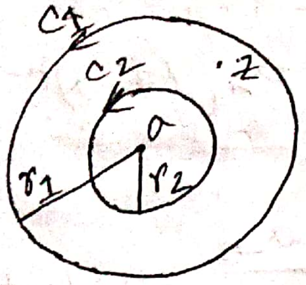
08.09.2022
Thursday

Lowronds Theorem



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2'} \frac{f(z)}{z-z_0}\,dz$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \quad \text{when } r_1 > r_2$$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw \; ; \quad a_{-n} = \oint \frac{1}{2\pi i} \frac{f(w)\,dw}{(w-a)^{-n+1}}$$

$n = 0 \dots$ $\qquad\qquad$ $n = 1 \dots$

proof :

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z)}\,dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-z)}\,dw \quad \text{—— (1)}$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$= \frac{1}{(w-a)}\left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{(w-a)}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3 + \dots\right.$$

or, $\frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{\omega - z} dz = \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + v_n$

$$f(z) = \left[a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{n+1}(z-a)^{n-1}\right]$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + U_n + v_n$$

Now,

$$U_n = \frac{1}{2\pi i} \oint_{C_1} \left(\frac{z-a}{\omega-a}\right)^n \frac{f(\omega)}{\omega - z}$$

$\lim_{n\to\infty} |u_n| \to 0$

or, $|U_n| = \left| \frac{1}{2\pi i} \oint \left(\frac{z-a}{\omega - a}\right)^n \frac{f(\omega)}{\omega - z} \right|$

or, $|U_n| \le \frac{1}{2\pi} 2\pi r_1 \frac{M \gamma^n}{|w-z|}$

or, $|U_n| < \frac{\gamma^n M r_1}{|w-z|}$

or, $|U_n| < \frac{\gamma^n u_n}{r_1 - |z-a|}$

$\mathop{Lt}_{n\to\infty} |U_n| < 0$

Modulus is always ~~a~~positive number. It can't be less than 0. So $U_n = 0$

$$|v_n| = \left| \frac{1}{2\pi i} \oint_{C_2} \left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w} dw \right|$$

$$\text{or, } |v_n| \leq \frac{1}{2\pi} \times 2\pi r_2 \times \frac{m\gamma^n}{|z-w|}$$

$$\text{or, } |v_n| \leq \frac{m\gamma^n r_2}{|z-w|}$$

$$\text{or, } |v_n| \leq \frac{m\gamma^n r_2}{|(z-a)-r_2|}$$

$$|z-w| = |(z-a)-(w-a)|$$
$$= |(z-a)-r_2|$$

$$\text{or, } |v_n| \leq \frac{m\gamma^n r_2}{|z-a|-r_2}$$

So,

$$\text{or, } \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)\,dw}{w-z} = a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots$$
$$\cdots + (z-a)^{n-1} a_{n-1} + u_n$$

$$\therefore \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)\,dw}{w-z} = a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots$$
$$\cdots + (z-a)^{n-1} a_{n-1} + u_n$$

here,
$$\frac{-1}{\omega - z} = \frac{1}{z-\omega} = \frac{1}{(z-a)-(\omega-a)}$$

$$= \frac{1}{(z-a)\left(1-\frac{\omega-a}{z-a}\right)} = \frac{1}{z-a}\left(1-\frac{\omega-a}{z-a}\right)^{-1}$$

$$= \frac{1}{z-a} + \frac{\omega-a}{(z-a)^2} + \frac{(\omega-a)^2}{(z-a)^3} + \cdots + \frac{(\omega-a)^{n-1}}{(z-a)^n}$$

$$+ \frac{(\omega-a)^n}{(z-a)^{n+1}}$$

or, $$\frac{-1}{2\pi i}\oint \frac{f(\omega)\,d\omega}{\omega - z} = \frac{1}{2\pi i}\oint_{c_2} \frac{f(\omega)\,d\omega}{z-a} + \frac{(\omega-a)}{2\pi i}\oint_{c_2} \frac{f(\omega)\,d\omega}{(z-a)^2}$$

$$+ \frac{(\omega-a)^2}{2\pi i}\oint_{c_2} \frac{f(\omega)\,d\omega}{(z-a)^3} + \cdots + \frac{(\omega-a)^{n-1}}{2\pi i}\oint_{c_2} \frac{f(\omega)\,d\omega}{(z-a)^n}$$

$+ v_n$

12-09-2022
Monday

P-6.22

EX-6.26

$\frac{e^{2z}}{(z-1)^3}$; $z=1$ find lorenge series

let $u=z-1$
$\therefore z=u+1$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+u)}}{u^3} = \frac{e^2 \cdot e^{2u}}{u^3}$$

$$= \frac{e^2}{u^3}\left[1+\frac{2u}{1!}+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\cdots\right]$$

$$= \frac{e^2}{u^3}\left[1+\frac{2u}{1}+\frac{2u^2}{1}+\frac{4u^3}{3}+\frac{2u^4}{3}+\cdots\right]$$

$$= e^2\left[\frac{1}{u^3}+\frac{2}{u^2}+\frac{2}{u}+\frac{4}{3}+\frac{2u}{3}+\cdots\right]$$

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{(z-1)}+\frac{4e^2}{3}$$

$$+\frac{2e^2}{3}(z-1)+\cdots$$

6.26 (6)

$$(z-3)\sin\frac{1}{z+2} \quad ; \quad z=-2$$

let,

$z+2=u$

or, $z=u-2$

$$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u}$$

$$= (u-5)\left[\frac{1}{u} - \frac{1}{3!\,u^3} + \frac{1}{5!\,u^5} - \frac{1}{7!\,u^7} + \cdots\right]$$

$$= \frac{u-5}{u} - \frac{u-5}{6u^3} + \frac{u-5}{120u^5} - \frac{u-5}{5040u^7} + \cdots$$

$$= 1 - \frac{5}{u} - \frac{1}{6u^2} + \frac{5}{6u^3} + \frac{1}{120u^4} - \frac{1}{24u^5} - \frac{1}{5040u^6} + \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \frac{1}{24(z+2)^4} - \frac{1}{5040(z+2)}$$

$z=-2$ the series is essential singularity conergence

P-6.24
Ex-6.27

b) $f(z) = \dfrac{1}{(z+1)(z+3)}$

$0 < |z+1| < 2$

$u = z+1$

$0 < |u| < 2$

$\therefore z = u-1$

$0 < |u| < 2$

$u < 2$

$\frac{u}{2} < 1$

$= \dfrac{1}{u(u+2)}$

$= \dfrac{1}{2u\left(1+\frac{u}{2}\right)}$

$= \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$

$= \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \frac{u^4}{16} - \cdots\right)$

$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \frac{u^3}{32} - \cdots$

$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} - \cdots$

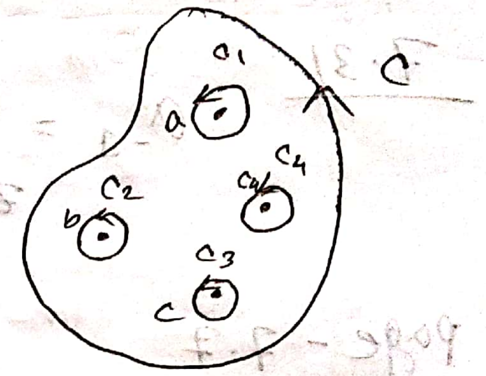
Chapter – 7

[$a_{-1} \Rightarrow$ differentiate [illegible] point [illegible] analiticity fail]

Page – 7.7 Theorem – 7.2

Solution: with centers at a, b, c — respectively constructs circle C_1, C_2 — that is lie entirely inside C as shown this can be done science a, b, c are interior points, we have

$$\oint_C f(z) = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + \cdots \right)$$



$$a_{-1} = \frac{1}{2\pi i} \oint_{C_1} f(z)\, dz$$

$$\oint_{C_1} f(z)\, dz = 2\pi i\, a_{-1}$$

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \oint_{C_3} f(z)\, dz + \cdots$$

$$a_{-1} = \oint_{C_1} \frac{1}{2\pi i} f(z)\, dz$$

$$\oint_{C_1} f(z)\, dz = 2\pi i\, a_{-1}$$

$$\oint_{C_2} f(z)\, dz = 2\pi i\, b_{-1}$$

$$\oint_{C_3} f(z)\, dz = 2\pi i\, c_{-1}$$

$$\oint_{C_4} f(z)\, dz = 2\pi i\, d_{-1}$$

$$\oint_C f(z)\,dz = 2\pi i a_{-1} + 2\pi i b_{-1} + 2\pi i c_{-1} + 2\pi i d_{-1} + \cdots$$

$$= 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + d_{-1} + \cdots\right)$$

7.31 $$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \times \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

page - 7.7

Theorem: 7.1

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots$$

or, $$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-2}(z-a) + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + a_2(z-a)^{m+2} + a_3(z-a)^{m+3} + \cdots$$

$$\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z) = \lfloor(m-1)\, a_{-1} + a_0 (z-a)\, m(m-1)$$

$$+ \cdots \quad 3 + \cdots$$

$$\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} = a_{-1} \lfloor m-1$$

$$\therefore a_{-1} = \lim_{z \to a} \frac{1}{\lfloor m-1} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

Page - 7.8 | math 7.4 (a) 19.09.22

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$$

$(z-a)^m$

$z = a$ is a pole of order

$(z+1)^2 (z^2+4) = 0$

$\therefore z = -1, -1, -2i, +2i$

Hence, $z = 1$ the double pole and $z = 2i$ and $z = -2i$ are single pole.

Residue at $z = -1$ is a_{-1}

$$a_{-1} = \lim_{z \to -1} \frac{1}{\lfloor 1} \times \frac{d}{dz} (z+1)^2 \times \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$$

$$= \lim_{z \to -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{(z^2+4)} \right)$$

$$= \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2-2z) \cdot 2z}{(z^2+4)^2}$$

$$= \lim_{z \to -1} \frac{(1+4)(-2-2) - (1+2) \cdot 2(-1)}{(1+4)^2}$$

$$= \frac{-20+6}{25} = \frac{-14}{25}$$

$z^2 + 4 = 0$

or, $z^2 = 4i^2$

or, $z = \pm 2i$

or, $(z+2i)(z-2i) = 0$

Residue at $z = 2i$ is b_{-1}

$$b_{-1} = \lim_{z \to 2i} \left\{ (z-2i) \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right\}$$

$$= \lim_{z \to 2i} \frac{\cancel{(z-2i)}(z^2-2z)}{(z+1)^2 \cancel{(z-2i)}(z+2i)}$$

$$= \frac{4i^2 - 2.2i}{(2i+1)^2 (2i+2i)}$$

$$= \frac{4i^2 - 4i}{(4i^2+4i+1)(4i)}$$

$$= \frac{-4(1+i)}{(-3+4i) \cdot 4i}$$

$$= \frac{-(1+i)}{(4i-3)i}$$

$$= \frac{-(1+i)}{4i^2-3i} = \frac{-(1+i)}{-4-3i} = \frac{1+i}{4+3i}$$

$$= \frac{(1+i)(4-3i)}{(4)^2-(3i)^2} = \frac{4-3i+4i+3}{16+9} = \frac{7+i}{25}$$

Residue at $z=-2i$ is c_{-1}

$$c_{-1} = \lim_{z \to -2i} \left\{ (z+2i) \frac{(z^2-2z)}{(z+1)^2(z^2+4)} \right\}$$

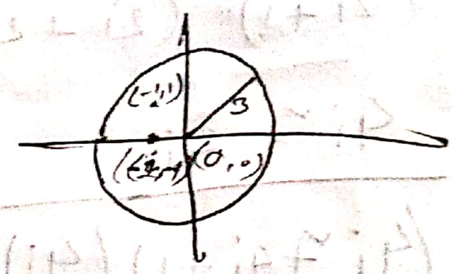
$$= \frac{7-i}{25}$$

Page − 7.10

Ex − 7.6

~~Ea~~ Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$; $|z|=3$

$z^2(z^2+2z+2)=0$



$z=0$, is double pole

$z^2+2z+2=0$

$\therefore z = \frac{-2\pm\sqrt{4-4\times1\times2}}{2\cdot1} = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2}$

$= \frac{-2\pm 2i}{2} = -1\pm i$

$\therefore z=-1+i$ and $z=-1-i$ are simple pole.
$(-1,1)$ $(-1,-1)$

All the poles are inside $|z|=3$

residue at $z=0$ is a_{-1} why $m=2$ ~~(a=0)~~

$$a_{-1} = \lim_{z\to 0} \frac{1}{\lfloor 1} \frac{d}{dz}(z-0)^2 \frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$= \lim_{z \to 0} \frac{d}{dz} \frac{e^{zt}}{z^2+2z+2}$$

$$= \lim_{z \to 0} \frac{(z^2+2z+2)\, t.e^{zt} - e^{zt}(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{2t - 1.2}{2^2} = \frac{2(t-1)}{4} = \frac{t-1}{2}$$

residue at $z = -1+i$ is b_{-1},

$$b_{-1} = \lim_{z \to -1+i} \left[\{z-(-1+i)\} \frac{e^{zt}}{z^2(z^2+2z+2)} \right] \text{ form } \frac{0}{0}$$

$$= \lim_{z \to -1+i} \{z-(-1+i)\} \frac{e^{zt}}{z^2\{z-(-1+i)\}\{z-(-1-i)\}}$$

$$= \lim_{z \to -1+i} \frac{e^{zt}}{z^2\{z-(-1-i)\}}$$

$$= \cancel{\lim_{z \to -1+i}} \frac{e^{(-1+i)t}}{(-1+i)^2\{-1+i+1+i\}} = \frac{e^{-t+it}}{(i-1)^2 2i}$$

$$= \frac{e^{-t} e^{it}}{(i^2-2i+1) \cdot 2i} = \frac{e^{-t} \cdot e^{it}}{-2i \cdot 2i}$$

$\therefore b_{-1} = \frac{t e^{(-1+i)t}}{4}$

residue at $z = -1-i$ is c_{-1}

$c_{-1} = \frac{t e^{(-1-i)t}}{4}$

$$\oint \frac{e^{zt}}{z^2(z^2+2z+2)} dz = 2\pi i (a_{-1} + b_{-1} + c_{-1})$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz = a_{-1} + b_{-1} + c_{-1}$$

$$= \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2} + \frac{t}{4}\{e^{-t} \cdot e^{it} + e^{-t} \cdot e^{-it}\}$$

$$= \frac{t-1}{2} + \frac{t}{4} \cdot e^{-t}\{e^{it} + e^{-it}\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\{e^{it} + e^{-it}\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\left\{\frac{e^{it} + e^{-it}}{2}\right\}$$

$$= \frac{1-1}{2} + \frac{e^{-1}}{2} \quad \cos 1$$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

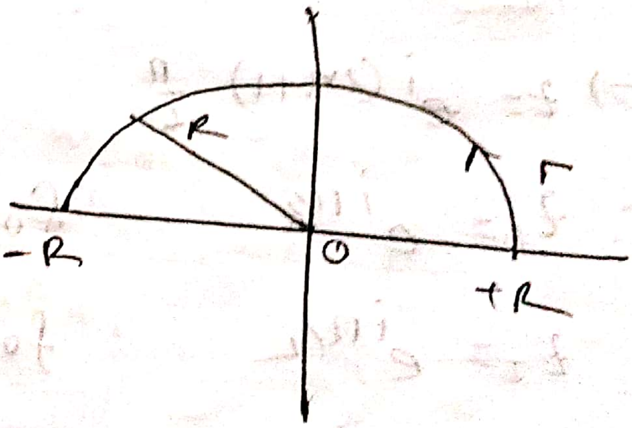
$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

P-7.11

Ex-7.7

$$\lim_{R\to\infty} \int_\Gamma F(z)\, dz = 0$$



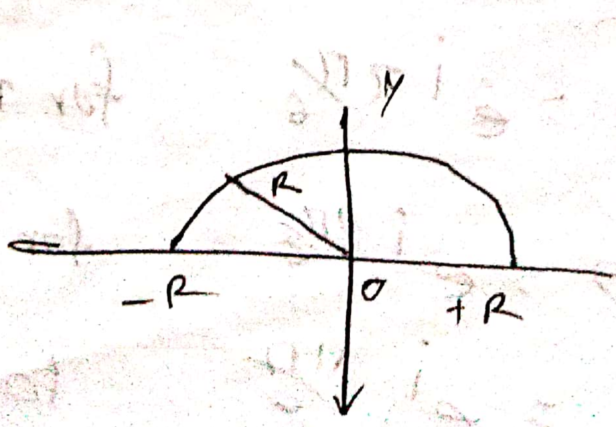
$$\boxed{\lim_{R\to\infty} \int_\Gamma F(z)\, dz = 0}$$

Ex-7.9

$$\int_0^\infty \frac{dz}{x^6+1}$$

let,

$$\oint_C \frac{dz}{z^6+1}$$



here, $z^6+1=0$

or, $z^6=-1 \Rightarrow z^6=\cos\pi+i\sin\pi$

or, $z^6=\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)$

or, $z^6=\cos(2n+1)\pi+i\sin(2n+1)\pi$

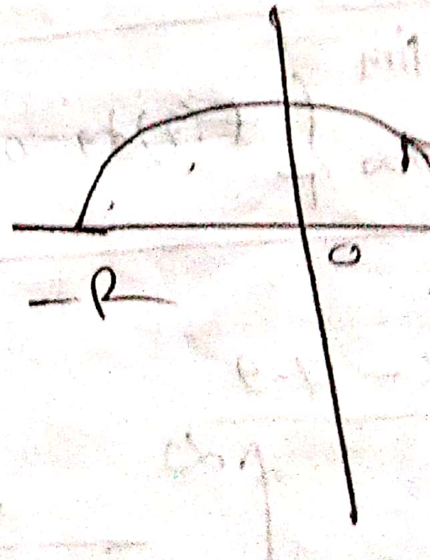
or, $z=[\cos(2n+1)\pi+i\sin(2n+1)\pi]^{1/6}$

or, $z=\cos(2n+1)\frac{\pi}{6}+i\sin(2n+1)\frac{\pi}{6}$

when $n=0,1,2\cdots5$

$\Rightarrow z=e^{i(2n+1)\frac{\pi}{6}}$

$\therefore z_1=e^{i\pi/6}$ for $n=0$



$z_2=e^{i\pi/2}$ for $n=1$

$z_3=e^{i5\pi/6}$ for $n=2$

$z_4=e^{i7\pi/6}$ for $n=3$

$z_5=e^{i9\pi/6}$ for $n=4$

$z_6=e^{i11\pi/6}$ for $n=5$

The point z_1, z_2, z_3, ~~z_4~~ is inside c and z_4, z_5 and z_6 is outside c. z_1, z_2, z_3 are simple pole. Residue at $z_1 = e^{i\pi/6}$ is a_{-1}

$$a_{-1} = \lim_{z \to \frac{\pi i}{6}} \left\{ \frac{z - e^{i\pi/6}}{z^6+1} \right\} \text{ ; form } \frac{0}{0}$$

$$= \lim_{z \to \frac{i\pi}{6}} \frac{1}{6z^5} \quad (1)$$

$$= \frac{1}{6\left(\frac{i\pi}{6}\right)^5} = \frac{1}{6} e^{-5i\pi/6}$$

$$b_{-1} = \lim_{z \to \frac{3i\pi}{6}} \left\{ \frac{z - e^{i3\pi/2}}{z^6+1} \right\} \text{ form } \frac{0}{0}$$

$$= \lim_{z \to \frac{i\pi}{2}} \frac{1}{6z^5} \quad (1)$$

$$= \frac{1}{6} e^{-5i\pi/2}$$

$$c_1 = \frac{1}{6} e^{-25i\pi/6}$$

$$\oint_c \frac{dz}{z^6+1} = 2\pi i \left(a_1 + b_1 + c_{-1} \right)$$

$$= 2\pi i \left(\frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5i\pi/2} + \frac{1}{6} e^{-25i\pi/6} \right)$$

$$\therefore \oint_c \frac{dz}{z^6+1} = \frac{\pi i}{3} \left(e^{-5\pi i/6} + e^{-5\pi i/2} + e^{-25i\pi/6} \right)$$

$$\text{or } \int_{-R}^{R} \frac{dz}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{\pi i}{3} \left[\cos\frac{5\pi}{6} \right.$$

$$- i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2} + \cos\frac{25\pi}{6}$$

$$\left. - i\sin\frac{25}{6} \right]$$

$$= \frac{\pi i}{3} \left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i + 0 - i + \frac{\sqrt{3}}{2} - \frac{1}{2}i \right]$$

$$= \frac{2\pi i^2}{3} = \frac{2\pi}{3}$$

$$\therefore \int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} + \lim_{R\to\infty} \int_{\Gamma} \frac{dz}{z^6+1} = \lim_{R\to\infty} \frac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \rightarrow$$

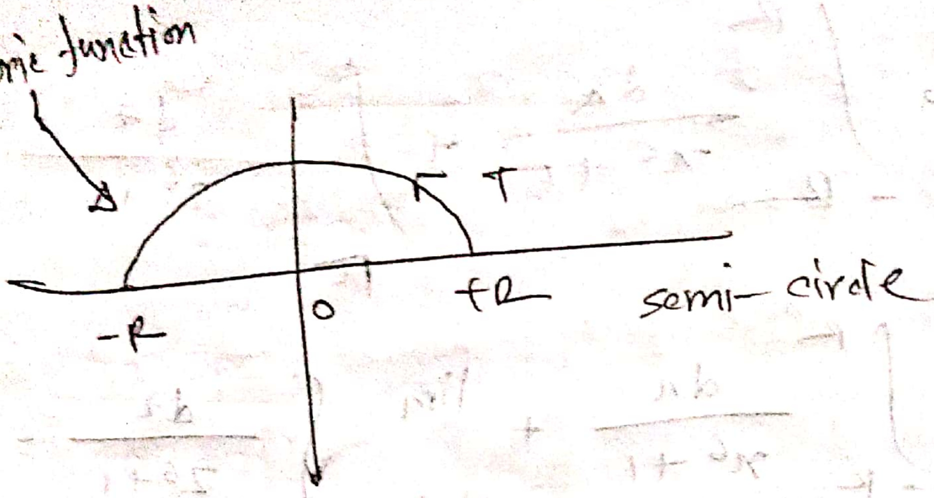
$\int_{-a}^{a} f(x)\,dx$

$= 2\int_0^a f(x)\,dx$ if f(x

$= 0$ if f(x) is

$$\therefore \int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

Ex→ 7.10, 7.11



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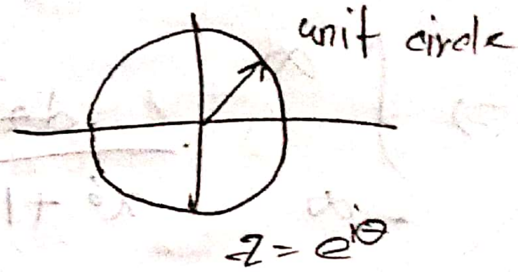


$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Page - 7.15

E - 7.14

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$$

$$= \int_0^{2\pi} \frac{d\theta}{3 - 2\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$e^{i\theta} = z$

$ie^{i\theta}\, d\theta = dz$

$d\theta = \frac{dz}{ie^{i\theta}}$

$$= \oint_C \frac{\frac{dz}{iz}}{3-(z+z^{-1})+(z-z^{-1})\frac{i}{2i}}$$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$= \frac{z+z^{-1}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$= \frac{z-z^{-1}}{2i}$

$$= \oint_C \frac{dz}{\cancel{iz}\, z\left[\frac{6i-2zi-2iz^{-1}+z-z^{-1}}{2\cancel{i}}\right]}$$

$$= \oint_C \frac{2dz}{z\left[6i-2iz-\frac{2i}{z}+z-\frac{1}{z}\right]}$$

$$= \oint_C \frac{2dz}{z\left[\frac{6iz-2iz^2-2i+z^2-1}{z}\right]}$$

$$= \oint_C \frac{2dz}{z^2(1-2i)+6iz-1-2i}$$

here,

$$z^2(1-2i)+6iz-1-2i=0$$

$$\therefore\ z = \frac{-6i \pm \sqrt{(6i)^2+4(1+2i)(1-2i)}}{2(1-2i)}$$

$$z = \frac{-6i \pm \sqrt{-636+4(1+4)}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)}$$

$$= \frac{-3i \pm 2i}{(1-2i)}$$

(+)ve

$$z = \frac{-3i+2i}{1-2i}$$

$$= \frac{-i}{1-2i}$$

$$= \frac{-i(1+2i)}{1+4}$$

$$= \frac{-i(1+2i)}{5}$$

$$= \frac{2-i}{5}$$

(-)ve →

$$z = \frac{-3i-2i}{1-2i}$$

$$\Rightarrow \frac{-5i}{(1-2i)}$$

$$= \frac{-5i(1+2i)}{1+4}$$

$$= \frac{-5i(1+2i)}{5}$$

$$\Rightarrow \cancel{2+}\ 2-i$$

Here, at $z = \frac{2-i}{5}$

$$\alpha_{-1} = \lim_{z \to \frac{2-i}{5}} \left[\left(z - \frac{2-i}{5}\right) \frac{2}{(1-2i)z^2 + 6iz - 2i} \right]$$

form $\frac{0}{0}$

$$= \lim_{z \to \frac{2-i}{5}} \left[\frac{2}{2z(1-2i) + 6i} \right]$$

$$= \frac{2}{2\left[\left(\frac{2-i}{5}\right)(1-2i) + 3i\right]}$$

$$= \frac{1}{\frac{1}{5}\left[(2-i)(1-2i) + 15i\right]}$$

$$= \frac{5}{2 - 4i - i - 2 + 15i}$$

$$= \frac{5}{-5i + 15i}$$

$$= \frac{5}{10i} = \frac{1}{2i}$$

P-7.16 , EX-7.15

P-7.17 Ex-7.16 , 7.17

Ex: 5.2 If $f(z)$ be analytic inside and on boundary c of a simply connected region $\mathbb{R}$, prove that,

$$f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^2} dz$$

Solution:

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)} dz$$

$$f(a+h) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-(a+h)} dz$$

$$f'(a) = \frac{f(a+h)-f(a)}{h} = \frac{1}{h}\left\{\frac{1}{2\pi i} \oint \frac{f(z)}{z-(a+h)} dz - \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz\right\}$$

$$= \frac{1}{2\pi i} \oint_c \frac{1}{h}\left[\frac{1}{z-(a+h)} - \frac{1}{z-a}\right] f(z) dz$$

$$= \frac{1}{2\pi i} \oint_c \frac{1}{h}\left[\frac{\cancel{z}-\cancel{a}-\cancel{z}+\cancel{a}+h}{(z-a-h)(z-a)}\right] f(z) dz$$

$$= \frac{1}{2\pi i} \oint_c \frac{1}{\cancel{h}} \frac{\cancel{h}}{(z-a-h)(z-a)} f(z) dz$$

$$= \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a-h)(z-a)} dz$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^2} dz + \frac{h}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-a-h)(z-a)^2}$$

$$P = \lim_{h \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-a)^2} + \frac{h}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-a-h)(z-a)^2}$$

$$f'(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^2} dz$$

3.52| Prove that in polar form the Cauchy-Riemann equations can be written.

$$\frac{\delta u}{\delta r} = \frac{1}{r} \frac{\delta v}{\delta \theta} \qquad \frac{\delta v}{\delta r} = -\frac{1}{r} \frac{\delta u}{\delta \theta}$$

Solution: We have $x = r\cos\theta$ $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$ Then,

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta r} \cdot \frac{\delta r}{\delta x} + \frac{\delta u}{\delta \theta} \frac{\delta \theta}{\delta x} = \frac{\delta}{\delta r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\delta u}{\delta \theta}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{\delta u}{\delta r} \cos\theta - \frac{1}{r} \frac{\delta u}{\delta \theta} \sin\theta$$

$$\frac{\delta u}{\delta y} = \frac{\delta u}{\delta r} \frac{\delta r}{\delta y} + \frac{\delta u}{\delta \theta} \cdot \frac{\delta \theta}{\delta y} = \frac{\delta}{\delta r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\delta u}{\delta \theta}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{\delta u}{\delta r} \sin\theta + \frac{1}{r} \frac{\delta u}{\delta \theta} \cos\theta$$

Similarly,

$$\frac{\delta v}{\delta x} = \frac{\delta v}{\delta r}\cdot\frac{\delta r}{\delta x} + \frac{\delta v}{\delta \theta}\cdot\frac{\delta \theta}{\delta x} = \frac{\delta v}{\delta r}\cos\theta - \frac{1}{r}\frac{\delta v}{\delta \theta}\sin\theta \quad \text{(iii)}$$

$$\frac{\delta v}{\delta y} = \frac{\delta v}{\delta r}\frac{\delta r}{\delta y} + \frac{\delta v}{\delta \theta}\frac{\delta \theta}{\delta y} = \frac{\delta v}{\delta r}\sin\theta + \frac{1}{r}\frac{\delta v}{\delta \theta}\cos\theta \quad \text{(iv)}$$

From the Cauchy–Rieman equation $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ we have, using (i) and (iv)

$$\left(\frac{\delta u}{\delta r} - \frac{1}{r}\frac{\delta v}{\delta \theta}\right)\cos\theta - \left(\frac{\delta v}{\delta r} + \frac{1}{r}\frac{\delta u}{\delta \theta}\right)\sin\theta = 0 \quad \text{(v)}$$

From the Cauchy–Rieman equation $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ we have, using (ii) and (iii)

$$\left(\frac{\delta u}{\delta r} - \frac{1}{r}\frac{\delta v}{\delta \theta}\right)\sin\theta + \left(\frac{\delta v}{\delta r} + \frac{1}{r}\frac{\delta u}{\delta \theta}\right)\cos\theta = 0 \quad \text{(vi)}$$

Multiplying (v) by $\cos\theta$, (vi) by $\sin\theta$ and adding yields

$$\frac{\delta u}{\delta r} - \frac{1}{r}\frac{\delta v}{\delta \theta} = 0 \quad \text{or} \quad \frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \theta}$$

Multiplying (v) by $-\sin\theta$ (vi) by $\cos\theta$ and adding yields

$$\frac{\delta v}{\delta r} + \frac{1}{r}\frac{\delta u}{\delta \theta} = 0 \quad \text{or} \quad \frac{\delta v}{\delta r} = -\frac{1}{r}\frac{\delta u}{\delta \theta}$$

E. 7.22 $\quad \int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m}$; $m>0$

$e^{ix} = \cos x + i\sin x$

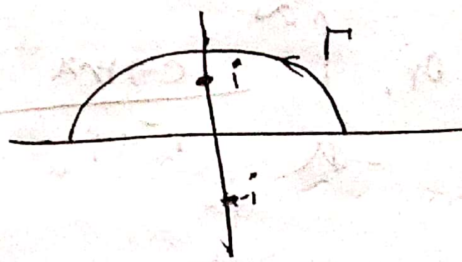
$\cos x = \text{Real}(e^{ix})$

Solution:

Consider $\oint \{ e^{imz} / (z^2+1) \} dz$

$z^2+1=0$

$\therefore z = \pm i$



Residue at $z=i$ is

$$a_{-1} = \lim_{z\to i} \left[(z-i) \frac{e^{imz}}{z^2+1} \right]$$

$$= \lim_{z\to i} \frac{(z-i)\, e^{imz}}{(z+i)(z-i)}$$

$$= \cancel{e^{imz}} = \frac{1}{2i} e^{-m}$$

$$a_{-1} = \lim_{z\to a} \frac{1}{\lfloor m-1} \left[(z-a)^m f(\right.$$

$$\therefore \oint_C \frac{e^{imz}}{z^2+1}\,dz = 2\pi i\, a_{-1}$$

$$= 2\pi i \frac{e^{-m}}{2i}$$

$$= \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{im}}{x^2+1}\,dx + \int_\Gamma \frac{e^{imz}}{z^2+1} = \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{R} \frac{\cos mx + i\sin mx}{x^2+1}\,dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\,dz = \pi e$$

$$\text{or, } \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\text{or, } \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + \lim_{R\to\infty} \int \frac{i \sin mx}{x^2+1}\,dx + \lim_{R\to\infty} \int \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\text{or, } \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx + \cancel{\lim_{R\to\infty}}\, i\int_{-\infty}^{\infty} \frac{\sin mx}{x^2+1}\,dx + 0 = \pi e^{-m}$$

Equating real part

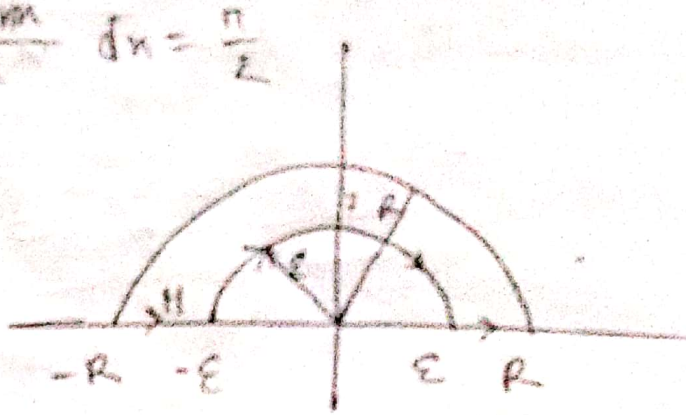
$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

$$\text{or, } 2\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

$$\text{or. } \int_{0}^{\infty} \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m}$$

Page - 7.25

Ex - 7.27 Prove that $\int_0^{\alpha} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$



Here,

$$\oint_C \frac{e^{iz}}{z}\, dz = 0$$

or, $\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{HJA} \frac{e^{iz}}{z}\, dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} + \int_{BDEFG} \frac{e^{iz}}{z}\, dz = 0$ $\quad R \to \alpha$, $\varepsilon \to 0$

or, $-\int_{\varepsilon}^{R} \frac{e^{-ix}}{x}(dx) + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{HJA} \frac{e^{iz}}{z}\, dz + \int_{BDEFG} \frac{e^{iz}}{z}\, dz = 0$

or, $\int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\, dx + \int_{HJA} \frac{e^{iz}}{z}\, dz + \int_{BDEFG} \frac{e^{iz}}{z}\, dz = 0$

$x = -x$
$dx = -dx$
$x = -R$
$-x = R$
$x = -\varepsilon, -x$

or, $2i \int_{\varepsilon}^{R} \frac{\sin x}{x} = -\int_{HJA} \frac{e^{iz}}{z}\, dz - \int_{BDEFG} \frac{e^{iz}}{z}\, dz$

Let, $R \to \alpha$ $\varepsilon \to 0$ then $\int_{BDEFG} \frac{e^{iz}}{z}\, dz = 0$

$z = \varepsilon e^{i\theta}$

or, $dz = i\varepsilon e^{i\theta}\, d\theta \quad (\theta = \pi \text{ to } 0)$

Now, $2i \int_0^{\alpha} \frac{\sin x}{x} dx = \lim_{\varepsilon \to 0} \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta$

$$= -\lim_{\varepsilon \to 0} i \int_{\pi}^{0} e^{i\varepsilon e^{i\theta}} d\theta$$

$$= -i \int_{\pi}^{0} e^{0} d\theta$$

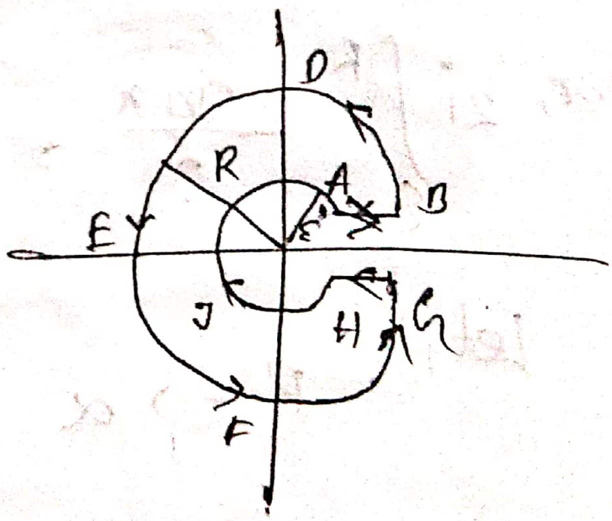
$$= -i [\theta]_{\pi}^{0} = \pi i$$

$$\int_0^{\alpha} \frac{\sin x}{x} = \frac{\pi i}{2i} = \frac{\pi}{2}$$

Ex: 7.29

Show that $\int_0^{\alpha} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}, \; 0 < p < 1$

$$\oint_C \frac{z^{p-1}}{1+z} dz = 2\pi i$$



$1+z=0 \quad \therefore z = -1$

$$a_{-1} = \lim_{z \to -1} \left[(z+1) \frac{z^{(p-1)}}{(1+z)} \right]$$

$$= (-1)^{p-1}$$

$$= (\cos\pi + i\sin\pi)^{p-1} = (e^{i\pi})^{p-1} = e^{(p-1)2\pi}$$

$$\therefore \oint_C \frac{z^{p-1}}{1+z}\,dz = 2\pi i\, e^{(p-1)\pi i}$$

$$\text{or, } \int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i\, e^{(p-1)\pi i}$$

$$\text{or, } \int_{\varepsilon}^{R} \frac{x^{p-1}}{1+x}\,dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1} Ri\,e^{i\theta}\,d\theta}{(1+Re^{i\theta})} + \int_R^{\varepsilon} \frac{x e^{2\pi i}}{1+Re^{2\pi i}}\,dx$$

$$+ \int_{2\pi}^{0} \frac{(\varepsilon e^{i\theta})^{p-1}\, i\varepsilon e^{i\theta}\,d\theta}{1+\varepsilon e^{i\theta}} = 2\pi i\, e^{(p-1)\pi i}$$

$$\int_0^{2\pi} \frac{x^{p-1}}{1+x}\,dx - \int_{dx}^{0} \frac{x^{p-1} e^{(p-1)2\pi i}}{x+1}\,dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\therefore \int_0^{\alpha} \frac{x^{p-1}}{1+x}\,dx - \int_0^{\alpha} \frac{x^{p-1} e^{(p-1)2\pi i}}{1+x}\,dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\text{or, } \{1 - e^{(p-1)2\pi i}\} \int_0^{\alpha} \frac{x^{p-1}}{x+1}\,dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\text{or, } \int_0^{\alpha} \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i\, e^{(p-1)\pi i}}{1 - e^{(p-1)2\pi i}}$$

$$= \frac{2\pi i\, e^{(p-1)\pi i}\, e^{-(p-1)\pi i}}{e^{-(p-1)\pi i} - e^{(p-1)\pi i} e^{(p-1)2\pi i}}$$

$$= \frac{2\pi i}{e^{-p\pi i} \cdot e^{\pi i} - e^{-p\pi i} e^{\pi i} e^{2\pi i} \cdot e^{-2\pi i}}$$

$$= \frac{2\pi i}{-e^{-p\pi i} + e^{p\pi i}} = \frac{\pi}{\frac{e^{p\pi i} - e^{-p\pi i}}{2i}}$$

$$= \frac{2\pi i}{\sin p i} \; ; \; 0 < p < 1$$

7-3

$$\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1} \, dx$$

$$\Rightarrow \oint_c \frac{\ln(z+i)}{z^2+1} \, dz$$

$z^2 + 1 = 0$

$\therefore z = \pm i$

z^2+1

$= (z+i)(z-i)$

$= z^2 - i^2$

$= z^2 + 1 = x^2 + 1$

$\ln(z+i) + \ln(z-i)$

$= \ln(z^2+1)$

$= \ln(x^2+1)$

Residul at $z = i$ is

$$a_{-1} = \lim_{z \to i} \left[\frac{(z-i) \ln(z+i)}{(z+i)(z-i)} \right]$$

$$a_{-1} = \frac{\ln(2i)}{2i}$$

$$\oint_C \frac{\ln(z+i)}{z^2+1}\,dz = 2\pi i\, a_{-1}$$

$$= 2\pi i \frac{\ln(2i)}{2i}$$

$$= \pi \ln(2i)$$

$$= \pi(\ln 2 + \ln i)$$

$$= \pi \ln 2 + \pi \ln i$$

$$= \pi \ln 2 + \pi \ln e^{(i\pi/2)}$$

$$= \pi \ln 2 + \pi \frac{\pi}{2} = \pi \ln 2 + \frac{\pi^2}{2}$$

$$\Rightarrow \int_{-R}^{R} \frac{\ln(x+i)}{x^2+1}\,dx = \pi \ln 2 + \frac{\pi^2}{2} + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz$$

$$\Rightarrow \int_{-R}^{0} \frac{\ln(x+i)}{x^2+1}\,dx + \int_0^R \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \frac{\pi^2}{2}$$

$$\Rightarrow \int_{R}^{0} -\frac{\ln(i-x)}{x^2+1}\,dx + \int_0^R \frac{\ln(i+x)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \frac{\pi^2}{2}$$

$$\Rightarrow \int_0^R \frac{\ln(i-x)(i+x)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \frac{\pi^2}{2}$$

$$\Rightarrow \int_0^R \frac{\ln(i^2-x^2)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \frac{\pi^2}{2}$$

or, $\int_0^R \frac{\ln\{-1(1+x^2)\}}{x^2+1}dx + \int_\Gamma \frac{\ln(z+i)}{z^2+1}dz = \pi\ln 2 + \frac{\pi^2}{2}$

or, $\int_0^R \frac{\ln(x^2+1)+\ln(-1)}{x^2+1}dx + \int_\Gamma \frac{\ln(z+i)}{z^2+1}dz = \pi\ln 2 + \frac{\pi^2}{2}$

or, $\int_0^R \frac{\ln(x^2+1)}{x^2+1}dx + \int_0^R \frac{\ln e^{i\pi}}{x^2+1}dx = \pi\ln 2 + \frac{\pi^2}{2}$

or, $\lim_{R\to\alpha}\int_0^R \frac{\ln(x^2+1)}{x^2+1}dx + i\lim_{R\to\alpha}\int_0^R \frac{\ln e^{\pi i}}{x^2+1}dx = \pi\ln 2 + \frac{\pi^2}{2}$

or, $\int_0^\alpha \frac{\ln(x^2+1)}{x^2+1}dx + i\int_0^\alpha \frac{\pi}{x^2+1}dx = \pi\ln 2 + i\frac{\pi^2}{2}$

or, $\int_0^\alpha \frac{\ln(x^2+1)}{x^2+1}dx + i\int_0^\alpha \frac{\pi}{x^2+1}dx = \pi\ln 2 + i\frac{\pi^2}{2}$

$\ln\{(-1)(x^2+1)\}$
$= \ln(-1) + \ln(x^2+1)$
$= \ln e^{\pi i}$
$\therefore$
$-1 = \cos\pi + i\sin\pi$
$= e^{i\pi}$

Equating real part from both side

$\therefore \int_0^\alpha \frac{\ln(x^2+1)}{x^2+1}dx = \pi\ln 2$

[proved]

Fourier series: odd, even

$f(-x) = -f(x) \Rightarrow \sin x, x, x^3 \ldots$

$f(-x) = f(x) \Rightarrow \cos x, x^2, x^4 \ldots$

Periodic function

$f(T+x) = f(x)$

$f(x) = \sin x$

$f(2\pi + x) = \sin(2\pi + x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

Fourier series

odd function $\to f(-x) = -f(x) \to \sin x, x, x^3$

even function $\to f(-x) = f(x) \to \cos x, x^2, x^4$

perodic function $\to f(T+x) = f(x)$

$f(x) = \sin x$

$f(2\pi + x) = \underbrace{\sin(2\pi + x)}_{\text{period}} = \sin x = f(x)$

$$\int_{-\pi}^{\pi} \sin x \, dx = 0$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

$a_0, a_n, b_n \longrightarrow$ fourier co-efficient

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}}$$

If the function is even, then $\boxed{b_n = 0}$

27-10-22
Tuesday

** $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$f(x+\pi) = f(x)$

$\pi \leq x \leq \pi \rightarrow 2\pi$

** $\int_{-\pi}^{\pi} \sin nx \, dx = 0$

** $\int_{-\pi}^{\pi} \cos nx \, dx = 0$

$\boxed{161 - 299 \text{ Page}}$

① $f(x) = x^2$; $-\pi \leq x \leq \pi$

$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

1st $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$

$= \frac{1}{6\pi} \left[\pi^3 - (-\pi)^3\right]$

$= \frac{1}{6\pi} \left[\pi^3 + \pi^3\right]$

$= \frac{1}{6\pi} 2\pi^3 = \frac{1}{3}\pi^2 = \frac{\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\left\{\left[x^2 \frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin nx}{n} dx\right.$$

$$= \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{2}{n} x \sin nx \, dx$$

$$= \frac{1}{n\pi}\left[x^2 \sin nx\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{n\pi} \times 0 - \frac{2}{n\pi}\left(\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n\pi}\int_{-\pi}^{\pi}\left[-\frac{\cos nx}{n}\right] dx\right.$$

$$= \frac{2}{n^2\pi}\left[\pi \cos^n \pi + \pi \cos^n \pi\right] + 0$$

$$= \frac{2}{n^2\pi} \times 2\pi \cos^n \pi$$

$$= \frac{4}{n^2} \times (-1)^n$$

$f(x) = x^2$

$f(-x) = (-x^2) = x^2$

$f(x)$ is an even function so, $b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4 \times (-1)^n}{n^2} \cos nx + 0 \right]$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \times (-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \right)$$

Here, when $x = 0$ then $f(0) = 0$

$$0 = \frac{\pi^2}{3} - 4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \right)$$

Or, $4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots \right) = \frac{\pi^2}{3}$

Or, $4 \left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right] - 2 \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \right] \right.$

$$\text{or, } 4\left[\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)-\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)\right]=\frac{\pi^2}{3}$$

$$\text{or, } 4\left[\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)\right]=\frac{\pi^2}{3}$$

$$\text{or, } 2\times\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}$$

$$\text{or } \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Even function: $b_n = 0 \longrightarrow$ cosine series

Odd function: $a_0 = 0$, $a_n = 0 \longrightarrow$ sine series

$$f(x)=\begin{cases}0 & ; -\pi < x \le 0\\ x & ; 0 < x \le \pi\end{cases}$$

We know, $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 0 + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{4\pi}\left[x^2\right]_0^{\pi} = \frac{1}{4\pi}\left[\pi^2\right] = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi} 1\cdot\frac{\sin nx}{n}\, dx$$

$$= \frac{1}{n\pi}\times 0 + \frac{1}{n\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{n^2\pi}\times\left[(-1)^n - 1\right]$$

$$\boxed{\begin{array}{l}\sin n\pi = 0 \\ \cos n\pi = (-1)^n\end{array}}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\, dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} x\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{\cos nx}{n}\, dx$$

$$= \frac{-1}{n\pi}\left[\pi\cos n\pi - 0\right] + 0$$

$$= -\frac{1}{n\pi} \times \pi\cos n\pi$$

$$= -\frac{1}{n}\times(-1)^n = \frac{1}{n}\times(-1)^{n+1}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\left[(-1)^n - 1\right]\cos nx + \frac{1}{n}(-1)^{n+1}\sin nx\right\}$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left\{\frac{-2}{1^2}\cos x + 0 + \frac{-2}{3^2}\cos 3x + 0 + \frac{-2}{5^2}\cos 5x + 0 + \cdots\right\} + \left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\} +$$

$$\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = ?}$$

putting $x=0$ in (*) $f(0)=0$

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + 0$$

$$\text{or, } \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi}{4}$$

$$\text{or, } \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi}{4}$$

$$\text{or, } \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{8}$$

$$\text{or, } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Half range fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

P – 217, Ex – 17

$f(x) = x(\pi - x) \quad 0 \le x \le \pi$

$f(x) = \pi x - x^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2)\, dx$$

$$= \frac{2}{\pi} \left[\pi \frac{x^2}{2}\right]_0^{\pi} - \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= [x^2]_0^{\pi} - \frac{2}{3\pi} \times \pi^3$$

$$= \pi^2 - \frac{2}{3}\pi^2 = \frac{3\pi^2 - 2\pi^2}{3} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx\, dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2) \cdot \frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} (\pi - 2x) \frac{\sin nx}{n}$$

$$= 0 - \frac{2}{n\pi}\left[(\pi-2x)\frac{-\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}(0-2)\frac{-\cos nx}{n}\,dx$$

$$= \frac{2}{n^2\pi}\left[-\pi(-1)^n - \pi\right]$$

$$= \frac{2}{n^2}\left[(-1)\cdot\{(-1)^n+1\}\right]$$

$$= \frac{-2}{n^2}\left[(-1)^n+1\right]$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = 0$$

$$f(x) = \frac{a_0}{2} + \cancel{f} \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{2}{n^2}\left[(-1)^n+1\right]\cos nx$$

$$= \frac{\pi^2}{6} - \Big[0 + 2\cdot\frac{2}{2^2}\cos 2x + 0 + 2\cdot\frac{2}{4^2}\cos 4x + 0$$
$$+ 2\cdot\frac{2}{6^2}\cos 6x + \cdots\Big]$$

$$= \frac{\pi^2}{6} - \left[\frac{\cos 2\cdot 1x}{1^2} + \frac{\cos 2\cdot 2x}{2^2} + \frac{\cos 2\cdot 3x}{3^2} + \cdots\right]$$

$$= \frac{\pi^2}{\;} - \sum^{\infty}\frac{\cos 2mx}{\;}$$

Vector Analysis

$$A = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\underline{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\underline{A}.\underline{B} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\underline{A}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$A \times B = \begin{vmatrix} i & j & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a.(b \times c) = [\underline{a}\ \underline{b}\ \underline{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\underline{\nabla} = i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}$$

$$= \Sigma i \frac{\delta}{\delta x}$$

$$\underline{\nabla} p = i\frac{\delta\phi}{\delta x} + j\frac{\delta\phi}{\delta y} + k\frac{\delta\phi}{\delta z}$$

$$= \Sigma i \frac{\delta\phi}{\delta x}$$

$$\underbrace{\underline{\nabla}.\underline{V}}_{\text{divergence of } V} = \frac{\delta V_1}{\delta x} + \frac{\delta V_2}{\delta y} + \frac{\delta V_3}{\delta z} = \Sigma \frac{\delta V_1}{\delta x}$$

$V = V_1 i + V_2 j + V_3 k$

$$\text{curl} = \nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i\left(\frac{\delta V_3}{\delta y} - \frac{\delta V_2}{\delta z}\right) + j\left(\frac{\delta V_1}{\delta z} - \frac{\delta V_3}{\delta x}\right) + k\left(\frac{\delta V_2}{\delta x} - \frac{\delta x}{\delta y}\right)$$

$$= \Sigma\, i\left(\frac{\delta V_3}{\delta y} - \frac{\delta V_2}{\delta z}\right)$$

Solenoidal $\rightarrow \nabla \cdot \underline{V} = 0$

Irrotational $\rightarrow \underline{\nabla} \times \underline{V} = 0$

Directional derivation $\underline{\nabla}\, \phi\, \underline{a}$

$$\bar{a} \rightarrow n = \frac{\vec{a}}{|\bar{a}|}$$

Page - 638

θ $\vec{A}, \vec{B}$ θ

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$

$\phi(x,y,z) = x^2+y^2+z^2-9$

$\psi(x,y,z) = x^2+y^2-3-z$

Let, $\vec{A} = \vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ | $\vec{B} = \vec{\nabla}\psi = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let, θ be the angle between the ϕ and ψ

$$\cos\theta = \frac{\vec{\nabla}\phi \cdot \vec{\nabla}\psi}{|\vec{\nabla}\phi||\vec{\nabla}\psi|}$$

$$= \frac{(2x\hat{i}+2y\hat{j}+2z\hat{k})(2x\hat{i}+2y\hat{j}-\hat{k})}{\sqrt{(4x^2+4y^2+4z^2)(4x^2+4y^2+1)}}$$

$$= \frac{4x^2+4y^2-2z}{2\sqrt{(x^2+y^2+z^2)(4x^2+4y^2+1)}}$$

At the point $(2,-1,2)$ $\cos\theta = \dfrac{16+4-4}{2\sqrt{(4+4+1)(16+4+2)}}$

$$\text{or, } \cos\theta = \frac{16}{2.3\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

[I41] $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{(x^2+y^2+z^2)}$$

$$\text{or, } r^n = (x^2+y^2+z^2)^{\frac{n}{2}}$$

$\vec{\nabla} \times (r^n . \vec{r}) = (\vec{\nabla} . r^n) \times \vec{r} + r^n (\vec{\nabla} \times \vec{r})$ [formula B23 pa

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x & y & z \end{vmatrix} \quad \text{———} (1)$$

$$= i(0-0) + y(0-0) + z(0-0)$$

$$= 0$$

$$\vec{\nabla} . r^n = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} . 2x + \hat{j}\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} . 2y +$$

$$\hat{k}\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} . 2z$$

$= n(x^2+y^2+z^2)^{n/2-1} (\hat{i}x + \hat{j}y + \hat{k}z)$

$= n(x^2+y^2+z^2)^{\frac{n-2}{2}} . \vec{r}$

$= n r^{n-2} . \vec{r}$ $\quad [\because r = (x^2+y^2+z^2)^{1/2}]$

putting these value in eq-(1) we get

$\Rightarrow \nabla \times (r^n \vec{r}) = n r^{n-2} . \vec{r} \times \vec{r} + 0$

$= n r^{n-2} . 0 = 0$

Hence, $r^n \vec{r}$ is a irrotional

(4)

$r^n . \vec{r} \rightarrow n = -3$

$\Rightarrow \nabla (r^n \vec{r}) = 0$

$\Rightarrow \nabla (\phi \vec{V}) = (\vec{\nabla} . \phi) . \vec{V} + \phi(\vec{\nabla} . \vec{V})$ —— (i)

$\Rightarrow \nabla (r^n . \vec{r}) = (\vec{\nabla} . r^n) . \vec{r} + r^n(\vec{\nabla} . \vec{r})$ —— (1)

$\vec{\nabla} . \vec{r} = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$

$= (1+1+1) = 3$ —— (2)

putting the value of (2) in equation (1)

$\Rightarrow (r^n \cdot \vec{r}) = n r^{n-2} \vec{r} \cdot \vec{r} + r^n \cdot 3$

$= n r^{n-2} \cdot r^2 + 3r^n$

$= r^n (n+3)$

when, $n=-3$ then $\Rightarrow (r^n \cdot \vec{r}) = 0$

Hence, when $n=-3$ $r^n \cdot \vec{r}$ is solinoidal.

line, surface, volume

$\int_C F(R)\, dR$

$= \int_C (f\,dx + g\,dy)$

$= (fi + gj)(i\,dx + j\,dy)$

$F(R) = if + jg$

$dR = i\,dx + j\,dy$

$\iint_S F \cdot ds$

$\iint_S F \cdot n \cdot ds = \iint_R F \cdot n \frac{\delta x \delta y}{|n \cdot k|}$

Page - 661 Math 32

$$\iint_R \sqrt{x^2+y^2}\, dx\, dy$$

$$\int_{x=6}^{6} \int_{y=\sqrt{36-x^2}}^{\sqrt{36-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$$= 4 \int_0^6 \int_0^{\sqrt{36-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$$= 4 \int_0^6 \left[y \frac{\sqrt{x^2+y^2}}{2} + \frac{x^2}{2} \log\left(y + \sqrt{x^2+y^2}\right) \right]_0^{\sqrt{36-x^2}} dx$$

$xy = x^2+y^2 = 36$

$\Rightarrow y^2 = 36 - x^2$

$\Rightarrow y = \pm \sqrt{36-x^2}$

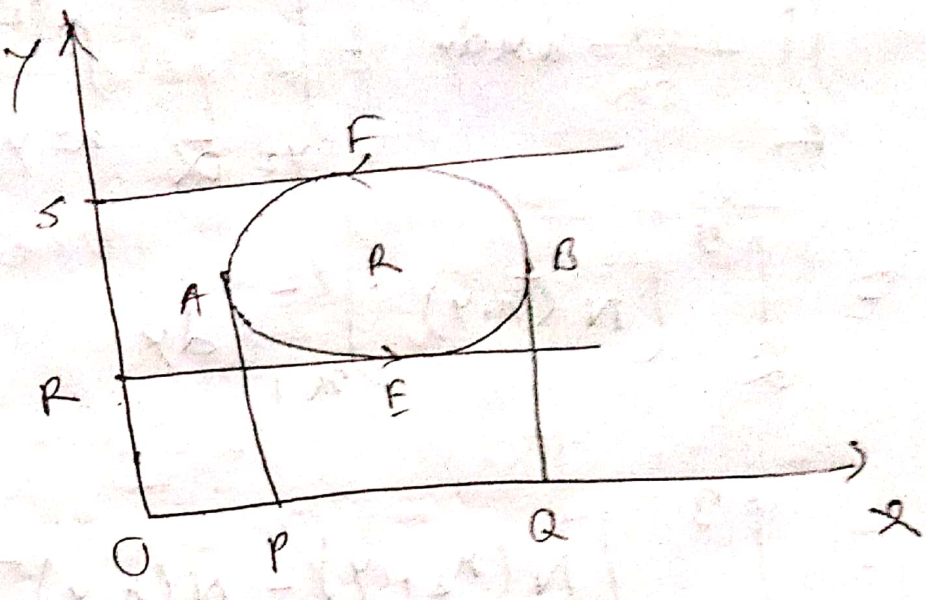
$\Rightarrow x = \pm \sqrt{36-x^2}$

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$$

even function

$$\int \sqrt{a^2+x^2}\, dx = x.\frac{\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \log\left(x+\sqrt{x^2+a^2}\right)$$

Green's Theorem :-



$$\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta m}{\delta y} \right) dx\,dy$$

Here, $$\int \frac{\delta m}{\delta y}\, dx\,dy = \int_{x=a}^{b} \left[\int_{y=y_1(x)}^{y_2(x)} \frac{\delta m}{\delta y}\, dy \right] dx$$

$$= \int_a^b \Big[m(x,y) \Big]_{y_1(x)}^{y_2(x)} dx$$

$$= -\int_a^b m(x, y_1(x))\, dx + \int_a^b M(x, y_2(x))\, dx$$

$$= -\left[\int_a^b m(x, y_1(x))\, dx + \int_b^a M(x, y_2(x))\, dx \right.$$

$$= -\oint_c m(x,y)\, dx$$

$$\therefore \oint_c m\,dx = -\iint_R \frac{\delta m}{\delta y}\, dx\,dy$$

$$\iint_R \frac{\delta N}{\delta x}\, dx\, dy = \int_{y=\alpha}^{\beta} \left[\int_{x=y_1(x)}^{y_2(x)} \frac{\delta N}{\delta x}\, dx \right] dy$$

$$= \int_{\alpha}^{\beta} \left[N(x,y) \right]_{x_1}^{y_2} dy$$

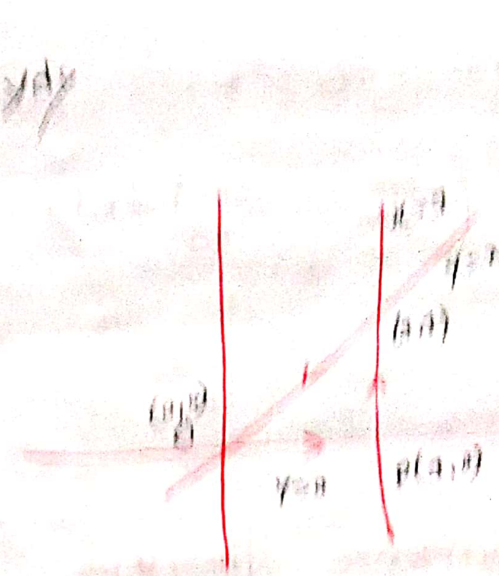
$$= \int_{\alpha}^{\beta} \left[N(x_2, y) - N(x, y) \right] dy$$

$$= \int_{\alpha}^{\beta} N(x_2, y)\, dy + \int_{\beta}^{\alpha} N(x, y)\, dy$$

$$= \oint_C N(x, y)\, dy$$

Page-309

9.8) $\oint_C (xy - x^2)\,dx + x^2 y\,dy$



$$= \int_0^1 (0 - x^2)\,dx + 0\,dy$$

$$= -\left[\frac{x^3}{3}\right]_0^1 = -\frac{1}{3}$$

$$\int_{y=0}^{1} \left[(1\cdot y - 1^2)\cdot 0 + 1^2 y\,dy\right]$$

$$= \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\int_{y=1}^{0} \left[(y\cdot y - y^2)\,dy + y^2 y\,dy\right]$$

$$= \int_1^0 y^3\,dy = \left[\frac{y^4}{4}\right]_1^0 = -\frac{1}{4}$$

and, $I = -\frac{1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{-4+6-3}{12} = \frac{-1}{12}$

(b) $\oint_C (xy - x^2)\,dx + x^2 y\,dy$

$N = x^2 y$ $\qquad M = xy - x^2$

$\frac{\delta N}{\delta x} = 2xy$ $\qquad \frac{dM}{dy} = x$

~~Here,~~ $\cancel{N = \frac{x^2}{2}}$ $\cancel{M = xy - x^2}$

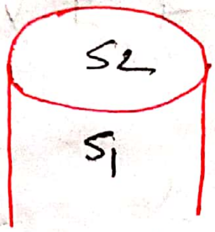
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} (2xy - x)\,dy\,dx$$

$x = const$
$y = variable$

$$= -\frac{1}{12}$$

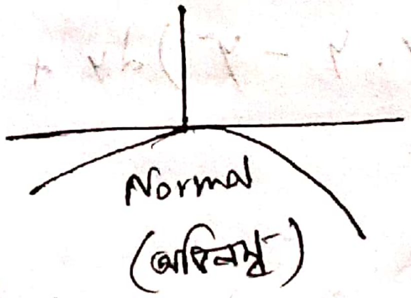
※ divergence theorem

$\boxed{P.715}$



relation between surface and volume integral

volume $v \Rightarrow \iiint$ surface $\Rightarrow \iint$



↑ positive

↓ negative

<u>math form:</u>

state and prove Gauss's divergence theorem.

$$\iiint_V \vec{\nabla}\cdot \vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

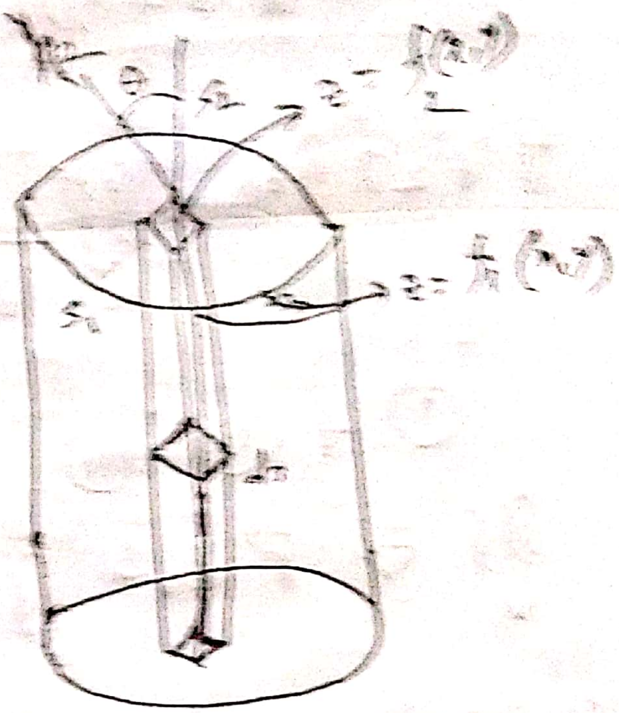
$\hat{n}$ = unit normal is outward normal.

$\vec{F} = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \left(\frac{\delta F_1}{\delta x}+\frac{\delta F_2}{\delta y}+\frac{\delta F_3}{\delta z}\right) dx\,dy\,dz$$

$z = f_1(x,y,z)$

$z = f_2(x,y,z)$



Here,

$$\iiint_V \frac{\delta F_3}{\delta z}\, dz\,dy\,dx$$

$$= \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\delta F_3}{\delta z}\, dz\right] dy\,dx$$

$$= \iint_R \Big[F_3(x,y,z)\Big]_{f_1(x,y)}^{f_2(x,y)} dy\,dx$$

$$= \iint_R \left[F_3(x,y,f_2) - F_3(x,y,f_1)\right] dy\,dx$$

$$= \iint_S F_3 \cos\theta_2\, ds_2 + \iint_S F_3\cos\theta_1\, ds_1 .$$

for S_2 surface

$dy\,dx = \cos\theta_2\, ds_2$

$\hat{k}\cdot\hat{n}_2 = |k||n_2| \cos\theta$

$\therefore \hat{k}\cdot\hat{n}_2\, ds_2 = \cos\theta_2\, ds_2$

$\boxed{\therefore \hat{k}\cdot\hat{n}_2\, ds_2 = \cos\theta_2\, ds_2}$

for S_1 surface

$dy\,dx = -\cos\theta_1\, ds_1$

$$\iint_S F_3 \hat{k}\,\hat{\eta}\, ds = \iiint_V \frac{\delta F_3}{\delta z}\, dz\, dy\, dx \quad \text{—— (i)}$$

$$\iint_S F_1 \hat{i}\,\hat{\eta}\, ds = \iiint_V \frac{\delta F_1}{\delta x}\, dx\, dy\, dz \quad \text{—— (ii)}$$

$$\iint_S F_2 \hat{j}\,\hat{\eta}\, ds = \iiint_V \frac{\delta F_2}{\delta y}\, dy\, dx\, dz \quad \text{—— (iii)}$$

(i) + (ii) + (iii)

$$\iiint_V \vec{\nabla}\cdot\vec{F}\, dv = \iint_S \vec{F}\cdot\hat{\eta}\, ds$$

P-7201

Ex-52

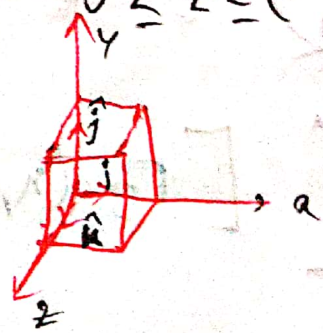
verify divergence theorem

$$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}\ ; \quad 0 \le x \le a$$

$$0 \le y \le b$$

$$0 \le z \le c$$

$$\iiint_V \vec{\nabla}\cdot\vec{F}\, dv = \iint_S \vec{F}\cdot\hat{\eta}\, ds$$



$$\vec{\nabla}\cdot\vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

$$\vec{\nabla}\cdot\vec{F}\, dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} 2(x+y+z)\, dz\, dy\, dx$$

$$= abc\,(a + b + c)$$

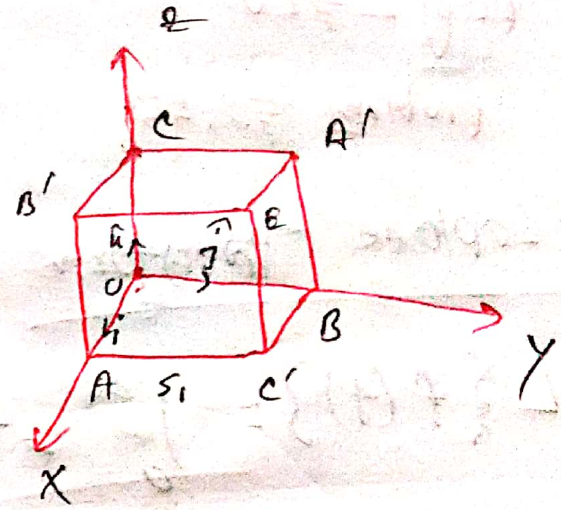
$$\iint_{S_1} F \cdot \hat{\eta}\, ds_1 \quad ; \eta = -k$$

$$= \iint_{S_1} F \cdot (-\hat{k})\, ds_1$$

$$= -\iint_{S_1} (z^2 - xy)\, ds_1$$

$$= \int_{y=0}^{b} \int_{x=0}^{a} xy \; dx\, dy = \frac{a^2 b^2}{4}$$

$$= \underline{\qquad}$$



pag - 710

problem 53, 55

Laplace Transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad s>0$$

$$L\{1\} = \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

$$L\{t\} = \int_0^\infty e^{-st}.t\,dt$$

$$= \left[t\cdot\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty 1.\frac{e^{-st}}{-s}\,dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s^2}$$

$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \; dt$

$\cancel{\left[e^{-st} \cdot -\frac{\cos at}{a}\right]_0^\infty - \int_0^\infty -se^{-st} \cdot -\frac{\cos at}{a} \, dt}$

$\cancel{= \left[0 + \frac{1}{a}\right]}$

Or, " $= \left[\sin at \, \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty a \cos at \, \frac{e^{-st}}{-s} \, dt$

Or, " $= 0 - 0 + \frac{a}{s} \int_0^\infty \cos at \; e^{-st} \, dt$

Or, " $= \frac{a}{s} \left[\cos at \, \frac{e^{-st}}{-s}\right]_0^\infty - \frac{a}{s} \int_0^\infty -a \sin at \, \frac{e^{-st}}{-s} \, dt$

Or, " $= \frac{a}{s^2} - \frac{a^2}{s^2} \int_0^\infty \sin at \; e^{-st} \, dt$

Or, " $= \frac{a}{s^2} - \frac{a^2}{s^2} L\{\sin at\}$

Or, $\left(1 + \frac{a^2}{s^2}\right) L\{\sin at\} = \frac{a}{s^2}$

Or, $\frac{s^2+a^2}{s^2} L\{\sin at\} = \frac{a}{s^2}$

Or, $L\{\sin at\} = \frac{a}{s^2+a^2}$

Or, $L\{\cos at\} = \frac{s}{s^2+a^2}$

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= 0 + \frac{1}{s-a}$$

$$= \frac{1}{s-a}$$

~~$L\{e^{at}$~~ $L\{F(t)\} = f(s)$

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty - \int_0^\infty -se^{-st} . F(t)\, dt$$

$$= 0 - F(0) + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= -F(0) + s\, L\{F(t)\}$$

$$= -F(0) + s f(s)$$

$$\boxed{\therefore\ L\{F'(t)\} = sf(s) - F(0) \quad \text{---} \ (1)}$$

$$L\{F''(t)\} = f(s)$$

$$L\{F''(t)\} = \int_0^\infty e^{-st} F''(t)\, dt$$

$$= \left[e^{-st} F'(t)\right]_0^\infty - \int_0^\infty -se^{-st} F'(t)\, dt$$

$$= 0 - F'(0) + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= - F'(0) + s\left[sf(s) - F(0)\right]$$

$$= - F'(0) + s^2 f(s) - sF(0)$$

$$= s^2 f(s) - sF(0) - F'(0)$$

$$= s^2 f(s) - F'(0) - sF(0)$$

$$\boxed{L\{F''(t)\} = s^2 f(s) - F'(0) - sF(0) \text{ --- } (11)}$$

$$L\{F'''(t)\} = \int_0^\infty e^{-st} F'''(t)\, dt$$

$$= \left[e^{-st} F''(t)\right]_0^\infty - \int_0^\infty -se^{-st} F''(t)\, dt$$

$$= - F''(0) + s\int_0^\infty e^{-st} F''(t)\, dt$$

$$\cancel{= - F''(0) + s\left[s^2 f(s) - sF(0) - F'(0)\right]}$$

$$= -F''(0) + s\left[e^{-st}F'(t) - \int_0^\infty -se^{-st}F'(t)\,dt\right]_0^\infty$$

$$= -F''(0) + s\left[e^{-st}F'(t) + s\left[e^{-st}F(t) - \int_0^\infty -se^{-st}F(t)\right]\right]_0^\infty$$

$$= -F''(0) + s\left[e^{-st}F'(t) + s\left[e^{-st}F(t) + s\left[f(s) - F(0)\right]\right]\right]_0^\infty$$

$$= -F''(0) + s\left[s^2 f(s) - sF(0) - F'(0)\right]$$

or, $$\boxed{L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)}$$

$$L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0)$$

$$\boxed{+ \dots - sF^{n-2}(0) - F^{n-1}(0)}$$

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n F^n(s)$$

$$L\{tF(t)\} = -f'(s)$$

$$L\{t^2 F(t)\} = F''(s)$$