c\cdot e^{kt} \quad \cdots (2)$

Let when $t=0$ then $y=y_0$, Now putting $t=0$ and $y=y_0$ in (2) we get.

$y_0 = ce^0 \Rightarrow y_0 = c$

$\therefore$ (2) $\Rightarrow y = y_0 e^{kt} \quad \cdots (3)$

when $t=4$ then $y=2y_0$ ∴ Now putting the values of t and y in (3) we get,

$$2y_0 = y_0 e^{4K}$$

$$\Rightarrow e^{4K} = 2 \cdots (4)$$

let when $t=12$ then the number of bacteria $y=y_1$

$$\therefore (3) \Rightarrow y_1 = y_0 e^{12K}$$

$$\Rightarrow y_1 = y_0 (e^{4K})^3$$

$$\Rightarrow y_1 = y_0 (2)^3$$

$$\Rightarrow y_1 = 8y_0$$

∴ After 12 hours, the numbers of bacteria will be 8 times.

part: 2) when $t=3$ then $y=10^4$

when ~~$y=4$~~ $t=5$ then $y = 4\times10^4$.

Integrating (1) by taking the limits of t and y,

$$\int_{10^4}^{4\times10^4} \frac{dy}{y} = K\int_3^5 dt$$

$$\Rightarrow \left[\ln y\right]_{10^4}^{4\times10^4} = K\left[t\right]_3^5$$

$$\Rightarrow \ln 4\times10^4 - \ln 10^4 = K(5-3)$$

$$\Rightarrow 2K = \ln\frac{4\times10^4}{10^4}$$

$$\Rightarrow 2K = \ln 4$$

$\Rightarrow 2k = \ln 2^2$

$\Rightarrow 2k = 2\ln 2$

$\Rightarrow k = \ln 2$

$\therefore$ At primary state when $t = 3$, then $y = 10^4$;

Now taking the limits of t and y and integrating ①,

$$\int_{y_0}^{10^4} \frac{dy}{y} = k\int_0^3 dt$$

$$\Rightarrow \left[\ln y\right]_{y_0}^{10^4} = k\left[t\right]_0^3$$

$$\Rightarrow \ln 10^4 - \ln y_0 = k(3-0)$$

$$\Rightarrow \frac{10^4}{y_0} = 3k$$

$$\Rightarrow y_0 = e^{-3k} \cdot 10^4$$

$$\Rightarrow y_0 = 10^4 \cdot (e^k)^{-3}$$

$$\Rightarrow y_0 = 10^4 (e^{\ln 2})^{-3} \text{ [by 5]}$$

$$\Rightarrow y_0 = 10^4 \cdot (2)^{-3}$$

$$\Rightarrow y_0 = \frac{100 \times 100}{2^2 \times 2}$$

$$\Rightarrow y_0 = 25 \times 50$$

$$\Rightarrow y_0 = 1250$$

$\therefore$ Initially the number of bacteria was 1250.

উদাহরণ: 6 তেজস্ক্রিয় কণার একটি নমুনাতে নমুনায় তেজস্ক্রিয় কণার পরিমাণের হ্রাস উক্ত সময়ে কণার সংখ্যার সমানুপাতিক। 1500 বছর পর নমুনাটির তেজস্ক্রিয় কণার পরিমাণ আদি সংখ্যার অর্ধেক দেখা যায়। 4500 বছর পর তেজস্ক্রিয় কণার শতকরা কত অংশ অবশিষ্ট থাকবে? এক দশমাংশ হবে কত বছর পর কণার সংখ্যার

[The rate at which radioactive nuclei decay in proportional to the number of such nuclei that are present in a given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1500 years. What percentage of the original radioactive nuclei will remain 4,500 years? In how many years will only one tenth of the original number remain?]

Solution: Let at time t, quantity of nuclei radioactive nuclei is y, then the rate of radioactive decay is $\frac{dy}{dt}$

Since $\frac{dy}{dt} \propto y$

$\therefore \frac{dy}{dt} = -ky$ [যেহেতু তেজস্ক্রিয়তার হ্রাস পায় তাই $k>0$

$\Rightarrow \frac{dy}{y} = -k\,dt \quad \text{--- (1)}$

Integrating this we get,

$\ln y = -Kt + \ln c$

$\Rightarrow \ln y - \ln c = -Kt$

$\Rightarrow \ln \frac{y}{c} = -Kt$

$\Rightarrow \frac{y}{c} = e^{-Kt}$

$\Rightarrow y = c.e^{-Kt}$ --(2)

Let initially, i.e., when $t=0$ then $y=y_0$

Now putting $t=0$ and $y=y_0$ in (2)

$y_0 = ce^0 \Rightarrow c = y_0$

$\therefore$ (2) $\Rightarrow y = y_0 \cdot e^{-Kt}$ --(3)

When $t = 1500$ then $y = \frac{y_0}{2}$

Now putting the values of t and y in (3),

$\frac{y_0}{2} = y_0 e^{-1500K}$

$\Rightarrow e^{-1500K} = \frac{1}{2}$ --(4)

$\Rightarrow e^{1500K} = 2$

$\Rightarrow 1500K = \ln 2$ --(4)

Now putting $t = 4500$ in (3),

$y = y_0 e^{-4500K}$

or, $y = y_0 (e^{-1500K})^3$

$= y_0 \left(\frac{1}{2}\right)^3 = \frac{y_0}{8}$

∴ ~~After 4500 years,~~ ও সুতরাং 4500 বছর পর তেজস্ক্রিয় পদার্থ $\frac{1}{8}$ অংশ অর্থাৎ 12.50%. অবশিষ্ট থাকবে।

<u>দ্বিতীয় অংশ:</u> Let after t_1 years, the number of nuclei will be $y = \frac{y_0}{10}$.

$$\therefore (3) \Rightarrow \frac{y_0}{10} = y_0 \cdot e^{-kt_1}$$

$$\Rightarrow e^{-kt_1} = 10$$

$$\Rightarrow kt_1 = \ln 10$$

$$\Rightarrow t_1 = \frac{\ln 10}{k} = \frac{1500 \ln 10}{1500k}$$

$$= \frac{1500 \ln 10}{\ln 2}$$

$$= \frac{3453.877}{0.693}$$

$$\therefore t_1 = 4984 \text{ year(approxi.)}$$

[by 4)

<u>উদাহরণ:-7</u>: কোন পদার্থের তেজস্ক্রিয়তার হ্রাসের হার পদার্থের পরিমাণের সমানুপাতিক। এবং, 6 ঘন্টা পরে পদার্থটির পরিমাণ 3% কমে যায়, প্রাথমিক অবস্থায় পদার্থের পরিমাণ 100 gm হলে 24 ঘন্টা পরে পদার্থের

পরিমাণ নিরূপণ কর, কত সময়ে উক্ত পদার্থের অর্ধেক হ্রাস পাবে।

[The rate at which radioactive nuclei decay is proportional to the quantity of the nuclei, 3% of orginal nuclei have undergone, dis-integration in a period of 6 hour. If initial quantity of nuclei is 100 gm then find the quantity of nuclei after 24 hour. In how many times, half of the original nuclei lost ?

Solution:

Let at time t, the quantity of nuclei y, then the rate of radioactive nuclei decay is $\frac{dy}{dt}$.

$$\text{Since, } \frac{dy}{dt} \propto y$$

$$\therefore \frac{dy}{dt} = -ky$$ যেহেতু, তেজস্ক্রিয়তার হ্রাস পায় এবং $k>0$

$$\Rightarrow \frac{dy}{y} = -k\,dt \quad \cdots (1)$$

Now Integrating this we get,

$$\ln y = -kt + \ln c$$

$\Rightarrow \ln y - \ln c = -kt$

$\Rightarrow \ln \frac{y}{c} = -kt$

$\Rightarrow \frac{y}{c} = e^{-kt}$

$\Rightarrow y = ce^{-kt}$ --(2)

When $t=0$, then $y = y_0$ 100

Now putting $t=0$ and $y=100$ in (2)

$100 = ce^{0}$

$\Rightarrow c = 100$.

$\therefore (2) \Rightarrow y = 100e^{-kt}$ --- (3)

When $t=6$ then, $y = 100 - 100 \times \frac{3}{100} = 97$

When $t=0$ then, $y=100$

Integrating (1) by taking the limits of

t and y,

$$\int_{100}^{97} \frac{dy}{y} = -k \int_{0}^{6} dt$$

$$\Rightarrow [\ln y]_{100}^{97} = -k [t]_0^6$$

$$\Rightarrow \ln 97 - \ln 100 = -k(6-0)$$

$$\Rightarrow \ln \frac{97}{100} = -6k \text{ --(4)}$$

$\Rightarrow 6K = \ln \frac{100}{97}$

$\Rightarrow 6K = \ln(1.03092) = 0.03045 \cdots (5)$

Let when $t = 24$ then $y = x$

$\therefore (2) \Rightarrow x = 100 \cdot e^{-24K}$

$\Rightarrow x = 100 \cdot (e^{-6K})^4 = 100 (e^{\ln(97/100)})^4$; by (4)

$\Rightarrow x = 100 \left(\frac{97}{100}\right)^4 = 100 \left(\frac{97}{100}\right)\left(\frac{97}{100}\right)^3$

$\Rightarrow x = 97(0.912673) = 88.53$ gm

দ্বিতীয় অংশ: Let at time $t = t_1$, the quantity of nuclei $y = 50$ then

$(3) \Rightarrow 50 = 100 \cdot e^{-Kt_1}$

$\Rightarrow e^{Kt_1} = 2$

$\Rightarrow Kt_1 = \ln 2$

$\Rightarrow t_1 = \frac{\ln 2}{K} = \frac{6 \ln 2}{6K} = \frac{4.15888}{0.03045}$ by (5)

$\therefore t_1 = 136.58$ ঘন্টা।

উদাহরণ 18(i)

যখন বাতাসের তাপমাত্রা 290°C তখন একটি বস্তু 10 মিনিটে 370°C হতে 330°C এ শীতল হয়। 40 মিনিট পর উহার তাপমাত্রা কত হবে?

[A body cools from 370°C to 330°C in air which is maintained to 290°C. What is the temperature after 40 minutes?]

Solution:

Let at time t, the temperature of the body be T°C and temperature of air is 290°C.

Difference between the temperature of air and the body is $= T - 290$

$\therefore$ Rate of change of temperature of the body $= \frac{dT}{dt}$

$\therefore \frac{dT}{dt} \propto T - 290$

$\Rightarrow \frac{dT}{dt} = -K(T-290)$ [when K ধ্রুবক সমানুপাতিক ধ্রুবক]

$$\Rightarrow \frac{dT}{T-290} = -k\,dt \quad \cdots (1)$$

When $t=0$ then $T=370$

When $t=10$ then $T=330$.

Now, integrating ① by taking limits of t and T, i.e.,

$$\int_{370}^{330} \frac{dT}{T-290} = -K\int_0^{10} dt$$

$$\Rightarrow \left[\ln(T-290)\right]_{370}^{330} = -K\left[t\right]_0^{10}$$

$$\Rightarrow \ln 40 - \ln 80 = -10K$$

$$\Rightarrow 10K = \ln 80 - \ln 40$$

$$\Rightarrow 10K = \ln \frac{80}{40}$$

$$\Rightarrow 10K = \ln 2 \quad \text{--- (2)}$$

When $t=0$ then $T=370°$

let the temperature of the body will be x after 40 minute.

Now integrating ① by taking limits of t and T. i.e.,

$$\int_{370}^{x} \frac{dT}{T-290} = -K\int_0^{40} dt$$

$$\Rightarrow \left[\ln(T-290)\right]_{370}^{x} = -k\left[t\right]_{0}^{40}$$

$$\Rightarrow \ln(x-290) - \ln 80 = -40k$$

$$\Rightarrow \ln\frac{x-290}{80} = -4\,(10k) = -4\ln 2;\ \text{by (1)}$$

$$\Rightarrow \ln\frac{x-290}{80} = \ln 2^{-4}$$

$$\Rightarrow \ln\frac{x-290}{80} = \ln\left(\frac{1}{2^4}\right)$$

$$\Rightarrow \frac{x-290}{80} = \frac{1}{2^4}$$

$$\Rightarrow x-290 = \frac{80}{16}$$

$$\Rightarrow x = 290+5$$

$$\therefore x = 295^\circ C$$

<u>উদাহরণ - 8(ii)।</u> বাতাসের তাপমাত্রা যখন 20°C তখন কোনো বস্তুর তাপমাত্রা 10 মিনিটে ঠান্ডা হইয়া 100°C হইতে 60°C হয়। 40 মিনিটে পর বস্তুর তাপমাত্রা নির্ণয় কর।

[i.e., when the temperature of the air is 20°C, a certain substance cools in 10 minutes from 100°C to 60°C. find the temperature after 40 minutes.]

সমাধান: মনে করি, t সময়ে বস্তুটির তাপমাত্রা $T°C$ এবং বাতাসের তাপমাত্রা $20°C$.

∴ বস্তু ও বাতাসের তাপমাত্রার পার্থক্য $= T-20°C$

∴ বস্তুর তাপমাত্রার পরিবর্তনের হার $= \frac{dT}{dt}$

$$\therefore \frac{dT}{dt} \propto T-20$$

$$\Rightarrow \frac{dT}{dt} = -K(T-20)$$

$$\Rightarrow \frac{dT}{T-20} = -K\,dt \quad \text{---(1)}$$

When $t=0$, then $T=100°$ and When $t=10$ then $T=60°$

এখন t এবং T এর সীমা লইয়া (1) কে ইন্টিগ্রেট করি,

$$\int_{100}^{60} \frac{dT}{T-20} = -K \cdot \int_0^{10} dt$$

$$\Rightarrow \Big[\ln(T-20)\Big]_{100}^{60} = -K \cdot [t]_0^{10}$$

$$\Rightarrow \ln 40 - \ln 80 = -10K$$

$$\Rightarrow 10K = \ln 80 - \ln 40 = \ln\frac{80}{40}$$

$$\Rightarrow 10K = \ln 2.$$

~~When $t=0$ then $T=370°$~~

Let the temperature of the body will be x after 40 minute.

যখন $t=0$ তখন $T=100$. মনে করি, $t=40$ মিনিট পর বস্তুটির তাপমাত্রা x হবে।

এখন t এবং T -এর সীমা সহকারে ① নং -কে ইন্টিগ্রেট করি,

$$\int_{100}^{x} \frac{dT}{T-20} = -K \cdot \int_{0}^{40} dt$$

$$\Rightarrow \left[\ln(T-20)\right]_{100}^{x} = -K\left[t\right]_{0}^{40}$$

$$\Rightarrow \ln(x-20) - \ln 80 = -40K = -4(10K)$$

$$\Rightarrow \ln\frac{x-20}{80} = -4\ln 2 = \ln 2^{-4} = \ln\left(\frac{1}{2^4}\right)$$

$$\Rightarrow \frac{x-20}{80} = \frac{1}{2^4}$$

$$\Rightarrow x-20 = \frac{80}{16}$$

$$\Rightarrow x = 20+5$$

$$\therefore x = 25^\circ C$$

<u>সমস্যা: 03</u>| বাতাসের তাপমাত্রা 290°C হলে একটি বস্তু 10 মিনিটে 370°C হতে

330°C -এ শীতল হয়। কখন ইহার তাপমাত্রা 295°C হইবে?

[A body cools from 370°C to 330°C in 10 minute in air which is maintained to 290°C. When will the temperature of the body is 295°C]

<u>সমাধান:</u> মনে করি, t সময়ে বস্তুটির তাপমাত্রা T°C ~~in 10 minute~~ এবং বাতাসের তাপমাত্রা 290°C.

∴ বস্তু ও বাতাসের তাপমাত্রার পার্থক্য $= T - 290$ এবং

বস্তুর তাপমাত্রার পরিবর্তনের হার $= \frac{dT}{dt}$

যেহেতু, $\frac{dT}{dt} \propto T - 290^{0}$

$$\Rightarrow \frac{dT}{dt} = -K(T-290)$$

$$\Rightarrow \frac{dT}{T-290} = -K\,dt \quad --(1)$$

When $t=0$, then $T=370$, when $t=10$ then $T=330$

এখন t এবং T এর সীমার সাহায্যে (1) নং কে ইন্টিগ্রেট করি,

$$\text{i.e. } \int_{370}^{330} \frac{dT}{T-290} = -k\int_0^{10} dt$$

$$\text{or, } \left[\ln(T-290)\right]_{370}^{330} = -k\left[t\right]_0^{10}$$

$$\text{or, } \ln 40 - \ln 80 = -10k$$

$$\text{or, } 10k = \ln 80 - \ln 40$$

$$\text{or, } 10k = \ln 2 = 0.693 \quad \text{---(2)}$$

When $t=0$, then $T=370$

মনে করি, $t = t_1$ সময়ে $T = 295°C$ হবে,

এখন t এবং T এর সীমার সাহায্যে (1) নং কে ইন্টিগ্রেট করি।

$$\int_{370}^{295} \frac{dT}{T-290} = -k\int_0^{t_1} dt$$

$$\text{or, } \left[\ln(T-290)\right]_{370}^{295} = -k\left[t\right]_0^{t_1}$$

$$\text{or, } \ln 5 - \ln 80 = -kt_1$$

$$\text{or, } kt_1 = \ln 80 - \ln 5$$

$$\text{or, } kt_1 = \ln\frac{80}{5}$$

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$$\Rightarrow t_1 = \frac{\ln 16}{k} = \frac{10\ln 16}{10k} = \frac{27.725}{0.693}$$

$$\approx 40, \text{ by } (2)$$

$\Rightarrow t_1 = 40$ min পরে $T = 295°C$ হইবে,

উদাহরণ:10। বাতাসের তাপমাত্রা 300°C এবং বস্তুটির তাপমাত্রা 15 মিনিটে 370°C হইতে 340°C শীতল হয়। বস্তুটির তাপমাত্রা কখন 310°C হইবে?

[A body cools from 370°C to 340°C in 15 minute in air which is maintained to 300°C. When will the temperature of the body be 310°C.

সমাধান: Let at time t, temperature of the body be $T°C$ and temperature of the air be 300°C. Difference between the temperature of the body and the air

$$= T - 300.$$

$\therefore$ Rate of change of temperature of the body

$$= \frac{dT}{dt}$$

Since, $\frac{dT}{dt} \propto T-300$

$\Rightarrow \frac{dT}{dt} = -K(T-300)$. এখানে $K>0$, যেহেতু তাপমাত্রা হ্রাস পায়.

$\Rightarrow \frac{dT}{T-300} = -K\,dt$ — (1)

When $t=0$ then $T = 370°C$

When $t=15$ then $T = 340°C$

Now integrating by taking limits of t and T

$$\int_{370}^{340} \frac{dT}{T-300} = -K\int_0^{15} dt$$

or, $\left[\ln(T-300)\right]_{370}^{340} = -K[t]_0^{15}$

or, $\ln 40 - \ln 70 = -15K$

or, $\ln 70 - \ln 40 = 15K$

or, $15K = \ln 7/4$

or, $K = 0.5596$ — (2)

When $t=0$ then $T = 370°C$

Let at time $t = t_1$, the temperature of the body will be $310°C$

Now, integrating (1) by taking limits of t and T,

$$\int_{370}^{310} \frac{dT}{T-300} = -k\int_0^{t_1} dt$$

$$\Rightarrow \left[\ln(T-300)\right]_{370}^{310} = -k[t]_0^{t_1}$$

$$\Rightarrow \ln 10 - \ln 70 = -kt_1$$

$$\text{or, } kt_1 = \ln 70 - \ln 10$$

$$\text{or, } kt_1 = \ln\frac{70}{10}$$

$$\text{or, } kt_1 = \ln 7$$

$$\Rightarrow t_1 = \frac{\ln 7}{k}$$

$$= \frac{15\ln 7}{15k} = \frac{29.1886}{0.5596}$$

$$\approx 52 \text{ min (by 2)}$$

∴ 52 মিনিট পর বস্তুটির তাপমাত্রা 310°C হবে।

$\boxed{\text{উদাহরণ-11}}$ 30 ভোল্টের বৈদ্যুতিক শক্তি 0.2 হেনরী আবেশ ও 50 ওহম রোধ সম্বলিত সার্কিট ধারায় প্রয়োগ করা হইল। যদি $I(0)=0$ হয়, তবে বিদ্যুৎ $I(t)$ নির্ণয় কর। অনেক সময় পর বিদ্যুৎ নির্ণয় কর।

[A 30 V electromotive force is applied to an LR series circuit in which the inductance is 0.2 henry and the resistance is 50 ohms. Find the current $I(t)$ if $I(0)=0$. determine the current after a long time]

$$V(t) = L\frac{dI}{dt} + IR.$$

সমাধান:

Electric circuit differential equation is,

$$L\cdot\frac{dI}{dt} + RI = V(t) \quad \text{---} \; (1)$$

প্রশ্নমতে, $L = 0.2 = \frac{2}{10} = \frac{1}{5}$, $R = 50$ এবং $V(t) = 30$

$$\therefore (1) \Rightarrow \frac{1}{5}\cdot\frac{dI}{dt} + 50I = 30$$

$$\Rightarrow \frac{dI}{dt} + 250I = 150 \quad \text{---} \; (2)$$

Here, $\quad I.F. = e^{\int 250\,dt} = e^{250t}$.

(2) নং কে e^{250t} দ্বারা গুণ করে পাওয়া যায়

[Multiplying (2) by e^{250t}, we get.]

$$\frac{d}{dt}\left(I e^{250t}\right) = 150 \cdot e^{250t} \quad \text{--- (3)}$$

Integrating (3) with respect to t by taking limits of t from 0 to t,

$$\left[I(t)\, e^{250t}\right]_0^t = 150 \int_0^t e^{250t}\, dt$$

$$\text{or, } I(t) \cdot e^{250t} - I(0) = 150 \left[\frac{e^{250t}}{250}\right]_0^t$$

$$\text{or, } I(t)\, e^{250t} - 0 = \frac{150}{250}\left[e^{250t} - e^{0}\right]$$

since $I(0) = 0$

$$\text{or, } I(t) \cdot e^{250t} = \frac{3}{5}\left[e^{250t} - 1\right]$$

$$\text{or, } I(t) = \frac{3}{5}\left(1 - e^{-250t}\right)$$

<u>দ্বিতীয় অংশঃ</u> $\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{3}{5}\left(1 - e^{-250t}\right)$

$$= \frac{3}{5}\left[1 - 0\right]$$

$$= \frac{3}{5}$$

উদাহরণ: 12 | 12 ভোল্টের বিদ্যুৎ চালক শক্তি $\frac{1}{3}$ হেনরী আবেশক, 10 ওহম রোধ সম্বলিত সার্কিট ধারায় প্রয়োগ করা হয়, যদি $I(0)=0$ হয়, তবে $I(t)$ নির্ণয় কর। অনেক সময় পর বিদ্যুৎ কত হবে?

[A 12 volt electromotive force is applied to an LR series circuit in which the inductance is $\frac{1}{3}$ henry, the resistance is 10 ohms. Find the current $I(t)$ if $I(0)=0$. Determine the current after a long time.]

সমাধান: Electric circuit differential equation is,

$$L\cdot\frac{dI}{dt}+RI=V(t) \quad ---(1)$$

$$\therefore (1) \Rightarrow \frac{1}{3}\cdot\frac{dI}{dt}+10I=12$$

$$\Rightarrow \frac{dI}{dt}+30I=36 \quad --(2)$$

$$I.F.=e^{\int 30dt}$$

$$=e^{30t}$$

Multiplying (2) by e^{30t} we get,

$$\therefore \frac{d}{dt}\left(b \cdot e^{30t}\right) = 36 \cdot e^{30t} \quad \text{---(3)}$$

Integrating (3) with respect to t by taking limits from $t=0$ to $t=t$.

$$\left[b(t) \cdot e^{30t}\right]_0^t = 36 \int_0^t e^{30t}\, dt$$

$$\text{or, } b(t) \cdot e^{30t} - b(0) = 36 \left[\frac{e^{30t}}{30}\right]_0^t$$

$$\text{or, } b(t) \cdot e^{30t} - 0 = \cancel{36} \frac{6}{5} \cancel{\left[\frac{e^{30t}}{30}\right]_0^t}$$

$$= \frac{6}{5}\left[e^{30t} - 1\right]$$

$$\text{or, } b(t) = \frac{6}{5}\left[1 - e^{-30t}\right]$$

যেহেতু, $b(0) = 0$

দ্বিতীয় অংশ :

$$\lim_{t \to \infty} b(t) = \lim_{t \to \infty} \frac{6}{5}\left(1 - e^{-30t}\right)$$

$$= \frac{6}{5}(1-0) = \frac{6}{5}$$

উদাহরণ: 13

$t=0$ সময়ে একটি ট্যাংকে বিশুদ্ধ 100 গ্যালন পানিতে 4 পাউন্ড লবণ দ্রবীভূত হয়। মনে করি, গ্যালন প্রতি 2 পাউন্ড লবণ মিশ্রিত পানি ~~ট্যাংক হতে বাইরে সরিয়ে দেওয়া হইল~~। প্রতি মিনিটে 5 গ্যালন হারে ট্যাংকে প্রবেশ করিতে দেওয়া হইল এবং লবণ মিশ্রিত পানি একই হারে ট্যাংক হইতে বাইরে সরিয়ে দেওয়া হয়। নির্ণয় কর 10 মিনিট পরে ট্যাংকে কি পরিমাণ লবণ থাকবে।

[At time $t=0$, a tank contains 4 lb of salt dissolved in 100 gal of water. Suppose that brine containing 2 lb of salt per gallon of brine is allowed to enter the tank at a rate of 5 gal/min and that the mixed solution is drained (বাহির হওয়া) from the tank at the same rate. Find the amount of salt in the tank after 10 minutes.

Solution:

let $y(t)$ lb be the amount of salt in the tank after t time then the rate of change of salt in the tank is $\frac{dy}{dt}$.

So, $\frac{dy}{dt}$ = rate in − rate out ...(1)

rate in = (2lb/gal) (5 gal/min) = 10 lb/min

Since $y(t)$ lb of salt per 100 gal of brine at time t, so the density of brine is $\frac{y(t)}{100}$ lb/gal

∴ rate out = $\left(\frac{y(t)}{100}\text{ lb/gal}\right)$ (5 gal/min)

$= \frac{y(t)}{20}$ lb/min.

∴ (1) ⇒ $\frac{dy}{dt} = 10 - \frac{y}{20}$

⇒ $\frac{dy}{dt} + \frac{y}{20} = 10$ ...(2)

∴ I.F. $= e^{\int (1/20)\,dt} = e^{t/20}$

(2) × $e^{t/20}$ ⇒ $\frac{d}{dt}(y \times e^{t/20}) = 10 \times e^{t/20}$

Integrating this w.r.to t,

$$y(t)\cdot e^{t/20} = 10\times 20\, e^{t/20} + c$$

$$\Rightarrow y(t) = 200 + c e^{-t/20}$$

Given that when $t=0$ then $y=4$

Now putting $t=0$ and $y=4$ in (3),

$$4 = 200 + c e^{0}$$

$$\Rightarrow c = -196$$

$$\therefore (3) \Rightarrow y(t) = 200 - 196 e^{-t/20}$$

After $t=10$ minutes the amount of salt in the tank is,

$$y(10) = 200 - 196 e^{-10/20}$$

$$\Rightarrow y(10) = 200 - 196\cdot e^{-1/2}$$

$$= 200 - \frac{196}{\sqrt{e}}$$

$$= 200 - \frac{196}{1.64899}$$

$$= 200 - 118.9002$$

$$= 81.09971 \approx 8$$

উদাহরণ : 14 প্রাথমিক ভাবে একটি ট্যাংকে 50 গ্যালন বিশুদ্ধ পানি আছে। যখন $t=0$ তখন গ্যালন প্রতি 2 পাউন্ড লবণ মিশ্রিত পানি মিনিটে 3 গ্যালন হিসেবে ট্যাংকে প্রবেশ করতে দেওয়া হলো এবং ট্যাংকের সুমিশ্রিত লবণ-অন্য একটি পাইপ দ্বারা একই হারে বাহির-করে দেওয়া হয় যাতে ট্যাংকের পানি বৃদ্ধি না পায়।

(i) $t>0$ হলে ট্যাংকে কতটুকু লবণ থাকিবে?

(ii) 25 মিনিট পরে ট্যাংকে কতটুকু লবণ থাকবে?

(iii) দীর্ঘ সময় পরে ট্যাংকে কতটুকু লবণ থাকবে।

[A tank initially contains 50 gal of pure water. When $t=0$ then a brine containing 2lb of dissolved salt per gallon flows into the tank at the rate 3 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows from the tank at the same rate.

(i) How much salt is in the tank at any time $t>0$?

(ii) How much salt is present at the end of 25 minute?

(iii) How much salt is present after a long time?

Solution:

Let at time t, $y(t)$ lb be the amount of salt in the tank, then the rate of change of salt in the tank is $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} \quad \text{---}(1)$$

$$\text{rate in} = (2\,\text{lb/gal})\,(3\,\text{gal/min})$$
$$= 6\,\text{lb/min}.$$

Since at time t, $y(t)$ lb of salt is in 50 gallon of water,

so the density of brine is

$$\frac{y(t)}{50}\ \text{lb/gal}$$

rate out $= \left(\frac{y(t)}{50} \text{ lb/gal} \right) =$

$= \frac{3y}{50}$ lb/min.

$\therefore (1) \Rightarrow \frac{dy}{dt} = 6 - \frac{3y}{50}$

$\Rightarrow \frac{dy}{dt} + \frac{3y}{50} = 6$ ---(2)

$I.F. = e^{\int \frac{3}{50} dt} = e^{3t/50}$

$\therefore (2) \times e^{3t/50} \Rightarrow \frac{d}{dt}\left(y e^{3t/50}\right) = 6 e^{3t/50}$

Integrating this with respect to t,

$y \cdot e^{3t/50} = 6 \cdot \frac{e^{3t/50}}{3/50} + c$

$\Rightarrow y = 100 + c e^{-3t/50}$ ---(3)

When $t=0$ then $y=2$ that is, putting $t=0$, $y=2$ in (3),

$2 = 100 + c \Rightarrow c = -98$

$\therefore (3) \Rightarrow y(t) = 100 - 98 e^{-3t/50}$

(i) When $t>0$, the amount of salt in the tank is, $y(t) = 100 - 98 e^{-3t/}$

(ii) After 25 minutes, the amount of salt in the tank is,

$$y(25) = 100 - 98\, e^{-75/50}$$

$$= 100 - \frac{98}{e^{3/2}} = 100 - \frac{98}{4.479}$$

$$= 100 - 21.8799$$

$$= 78.12\ \text{lb}$$

(iii) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (100 - 98 e^{-3t/50})$

$$= 100 - \cancel{x}\, 98 \cdot e^{-\infty}$$

$$= 100 - 0$$

$= 100$ পাউন্ড।

Picard's Method

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

$$x \to x_0$$

$$y(x) \to y$$

Formula: $\boxed{y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx}$

উদাহরণ : 1 পিকার্ডের পদ্ধতি প্রয়োগ করিয়া নিম্নে উল্লেখিত আদিমান সমস্যাগুলোর তৃতীয় আসন্ন মান পর্যন্ত সমাধান কর।

Apply picard's method upto third approximation to solve the following initial value problems.

(i) $\frac{dy}{dx} = x+y$; $y(0) = -1$.

Solution: Given initial value problem is

$$\frac{dy}{dx} = x+y, \; y(0) = -1 \quad \text{--- (1)}$$

Comparing (1) with $\frac{dy}{dx} = f(x, y), y(x_0) =$

we get, $f(x,y) = x+y$, $x_0 = 0$, $y_0 = -1$

The first approximation y_1 is given by.

$$y_1 = y_0 + \int_{x_0}^{x} f(x\, y_0)\, dx$$

$$= -1 + \int_0^x (x + y_0)\, dx \quad [\text{by } 2]$$

$$= -1 + \int_0^x (x-1)\, dx \quad [\text{Since } y_0 = -1]$$

$$= -1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$= -1 + \frac{x^2}{2} - x \quad \text{---- (3)}$$

The second approximation y_2 is given by;

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= y_0 + \int_{x_0}^{x} (x + y)\, dx$$

$$= -1 + \int_0^x \left(x - 1 + \frac{x^2}{2} - x\right) dx$$

[by (2) and (3)]

$$= -1 + \int_0^x \left(-1 + \frac{x^2}{2}\right) dx$$

$$= -1 + \left[-x + \frac{x^3}{3 \cdot 2}\right]_0^x$$

$$= -1 - x + \frac{x^3}{6} \quad \text{---- (4)}$$

The third approximation y_3 is given by

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$= -1 + \int_{0}^{x} f(x+y_2)\, dx \quad [\text{by } ②]$$

$$= -1 + \int_{0}^{x} \left(x + -1 - x + \frac{x^3}{6}\right) dx$$

$$= -1 + \int_{0}^{x} \left(-1 + \frac{x^3}{6}\right) dx$$

$$= -1 + \left[-x + \frac{x^4}{24}\right]_0^x$$

$$= -1 - \left(x - \frac{x^4}{24}\right)$$

$$= -1 - x + \frac{x^4}{24} \quad (\text{Ans.})$$

উদাহরণ – 1(ii)

Given initial value problem is,

$$\frac{dy}{dx} = y - x, \quad y(0) = 2 \text{ --- } (1)$$

Comparing (1) with $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$

we get,

$$f(x, y) = y - x, \quad x_0 = 0, \quad y_0 = 2 \text{ --- } (2)$$

The first approximation y_1 is given by,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$\Rightarrow y_1 = y_0 + \int_0^x (y_0 - x)\, dx \ (\text{by } 2)$$

$$\Rightarrow y_1 = 2 + \int_0^x (2-x)\, dx \ [\text{by } 2]$$

$$\Rightarrow y_1 = 2 + \left[2x - \frac{x^2}{2}\right]_0^x$$

$$\therefore y_1 = 2 + 2x - \frac{x^2}{2} \ \text{---} (3)$$

The second approximation y_2 is given by,

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1)\, dx$$

$$= y_0 + \int_{x_0}^x (y_1 - x)\, dx \ [\text{by } ②$$

$$= 2 + \int_0^x \left(2 + 2x - \frac{x^2}{2} - x\right) dx$$

$$= 2 + \int_0^x \left(2 + x - \frac{x^2}{2}\right) dx$$

$$= 2 + \left[2x + \frac{x^2}{2} - \frac{x^3}{6}\right]_0^x$$

$$= 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$$

The third approximation y_3 is g

by,

$$y_3 = y_0 + \int_{x_0}^x f(x, y_2)\, dx$$

$$= y_0 + \int_{x_0}^x (y_2 - x)\, dx$$

$$= 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} - \frac{x^3}{6} - x\right) dx \quad$$ [by ② and ③]

$$= 2 + \left[2x + \frac{2x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^2}{2}\right]_0^x$$

$$= 2 + 2x + x^2 + \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^2}{2}$$

$$= 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} \text{ . (Ans.)}$$

<u>Initial value problem</u> :

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$$\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = -y$$

$$\Rightarrow dy \cdot \left(-\frac{1}{y}\right) = dx$$

$$\Rightarrow \int \frac{dy}{-y} = \int dx$$

$$\Rightarrow -\ln y = x + c$$

$$\Rightarrow \ln y = \ln c - x$$

$$\Rightarrow \ln y = \ln c + \ln e^{-x}$$

$$\Rightarrow \ln y = \ln c e^{-x}$$

$$\Rightarrow y = c e^{-x} \qquad \Rightarrow c = 1$$

$$\therefore y = e^{-x}$$

Problem-1: Solve the following initial value problems:

(i) $2x^3y' = y(y^2+3x^2)$, $y(1)=1$.

Solution:

Given initial value problem is,

$$2x^3y' = y(y^2+3x^2) \;\text{---}\; (1), \quad y(1)=1 \;\text{---}\; (2)$$

$$\Rightarrow 2x^3\frac{dy}{dx} = y^3+3x^2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^3+3x^2y}{2x^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^3}{2x^3} + \frac{3y}{2x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{3y}{2x} = \frac{y^3}{2x^3}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} - \frac{3y^{-2}}{2x} = \frac{1}{2x^3} \;\text{---}\; (3)$$

We put $y^{-2} = v$ then $-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

$$\text{Or, } y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

$$\therefore (3) \Rightarrow -\frac{1}{2}\frac{dv}{dx} - \frac{3v}{2x} = \frac{1}{2x^3}$$

$$\text{Or, } \frac{dv}{dx} + \frac{3v}{x} = -\frac{1}{x^3} \;\text{---}\; (4)$$

$\therefore$ I.F. $= e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

Multiplying both sides of ④ by x^3 we get,

$$\frac{d}{dx}(vx^3) = -1$$

Integrating both sides with respect to x we get,

$$vx^3 = -x + A$$

$$\Rightarrow v = -\frac{1}{x^2} + \frac{A}{x^3}$$

$$\Rightarrow y^{-2} = -\frac{1}{x^2} + \frac{A}{x^3}$$

$$\Rightarrow \frac{1}{y^2} = -\frac{1}{x^2} + \frac{A}{x^3} \quad \text{--- (5)}$$

putting $x = 1$, in ⑤ we get,

$$\frac{1}{\{y(1)\}^2} = -\frac{1}{1^2} + \frac{A}{1^3}$$

$$\Rightarrow \frac{1}{1^2} = -1 + A \quad [\text{by } ②]$$

$$\Rightarrow A = 2$$

Now, putting the value of A in ⑤ we get,

$$\frac{1}{y^2} = -\frac{1}{x^2} + \frac{2}{x^3}$$

সমাধান :

1(ii) Given initial value problem is,

$x dy - y dx - (1-x^2) dx = 0$ ... (1)

$y(1) = 1$ ... (2)

Dividing both sides of (1) by x^2 we get,

$$\frac{x dy - y dx}{x^2} - \left(\frac{1}{x^2} - 1\right) dx = 0$$

or, $d\left(\frac{y}{x}\right) - \left(\frac{1}{x^2} - 1\right) dx = 0$

Integrating this, we get,

$$\frac{y}{x} - \left(-\frac{1}{x} - x\right) = A$$

$$\Rightarrow \frac{y}{x} + \frac{1}{x} + x = A$$

$$\Rightarrow y + 1 + x^2 = Ax$$

$$\Rightarrow y(x) = Ax - 1 - x^2 \quad \text{... (3)}$$

putting $x = 1$, in (3) we get,

$$y(1) = A - 1 - 1$$

$$\Rightarrow 1 = A - 2$$

$$\Rightarrow A = 3.$$

$\therefore (4) \Rightarrow y(x) = 3x - 1 - x^2$

সমাধান (iii) Given initial value problem

is,

$$y \ln y \, dx + (x - \ln y)\, dy = 0 \quad \text{--- (1)}$$

$$x(2) = 1 \quad \text{--- (2)}$$

$$(1) \Rightarrow y \ln y \, dx = (\ln y - x)\, dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{\ln y - x}{y \ln y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{y} - \frac{x}{y \ln y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y \ln y} = \frac{1}{y} \quad \text{--- (3)}$$

$$\therefore I.F. = e^{\int \frac{1}{y \ln y} \cdot dy} = e^{\int \frac{d(\ln y)}{\ln y}} = e^{\ln(\ln}$$

$$= \ln y$$

$$= \ln($$

Multiplying both sides of (3) by $\ln y$ we get,

$$\frac{d}{dy}(x \ln y) = \frac{\ln y}{y}$$

Integrating this w.r. to y we ge

$$x \ln y = \int \frac{\ln y}{y}\, dy + A$$

$\Rightarrow x\ln y = \int \ln y \, d(\ln y) + A$

$\Rightarrow x\ln y = \frac{(\ln y)^2}{2} + A \quad \text{--- (4)}$

$\Rightarrow x = \frac{\ln y}{2} + \frac{A}{\ln y}$

$\Rightarrow x(y) = \frac{\ln y}{2} + \frac{A}{\ln y} \quad \text{--- (5)}$

putting $y = 2$, in (5) we get,

$x(2) = \frac{\ln 2}{2} + \frac{A}{\ln 2}$

$\Rightarrow 1 = \frac{1}{2} \ln 2 + \frac{A}{\ln 2} \quad$ [by (2)]

$\Rightarrow 1 - \frac{1}{2} \ln 2 = \frac{A}{\ln 2}$

$\Rightarrow \ln 2 - \frac{1}{2} (\ln 2)^2 = A$

$\therefore A = \ln 2 - \frac{1}{2} (\ln 2)^2$

$\therefore (4) \Rightarrow x\ln y = \frac{1}{2} (\ln y)^2 + \ln 2 - \frac{1}{2} (\ln 2)^2$

Example:02 Solve the following initial value problems.

(i) $2 \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0$, $y(0) = 2$, $y'(0) = \frac{-7}{2}$

Given initial value problem is,

$$2 \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0 \quad \cdots (1)$$

$$y(0) = 2, \; y'(0) = -7/2 \quad \cdots (2)$$

Let $y = e^{mx}$ be a trial solution of (1)

then,

$$(1) \Rightarrow (2m^2 - m - 3)e^{mx} = 0$$

$$\therefore A.E. = 2m^2 - m - 3 \; ; \text{ since } e^{mx} \neq 0$$

$$= 2m^2 - 3m + 2m - 3$$

$$= m(2m-3) + 1(2m-3)$$

$$= (2m-3)(m+1)$$

$$\therefore m = -1, 3/2$$

$\therefore$ General solution is,

$$y(x) = c_1 \cdot e^{-x} + c_2 \cdot e^{3/2 x} \quad \cdots$$

$\therefore \quad y'(x) = c_1 \cdot \frac{3}{2} e^{3x/2} - c_2 e^{-x} \cdots (4)$

(5) putting successively $x=0$ in (3) and (4) we get,

$y(0) = c_1 + c_2$

$\Rightarrow 2 = c_1 + c_2 \quad [\text{by } (2)]$

$\Rightarrow c_1 = 2 - c_2 \cdots (5)$

and $y'(0) = \frac{3c_1}{2} - c_2$

$\Rightarrow -\frac{7}{2} = \frac{3c_1}{2} - c_2 \quad [\text{by } (2)]$

$\Rightarrow -\frac{7}{2} = \frac{3}{2}(2-c_2) - c_2 \quad [\text{by } 5)$

$\Rightarrow -\frac{7}{2} = 3 - \frac{3c_2}{2} - c_2$

$\Rightarrow -\frac{7}{2} - 3 = -\frac{5c_2}{2}$

$\Rightarrow -\frac{13}{2} = -\frac{5c_2}{2}$

$\Rightarrow 5c_2 = 13$

$\Rightarrow c_2 = \frac{13}{5}$

~~उदाहरण~~ putting the value of c_2 in 5 we get,

$$c_1 = 2 - \frac{13}{5}$$

$$\Rightarrow c_1 = -\frac{3}{5}$$

Now putting the values of c_1 and c_2 in 3 we get,

$$y(x) = -\frac{3}{5} e^{3x/2} + \frac{13}{5} e^{-x}$$

$$\text{or, } y(x) = \frac{1}{5}\left(13e^{-x} - 3e^{3x/2}\right)$$

(Ans)

$\frac{dy_2}{dt} = f(t,y)$, $y_2(t_0) = y_0$, then

on the interval $(t_0 - \epsilon, t_0 + \epsilon)$.

$y_1(t) = y_2(t)$ on this interval.

4) States the conditions for which the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ will have a unique solution.

Solution: Given initial value problem in,

$y' = f(x,y)$, $y(x_0) = y_0$ -- (1)

The conditions for existence of a unique solution of initial value problem (1) are

(i) In rectangular region,

$R = \{(x,y) ; |x - x_0| \leq a, |y - y_0| \leq$

If $f(x,y)$ is continuous a

bounded, i.e. $|f(x,y)| < M$ where M is constant

(ii) $|f(x, y_1) - f(x, y_2)| \leq A\,|y_1 - y_2|$

যখন A ধ্রুবক এবং $\forall\ (x, y_1), (x, y_2) \in R$

<u>উদাহরণ: 5</u> Discuss the unique solution of the initial value problem $\frac{dy}{dx} = xy^3$, $y(0) = 1$. Find its solution and determine its interval

<u>Solution:</u>

Given initial value problem is,

$$\frac{dy}{dx} = xy^3, \quad y(0) = 1$$

Here $f(x,y) = xy^3$ --- (1), $x_0 = 0$, $y_0 = 1$

We put $a = 1$, $b = 1$ then the rectangular region is,

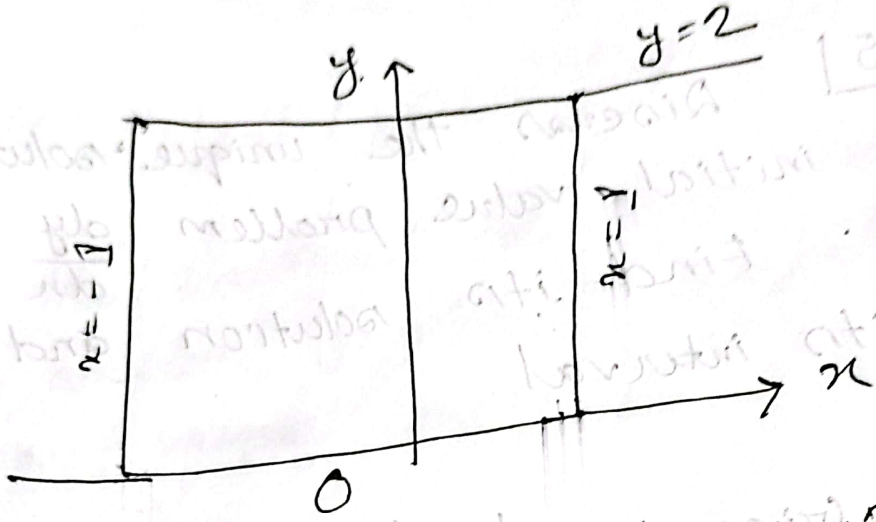
$$R = \{(x,y) : |x - x_0| \leq a,\ |y - y_0| \leq b\}$$

$$= \{(x,y) : |x - 0| \leq 1,\ |y - 1| \leq 1\}$$

$$= \{(x,y) : -1 \leq x \leq 1,\ -1 \leq y - 1 \leq$$

$= \{(x,y) : -1 \le x \le 1,\ -1+1 \le y \le 1+1\}$

$= \{(x,y) : -1 \le x \le 1,\ 0 \le y \le 2\} \quad \dots (1)$



It is evident that $f(x,y) = xy^3$ is continuous in rectangular region R.

Let $(x, y_1), (x, y_2) \in R$ then

$|f(x,y_1) - f(x,y_2)| = |xy_1^3 - xy_2^3|$, [by (1)]

$\Rightarrow |f(x,y_1) - f(x,y_2)| = |x(y_1^3 - y_2^3)|$

$\Rightarrow |f(x,y_1) - f(x,y_2)| = |x(y_1 - y_2)(y_1^2 + y_1 y_2 + y_2^2)|$

$\Rightarrow f(x,y_1) - f(x,y_2) = |x| \, |y_1^2 + y_1 y_2 + y_2^2| \, |y_1 - y_2|$

$\Rightarrow f(x,y_1) - f(x,y_2) \le 1 \cdot |2^2 + 2\cdot 2 + 2^2| \, |y_1 - y_2|$; [by ③]

$\Rightarrow |f(x,y_1) - f(x,y_2)| \le 12 \, |y_1 - y_2|$

$\Rightarrow |f(x,y_1) - f(x,y_2)| \le A \, |y_1 - y_2|$

when $A = 12$

Hence $f(x,y)$ satisfies Lipschitz condition. Since $f(x,y)$ is continuous and satisfies Lipschitz condition so the given initial value problem has a unique solution

<u>2nd part:</u>

$$\frac{dy}{dx} = xy^3$$

$$\Rightarrow y^{-3}\,dy = x\,dx$$

Integrating this we get,

$$\frac{y^{-2}}{-2} = \frac{x^2}{2} + \frac{c}{2}$$

$$\Rightarrow -\frac{1}{y^2} = x^2 + C \quad \text{---} (3)$$

Applying $y(0) = 1$ in (3) i.e. putting $x = 0$, $y = 1$ in (3) we get,

$$-\frac{1}{1^2} = 0 + C$$

$$\Rightarrow C = -1$$

$$\therefore (3) \Rightarrow -\frac{1}{y^2} = x^2 - 1$$

$$\Rightarrow \frac{1}{y^2} = 1 - x^2$$

$$\Rightarrow y^2 = \frac{1}{1-x^2}$$

$$\therefore y = \frac{1}{\sqrt{1-x^2}}$$

When $|x| < 1$ i.e. $-1 < x < 1$ then the solution is valid.

Hence the interval solution is $-1 < x < 1$.

* Show that if $f(x,y) = xy^2$, $\forall (x,y)$ then f satisfying Lipschitz condition with respect to y on any bounded region of the xy plane.

<u>Solution:</u> $f(x,y)$ is said to satisfy Lipschitz condition in a bounded region of the xy plane if

$|f(x, y_1) - f(x, y_2)| \leq A |y_1 - y_2|$ where A is constant.

Let $R = \{(x,y) : |x| \leq a, |y - y_0| \leq b\}$ ...(1)

i.e. a bounded region and

$f(x,y) = xy^2$ ... (2)

$\therefore |f(x, y_1) - f(x, y_2)| = |xy_1^2 - xy_2^2|$

$\Rightarrow |f(x, y_1) - f(x, y_2)| = |x(y_1+y_2)(y_1-y_2)|$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| = |x|\, |y_1+y_2|\, |y_1-y_2|$$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \le a\, |y_1+y_2|\, |y_1-y_2| \quad \text{--- (3)} \quad [\text{by (1)}]$$

Now, $|y_1+y_2| = |y_1-y_0+y_2-y_0+2y_0|$

$$\Rightarrow |y_1+y_2| \le |y_1-y_0| + |y_2-y_0| + |2y_0|$$

$$\Rightarrow |y_1+y_2| \le b+b+2y_0 \quad [\text{by (1)}]$$

$$\Rightarrow |y_1+y_2| \le 2(b+y_0) \quad \text{--- (4)}$$

From (3) and (4) we get,

$$|f(x,y_1) - f(x,y_2)| \le a \cdot 2(b+y_0)\, |y_1-y_2|$$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \le 2a(b+y_0)\, |y_1-y_2|$$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \le A\, |y_1-y_2|$$

where $A = 2a(b+y_0)$

Hence the function $f(x,y) = xy^2$ satisfies Lipschitz condition.

State existence and uniqueness theorem and examine for $\frac{dx}{dt} = \frac{\sqrt{x}}{t}$, $x(1) = 0$

$t_0 = 1$	$y(x_1) = y_0$
$x_0 = 0$	$x(t_0) = x_0$

Solution:

If in rectangular region,

$$R = \{(t,x) : |t - t_0| \leq a, |x - x_0| \leq b\}$$

(i) $f(t,x)$ and $\frac{\partial f}{\partial x}$ are continuous at $(1,0)$

(ii) $f(t,x)$ and $\frac{\partial f}{\partial x}$ are bounded

$|f|x,t|| \leq A$.

$$f(t,x) = \frac{\sqrt{x}}{t}$$ // $t=0$ -2 (no discontinuous ~~at~~ ([illegible])

$$\Rightarrow \frac{df}{dx} = \frac{1}{2\sqrt{x}\,t}$$

$$R = \{(t,x) : |t - t_0| \leq a, |x - x_0| \leq b\}$$

$$= \{(t,x) : |t - 1| \leq a, |x - 0| \leq b\}$$

$= \{(t,x) : |t-1| \le 1, |x| \le 1\}$

$= \{(t,x) : -1 \le t-1 \le 1, -1 \le x \le 1\}$

$R = \{(t,x) : -1+1 \le t \le 1+1, -1 \le x \le 1\}$

$= \{(t,x) : 0 \le t \le 2, -1 \le x \le 1\}$

It shows that $f(t,x)$ and $\frac{df}{dx}$ both discontinuous in the region R.

Hence, the initial value problem has no unique solution.

Show that the function $f(x,y) = \sqrt{y}$ doesn't satisfy the Lipschitz condition over the rectangle

$R = \{(x,y) : 0 \le x \le a, |y-1| \le 1\}$

Solution: we have,

$$\frac{dy}{dx} = f(x,y) = \sqrt{y}$$

$R = \{(x,y) : 0 \le x \le a, |y-1| \le 1\}$

$= \{(x,y) : 0 \le x \le a, -1 \le (y-1) \le 1\}$

$= \{(x,y) : 0 \le x \le a, 0 \le y \le 2\}$

Now, $f(x, y_1) = \sqrt{y_1}$

$f(x, y_2) = \sqrt{y_2}$

Now, $|f(x,y_1) - f(x,y_2)| = |\sqrt{y_1} - \sqrt{y_2}|$

$$= \left| \frac{(\sqrt{y_1} - \sqrt{y_2})(\sqrt{y_1} + \sqrt{y_2})}{(\sqrt{y_1} + \sqrt{y_2})} \right|$$

$$= \frac{y_1 - y_2}{\sqrt{y_1} + \sqrt{y_2}}$$

we have, $y_1 = 0$, $y_2 \to 0$.

$|f(x, y_1) - f(x, y_2)| \nleq A(y_1 - y$

Hence the function doesn't satisfy Lipschitz condition.

* $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

$R = D = \{ (x,y) : |x - x_0| \le a, |y - y_0| \le b \}$

$f(x, y_1)$, $f(x, y_2)$ $y_1, y_2 \in R$, $A > 0$

$$\boxed{|f(x,y_1) - f(x,y_2)| \le A\,|y_1 - y_2|}$$

→ Lipschitz condition.

A = Lipchitz constant.

☐ unique solution পাওয়ার condition হবে

① $f(x,y) \rightarrow$ continuous

② $\frac{\partial f}{\partial y} \rightarrow$ "

→ Lipschitz condition.

Cauchy - Euler

$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$ — (I)

Let

$x = e^z$

$\Rightarrow z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$ — (II)

$\Rightarrow x \cdot Dy = D_1 y$ $\left[D = \frac{d}{dx} \quad D_1 = \frac{d}{dz} \right]$

$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right)$

$= \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right) + \left(-\frac{1}{x^2}\right)\frac{dy}{dz}$

$= \frac{1}{x} \cdot \frac{d}{dz} \cdot \left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{d}{d}$

$= \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} - \frac{1}{x^2} \cdot \frac{dy}{dz}$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} - \frac{1}{x^2} \frac{dy}{dz}$

$\Rightarrow x^2 \frac{d^2y}{dx^2} - \frac{d^2y}{dz^2} - \frac{dy}{dz} = 0$

$\Rightarrow x^2 D^2 y = D_1^2 y - D_1 y$

$= D_1 (D_1 - 1) y$

$\therefore x \cdot \frac{dy}{dx} = x \cdot Dy = D_1 y$

$\boxed{x^2 \frac{d^2y}{dx^2}} = x^2 D^2 y = D_1(D_1-1)y$

$x^3 \cdot \frac{d^3y}{dx^3} = x^3 D^3 y = D_1(D_1-1)(D_1-2)y$

from (1),

$D_1(D_1-1)(D_1-2)y + 3D_1(D_1-1)y - 2D_1 y + 2y = 0$

$D_1(D_1^2 - 3D_1 + 2)y + 3D_1^2 y - 3D_1 y - 2D_1 y + 2y = 0$

$\Rightarrow (D_1^3 - 3D_1^2 + 2D_1 + 3D_1^2 - 3D_1 - 2D_1 + 2)y$
$= 0$

$\therefore (D_1^3 - 3D_1 + 2)\, y = 0$

let $y = e^{mz}$ be the trial solution of ①,

$\therefore$ A.E. is,

$m^3 - 3m + 2 = 0, \; e^{mz} \neq 0$

$\Rightarrow m^2(m-1) + m(m-1) - 2(m-1) = 0$

$\Rightarrow (m-1)\;(m^2 + m - 2) = 0$

$\Rightarrow (m-1)\;\{m\;(m-1) + 2\;(m-1)\} = 0$

$\Rightarrow (m-1)\;(m-1)\;(m+2) = 0$

$\therefore m = 1, 1, -2.$

$\therefore$ Solution ; $y = (c_1 + c_2 z)e^{z} + c_3 e^{-2z}$

$= \left(c_1 + c_2 \log x\right) x + c_3 \cdot \frac{1}{x^2}$