

CSER 2207: Numerical Analysis

Lecture-7 Interpolation and Polynomial Approximation

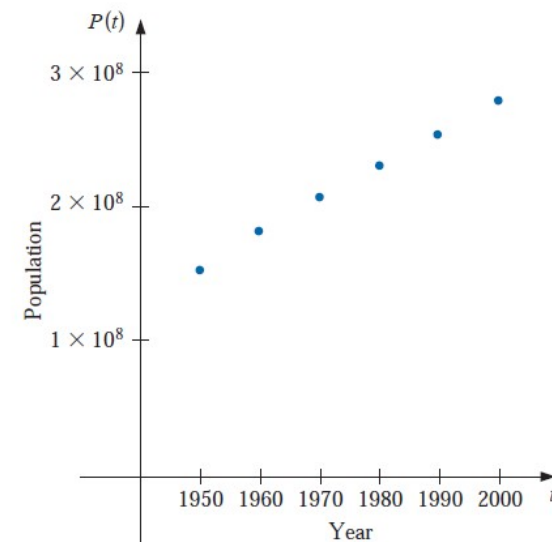
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Introduction

A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000, and the data are also represented in the figure.

Year	1950	1960	1970	1980	1990	2000
Population (in thousands)	151,326	179,323	203,302	226,542	249,633	281,422

In reviewing these data, we might ask whether they could be used to provide a reasonable estimate of the population, say, in 1975 or even in the year 2020. Predictions of this type can be obtained by using a function that fits the given data. This process is called *interpolation* and is the subject of this chapter.



Finite Differences

Assume that we have a table of values (x_i, y_i) , $i = 0, 1, 2, \dots, n$ of any function $y = f(x)$, the values of x being equally spaced, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, n$. Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x , or to obtain the derivative of $f(x)$ for some x in the range $x_0 \leq x \leq x_n$. The methods for the solution to these problems are based on the concept of the 'differences' of a function which we now proceed to define.

3.3.1 Forward Differences

If $y_0, y_1, y_2, \dots, y_n$ denote a set of values of y , then $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ are called the *differences* of y . Denoting these differences by $\Delta y_0, \Delta y_1, \dots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \dots, \quad \Delta y_{n-1} = y_n - y_{n-1},$$

where Δ is called the *forward difference operator* and $\Delta y_0, \Delta y_1, \dots$ are called *first forward differences*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \dots$. Similarly, one can define *third forward differences*, *fourth forward differences*, etc. Thus,

Cont...

$$\begin{aligned}\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) \\ &= y_2 - 2y_1 + y_0,\end{aligned}$$

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

$$\begin{aligned}\Delta^4 y_0 &= \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0) \\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.\end{aligned}$$

It is therefore clear that any higher-order difference can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.

Table 3.1 shows how the forward differences of all orders can be formed:

Forward Difference Table

Table 3.1 Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Difference

3.3.2 Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ are called *first backward differences* if they are denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ respectively, so that $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$, where ∇ is called the *backward difference operator*. In a similar way, one can define backward differences of higher orders. Thus we obtain

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0, \text{ etc.}$$

With the same values of x and y as in Table 3.1, a backward difference table can be formed:

Backward Difference Table

Table 3.2 Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Central Differences

3.3.3 Central Differences

The *central difference operator* δ is defined by the relations

$$y_1 - y_0 = \delta y_{1/2}, \quad y_2 - y_1 = \delta y_{3/2}, \dots, \quad y_n - y_{n-1} = \delta y_{n-1/2}.$$

Similarly, higher-order central differences can be defined. With the values of x and y as in the preceding two tables, a central difference table can be formed:

Table 3.3 Central Difference Table

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0	$\delta y_{1/2}$					
x_1	y_1	$\delta y_{3/2}$	$\delta^2 y_1$	$\delta^3 y_{3/2}$			
x_2	y_2	$\delta y_{5/2}$	$\delta^2 y_2$	$\delta^3 y_{5/2}$	$\delta^4 y_2$		
x_3	y_3	$\delta y_{7/2}$	$\delta^2 y_3$	$\delta^3 y_{7/2}$	$\delta^4 y_3$	$\delta^5 y_{5/2}$	$\delta^6 y_3$
x_4	y_4	$\delta y_{9/2}$	$\delta^2 y_4$	$\delta^3 y_{9/2}$	$\delta^4 y_4$	$\delta^5 y_{7/2}$	
x_5	y_5	$\delta y_{11/2}$	$\delta^2 y_5$				
x_6	y_6						

It is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward or central differences. Thus we obtain

$$\Delta y_0 = \nabla y_1 = \delta y_{1/2}, \quad \Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2}, \dots$$

Example

x	y	Δ	Δ^2	Δ^3	Δ^4
1	3010				
		414			
2	3424		-36		
		378		-39	
3	3802		-75		+178
		303		+139	
4	4105		+64		-271
		367		-132	
5	4472		-68		+181
		299		+49	
6	4771		-19		-46
		280		+3	
7	5051		-16		
		264			
8	5315				

Thank You