CHAPTER TEN

VECTOR ANALYSIS

10.1. Introduction and definitions :

In Applied Mathematics, Physics, Geometry and Mechanics we generally deal with different plysical quantities as for examples length, mass, volume, temperature, weight, velocity, acceleration, force etc. Considering various properties these physical quantities can be classified into two types viz, scalar and vector quantities as defined below :

Some physical quantities which are completely determined when their magnitudes are given in terms of a specfic unit are called **scalars.** e. g. mass, length, volume, temperature, density, time and also any real number etc.

On the other hand, some physical quantities which are completely determined when their magnitudes as well as directions are given are called **vectors.** e.g.

weight, displacement, velocity, acceleration, tension, force and electric current density etc.

Vectors are generally represented by a directed line segment whose direction represents the direction of the vector and whose length, in terms of some chosen unit, represents its magnitude.

Generally vectors are denoted by ***a*, *b*, *c*,** - - -

(i.e Letters printed in bold types) or $\vec{a} \cdot \vec{b} \cdot \vec{c}$ - - - -

(i.e by putting arrows over the letters) and scalars are denoted by $a, b, c, x, y, z, \alpha, \beta, \gamma, \delta,$ - - - - .

If **v** is a vector, then its magnitude is represented by $|\mathbf{v}|$ (modulus of **v**).

Definition Unit vector

A vector whose length or magnitude is 1 (unity) is called a **unit vector**, that is, if $\mathbf{v}$ is a vector then $\frac{\mathbf{v}}{|\mathbf{v}|}$ is called a **unit vector** in the direction of $\mathbf{v}$.

If $\mathbf{v}$ is a vector and α is a scalar, then $\alpha\mathbf{v}$ or $\mathbf{v}\,\alpha$ is a vector whose magnitude is α times the magnitude of the original vector $\mathbf{v}$. If α is positive, the direction of $\alpha\mathbf{v}$ or $\mathbf{v}\alpha$ will be in the direction of $\mathbf{v}$ and if α is negative, the direction of $\alpha\mathbf{v}$ or $\mathbf{v}\alpha$ will be in the opposite direction of $\mathbf{v}$.

Any vector $\mathbf{v}$ can be expressed by a unit vector $\mathbf{i}$ in the direction of $\mathbf{v}$ multiplied by the magnitude of $\boldsymbol{v}$ i.e $\boldsymbol{v} = \mathbf{i} \times |\boldsymbol{v}|$.

If α, β are scalars and $\mathbf{v}$ is a vector, then we have the following properties :

(i) $\alpha(-\boldsymbol{v}) = (-\alpha)\ \boldsymbol{v} = -\alpha\,\boldsymbol{v}$

(ii) $(-\alpha)(-\boldsymbol{v}) = \alpha\,\boldsymbol{v}$

(iii) $(\alpha + \beta)\,\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\,\boldsymbol{v}$

(iv) $(\alpha\beta)\,\boldsymbol{v} = \alpha\,(\beta\boldsymbol{v}) = \beta(\alpha\,\boldsymbol{v})$

(v) $o\,\boldsymbol{v} = \mathbf{O}$ where $\alpha = o$ (scalar) and $\mathbf{O}$ is a zero vector.

Definition Zero vector (or, Null vector)

A vector whose magnitude (or modulus) is zero is called the **zero vector (or null vector)**. whose direction is indeterminate. The null vector is represented by $\vec{0}$ or, **O.** The null vector can also be defined as a vector whose initial and terminal points coincide.

Definition Equal vectors

Equal vectors are those vectors which have equal magnitude, same direction (parallel) and same sense (arrow).

Definition Negative (or oppisite) vectors.

Negative vector is a vector whose magnitude is equal to that of the given vector, same direction (parallel) but opposite sense (arrow).

Definition Like and unlike vectors

Like vectors are those which have the same direction (parallel), same sense (arrow) and the magnitude may be different.

Unlike vectors are those vectors which have the same direction (parallel), opposite sense (arrow) and the magnitude may be different.

Definition collinear vectors

The vectors parallel to the same line, regardless of their magnitudes and sense of directions are **called collinear vectors.**

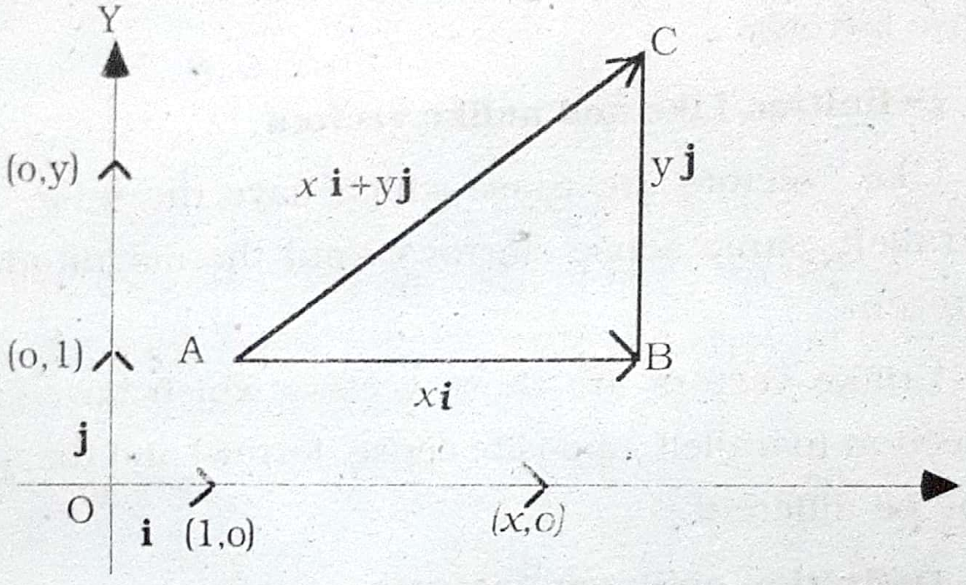
Any verctor $\mathbf{v}$ collinear with the vector $\mathbf{u}$ can be expressed as $\mathbf{v} = \lambda\, \mathbf{u}$ where λ is a scalar.

Definition Coplanar vectors.

Three or more vectors which are parallel to the same plane are called **coplanar vectors.** Evidently, any two vectors are always coplanar.

10. 2. Vectors in plane (i. e. in R^2)

Vectors can be expressed in terms of its components parallel to the axes of cartesian co-ordinate system. This is accomplished by using the same unit of length on the two axes with vectors of unit length along the axes used as basic vectors in terms of which the other vectors in the plane may be expressed.



We choose the unit vector **i** along the x-axis from (0, 0) to (1,0) and along the y- axis we choose the unit vector **j** from (0, 0) to (0, 1). Then $x\mathbf{i}$, when x is a scalar, represents a vector parallel to the x-axis having magnitude $|x|$ and $x\mathbf{i}$ have the same direction as **i** if x is positive or the opposite direction if x is negative. Similarly, $y\mathbf{j}$ is a vector parallel to the y-axis and having the same direction as **j** if y is positive or the opposite direction if y is negative.

Now a vector $\boldsymbol{v} = \overrightarrow{AC}$ in the plane of X and Y can be expressed as a multiple of **i** plus a multiple of **j**, that is,

$\boldsymbol{v} = x\mathbf{i} + y\mathbf{j}$.

The **length** or **magnitude** of $\boldsymbol{v}$ is $|\boldsymbol{v}|$ ...

Two vectors are **equal** if they have the same direction and same magnitude.

If two vectors **u** and $\boldsymbol{v}$ are given in terms of components in plane.

$\mathbf{u} = x_1\mathbf{i} + y_1\boldsymbol{j}$ *ans* $\boldsymbol{v} = x_2 + y_2\boldsymbol{j}$

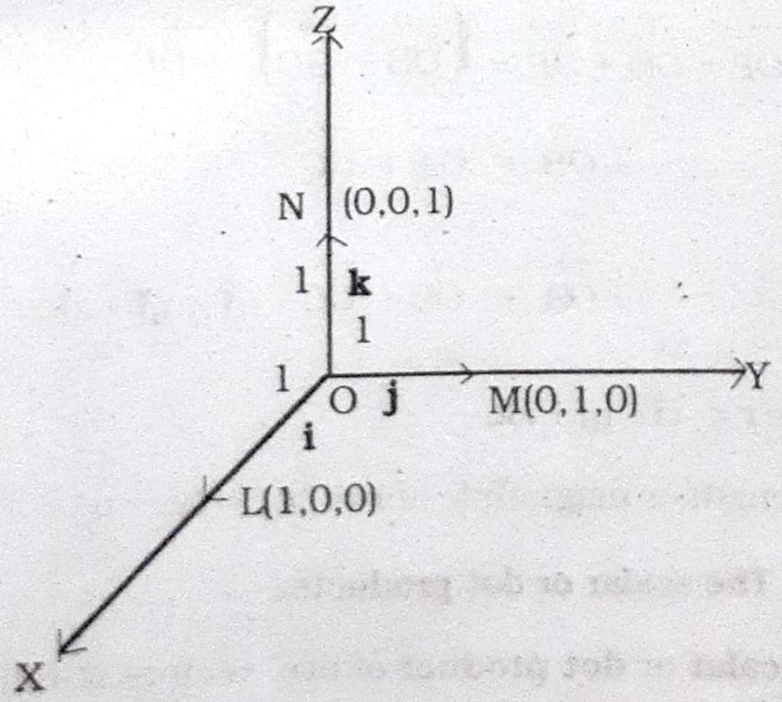
then addition of $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{\mu} + \boldsymbol{v} = (x_1 + x_2)\,\mathbf{i} + (y_1 + y_2)\,\boldsymbol{j}$ and the sutraction of $\boldsymbol{v}$ from **u** is

$\mathbf{u} - \boldsymbol{v} = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j}$

Any vector whose lengh or magnitude is zero is called a **zero vector, O**, the vector $x\mathbf{i} + y\mathbf{j} = 0$ if and only if $x = y = 0$.

10. 3. Unit vectors in space (or in R³)

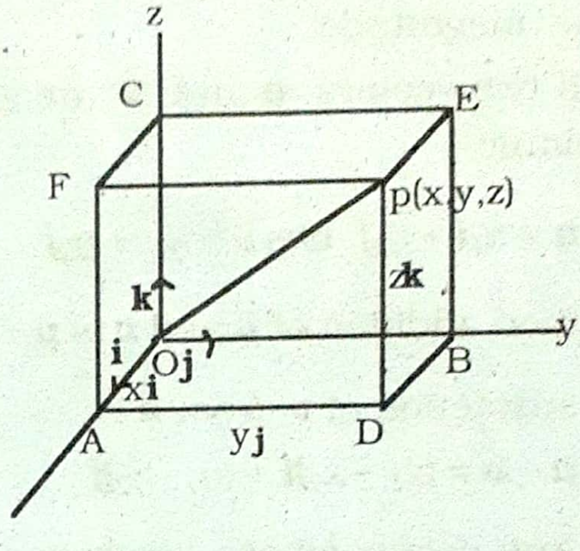
OL = OM = ON = 1 (unity).



The vectors from the origin to the points L(1,0,0) M(0, 1, 0) and N(0, 0, 1) are basic unit vectors and are generally denoted by $\vec{OL} = \mathbf{i}$, $\vec{OM} = \boldsymbol{j}$ *and* $\vec{ON} = \mathbf{k}$.

10.4. Vectors in space (i. e. in R^3).

Let O be the origin and P be any point in space having coordinates (x, y, z), Let **i**, **j**, **k** be the unit vectors along the three axes of x, y, z respectively. Let Op=r,



Construct a rectangular parallelopiped with OP as diagonal and OA, OB, OC as edges along the three respective axes. Then

$$\vec{OA} = x\mathbf{i}, \vec{OB} = y\mathbf{j} \text{ and } \vec{OC} = z\mathbf{k},$$

$$\text{Now } \vec{OP} = \vec{OD} + \vec{DP} = \left(\vec{OB} + \vec{BD}\right) + \vec{DP}$$

$$= \vec{OB} + \vec{OA} + \vec{OC}$$

$$= \vec{OA} + \vec{OB} + \vec{OC} = x\hat{\mathbf{i}} + y\mathbf{j} + z\mathbf{k}.$$

Hence $\mathbf{r} = x\hat{\mathbf{i}} + y\mathbf{j} + z\mathbf{k}.$

The length or magnitude of **r** is $|\mathbf{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

10.5. The scalar or dot products.

The **scalar** or **dot product** of two vectors **u** and $\boldsymbol{v}$ is a scalar defined by the equation $\mathbf{u}.\boldsymbol{v} = |\mathbf{u}|\,|\boldsymbol{v}| \cos\theta$ where θ is the angle between **u** and $\boldsymbol{v}$ when their initial points coincide and $0 \le \theta \le \pi$.

The **scalar** or **dot product** has the following properties : If **u**, v, **w** be the vectors with the same number of components then

(i) $\mathbf{u}.\mathbf{u} \geq 0$ *and* $\mathbf{u}.\ \mathbf{u} = 0$ *if and only if* $\mathbf{u}=0$.

(ii) $u.v = v.u$

(iii) $u.\ (v+w) = u.v + u.w$

(iv) $\alpha\ (\mathbf{u}.\mathbf{v}) = (\alpha\ \mathbf{u}).\ \mathbf{v} = \mathbf{u}.(\alpha\mathbf{v}) = (\mathbf{u}.\mathbf{v})\ \alpha$.*where* α *is a scalar.*

(v) $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\ \mathbf{k} = 1$ *and* $\mathbf{i}.\ \mathbf{j} = \mathbf{j}.\ \mathbf{k} = \mathbf{k}.\ \mathbf{i} = 0$

(vi) *if* $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ *and* $\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

then $\mathbf{u}.\ \mathbf{v} = x_1x_2 + y_1y_2 + z_1z_2$

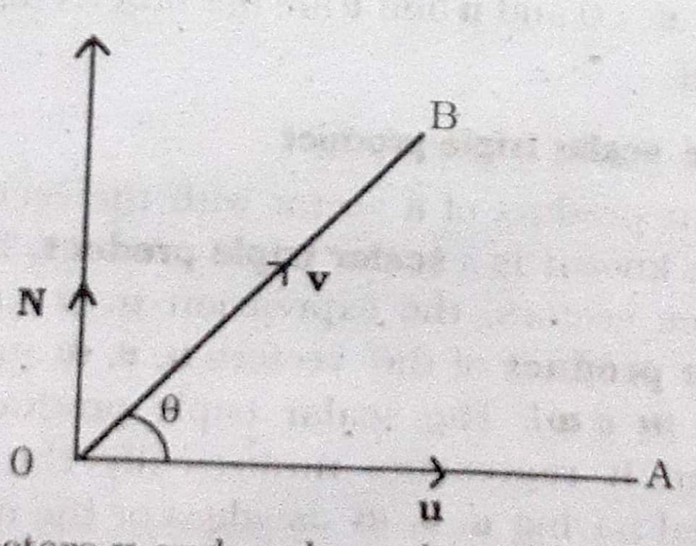
$$\mathbf{u}.\mathbf{u} = |\mathbf{u}|^2 = x_1^2 + y_1^2 + z_1^2$$

$$\mathbf{v}.\mathbf{v} = |\mathbf{v}|^2 = x_2^2 + y_2^2 + z_2^2.$$

(vii) If $\mathbf{u}.v = 0$ and **u** and v are not null vectors, then **u** and v are perpendicular. Perpendicular vectors are also known as **orthogonal vectors.**

10.6. The vector or cross product.

$\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\ \sin\theta\,\mathbf{N}$



Given two vectors **u** and v whose directions are inclined at an angle θ, their **vector product** or **cross product** is defined to be a vector $\mathbf{u} \times v$, whose module is $|\mathbf{u}|\ |v|\ \sin\theta$ and whose direction is perpendicular to both **u** and $\boldsymbol{v}$, being positive relative to a rotation from **u** to $\boldsymbol{v}$,

that is, $\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\ |\boldsymbol{v}|\ \sin\theta\ \mathbf{N}$

where **N** is the unit vector perpendicular to the plane of **u** and $\boldsymbol{v}$ and has the same direction as is obtained by the motion of a right handed screw due to rotation from **u** to $\boldsymbol{v}$.

The vector product or cross product satisfies the following properties :

(i) $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$

(ii) $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$

(iii) $\alpha(\boldsymbol{u} \times \boldsymbol{v}) = (\alpha\boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (\alpha\boldsymbol{v}) = (\boldsymbol{u} \times \boldsymbol{v})\alpha$.

where α is a scalar

(iv) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Also $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

(v) If $\boldsymbol{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\boldsymbol{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$,

$$\text{then } \boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

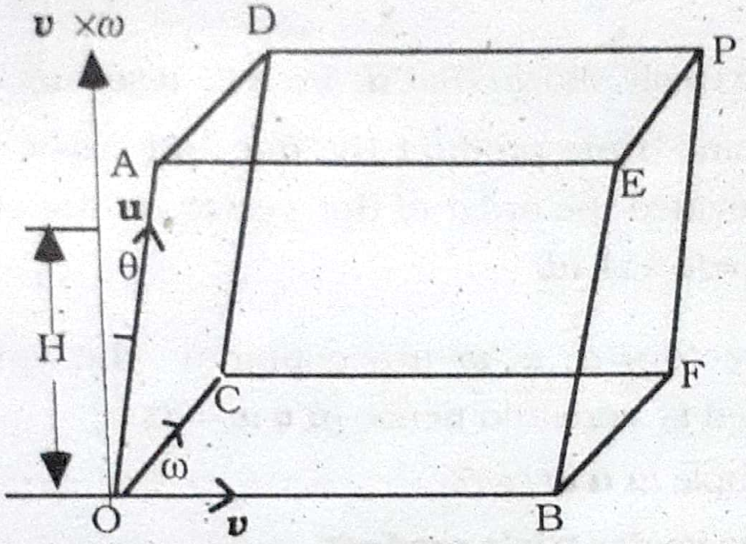
(vi) $|\boldsymbol{u} \times \boldsymbol{v}|$ = the area of a parallelogram with sides **u** and $\boldsymbol{v}$

(vii) If $\boldsymbol{u} \times \boldsymbol{v} = 0$ and **u** and $\boldsymbol{v}$ are not null vectors, then **u** and **v** are parallel.

10.7. The scalar triple product

The scalar product of a vector with the vector product of two vectors is known as a **scalar triple product.** Thus if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are any three vectors, the expression $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})$ is called a **scalar triple product** of the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and is generally donoted by $[\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}]$. The scalar triple product is a scalar quantity and it represents numerically the volume of a parallelopiped having $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as edges or the negative of this volume according as $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ do or do not form a right handed system.

Consider the parallelopiped having concurrent edges OA, OB, OC, such that $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OB} = \mathbf{v}$ $\overrightarrow{OC} = \mathbf{w}$



Now the vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to the face OBFC and its modulus $|\mathbf{v} \times \mathbf{w}|$ is the area of that face.

Let A = $|\mathbf{v} \times \mathbf{w}|$ and H be the altitude of the parallelopiped (Or the box) then its volume is V = AH.

Le θ be the angle between the direction of $\mathbf{v} \times \mathbf{w}$ and $\mathbf{u}$.

Then $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = |\mathbf{u}| \, . \, |\mathbf{v} \times \mathbf{w}| \cos\theta$

$$= A \, |\mathbf{u}| \cos\theta$$

Now from the above *fiqure* $|\mathbf{u}| \cos\theta = H$

$\therefore \mathbf{u}.(\mathbf{v} \times \mathbf{w}) = AH = V.$

Hence $\mathbf{u}.(\mathbf{v} \times \mathbf{w})$ is numerically equal to the volume of the parallelopiped (or the box) having vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as concurrent edges. Scalar triple product is also known as the **box product** and is written as $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

If $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

$\mathbf{w} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$

Then $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

It can be easily shown that $\mathbf{u}.\ (\mathbf{v} \times \mathbf{w}) = \mathbf{v}.(\mathbf{w} \times \mathbf{u}) = \mathbf{w}.(\mathbf{u} \times \mathbf{v})$

In a scalar triple product the dot and cross may be inter changed provided the order of the vectors is not changed, that is, $\mathbf{u}.\ (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}).\ \mathbf{w}$.

If the vectors $\mathbf{u}$, $\mathbf{u}$, $\mathbf{w}$ are coplanar, the volume of the parallelopiped is zero and hence $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = 0$.

For example $[\mathbf{u}\ \mathbf{u}\ \mathbf{w}] = 0$.

10.8. The vector triple product

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are three vectors, then the vector product or cross product of $\mathbf{u}$ with the vector $\mathbf{v} \times \mathbf{w}$ is called the **vector triple product** and is usually written as $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

The vector $\mathbf{p} = \mathbf{v} \times \mathbf{w}$ is perpendicular to the plane of the vectors $\mathbf{v}$ and $\mathbf{w}$. Hence the vector $\mathbf{u} \times \mathbf{p} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ being perpendicular to the vector $\mathbf{v} \times \mathbf{w}$, is coplanar with $\mathbf{v}$ and $\mathbf{w}$. Thus $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ and is expressed as $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \alpha\mathbf{v} + \beta\mathbf{w}$ (1) where α and β are scalars. Again, the vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is perpendicular to $\mathbf{u}$. Therefore, $\mathbf{u}.\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = 0$

Hence from (1) we get $\mathbf{u}.(\alpha\mathbf{u} + \beta\mathbf{w}) = 0$

Or, $\quad \alpha(\mathbf{u}.\mathbf{v}) + \beta(\mathbf{u}.\mathbf{w}) = 0$

Or, $\quad \dfrac{\alpha}{\mathbf{u}.\mathbf{w}} = -\dfrac{\beta}{\mathbf{u}.\mathbf{v}} = \lambda$ (say), λ being a scalar.

$\therefore \quad \alpha = \lambda\ (\mathbf{u}.\mathbf{w}),\ \beta = -\lambda(\mathbf{u}.\mathbf{v})$.

Substituting these values in (1), we get

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \lambda(\mathbf{u}.\mathbf{w})\mathbf{v} - \lambda(\mathbf{u}.\mathbf{v})\,\mathbf{w}$ (2).

which is true for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Now to find the value of λ, consider the special case for which $\mathbf{u} = \mathbf{w} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$.

Then $\mathbf{v} \times \mathbf{w} = \mathbf{j} \times \mathbf{i} = -\mathbf{k}$; $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{i} \times -\mathbf{k} = \mathbf{j}$

$\mathbf{u}.\mathbf{w} = \mathbf{i}.\mathbf{i} = 1$, $\mathbf{u}.\mathbf{v} = \mathbf{i}.\mathbf{j} = 0$.

Putting these values in (2), we get $\mathbf{j} = \lambda\,\{1(\mathbf{j}) - 0.\mathbf{i}\}$

Or, $\lambda = 1$.

Hence $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{u}.\mathbf{v})\,\mathbf{w}$ (3)

Similarly, we can show that

$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u}$ (4)

Alternative Proof

Let $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

$\mathbf{w} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$

$$\text{Then } \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$= (y_2z_3 - y_3z_2)\,\mathbf{i} + (z_2x_3 - z_3x_2)\,\mathbf{j} + (x_2y_3 - x_3y_2)\mathbf{k}$

$$\therefore \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ y_2z_3 - y_3z_2 & z_2x_3 - z_3x_2 & x_2y_3 - x_3y_2 \end{vmatrix}$$

$$= \{y_1(x_2y_3 - x_3y_2) - z_1(z_2x_3 - z_3x_2)\}\mathbf{i} +$$

$$\{z_1(y_2z_3 - y_3z_2) - x_1(x_2y_3 - x_3y_2)\}\mathbf{j} +$$

$$\{x_1(z_2x_3 - z_3x_2) - y_1(y_2z_3 - y_3z_2)\}\mathbf{k}.$$

Again $(\mathbf{u}.\mathbf{w})\mathbf{v} = (x_1x_3 + y_1y_3 + z_1z_3)\,(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$

$= x_1x_2x_3\mathbf{i} + x_2y_1y_3\mathbf{i} + x_2z_1z_3\mathbf{i} + x_1x_3y_2\mathbf{j} + y_1y_2y_3\mathbf{j} + y_2z_1z_3\mathbf{j} +$

$x_1x_3z_2\mathbf{k} + y_3y_1z_1\mathbf{k} + z_1z_2z_3\mathbf{k}$ and

$(\mathbf{u}.\mathbf{v})\mathbf{w} = (x_1x_2 + y_1y_2 + z_1z_2)\,(x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k})$

$= x_1x_2x_3\mathbf{i} + x_3y_1y_2\mathbf{i} + z_1z_2x_3\mathbf{i} + x_1x_2y_3\mathbf{j} + y_1y_2y_3\,\mathbf{j} + y_3z_1z_2\mathbf{j}$

$+ x_1x_2z_3\mathbf{k} + y_1y_2z_3\mathbf{k} + z_1z_2z_3\mathbf{k}.$

Therefore, $(\mathbf{u}.\mathbf{w})\,\mathbf{v} - (\mathbf{u}.\mathbf{v})\mathbf{w}$

$= (x_2y_1y_3\mathbf{i} + x_2z_1z_3\mathbf{i} + x_1x_3y_2\mathbf{j} + y_2z_1z_3\mathbf{j} + x_1x_3z_2\,\mathbf{k}$

$+ y_1y_3z_2\mathbf{k}) - (x_3y_1y_2\mathbf{i} + z_1z_2x_3\mathbf{i} + x_1x_2y_3\mathbf{j} + y_3z_1z_2\mathbf{j}$

$+ x_1x_2z_3\mathbf{k} + y_1y_2z_3\mathbf{k})$

$= \{y_1(x_2y_3 - x_3y_2) - z_1(z_2x_3 - z_3x_2)\}\mathbf{i} + \{z_1(y_2z_3 - y_3z_2)$

$- x_1(x_2y_3 - x_3y_2)\}\mathbf{j} + \{x_1(z_2x_3 - z_3x_2) - y_1(y_2z_3 - y_3z_2)\}\mathbf{k}.$

Hence $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{u}.\mathbf{v})\mathbf{w}.$

WORKED OUT EXAMPLES

Example 1. Find the angle between the vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and also find a unit vector perpendicular to the above two vectors. **[D.U.P. 1980]**

Solution : First portion

Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

then $\mathbf{a}.\mathbf{b} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}).(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$= 2\mathbf{i}.\mathbf{i} - \mathbf{j}.\mathbf{j} + 2\mathbf{k}.\mathbf{k} + 0 = 2 - 1 + 2 = 3.$

Also $|\mathbf{a}| = |2\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$|\mathbf{b}| = |\mathbf{i} - \mathbf{j} + 2\mathbf{k}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

Let θ be the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$ then

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2} = \cos 60^\circ \ \therefore\ \theta = 60^\circ$$

Second portion

Again $\mathbf{a} \times \mathbf{b} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$= 2\mathbf{i} \times \mathbf{i} - 2\mathbf{i} \times \mathbf{j} + 4\mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{i} - \mathbf{j} \times \mathbf{j}$

$+ 2\mathbf{j} \times \mathbf{k} + \mathbf{k} \times \mathbf{i} - \mathbf{k} \times \mathbf{j} + 2\mathbf{k} \times \mathbf{k}$

$= 0 - 2\mathbf{k} - 4\mathbf{j} - \mathbf{k} - 0 + 2\mathbf{i} + \mathbf{j} + \mathbf{i} + 0$

$= 3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$

Now the required unit vector $\mathbf{e}$ is given by

$$\mathbf{e} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{|3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}|}{\sqrt{(3)^2 + (-3)^2 + (-3)^2}} = \frac{3(\mathbf{i} - \mathbf{j} - \mathbf{k})}{3\sqrt{3}}$$

$$= \frac{\mathbf{i}}{\sqrt{3}} - \frac{\mathbf{j}}{\sqrt{3}} - \frac{\mathbf{k}}{\sqrt{3}}.$$

Example 2. *If* $\mathbf{a} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{K}$ *and* $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ then find the value of $|\mathbf{a} - \mathbf{b} - \mathbf{c}|$ and a unit vector parallel to $2\mathbf{a} - \mathbf{b} - \mathbf{c}$. **[D.U.P. 1979]**

Solution : $\mathbf{a}-\mathbf{b}-\mathbf{c} = 4\mathbf{i}+\mathbf{j}-\mathbf{k}-3\mathbf{i}+2\mathbf{j}-2\mathbf{k}+\mathbf{i}+2\mathbf{j}-\mathbf{k}$

$$= 2\mathbf{i}+5\mathbf{j}-4\mathbf{k}.$$

$$\therefore \ |\mathbf{a}-\mathbf{b}-\mathbf{c}| = |2\mathbf{i}+5\mathbf{j}-4\mathbf{k}| = \sqrt{2^2+5^2+(-4)^2} = \sqrt{45}$$

Again $2\mathbf{a}-\mathbf{b}-\mathbf{c} = 8\mathbf{i}+2\mathbf{j}-2\mathbf{k}-3\mathbf{i}+2\mathbf{j}-2\mathbf{k}+\mathbf{i}+2\mathbf{j}-\mathbf{k}$

$$= 6\mathbf{i}+6\mathbf{j}-5\mathbf{k}$$

$$|2\mathbf{a}-\mathbf{b}-\mathbf{c}| = |6\mathbf{i}+6\mathbf{j}-5\mathbf{k}|$$

$$\sqrt{6^2+6^2+(-5)^2} = \sqrt{97}$$

Hence the unit vector parallel to $2\mathbf{a}-\mathbf{b}-\mathbf{c}$ is given by

$$\frac{2\mathbf{a}-\mathbf{b}-\mathbf{c}}{|2\mathbf{a}-\mathbf{b}-\mathbf{c}|} = \frac{6\mathbf{i}+6\mathbf{j}-5\mathbf{k}}{\sqrt{97}}$$

$$= \frac{6}{\sqrt{97}}\mathbf{i}+\frac{6}{\sqrt{97}}\mathbf{j}-\frac{5}{\sqrt{97}}\mathbf{k}.$$

Example 3. Show that the three points whose position vectors are given by $-2\mathbf{a}+3\mathbf{b}+5\mathbf{c}$, $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$ and $7\mathbf{a}-\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non — coplanar, are collinear. **[D.U.P.1980]**

Proof : Let the three points be denoted by A, B and C respectively. Then

$$\vec{AB} = (\mathbf{a}+2\mathbf{b}+3\mathbf{c})-(-2\mathbf{a}+3\mathbf{b}+5\mathbf{c})$$
$$= \mathbf{a}+2\mathbf{b}+3\mathbf{c}+2\mathbf{a}-3\mathbf{b}-5\mathbf{c} = 3\mathbf{a}-\mathbf{b}-2\mathbf{c}.$$

$$\vec{BC} = (7\mathbf{a}-\mathbf{c})-(\mathbf{a}+2\mathbf{b}+3\mathbf{c})$$

$$= 7\mathbf{a}-\mathbf{c}-\mathbf{a}-2\mathbf{b}-3\mathbf{c}$$

$$= 6\mathbf{a}-2\mathbf{b}-4\mathbf{c} = 2(3\mathbf{a}-\mathbf{b}-2\mathbf{c}) = 2\vec{AB}$$

$\overrightarrow{AC} = (7\mathbf{a} - \mathbf{c}) - (-2\mathbf{a} + 3\mathbf{b} + 5\mathbf{c})$

$= 7\mathbf{a} - \mathbf{c} + 2\mathbf{a} - 3\mathbf{b} - 5\mathbf{c}$

$= 9\mathbf{a} - 3\mathbf{b} - 6\mathbf{c} = 3(3\mathbf{a} - \mathbf{b} - 2\mathbf{c}) = 3\overrightarrow{AB}$

Now $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + 2\overrightarrow{AB} = 3\overrightarrow{AB} = \overrightarrow{AC}$

Thus the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear.

Let 0 be the origin, then $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are the position vectors respectively of the points A, B & C.

Now we have $\overrightarrow{BC} = 2\overrightarrow{AB}$

Or, $\overrightarrow{OC} - \overrightarrow{OB} = 2\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = 2\overrightarrow{OB} - 2\overrightarrow{OA}$

Or, $2\overrightarrow{OA} - 3\overrightarrow{OB} + \overrightarrow{OC} = 0$

The sum of the coefficients of $\overrightarrow{OA}$, $\overrightarrow{OB}$ & $\overrightarrow{OC}$ is

$2 + (-3) + 1 = 2 - 3 + 1 = 0.$

Hence the three points A, B & C are collinear.

Example 4. If **a**, **b**, **c** are three non-coplanar vectors, then show that the four points whose position vectors are given by $-\mathbf{a}+4\mathbf{b}-3\mathbf{c}$, $3\mathbf{a}+2\mathbf{b}-5\mathbf{c}$, $-3\mathbf{a}+8\mathbf{b}-5\mathbf{c}$ and $-3\mathbf{a}+2\mathbf{b}+\mathbf{c}$ are coplanar.

Proof : Let O be the origin and A, B, C, D be four given points. Then

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(3\mathbf{a}+2\mathbf{b}-5\mathbf{c})-(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})$$

$$=4\mathbf{a}-2\mathbf{b}-2\mathbf{c}.$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(-3\mathbf{a}+8\mathbf{b}-5\mathbf{c})-(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})$$

$$=-2\mathbf{a}+4\mathbf{b}-2\mathbf{c}.$$

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(-3\mathbf{a}+2\mathbf{b}+\mathbf{c})-(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})$$

$$=-2\mathbf{a}-2\mathbf{b}+4\mathbf{c}.$$

Now, if the three vectors be linearly connected then let it be $l\overrightarrow{AB}+m\overrightarrow{AC}=\overrightarrow{AD}$ where l and m are scalars.

$$l(4\mathbf{a}-2\mathbf{b}-2\mathbf{c})+m(-2\mathbf{a}+4\mathbf{b}-2\mathbf{c})=-2\mathbf{a}-2\mathbf{b}+4\mathbf{c}$$

$$\text{Or, } (4l-2m)\mathbf{a}+(-2l+4m)\mathbf{b}+(-2l-2m)\mathbf{c}=-2\mathbf{a}-2\mathbf{b}+4\mathbf{c}.$$

Now comparing the coefficients of **a**, **b**, **c** from both sides of the above equation, we get $4l-2m=-2$, $-2l+4m=-2$, $-2l-2m=4$.

Solving first two equations, we get $l = m = -1$ and these values satisfy the third equation.

Hence the vectors $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar.

$$\text{Thus } -\vec{AB} - \vec{AC} = AD.$$

Or, $-\left(\vec{OB} - \vec{OA}\right) - \left(\vec{OC} - \vec{OA}\right)$

$= \vec{OD} - \vec{OA}$

Or, $\quad -\vec{OB} + \vec{OA} - \vec{OC} + \vec{OA} = \vec{OD} - \vec{OA}$

Or, $\quad 3\vec{OA} - \vec{OB} - \vec{OC} - \vec{OD} = 0.$

Here the sum of the coefficients of $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ *and* $\vec{OD}$

is $3 + (-1) + (-1) + (-1) = 3 - 3 = 0.$

Hence the four points A, B, C, D are coplanar.

Example 5. What is the unit vector perpendicular to each of the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{v} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, Calculate the sine of the angle between these vectors.

Solution : $\mathbf{u} \times \mathbf{v} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (5\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & -1 & 2 \end{vmatrix} = 7\mathbf{i} + 13\mathbf{j} - 11\mathbf{k}$$

$$\therefore \ |\mathbf{u} \times \mathbf{v}| = |7\mathbf{i} + 13\mathbf{j} - 11\mathbf{k}| = \sqrt{7^2 + (13)^2 + (-11)^2} = \sqrt{339}$$

If n be a unit vector perpendicular to the plane of u and v, then since u, v is also perpendicular to the plane of u and v, we have

$$n = \frac{u \times v}{|u \times v|} = \frac{7i + 13j - 11k}{\sqrt{339}} = \frac{7}{\sqrt{339}} i + \frac{13}{\sqrt{339}} j - \frac{11}{\sqrt{339}} k.$$

Again $|u| = |i + 2j + 3k| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$$|v| = |5i - j + 2k| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{30}$$

Now let θ be the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$ then $u \times v = |u|\ |v| \sin\theta\, n$.

$$\text{Or, } \sin\theta = \frac{u \times v}{|u|\ |v|\ n} \left\{ \text{since } n = \frac{u \times v}{|u \times v|} \right.$$

$$\text{Or, } \sin\theta = \frac{|u \times v|}{|u|\,|v|} = \frac{\sqrt{339}}{\sqrt{14}\sqrt{30}}$$

$$\text{Or, } \sin\theta = \sqrt{\frac{339}{420}}$$

Example 6. Find the angle betwen the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\lambda^2 i - 2\lambda j + k$. For what value of λ will the vectors be perpendicular? **[D.U.P. 1979]**

Solution : Let $a = i + j + k$ and $\mathbf{b} = \lambda^2 i - 2\lambda j + k$ then

$$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3},\ |\mathbf{b}| = \sqrt{(\lambda^2)^2 + (-2\lambda)^2 + 1^2}$$

$$= \sqrt{\lambda^4 + 4\lambda^2 + 1}.$$

Let θ be the angle between the two vectors

$$\text{then } \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|a|\ \ |b|} = \frac{1.\ \lambda^2 + 1.(-2\lambda) + 1.1}{\sqrt{3}\sqrt{\lambda^4 + 4\lambda^2 + 1}}$$

Or, $\cos\theta = \dfrac{\lambda^2 - 2\lambda + 1}{\sqrt{3}\sqrt{\lambda^4 + 4\lambda^2 + 1}} = \dfrac{(\lambda - 1)^2}{\sqrt{3}\sqrt{\lambda^4 + 4\lambda^2 + 1}}$

$\therefore \theta = \cos^{-1}\left\{\dfrac{(\lambda - 1)^2}{\sqrt{3}\sqrt{\lambda^4 + 4\lambda^2 + 1}}\right\}.$

When $\theta = 90^\circ$, $\cos\theta = \cos 90^\circ = 0$.

Therefore, $\dfrac{(\lambda - 1)^2}{\sqrt{3}\sqrt{\lambda^4 + 4\lambda^2 + 1}} = 0$ or, $(\lambda - 1)^2 = 0$

i.e. $\lambda - 1 = 0$ or, $\lambda = 1$.

Example 7. Prove that $[\mathbf{c} \times \mathbf{a} \;\; \mathbf{a} \times \mathbf{b} \;\; \mathbf{b} \times \mathbf{c}] = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]^2$

Proof : $[\mathbf{c} \times \mathbf{a} \;\; \mathbf{a} \times \mathbf{b} \;\; \mathbf{b} \times \mathbf{c}] = (\mathbf{c} \times \mathbf{a}).\{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})\}$

$= \{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})\} . (\mathbf{b} \times \mathbf{c})$

Let $\mathbf{c} \times \mathbf{a} = \mathbf{P}$ then we have

$(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = \mathbf{P} \times (\mathbf{a} \times \mathbf{b})$

$= (\mathbf{p}.\mathbf{b})\mathbf{a} - (\mathbf{p}.\mathbf{a})\mathbf{b}$

Now putting the value of **P** in the above equation, we have

$(\mathbf{c}\times\mathbf{a}) \times (\mathbf{a}\times\mathbf{b}) = (\mathbf{c} \times \mathbf{a}.\mathbf{b})\mathbf{a} - (\mathbf{c} \times \mathbf{a}.\,\mathbf{a})\mathbf{b}$

$= [\mathbf{cab}]\mathbf{a} - 0$ since $\mathbf{c} \times \mathbf{a}.\mathbf{a} = 0$

$= [\mathbf{cab}]\,\mathbf{a}$.

$[\mathbf{c} \times \mathbf{a} \;\; \mathbf{a} \times \mathbf{b} \;\; \mathbf{b} \times \mathbf{c}] = \{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}\}.\,(\mathbf{b} \times \mathbf{c})$

$= [\mathbf{cab}]\mathbf{a}.\,(\mathbf{b} \times \mathbf{c})$

$= (\mathbf{cab}]\,[\mathbf{abc}]$ since $[\mathbf{cab}] = [\mathbf{abc}]$

$= [\mathbf{abc}]\,[\mathbf{abc}] = [\mathbf{abc}]^2$.

Hence $[\mathbf{c} \times \mathbf{a} \;\; \mathbf{a} \times \mathbf{b} \;\; \mathbf{b} \times \mathbf{c}] = [\mathbf{abc}]^2$.

Example 8. Prove that

$$[\mathbf{c}+\mathbf{a}\ \ \mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}] = 2[\mathbf{abc}]$$

Proof : $[\mathbf{c}+\mathbf{a}\ \ \mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}]$

$= (\mathbf{c}+\mathbf{a}).\{(\mathbf{a}+\mathbf{b})\times(\mathbf{b}+\mathbf{c})\}$

$= (\mathbf{c}+\mathbf{a}).\{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}\}$

$= (\mathbf{c}+\mathbf{a}).\{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}\}$ since $\mathbf{b}\times\mathbf{b}=0$

$= \mathbf{c}.\mathbf{a}\times\mathbf{b}+\mathbf{c}.\mathbf{a}\times\mathbf{c}+\mathbf{c}.\mathbf{b}\times\mathbf{c}.+\mathbf{a}.\mathbf{a}\times\mathbf{b}+\mathbf{a}.\mathbf{a}\times\mathbf{c}+\mathbf{a}.\mathbf{b}\times\mathbf{c}$

$= [\mathbf{cab}]+0+0+0+0+[\mathbf{abc}] = [\mathbf{abc}]+[\mathbf{abc}] = 2[\mathbf{abc}].$

Hence $[\mathbf{c}+\mathbf{a}\ \mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}] = 2[\mathbf{abc}].$

Example 9. Prove that

$\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k}) = 2\mathbf{a}.$ **[D. U. S. 1986]**

Proof : Let $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ then

$\mathbf{a}.\mathbf{i} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\mathbf{i} = a_1\mathbf{i}.\mathbf{i}+a_2\mathbf{j}.\mathbf{i}+a_3\mathbf{k}.\mathbf{i} = a_1+0+0 = a_1$

Similarly, $\mathbf{a}.\mathbf{j} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\mathbf{j}$

$= a_1\mathbf{i}.\mathbf{j}+a_2\mathbf{j}.\mathbf{j}+a_3\mathbf{k}.\mathbf{j}$

$= 0+a_2+0 = a_2$

$\mathbf{a}.\mathbf{k} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\mathbf{k}$

$= a_1\mathbf{i}.\mathbf{k}+a_2\mathbf{j}.\mathbf{k}+a_3\mathbf{k}.\mathbf{k}$

$= 0+0+a_3 = a_3.$

$\therefore \mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$

$= (\mathbf{a}.\mathbf{i})\mathbf{i}+(\mathbf{a}.\mathbf{j})\mathbf{j}+(\mathbf{a}.\mathbf{k})\mathbf{k}.$

Now $\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})$

$= (\mathbf{i}.\mathbf{i})\mathbf{a}-(\mathbf{i}.\mathbf{a})\mathbf{i}+(\mathbf{j}.\mathbf{j})\mathbf{a}-(\mathbf{j}.\mathbf{a})\mathbf{j}+(\mathbf{k}.\mathbf{k})\mathbf{a}-(\mathbf{k}.\mathbf{a})\mathbf{k}.$

$= \mathbf{a}-(\mathbf{a}.\mathbf{i})\mathbf{i}+\mathbf{a}-(\mathbf{a}.\mathbf{j})\mathbf{j}+\mathbf{a}-(\mathbf{a}.\mathbf{k})\mathbf{k}$

$= 3\mathbf{a}-\{(\mathbf{a}.\mathbf{i})\mathbf{i}+(\mathbf{a}.\mathbf{j})\mathbf{j}+(\mathbf{a}.\mathbf{k})\mathbf{k}.\}$

$= 3\mathbf{a}-\{a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\} = 3\mathbf{a}-\mathbf{a} = 2\mathbf{a}.$

Hence $\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k}) = 2\mathbf{a}.$

Example 10. Prove that

$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d}) = 0$

Proof : (i) Let us consider first $(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d})$.

Let $\mathbf{P} = \mathbf{a} \times \mathbf{b}$ then

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = \mathbf{p}.(\mathbf{c} \times \mathbf{d}) = (\mathbf{p} \times \mathbf{c}).\mathbf{d}$$

Since dot and cross are interchangable.

Now putting the value of $\mathbf{P} = \mathbf{a} \times \mathbf{b}$, we have

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = \{(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\}.\mathbf{d}$$

$$= \{(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}\}.\mathbf{d}$$

$$= (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{d}) - (\mathbf{b}.\mathbf{c})(\mathbf{a}.\mathbf{d}) \quad (1)$$

(ii) Again let us consider $(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d})$

Let $\mathbf{Q} = \mathbf{b} \times \mathbf{c}$ then we have

$$(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) = \mathbf{Q}.(\mathbf{a} \times \mathbf{d}) = (\mathbf{Q} \times \mathbf{a}).\mathbf{d}$$

Since dot and cross are interchangable.

$$= \{(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}\}.\mathbf{d}$$

$$= \{(\mathbf{b}.\mathbf{a})\mathbf{c} - (\mathbf{c}.\mathbf{a})\mathbf{b}\}.\mathbf{d}$$

$$= (\mathbf{b}.\mathbf{a})(\mathbf{c}.\mathbf{d}) - (\mathbf{c}.\mathbf{a})(\mathbf{b}.\mathbf{d}) \quad (2)$$

(iii) Lastly, let us consider $(\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d})$

Let $\mathbf{R} = \mathbf{c} \times \mathbf{a}$ then we have

$$(\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d}) = \mathbf{R}.(\mathbf{b} \times \mathbf{d}) = (\mathbf{R} \times \mathbf{b}).\mathbf{d}$$

$$= \{(\mathbf{c} \times \mathbf{a}) \times \mathbf{b}\}.\mathbf{d} \begin{cases} \text{since dot and cross are} \\ \text{interchangable.} \end{cases}$$

$$= \{(\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{a}.\mathbf{b})\mathbf{c}\}.\mathbf{d} = (\mathbf{c}.\mathbf{b})(\mathbf{a}.\mathbf{d}) - (\mathbf{a}.\mathbf{b})(\mathbf{c}.\mathbf{d}) \quad (3)$$

Now adding (1), (2) & (3), we get

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d})$$

$$= \{(\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{d}) - (\mathbf{b}.\mathbf{c})(\mathbf{a}.\mathbf{d})\} + \{(\mathbf{b}.\mathbf{a})(\mathbf{c}.\mathbf{d}) - (\mathbf{c}.\mathbf{a})(\mathbf{b}.\mathbf{d})\}$$
$$+ \{(\mathbf{c}.\mathbf{b})(\mathbf{a}.\mathbf{d}) - (\mathbf{a}.\mathbf{b})(\mathbf{c}.\mathbf{d})\}$$

$$= \{(\mathbf{c}.\mathbf{a})(\mathbf{b}.\mathbf{d}) - (\mathbf{c}.\mathbf{a})(\mathbf{b}.\mathbf{d})\} + \{(\mathbf{b}.\mathbf{c})(\mathbf{a}.\mathbf{d}) - (\mathbf{b}.\mathbf{c})(\mathbf{a}.\mathbf{d})\}$$
$$+ \{(\mathbf{a}.\mathbf{b})(\mathbf{c}.\mathbf{d}) - (\mathbf{a}.\mathbf{b})(\mathbf{c}.\mathbf{d})\} = 0 + 0 + 0 = 0.$$

Example 11. Prove that

$$[\mathbf{l}\ \mathbf{m}\mathbf{n}]\,[\mathbf{abc}] \begin{vmatrix} \mathbf{l}.\mathbf{a} & \mathbf{l}.\mathbf{b} & \mathbf{l}.\mathbf{c} \\ \mathbf{m}.\mathbf{a} & \mathbf{m}.\mathbf{b} & \mathbf{m}.\mathbf{c} \\ \mathbf{n}.\mathbf{a} & \mathbf{n}.\mathbf{b} & \mathbf{n}.\mathbf{c} \end{vmatrix}$$

and hence deduce the value of $[\mathbf{abc}]^2$.

Proof : *Let* $\left.\begin{matrix} \mathbf{l} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k} \\ \mathbf{m}=m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} \\ \mathbf{n}= n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \end{matrix}\right\}$ *and* $\left.\begin{matrix} \mathbf{a}=a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j} + b_3\mathbf{k} \\ \mathbf{c}=c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \end{matrix}\right\}$

$$[\mathbf{lmn}]\,[\mathbf{abc}] = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} l_1a_1+l_2a_2+l_3a_3 & l_1b_1+l_2b_2+l_3b_3 & l_1c_1+ l_2c_2+l_3c_3 \\ m_1a_1+m_2a_2+m_3a_3 & m_1b_1+m_2b_2+m_3b_3 & m_1c_1+m_2c_2+m_3c_3 \\ n_1a_1+n_2a_2+n_3a_3 & n_1b_1+n_2b_2+n_3b_3 & n_1c_1+n_2c_2+n_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{l}.\mathbf{a} & \mathbf{l}.\mathbf{b} & \mathbf{l}.\mathbf{c} \\ \mathbf{m}\mathbf{a} & \mathbf{m}\mathbf{b} & \mathbf{m}.\mathbf{c} \\ \mathbf{n}\mathbf{a} & \mathbf{n}\mathbf{b} & \mathbf{n}\mathbf{c} \end{vmatrix} \quad \begin{matrix}\textit{where} \\ (1)\end{matrix} \left.\begin{matrix} \mathbf{l}.\mathbf{a}=l_1a_1+l_2a_2+l_3a_3 \\ \mathbf{l}.\mathbf{b}=l_1b_1+l_2b_2+l_3b_3 \\ \mathbf{l}.\mathbf{c}=l_1c_1+l_2c_2+l_3c_3 \end{matrix}\right\}$$

$$\left.\begin{matrix} \mathbf{m}.\mathbf{a}=m_1a_1+m_2a_2+m_3a_3 \\ \mathbf{m}.\mathbf{b}=m_1b_1+m_2b_2+m_3b_3 \\ \mathbf{m}.\mathbf{c}=m_1c_1+m_2c_2+m_3c_3 \end{matrix}\right\} \text{ and } \left.\begin{matrix} \mathbf{n}.\mathbf{a}=n_1a_1+n_2a_2+n_3a_3 \\ \mathbf{n}.\mathbf{b}=n_1b_1+n_2b_2+n_3b_3 \\ \mathbf{n}.\mathbf{c}=n_1c_1+n_2c_2+n_3c_3 \end{matrix}\right\}$$

Putting $\mathbf{l}=\mathbf{a}$, $\mathbf{m}= \mathbf{b}$, $\mathbf{n}= \mathbf{c}$ *in (1), we get*

$$[\mathbf{abc}]\,[\mathbf{abc}] = \begin{bmatrix} \mathbf{a}.\mathbf{a} & \mathbf{a}.\mathbf{b} & \mathbf{a}.\mathbf{c} \\ \mathbf{b}.\mathbf{a} & \mathbf{b}.\mathbf{b} & \mathbf{b}.\mathbf{c} \\ \mathbf{c}.\mathbf{a} & \mathbf{c}.\mathbf{b} & \mathbf{c}.\mathbf{c} \end{bmatrix}$$

$$\text{or, } [\mathbf{abc}]^2 = \begin{bmatrix} \mathbf{a}.\mathbf{a} & \mathbf{a}.\mathbf{b} & \mathbf{a}.\mathbf{c} \\ \mathbf{b}.\mathbf{a} & \mathbf{b}.\mathbf{b} & \mathbf{b}.\mathbf{c} \\ \mathbf{c}.\mathbf{a} & \mathbf{c}.\mathbf{b} & \mathbf{c}.\mathbf{c} \end{bmatrix}$$

EXERCISES 10 (A)

1 (i) Find the scalar product of the two vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ *and* $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

(ii) Find the cross product of the two vectors $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ *and* $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(iii) Find the angle between the vectors

$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

(iv) If $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, find the projection of $\mathbf{u}+\mathbf{w}$ in the direction of $\mathbf{v}$.

Answer : (i) $\mathbf{u}.\mathbf{v} = 1$ (ii) $-\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ (iii) $\cos^{-1}\left(\frac{16}{7\sqrt{10}}\right)$. (iv) $\frac{17}{3}$.

2. (i) Find the unit vector perpendicular to both

$\mathbf{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

(ii) Find a unit vector parallel to the vector $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

(iii) If $\mathbf{u} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ find a unit vector parallel to $\mathbf{u} \times \mathbf{v}$.

(iv) Find a unit vector parallel to the resultant of the vectors

$\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Answer : (i) $\frac{1}{\sqrt{35}}(\mathbf{i} - 5\mathbf{j} - 3\mathbf{k})$ (ii) $\frac{1}{\sqrt{50}}(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$

(iii) $\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ (iv) $\frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$.

3 (i) What is the unit vector perpendicular to each of the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$?

Calculate the sine of the angle between these vectors.

(ii) What is the unit vector perpendicular to each of the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = -6\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$? Calculate the sine of the angle between these vectors.

(iii) What is the unit vector perpendicular to each of the vectors $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$? Also determine the sine of the angle between the two given vectors.

Answer : (i) $\frac{1}{\sqrt{155}}(-3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k})$. $\sin\theta = \sqrt{\frac{155}{156}}$

(ii) $\frac{1}{\sqrt{14}}(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$, $\sin\theta=\frac{2\sqrt{2}}{\sqrt{15}}$

(iii) $\frac{1}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$, $\sin\theta=\frac{5}{\sqrt{26}}$

4 (i) Find the volume of the box in space having the vectors $\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $2\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $-3\mathbf{i}+5\mathbf{j}-2\mathbf{k}$ as coternious edges.

(ii) If $\mathbf{u}=\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{v}=-2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{w}=4\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, find $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

(iii) If $\mathbf{a}=2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ and $\mathbf{c}=-5\mathbf{i}+\mathbf{j}+3\mathbf{k}$. compute $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. **[R. U. P. 1974]**

Answer : (i) volume of the box = 3.

(ii) $-4\mathbf{i}-2\mathbf{j}+6\mathbf{k}$, (iii) $-39\mathbf{i}-21\mathbf{j}+30\mathbf{k}$.

5 (a) If $\mathbf{A}=3\mathbf{i}-\mathbf{j}-4\mathbf{k}$, $\mathbf{B}=-2\mathbf{i}+4\mathbf{j}-3\mathbf{k}$ and $\mathbf{C}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$. find a unit vector parallel to $3\mathbf{A}-2\mathbf{B}+4\mathbf{C}$. **[D. U. P. 1985]**

(b) Find the volume of the parallelopiped whose edges are given by the vectors $\mathbf{A}=2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\mathbf{B}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{C}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

Answer : (i) $\frac{1}{\sqrt{398}}(17\mathbf{i}-3\mathbf{j}-10\mathbf{k})$ (ii) volume=7.

6. Prove that

(i) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})=0$

(ii) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$ **[D. U. P. 1984]**

7. Prove that

(i) $[\mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}\ \ \mathbf{a}\times\mathbf{b}]=[\mathbf{abc}]^2$ **[R. U. P '84, D. U. P. 1984]**

(ii) $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$ **[R. U. H 1969]**

8. (i) If $\mathbf{a}=3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ and $\mathbf{c}=\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, prove that $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.

(ii) Prove that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})^2$

9. Prove that

(i) $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}] = 2\,[\mathbf{abc}]$

(ii) $[\mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}\ \mathbf{a}+\mathbf{b}] = 2\,[\mathbf{abc}]$.

10. *Prove that* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *if* $(\mathbf{a}\times\mathbf{c})\times\ \mathbf{b} = 0$.

[R. U. P. 1966]

(ii) Prove that $\mathbf{a}.\,(\mathbf{b}\times\mathbf{c}) = \mathbf{b}.\,(\mathbf{c}\times\mathbf{a}) = \mathbf{c}.\,(\mathbf{a}\times\mathbf{b})$.

(iii) Prove that $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{b}\,(\mathbf{a}.\ \mathbf{c}\times\mathbf{d}) - \mathbf{a}\,(\mathbf{b}.\ \mathbf{c}\times\mathbf{d})$

$$= \mathbf{c}\,(\mathbf{a}.\mathbf{b}\times\mathbf{d}) - \mathbf{d}\,(\mathbf{a}.\ \mathbf{b}\times\mathbf{c}).$$

11. Show that any vector r can be expressed in terms of three other vectors **a, b, c** in the form

$$\mathbf{r} = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{a} + [\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{b} + [\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\,\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

12. Show that the vectors $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}-3\mathbf{j}-5\mathbf{k}$ *and* $3\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ are coplanar.

13. (i) Show that the points $-6\mathbf{a}+3\mathbf{b}+2\mathbf{c}$, $3\mathbf{a}-2\mathbf{b}+4\mathbf{c}$, $5\mathbf{a}+7\mathbf{b}+3\mathbf{c}$ and $-13\mathbf{a}+17\mathbf{b}-\mathbf{c}$ are coplanar, **a**, **b**, **c**, being three non-coplanar vectors.

(ii) Prove that the four points

$4\mathbf{i}+5\mathbf{j}+\mathbf{k}$, $-(\mathbf{j}+\mathbf{k})$, $3\mathbf{i}+9\mathbf{j}+4\mathbf{k}$ *and* $4(-\mathbf{i}+\mathbf{j}+\mathbf{k})$ *are* coplanar.

10. 9 Vector function of scalar variables.

Let us consider the vector function of a single and of several scalar variables and those of their limits, continuity and derivability.

General rules of derivatives will also be considered.

Definition **R** is a **vector function** of a single scalar variable t if to each t in some interval (a, b) there corresponds a vector **R** and is written as **R** = **F** (t) where **F** denotes the law of correspondence. Also as usual, **F**(c) denotes the particular vecto which the law associates to the value c of t.

If **i**, **j**, **k** denote a fixed triad of mutually orthogonal unit vectors, then the vector function **R** = **F** (t) can be expressed as a linear combination of **i**, **j** and **k** such as **R** = $F_1(t)$ **i** + $F_2(t)$**j** + $F_3(t)$**k** where $F_1(t)$, $F_2(t)$ and $F_3(t)$ are three scalar functions of t.

10.10. Limit and continuity of a vector function.

Definition A vector function **F** (t) is said to have a **limit L** when $t \to c$ if to any preassigned positive number ε there corresponds a positive number δ such that

$$|\mathbf{F}(t) - \mathbf{L}| < \varepsilon \text{ when } 0 < |t-c| \le \delta.$$

If **F**(t) $\to$ **L** *when* $t \to c$, *we say that* $\lim_{t\to c} \mathbf{F}(t)$ exists and we write $\lim_{t\to c} \mathbf{F}(t) = \mathbf{L}$.

Definition A vector function **F** (t) is said to be **continuous** for a value c if $\lim_{t\to c} \mathbf{F}(t) = \mathbf{F}(c)$.

Also F(t) is said to be **continuous function** if it is continous for every value of the interval of definition of the function.

10.11. Differentiation of vectors

Just as in scalar calculus, we define the derivative of a vector function **R** = **F**(t) as

$\lim\limits_{\delta t \to 0} \dfrac{\mathbf{F}(t+\delta t)-\mathbf{F}(t)}{\delta t}$ and is written as $\dfrac{d\mathbf{R}}{dt}$.

or $\dfrac{d\mathbf{F}}{dt}$ or $\mathbf{F}'(t)$.

If $\mathbf{R} = \mathbf{F}(t) = \mathbf{i}\, F_1(t) + \mathbf{j}\, F_2(t) + \mathbf{k} F_3(t)$

where **i**, **j**, **k** are fixed unit vectors and $F_1(t)$, $F_2(t)$ and $F_3(t)$ are functions of t, then we define derivative

$\dfrac{d\mathbf{R}}{dt}$ by the equation $\dfrac{d\mathbf{R}}{dt} = \mathbf{i}\dfrac{dF_1}{dt} + \mathbf{j}\dfrac{dF_2}{dt} + \mathbf{k}\dfrac{dF_3}{dt}$.

Thus the derivative of a vector **R** means a vector whose components are the derivatives of the components of **R.**

Let (x, y, z) be the co-ordinates of the moving particle at time t; then x, y, z are functions of t. The vector displacement of the particle from the origin at time t is $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, where **r** is a vector from the origin to the particle at time t. we say that **r** is the **position vector** or **vector coordinate** of the particle.

The components of the velocity of the particle at time t are $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}$, so the velocity vector is

$\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \mathbf{i}\dfrac{dx}{dt} + \mathbf{j}\dfrac{dy}{dt} + \mathbf{k}\dfrac{dz}{dt}$ and this acceleration vector is

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \mathbf{i}\dfrac{d^2x}{dt^2} + \mathbf{j}\dfrac{d^2y}{dt^2} + \mathbf{k}\dfrac{d^2z}{dt^2}$.

General rules of differentiation are similar to those of ordinary calculus provided the order of factors in vector product is maintained. Thus if **u**, **v**. **w** are differentiable vector functions of a scalar variable t and φ is a differentiable scalar function of t, then

(i) $\dfrac{d}{dt}(\mathbf{u}+\mathbf{v}) = \dfrac{d\mathbf{u}}{dt} + \dfrac{d\mathbf{v}}{dt}$

(ii) $\dfrac{d}{dt}(\varphi\,\mathbf{u}) = \varphi\dfrac{d\mathbf{u}}{dt} + \dfrac{d\varphi}{dt}\mathbf{u}$

(iii) $\dfrac{d}{dt}(\mathbf{u}.\,\mathbf{v}) = \mathbf{u}.\dfrac{d\mathbf{v}}{dt} + \dfrac{d\mathbf{u}}{dt}.\mathbf{v}$

(iv) $\dfrac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times\dfrac{d\mathbf{v}}{dt} + \dfrac{d\mathbf{u}}{dt}\times\mathbf{v}$

(v) $\dfrac{d}{dt}(\mathbf{u}.\mathbf{v}\times\mathbf{w}) = \mathbf{u}.\,\mathbf{v}\times\dfrac{d\mathbf{w}}{dt} + \mathbf{u}.\dfrac{d\mathbf{v}}{dt}\times\mathbf{w} + \dfrac{d\mathbf{u}}{dt}.\,\mathbf{v}\times\mathbf{w}$

(vi) $\dfrac{d}{dt}(\mathbf{u}\times(\mathbf{v}\times\mathbf{w})) = \mathbf{u}\times\left(\mathbf{v}\times\dfrac{d\mathbf{w}}{dt}\right) + \mathbf{u}\times\left(\dfrac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \dfrac{d\mathbf{u}}{dt}\times(\mathbf{v}\times\mathbf{w})$

As an illustration, let us prove (iv) while the others can be proved similarly,

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \underset{\delta t\to 0}{Lt}\ \frac{(\mathbf{u}+\delta\mathbf{u})\times(\mathbf{v}+\delta\mathbf{v})-(\mathbf{u}\times\mathbf{v})}{\delta t}$$

$$= \underset{\delta t\to 0}{Lt}\left[\frac{\mathbf{u}\times\delta\mathbf{v}+\delta\mathbf{u}\times\mathbf{v}+\delta\mathbf{u}\times\delta\mathbf{v}}{\delta t}\right]$$

$$= \underset{\delta t\to 0}{Lt}\left[\mathbf{u}\times\frac{\delta\mathbf{v}}{\delta t}+\frac{\delta\mathbf{u}}{\delta t}\times\mathbf{v}+\frac{\delta\mathbf{u}}{\delta t}\times\delta\mathbf{v}\right]$$

$$= \mathbf{u}\times\frac{d\mathbf{v}}{dt}+\frac{d\mathbf{u}}{dt}\times\mathbf{v}.\ \left\{\begin{array}{l}\text{since } \delta\mathbf{v}\to 0 \text{ as } \delta t\to 0\\ \therefore \dfrac{d\mathbf{u}}{dt}\times\delta\mathbf{v}=0.\end{array}\right.$$

10.12 Partial derivatives of vectors

Let $\mathbf{v}$ be a vector function of more than one scalar independent variables, say, x, y, z; then we write $\mathbf{v} = \mathbf{F}(x, y, z)$ and the partial derivatives of $\mathbf{v}$ with respect to x is defined as

$$\frac{\delta\mathbf{v}}{\delta x} = \underset{\delta x\to 0}{Lt}\ \frac{\mathbf{F}(x+\delta x, y, z) - \mathbf{F}(x, y, z)}{\delta x}$$

provided this limit exists.

Similarly, partial derivatives of $\boldsymbol{v}$ with respect to y and z can be written respectively as

$$\frac{\delta \boldsymbol{v}}{\delta y} = \underset{\delta y \to 0}{Lt} \frac{\mathbf{F}(x, y+\delta y, z) - \mathbf{F}(x, y, z)}{\delta y} \text{ and}$$

$$\frac{\delta \boldsymbol{v}}{\delta z} = \underset{\delta z \to 0}{Lt} \frac{\mathbf{F}(x, y, z+\delta z) - \mathbf{F}(x, y, z)}{\delta z}$$

provided these limits exist.

Higher order partial derivatives of vector can be defined as in the ordinary calculus. Thus for example,

$$\frac{\delta^2 \mathbf{v}}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta \mathbf{v}}{\delta x}\right), \frac{\delta^2 \boldsymbol{v}}{\delta y^2} = \frac{\delta}{\delta y}\left(\frac{\delta \boldsymbol{v}}{\delta y}\right)$$

$$\frac{\delta^2 \boldsymbol{v}}{\delta z^2} = \frac{\delta}{\delta z}\left(\frac{\delta \boldsymbol{v}}{\delta z}\right), \frac{\delta^2 \boldsymbol{v}}{\delta x \delta y} = \frac{\delta}{\delta x}\left(\frac{\delta \boldsymbol{v}}{\delta y}\right)$$

$$\frac{\delta^2 \boldsymbol{v}}{\delta y \delta x} = \frac{\delta}{\delta y}\left(\frac{\delta \boldsymbol{v}}{\delta x}\right), \frac{\delta^3 \boldsymbol{v}}{\delta y \delta z^2} = \frac{\delta}{\delta y}\left(\frac{\delta^2 \boldsymbol{v}}{\delta z^2}\right) \ldots \text{etc.}$$

If $\boldsymbol{v}$ has continuous partial derivatives of the second order at least, then $\frac{\delta^2 \boldsymbol{v}}{\delta x \delta y} = \frac{\delta^2 \boldsymbol{v}}{\delta y \delta x}$

Rules for partial differentiation of vectors are similar to those used in ordinary calculus for scalar functions. Thus if **u** and $\boldsymbol{v}$ are functions of x, y, z, then.

(i) $$\frac{\delta}{\delta x}(\mathbf{u}.\boldsymbol{v}) = \mathbf{u}.\frac{\delta \boldsymbol{v}}{\delta x} + \frac{\delta \mathbf{u}}{\delta x}.\boldsymbol{v}$$

(ii) $$\frac{\delta}{\delta x}(\mathbf{u} \times \boldsymbol{v}) = \mathbf{u} \times \frac{\delta \boldsymbol{v}}{\delta x} + \frac{\delta \mathbf{u}}{\delta x} \times \boldsymbol{v}$$

(iii) $$\frac{\delta^2}{\delta y \delta x}(\mathbf{u}.\boldsymbol{v}) = \frac{\delta}{\delta y}\left\{\frac{\delta}{\delta y}(\mathbf{u}.\boldsymbol{v})\right\} = \frac{\delta}{\delta y}\left\{\mathbf{u}.\frac{\delta \boldsymbol{v}}{\delta x} + \frac{\delta \mathbf{u}}{\delta x}.\boldsymbol{v}\right\}$$

$$= \mathbf{u}\frac{\delta^2 \boldsymbol{v}}{\delta y \delta x} + \frac{\delta \mathbf{u}}{\delta y}.\frac{\delta \boldsymbol{v}}{\delta x} + \frac{\delta \mathbf{u}}{\delta x}.\frac{\delta \boldsymbol{v}}{\delta y} + \frac{\delta^2 \boldsymbol{v}}{\delta x^2}.\boldsymbol{v}$$

(iv) $$\frac{\delta}{\delta y \delta x}(\mathbf{u} \times \boldsymbol{v}) = \frac{\delta}{\delta y}\left\{\frac{\delta}{\delta x}(\mathbf{u} \times \boldsymbol{v})\right\}$$

$$= \frac{\delta}{\delta y}\left\{\mathbf{u} \times \frac{\delta \boldsymbol{v}}{\delta x} + \frac{\delta \mathbf{u}}{\delta x} \times \boldsymbol{v}\right\}$$

$$= \mathbf{u} \times \frac{\delta^2 \boldsymbol{v}}{\delta y \delta x} + \frac{\delta \mathbf{u}}{\delta y} \times \frac{\delta \boldsymbol{v}}{\delta x} + \frac{\delta \mathbf{u}}{\delta x} \times \frac{\delta \boldsymbol{v}}{\delta y} + \frac{\delta^2 \mathbf{u}}{\delta y \delta x} \times \boldsymbol{v} \ldots \text{etc.}$$

Differentials of vectors follow rules similar to those of elementary calculus.

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *then*

(i) $d\mathbf{v} = dv_1\mathbf{i} + dv_2\mathbf{j} + dv_3\mathbf{k}$

(ii) $d(\mathbf{u}.\mathbf{v}) = \mathbf{u}.\, d\mathbf{v} + d\mathbf{u}.\, \mathbf{v}$

(iii) $d(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times d\mathbf{v} + d\mathbf{u} \times \mathbf{v}$

(iv) *If* $\mathbf{v} = \mathbf{v}\,(x, y, z)$, *then*

$$d\mathbf{v} = \frac{\delta \mathbf{v}}{\delta x}dx + \frac{\delta \mathbf{v}}{\delta y}dy + \frac{\delta \mathbf{v}}{\delta z}dz \ \ldots\text{etc.}$$

WORKED OUT EXAMPLES

Example 12. *If* $\mathbf{v} = (3x^2 - 2xy)\,\mathbf{i} + (2x - 5y)\,\mathbf{j} + (\sin xy)\mathbf{k}$

find $\frac{\delta \mathbf{v}}{\delta x}, \frac{\delta \mathbf{v}}{\delta y}, \frac{\delta^2 \mathbf{v}}{\delta x^2}, \frac{\delta^2 \mathbf{v}}{\delta y^2}, \frac{\delta^2 \mathbf{v}}{\delta y \delta x}, \frac{\delta^2 \mathbf{v}}{\delta x \delta y}$.

Solution : $\mathbf{v} = (3x^2 - 2xy)\mathbf{i} + (2x - 5y)\mathbf{j} + (\sin xy)\mathbf{k}$

$$\frac{\delta \mathbf{v}}{\delta x} = \frac{\delta}{\delta x}(3x^2 - 2xy)\,\mathbf{i} + \frac{\delta}{\delta x}(2x - 5y)\mathbf{j} + \frac{\delta}{\delta x}(\sin xy)\mathbf{k}$$

$$= (6x - 2y)\,\mathbf{i} + 2\mathbf{j} + (y \cos xy)\mathbf{k}.$$

$$\frac{\delta \mathbf{v}}{\delta y} = \frac{\delta}{\delta y}(3x^2 - 2xy)\mathbf{i} + \frac{\delta}{\delta y}(2x - 5y)\,\mathbf{j} + \frac{\delta}{\delta y}(\sin xy)\mathbf{k}$$

$$= -2x\mathbf{i} - 5\mathbf{j} + (x \cos xy)\mathbf{k}.$$

$$\frac{\delta^2 \mathbf{v}}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta \mathbf{v}}{\delta x}\right) = \frac{\delta}{\delta x}(6x - 2y)\,\mathbf{i} + \frac{\delta}{\delta x}(2)\,\mathbf{j} + \frac{\delta}{\delta x}(y \cos xy)\,\mathbf{k}$$

$$= 6\mathbf{i} + 0 - (y^2 \sin xy)\mathbf{k} = 6\mathbf{i} - (y^2 \sin xy)\,\mathbf{k}$$

$$\frac{\delta^2 \mathbf{v}}{\delta y^2} = \frac{\delta}{\delta y}\left(\frac{\delta \mathbf{v}}{\delta y}\right) = \frac{\delta}{\delta y}(-2x)\,\mathbf{i} + \frac{\delta}{\delta y}(-5)\,\mathbf{j} + \frac{\delta}{\delta y}(x \cos xy)\mathbf{k}$$

$$= 0 + 0 - (x^2 \sin xy)\,\mathbf{k} = -(x^2 \sin xy)\,\mathbf{k}$$

$$\frac{\delta^2 \mathbf{v}}{\delta y \delta x} = \frac{\delta}{\delta y}\left(\frac{\delta \mathbf{v}}{\delta x}\right) = \frac{\delta}{\delta y}(6x - 2y)\,\mathbf{i} + \frac{\delta}{\delta y}(2)\,\mathbf{j} + \frac{\delta}{\delta y}(y \cos xy)\,\mathbf{k}$$

$$= -2\mathbf{i} + 0 + (-xy \sin xy + \cos xy)\mathbf{k}$$

$$= -2\mathbf{i} - (xy \sin xy - \cos xy)\,\mathbf{k}.$$

$$\frac{\delta^2 \mathbf{v}}{\delta x \delta y} = \frac{\delta}{\delta x}\left(\frac{\delta \mathbf{v}}{\delta y}\right) = \frac{\delta}{\delta x}(-2x)\,\mathbf{i} + \frac{\delta}{\delta x}(-5)\,\mathbf{j} + \frac{\delta}{\delta x}(x\cos xy)\mathbf{k}.$$

$$= -2\mathbf{i} + 0 + (-xy\sin xy + \cos xy)\mathbf{k}$$

$$= -2\mathbf{i} - (xy\sin xy - \cos xy)\,\mathbf{k}.$$

Example13. *If* $\mathbf{u} = x^2yz\,\mathbf{i} - 2xz^3\mathbf{j} + xz^2\mathbf{k}$ *and*

$\mathbf{v} = 2z\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}.$

find $\frac{\delta^2}{\delta y \delta x}(\mathbf{u} \times \mathbf{v})$ *at* $(1, 0, -2)$.

Solution : $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$.

$= (2x^3z^3 - xyz^2)\,\mathbf{i} + (2xz^3 + x^4yz)\,\mathbf{j} + (x^2y^2z + 4xz^4)\,\mathbf{k}$

$$\frac{\delta}{\delta x}(\mathbf{u} \times \mathbf{v}) = \frac{\delta}{\delta x}(2x^3z^3 - xyz^2)\,\mathbf{i} + \frac{\delta}{\delta x}(2xz^3 + x^4yz)\mathbf{j} + \frac{\delta}{\delta x}(x^2y^2z + 4xz^4)\,\mathbf{k}$$

$$= (6x^2z^3 - yz^2)\mathbf{i} + (2z^3 + 4x^3yz)\mathbf{j} + (2xy^2z + 4z^4)\mathbf{k}$$

$$\frac{\delta^2(\mathbf{u} \times \mathbf{v})}{\delta y \delta x} = \frac{\delta}{\delta y}\left\{\frac{\delta}{\delta x}(\mathbf{u} \times \mathbf{v})\right\} = \frac{\delta}{\delta y}(6x^2z^3 - yz^2)\,\mathbf{i} + \frac{\delta}{\delta y}(2z^3 + 4x^3yz)\,\mathbf{j} + \frac{\delta}{\delta y}(2xy^2z + 4z^4)\mathbf{k}$$

$$= -z^2\,\mathbf{i} + 4x^3z\mathbf{j} + 4xyz\mathbf{k}$$

When $x = 1$, $y = 0$, $z = -2$, *we have*

$$\frac{\delta^2}{\delta y \delta x}(\mathbf{u} \times \mathbf{v}) = -4\mathbf{i} - 8\mathbf{j} + 0 = -4\mathbf{i} - 8\mathbf{j}.$$

Example 14. *If* $\mathbf{u} = \mathbf{F}(t) = 3t^2\mathbf{i} - (t+4)\,\mathbf{j} + (t^2 - 2t)\mathbf{k}$

and $\mathbf{v} = \mathbf{G}(t) = \sin t\,\mathbf{i} + 3e^{-t}\mathbf{j} - 3\cos t\,\mathbf{k}$

find $\frac{d^2}{dt^2}(\mathbf{u} \times \mathbf{v})$ *at* $t = o$.

Solution : $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 & -(t+4) & (t^2 - 2t) \\ \sin t & 3e^{-t} & -3\cos t \end{vmatrix}$

$= \left\{3(t+4)\cos t - 3(t^2-2t)e^{-t}\right\} \mathbf{i} +$

$\{(t^2-2t)\sin t + 9t^2\cos t\}\mathbf{j} + \{9t^2e^{-t} + (t+4)\sin t\}\mathbf{k}$

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \left[-3(t+4)\sin t + 3\cos t + 3(t^2-2t)e^{-t} - 3(2t-2)e^{-t}\right]\mathbf{i}$$

$$+[(t^2-2t)\cos t + (2t-2)\sin t - 9t^2\sin t + 18t\cos t]\mathbf{j}$$

$$+\left[-9t^2e^{-t} + 18te^{-t} + (t+4)\cos t + \sin t\right]\mathbf{k}$$

$$\frac{d^2}{dt^2}(\mathbf{u}\times\mathbf{v}) = \left[-3(t+4)\cos t - 3\sin t - 3\sin t - 3(t^2-2t)e^{-t}\right.$$

$$\left. + 3(2t-2)e^{-t} + 3(2t-2)e^{-t} - 6e^{-t}\right]\mathbf{i} +$$

$[-(t^2-2t)\sin t + (2t-2)\cos t + (2t-2)\cos t + 2\sin t$

$-9t^2\cos t - 18t\sin t - 18t\sin t - 18\cos t]\,\mathbf{j} +$

$[9t^2e^{-t} - 18te^{-t} - 18te^{-t} + 18te^{-t} - (t+4)\sin t + 2\cos t]\,\mathbf{k}]$

When $t = 0$, we have

$$\frac{d^2}{dt^2}(\mathbf{u}\times\mathbf{v}) = [-12 + 0 - 6 - 6 - 6]\,\mathbf{i} +$$

$[-2 - 2 + 0 + 18]\,\mathbf{j} + [0 + 18 + 0 + 2]\,\mathbf{k}$

$= -30\mathbf{i} + 14\mathbf{j} + 20\mathbf{k}.$

Example 15. *If* $\mathbf{u} = \cos t\,\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$, $\mathbf{v} = \sin t\,\mathbf{i} - 2t\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, *find*

$$\frac{d}{dt}\{(\mathbf{u}\times\mathbf{v})\times\mathbf{w}\} \text{ at } t = 0.$$

Solution : $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & t^2 & 2t \\ \sin t & -2t & 3 \end{vmatrix}$

$= (3t^2 + 4t^2)\,\mathbf{i} + (2t\sin t - 3\cos t)\,\mathbf{j} + (-2t\cos t - t^2\sin t)\mathbf{k}$

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7t^2 & 2t\sin t - 3\cos t & -2t\cos t - t^2\sin t \\ 5 & 3 & -2 \end{vmatrix}$$

$$= [-4t\sin t + 6\cos t + 6t\cos t + 3t^2\sin t]\,\mathbf{i} +$$

$$[-10t\cos t - 5t^2\sin t + 14t^2]\,\mathbf{j} +$$

$$[21t^2 - 10t\sin t + 15\cos t]\,\mathbf{k}$$

$$\frac{d}{dt}\{(\mathbf{u}\times\mathbf{v})\times\mathbf{w}\} = [-4t\cos t - 4\sin t - 6\sin t - 6t\sin t + 6\cos t +$$

$$3t^2\cos t + 16\sin t]\,\mathbf{i} +$$

$$[10t\sin t - 10\cos t - 5t^2\cos t - 10t\sin t + 28t]\,\mathbf{j}$$

$$+ [42t - 10t\cos t - 10\sin t - 15\sin t]\,\mathbf{k}$$

when $t = 0$, we have

$$\frac{d}{dt}\{(\mathbf{u}\times\mathbf{v})\times\mathbf{w}\} = o + 6\mathbf{i} + o - 10\mathbf{j} + o = 6\mathbf{i} - 10\mathbf{j}.$$

Example 16. *If* $\varphi(x, y, z) = x^2yz$ *and* $\mathbf{u} = 3x^2y\mathbf{i} + yz^2\mathbf{j} - xz\mathbf{k}$, *find* $\frac{\delta^2}{\delta y\delta z}(\varphi\mathbf{u})$ *at the point* $(1, -2, -1)$.

Solution : $\varphi\mathbf{u} = (x^2yz)(3x^2y\mathbf{i} + yz^2\mathbf{j} - xz\mathbf{k})$

$$= 3x^4y^2z\mathbf{i} + x^2y^2z^3\mathbf{j} - x^3yz^2\mathbf{k}$$

$$\frac{\delta}{\delta z}(\varphi\mathbf{u}) = \frac{\delta}{\delta z}(3x^4y^2z)\,\mathbf{i} + \frac{\delta}{\delta z}(x^2y^2z^3)\mathbf{j} - \frac{\delta}{\delta z}(x^3yz^2)\mathbf{k}$$

$$= 3x^4y^2\mathbf{i} + 3x^2y^2z^2\mathbf{j} - 2x^3yz\mathbf{k}$$

$$\frac{\delta^2}{\delta y\delta z}(\varphi\mathbf{u}) = \frac{\delta}{\delta y}\left(\frac{\delta}{\delta z}(\varphi\mathbf{u})\right) = \frac{\delta}{\delta y}(3x^4y^2)\,\mathbf{i} + \frac{\delta}{\delta y}(3x^2y^2z^2)\,\mathbf{j}$$

$$- \frac{\delta}{\delta y}(2x^3yz)\mathbf{k}$$

$$= 6x^4y\mathbf{i} + 6x^2yz^2\mathbf{j} - 2x^3z\mathbf{k}$$

When $x = 1$, $y = -2$, $z = -1$, we have

$$\frac{\delta^2}{\delta y\delta z}(\varphi\mathbf{u}) = -12\mathbf{i} - 12\mathbf{j} + 2\mathbf{k}.$$

10.13 Gradient, Divergence and Curl

In vector calculus a certain **vector differential operator** ∇ (read 'del') defined by $\nabla = \mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}$ plays a very prominent role.

The operator ∇ is also known as **nabla.** Using this ∇ we define the following three important quantities :

(i) Gradient

The **gradient** of a scalar function $\varphi(x, y, z)$, written as $\nabla\varphi$ or grad φ, is defined by

$$\nabla\varphi = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\varphi = \mathbf{i}\frac{\delta\varphi}{\delta z} + \mathbf{j}\frac{\delta\varphi}{\delta y} + \mathbf{k}\frac{\delta\varphi}{\delta z}.$$

The component of $\nabla\varphi$ in the direction of a unit vector **a** is given by $\nabla\varphi.\,\mathbf{a}$ and is called the **directional derivative** of φ in the direction **a.**

(ii) Divergence

The scalar product of the vector operator ∇ and vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ gives a scalar which is called the **divergence** of **v**, that is,

$$\nabla.\,\mathbf{v} = \operatorname{div}\mathbf{v} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \frac{\delta v_1}{\delta x} + \frac{\delta v_2}{\delta y} + \frac{\delta v_3}{\delta z}.$$

It is to be noted that $\nabla.\,\mathbf{v} \neq \mathbf{v}.\,\nabla$.

(iii) Curl

If $\mathbf{v} = \mathbf{v}(x, y, z)$ is a vector function of x, y, z then the **Curl** or **rotation** of **v**, written as $\nabla \times \mathbf{v}$ or curl **v** or rot **v** is defined by

$$\nabla \times \mathbf{v} = \operatorname{curl}\mathbf{v} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right)\mathbf{i} + \left(\frac{\delta v_1}{\delta z} - \frac{\delta v_3}{\delta x}\right)\mathbf{j} + \left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)\mathbf{k}$$

10.14 . Formulae involving ∇

If **u** and **v** are differentiable vector functions and φ and Ψ are differentiable scalar functions of position (x, y, z) then

(i) $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$ or $grad\,(\varphi+\psi) = grad\,\varphi + grad\,\psi$

(ii) $\nabla.(\mathbf{u} + \mathbf{v}) = \nabla.\mathbf{u} + \nabla.\mathbf{v}$ or $div\,(\mathbf{u} + \mathbf{v}) = div\,\mathbf{u} + div\,\mathbf{v}$

(iii) $\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times \mathbf{u} + \nabla \times \mathbf{v}$ or $curl\,(\mathbf{u}+\mathbf{v}) = curl\,\mathbf{u} + curl\,\mathbf{v}$

(iv) $\nabla.(\varphi\mathbf{u}) = (\nabla\varphi).\mathbf{u} + \varphi(\nabla.\mathbf{u})$

(v) $\nabla \times (\varphi\mathbf{u}) = (\nabla\varphi) \times \mathbf{u} + \varphi(\nabla \times \mathbf{u})$

(vi) $\nabla.(\mathbf{u} \times \mathbf{v}) = \mathbf{v}.(\nabla \times \mathbf{u}) - \mathbf{u}.(\nabla \times \mathbf{v})$

(vii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v}.\nabla)\mathbf{u} - \mathbf{v}(\nabla.\mathbf{u}) - (\mathbf{u}.\nabla)\mathbf{v} + \mathbf{u}(\nabla.\mathbf{v})$

(viii) $\nabla(\mathbf{u}.\mathbf{v}) = (\mathbf{v}.\nabla)\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}.)$

If φ and **u** have continuous second partial derivatives. then

(ix) $\nabla.(\nabla\varphi) = \nabla^2\varphi = \frac{\delta^2\varphi}{\delta y^2} + \frac{\delta^2\varphi}{\delta y^2} + \frac{\delta^2\varphi}{\delta z^2}$ where

$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta x^2}$ is called the **Laplacian operator.**

(x) $\nabla \times (\nabla\varphi) = 0$ i. e. the curl of the gradient of φ is zero.

(xi) $\nabla.(\nabla \times \mathbf{u}) = 0$ i. e. the divergence of the curl of **u** is zero

(xii) $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2\mathbf{u}$.

Proof of the formulae.

(i) $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$

$$\nabla(\varphi + \psi) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(\varphi + \psi)$$

$$= \mathbf{i}\frac{\delta}{\delta x}(\varphi+\psi) + \mathbf{j}\frac{\delta}{\delta y}(\varphi+\psi) + \mathbf{k}\frac{\delta}{\delta z}(\phi+\psi)$$

$$= \mathbf{i}\frac{\delta\varphi}{\delta x} + \mathbf{i}\frac{\delta\psi}{\delta x} + \mathbf{j}\frac{\delta\varphi}{\delta y} + \mathbf{j}\frac{\delta\psi}{\delta y} + \mathbf{k}\frac{\delta\varphi}{\delta z} + \mathbf{k}\frac{\delta\psi}{\delta z}$$

$$= \left(\mathbf{i}\frac{\delta\varphi}{\delta x} + \mathbf{j}\frac{\delta\varphi}{\delta y} + \mathbf{k}\frac{\delta\varphi}{\delta z}\right) + \left(\mathbf{i}\frac{\delta\psi}{\delta x} + \mathbf{j}\frac{\delta\psi}{\delta y} + \mathbf{k}\frac{\delta\psi}{\delta z}\right)$$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\varphi + \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\psi$$

$$= \nabla\varphi + \nabla\psi.$$

(ii) $\nabla.(\mathbf{u}+\mathbf{v}) = \nabla.\,\mathbf{u} + \nabla.\mathbf{v}$

Let $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\mathbf{k}$ *and* $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$ *then*

$\mathbf{u} + \mathbf{v} = (u_1 + v_1)\,\mathbf{i} + (u_2+v_2)\,\mathbf{j} + (u_3 + v_3)\mathbf{k}$

$$\nabla.\,(\mathbf{u}+\mathbf{v}) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\,[(u_1 + v_1)\,\mathbf{i} + (u_2 + v_2)\,\mathbf{j} + (u_3 + v_3)\mathbf{k}]$$

$$= \frac{\delta}{\delta x}(u_1 + v_1) + \frac{\delta}{\delta y}(u_2+v_2) + \frac{\delta}{\delta z}(u_3 + v_3)$$

$$= \frac{\delta u_1}{\delta x} + \frac{\delta v_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta v_2}{\delta y} + \frac{\delta u_3}{\delta z} + \frac{\delta v_3}{\delta z}$$

$$= \left(\frac{\delta u_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta u_3}{\delta z}\right) + \left(\frac{\delta v_1}{\delta x} + \frac{\delta v_2}{\delta y} + \frac{\delta v_3}{\delta z}\right)$$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\,(u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\mathbf{k})$$

$$+ \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\,(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k})$$

$$= \nabla.\,\mathbf{u} + \nabla.\,\mathbf{v}.$$

(iii) $\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times \mathbf{u} + \nabla \times \mathbf{v}$

Let $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ *and* $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$.

then $\mathbf{u} + \mathbf{v} = (u_1 + v_1)\,\mathbf{i} + (u_2 + v_2)\,\mathbf{j} + (u_3 + v_3)\,\mathbf{k}$.

$$\nabla \times (\mathbf{u} + \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ u_1 + v_1 & u_2 + v_2 & u_3 + v_3 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\delta}{\delta y}(u_3+v_3) - \frac{\delta}{\delta z}(u_2+v_2)\right] +$$

$$\mathbf{j}\left[\frac{\delta}{\delta z}(u_1+v_1) - \frac{\delta}{\delta x}(u_3+v_3)\right] +$$

$$\mathbf{k}\left[\frac{\delta}{\delta x}(u_2+v_2) - \frac{\delta}{\delta y}(u_1+v_1)\right]$$

$$= \mathbf{i}\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right) + \mathbf{j}\left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x}\right) + \mathbf{k}\left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right)$$

$$+\mathbf{i}\left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right) + \mathbf{j}\left(\frac{\delta v_1}{\delta z} - \frac{\delta v_3}{\delta x}\right) + \mathbf{k}\left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ u_1 & u_2 & u_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \nabla \times \mathbf{u} + \nabla \times \mathbf{v}.$$

(iv) $\nabla . (\varphi \mathbf{u}) = (\nabla \varphi) . \mathbf{u} + \varphi (\nabla . \mathbf{u})$

Let $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ *then* $\varphi\mathbf{u} = \varphi u_1 \mathbf{i} + \varphi u_2 \mathbf{j} + \varphi u_3 \mathbf{k}$

$$\nabla . (\varphi \mathbf{u}) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) . (\varphi \mathbf{u}_1 \mathbf{i} + \varphi \mathbf{u}_2 \mathbf{j} + \varphi \mathbf{u}_3 \mathbf{k})$$

$$= \frac{\delta}{\delta x}(\varphi u_1) + \frac{\delta}{\delta y}(\varphi u_2) + \frac{\delta}{\delta z}(\varphi u_3)$$

$$= \frac{\delta \varphi}{\delta x} u_1 + \varphi \frac{\delta u_1}{\delta x} + \frac{\delta \varphi}{\delta y} u_2 + \varphi \frac{\delta u_2}{\delta y} + \frac{\delta \varphi}{\delta z} u_3 + \varphi \frac{\delta u_3}{\delta z}$$

$$= \left(\frac{\delta \varphi}{\delta x} u_1 + \frac{\delta \varphi}{\delta y} u_2 + \frac{\delta \varphi}{\delta z} u_3\right) + \varphi\left(\frac{\delta u_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta u_3}{\delta z}\right)$$

$$= \left(\mathbf{i}\frac{\delta \varphi}{\delta x} + \mathbf{j}\frac{\delta \varphi}{\delta y} + \mathbf{k}\frac{\delta \varphi}{\delta z}\right) . (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) +$$

$$\varphi \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) . (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k})$$

$$= (\nabla \varphi) . \mathbf{u} + \varphi (\nabla . \mathbf{u})$$

(v) $\nabla \times (\varphi \mathbf{u}) = \nabla \varphi \times \mathbf{u} + \varphi(\nabla \times \mathbf{u})$

Let $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ *then* $\varphi \mathbf{u} = \varphi u_1 \mathbf{i} + \varphi u_3 \mathbf{j} + \varphi u_3 \mathbf{k}$

$$\nabla \times (\varphi \mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ \varphi u_1 & \varphi u_2 & \varphi u_3 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\delta}{\delta y}(\varphi u_3) - \frac{\delta}{\delta z}(\varphi u_2)\right] + \mathbf{j}\left[\frac{\delta}{\delta z}(\varphi u_1) - \frac{\delta}{\delta x}(\varphi u_3)\right]$$

$$+ \mathbf{k}\left[\frac{\delta}{\delta x}(\varphi u_2) - \frac{\delta}{\delta y}(\varphi u_1)\right]$$

$$= \mathbf{i}\left[\frac{\delta\varphi}{\delta y}u_3 + \varphi\frac{\delta u_3}{\delta y} - \frac{\delta\varphi}{\delta z}u_2 - \varphi\frac{\delta u_2}{\delta z}\right] +$$

$$\mathbf{j}\left[\frac{\delta\varphi}{\delta z}u_1 + \varphi\frac{\delta u_1}{\delta z} - \frac{\delta\varphi}{\delta x}u_3 - \varphi\frac{\delta u_3}{\delta x}\right] +$$

$$\mathbf{k}\left[\frac{\delta\varphi}{\delta x}u_2 + \varphi\frac{\delta u_2}{\delta x} - \frac{\delta\varphi}{\delta y}u_1 - \varphi\frac{\delta u_1}{\delta y}\right]$$

$$= \left[\left(\frac{\delta\varphi}{\delta y}u_3 - \frac{\delta\varphi}{\delta z}u_2\right)\mathbf{i} + \left(\frac{\delta\varphi}{\delta x}u_1 - \frac{\delta\varphi}{\delta x}u_3\right)\mathbf{j} + \left(\frac{\delta\varphi}{\delta x}u_2 - \frac{\delta\varphi}{\delta y}u_1\right)\mathbf{k}\right]$$

$$+ \varphi\left[\mathbf{i}\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right) + \mathbf{j}\left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x}\right) + \mathbf{k}\left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right)\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta\varphi}{\delta x} & \dfrac{\delta\varphi}{\delta y} & \dfrac{\delta\varphi}{\delta z} \\ u_1 & u_2 & u_3 \end{vmatrix} + \varphi \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$= \nabla\varphi \times \mathbf{u} + \varphi\,(\nabla \times \mathbf{u})$.

(vi) $\nabla.\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v}.\,(\nabla \times \mathbf{u}) - \mathbf{u}.\,(\nabla \times \mathbf{v})$

Let $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ *and* $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$

then $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$= (u_2v_3 - u_3v_2)\,\mathbf{i} + (u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

$$\nabla.\,(\mathbf{u} \times \mathbf{v}) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\,[(u_2v_3 - u_3v_2)\,\mathbf{i} +$$

$$(u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}]$$

$$= \frac{\delta}{\delta x}(u_2v_3 - u_3v_2) + \frac{\delta}{\delta y}(u_3v_1 - u_1v_3) + \frac{\delta}{\delta z}(u_1v_2 - u_2v_1).$$

$$= \left[u_2\frac{\delta v_3}{\delta x} + v_3\frac{\delta u_2}{\delta x} - u_3\frac{\delta v_2}{\delta x} - v_2\frac{\delta u_3}{\delta x}\right] +$$

$$\left[u_3\frac{\delta v_1}{\delta y} + v_1\frac{\delta u_3}{\delta y} - u_1\frac{\delta v_3}{\delta y} - v_3\frac{\delta u_1}{\delta y}\right] +$$

$$\left[u_1\frac{\delta v_2}{\delta z} + v_2\frac{\delta u_1}{\delta z} - u_2\frac{\delta v_1}{\delta z} - v_1\frac{\delta u_2}{\delta z}\right]$$

$$= \left[v_1\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right) + v_2\left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x}\right) + v_3\left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right)\right]$$

$$-\left[u_1\left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right) + u_2\left(\frac{\delta v_1}{\delta z} - \frac{\delta v_3}{\delta x}\right) + u_3\left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)\right]$$

$$= (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}).\left[\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right)\mathbf{i} + \left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x}\right)\mathbf{j} + \left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right)\mathbf{k}\right]$$

$$-(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}).\left[\left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right)\mathbf{i} + \left(\frac{\delta v_1}{\delta z} - \frac{\delta v_3}{\delta x}\right)\mathbf{j} + \left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right)\mathbf{k}\right]$$

$$= \mathbf{v}\,.\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ u_1 & u_2 & u_3 \end{vmatrix} - \mathbf{u}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \mathbf{v}.\ (\nabla \times \mathbf{u}) - \mathbf{u}.\ (\nabla \times \mathbf{v}).$$

(vii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v}.\ \nabla)\,\mathbf{u} - \mathbf{v}(\nabla\,.\,\mathbf{u}) - (\mathbf{u}.\ \nabla)\,\mathbf{v} + \mathbf{u}(\nabla.\ \mathbf{v})$

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\,\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\,k$

then $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_i & v_2 & v_3 \end{vmatrix}$

$$= (u_2\,v_3 - u_3\,v_2)\,\mathbf{i} + (u_3\,v_1 - u_1\,v_3)\,\mathbf{j} + (u_1\,v_2 - u_2\,v_1)\,\mathbf{k}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ u_2\,v_3 - u_3\,v_2 & u_3\,v_1 - u_1\,v_3 & u_1\,v_2 - u_2\,v_1 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\delta}{\delta y}(u_1\,v_2 - u_2\,v_1) - \frac{\delta}{\delta z}(u_3\,v_1 - u_1\,v_3)\right] +$$

$$\mathbf{j}\left[\frac{\delta}{\delta z}(u_2 v_3 - u_3 v_2) - \frac{\delta}{\delta x}(u_1 v_2 - u_2 v_1)\right] +$$

$$\mathbf{k}\left[\frac{\delta}{\delta x}(u_3 v_1 - u_1 v_3) - \frac{\delta}{\delta y}(u_2 v_3 - u_3 v_2)\right]$$

$$= \mathbf{i}\left[u_1 \frac{\delta v_2}{\delta y} + v_2 \frac{\delta u_1}{\delta y} - u_2 \frac{\delta v_1}{\delta y} - v_1 \frac{\delta u_2}{\delta y}\right.$$

$$\left. - u_3 \frac{\delta v_1}{\delta z} - v_1 \frac{\delta u_3}{\delta z} + u_1 \frac{\delta v_3}{\delta z} + v_3 \frac{\delta u_1}{\delta z}\right]$$

$$+ \mathbf{j}\left[u_2 \frac{\delta v_3}{\delta z} + v_3 \frac{\delta u_2}{\delta z} - u_3 \frac{\delta v_2}{\delta z} - v_2 \frac{\delta u_3}{\delta z}\right.$$

$$\left. - u_1 \frac{\delta v_2}{\delta x} - v_2 \frac{\delta u_1}{\delta x} + u_2 \frac{\delta v_1}{\delta x} + v_1 \frac{\delta u_2}{\delta x}\right]$$

$$+ \mathbf{k}\left[u_3 \frac{\delta v_1}{\delta x} + v_1 \frac{\delta u_3}{\delta x} - u_1 \frac{\delta v_3}{\delta x} - v_3 \frac{\delta u_1}{\delta x}\right.$$

$$\left. - u_2 \frac{\delta v_3}{\delta y} - v_3 \frac{\delta u_2}{\delta y} + u_3 \frac{\delta v_2}{\delta y} + v_2 \frac{\delta u_3}{\delta y}\right]$$

$$= v_1\left(\frac{\delta u_2}{\delta x}\mathbf{j} + \frac{\delta u_3}{\delta x}\mathbf{k}\right) + v_2\left(\frac{\delta u_1}{\delta y}\mathbf{i} + \frac{\delta u_3}{\delta y}\mathbf{k}\right)$$

$$+ v_3\left(\frac{\delta u_1}{\delta z}\mathbf{i} + \frac{\delta u_2}{\delta z}\mathbf{j}\right)$$

$$- v_1\left(\frac{\delta u_2}{\delta y}\mathbf{i} + \frac{\delta u_3}{\delta z}\mathbf{i}\right) - v_2\left(\frac{\delta u_1}{\delta x}\mathbf{j} + \frac{\delta u_3}{\delta z}\mathbf{j}\right)$$

$$- v_3\left(\frac{\delta u_1}{\delta x}\mathbf{k} + \frac{\delta u_2}{\delta y}\mathbf{k}\right)$$

$$- u_1\left(\frac{\delta v_2}{\delta x}\mathbf{j} + \frac{\delta v_3}{\delta x}\mathbf{k}\right) - u_2\left(\frac{\delta v_1}{\delta y}\mathbf{i} + \frac{\delta v_3}{\delta y}\mathbf{k}\right)$$

$$- u_3\left(\frac{\delta v_1}{\delta z}\mathbf{i} + \frac{\delta v_2}{\delta z}\mathbf{j}\right)$$

$$+ u_1\left(\frac{\delta v_2}{\delta y}\mathbf{i} + \frac{\delta v_3}{\delta z}\mathbf{i}\right) + u_2\left(\frac{\delta v_1}{\delta x}\mathbf{j} + \frac{\delta v_3}{\delta z}\mathbf{j}\right)$$

$$+ u_3\left(\frac{\delta v_1}{\delta x}\mathbf{k} + \frac{\delta v_2}{\delta y}\mathbf{k}\right)$$

$$= v_1\left(\frac{\delta u_1}{\delta x}\,\boldsymbol{i}+\frac{\delta u_2}{\delta x}\,\boldsymbol{j}+\frac{\delta u_3}{\delta x}\,\boldsymbol{k}\right)+v_2\left(\frac{\delta u_1}{\delta y}\,\mathbf{i}+\frac{\delta u_2}{\delta y}\,\mathbf{j}+\frac{\delta u_3}{\delta y}\,\mathbf{k}\right)$$

$$+ v_3\left(\frac{\delta u_1}{\delta z}\,\boldsymbol{i}+\frac{\delta u_2}{\delta z}\,\boldsymbol{j}+\frac{\delta u_3}{\delta z}\,\mathbf{k}\right)$$

$$-v_1\left(\frac{\delta u_1}{\delta x}\,\mathbf{i}+\frac{\delta u_2}{\delta y}\,\mathbf{i}+\frac{\delta u_3}{\delta z}\,\mathbf{i}\right)-v_2\left(\frac{\delta u_1}{\delta x}\,\mathbf{j}+\frac{\delta u_2}{\delta y}\,\mathbf{j}+\frac{\delta u_3}{\delta z}\,\mathbf{j}\right)$$

$$-v_3\left(\frac{\delta u_1}{\delta x}\,\mathbf{k}+\frac{\delta u_2}{\delta y}\,\mathbf{k}+\frac{\delta u_3}{\delta z}\,\mathbf{k}\right)$$

$$-u_1\left(\frac{\delta v_1}{\delta x}\,\mathbf{i}+\frac{\delta v_2}{\delta x}\,\mathbf{j}+\frac{\delta v_3}{\delta x}\,\mathbf{k}\right)-u_2\left(\frac{\delta v_1}{\delta y}\,\mathbf{i}+\frac{\delta v_2}{\delta y}\,\mathbf{j}+\frac{\delta v_3}{\delta y}\,\mathbf{k}\right)$$

$$-u_3\left(\frac{\delta v_1}{\delta z}\,\mathbf{i}+\frac{\delta v_2}{\delta z}\,\mathbf{j}+\frac{\delta v_3}{\delta z}\,\mathbf{k}\right)$$

$$+u_1\left(\frac{\delta v_1}{\delta x}\,\mathbf{i}+\frac{\delta v_2}{\delta y}\,\mathbf{i}+\frac{\delta v_3}{\delta z}\,\mathbf{i}\right)+u_2\left(\frac{\delta v_1}{\delta x}\,\mathbf{j}+\frac{\delta v_2}{\delta y}\,\mathbf{j}+\frac{\delta v_3}{\delta z}\,\mathbf{j}\right)$$

$$+u_3\left(\frac{\delta v_1}{\delta x}\,\mathbf{k}+\frac{\delta v_2}{\delta y}\,\mathbf{k}+\frac{\delta v_3}{\delta z}\,\mathbf{k}\right)$$

$$=\left(v_1\frac{\delta}{\delta x}+v_2\frac{\delta}{\delta y}+v_3\frac{\delta}{\delta z}\right).(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})$$

$$-(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}).\left(\frac{\delta u_1}{\delta x}+\frac{\delta u_2}{\delta y}+\frac{\delta u_3}{\delta z}\right)$$

$$-\left(u_1\frac{\delta}{\delta x}+u_2\frac{\delta}{\delta y}+u_3\frac{\delta}{\delta z}\right).(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})$$

$$+(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\left(\frac{\delta v_1}{\delta x}+\frac{\delta v_2}{\delta y}+\frac{\delta v_3}{\delta z}\right)$$

$$\left[(v_1\,\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}).\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right)\right]\boldsymbol{u}$$

$$-\boldsymbol{v}\left[\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right).(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\right]$$

$$-\left[(u_1\mathbf{i}+u_2\mathbf{j}+j_3\mathbf{k}).\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right)\right]\mathbf{v}$$

$$+\mathbf{u}\left[\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right).(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\right]$$

$$=(\mathbf{v}.\nabla)\,\mathbf{u}-\mathbf{v}(\nabla.\,\mathbf{u})-(\mathbf{u}\,.\,\nabla)\mathbf{v}+\mathbf{u}\,(\nabla.\,\mathbf{v}).$$

(viii) $\nabla(\mathbf{u}.\mathbf{v})=(\mathbf{v}.\nabla)\mathbf{u}+(\mathbf{u}.\nabla)\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{u})+\mathbf{u}\times(\nabla\times\mathbf{v})$

Let $\mathbf{u}=u_1\mathbf{i}\div u_2\mathbf{j}+u_3\mathbf{k}$ and $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. then

$$\mathbf{v}.\ \nabla=(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}).\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right)$$

$$\mathbf{v}_1\frac{\delta}{\delta x}+v_2\frac{\delta}{\delta y}+\mathbf{v_3}\frac{\delta}{\delta z}$$

$$\therefore\ (\mathbf{v}.\nabla)\mathbf{u}=\left(v_{\mathbf{i}}\frac{\delta}{\delta x}+\mathbf{v_2}\frac{\delta}{\delta y}+\mathbf{v_2}\frac{\delta}{\delta z}\right)(\mathbf{u}_1\mathbf{i}+\mathbf{u}_2\mathbf{j}+\mathbf{u}_3\mathbf{k})$$

$$=\mathbf{i}v_1\frac{\delta\mathbf{u}_1}{\delta x}+\mathbf{j}v_1\frac{\delta\mathbf{u}_2}{\delta x}+\mathbf{k}v_1\frac{\delta\mathbf{u}_3}{\delta x}+\mathbf{i}v_2\frac{\delta\mathbf{u}_1}{\delta y}+\mathbf{j}v_2\frac{\delta\mathbf{u}_2}{\delta y}+\mathbf{k}v_2\frac{\delta\mathbf{u}_3}{\delta y}$$

$$+\mathbf{i}v_3\frac{\delta\mathbf{u}_1}{\delta z}+\mathbf{j}v_3\frac{\delta\mathbf{u}_2}{\delta z}+\mathbf{k}v_3\frac{\delta\mathbf{u}_3}{\delta z}\quad(1)$$

Similarly,

$$(\mathbf{u}.\ \nabla)\ \mathbf{v}=\left(u_{\mathbf{i}}\frac{\delta}{\delta x}+u_2\frac{\delta}{\delta y}+u_3\frac{\delta}{\delta z}\right)(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})$$

$$=\mathbf{i}u_1\frac{\delta v_1}{\delta x}+\mathbf{j}u_1\frac{\delta v_2}{\delta x}+\mathbf{k}u_1\frac{\delta v_3}{\delta x}+\mathbf{i}u_2\frac{\delta v_1}{\delta y}+\mathbf{j}u_2\frac{\delta v_2}{\delta y}+\mathbf{k}u_2\frac{\delta v_3}{\delta y}$$

$$+\mathbf{i}u_3\frac{\delta v_1}{\delta z}+\mathbf{j}u_3\frac{\delta v_2}{\delta z}+\mathbf{k}u_3\frac{\delta v_3}{\delta z}\quad(2)$$

$$(\nabla\times\mathbf{u})=\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z}\\ u_1 & u_2 & u_3\end{bmatrix}$$

$$=\left(\frac{\delta u_3}{\delta y}-\frac{\delta u_2}{\delta z}\right)\mathbf{i}+\left(\frac{\delta u_1}{\delta z}-\frac{\delta u_3}{\delta x}\right)\mathbf{j}+\left(\frac{\delta u_2}{\delta x}-\frac{\delta u_1}{\delta y}\right)\mathbf{k}$$

$$\mathbf{v}\times(\nabla\times\mathbf{u})=\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ v_1 & v_2 & v_3\\ \dfrac{\delta u_3}{\delta y}-\dfrac{\delta u_2}{\delta z} & \dfrac{\delta u_1}{\delta z}-\dfrac{\delta u_3}{\delta x} & \dfrac{\delta u_2}{\delta x}-\dfrac{\delta u_1}{\delta y}\end{bmatrix}$$

$$= \mathbf{i}\left[\left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right) v_2 - \left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta y}\right) v_3\right]$$

$$+ \mathbf{j}\left[\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right) v_3 - \left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right) v_1\right]$$

$$+ \mathbf{k}\left[\left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x}\right) v_1 - \left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right) v_2\right]$$

$$= \mathbf{i} v_2 \frac{\delta u_2}{\delta x} - \mathbf{i} v_2 \frac{\delta u_1}{\delta y} - \mathbf{i} v_3 \frac{\delta u_1}{\delta z} + \mathbf{i} v_3 \frac{\delta u_3}{\delta x}$$

$$+ \mathbf{j} v_2 \frac{\delta u_3}{\delta y} - \mathbf{j} v_3 \frac{\delta u_2}{\delta z} - \mathbf{j} v_1 \frac{\delta u_2}{\delta x} + \mathbf{j} v_1 \frac{\delta u_1}{\delta y}$$

$$+ \mathbf{k} v_1 \frac{\delta u_1}{\delta z} - \mathbf{k} v_1 \frac{\delta u_3}{\delta x} - \mathbf{k} v_2 \frac{\delta u_3}{\delta y} + \mathbf{k} v_2 \frac{\delta u_2}{\delta z} \quad (3)$$

Similarly, interchanging **u** and **v**, we have

$$\mathbf{u} \times (\nabla \times \mathbf{v}) = \mathbf{i} u_2 \frac{\delta v_2}{\delta x} - \mathbf{i} u_2 \frac{\delta v_1}{\delta y} - \mathbf{i} u_3 \frac{\delta v_1}{\delta z} + \mathbf{i} u_3 \frac{\delta u_3}{\delta x}$$

$$+ \mathbf{j} u_3 \frac{\delta v_3}{\delta y} - \mathbf{j} u_1 \frac{\delta v_2}{\delta z} - \mathbf{j} u_i \frac{\delta v_2}{\delta x} + \mathbf{j} u_1 \frac{\delta v_1}{\delta y}$$

$$+ \mathbf{k} u_1 \frac{\delta v_1}{\delta z} - \mathbf{k} u_1 \frac{\delta v_3}{\delta x} - \mathbf{k} u_2 \frac{\delta v_3}{\delta y} + \mathbf{k} u_2 \frac{\delta v_2}{\delta z} \quad (4)$$

Adding (1), (2) (3), (4) and cancelling the equal positive and negative terms, we get $(\mathbf{v}.\ \nabla)\mathbf{u} + (\mathbf{u}.\ \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v})$

$$= \mathbf{i}\left(u_1 \frac{\delta v_1}{\delta x} + v_1 \frac{\delta u_1}{\delta x} + u_2 \frac{\delta v_2}{\delta x} + v_2 \frac{\delta u_2}{\delta x} + u_3 \frac{\delta v_3}{\delta x} + v_3 \frac{\delta u_3}{\delta x}\right)$$

$$+ \mathbf{j}\left(u_1 \frac{\delta v_1}{\delta y} + v_1 \frac{\delta u_1}{\delta y} + u_2 \frac{\delta v_2}{\delta y} + v_2 \frac{\delta u_2}{\delta y} + u_3 \frac{\delta v_3}{\delta y} + v_3 \frac{\delta u_3}{\delta y}\right)$$

$$+ \mathbf{k}\left(u_1 \frac{\delta v_1}{\delta z} + v_1 \frac{\delta u_1}{\delta z} + u_2 \frac{\delta v_2}{\delta z} + v_2 \frac{\delta u_2}{\delta z} + u_3 \frac{\delta v_3}{\delta z} + v_3 \frac{\delta u_3}{\delta z}\right)$$

$$= \mathbf{i}\frac{\delta}{\delta x}(u_1v_1 + u_2v_2 + u_3v_3) + \mathbf{j}\frac{\delta}{\delta y}(u_1v_1 + u_2v_2 + u_3v_3)$$

$$+ \mathbf{k}\frac{\delta}{\delta z}(u_1v_1 + u_2v_2 + u_3v_3)$$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)[(u_1v_1 + u_2v_2 + u_3v_3).$$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)[u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}).(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})]$$

$$= \nabla(\mathbf{u}.\ \mathbf{v}).$$

(ix) $\nabla.(\nabla\varphi) = \nabla^2\varphi = \frac{\delta^2\varphi}{\delta x^2} + \frac{\delta^2\varphi}{\delta y^2} + + \frac{\delta^2\varphi}{\delta z^2}.$

$$\nabla.\nabla\varphi = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\left[\left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\varphi\right]$$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\left[\mathbf{i}\frac{\delta\varphi}{\delta x} + \mathbf{j}\frac{\delta\varphi}{\delta y} + \mathbf{k}\frac{\delta\varphi}{\delta z}\right]$$

$$= \frac{\delta^2\varphi}{\delta x^2} + \frac{\delta^2\varphi}{\delta y^2} + \frac{\delta^2\varphi}{\delta z^2}$$

$$= \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)\varphi = \nabla^2\varphi.$$

(x) $\nabla \times (\nabla\varphi) = 0$ **i**.*e*. *the curl of the gradient of* φ **is** *zero*.

$$\nabla \times (\nabla\varphi) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times \left[\left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\varphi\right]$$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times \left(\mathbf{i}\frac{\delta\varphi}{\delta x} + \mathbf{j}\frac{\delta\varphi}{\delta y} + \mathbf{k}\frac{\delta\varphi}{\delta z}\right)$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{\delta\varphi}{\delta x} & \frac{\delta\varphi}{\delta y} & \frac{\delta\varphi}{\delta z} \end{bmatrix}$$

$$= \mathbf{i}\left[\frac{\delta^2\varphi}{\delta y\delta z} - \frac{\delta^2}{\delta z\delta y}\right] + \mathbf{j}\left[\frac{\delta^2\varphi}{\delta z\delta x} - \frac{\delta^2\varphi}{\delta x\delta z}\right] + \mathbf{k}\left[\frac{\delta^2\varphi}{\delta x\delta y} - \frac{\delta^2\varphi}{\delta y\delta x}\right]$$

since $\frac{\delta^2\varphi}{\delta y\delta z} = \frac{\delta^2\varphi}{\delta z\delta y}, \frac{\delta^2\varphi}{\delta z\delta x} = \frac{\delta^2\varphi}{\delta x\delta z}$ *and* $\frac{\delta^2\varphi}{\delta x\delta y} = \frac{\delta^2\varphi}{\delta y\ \delta x}$

Thus $\nabla. \times (\nabla\varphi) = 0$.

(xi) $\nabla.(\nabla\times u) = \mathbf{0}$ i. e.Divcurl $\mathbf{u} = 0$. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$.

$$\nabla.(\nabla\times\mathbf{u})$$

$$=\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right).\left[\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right)\times(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\right]$$

$$=\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right).\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z}\\ u_1 & u_2 & u_3\end{bmatrix}$$

$$=\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right).\left[\mathbf{i}\left(\frac{\delta u_3}{\delta y}-\frac{\delta u_2}{\delta z}\right)+\mathbf{j}\left(\frac{\delta u_1}{\delta z}-\frac{\delta u_3}{\delta x}\right)+\mathbf{k}\left(\frac{\delta u_2}{\delta x}-\frac{\delta u_1}{\delta y}\right)\right]$$

$$=\frac{\delta}{\delta x}\left(\frac{\delta u_3}{\delta y}-\frac{\delta u_2}{\delta z}\right)+\frac{\delta}{\delta y}\left(\frac{\delta u_1}{\delta z}-\frac{\delta u_3}{\delta x}\right)+\frac{\delta}{\delta z}\left(\frac{\delta u_2}{\delta x}-\frac{\delta u_1}{\delta y}\right)$$

$$=\frac{\delta^2 u_3}{\delta x\delta y}-\frac{\delta^2 u_2}{\delta x\delta z}+\frac{\delta^2 u_1}{\delta y\delta z}-\frac{\delta^2 u_3}{\delta y\delta x}+\frac{\delta^2 u_2}{\delta z\delta x}-\frac{\delta u_1}{\delta z\delta y}$$

$$=0\left[\text{since }\frac{\delta^2 u_3}{\delta x\delta y}=\frac{\delta^2 u_3}{\delta y\delta x}\ \ldots\text{ etc}\right]$$

(xi) $\nabla.\times(\nabla\times\mathbf{u}) = \nabla(\nabla.\mathbf{u})\cdot\nabla^2\mathbf{u}$

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ then

$$\nabla\times\mathbf{u} = \left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right)\times(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})$$

$$\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z}\\ u_1 & u_2 & u_3\end{bmatrix}=\mathbf{i}\left(\frac{\delta u_3}{\delta y}-\frac{\delta u_2}{\delta z}\right)+\mathbf{j}\left(\frac{\delta u_1}{\delta z}-\frac{\delta u_3}{\delta x}\right)+\mathbf{k}\left(\frac{\delta u_2}{\delta x}-\frac{\delta u_1}{\delta y}\right)$$

$$\therefore \nabla\times(\nabla\times u) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z}\\ \frac{\delta u_3}{\delta y}-\frac{\delta u_2}{\delta z} & \frac{\delta u_r}{\delta z}-\frac{\delta u_3}{\delta x} & \frac{\delta u_2}{\delta x}-\frac{\delta u_1}{\delta y}\end{vmatrix}$$

$$= \boldsymbol{i}\left[\frac{\delta}{\delta y}\left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right) - \frac{\delta}{\delta z}\left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x}\right)\right]$$

$$+ \boldsymbol{j}\left[\frac{\delta}{\delta z}\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right) - \frac{\delta}{\delta x}\left(\frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y}\right)\right]$$

$$+ \boldsymbol{k}\left[\frac{\delta}{\delta x}\left(\frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta y}\right) - \frac{\delta}{\delta y}\left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z}\right)\right]$$

$$= \boldsymbol{i}\,\frac{\delta^2 u_2}{\delta y \delta x} - \boldsymbol{i}\,\frac{\delta^2 u_1}{\delta y^2} - \boldsymbol{i}\,\frac{\delta^2 u_1}{\delta z^2} + \boldsymbol{i}\,\frac{\delta^2 u_3}{\delta z \delta x}$$

$$+ \boldsymbol{j}\,\frac{\delta^2 u_3}{\delta z \delta y} - \boldsymbol{j}\,\frac{\delta^2 u_2}{\delta z^2} - \boldsymbol{j}\,\frac{\delta^2 u_2}{\delta x^2} + \boldsymbol{j}\,\frac{\delta^2 u_1}{\delta x \delta y}$$

$$+ \boldsymbol{k}\,\frac{\delta^2 u_1}{\delta x \delta z} - \boldsymbol{k}\,\frac{\delta^2 u_3}{\delta x^2} - \boldsymbol{k}\,\frac{\delta^2 u_3}{\delta y^2} + \boldsymbol{k}\,\frac{\delta^2 u_2}{\delta y \delta z}$$

$$= \boldsymbol{i}\,\frac{\delta^2 u_1}{\delta x^2} + \boldsymbol{i}\,\frac{\delta^2 u_2}{\delta x \delta y} + \boldsymbol{i}\,\frac{\delta^2 u_3}{\delta x \delta z} - \boldsymbol{i}\,\frac{\delta^2 u_1}{\delta x^2} - \boldsymbol{i}\,\frac{\delta^2 u_1}{\delta y^2} - \boldsymbol{i}\,\frac{\delta^2 u_1}{\delta z^2}$$

$$+ \boldsymbol{j}\,\frac{\delta^2 u_1}{\delta y \delta x} + \boldsymbol{j}\,\frac{\delta^2 u_2}{\delta y^2} + \boldsymbol{j}\,\frac{\delta^2 u_3}{\delta y \delta z} - \boldsymbol{j}\,\frac{\delta^2 u_2}{\delta x^2} - \boldsymbol{j}\,\frac{\delta^2 u_2}{\delta y^2} - \boldsymbol{j}\,\frac{\delta^2 u_2}{\delta z^2}$$

$$+ k\,\frac{\delta^2 u_1}{\delta z \delta x} + k\,\frac{\delta^2 u_2}{\delta z \delta y} + k\,\frac{\delta^2 u_3}{\delta z^2} - k\,\frac{\delta^2 u_3}{\delta x^2} - k\,\frac{\delta^2 u_3}{\delta y^2} - k\,\frac{\delta^2 u_3}{\delta z^2}$$

$$= \boldsymbol{i}\,\frac{\delta}{dx}\left(\frac{\delta u_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta u_3}{\delta z}\right) + \boldsymbol{j}\,\frac{\delta}{dy}\left(\frac{\delta u_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta u_3}{\delta z}\right)$$

$$+ \boldsymbol{k}\,\frac{\delta}{\delta z}\left(\frac{\delta u_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta u_3}{\delta z}\right) - \boldsymbol{i}\left(\frac{\delta^2 u_1}{\delta x^2} + \frac{\delta^2 u_1}{\delta y^2} + \frac{\delta^2 u_1}{\delta z^2}\right)$$

$$- \boldsymbol{j}\left(\frac{\delta^2 u_2}{\delta x^2} + \frac{\delta^2 u_2}{\delta y^2} + \frac{\delta^2 u_2}{\delta z^2}\right) - \boldsymbol{k}\left(\frac{\delta^2 u_3}{\delta x^2} + \frac{\delta^2 u_3}{\delta y^2} + \frac{\delta^2 u_3}{\delta z^2}\right)$$

$$= \left(\boldsymbol{i}\,\frac{\delta}{\delta x} + \boldsymbol{j}\,\frac{\delta}{\delta y} + \boldsymbol{k}\,\frac{\delta}{\delta z}\right)\left(\frac{\delta u_1}{\delta x} + \frac{\delta u_2}{\delta y} + \frac{\delta u_3}{\delta z}\right)$$

$$- \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})$$

$$= \nabla\left[\left(\boldsymbol{i}\,\frac{\delta}{\delta x} + \boldsymbol{j}\,\frac{\delta}{\delta y} + \boldsymbol{k}\,\frac{\delta}{\delta z}\right).(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})\right] - \nabla^2\boldsymbol{u}$$

$$= \nabla(\nabla.\,\boldsymbol{u}) - \nabla^2\boldsymbol{u}.$$

Definition Solenoidal vector

If the divergence of a vector $\boldsymbol{v}$ is zero i.e. $\nabla . \boldsymbol{v} = 0$, then $\boldsymbol{v}$ is called a **Solenoidal vector.**

Definition Irrotational vector.

If the curl of a vector $\mathbf{v}$ is zero i.e curl $\mathbf{v} = 0$ or $\nabla \times \mathbf{v} = 0$ then $\mathbf{v}$ is called the **irrotational vector,**

WORKED OUT EXAMPLES

Example 17. *If $\varphi(x, y, z) = 3x^2y - y^3z^2$, find grad$\varphi$ at the point $(1, -2, -1)$.* **[D. U. P. 1983]**

Solution : grad $\varphi = \nabla \varphi \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \ (3x^2y - y^3z^2)$

$$= \mathbf{i}\frac{\delta}{\delta x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\delta}{\delta y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\delta}{\delta z}(3x^2y - y^3z^2)$$

$$= 6xy\,\mathbf{i} + (3x^2 - 3y^2z^2)\,\mathbf{j} - 2y^2z\mathbf{k}$$

$\therefore$ At the point $(1, -2, -1)$, we have

$$\text{grad}\varphi = 6(-2)\,\mathbf{i} + (3 - 12)\,\mathbf{j} - 16\mathbf{k}$$

$$= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}.$$

Example 18. *If $\mathbf{u} = 3x^2y\mathbf{i} + 5xy^2z\mathbf{j} + xyz^3\mathbf{k}$ find divergence of $\mathbf{u}$ at the point $(1, 2, 3)$.*

Solution : div $\mathbf{u} = \nabla . \mathbf{u}$.

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) . (3x^2y\mathbf{i} + 5xy^2z\mathbf{j} + xyz^3\mathbf{k})$$

$$= \frac{\delta}{\delta x}(3x^2y) + \frac{\delta}{\delta y}(5xy^2z) + \frac{\delta}{\delta z}(xyz^3)$$

$$= 6xy + 10xyz + 3xyz^2.$$

So at the point $(1, 2, 3)$, we have

$$\text{div}\,\mathbf{u} = 12 + 60 + 54 = 126$$

Example 19. *If* $\mathbf{A} = xz^3\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\mathbf{k}$ *find curl* $\mathbf{A}$ *at the point (1, —1, 1)* **[D.U.P.1983]**

Solution : Curl $\mathbf{A} = \nabla \times \mathbf{A}$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times (xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\delta}{\delta y}(2yz^4) - \frac{\delta}{\delta z}(-2x^2yz)\right\} + \mathbf{j}\left\{\frac{\delta}{\delta z}(xz^3) - \frac{\delta}{\delta x}(2yz^4)\right\}$$

$$+ \mathbf{k}\left\{\frac{\delta}{\delta x}(-2x^2yz) - \frac{\delta}{\delta y}(xz^3)\right\}$$

$$= (2z^4 + 2x^2y)\mathbf{i} + (3xz^2)\mathbf{j} - 4xyz\mathbf{k}$$

So that the point (1, –1, 1), we have

$$\text{Curl } \mathbf{A} = (2-2)\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} = 3j + 4\mathbf{k}$$

Example 20. Find the directional derivative of $\varphi\ (x, y, z) = 4xz^3 - 3x^2y^2z$ *at the point* $(2, -1, 2)$ in the direction of $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

Solution : $\varphi\,(x, y, z) = 4xz^3 - 3x^2y^2z$

$$\nabla\varphi = \left\{\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right\}(4xz^3 - 3x^2y^2z)$$

$$= \mathbf{i}\frac{\delta}{\delta x}(4xz^3 - 3x^2y^2z) + \mathbf{j}\frac{\delta}{\delta y}(4xz^3 - 3x^2y^2z)$$

$$+ \mathbf{k}\frac{\delta}{\delta z}(4xz^3 - 3x^2y^2z)$$

$$= (4z^2 - 6xy^2z)\,\mathbf{i} - (6x^2yz)\mathbf{j} + (12xz^2 - 3x^2y^2)\mathbf{k}$$

So at the point (2, –1, 2) we have

$$\nabla\varphi = (32 - 24)\mathbf{i} - (-48)\mathbf{j} + (96 - 12)\mathbf{k}$$

$$= 8\mathbf{i} + 48\mathbf{j} + 84\mathbf{k}.$$

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Again the unit vector in the direction of

$$2\mathbf{i}-3\mathbf{j}+6\mathbf{k} \text{ is } \mathbf{a}=\frac{2\mathbf{i}-3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+(-3)^2+6^2}}=\frac{2\mathbf{i}-3\mathbf{j}+6\mathbf{k}}{\sqrt{4+9+36}}$$

$$=\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}.$$

Thus the required directional derivative is

$$\nabla\varphi.\mathbf{a}=(8\mathbf{i}+48\mathbf{j}+84\mathbf{k}).\left(\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}\right)$$

$$=\frac{16}{7}-\frac{144}{7}+\frac{504}{7}=\frac{376}{7}.$$

Since this is positive, φ is increasing in the direction of $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$.

Example 21(a). Find the unit normal to the surface $xy^3z^2=4$ at the point $(-1,-1,2)$.

Solution : Let us regard the given surface as a particular level surface of the function $\varphi(x,y,z)=xy^3z^2$. Then the gradient of this function at the point $(-1,-1,2)$ will be perpendicular to the level surface through $(-1,-1,2)$ which is the given surface.

$$\nabla\varphi=\left\{\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right\}(xy^3z^2)$$

$$=\mathbf{i}\frac{\delta}{\delta x}(xy^3z^2)+\mathbf{j}\frac{\delta}{\delta z}(xy^3z^2)+\mathbf{k}\frac{\delta}{\delta z}(xy^3z^2)$$

$$=y^3x^2\mathbf{i}+3xy^2z^2\mathbf{j}+2xy^3\mathbf{k}$$

So at the point $(-1,-1,2)$,

$$\nabla\varphi=-4\mathbf{i}-12\mathbf{j}+4\mathbf{k}.$$

$$\therefore\ |\nabla\varphi|=\sqrt{(-4)^2+(-12)^2+4^2}=4\sqrt{11}.$$

Thus a unit normal to the given surface is

$$\frac{\nabla\varphi}{|\nabla\varphi|}=\frac{-4\mathbf{i}-12\mathbf{j}+4\mathbf{k}}{4\sqrt{11}}=-\frac{1}{\sqrt{11}}\mathbf{i}-\frac{3}{\sqrt{11}}\mathbf{j}+\frac{1}{\sqrt{11}}\mathbf{k}.$$

Another unit normal is $\frac{1}{\sqrt{11}}\mathbf{i}+\frac{3}{\sqrt{11}}\mathbf{j}-\frac{1}{\sqrt{11}}\mathbf{k}$,

having direction opposite to that above.

Example 21(a) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ *and* $z = x^2 + y^2 - 3$ *at the point* $(2, -1, 2)$.

Solution : The angle between the two surfaces at a point is equal to the angle between the normals to the surfaces at that point.

Let $\varphi_1(x,y,z) = x^2 + y^2 + z^2 - 9$ *and* $\varphi_2(x, y, z) = x^2 + y^2 - z - 3$

then $\nabla\varphi_1 = \nabla(x^2 + y^2 + z^2) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2 + z^2)$

$= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ *and*

$$\nabla\varphi_2 = \nabla(x^2 + y^2 - z) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2 - z)$$

$$= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$$

Therefore, the normal to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$ *is* $\nabla\varphi_1 = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and the normal to the surface $x^2 + y^2 - z = 3$ *at the point* $(2, -1, 2)$ *is*

$\nabla\varphi_2 = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Let θ be the angle between the two surfaces (or normals) then $(\nabla\varphi_1) . (\nabla\varphi_2) = |\nabla\varphi_1| . |\nabla\varphi_2| \cos\theta$ (1)

Now $|\nabla\varphi_1| = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}| = \sqrt{4^2 + (-2)^2 + 4^2} = 6$

$|\nabla\varphi_1| = |4\mathbf{i} - 2\mathbf{j} - \mathbf{k}| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{21}$

$(\nabla\varphi_1) . (\nabla\varphi_2)\ (4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) . (4\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

$= 16 + 4 - 4 = 16$

Putting these values in equation (1), we get

$16 = 6\sqrt{21}\cos\theta$

Or, $8 = 3\sqrt{21}\cos\theta$

Or, $\cos\theta = \dfrac{8}{3\sqrt{21}}$ $\therefore\ \theta = \cos^{-1}\dfrac{8}{3\sqrt{21}}$

which is the required angle between the two given surfaces.

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Example 22. Find the unit vector perpendicular to the surface $x^2 + y^2 + z^2 = 3$ at the point (1, 1, 1) and derive the equation of the plane tangential to the surface at (1, 1, 1).

Solution : Let $\varphi = x^2 + y^2 + z^2 - 3$. then

$$\nabla\varphi = \nabla(x^2 + y^2 + z^2)$$

$$\left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2 + z^2)$$

$$= \mathbf{i}\frac{\delta}{\delta x}(x^2 + y^2 + z^2) + \mathbf{j}\frac{\delta}{\delta y}(x^2 + y^2 + z^2) + \mathbf{k}\frac{\delta}{\delta z}(x^2 + y^2 + z^2)$$

$$= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

So at the point (1, 1, 1) $\nabla\varphi = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Thus a unit vector, say **N**, perpendi o the surface is

$$\mathbf{N} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{4+4+4}} = \qquad \frac{\mathbf{k}}{3}$$

Now the position vector to 1) is

$\mathbf{r}_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and the position v

to any point on the plane is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then $\mathbf{r} - \mathbf{r}_0$ is perpendicular to **N** and the required equation of the tangent plane is $(\mathbf{r} - \mathbf{r}_0) . \mathbf{N} = 0$ that is,

$$\{(x-1)\mathbf{i} + (y-1)\mathbf{j} + (z-1)\mathbf{k}\} . \left(\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}\right) = 0$$

$$\text{Or, } \frac{x-1}{\sqrt{3}} + \frac{y-1}{\sqrt{3}} + \frac{z-1}{\sqrt{3}} = 0$$

$$\text{Or, } x - 1 + y - 1 + z - 1 = 0$$

Or, $x + y + z = 3$ which is the equation of the required tangent plane.

Example 23. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$ then show tha $r^n\mathbf{r}$ is an **irrotational** vector for any value of n, but $r^n\mathbf{r}$ is **solenoidal** only if $n = -3$.

Proof : First portion

We know that $\nabla \times (\varphi \mathbf{u}) = (\nabla\varphi) \times \mathbf{u} + \varphi(\nabla \times \mathbf{u})$.

Let $\mathbf{u} = r^n\mathbf{r}$ then $\mathbf{u}$ is irrotational if curl$\mathbf{u} = 0$.

$$\text{Curl } \mathbf{u} = \text{Curl } (r^n\mathbf{r}) = \nabla \times (r^n\mathbf{r}) = (\nabla r^n) \times \mathbf{r} + r^n (\nabla \times \mathbf{r})$$

$$\text{Now } \nabla r^n = \left(\mathbf{i}\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \{(x^2 + y^2 + z^2)^{\frac{n}{2}}\}$$

$$= \mathbf{i}\frac{\delta}{\delta x}\{(x^2 + y^2 + z^2)^{\frac{n}{2}}\} + \mathbf{j}\frac{\delta}{\delta y}\{(x^2 + y^2 + z^2)^{\frac{n}{2}}\}$$

$$+ \mathbf{k}\frac{\delta}{\delta z}\{(x^2 + y^2 + z^2)^{\frac{n}{2}}\}$$

$$= \mathbf{i}\, nx(x^2 + y^2 + z^2)^{\frac{n-2}{2}} + \mathbf{j}\, ny(x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

$$+ \mathbf{k}\, nz(x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

$$= n(x^2 + y^2 + z^2)^{\frac{n-2}{2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nr^{n-2}\mathbf{r}$$

$$\text{Again Curl } \mathbf{r} = \nabla \times \mathbf{r} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x & y & z \end{vmatrix} = 0$$

$$\therefore \text{ curl } u = nr^{n-2}\, \mathbf{r}\times\mathbf{r} + 0 \text{ since } \nabla \times r = 0.$$

$$= nr^{n-2}\, \mathbf{r}\times\mathbf{r} = 0 \text{ since } \mathbf{r}\times\mathbf{r} = 0.$$

Thus curl$\mathbf{u}$ = curl $(r^n\mathbf{r}) = 0$. So the vector $\mathbf{u}$ is **irrotational** for any value of n.

Second portion

We also know that $\nabla.(\varphi\mathbf{u}) = (\nabla\varphi).\mathbf{u} + \varphi(\nabla.\mathbf{u})$

$\therefore$ $div\,(r^n\mathbf{r}) = \nabla.(r^n\mathbf{r})$

$= (\nabla r^n).\mathbf{r} + r^n\,(\nabla.r)$

$= nr^{n-2}\,\mathbf{r}.\mathbf{r} + 3r^n$

[Since $\nabla r^n = nr^{n-2}\,\mathbf{r}$ and $\nabla.r = 3$]

$= nr^{n-2}\,\mathbf{r}^2 + 3r^n$

$= nr^n + 3r^n = (n+3)r^n = 0$ if $n + 3 = 0$ i.e. $n = -3$.

This shows that $r^n\mathbf{r}$ is solenoidal when $n = -3$.

Example 24. If $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ and $r = |\mathbf{r}|$

find (i) div gradr^n (ii) curl grad r^n.

Solution : Given $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and

$r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

$$gard\ r^n = \nabla r^n = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)\left\{(x^2 + y^2 + z^2)^{n/2}\right\}$$

$$= \mathbf{i}\frac{\delta}{\delta x}\left\{(x^2 + y^2 + z^2)^{n/2}\right\} + \mathbf{j}\frac{\delta}{\delta y}\left\{(x^2 + y^2 + z^2)^{n/2}\right\}$$

$$+ k\frac{\delta}{\delta z}\left\{(x^2 + y^2 + z^2)^{n/2}\right\}$$

$$= (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})n\,(x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

$$= (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,nr^{n-2}$$

(i) $div\ grad\ r^n = \nabla\cdot\nabla\ r^n$

$$= \left[\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right](x\mathbf{i} + y\mathbf{j} + z\mathbf{k})nr^{n-2}$$

$$= \frac{\delta}{\delta x}(nxr^{n-2}) + \frac{\delta}{\delta y}(nyr^{n-2}) + \frac{\delta}{\delta z}(nzr^{n-2})$$

Now $\frac{\delta}{\delta x}[nxr^{n-2}] = nr^{n-2} + nx(n-2)r^{n-3}\frac{\delta r}{\delta x}$

Since $r = \sqrt{x^2+y^2+z^2}$

$\frac{\delta r}{\delta x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}.2x = \frac{x}{r}$

Similarly, we have $\frac{\delta r}{\delta y} = \frac{y}{r}, \frac{\delta r}{\delta z} = \frac{z}{r}$

Thereore, $\frac{\delta}{\delta x}\left[nxr^{n-2}\right] = nr^{n-2} + nx(n-2)r^{n-2}\frac{x}{r}$

$= nr^{n-2} + n(n-2)r^{n-4}x^2$. (1)

Similarly, $\frac{\delta}{\delta y}\left(nyr^{n-2}\right) = nr^{n-2} + n(n-2)r^{n-4}y^2$ (2)

and $\frac{\delta}{\delta z}\left(nzr^{n-2}\right) = nr^{n-2} + n(n-2)r^{n-4}z^2$ (3)

Adding (1), (2) and (3), we have

div grad $r^n = 3rn^{n-2} + n(n-2)r^{n-4}(x^2+y^2+z^2)$

$= 3nr^{n-2} + n(n-2)r^{n-4}r^2$

$= 3nr^{n-2} + n(n-2)r^{n-2}$

$= nr^{n-2}\{3+n-2\}$

$= n(n+1)r^{n-2}$

(ii) Curl grad $r^n = \nabla \times \nabla r^n$

$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times \left\{\left(nr^{n-2}x\right)\mathbf{i} + \left(nr^{n-2}y\right)\mathbf{j} + (nr^{n-2}z)\mathbf{k}\right\}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ nr^{n-2}x & nr^{n-2}y & nr^{n-2}z \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\delta}{\delta y}\left(nr^{n-3}\,z\right) - \frac{\delta}{\delta z}\left(nr^{n-2}\,y\right)\right\}$$

$$+\mathbf{j}\left\{\frac{\delta}{\delta z}\left(nr^{n-2}\,x\right) - \frac{\delta}{\delta x}\left(nr^{n-2}\,z\right)\right\}$$

$$+\mathbf{k}\left\{\frac{\delta}{\delta x}\left(nr^{n-2}\,y\right) - \frac{\delta}{\delta y}\left(nr^{n-2}\,x\right)\right\}$$

$$= \mathbf{i}\left\{ n(n-2)\, r^{n-2}\,\frac{\delta r}{\delta y}\, z - n(n-2)\, r^{n-3}\,\frac{\delta r}{\delta z}\, y\right\}$$

$$+\mathbf{j}\left\{ n(n-2)\, r^{n-3}\,\frac{\delta r}{\delta z}\, x - n(n-2)\, r^{n-3}\,\frac{\delta r}{\delta x}\, z\right\}$$

$$+\mathbf{k}\left\{ n(n-2)\, r^{n-3}\,\frac{\delta r}{\delta x}\, y - n(n-2)\, r^{n-3}\,\frac{\delta r}{\delta y}\, x\right\}$$

Since $\dfrac{\delta r}{\delta x} = \dfrac{x}{r}, \dfrac{\delta r}{\delta y} = \dfrac{y}{r}$ *and* $\dfrac{\delta r}{\delta z} = \dfrac{z}{r}$.

$$= n(n-2)\, r^{n-3}\left[\mathbf{i}\left(\frac{yz}{r} - \frac{yz}{r}\right) + \mathbf{j}\left(\frac{zx}{r} - \frac{zx}{r}\right) + \mathbf{k}\left(\frac{xy}{r} - \frac{xy}{r}\right)\right]$$

$$= n(n-2)\, r^{n-3}\,[o + o + 0]$$

$$= \left\{ n(n-2)\, r^{n-3}\right\} 0 = 0$$

Thus curl grad $r^n = 0$.

Example 25. Show that

$$\text{Curl } \frac{\boldsymbol{a}\times\boldsymbol{r}}{r^3} = -\frac{\boldsymbol{a}}{r^3} + 3\frac{r}{r^3}\,(a\,.\,r)$$

where **a** is a constant vector and $\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

[D. U. S. 1986]

Proof : $Curl\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = \nabla \times \left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times \left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)$$

$$= \mathbf{i}\frac{\delta}{\delta x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) + \mathbf{j}\frac{\delta}{\delta y}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) + \mathbf{k}\frac{\delta}{\delta z}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) \quad (1)$$

Now $\frac{\delta}{\delta x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3}{r^4}\frac{\delta r}{\delta x}(\mathbf{a}\times\mathbf{r}) + \frac{1}{r^3}\left(\mathbf{a}\times\frac{\delta \mathbf{r}}{\delta x}\right)$

$$= -\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r}) + \frac{\mathbf{a}\times\mathbf{i}}{r^3}$$

Since $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}, \frac{\delta \mathbf{r}}{\delta x} = \mathbf{i}$ and

$$r^2=x^2+y^2+z^2 \text{ gives } \frac{\delta r}{\delta x} = \frac{x}{r}$$

$$\therefore \mathbf{i} \times \frac{\delta}{\delta x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^5}\right)$$

$$= -\frac{3x}{r^5}\{\mathbf{i}\times(\mathbf{a}\times\mathbf{r})\} + \frac{1}{r^3}\{\mathbf{i}\times(\mathbf{a}\times\mathbf{i})\}$$

Since $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}.\,\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B})$

$$= -\frac{3x}{r^5}[\mathbf{a}(\mathbf{i}.\,\mathbf{r}) - \mathbf{r}(\mathbf{i}.\,\mathbf{a})] + \frac{1}{r^3}[\mathbf{a}(\mathbf{i}.\,\mathbf{i}) - \mathbf{i}(\mathbf{i}.\,\mathbf{a})]$$

$$= -\frac{3x}{r^5}[x\mathbf{a} - \mathbf{r}(\mathbf{i}.\mathbf{a})] + \frac{1}{r^3}[\mathbf{a} - \mathbf{i}(\mathbf{i}.\,\mathbf{a})]$$

Since $\mathbf{i}.\,\mathbf{r} = x$ & $\mathbf{i}.\,\mathbf{i} = 1$

$$= -\frac{3x^2}{r^5}\mathbf{a} + \frac{3\mathbf{r}}{r^5}(x\ \mathbf{i}.\,\mathbf{a}) + \frac{\mathbf{a}}{r^3} - \frac{\mathbf{i}(\mathbf{i}.\mathbf{a})}{r^3} \quad (2)$$

Similarly, we have

$$\mathbf{j}\times\frac{\delta}{\delta y}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3y^2}{r^5}\mathbf{a} + \frac{3\mathbf{r}}{r^6}(y\,\mathbf{j}.\,\mathbf{a}) + \frac{\mathbf{a}}{r^3} - \frac{\mathbf{j}(\mathbf{j}.\,\mathbf{a})}{r^3} \quad (3)$$

$$\mathbf{k}\times\frac{\delta}{\delta z}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3z^2}{r^5}\mathbf{a} + \frac{3\mathbf{r}}{r^5}(z\,\mathbf{k}.\mathbf{a}) + \frac{\mathbf{a}}{r^3} - \frac{\mathbf{k}(\mathbf{k}.\mathbf{a})}{r^3} \quad (4)$$

Adding (2), (3) and (4), we get

$$Curl\left(\frac{\mathbf{a}\times\mathbf{r}}{r}\right) = -\frac{3}{r^5}(x^2+y^2+z^2)\,\mathbf{a} + \frac{3\mathbf{r}}{r^5}\{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).\mathbf{a}\}$$

$$+\frac{3\mathbf{a}}{r^3} - \frac{1}{r^3}[\mathbf{i}(\mathbf{i}.\,\mathbf{a}) + \mathbf{j}(\mathbf{j}.\,\mathbf{a}) + \mathbf{k}(\mathbf{k}.\,\mathbf{a})]$$

Proof : $\text{Curl}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = \nabla \times \left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times \left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)$$

$$= \mathbf{i}\frac{\delta}{\delta x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) + \mathbf{j}\frac{\delta}{\delta y}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) + \mathbf{k}\frac{\delta}{\delta z}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) \quad (1)$$

Now $\frac{\delta}{\delta x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3}{r^4}\frac{\delta r}{\delta x}(\mathbf{a}\times\mathbf{r}) + \frac{1}{r^3}\left(\mathbf{a}\times\frac{\delta \mathbf{r}}{\delta x}\right)$

$$= -\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r}) + \frac{\mathbf{a}\times\mathbf{i}}{r^3}$$

Since $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}, \frac{\delta \mathbf{r}}{\delta x} = \mathbf{i}$ and

$$r^2 = x^2 + y^2 + z^2 \text{ gives } \frac{\delta r}{\delta x} = \frac{x}{r}$$

$$\therefore \mathbf{i} \times \frac{\delta}{\delta x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^5}\right)$$

$$= -\frac{3x}{r^5}\{\mathbf{i}\times(\mathbf{a}\times\mathbf{r})\} + \frac{1}{r^3}\{\mathbf{i}\times(\mathbf{a}\times\mathbf{i})\}$$

Since $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B})$

$$= -\frac{3x}{r^5}[\mathbf{a}(\mathbf{i}.\mathbf{r}) - \mathbf{r}(\mathbf{i}.\mathbf{a})] + \frac{1}{r^3}[\mathbf{a}(\mathbf{i}.\mathbf{i}) - \mathbf{i}(\mathbf{i}.\mathbf{a})]$$

$$= -\frac{3x}{r^5}[x\mathbf{a} - \mathbf{r}(\mathbf{i}.\mathbf{a})] + \frac{1}{r^3}[\mathbf{a} - \mathbf{i}(\mathbf{i}.\mathbf{a})]$$

Since $\mathbf{i}.\mathbf{r} = x$ & $\mathbf{i}.\mathbf{i} = 1$

$$= -\frac{3x^2}{r^5}\mathbf{a} + \frac{3\mathbf{r}}{r^5}(x\ \mathbf{i}.\mathbf{a}) + \frac{\mathbf{a}}{r^3} - \frac{\mathbf{i}(\mathbf{i}.\mathbf{a})}{r^3} \quad (2)$$

Similarly, we have

$$\mathbf{j}\times\frac{\delta}{\delta y}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3y^2}{r^5}\mathbf{a} + \frac{3\mathbf{r}}{r^6}(y\,\mathbf{j}.\mathbf{a}) + \frac{\mathbf{a}}{r^3} - \frac{\mathbf{j}(\mathbf{j}.\mathbf{a})}{r^3} \quad (3)$$

$$\mathbf{k}\times\frac{\delta}{\delta z}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3z^2}{r^5}\mathbf{a} + \frac{3\mathbf{r}}{r^5}(z\,\mathbf{k}.\mathbf{a}) + \frac{\mathbf{a}}{r^3} - \frac{\mathbf{k}(\mathbf{k}.\mathbf{a})}{r^3} \quad (4)$$

Adding (2), (3) and (4), we get

$$\text{Curl}\left(\frac{\mathbf{a}\times\mathbf{r}}{r}\right) = -\frac{3}{r^5}(x^2+y^2+z^2)\,\mathbf{a} + \frac{3\mathbf{r}}{r^5}\{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).\mathbf{a}\}$$

$$+\frac{3\mathbf{a}}{r^3} - \frac{1}{r^3}[\mathbf{i}(\mathbf{i}.\mathbf{a}) + \mathbf{j}(\mathbf{j}.\mathbf{a}) + \mathbf{k}(\mathbf{k}.\mathbf{a})]$$

Now if $\mathbf{a}=a_1\,\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, *then*

$$\mathbf{i}.\,\mathbf{a}=a_1,\ \mathbf{j}.\,a=a_2,\ \mathbf{k}.\,\mathbf{a}=a_3$$

$$\therefore\quad \mathbf{i}\,(\mathbf{i}.\,\mathbf{a})+\mathbf{j}\,(\mathbf{j}.\,\mathbf{a})+\mathbf{k}\,(\mathbf{k}.\,\mathbf{a})=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}$$

$$\text{Thus Curl}\left(\frac{a\times\mathbf{r}}{r^3}\right)=-\frac{3r^2}{r^5}\,\mathbf{a}+\frac{3\mathbf{r}}{r^5}(\mathbf{r}.\,\mathbf{a})+\frac{3\mathbf{a}}{r^3}-\frac{\mathbf{a}}{r^3}$$

$$=-\frac{3\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^5}(\mathbf{a}.r)+\frac{2\mathbf{a}}{r^3}$$

$$=-\frac{\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^5}(\mathbf{a}.\mathbf{r}).$$

EXERCISES 10 (B)

1. *If* $\mathbf{u}=(e^{xy}-y^2)\,\mathbf{i}+(2x^3y-x^4\cos x)\,\mathbf{j}+(y^2\sin x)\,\mathbf{k}$

find $\dfrac{\delta\mathbf{u}}{\delta x},\dfrac{\delta\mathbf{u}}{\delta y},\dfrac{\delta^2\mathbf{u}}{\delta x^2},\dfrac{\delta^2\mathbf{u}}{\delta x\delta y},\dfrac{\delta^2\mathbf{u}}{\delta y\delta x}.$

Answer : $\dfrac{\delta\mathbf{u}}{\delta x}=\left(ye^{xy}\right)\mathbf{i}+(6x^2y+x^4\sin x-4x^3\cos x)\mathbf{j}+(y^2\cos x)\mathbf{k}$

$$\frac{\delta\mathbf{u}}{\delta y}=\left(xe^{xy}-2y\right)\ \mathbf{i}+2x^3\mathbf{j}+(2y\sin x)\mathbf{k};$$

$$\frac{\delta^2\mathbf{u}}{\delta x^2}=\left(y^2e^{xy}\right)\ \mathbf{i}+(12xy+x^4\cos x+8x^3\sin x-12x^2\cos x)\,\mathbf{j}$$

$$-(y^2\sin x)\mathbf{k},$$

$$\frac{\delta^2\mathbf{u}}{\delta y^2}=(x^2e^{xy}-2)\,\mathbf{i}+2\sin x\mathbf{k};$$

$$\frac{\delta^2\mathbf{u}}{\delta x\delta y}=\frac{\delta}{\delta x}\left(\frac{\delta\mathbf{u}}{\delta y}\right)=e^{xy}\,(xy+1)\,\mathbf{i}+6x^2\mathbf{j}+(2y\cos x)\,\mathbf{k};$$

$$\frac{\delta^2\mathbf{u}}{\delta y\delta x}=\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right)=e^{xy}\,(xy+1)\,\mathbf{i}+6x^2\mathbf{j}+(2y\cos x)\mathbf{k}.$$

2. *If* $\mathbf{u}=t\,\mathbf{i}-\sin t\mathbf{k}$ *and* $\mathbf{v}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+\mathbf{k}$,

find $\dfrac{d}{dt}(\mathbf{u}.\,\mathbf{v})$.

Answer : $-t\sin t$.

3. *If* $u=\cos t\,\mathbf{i}-2t^2\mathbf{j}+e^{5t}\mathbf{k}$ *and*

$$\mathbf{v}=3t^2\mathbf{i}-2\sin t\,\mathbf{j}+5t\mathbf{k}$$

find $\dfrac{d}{dt}(\mathbf{u}\times\mathbf{v})$ *at* $t=0$.

Answer : $2\mathbf{i}-5\mathbf{j}-2\mathbf{k}$.

4. *If* $\mathbf{u} = x^2y\mathbf{i} - 2y^2z\mathbf{j} + xy^2z^2\mathbf{k}$, *find*

$\left[\dfrac{\delta^2\mathbf{u}}{\delta x^2} \times \dfrac{\delta^2\mathbf{u}}{\delta y^2}\right]$ *at the point* $(2, 1, -2)$.

Answer : $16\sqrt{5}$.

5. *If* $\mathbf{u} = x^2\mathbf{i} - y\mathbf{j} + xz\mathbf{k}$ *and* $\mathbf{v} = y\mathbf{i} + x\mathbf{j} - xyz\mathbf{k}$

find $\dfrac{\delta^2}{\delta y \delta x}(\mathbf{u} \times \mathbf{v})$ *at the point* $(1, -1, 2)$.

Answer : $-4\mathbf{i} + 8\mathbf{j}$.

6. *If* $\mathbf{u} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, $\mathbf{v} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} - 3\mathbf{k}$ *and*

$\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ *find* $\dfrac{d}{dt}(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))$ *at* $t = 0$.

Answer : $7\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$.

7. *Find the gradient of* φ *i. e* $\nabla\varphi$, *where*

(i) $\varphi(x, y, z) = (x^2 + y^2 + z^2)^2$

(ii) $\varphi(x, y, z) = \sin(x^2 + y^2 + z^2)$.

Answer : (i) $4(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})(x^2 + y^2 + z^2)$

(ii) $2\cos(x^2 + y^2 + z^2)(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

8. *Find the divergence of the vector functions*

(i) $xyz(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

(ii) $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}/\sqrt{x^2 + y^2 + z^2}$.

Answer : (i) $6xyz$ (ii) $\dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$

9. *Find curl* $\mathbf{v}$ *at the point* $(1, 1, 1)$

$\mathbf{v} = 2xz^2\mathbf{i} - yz\mathbf{j} + 3xz^2\mathbf{k}$ **[D. U. S. 1983]**

Answer : $\mathbf{i}. + \mathbf{j}$.

10. *Find the scalar function* φ *such that*

$\nabla\varphi = 6x\mathbf{i} - 4y\hat{\mathbf{j}} + 2z\mathbf{k}$.

Answer : $3x^2 - 2y^2 + z^2$.

11. If $\mathbf{u} = (2xy+z^3)\,\mathbf{i} + (x^2 + 2y)\,\mathbf{j} + (3xz^2 - 2)\mathbf{k}$ find a scalar function φ such that $\mathbf{u} = \nabla\varphi$.

Answer : $\varphi = x^2y + xz^2 + y^2 - 2z$.

12. If $\mathbf{u} = xz\mathbf{i} + (2x^2 - y)\mathbf{j} - yz^2\mathbf{k}$ and

$\varphi(x, y, z) = 3x^2y + y^2z^3$, find (a) $\nabla\varphi$ (b) $\nabla .\ \mathbf{u}$ and (c) $\nabla \times \mathbf{u}$.

at the point $(1, -1, 1)$.

Answer : (a) $-6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ (b) 2 (c) $-\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

13. Compute the divergence and curl of each of the following vectors.

(i) $y^2\mathbf{i} + 2x^2z\mathbf{j} - xyz\mathbf{k}$

(ii) $xyz\mathbf{i} + 3x^2y\mathbf{j} + (xz^2 - y^2z)\,\mathbf{k}$.

Answer : (i) $-xy$, $-x(2x+z)\,\mathbf{i} + yz\mathbf{j} + 2(2xz - y)\,\mathbf{k}$.

(ii) $yz + 3x^2 + 2xz - y^2$, $-2yz\mathbf{i} + (xy - z^2)\,\mathbf{j} + x(6y - z)\mathbf{k}$

14. If $f(x, y, z) = x + y + z$, $g(x, y, z) = x + y$ and

$h(x, y, z) = -2xz - 2yz - z^2$, show that

$[\nabla f\ \nabla g\ \nabla h] = 0$.

15. If $\mathbf{v} = (2x^2 - yz)\mathbf{i} + (y^2 - 2xz)\mathbf{j} + x^2z^3\mathbf{k}$ and

$\varphi(x, y, z) = x^2y - 3xz^2 + 2xyz$, show directly that

div curl $\mathbf{v} = 0$ and curl grad $\varphi = 0$.

16. If $\varphi(x, y, z) = xy + yz + zx$ and

$u = x^2y\mathbf{i} + y^2z\mathbf{j} + z^2x\mathbf{k}$, find (a) $\mathbf{u}.\ \nabla\varphi$ and (b) $\nabla\varphi \times \mathbf{u}$ at the point $(3, -1, 2)$.

Answer : (a) 25 (b) $56\mathbf{i} - 30\mathbf{j} + 47\mathbf{k}$.

17. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Answer : $-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

18. (i) Find an equation to the tangent plane to the surface $2xz^2-3xy-4x=7$ *at the point* $(1,-1,.2)$

(ii) Find an equation for the tangent plane to the surface $xz^2+x^2y=z-1$ *at the point* $(1,-3,2)$

(iii) Find an equation for the tangent plane to the surface $z=x^2+y^2$ *at the point* $(1,-1,2.)$

Answer : (i) $7x-3y+8z-26=0.$

(i) $2x-y-3z+1=0$

(iii) $2x-2y-z-2=0.$

19. Find the directional derivative of the function

$\varphi(x, y. z) = xy^2+yz^3$ *at the point* $(2,-1,1)$ in the direction of the vector $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

Answer : $-\frac{11}{3}$.

20. Find the directional derivative of

$\varphi(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ *at the point* $(1, 2, 3)$ in the direction of $4\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

Answer : $\frac{-4}{49\sqrt{6}}$

21. Find the directional derivative of

$\varphi(x, y. z) = x^2yz+4xz^2$ at the point $(1,-2,-1)$ in the direction of $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.

Answer : $\frac{37}{3}$.

22. Find the directional derivative of $\nabla.(\nabla\varphi)$ at the point $(1,-2,1)$ in the direction of the normal to the surface

$xy^2z-3x-z^2=0$ *where* $\varphi(x, y, z)=2x^3y^2z^4$.

Answer : $\frac{1724}{\sqrt{21}}$.

23. (i) Prove that the vector $\nabla\varphi$ is perpendicular to the surface $\varphi(x, y, z)$ =constant.

(ii) Find the value of ∇u if $\mathbf{u} = \log(x^2 + y^2 + z^2)$.

Answer : (ii) $\frac{2}{x^2 + y^2 + z^2}\{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\}$

24. Find the divergence and curl of the vector field

$x^2 y\mathbf{i} + xz\,\mathbf{j} + yz\mathbf{k}$ at the point (4, 3, 2) **[R. U. P. 1970]**

Answer : Divergence=27

Curl = $-2\mathbf{i} - 14\mathbf{k}$.

25. If $\mathbf{A} = x^2 y\mathbf{i} - 2xz\mathbf{j} + 2yz\mathbf{k}$

Find Curl Curl $\mathbf{A}$ i. e $\nabla \times (\nabla \times \mathbf{A})$. **[R. U. P. 1974]**

Answer : $(2x+2)\,\mathbf{j}$

26. If $\mathbf{F} = (3x^2 y - z)\mathbf{i} + (xz^3 + y^4)\,\mathbf{j} - 2x^3 z^2\mathbf{k}$.

find grad (div $\mathbf{F}$) at the point (2, −1, o). **[D. U. S 1964]**

Answer : $-6\mathbf{i} + 24\mathbf{j} - 32\mathbf{k}$.

27. Given $\varphi(x, y, z) = 2x^3 y^2 z^4$, find $\nabla.(\nabla\varphi)$.

Answer : $12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$. **[R. U. P. 1964]**

28. If $\mathbf{A} = 2x^2\mathbf{i} - yz\mathbf{j} + xz^2\mathbf{k}$ and

$\varphi(x, y, z) = 2z - x^3 y$ then find $\mathbf{A}.(\nabla\varphi)$

and $\mathbf{A} \times (\nabla\varphi)$ at the point (1, −1, 1). **[D. U. S 1985]**

Answer : $\mathbf{A}.\nabla\varphi = 7$.

$\mathbf{A} \times \nabla\varphi = 3\mathbf{i} - \mathbf{j} - 5\mathbf{k}$.

29. If $\mathbf{u} = 3xz^2\mathbf{i} - yz\mathbf{j} + (x + 2z)\,\mathbf{k}$ find curl curl$\mathbf{u}$.

Answer : $-6x\mathbf{i} + (6z - 1)\,\mathbf{k}$.

30. If $\mathbf{u} = yz^2\mathbf{i} - 3xz^2\mathbf{j} + 2xyz\mathbf{k}$,

$\mathbf{v} = 3x\mathbf{i} + 4z\mathbf{j} - xy\mathbf{k}$ and $\varphi(x, y, z) = xyz$,

find (a) $\mathbf{u} \times (\nabla\varphi)$ (b) $(\mathbf{u} \times \nabla)\,\varphi$ (c) $(\nabla \times \mathbf{u}) \times \mathbf{v}$.

Answer : (a) $-5x^2yz^2\mathbf{i} + xy^2z^2\mathbf{j} + 4xyz^3\mathbf{k}$

(b) $-5x^2yz^2\mathbf{i} + xy^2z^2\mathbf{j} + 4xyz^3\mathbf{k}$

(c) $16z^3\mathbf{i} + (8x^2yz - 12xz^2)\,\mathbf{j} + 32xz^2\,\mathbf{k}$.

31. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

then prove that (i) $\nabla.\,(\mathbf{u} \times \mathbf{r}) = 0$

(ii) $\nabla \times (\mathbf{u} \times \mathbf{r}) = 2\mathbf{u}$.

32. Find the acute angle between the surfaces

$xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$.

Answer : $\theta = \cos^{-1}\left(\dfrac{\sqrt{3}}{7\sqrt{2}}\right)$.

33. Find the constants a and b so that the surface

$ax^2 - byz = (a+2)\,x$ will be orthogonal to the surface

$4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

Answer : $a = \frac{5}{2}, b = 1$.

34. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$, then prove that $\mathbf{F} = \dfrac{\mathbf{r}}{r^2}$ is irrotational.

35. Show that $\mathbf{u} = (6xy + z^3)\,\mathbf{i} + (3x^2 - z)\mathbf{j} + (3xz^2 - y)\,\mathbf{k}$ is irrotational. Find φ such that $\mathbf{u} = \nabla\varphi$.

Answer : $\varphi = 3x^2y + xz^3 - yz + \text{constant}$.

36. Show that $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\,\mathbf{j} - (4y^2z^2 + 2x^3z)\,\mathbf{k}$ is not solenoidal but $B = xyz^2\,\mathbf{A}$ is solenoidal.

[R. U. P. 1970]

37. Show that the vector

$$\mathbf{u} = (x+3y)\,\mathbf{i} + (y-3z)\,\mathbf{j} + (x-2z)\,\mathbf{k} \text{ is solenoidal.}$$

38. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$, then show that

$$\nabla^2 r^n = n(n+1)\, r^{n-2}, \text{ where n is a constant.}$$

39. If $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$, *show that* $\nabla^2 (\log r) = \dfrac{1}{r^2}$

40. If $\mathbf{A} = \dfrac{\mathbf{r}}{r}$, *where* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = |\mathbf{r}|$, *find* $\nabla(\nabla.\mathbf{A})$.

Answer : $-\dfrac{2\mathbf{r}}{r^3}$.

41. *If* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = |\mathbf{r}|$, *then show that*

(a) $\nabla^2 \left(\dfrac{1}{r}\right) = 0$

(b) $\nabla \times \left(\dfrac{\mathbf{r}}{r^2}\right) = 0.$

42. *Show that* $\nabla^2 \times (r^2\mathbf{r}) = 0$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = |\mathbf{r}|$.

43. *Find* $\nabla \left(\dfrac{1}{r}\right)$ *where* $r = \sqrt{x^2+y^2+z^2}$.

Answer : $-\dfrac{\mathbf{r}}{r^3}$.

44. *If* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = |\mathbf{r}|$

find (a) ∇r^3 (b) $\nabla\left(r^2 e^{-r}\right)$

Answer : (a) $3r\mathbf{r}$ (b) $(2-r)\, e^{-r}\, \mathbf{r}$.

45 *If* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $r = |\mathbf{r}|$

prove that (a) $\nabla.\left(\dfrac{\mathbf{r}}{r^3}\right) = 0$ (b) $\nabla(r^3\mathbf{r}) = 6r^3$

10.15. VECTOR INTEGRATION

If there is a vector $\mathbf{u} = \mathbf{u}\ (t)$,.where t is a scalar variable then we define the **vector integration** as

$$\int \mathbf{u}\ (t)\ dt = \int [u_1\ (t)\ \mathbf{i} + u_2\ (t)\ \mathbf{j} + u_3\ (t)\ \mathbf{k}]\ dt$$

$$= \mathbf{i} \int u_1\ (t)\ dt + \mathbf{j} \int u_2\ (t)\ dt + \mathbf{k} \int u\ (t)\ dt$$

where **i**, **j**, **k** are the unit vectors in three mutually perpendicular directions, taken as axes of reference.

If there exists a vector $\mathbf{v} = \mathbf{v}\ (t)$ such that

$$\mathbf{u}\ (t) = \frac{d}{dt}\ \{\ \mathbf{v}\ (t)\ \}\ ,\ \textit{then}$$

$$\int \mathbf{u}\ (t)\ dt = \int \frac{d}{dt}\ \{\ \mathbf{v}\ (t)\ \}\ dt = \mathbf{v}\ (t) + \mathbf{c},$$

where **c** is an arbitrary constant vector independent of t.

The **definite integral** between limits t=a and t=b can in such case be written as

$$\int_a^b \mathbf{u}(t)\ dt = \int_a^b \frac{d}{dt}\ \{\mathbf{v}(t)\}\ dt = \Big[\mathrm{v}(t)+\mathrm{c}\Big]_a^b$$

which can also be written as the limit of a sum.

The most important integrals used in vector calculus are the **line integral**, **the surface integral** and **the volume integral.**

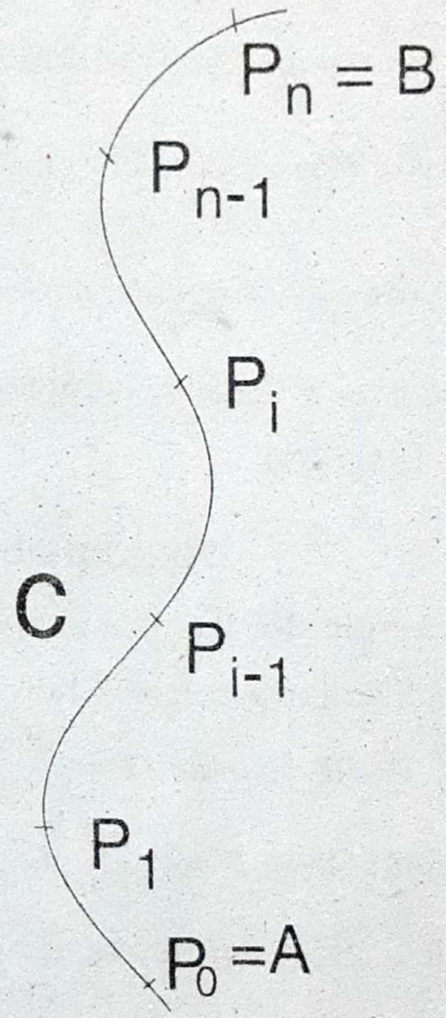
10.16 The line integral.

The integration of a vector along a curve is known as the **line integral.**

Let us consider a continuous vector function $\mathbf{F}(\mathbf{R})$ which is defined at each point of a curve C in space. Divide C into n parts at the points $A=P_o, P_1, \ldots P_{i-1}, P_i, \ldots, P_{n-1}, P_n = B$.

Let their position vectors be $\mathbf{R}_o, \mathbf{R}_1, \ldots, \mathbf{R}_{i-1}, \mathbf{R}_i, \ldots, \mathbf{R}_{n-1}, \mathbf{R}_n$.

Let $\mathbf{u}_i$ be the position vector of any poiont on the arc $P_{i-1} P_i$.



Now consider the sum

$$S = \sum_{i=o}^{n} F(\mathbf{u}_i)\, \delta \mathbf{R}_i$$

where $\delta \mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.

The limit of this sum as $n \to \infty$ in such a way that $|\delta \mathbf{R}_i| = o$, provided it exists, is called the **tangential line integral** of $\mathbf{F}(\mathbf{R})$ along C and symbolically written as

$$\int_C \mathbf{F}(\mathbf{R}).\, d\,\mathbf{R} \text{ or } \int_C \mathbf{F}.\frac{d\mathbf{R}}{dt}dt$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\mathbf{F}(\mathbf{R}) = \mathbf{i}f(x, y, z) + \mathbf{j}\, g(x, y, z) + \mathbf{k}h(x, y, z)$

and $d\mathbf{R} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz$

then $$\int_C F(\mathbf{R}).\, d\mathbf{R} = \int_C (f dx + g dy + h dz).$$

Theorem If $\mathbf{F} = \nabla \varphi$ everywhere in a region R of space defined by $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ and $c_1 \le z \le c_2$ where $\varphi(x, y, z)$ is single valued and has continuous derivatives in R, then

(i) $\int_A^b \mathbf{F} \cdot d\mathbf{R}$ is indepndent of the path C in R joining A and B.

(ii) $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ around any closed curve C in R.

In such case **F** is called a **conservative vector field** and φ is its scalar potential.

WORKED OUT EXAMPLES

Example 26. If C is a simple closed curve in the x–y plane not enclosing the origin, Show that

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = o \text{ where } \mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}.$$

Proof : Given $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ and

$d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j}$ where $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$. Since in the x–y plane $z = o$.

$\therefore \mathbf{F} \cdot d\mathbf{R} = \frac{-ydx + xdy}{x^2 + y^2}$, so that

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \oint_C \left(\frac{xdx-ydx}{x^2+y^2}\right).$$

Let us change the cartesian coordinates to polar coordinates by putting $x = r\cos\theta$ and $y = r\sin\theta$.

then $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

$$\therefore \quad d\theta = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{xdy - ydx}{x^2} = \frac{xdy - ydx}{x^2+y^2}$$

Now since the curve is closed, if there is a point P on it suc that lower limit of θ at P is φ, then its upper limit will also be φ.

$$\oint_C \mathbf{F}.\, d\mathbf{R} = \int_{\varphi}^{\varphi} d\theta = [\theta]_{\varphi}^{\varphi} = 0.$$

Example 27. Calculate the work done when a force $\mathbf{F}=3xy\mathbf{i}-y^2\mathbf{j}$ moves a particle in the x-y plane from (o, o) to $(1, 2)$ along the parabola $y = 2x^2$.

Solution : Since the particle moves in the x-y plane, we have z=o. and so $\mathbf{R}=x\mathbf{i}+y\mathbf{j}$.

Then the required work done = $\oint_C \mathbf{F}.\, d\mathbf{R}$, where C is the parabola $y = 2x^2$.

$$= \oint_C (3xy\mathbf{i} - y^2\mathbf{j}).(dx\mathbf{i} + dy\mathbf{j})$$

$$= \oint_C (3xydx - y^2dy) \quad (1)$$

Substituting $y = 2x^2$, where x goes from 0 to 1, equation (1) becomes

$$\text{Work done} = \int_{x=o}^{x=1} (3x(2x^2)dx - 4x^4.\, 4xdx)$$

$$= \int_{x=0}^{x=1} (6x^3dx - 16x^5dx)$$

$$= 6.\left[\frac{x^4}{4}\right]_0^1 - 16.\left[\frac{x^6}{6}\right]_0^1 = \frac{6}{4} - \frac{16}{6} = \frac{-7}{6}.$$

Example 28. *If* $\mathbf{F}=(2xz^3 + 6y)\mathbf{i} + (6x-2yz)\mathbf{j} + (3x^2z^2 - y^2)\mathbf{k}$ then prove that $\mathbf{F}$ is conservative. Also find the work done by it along the path C from $(1, -1, 1)$ to $(2, 1 - 1)$. **[R. U. H 1967]**

Proof : Ist Portion

If **F** is conservative, then $\nabla \times \mathbf{F} = 0$ *and* **F** can be expressed as the gradient of a scalar function φ i.e $\mathbf{F} = \nabla\varphi$.

$$\text{Now } \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xz^3+6y & 6x-2yz & 3x^2z^2-y^2 \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\delta}{\delta y}(3x^2z^2 - y^2) - \frac{\delta}{\delta z}(6x-2yz)\right\}$$

$$+ \mathbf{j}\left\{\frac{\delta}{\delta z}(2xz^3+6y) - \frac{\delta}{\delta x}(3x^2z^2-y^2)\right\}$$

$$+ \mathbf{k}\left\{\frac{\delta}{\delta x}(6x-2yz) - \frac{\delta}{\delta y}(2xz^3 + 6y)\right\}$$

$$= \mathbf{i}(-2y+2y) + \mathbf{j}(6xz^2-6xz^2) + \mathbf{k}(6-6) = 0.$$

Hence **F** is conservative.

Solution : 2nd portion.

Since **F** is conservative, we have $\mathbf{F} = \nabla\varphi$, *that is,*

$$(2xz^3 + 6y)\mathbf{i} + (6x - 2yz)\mathbf{j} + (3x^2z^2 - y^2)\mathbf{k} = \mathbf{i}\frac{\delta\varphi}{\delta x} + \mathbf{j}\frac{\delta\varphi}{\delta y} + \mathbf{k}\frac{\delta\varphi}{\delta z}$$

$$\therefore \quad \frac{\delta\varphi}{\delta x} = 2xz^3 + 6y, \ \frac{\delta\varphi}{\delta y} = 6x - 2yz, \ \frac{\delta\varphi}{\delta z} = 3x^2z^2 - y^2$$

$$\text{Again, } \mathbf{F}.\, d\mathbf{R} = \nabla\varphi.\, d\mathbf{R} = \frac{\delta\varphi}{\delta y}dx + \frac{\delta\varphi}{\delta y}dy + \frac{\delta\varphi}{\delta z}dz = d\varphi$$

$$\therefore \quad 2xz^3dx + 6ydx + 6xdy - 2yzdy + 3x^2z^2dz - y^2dz = d\varphi$$

$$\text{Or, } (3x^2z^2dz+2xz^3dx)+(6xdy+6ydx)-(2yzdy+y^2dz) = d\varphi$$

$$\text{Or, } d(x^2z^3) + d(xy) - d(y^2z) = d\varphi$$

$$\text{Or } d(x^2z^3 + xy - y^2z) = d\varphi.$$

$$\text{Work done} = \int_C \mathbf{F}.\, d\mathbf{R} = \int_{(1,\,1,\,-1)}^{(2,\,1,\,-1)} d(x^2z^3 + 6xy - y^2z)$$

$$= \left[x^2z^3 + 6xy - y^2z\right]_{(1,\,-1,\,1)}^{(2,\,1,\,-1)}$$

$$= (-4 + 12 + 1) - (1 - 6 - 1) = 9 + 6 = 15.$$

Example 29. If the vector field is given by

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k},$$

evaluate the line integral over a circular path given by

$$x^2 + y^2 = a^2 \quad z = 0.$$

Solution : Since z = o, the given curve (path) lies in the $x - y$ plance. Thus

$$\mathbf{F} = (2x - y)\mathbf{i} + (x + y)\mathbf{j} + (3x - 2y)\,\mathbf{k} \text{ and } d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j}$$

where $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$.

Then the line integral

$$\int_C \mathbf{F}.\,d\mathbf{R} = \int_C \{(2x- y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}\}.\,\{dx\mathbf{i}+dy\mathbf{j}\}$$

$$= \int_C \{(2x - y)\,dx + (x + y)\,dy\}$$

Let the parametric equations of the circular path be $x = a\cos\theta$ y = a sinθ where $0 \le \theta \le 2\pi$ in the positive direction.

Then $$\int_C \mathbf{F}.\,d\mathbf{R} = \int_0^{2\pi} \{(2a\cos\theta - a\sin\theta)d(a\cos\theta)$$

$$+ (a\cos\theta + a\sin\theta)\ \ d(a\sin\theta)\}$$

$$= \int_0^{2\pi} \{(2a\cos\theta - a\sin\theta)\,(-a\sin\theta)\,d\theta + (a\cos\theta + a\sin\theta)\,(a\cos\theta d\theta)\}$$

$$= a^2 \int_0^{2\pi} \{-2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + \sin\theta\cos\theta\}\,d\theta$$

$$= a^2 \int_0^{2\pi} (1 - \sin\theta\cos\theta)\,d\theta$$

$$= a^2 \left[\theta - \frac{\sin^2\theta}{2}\right]_0^{2\pi} = a^2.\,2\pi = 2a^2\pi.$$

Example 30. Given the force $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$, find the work done by the path given by $x = 2t^3$, $y = t^2$ from $(0, 0)$ to $(2, 1)$

Solution : since $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ on the $x - y$ plane.

we have $d\mathbf{R} = d\mathbf{i} + dy\mathbf{j}$,

$$\therefore \int_C \mathbf{F}.\, d\mathbf{R} = \int_C \{(xy\mathbf{i} - y^2\mathbf{j}).\ (dx\mathbf{i} + dy\mathbf{j})\}$$

$$= \int_C (xydx - y^2dy)\ (1)$$

For $x=2t^3$, $y=t^2$, we have $dx = 6t^2dt$, $dy = 2tdt$ and at the origin $t = 0$, and at the point $(2, 1)$ $t = 1$.

Substituting these values in (1) we get work done by the path

$$= \int_C \mathbf{F}.\, d\mathbf{R}$$

$$= \int_{t=o}^{t=1} (2t^3.\ t^2.\ 6t^2dt - t^4.\ 2tdt)$$

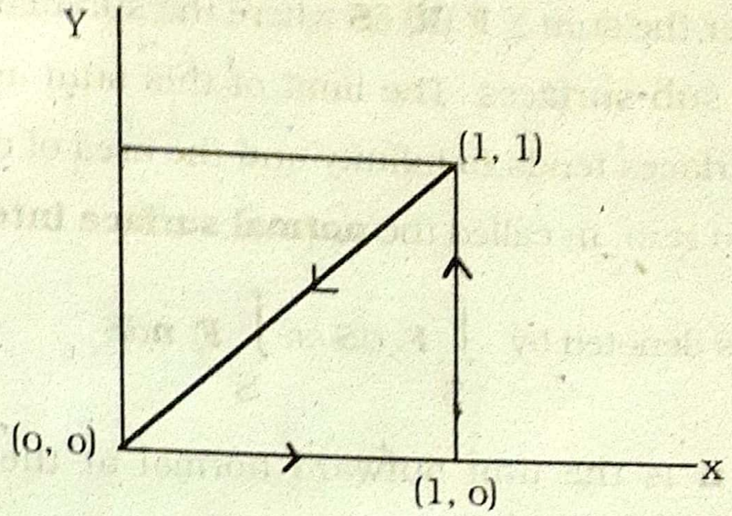
$$= \int_0^1 (12t^7dt - 2t^5dt)$$

$$= 12\left[\frac{t^8}{8}\right]_o^1 - 2\left[\frac{t^6}{6}\right]_o^1$$

$$= \frac{12}{8} - \frac{2}{6} = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}.$$

Example 31. Evaluate $\oint_C (y^2dx + x^2\, dy)$ where C is the triangle with verices $(1, 0)$, $(1,1)$, (o, o).

Solution : Here we have to compute three integrals. The first is the integral from (0, 0) to (1, 0), along this path y=0 and if x is the parameter then dy = o. Hence the first integral is o (zero).



The second integral is that form *(1, o), to (1, 1);* along this path $x = 1$ and if y is used as parameter then this reduces to

$$\int_0^1 dy = [y]_o^1 = 1 \text{ (since } dx = o)$$

For the third integral from *(1, 1) to (o, o)*, we have $y=x$,

$dx = dy$ and x can be used as parameter so that the integral is

$$\int_1^o 2x^2 dx = \frac{2}{3}\left[x^3\right]_1^o = \frac{2}{3}(o-1) = -\frac{2}{3}.$$

Thus finally, we have

$$\oint_C (y^2 dx + x^2 dy) = 0 + 1 - \frac{2}{3} = \frac{1}{3}.$$

10.17 The surface integral

Consider a continuous vector function **F**(**R**) and a two sided surface S. Divide S into a finite number of sub-surfaces. Let the surface element surrounding any point P(**R**) be $\delta\mathbf{S}$ which can be regarded as a vector; its magniude being the area and its direction being the outward normal to the element. Let us consider the sum $\sum \mathbf{F}(\mathbf{R})\,\delta\mathbf{S}$ where the summation extends over all the sub-surfaces. The limit of this sum as the number of sub-surfaces tends to infinity and the area of each sub-surface tends to zero, in called the **normal surface integral** of ***F*(*R*)** over S and is denoted by $\int_S \mathbf{F}.\,d\mathbf{S}$ or $\int_S \mathbf{F}.\,\mathbf{n}dS$

where **n** is the unit outward normal at the point P to the surface S.

These surface integrals are also written as

$$\iint_S \mathbf{F}.d\mathbf{S} \text{ or } \iint_S \mathbf{F}.\mathbf{n}dS.$$

When the surface S is expressed in the form $z=f(x, y)$, (x, y) in the region R(i.e: Rxy); the surface area element dS is given by $dS = \sqrt{1+z_x^2+z_y^2})\, dxdy$ where z_x and z_y are partial derivatives of $z=f(x, y)$ with respect to x and y respectively,

Thus $\iint_S \mathbf{F}.\,\mathbf{n}dS = \iint_R \mathbf{F}.\,\mathbf{n}\sqrt{1+z_x^2+z_y^2}),\, dxdy.$

When the surface S is closed; the notation $\oint\!\!\oint_S$ is used in place of $\iint_S$

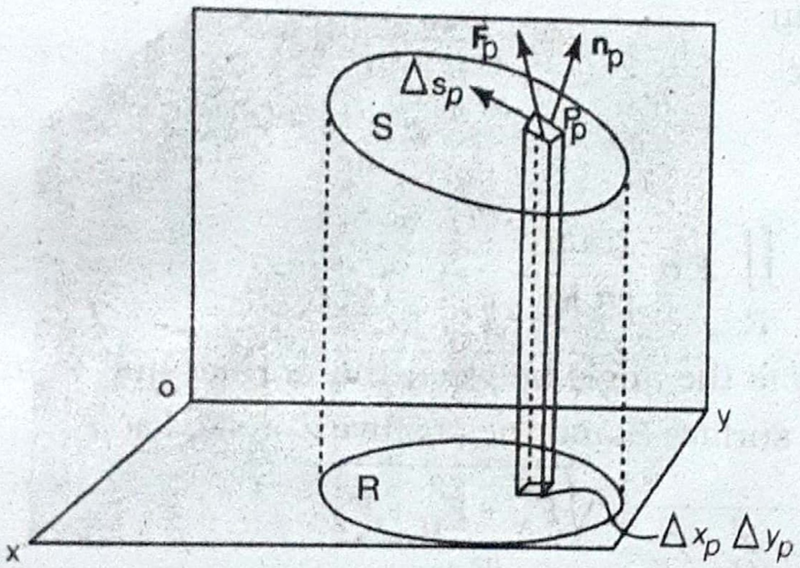
Other types of surface integrals are $\int_S \mathbf{F} \times d\mathbf{S}$ and $\int_S \varphi\, d\mathbf{S}$

where **F** is a continuous vector and φ is a continuous scalar point function. These two integrals are vectors.

Theorem If the surface S has projection R on the x–y plane

then $\iint_S \mathbf{F.n}dS = \iint_R \mathbf{F.n} \dfrac{dxdy}{|\mathbf{n.k}|}$

Proof :



Subdivide the area S into m elements of area ΔS_p where $p = 1, 2, 3, \ldots, m$. Choose any point P_p within ΔS_p whose coordinates are (x_p, y_p, z_p). Define $\mathbf{F}(x_p, y_p, z_p) = \mathbf{F}_p$ Let $\mathbf{n}_p$ be the positive unit normal to ΔS_p at P. Form the sum $\sum_{p=1}^{m} \mathbf{F}_p \cdot \mathbf{n}_p \Delta S_p$ where $\mathbf{F}_p \cdot \mathbf{n}_p$. is the normal component of $\mathbf{F}_p$ at P_p. Now the surface integral $\iint_S \mathbf{F.n}d\,S$ is the limit of the

sum $\sum_{p=1}^{m} \mathbf{F}_p \cdot \mathbf{n}_p \Delta S_p$ (1) when $m \to \infty$, $\Delta S_p \to 0$

The projection of ΔS_p on the x–y plane is

$|(\mathbf{n}_p \Delta S_p).\mathbf{k}|$ or $|\mathbf{n}_p.\mathbf{k}|\Delta S_p$ which is equal to

$\Delta x_p \Delta y_p$ so that $\Delta S_p = \dfrac{\Delta x_p \Delta y_p}{|\mathbf{n}_p . \mathbf{k}|}$ (2)

Thus the sum (1) becomes

$$\sum_{p=1}^{m} \mathbf{F}_p . \mathbf{n}_p \frac{\Delta x_p \Delta y_p}{|\mathbf{n}_p . \mathbf{k}|}$$

Now by the fundamental theorem of integral calculus the limit of this sum as $m \to \infty$ in such a manner that the largest Δx_p and Δy_p approaches zero is

$$\iint_R \mathbf{F}.\mathbf{n} \frac{dxdy}{|\mathbf{n}.\mathbf{k}|}$$

and hence

$$\iint_S \mathbf{F}.\mathbf{n}\, dS = \iint_R \mathbf{F}.\mathbf{n} \frac{dxdy}{|\mathbf{n}.\mathbf{k}|}$$

Theorem If φ is the angle between the normal line to any point (x, y, z) of a surface S and the positive z– axis, then

$$|\sec\varphi| = \sqrt{1 + z_x^2 + z_y^2} = \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}$$

according as equation for S is $z = f(x,y)$

or, $F(x,y,z) = 0$

Proof : If the equation of S is $F(x, y, z) = 0$,

a normal to S at (x, y, z) is $\nabla F = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

Then $\nabla F.\mathbf{k} = |\nabla \mathbf{F}|\,|\mathbf{k}| \cos\varphi$

or, $F_z = \sqrt{F_x^2 + F_y^2 + F_z^2} \cos\varphi$

$$\therefore \quad |\sec\varphi| = \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}$$

When the equation is $z = f(x, y)$, we can write

$F(x,y,z) = z - f(x, y) = 0$, from which, we have $F_x = -z_x$, $F_y = -z_y$, $F_z = 1$.

Thus we find $|sec\varphi| = \dfrac{\sqrt{(-z_x)^2 + (-z_y)^2 + 1^2}}{1}$

or, $|sec\varphi| = \sqrt{1 + z_x^2 + z_y^2}$.

Example 32. Evaluate $\iint\limits_R \sqrt{x^2 + y^2}\, dxdy$.

over the region R in the $x-y$ plane bounded by $x^2 + y^2 = 36$

[R. U. P 1975)

Solution : To cover the whole area of $x^2 + y^2 = 36$ y varies from $-\sqrt{36 - x^2}$ to $+\sqrt{36 - x^2}$ and x varies from -6 to $+6$.

$$\therefore \iint\limits_R \sqrt{x^2 + y^2}\, dxdy = \int_{x=-6}^{+6} \int_{y=-\sqrt{36-x^2}}^{\sqrt{36-x^2}} \sqrt{x^2 + y^2}\, dxdy$$

$$= 4 \int_0^6 \int_0^{\sqrt{16 - x^2}} \sqrt{x^2 + y^2} dxdy.$$

$$= 4 \int_0^6 \left[\frac{y\sqrt{x^2 + y^2}}{2} + \frac{x^2}{2} \log(y + \sqrt{(x^2 + y^2)}) \right]_0^{\sqrt{36 - x^2}} dx$$

$$= 4 \int_0^6 \left[\frac{\sqrt{36 - x^2}\,.6}{2} + \frac{x^2}{2} \log(\sqrt{36 - x^2} + 6) - \frac{x^2}{2} \log x \right] dx.$$

$$= 12 \int_0^6 \sqrt{6^2 - x^2}\, dx + 2\int_0^6 x^2 \log(\sqrt{36 - x^2} + 6)\, dx.$$

$$- 2 \int_0^6 x^2 \log x dx.$$

$$=12\left[\frac{x\sqrt{6^2-x^2}}{2}+\frac{36}{2}\sin^{-1}\frac{x}{6}\right]_0^6$$

$$+2\left[\log(\sqrt{36-x^2}+6).\frac{x^3}{3}\right]_0^6$$

$$-2\int_0^6\frac{1}{\sqrt{36-x^2}+6}.\frac{1}{2\sqrt{36-x^2}}.(-2x).\frac{x^3}{3}dx.$$

$$-2\left[\log x.\frac{x^3}{3}\right]_0^6+2\int_0^6\frac{1}{x}.\frac{x^3}{3}dx.$$

$$=0+216.\frac{\pi}{2}+2\log 6.\frac{216}{2}+\frac{2}{3}\int_0^6\frac{x^4dx}{(\sqrt{36-x^2}+6)\sqrt{36-x^2}}$$

$$-2\log 6\frac{216}{3}+\frac{2}{3}\int_0^6 x^2dx.$$

$$=108\pi+\frac{2}{3}\int_0^6\frac{x^4dx}{(\sqrt{36-x^2}+6)\sqrt{36-x^2}}+\frac{2}{3}\left[\frac{x^3}{3}\right]_0^6$$

$$=108\pi+\frac{2}{3}.\frac{216}{3}+\frac{2}{3}\int_0^6\frac{x^4dx}{(\sqrt{36-x^2}+6)\sqrt{36-x^2}}.$$

Now putting $x=6\sin\theta,\ dx=6\cos\theta d\theta$

$\sqrt{36-x^2}=\sqrt{36-36\sin^2\theta}=6\sqrt{1-\sin^2\theta}=6\cos\theta.$

limits $\left.\begin{matrix}x=0\\ \theta=0\end{matrix}\right\}$ $\left.\begin{matrix}x=6\\ \theta=\frac{\pi}{2}\end{matrix}\right\}$

$$\int_0^6\frac{x^4dx}{(\sqrt{36-x^2}+6)\sqrt{36-x^2}}=\int_0^{\pi/2}\frac{6^4\sin^4\theta}{(6\cos\theta+6)6\cos\theta}.6\cos\theta d\theta$$

$$=\int_0^{\pi/2}\frac{6^3\sin^4\theta}{1+\cos\theta}d\theta$$

$$=216\int_0^{\pi/2}\frac{\sin^4\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}d\theta.$$

$$=216\int_0^{\pi/2}\sin^2\theta(1-\cos\theta)d\theta$$

$$=216\left[\frac{1}{2}\int_0^{\pi/2} 2\sin^2\theta d\theta - \int_0^{\pi/2} \sin^2\theta\cos\theta d\theta\right]$$

$$=216\left[\frac{1}{2}\int_0^{\pi/2} (1-\cos 2\theta)\, d\theta - \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}\right]$$

$$=216\left[\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} - 216.\frac{1}{3}$$

$$= 216.\pi/4 - \frac{216}{3} = 54\pi - \frac{216}{3}.$$

$$\therefore \iint_{\mathbf{R}} \sqrt{x^2+y^2}dxdy = 108\pi + \frac{2}{3}.\frac{216}{3} + \frac{2}{3}.54\pi - \frac{2}{3}.\frac{216}{3}$$

$$= 108\pi + 36\pi = 144\pi.$$

Example 33. Evaluate the surface integral $\int_S (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})\, dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Solution : The sphere $x^2 + y^2 + z^2 = 1$ is parametrically written as $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$ and the part in the first octant corresponds to $0 \le \theta \le \pi$, $0 \le \varphi \le \pi/2$.

Thus the required surface integral

$(yzdydz + zxdzdx + xydxdy$

$$= \int_0^{\pi/2}\int_0^{\pi/2} yz\frac{\delta(y,z)}{\delta(\theta,\varphi)} + zx\frac{\delta(z,x)}{\delta(\theta,\varphi)} + xy\frac{(x,y)}{\delta(\theta,\varphi)})\, d\theta d\varphi$$

$$\text{Now} \quad \frac{\delta(y,z)}{\delta(\theta,\varphi)} = \begin{vmatrix} \frac{\delta y}{\delta\theta} & \frac{\delta y}{\delta\varphi} \\ \frac{\delta z}{\delta\theta} & \frac{\delta z}{\delta\varphi} \end{vmatrix} \cdot \begin{vmatrix} \cos\theta\sin\theta & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{vmatrix}$$

$$= \sin^2\theta\cos\varphi$$

$$\frac{\delta(z,x)}{\delta(\theta,\varphi)} = \begin{vmatrix} \frac{\delta z}{\delta\theta} & \frac{\delta z}{\delta\varphi} \\ \frac{\delta x}{\delta\theta} & \frac{\delta x}{\delta\varphi} \end{vmatrix} \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta\sin\varphi & -\sin\theta\sin\varphi \end{vmatrix}$$

$$= \sin^2\theta \sin\varphi$$

$$\frac{\delta(x,y)}{\delta(\theta,\varphi)} = \begin{vmatrix} \frac{\delta x}{\delta\theta} & \frac{\delta x}{\delta\varphi} \\ \frac{\delta y}{\delta\theta} & \frac{\delta y}{\delta\varphi} \end{vmatrix} \begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \end{vmatrix}$$

$$= \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi$$

$$= \sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi) = \sin\theta\cos\theta.$$

Thus the required integral

$$= \int_0^{\pi/2}\int_0^{\pi/2} [\sin\theta\sin\varphi\cos\theta\sin^2\theta\cos\varphi +$$

$$\cos\theta\sin\theta\cos\varphi\sin^2\theta\sin\varphi + \sin^2\theta\sin\varphi\cos\varphi\sin\theta\cos\theta]\, d\theta d\varphi.$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} 3\sin^3\theta\cos\theta\sin\varphi\cos\varphi\, d\theta d\varphi.$$

$$= 3\int_0^{\pi/2} \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi$$

$$= \frac{3}{4}\int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi$$

$$= \frac{3}{4}\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = \frac{3}{4}\left(\frac{1}{2} - 0\right) = \frac{3}{8}.$$

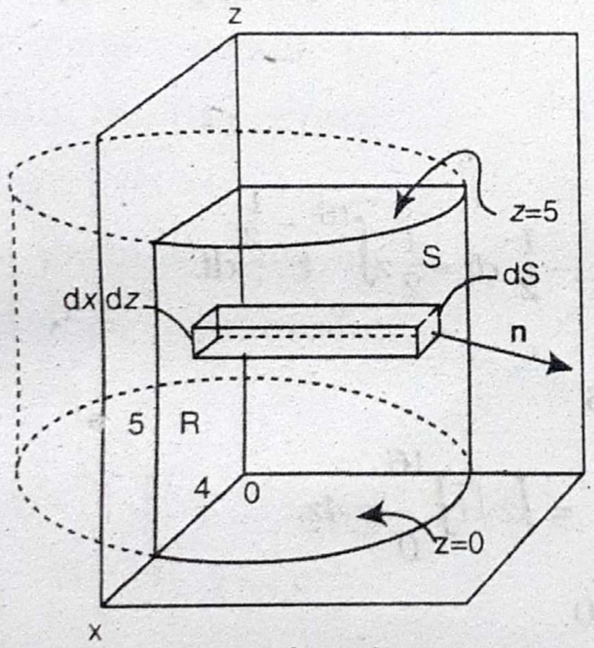
Example 34. If $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - 3y^2z\mathbf{k}$, *evaluate* $\iint_S \mathbf{F}.\,\mathbf{n}dS$. where S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = o$ *and* $z = 5$ and $\mathbf{n}$ is the unit normal to S.

[R. U. H. 1972]

Solution : We know that

$$\iint_S \mathbf{F}.\,\hat{\mathbf{n}}\,dS = \iint_S \mathbf{F}.\,\mathbf{n}\frac{dxdz}{[\mathbf{n}.\,\mathbf{j}]} \quad (1)$$

where R is the projection of S on the x–z plane.



Now a normal to $x^2 + y^2 = 16$ is

$$\nabla (x^2 + y^2) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2)$$

$$= \mathbf{i}\frac{\delta}{\delta x}(x^2 + y^2) + \mathbf{j}\frac{\delta}{\delta y}(x^2 + y^2) + 0 = 2x\mathbf{i} + 2y\mathbf{j}.$$

Thus the unit normal $\mathbf{n}$ to S is given by

$$\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{16}} = \frac{x\mathbf{i} + y\mathbf{j}}{4}$$

$$\mathbf{F}.\,\mathbf{n} = (z\mathbf{i} + x\mathbf{j} - 3y^2z\mathbf{k}).\left(\frac{x\mathbf{i} + y\mathbf{j}}{4}\right) = \frac{1}{4}(zx + xy)$$

$$\text{Also } \mathbf{n}.\,\mathbf{j} = \left(\frac{x\mathbf{i} + y\mathbf{j}}{4}\right).\,\mathbf{j} = \frac{y}{4}.$$

Thus the surface integral

$$\iint_S \mathbf{F}.\,\mathbf{n}dS = \iint_R \frac{zx + xy}{y}\,dxdz = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{zx}{\sqrt{16 - x^2}} + x\right)dxdz. \quad (2)$$

Now for $\int_0^4 \frac{zx}{\sqrt{16 - x^2}}\,dx$, let us put $16 - x^2 = t$.

$$-2xdx = dt. \qquad \textbf{limits} \left.\begin{matrix} x = 0 \\ t = 16 \end{matrix}\right\} \left.\begin{matrix} x = 4 \\ t = 0 \end{matrix}\right\}$$

$$\text{or, } xdx = -\frac{1}{2}dt.$$

$$\therefore \int_0^4 \frac{zx}{\sqrt{16 - x^2}}\,dx = \int_{16}^{0} \frac{z}{\sqrt{t}}.-\frac{1}{2}\,dt \approx \frac{1}{2}z\int_0^{16} t^{-\frac{1}{2}}\,dt.$$

$$= \frac{1}{2}z\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{6} = \left[z\sqrt{t}\right]_0^{16} = 4z.$$

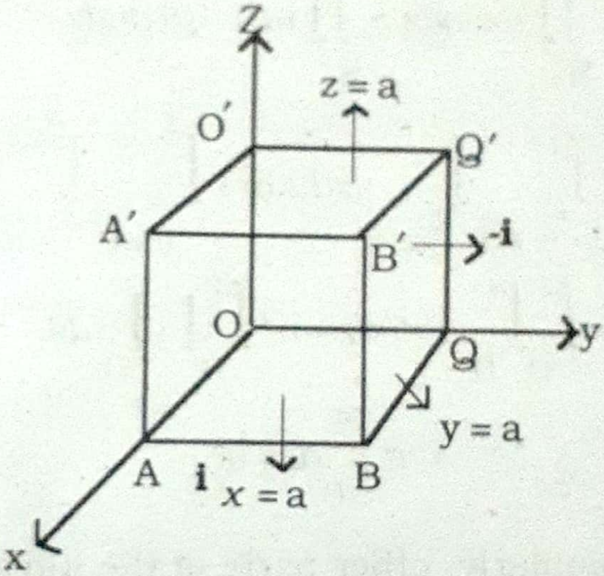
$$\text{Also } \int_0^4 xdx = \left[\frac{x^2}{2}\right]_0^4 = \frac{16}{2} = 8.$$

Therefore, from (2) we have $\iint_S \mathbf{F}.\,\mathbf{n}ds = \int_{z=0}^{5} (4z + 8)dz.$

$$= \left[4.\frac{z^2}{2} + 8z\right]_0^5 = 50 + 40 = 90.$$

Example 35. Evaluate $\iint_S [(x^3 - yz)\, dydz - 2x^2ydzdx + zdxdy]$ over the surface of a cube bounded by the coordinates planes and the plane $x = y = z = a$.

Solution : Here the surface is the cube in the positive octant. To evaluate the given integral let us project the given surface on the three coordinate planes.



Then $\iint_S [(x^3 - yz)dydz - 2x^2ydzdx + zdxdy]$

$$= \iint_S (x^3 - yz)\, dydz + \iint_S (-2x^2y)dzdx + \iint_S zdxdy$$

First let us find out the value of the first integral

$$\iint_S (x^3 - yz)\, dydz.$$

Here the unit normal vector to the face O QQ'O' in the outward direction is $-\mathbf{i}$ and the unit normal vector to the opposite face ABB'A' in the outward direction is $+\mathbf{i}$.

Thus $\iint_S (x^3 - yz)\, dydz$ [for $x = 0$ and $x = a$]

$$= \mathbf{i}.(-\mathbf{i}) \iint_S (x^3 - yz)\, dydz + \mathbf{i}.\mathbf{i} \iint_S (x^3 - yz)\, dydz$$

[for $x = o$] [for $\mathbf{x} = a$]

$$= \iint_S yzdydz + \iint_S (a^3 - yz)\, dydz$$

$$= \int_{y=0}^{a} \int_{z=0}^{a} yzdydz + \int_{y=0}^{a} \int_{z=0}^{a} (a^3 - yz)\, dydz.$$

$$= \int_0^a \int_0^a a^3 dydz = a^3 \int_0^a \Big[\, y \,\Big]_0^a dz.$$

$$= a^4 \int_0^a dz = a^5.$$

Similarly, other parts of the integral are

$$- \iint_S 2x^2 ydzdx \text{ [for } y = 0 \text{ and } y = a]$$

$$= -\left[\mathbf{j}\,(-\mathbf{j}) \iint_S 2x^2 ydzdx + \mathbf{j}.\mathbf{j} \iint_S 2x^2 ydzdx\right]$$

[for $y = 0$] [for $y = a$]

$$= -\left[0 + \iint_S 2ax^2 dzdx\right]$$

$$= -\int_{z=0}^{a} \int_{x=0}^{a} 2ax^2 dzdx$$

$$= -2a \int_{z=0}^{a} \left[\frac{x^3}{3}\right]_0^a dz = -\frac{2a^4}{3} \int_0^a dz = -\frac{2a^5}{3}.$$

and $\iint_S z\,dx\,dy$ [for $z = 0$ and $z = a$]

$$= \mathbf{k}.(-\mathbf{k}) \iint_S z\,dx\,dy + \mathbf{k}.\,\mathbf{k} \iint_S z\,dx\,dy$$

[for $z = 0$] [for $z = a$]

$$= 0 + \int_{x=0}^{a} \int_{y=0}^{a} a\,dx\,dy = \int_0^a a^2 dy = a^3.$$

Therefore, the value of the given integral

$$= a^5 - \frac{2}{3} a^5 + a^3 = \frac{1}{3} a^5 + a^3.$$

Example 36. Show that

$$\iint (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}).\,\mathbf{n}dS = \frac{4}{3}\pi\,(a + b + c)$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{n}$ is the unit normal to S.

Proof : The projection of $x^2 + y^2 + z^2 = 1$ on the plane $z = 0$ is $x^2 + y^2 = 1$.

A normal vector to the surface $x^2 + y^2 + z^2 = 1$

is $\nabla\varphi = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2 + z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$

Thus the unit normal vector along $2\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

is give by $\mathbf{n} = \dfrac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

since $x^2 + y^2 + z^2 = 1.$

$\therefore\ \mathbf{n}.\mathbf{k} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).\,\mathbf{k} = z$

so that $\mathbf{F}.\mathbf{n} = (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}).\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$$= ax^2 + by^2 + cz^2.$$

Now we know that $\iint\limits_S \mathbf{F}.\mathbf{n}\, dS = \iint\limits_R \mathbf{F}.\mathbf{n}\, \frac{dxdy}{[\mathbf{n}.\mathbf{k}]}$

$$\therefore \iint\limits_S (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}).\ \mathbf{n}dS$$

$$= \iint\limits_R (ax^2 + by^2 + cz^2)\frac{dxdy}{z}$$

$$= \iint\limits_R \frac{ax^2 + by^2 + c\,(1-x^2-y^2)}{\sqrt{1-x^2-y^2}}\, dxdy$$

$$= \iint\limits_R \left[\frac{ax^2}{\sqrt{1-x^2-y^2}} + \frac{by^2}{\sqrt{1-x^2-y^2}} + c\sqrt{1-x^2-y^2}\right] dxdy.$$

To cover the whole area of $x^2 + y^2 = 1$

y varies from $-\sqrt{1-x^2}$ to $+\sqrt{1-x^2}$ and

x varies from -1 to $+1$.

Now $\iint\limits_S (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}).\ \mathbf{n}dS$

$$= 2\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \left[\frac{ax^2}{\sqrt{1-x^2-y^2}} + \frac{-\frac{1}{2}by(-2y)}{\sqrt{1-x^2-y^2}} + c\sqrt{1-x^2-y^2}\right] dxdy$$

$$= 8\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{ax^2}{\sqrt{1-x^2-y^2}} + \frac{-\frac{1}{2}by(-2y)}{\sqrt{1-x^2-y^2}} + c\sqrt{1-x^2-y^2}\right] dxdy$$

$$= 8\int_0^1 \left[ax^2 . \sin^{-1} \frac{y}{\sqrt{1-x^2}} - \frac{1}{2} b\{y.2\sqrt{1-x^2-y^2}\} \right]_0^{\sqrt{1-x^2}} dx$$

$$+ 8\int_0^1 \left[\frac{1}{2} b.2\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy + c\int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}} \right] dx$$

$$= 8\int_0^1 \left[ax^2 . \frac{\pi}{2} - 0 \right] dx + 8(b+c)\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy \right] dx$$

$$= 4a\pi \left[\frac{x^3}{3} \right]_0^1 + 8(b+c)\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy \right] dx$$

$$= \frac{4a\pi}{3} + 8(b+c)\int_0^1 \left[\int_0 \frac{\sqrt{1-x^2}}{\sqrt{1-x^2-y^2 dy}} \right] dx.$$

Now for the integral $\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy.$

Putting $\left. \begin{aligned} y &= \sqrt{1-x^2} \sin\theta \\ dy &= \sqrt{1-x^2} \cos\theta d\theta. \end{aligned} \right\}$

limits $\left. \begin{aligned} y &= 0 \\ \theta &= 0 \end{aligned} \right\} \left. \begin{aligned} y &= \sqrt{1-x^2} \\ \theta &= \frac{\pi}{2} \end{aligned} \right\}$

$$\sqrt{1-x^2-y^2} = \sqrt{1-x^2-(1-x^2)\sin^2\theta}$$

$$= \sqrt{(1-x^2)(1-\sin^2\theta)} = \sqrt{1-x^2)}\cos\theta$$

$$\therefore \quad \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} = \int_0^{\pi/2} (1-x^2)\cos^2\theta d\theta$$

$$= (1-x^2)\frac{\pi}{4},$$

Therefore, $\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy \right] dx$.

$$= \int_0^1 (1-x^2)\frac{\pi}{4}\,dx = \frac{\pi}{4}\left[x - \frac{x^2}{3}\right]_0^1$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{3}\right) = \frac{\pi}{4}\cdot\frac{2}{3} = \frac{\pi}{6}.$$

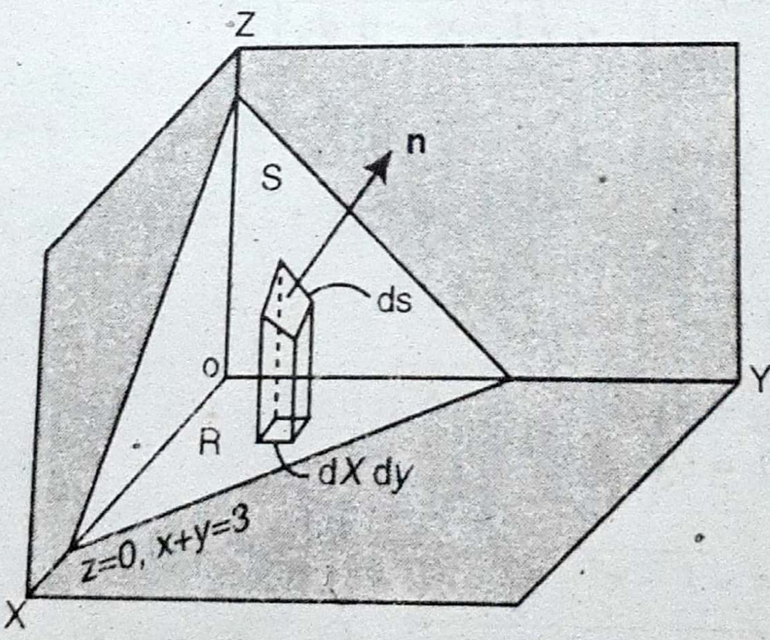
Thus $\iint_S (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}).\, \mathbf{n}dS.$

$$= \frac{4a\pi}{3} + 8(b+c)\frac{\pi}{6}$$

$$= \frac{4a\pi}{3} + \frac{4(b+c)\pi}{3} = \frac{4\pi}{3}(a+b+c).$$

Example 37. If $\mathbf{F} = xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}$, *evaluate* $\iint \mathbf{F}.\, \mathbf{n}dS.$ where S is that portion of the plane $2x + 2y + z = 6$ included in the first octant and $\mathbf{n}$ is the unit normal to S.

Solution : A normal to S is



$\nabla(2x + 2y + z - 6)$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(2x + 2y + z - 6)$$

$$= 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Therefore, unit normal $\boldsymbol{n} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}$

Then $\mathbf{F}.\ \boldsymbol{n} = \{xy\mathbf{i} - x^2\mathbf{j} + (x + z)\,\mathbf{k}\}.\left(\frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}\right)$

$$= \frac{2xy - 2x^2 + x + z}{3} \quad \{\text{since } z = 6 - 2x - 2y$$

$$= \frac{2xy - 2x^2 + x + 6 - 2x - 2y}{3}$$

$$= \frac{2xy - 2x^2 - x - 2y + 6}{3}.$$

Thus the required surface integral is

$$\iint_S \left(\frac{2xy - 2x^2 - x - 2y + 6}{3}\right) dS$$

$$= \iint_R \left(\frac{2xy - 2x^2 - x - 2y + 6}{3}\right) \sqrt{1 + z_x^2 + z_y^2}\; dxdy$$

since $dS = \sqrt{1 + z_x^2 + z_y^2}\; dxdy.$

Now $z = 6 - 2x - 2y$

$z_x = -2$ and $z_y = -2$

$\therefore \quad \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + (-2)^2 + (-2)^2} = 3$

$\therefore \quad$ Surface integral $\iint_S \mathbf{F}.\ \boldsymbol{n}\, dS.$

$$= \int_{x=0}^{3} \int_{y=0}^{3-x} (2xy - 2x^2 - x - 2y + 6)\; dxdy$$

$$= \int_{x=0}^{3} \left[2x\left[\frac{y^2}{2}\right]_0^{3-x} - (2x^2 + x)\left[y\right]_0^{3-x} - 2\left[\frac{y^2}{2}\right]_0^{3-x} + 6\left[y\right]_0^{3-x} \right] dx$$

$$= \int_0^3 \{x(3-x)^2 - (2x^2 + x)(3 - x) - (3 - x)^2 + 6(3 - x)\}\; dx.$$

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$$= \int_0^3 (9x - 6x^2 + x^3 - 6x^2 - 3x + 2x^3 + x^2 - 9 + 6x - x^2$$

$$+ 18 - 6x)dx.$$

$$= \int_0^3 (3x^3 - 12x^2 + 6x + 9)dx.$$

$$= 3\left[\frac{x^4}{4}\right]_0^3 - 12\left[\frac{x^3}{3}\right]_0^3 + 6\left[\frac{x^2}{2}\right]_0^3 + 9\left[x\right]_0^3$$

$$= \frac{3}{4}.\ 81 - 12.\frac{27}{3} + 6.\frac{9}{2} + 27$$

$$= \frac{243}{4} - 4.27 + 27 + 27$$

$$= \frac{9.27}{4} - 2.\ 27 = \frac{(9-8)\ 27}{4} = \frac{27}{4}.$$

10.18. The volume integral.

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ and a closed surface S in space enclosing a region V, Let us subdivide V into a finite a number of sub-regions $V_i, V_2, \ldots, V_n$. Let δV_i be the volume of the sub-region V_i enclosing any point whose position vector is $\mathbf{R}_i$.

Consider the sum $\sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\ \delta V_i$ where the summation extends over all the sub-regions. The limit of this sum if it exists when the number of sub-regions tends to infinity and the volume of each sub-region tends to zero is called the **volume integral** or **space integral** of $\mathbf{F}(\mathbf{R})$ over V and is symbolically written as

$$\int_V \mathbf{F}(\mathbf{R})\, dV \quad \text{or simply} \int_V \mathbf{F} dV \text{ or } \iiint_V \mathbf{F} dV$$

Introducing rectangular cartesian coordinates we may see that if $\mathbf{F}(\mathbf{R}) = \mathbf{i}\, f(x, y, z) + \mathbf{j}\, g(x, y, z) + \mathbf{k}\, h(x, y, z)$ so that $\delta V = \delta x \delta y\, \delta z$, then

$$\int_V \mathbf{F} dV = \mathbf{i} \iiint_V f dx dy dz + \mathbf{j} \iiint_V g dx dy dz + \mathbf{k} \iiint_V h dx dy dz.$$

If φ is a scalar point function in V then $\iiint_V \varphi dV$ *is also* known as **volume integral** or **space integral.**

Example 38. Let $\varphi = y^2 z$ and let V denote the region bounded by the plane $x + 4y + 2z = 4$, $x = 0$, $z = 0$; then evaluate $\iiint_V \varphi dV$.

Solution : Keeping y, z constant, let us first integrate φ from $x = 0$ to $x = 4 - 4y - 2z$. Now in the $y - z$ plane $x = 0$, so $x + 4y + 2z = 4$ becomes $4y + 2z = 4$ i. e $z = 2 - 2y$.

Let us take y as constant and integrate with respect to z from $z = 0$ to $z = 2 - 2y$.

Also finally we have to integrate with respect to y from $y = 0$ to $y = 1$.

$$\text{Hence } \iiint_V \varphi dV = \int_{y=0}^{1} \int_{z=0}^{2-2y} \int_{x=0}^{4-4y-2z} y^2 z dx dz dy$$

$$= \int_{y=0}^{1} \int_{z=0}^{2-2y} \Big[y^2 z x \Big]_0^{4-4y-2z} dz dy$$

$$= \int_0^1 \int_0^{2-2y} y^2 z(4 - 4y - 2z)\,dz\,dy$$

$$= \int_0^1 \int_0^{2-2y} (4y^2 z - 4y^3 z - 2y^2 z^2)\,dz\,dy$$

$$= \int_0^1 \left[\frac{4y^2z^2}{2} - \frac{4y^3z^2}{2} - \frac{2y^2z^3}{3} \right]_0^{2-2y} dy$$

$$= \int_0^1 \left[2y^2 (2 - 2y)^2 - 2y^3(2-2y)^2 - \frac{2}{3} y^2(2 - 2y)^3 \right] dy$$

$$= \int_0^1 \left[2y^2(4 - 8y + 4y^2) - 2y^3(4 - 8y + 4y^2) \right.$$

$$\left. - \frac{2}{3} y^2 (8 - 24y + 24y^2 - 8y^3) \right] dy$$

$$= \int_0^1 [8y^2 - 16y^3 + 8y^4 - 8y^3 + 16y^4 - 8y^5 - \frac{16}{3} y^2$$

$$+ 16y^3 - 16y^4 + \frac{16}{3} y^5]dy$$

$$= \int_0^1 \left(\frac{8}{3} y^2 - 8y^3 + 8y^4 - \frac{8}{3} y^5 \right) dy$$

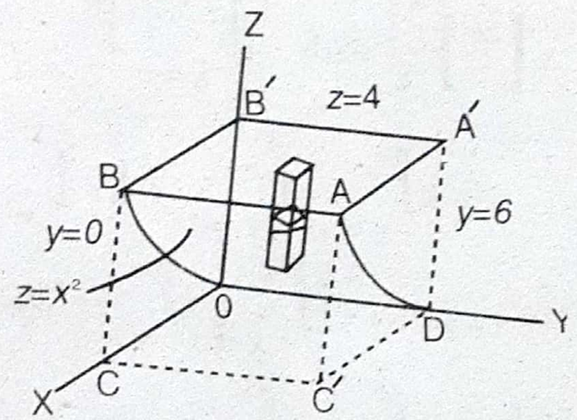
$$= \left[-\frac{8}{3} \cdot \frac{y^3}{3} - 8 . \frac{y^4}{4} + 8 . \frac{y^5}{5} - \frac{8}{3} \cdot \frac{y^6}{6} \right]_0^1$$

$$= \frac{8}{9} - 2 + \frac{8}{5} - \frac{4}{9}$$

$$= \frac{40 - 90 + 72 - 20}{45} = \frac{-50 + 52}{45} = \frac{2}{45}.$$

Example 39. Evaluate $\iiint_V \mathbf{F}\, dV$ where $\mathbf{F} = 2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}$ and V is the region bounded by the surface $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$.

Solution : The given solid is a parabolic cylinder with its axis parallel to y–axis. The part of the volume to be determined is shown in the adjacent figure.



If we subdivide the given volume into a large number of cubes and consider an elementary cube of volume δV, then the required integral is

$$\iiint_V F\, dV = \iiint_V (2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k})\, dx dy dz$$

$$= \mathbf{i}\iiint_V 2xz dx dy dz - \mathbf{j}\iiint_V x dx dy dz + \mathbf{k}\iiint_V y^2 dx dy dz.$$

Now to cover the whole volume, x varies from O to the line in which $z = x^2$ meets the plane $z = 4$ i. e x varies from O to $x^2 = 4$ or $x = 2$. Also y varies from O to 6 and z varies from x^2 to 4. Thus we have

$$\iiint_V \mathbf{F}dV = 2\mathbf{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} xz\,dx\,dy\,dz$$

$$-\mathbf{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz$$

$$+\mathbf{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz.$$

$$= 2\mathbf{i}\int_{x=0}^{2}\int_{y=0}^{6} x\left[\frac{z^2}{2}\right]_{x^2}^{4} dxdy - \mathbf{j}\int_{x=0}^{2}\int_{y=0}^{6} x[z]_{x^2}^{4}\,dxdy$$

$$+\mathbf{k}\int_{x=0}^{2}\int_{y=0}^{6} y[z]_{x^2}^{4}\,dxdy$$

$$= \mathbf{i}\int_{x=0}^{2}\int_{y=0}^{6} x(16-x^4)dxdy - \mathbf{j}\int_{x=0}^{2}\int_{y=0}^{6} x(4-x^3)\,dxdy$$

$$+\mathbf{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(4-x^2)dxdy$$

$$= \mathbf{i}\int_{0}^{2}\int_{0}^{6}(16x-x^5)dxdy - \mathbf{j}\int_{0}^{2}\int_{0}^{6}(4x-x^2)\,dxdy$$

$$+\mathbf{k}\int_{0}^{2}\int_{0}^{6}(4-x^2)y^2\,dxdy$$

$$= \mathbf{i}\left[8x^2-\frac{x^6}{6}\right]_0^2[y]_0^6 - \mathbf{j}\left[2x^2-\frac{x^4}{4}\right]_0^2[y]_0^6$$

$$+\mathbf{k}\left[4x-\frac{x^3}{3}\right]_0^2\left[\frac{y^3}{3}\right]_0^6$$

$$= \mathbf{i}\left(32-\frac{64}{6}\right)(6) - \mathbf{j}\left(8-\frac{16}{4}\right)(6) + k\left(8-\frac{8}{3}\right)(72)$$

$$= \mathbf{i}(192-64) - \mathbf{j}(24) + \mathbf{k}(576-192)$$

$$= 128\mathbf{i} - 24\mathbf{j} + 384\mathbf{k}.$$

Example 40. If $F = \frac{1}{3}(x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k})$, *evaluate* $\iiint\limits_V \nabla.\mathbf{F}\, dV$

where V is the closed region bounded by the planes

$x + y + z = a,\ x = 0,\ y = 0,\ z = 0.$

Solution : Given $\mathbf{F} = \frac{1}{3}(x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}) = \frac{1}{3}x^3\mathbf{i} + \frac{1}{3}y^3\mathbf{j} + \frac{1}{3}z^3\mathbf{k}$

$$\nabla.\mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) . \left(\frac{1}{3}x^3\mathbf{i} + \frac{1}{2}y^3\mathbf{j} + \frac{1}{3}z^3k\right)$$

$$= \frac{\delta}{\delta x}\left(\frac{1}{3}x^3\right) + \frac{\delta}{\delta y}\left(\frac{1}{3}y^3\right) + \frac{\delta}{\delta z}\left(\frac{1}{3}z^3\right) = x^2 + y^2 + z^2.$$

Here z varies from 0 to $a - x - y$, y varies from 0 to $a-x$ and x varies from 0 to a.

Therefore, $\iiint\limits_V (\nabla.\ \mathbf{F})dV = \iiint\limits_V (x^2 + y^2 + z^2)dxdydz$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} (x^2 + y^2 + z^2)\, dzdydx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[(x^2 + y^2)\,[z]_0^{a-x-y} + \left[\frac{z^3}{3}\right]_0^{a-x-y}\right] dydx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[(x^2 + y^2)(a - x - y) + \frac{1}{3}(a - x - y)^3\right] dydx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} [x^2(a - x) - x^2y + (a - x)y^2 - y^3 + \frac{1}{3}(a - x - y)^3]\, dydx$$

$$= \int_{x=0}^{a}\left[x^2(a - x)\,[y]_0^{a-x} - x^2\left[\frac{y^2}{2}\right]_0^{a-x} + \frac{a-x}{3}[y^3]_0^{a-x}\right.$$

$$\left. - \frac{1}{4}\left[y^4\right]_0^{a-x} - \frac{1}{12}\left[(a - x - y)^4\right]_0^{a-x}\right] dx$$

$$= \int_{x=0}^{a} \left[x^2 (a-x)^2 - \frac{1}{2} x^2 (a-x)^2 + \frac{1}{3} (a-x)^4 - \frac{1}{4} (a-x)^4 + \frac{1}{12} (a-x)^4 \right] dx$$

$$= \int_0^a \left[\frac{1}{2} x^2 (a-x)^2 + \frac{1}{6} (a-x)^4 \right] dx$$

$$= \frac{1}{2} \int_0^a (a^2x^2 - 2ax^3 + x^4) dx + \frac{1}{6} \int_0^a (a-x)^4 \, dx.$$

$$= \frac{1}{2} \left[\left[a^2 \frac{x^3}{3} \right]_0^a - 2a \left[\frac{x^4}{4} \right]_0^a + \left[\frac{x^5}{5} \right]_0^a \right] - \frac{1}{6} \left[\frac{(a-x)^5}{5} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{a^5}{3} - \frac{1}{2} a^5 + \frac{1}{5} a^5 \right] + \frac{1}{30} a^5$$

$$= \frac{10a^5 - 15a^5 + 6a^5}{60} + \frac{a^5}{30} = \frac{a^5}{60} + \frac{a^5}{30} = \frac{3a^5}{60} = \frac{a^5}{20}.$$

Example 41. If $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$,

evaluate $\iiint_V (\nabla \times F) \, dV$ *where* V is the closed region bounded by $x = 0$, $y = 0$, $z = 0$ *and* $2x + 2y + z = 4$.

Solution : Given $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$

$$(\nabla \times \mathbf{F}) = \left(\mathbf{i} \frac{\delta}{\delta x} + \mathbf{j} \frac{\delta}{\delta y} + \mathbf{k} \frac{\delta}{\delta z} \right) \times \{(2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}\}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \mathbf{i} \left\{ \frac{\delta}{\delta y} (-4x) + \frac{\delta}{\delta z} (2xy) \right\} + \mathbf{j} \left\{ \frac{\delta}{\delta z} (2x^2 - 3z) - \frac{\delta}{\delta x} (-4x) \right\}$$

$$+ \mathbf{k} \left\{ \frac{\delta}{\delta x} (-2xy) - \frac{\delta}{\delta y} (2x^2 - 3z) \right\}$$

$$= \mathbf{i}\,(0) + \mathbf{j}(-3 + 4) + \mathbf{k}(-2y - 0) = \mathbf{j} - 2y\mathbf{k}.$$

Now z varies 0 to $4-2x-2y$,

y varies from 0 to $2-x$ and

x varies from 0 to 2.

Therefore, $\iiint_V (\nabla \times F)dV = \iiint_V (\mathbf{j}-2y\mathbf{k})dxdydz$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}(\mathbf{j}-2y\mathbf{k})dzdydx.$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}(\mathbf{j}-2y\mathbf{k})\Big[\ z\ \Big]_{0}^{4-2x-2y}dydx$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}(\mathbf{j}-2y\mathbf{k})(4-2x-2y)dydx$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}[\mathbf{j}(4-2x-2y\mathbf{j}-2(4-2x)y\mathbf{k}+4y^2\mathbf{k}]dydx$$

$$=\int_0^2\Big[\ \mathbf{j}(4-2x)[y]_0^{2-x}-\mathbf{j}[y^2]_0^{2-x}-\mathbf{k}(4-2x)[y^2]_0^{2-x}+\mathbf{k}\frac{4}{3}[y^3]_0^{2-x}\Big]dx$$

$$=\int_0^2\Big[\ \mathbf{j}\ \{2(2-x)^2-(2-x)^2+\mathbf{k}\ \Big\{\frac{4}{3}(2-x)^3-2(2-x)^3\Big\}\Big]\ dx$$

$$=\int_0^2[\mathbf{j}(2-x)^2-\mathbf{k}\frac{2}{3}(2-x)^3]dx$$

$$=-\frac{1}{3}\mathbf{j}\Big[\ (2-x)^3\Big]_0^2+\frac{2}{12}\mathbf{k}\Big[\ (2-x)^4\Big]_0^2$$

$$=\frac{2^3}{3}\mathbf{j}-\frac{1}{6}\mathbf{k}.2^4$$

$$=\frac{8}{3}\mathbf{j}-\frac{8}{3}\mathbf{k}=\frac{8}{3}(\mathbf{j}-\mathbf{k}).$$

Example 42. Evaluate $\iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + x^2y^2\mathbf{k}).\, \mathbf{n}dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the x–y plane and **n** is the unit normal to S.

Solution : A normal to the sphere $x^2 + y^2 + z^2 = a^2$

is $\nabla(x^2 + y^2 + z^2 - a^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$.

Then unit mormal to S is

$$n = \frac{2xi + 2yi + 2zk}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} \qquad = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$$

$$= \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}.$$

Therefore, $(y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + x^2y^2\mathbf{k}).\mathbf{n}$

$$= (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + x^2y^2\mathbf{k}).\left(\frac{x}{a}\mathbf{i} + \frac{y}{a}j + \frac{z}{a}\mathbf{k}\right)$$

$$= \frac{1}{a}(y^2z^2x + z^2x^2y + x^2y^2z)$$

Now $$\iint_S F.\,\mathbf{n}\,dS = \iint_R F.\mathbf{n}\frac{dxdy}{|\mathbf{n.k}|}$$

$$= \frac{1}{a}\iint_R \frac{y^2z^2x + z^2x^2y + x^2y^2z}{\frac{z}{a}}\, dxdy \text{ ,Since } \mathbf{n.k} = \frac{z}{a}$$

$$= \iint_R (y^2zx + x^2yz + x^2y^2)\, dxdy$$

Let the projection of S on the x –y plane be R which is bounded by the circle $x^2 + y^2 = a^2$, $z = o$.

Then $\iint_S \mathbf{F}.\mathbf{n}dS = \iint_R \left(xy^2\sqrt{a^2 - x^2 - y^2} + x^2y\sqrt{a^2 - x^2 - y^2} + x^2y^2\right) dxdy$

$$= \iint_R (xy^2 + x^2y)\sqrt{a^2 - x^2 - y^2}\, dxdy + \iint_R x^2y^2\, dxdy$$

$$= \int_o^a \int_o^{\sqrt{a^2 - x^2}} (xy^2 + x^2y)\sqrt{a^2 - x^2 - y^2}\, dxdy + \int_o^a \int_o^{\sqrt{a^2 - x^2}} x^2y^2 dxdy.$$

$$\text{Let } I_1 = \int_o^a \int_o^{\sqrt{a^2 - x^2}} (xy^2 + x^2y)\sqrt{a^2 - x^2 - y^2}\, dxdy.$$

Putting $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\}$ $x^2 + y^2 = r^2$ and $y/x = \tan\theta$.

Also $dxdy = rdrd\theta$.

$$xy^2 + x^2y = r\cos\theta.\, r^2\sin^2\theta + r^2\cos^2\theta.\, r\sin\theta$$

$$= r^3\cos\theta\sin^2\theta + r^3\cos^2\theta.\, \sin\theta$$

$$= r^3(\cos\theta\sin^2\theta + \cos^2\theta\sin\theta).$$

$$\sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2}$$

limits $\left.\begin{matrix} \theta = o \\ \theta = \pi/2 \end{matrix}\right\}$ and $\left.\begin{matrix} r = o \\ r = a \end{matrix}\right\}$

Thus $I_1 = \int_0^{\pi/2} (\cos\theta \sin^2\theta + \cos^2\theta \sin\theta)\, d\theta . \int_0^a r^4 \sqrt{a^2 - r^2}\, dr$

(i) $\int_0^{\pi/2} \cos\theta \sin^2\theta d\theta + \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta$

$$= \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} - \left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$= \frac{1}{3}(1-0) - \frac{1}{3} + (0-1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

(ii) $\int_0^a r^4 \sqrt{a^2 - r^2}\, dr$ putting $r = a \sin\varphi$

$\therefore dr = a \cos\varphi d\varphi$ and

limits $\left.\begin{matrix} r = o \\ \varphi = o \end{matrix}\right\}$ $\left.\begin{matrix} r = a \\ \varphi = \frac{\pi}{2} \end{matrix}\right\}$ $\sqrt{a^2 - r^2} = \sqrt{a^2 - a^2 \sin^2\varphi}$

$= a\cos\varphi.$

Then $\int_0^a r^4 \sqrt{a^2 - r^2}\, dr$

$$= \int_0^{\pi/2} a^4 \sin^4\varphi \, a \cos\varphi . \, a\cos\varphi d\varphi$$

$$= a^6 \int_0^{\pi/2} \sin^4\varphi \cos^2\varphi \, d\varphi$$

$$= a^6 \int_0^{\pi/2} \sin^4\varphi \cos^2\varphi \, d\varphi$$

$$= a^6 \frac{\Gamma\left(\frac{4+1}{2}\right) \Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)} = a^6 \frac{\Gamma\left(\frac{5}{2}\right) . \Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$$

$$= a^6 . \frac{\frac{3}{2} . \frac{1}{2} \Gamma\left(\frac{1}{2}\right) . \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{2. \, 3. \, 2. \, 1.}$$

$$= a^6 . \frac{1}{2^5} \pi = \frac{\pi a^6}{32}.$$

Thus $I_1 = \int_0^a \int_0^{\sqrt{a^2-x^2}} (xy^2 + x^2y)\sqrt{a^2-x^2-y^2}\,dxdy$

$$= \frac{2}{3} \times \frac{\pi a^6}{32} = \frac{2\pi a^6}{96}.$$

Again let $I_2 = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2y^2 dxdy$

$$= \int_0^a x^2 . \left[\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{3}\int_0^a x^2 \left(\sqrt{a^2-x^2}\right)^3 dx.$$

Putting $x = a \sin\theta$, then $dx = a\cos\theta\, d\theta$ and

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

limits $\left.\begin{matrix} x = o \\ \theta = o \end{matrix}\right\}$ $\left.\begin{matrix} x = a \\ \theta = \pi/2 \end{matrix}\right\}$ $= a\cos\theta$

$$\therefore I_2 = \frac{1}{3}\int_0^{\pi/2} a^2\sin^2\theta .\, a^3\cos^3\theta\, a\cos\theta\, d\theta$$

$$= \frac{a^6}{3}\int \cos^4 \sin^2\theta d\theta$$

$$= \frac{a^6}{3} . \frac{\left\lceil\left(\frac{4+1}{2}\right) \left\lceil\left(\frac{2+1}{2}\right)\right.\right.}{2\left\lceil\frac{4+2+2}{2}\right.} \quad \left\{ \left\lceil\frac{1}{2}\right. = \sqrt{\pi} \right.$$

$$= \frac{a^6}{3} . \frac{\left\lceil\left(\frac{5}{2}\right) . \left\lceil\left(\frac{3}{2}\right)\right.\right.}{2\lceil 4} = \frac{\frac{a^6}{3} . \frac{3}{2} . \frac{1}{2}\left\lceil\left(\frac{1}{2}\right) \frac{1}{2}\left\lceil\left(\frac{1}{2}\right)\right.\right.}{2.\ 3.\ 2.\ 1,}$$

$$= \frac{\pi a^6}{96}.$$

$$\text{Therefore, } \iint_S \mathbf{F}.\,\mathbf{n}dS = \frac{2\pi a^6}{96} + \frac{\pi a^6}{96} + \frac{3\pi a^6}{96} = \frac{\pi a^6}{32};$$

$$\text{Hence } \iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + x^2y^2\mathbf{k}).\,\mathbf{n}dS = \frac{\pi a^6}{32}.$$

EXERCISES 10(C)

1. Evaluate the following line integrals :

(a) $\oint_C (y^2dx + xydy)$, where C is the square with vertices $(1,1), (-1,1), (-1,-1), (1,-1)$;

(b) $\oint_C (ydx - xdy)$, where C is the circle $x^2 + y^2 = 1$;

(c) $\oint_C (x^2y^2dx - xy^3dy)$, where C is the triangle with vertices $(0,0), (1,0), (1,1)$.

Answer : (a) 0 (b) -2π (c) $-1/4$.

2. Evaluate the line integral

$I = \int_C \{x^2ydx + (x-z)dy + xyzdz\}$ *where* C is the arc of the parabola $y = x^2$ *in the plane* $z = 2$ *from* $(0,0,2)$ *to* $(1,1,2)$.

Answer : $-\dfrac{17}{15}$.

3. Evalute the line integral

$\oint_C \{(x^2 + xy)dx + (x^2 + y^3)dy\}$

where C is the square formed by the lines $y = \pm 1$ *and* $x = \pm 1$

Answer : 0.

4. Evaluate the line integral

$\int_C (3x^2y^2dx + 2x^3ydy)$ in the positive direction around C.

where C is the ellipse $x^2 + 4y^2 = 4$.

Answer : 0

5. Let C be the graph of $x^2 + y^2 = a^2$.

Evaluate each of the following curvilinear integrals :

(i) $\oint_C (ydx + xdy)$ (ii) $\oint_C (ydx - xdy)$.

Answer : (i) 0 (ii) $-2\pi a^2$

6. Evaluate $\int_C [(x - y^2)\,dx + 2xydy]$ from (0,0) to (1,1)

(i) where C is an arc of the graph of $y = x$

(ii) where C is an arc of the graph of $y = x^2$.

Answer : (i) $\frac{5}{6}$ (ii) $\frac{11}{10}$

7. Evaluate $\oint_C (4ydx + 2xdy)$ where C is the graph of

$x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$

Answer : -2π

8. Evaluate $\oint_C (3ydx + xdy)$ where C is the boundary of the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).

Answer : -2.

9. Find the work done in moving a particle once in the positive direction around an ellipse C given by

$9x^2 + 16y^2 = 144$ in the $x - y$ plane and the force field is

$\mathbf{F} = (3x - 4y + 2z)\mathbf{i} + (4x + 2y - 3y^2)\mathbf{j} + (2xz - 4y^2 + z^3)\mathbf{k}$.

Answer : 96π.

10. If $\mathbf{F} = (5xy - 6x^2)\mathbf{i} + (2y - 4x)\mathbf{j}$, *find*

$\int_C \mathbf{F} \cdot d\mathbf{R}$ *along the curve C given by* $y = x^2$ *in the* $x - y$ plane

from the point $(1,1)$ *to* $(2, 8)$.

Answer : 35.

11. If $\mathbf{F} = (x^2 - y)\mathbf{i} + (y^2 - z)\mathbf{j} + (z^2 - x)\mathbf{k}$, evaluate

$\int_C F. dR$ along the path C given by $x = t$, $y = t^2$, $z = t^3$ from

$t = 0$ *to* $t = 1$.

Answer : $\dfrac{-29}{60}$.

12. If $\mathbf{F} = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}$.

evaluate $\int_C \mathbf{F}. d\mathbf{R}$ *along the path C given by*

$x = t, y = t^2, z = t^3$ *from* (o, o, o) *to* $(1, 1, 1)$.

Answer : 2

13. If $\mathbf{F} = (2y + 3)\mathbf{i} + xz\mathbf{j} + (yz - x)\mathbf{k}$, *evaluate*

$\int_C \mathbf{F}.d\mathbf{R}$ *along the path C given by* $x = 2t^2$, $y = t$, $z = t^2$ *from*

$t = o$ to $t = 1$

Answer : $\dfrac{288}{35}$

14. Evaluate $\iint_S (x^2 + y^2)\, dS$ where S is the surface of the

cone $z^2 = 3(x^2 + y^2)$ *bounded by* $z = 0$ *and* $z = 3$.

Answer : 9π.

15. Find the value of $\iint_S (xdydz + ydzdx + zdxdy)$ where S is the surface of the region bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3$.

Answer : 81π.

16. Evaluate $\iint_S \mathbf{F}.\,d\mathbf{S}$ over the faces of a cube given by $o \le x, y, z \le 1$ and $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

Answer : 3.

17. If $\mathbf{F} = 18\,z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$, evaluate $\iint_S \mathbf{F}.\mathbf{n}dS$ where S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant and **n** in the unit normal to S.

Answer : 24.

18. If $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, evaluate $\iint_S \mathbf{F}.\mathbf{n}dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ and **n** is the unit normal to S.

Answer : $3\pi a^4$.

19. If $\mathbf{F} = (x^2 - yz)\,\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}$, *prove that* $\iint_S \mathbf{F}.\mathbf{n}dS = \frac{1}{3}$,

where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ and $\mathbf{n}$ is the unit normal to S.

20. Evaluate $\iint_S (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}).\,\mathbf{n}dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{n}$ is the unit normal to S.

Answer : $\frac{12}{5}\pi$.

21. Find the area cut from the upper half of the sphere $x^2 + y^2 + z^2 = 1$ *by the cylinder* $x^2 + y^2 - y = 0$.

Answer : $\pi - 2$.

22. If $\mathbf{F} = y\mathbf{i} + (x - 2xz)\mathbf{j} - xy\mathbf{k}$, *evaluate* $\iint_S (\nabla \times \mathbf{F}).\,\mathbf{n}dS$

where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the $x - y$ plane.

Answer : 0

23. If $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$, *evaluate* $\iint_S \mathbf{F}.\mathbf{n}dS$ where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ and $\mathbf{n}$ is unit normal to S.

Answer : 3/2.

24. *If* $\mathbf{F} = 6z\mathbf{i} + (2x + y)\mathbf{j} - x\mathbf{k}$, *evaluate* $\iint_S \mathbf{F}.\mathbf{n}dS$ where S is the surface of the region bounded by the cylinder $x^2 + z^2 = 9$.

$x = 0, y = 0, z = 0, y = 8$ and $\mathbf{n}$ is the unit normal to S.

Answer : 18π.

25. $\mathbf{F} = 2y\mathbf{i} - z\mathbf{j} + x^2\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the lines

$y = 4$ *and* $z = 6$, *evaluate* $\iint_S \mathbf{F}.\mathbf{n}dS$.

Answer : 132.

26. If $\mathbf{F} = 4xz\mathbf{i} + xyz^2\mathbf{j} + 3z\mathbf{k}$, *evaluate* $\iint_S \mathbf{F}.\mathbf{n}dS$ *where* S is the surface of the region bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$ above the $x - y$ plane and $\mathbf{n}$ is the unit normal to S.

Answer : 320π **[R. U. P. 1975; 1978]**

27. Find $\iint_S \mathbf{F}.d\mathbf{S}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ *and* $x = 2$ *where* $\mathbf{F} = 2x^2y\mathbf{i} - y^2\mathbf{j} + 4xz^2\mathbf{k}$.

[R. U. H. 1976]

Answer : 180.

28. Evaluate $\iint_S \mathbf{F}.\,\mathbf{n}dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ and $\mathbf{F} = x^3\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$.

Answer : $4\pi a^5$.

29. Evaluate $\iint_R \sqrt{y^2 + z^2}\,dydz$ over the region R in the y – z plane bounded by $y^2 + z^2 = 4$.

Answer : $\frac{16}{3}\pi$.

30. If $\mathbf{F} = 2xy\mathbf{i} + yz^2\mathbf{j} + xz\,\mathbf{k}$ evaluate $\iint_S \mathbf{F}.\mathbf{n}\,ds$ where S is the surface of the region bounded by $x = 0,\ y = 0,\ y = 3,\ z = 0,$ $x + 2z = 6$ and $\mathbf{n}$ is the unit normal to S.

Answer : $\frac{351}{2}$.

31. Evaluate

$\iiint_V \varphi\,dV$ *where* $\varphi = x^2y$ *and* V denotes the closed region bounded by the planes $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0$.

Answer : $\frac{128}{45}$.

32. Evaluate $\iiint_V \varphi\,dV$ *where* $\varphi = z^2x$ *and* V denotes the closed region bounded by the planes $4z + 2x + y = 8,\ x = 0,\ y=0,$ $z=0$.

Answer : $\frac{128}{45}$

33. Evaluate $\iiint\limits_V (2x + y)\, dV$ *where* V is closed by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$, and $z = 0$.

Answer : $\frac{80}{3}$

34. Evaluate $\iiint\limits_V (2y + z)\, dV$ where V is closed by the cylinder $x = 4 - y^2$ and the planes $y = 0$, $z = 0$, $z = 2$ and $x = 0$.

Answer : $\frac{80}{3}$

35. If $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$ *Evaluate* $\iiint\limits_V (\nabla . \mathbf{F})\, dV$ where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

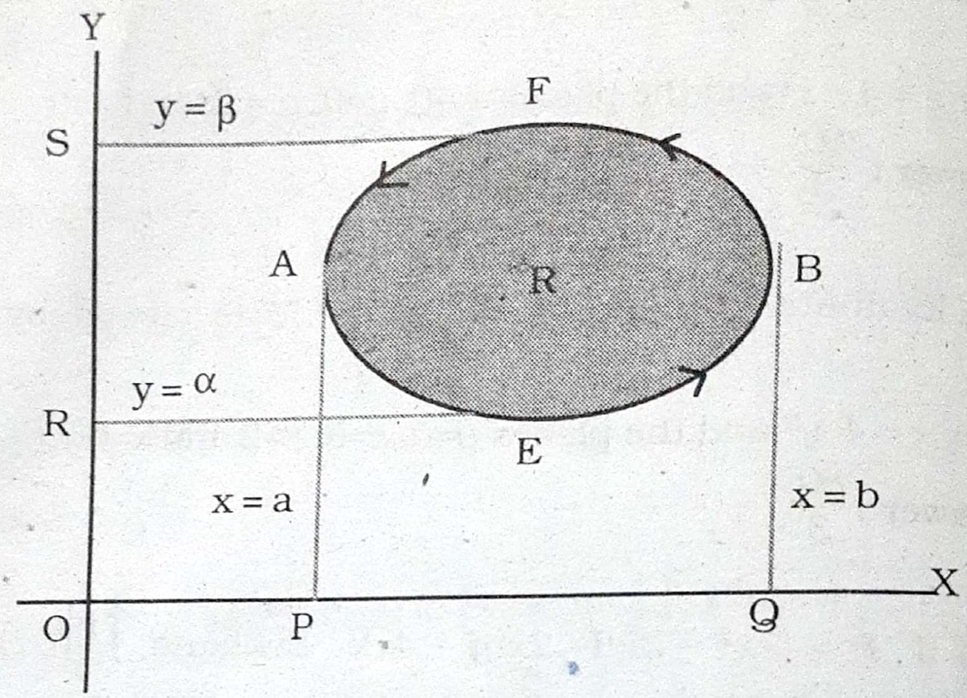
Answer : $\frac{8}{3}$

10.19. Green's theorem

Statement : Let C be a simple closed curve in the xy–plane such that a line parallel to either axis cuts C in at most two points. Let M, N, $\frac{\delta N}{\delta x}$ *and* $\frac{\delta M}{\delta y}$ be continuous functions of x and y in side and on C. Let R be the region inside C. Then

$$\oint_C (Mdx + Ndy) = \iint_S \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right) dxdy$$ where C is traversed in the positive (counter clockwise) direction.

Proof :



Let C be the closed curve AEBFA which is divided into two curves C_1 = AEB and C_2 = AFB and let the equations of the curves C_1 and C_2 be $y = Y_1(x)$ and $y = Y_2(x)$ respectively.

If R is the region inside C, then we have

$$\iint_R \frac{\delta M}{\delta y}\,dxdy = \int_{x=a}^{x=b}\left[\int_{y=Y_1(x)}^{y=Y_2(x)} \frac{\delta M}{\delta y}\,dy\right]dx$$

$$= \int_a^b \left[M(x,y)\right]_{Y_1}^{Y_2} dx$$

$$= \int_a^b [M(x,Y_2) - M(x,Y_1)]dx$$

$$= -\int_b^a [M(x,Y_2)\, dx - \int_a^b (x,Y_1)\, dx$$

$$= -\left[\int_a^b (x,Y_1)\, dx + \int_b^a M(x,Y_2)\, dx\right]$$

$$= -\int_C Mdx. \text{ Therefore, } \oint_c Mdx = -\iint_R \frac{\delta M}{\delta y}\, dxdy \quad (1)$$

Similarly, Let us divide C into two curves C_3= EAF and C_4= EBF and let the equations of the curves C_3 and C_4 be $x=X_1(y)$ *and* $x = X_2(y)$ respectiely. Then

$$\iint_R \frac{\delta N}{\delta x}\, dxdy = \int_{y=\alpha}^{y=\beta} \left[\int_{x=X_1(y)}^{x=X_2(y)} \frac{\delta N}{\delta x}\, dx\right] dy$$

$$= \int_\alpha^\beta [N(X_2,y) - N(X_1,y)]\, dy$$

$$= \int_\alpha^\beta N(X_2,y)dy - \int_\alpha^\beta N(X_1,y)\, dy$$

$$= \int_\alpha^\beta N(X_2,y)dy + \int_\beta^\alpha N(X_1,y)\, dy$$

$$= \oint_C Ndy.$$

Therefore, $\oint_C Ndy = \iint_R \frac{\delta N}{\delta x}\, dxdy$ (2)

Adding (1) and (2) we find that

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\delta N}{dx} - \frac{\delta M}{dy}\right) dxdy.$$

Hence the Green's theorem in the plane is proved.

10.20. Expression of Green's theorem in the plane in vector notation.

By Green's theorem in the plane we have

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\delta N}{dx} - \frac{\delta M}{dy}\right) dxdy \quad (1)$$

When $F = M\mathbf{i} + N\mathbf{j}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ and

$\mathbf{F}.d\mathbf{r} = (M\mathbf{i} + N\mathbf{j}).(dx\mathbf{i} + dy\mathbf{j}) = Mdx + Ndy$ (2)

Also $\nabla \times \mathbf{F}$

$$= \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) . (M\mathbf{i} + N\mathbf{j} + o.\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ M & N & O \end{vmatrix}$$

$$= -\frac{\delta N}{\delta z}\mathbf{i} + \frac{\delta M}{\delta z}\mathbf{j} + \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)\mathbf{k}.$$

so that $(\nabla \times \mathbf{F}).\mathbf{k} = \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)$ (3)

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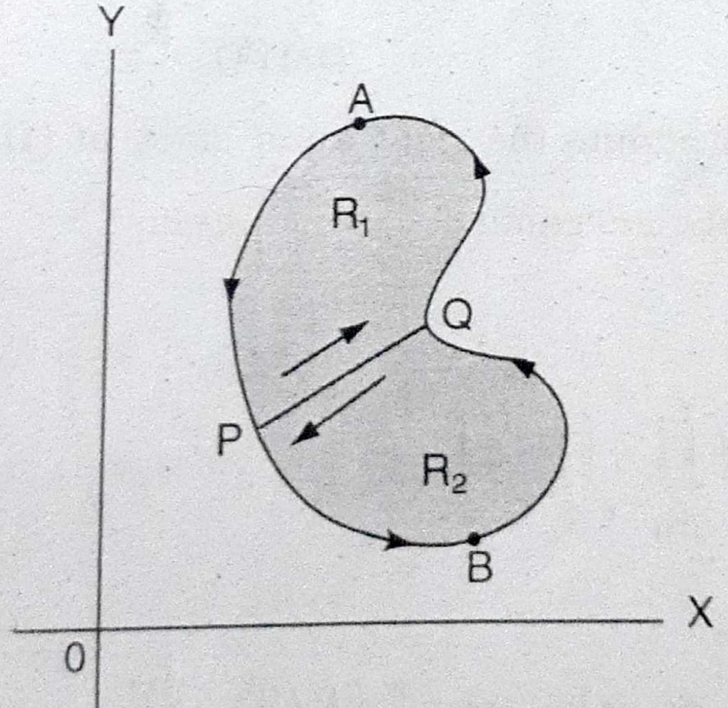
Now substituting the values of $Mdx + Ndy$ from (2) and $\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}$ from (3) in (1), we get

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}).\mathbf{k}\, dR \text{ where } dR = dxdy,$$

which is the required expression of Green's theorem in the plane in vector notation.

Corollary 1. Extension of the proof of Green's theorem in the plane to the curve C for which lines parallel to the co-ordinate axes may cut C in more than two points.

Proof : Let C be the closed curve as shown in the figure given below in which the lines parallel to the axes may cut C in more than two points.



Let R be the region bounded by the curve C. Now we divide the region R by the line PQ into two regions R_1 and R_2. Then by applying Green's theorem we get

$$\int_{PQAP} (Mdx+Ndy) = \iint_{R_1} \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy \quad (1)$$

and $$\int_{PBQP} (Mdx+Ndy) = \iint_{R_2} \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy \quad (2)$$

Adding the left hand sides of (1) and (2) and omitting the itegrand Mdx + Ndy in each case we have

$$\int_{PQAP} + \int_{PBQP} = \int_{PQ} + \int_{QAP} + \int_{PBQ} + \int_{QP}$$

$$= \int_{QAP} + \int_{PBQ} \quad \text{since} \int_{PQ} = -\int_{QP}$$

$$= \int_{QAPBQ} = \oint_C$$

Again adding the right hand sides of (1) and (2) and omitting the integrand $\left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$

we get $\iint_{R_1} + \iint_{R_2} = \iint_{R}$.

Thus $$\oint_{\mathbf{c}} (Mdx + Ndy) = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy.$$

Hence the corollary is proved.

Corollary 2. If C is a simple closed curve such that a line parallel to either axis cuts it in at most two points, then the area enclosed by C is equal to $\frac{1}{2}\oint_C (xdy - ydx)$.

Proof : From Green's theorem we hae

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy \quad (1)$$

Now if we take $M = -\frac{1}{2}y$ *and* $N = \frac{1}{2}x$ *in (1),*

we get $\oint_C (\frac{1}{2}x\,dy - \frac{1}{2}ydx)$

$$= \iint_R (\frac{1}{2} + \frac{1}{2})\,dxdy$$

$$= \iint dxdy$$

$$= \textit{Area of } R.$$

$\therefore$ Area of $R = \frac{1}{2}\oint_C (xdy - ydx)$.

Example 43. (i) Find the area of the circle $x = a\cos\theta$, $y = a\sin\theta$.

Solution : Area $= \frac{1}{2}\oint_C (xdy - ydx)$

$$= \frac{1}{2}\int_0^{2\pi} \{a\cos\theta\, d(a\sin\theta) - a\sin\theta\, d(a\cos\theta)\}$$

$$=\frac{1}{2}\int_0^{2\pi}\{a\cos\theta.\ (a\cos\theta d\theta)\ +\ a\sin\theta.\ a\sin\theta d\theta\}$$

$$=\frac{1}{2}a^2\int_0^{2\pi}(\cos^2\theta.\ +\sin^2\theta)d\theta=\frac{1}{2}a^2\int_0^{2\pi}1d\theta$$

$$=\frac{1}{2}a^2.\ [\theta]_0^{2\pi}=\frac{1}{2}a^2.2\pi=\pi a^2.$$

(ii) Find the area of the ellipse $x = 2\cos\varphi, y = 3\sin\varphi$.

Solution : Area $= \frac{1}{2}\oint_C (xdy - ydx)$

$$=\frac{1}{2}\int_0^{2\pi}\{(2\cos\varphi\ d(3\sin\varphi)-(3\sin\varphi)d\ (2\cos\varphi)\}$$

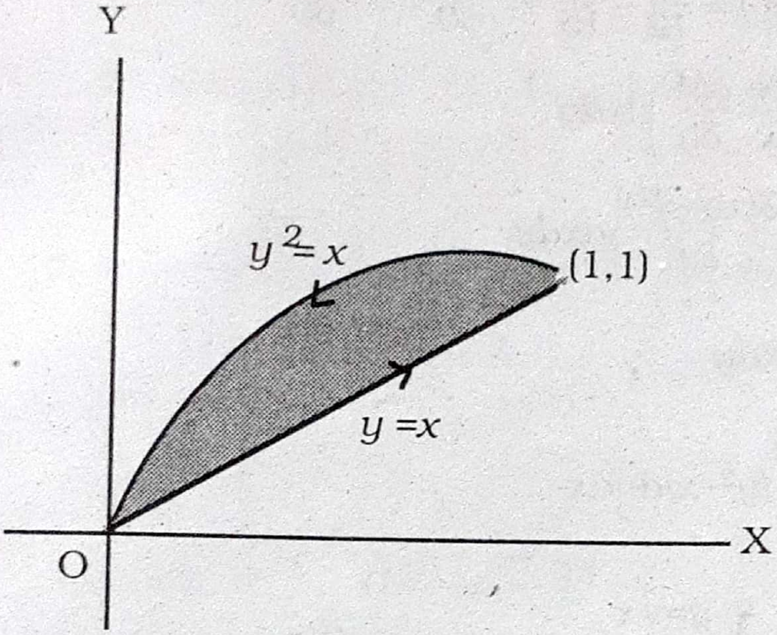
$$=\frac{1}{2}\int_0^{2\pi}\{2\cos\varphi.3\cos\varphi d\varphi+3\sin\varphi.\ 2\sin\varphi d\varphi\}$$

$$=\frac{1}{2}\int_0^{2\pi}6(\cos^2\varphi+\ \sin^2\varphi)d\varphi.$$

$$=3\int_0^{2\pi}d\varphi=3\ [\varphi]_0^{2\pi}=6\pi.$$

Example 44. Verify Green's theorem in the plane for $\oint_C \{(xy + x^2)dx + xy^2dy\}$ where C is the closed curve of the region bounded by $y = x$ and $y^2 = x$.

Verification : $y = x$ *and* $y^2 = x$ interest at *(0,0) and (1,1)*. The positive direction in traversing C is as shown in the figure.



Along the curve $y = x$, the line integral becomes

$$\int_0^1 \{(x^2 + x^2)dx + x^3dx\}$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1 = \frac{2}{3} + \frac{1}{4} = \frac{11}{12}.$$

Along the curve $y^2 = x$, the line integral becomes

$$\int_0^1 \{(x\sqrt{x} + x^2)\, dx + x^2.\frac{1}{2\sqrt{x}}dx\}$$

$$= \int_0^1 \{(x^{3/2} + x^2)dx + \frac{1}{2}x^{3/2}\, dx\}$$

$$= \frac{3}{2}\left[\frac{x^{5/2}}{5/2}\right]_1^0 + \left[\frac{x^3}{3}\right]_1^0 = -\frac{3}{5} - \frac{1}{3} = -\frac{14}{15}.$$

Then the required line integral

$$\oint_C (Mdx + Ndy) = \frac{11}{12} - \frac{14}{15} = \frac{55-56}{60} = -\frac{1}{60}.$$

Again $\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$

$$= \iint_R \left\{\frac{\delta(xy^2)}{\delta x} - \frac{\delta(xy+x^2)}{\delta y}\right\} dxdy.$$

$$= \iint_R (y^2 - x) dxdy$$

$$= \int_0^1 \int_{y=x}^{y=\sqrt{x}} (y^2 - x) dydx$$

$$= \int_O^I \left[\frac{y^3}{3} - xy\right]_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_O^I \left\{\frac{1}{3}x^{3/2} - x^{3/2} - \frac{1}{3}x^3 + x^2\right\} dx$$

$$= \int_O^I \left\{-\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 + x^2\right\} dx$$

$$= \left[-\frac{2}{3}\frac{x^{5/2}}{5/2} - \frac{1}{3}\frac{x^4}{4} + \frac{x^3}{3}\right]_o^1.$$

$$= -\frac{4}{15} - \frac{1}{12} + \frac{1}{3} = \frac{-16-5+20}{60} = -\frac{1}{60}.$$

Hence the theorem is verified.

Example 45. Evaluate $\oint_C \{(3x + 4y)\, dx + (2x - 3y)\, dy$ where C, a circle of radius 2 with centre at the origin of the xy–plane, is traversed in the positive sense.

Solution : Let the equation of the circle be $x = 2\cos\theta$, $y = 2\sin\theta$ where $o \le \theta \le 2\pi$.

$$\oint_C \{(3x + 4y)\, dx + (2x - 3y)\, dy\}$$

$$= \int_0^{2\pi} (6\cos\theta + 8\sin\theta)\, d(2\cos\theta) + (4\cos\theta - 6\sin\theta)\, d(2\sin\theta)$$

$$= \int_0^{2\pi} (6\cos\theta + 8\sin\theta)\,(-2\sin\theta d\theta) + (4\cos\theta - 6\sin\theta)\, 2\cos\theta d\theta$$

$$= -6\int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta - 8\int_0^{2\pi} 2\sin^2\theta d\theta + 4\int_0^{2\pi} 2\cos^2\theta\, d\theta$$

$$-6\int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta$$

$$= -6\int_0^{2\pi} \sin 2\theta d\theta - 8\int_0^{2\pi} (1-\cos 2\theta)\, d\theta$$

$$+ 4\int_0^{2\pi} (1+\cos 2\theta)\, d\theta - 6\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$= 6\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} - 8\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} + 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$+ 6\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

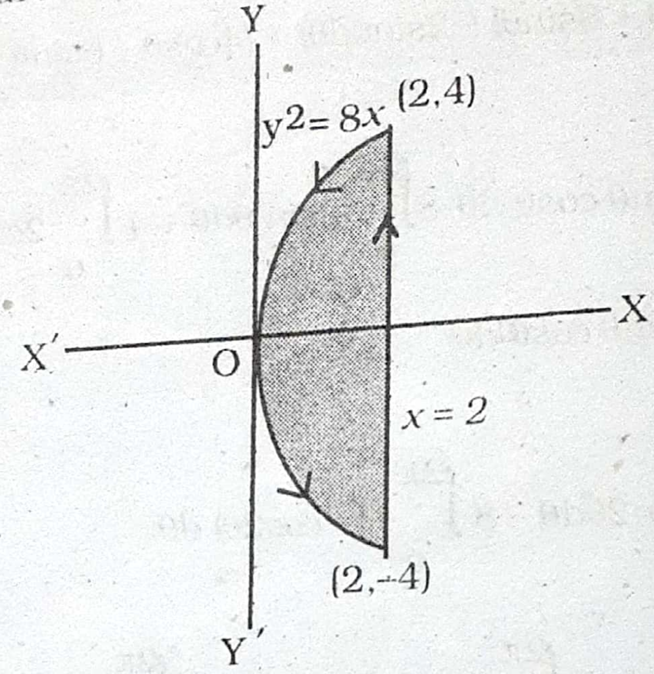
$$= 0 - 8(2\pi - 0) + 4(2\pi + 0) + 0$$

$$= -16\pi + 8\pi = -8\pi.$$

Example 46. Verify Green's theorem in the plane for $\oint_C (x^2 - 2xy)\, dx + (x^2y + 3)dy$, where C is the closed curve of the region bounded by $y^2 = 8x$ and $x = 2$.

Verification : $y^2=8x$ and $x=2$ intersect at (2,4) and (2,−4).

The positive direction in traversing C is shown in adjacent figure.



Along the path $y^2 = 8x$ i.e $x = \dfrac{y^2}{8}$, the line integral becomes.

$$\int_4^{-4}\left\{\left(\frac{y^4}{64} - 2\frac{y^2}{8}\cdot y\right)\frac{1}{8}\,2y\,dy + \left(\frac{y^4}{64}\cdot y + 3\right)dy\right\}$$

$$= \frac{1}{256}\int_4^{-4} y^5 dy - \frac{1}{16}\int_4^{-4} y^4 dy$$

$$+ \frac{1}{64}\int_4^{-4} y^5 dy + 3\int_4^{-4} dy$$

$$= \frac{1}{256}\left[\frac{y^6}{6}\right]_4^{-4} - \frac{1}{16}\left[\frac{y^5}{5}\right]_4^{-4} + \frac{1}{64}\left[\frac{y^6}{6}\right]_4^{-4} + 3[y]_4^{-4}$$

$$= 0 - \frac{1}{80}(-4^5 - 4^5) + 0 + 3(-4-4)$$

$$= \frac{1}{80}\times 2\times 4^5 - 24 = \frac{128}{5} - 24.$$

Along the curve $x = 2$, the line integral becomes

$$\int_{-4}^{4} \{(4-4y)0 + (4y+3)\,dy\}$$

$$= 0 + \left[4.\frac{y^2}{2} + 3y\right]_{-4}^{4}$$

$$= 2(16-16) + 3(4+4) = 0 + 24 = 24.$$

Then the required line integral $= \frac{128}{5} - 24 + 24 = \frac{128}{5}$.

Again $\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$

$$= \iint_R \left\{\frac{\delta(x^2y+3)}{\delta x} - \frac{\delta(x^2-2xy)}{\delta y}\right\} dxdy$$

$$= \iint_R (2xy + 2x)\,dxdy$$

$$= \int_4^{-4}\int_{x=2}^{x=\frac{y^2}{8}} (2xy+2x)\,dxdy$$

$$= \int_4^{-4} \left[x^2y + x^2\right]_{x=2}^{x=\frac{y^2}{8}}$$

$$= \int_4^{-4} \left(\frac{y^5}{64} + \frac{y^4}{64} - 4y - 4\right) dy$$

$$= \left[\frac{y^6}{6\times 64} + \frac{y^5}{64\times 5} - \frac{4y^2}{2} + 4y\right]_4^{-4}$$

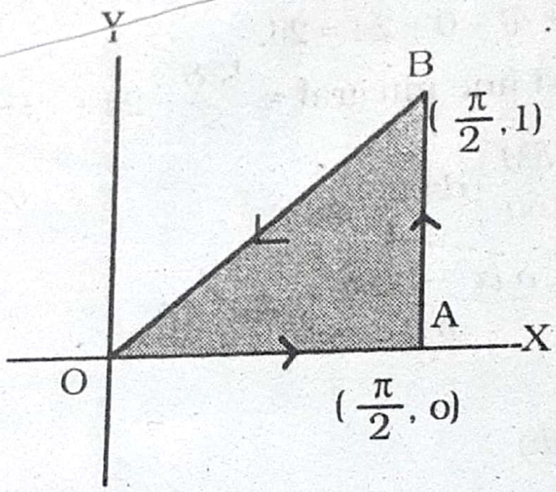
$$= 0 + \frac{1}{64\times 5}(-4^5 - 4^5) - 0 + 4(-4-4)$$

$$= -\frac{32}{5} + 32 = -\frac{-32+160}{5} = \frac{128}{5}.$$

Hence the Green's theorem is verified.

Example 47. Verify Green's theorem in the plane for $\int[(y-\sin x)\,dx + \cos x\,dy]$ where C is the triangle of adjoining figure.

Verification : Along oA, $y = 0$, $dy = 0$ and the integral becomes



$$\int_0^{\pi/2} (0 - \sin x)\,dx + (\cos x)(0)$$

$$= \int_0^{\pi/2} (-\sin x)\,dx = \Big[\cos x\Big]_0^{\pi/2} = 0 - 1 = -1.$$

Along the line AB, $x = \frac{\pi}{2}$, $dx = o$ and the integral becomes

$$\int_0^1 (y - 1)(0) + (0)\,dy = 0.$$

Along the line BO, $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}\,dx$ and the integral becomes $\int_{\pi/2}^{0} \left(\frac{2x}{\pi} - \sin x\right) dx + (\cos x)\frac{2}{\pi}\,dx$

$$= \left[\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right]_{\pi/2}^{0}$$

$$= 0 + 1 + 0 - \frac{\pi}{4} - 0 - \frac{2}{\pi} = 1 - \frac{\pi}{4} - \frac{2}{\pi}.$$

Then the required line integral along the curve C

$= -1 + o + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}.$

Again $M = y - \sin x$, $N = \cos x$

$\frac{\delta N}{\delta x} = -\sin x$ and $\frac{\delta M}{\delta y} = 1.$

Now $\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right) dxdy$

$= \iint_R (-\sin x - 1)\, dxdy$

$= \int_{x=0}^{x=\frac{\pi}{2}} \left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\, dy \right] dx$

$\int_0^{\pi/2} \left[-y \sin x - y \right]_0^{2x/\pi} dx$

$= \int_0^{\pi/2} \left(-\frac{2x}{\pi} \sin x - \frac{2x}{\pi} \right) dx$

$= -\frac{2}{\pi} \int_0^{\pi/2} (x \sin x + x)\, dx$

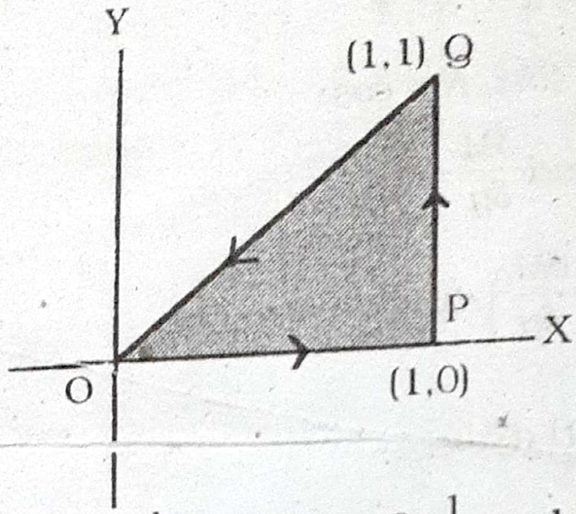
$= -\frac{2}{\pi} \left[-x \cos x \right]_0^{\pi/2} + \left(-\frac{2}{\pi} \right) \int_0^{\pi/2} \cos x dx - \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi/2}$

$= 0 - \frac{2}{\pi} \left[\sin x \right]_0^{\pi/2} - \frac{2}{\pi} \cdot \frac{\pi^2}{8}$

$= -\frac{2}{\pi} - \frac{\pi}{4}$. Hence the Green's theorem is verified .

Example 48. Compute $\oint (xy - x^2)\, dx + x^2 y dy)$ over the triangle bounded by the lines $y=0$, $x = 1$, $y = x$ and verify Green's theorem.

Solution : The line integral along OP, where y = 0 and x varies from 0 to 1



$$= \int_{x=0}^{x=1} (-x^2)dx = -\int_0^1 x^2 dx = -\left[\frac{x^3}{3}\right]_0^1 = -\frac{1}{3}.$$

Again, the line integral along PQ, for which $x = 1$ and y varies from 0 to 1

$$= \int_{y=0}^{y=1} \{(y-1)0 + ydy\} = \int_0^1 ydy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$

Also the line integral along QO for which $x = y$ and y varies from 1 to 0

$$= \int_1^0 (y^2 - y^2)\, dy + y^3 dy$$

$$= 0 + \int_1^0 y^3 dy = \left[\frac{y^4}{4}\right]_1^0 = 0 - \frac{1}{4} = -\frac{1}{4}.$$

Thus the total integral along C

$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{4} = -\frac{1}{12}.$$

Now by Green's theorem we have

$$\oint_C (Mdx + Ndy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy.$$

Now $\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$

$$= \iint_R \left[\frac{\delta}{\delta x}(x^2y) - \frac{\delta}{\delta y}(xy - x^2)\right] dxdy$$

$$= \iint_R [2xy - (x-0)]\ dxdy$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x} [2xy - x]dydx$$

$$= \int_0^1 \left[xy^2 - xy\right]_0^x dx$$

$$= \int_0^1 [x^3 - x^2]\ dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

$$\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy = -\frac{1}{12}.$$

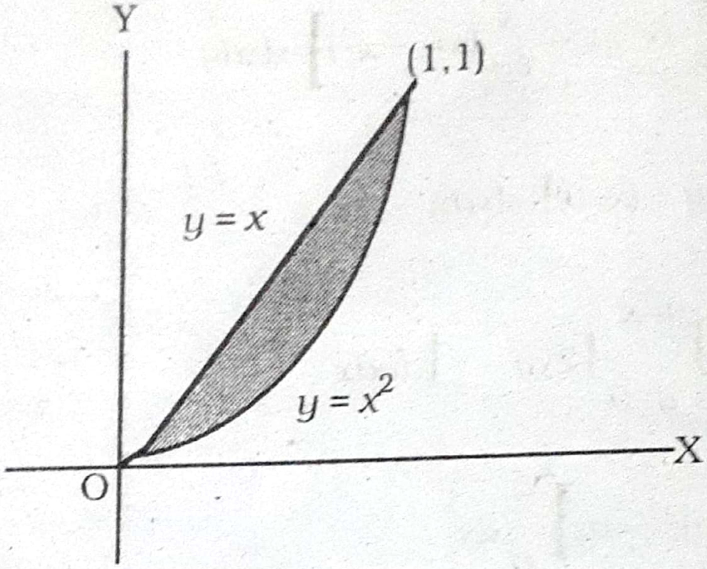
Hence the Green's theorem is verified.

Example 49. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy)$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

Verification :

$y = x$ and $y = x^2$ intersect at (0, 0) and (1, 1).

The positive direction in traversing C is as shown in the figure given below.



Along the curve $y = x^2$, the line integral equals

$$= \int_0^1 (x^3 + x^4 + 2x^3)\, dx$$

$$= \int_0^1 (3x^3 + x^4)\, dx = \left[\frac{3}{4}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}.$$

Along the curve $y = x$ from $(1,1)$ to $(0, 0)$, the line integral equals $\int_1^0 [\{x(x) + x^2\}\, dx + x^2\, dx]$

$$= \int_1^0 3x^2 dx = \frac{3}{3}\left[x^3\right]_1^0 = 0 - 1 = -1.$$

Hence the required line integral

$$= \frac{19}{20} - 1 = -\frac{1}{20}$$

Again $\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$

$$= \iint_R \left\{\frac{\delta(x^2)}{\delta x} - \frac{\delta(xy + y^2)}{\delta y}\right\} dxdy$$

$$= \iint_R \{2x - (x+2y)\} dxdy$$

$$= \iint_R (x - 2y) dxdy.$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dydx$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 [0 - x^3 + x^4] dx.$$

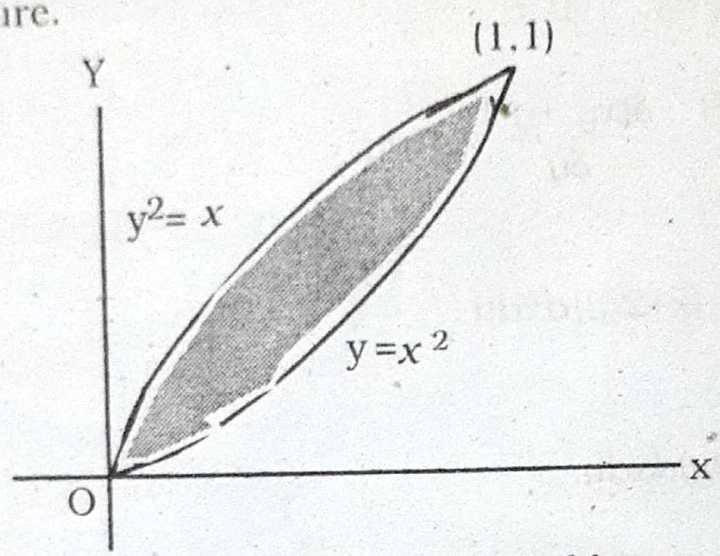
$$= \int_0^1 [x^4 - x^3] dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

Hence the Green's theorem is verified.

Example 50. Verify Green's theorem in the plane for $\oint_C (2xy - x^2)\, dx + (x + y^2) dy$ where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Verification : The plane curves $y = x^2$ and $y^2 = x$ intersect at (0,0) and (1,1).

The positive direction in traversing C is as shown in the adjacent figure.



Along the curve $y = x^2$, the line integral becomes

$$\int_0^1 \{2x)(x^2)-x^2\}dx + \{x+(x^2)^2\}d(x^2)$$

$$= \int_0^1 (2x^3 - x^2 + 2x^2 + 2x^5)\,dx$$

$$= \int_0^1 (2x^3 + x^2 + 2x^5)dx$$

$$= \left[\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{3+2+2}{6} = \frac{7}{6}.$$

Along the curve $y^2 = x$, the line integral becomes

$$\int_1^0 \{2(y^2)(y) - (y^2)^2\}d(y^2) + \{y^2 + y^2\}\,dy$$

$$= \int_1^0 \{(2y^3 - y^4)\,2ydy + 2y^2dy\}$$

$$= \int_1^0 (4y^4 - 2y^5 + 2y^2)dy.$$

$$= \left[\frac{4}{5}y^5 - \frac{2}{6}y^6 + \frac{2}{3}y^3\right]_1^0$$

$$= 0 - \frac{4}{5} + \frac{1}{3} - \frac{2}{3} = \frac{-12+5-10}{15} = \frac{-17}{15}.$$

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Thus the required line integral

$$\oint_C (Mdx + Ndy) = \frac{7}{6} - \frac{17}{15} = \frac{1}{30}.$$

Again, $\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$

$$= \iint_R \left\{ \frac{\delta}{\delta x}(x + y^2) - \frac{\delta}{\delta y}(2xy - x^2) \right\} dxdy$$

$$= \iint_R (1 - 2x) dxdy$$

$$= \int_0^1 \int_{y=x^2}^{\sqrt{x}} (1-2x) dydx$$

$$= \int_0^1 \Big[y - 2xy \Big]_{x^2}^{\sqrt{x}} dx.$$

$$= \int_0^1 (\sqrt{x} - 2x\sqrt{x} - x^2 + 2x^3)\, dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^2 + 2x^3)\, dx$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} - 2.\frac{2}{5} x^{\frac{5}{2}} - \frac{x^3}{3} + 2.\frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2}$$

$$= \frac{20 - 24 - 10 + 15}{30} = \frac{1}{30}.$$

Hence the Green's theorem is verified.

10. 21 Gauss's divergence theorem

(Relation between surface and volume integrals)

Statement : The surface integral of the normal component of a continuously differentiable vector **F** taken over a closed

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surface S is equal to the integral to the divergence of **F** taken over the volume V enclosed by the surface;

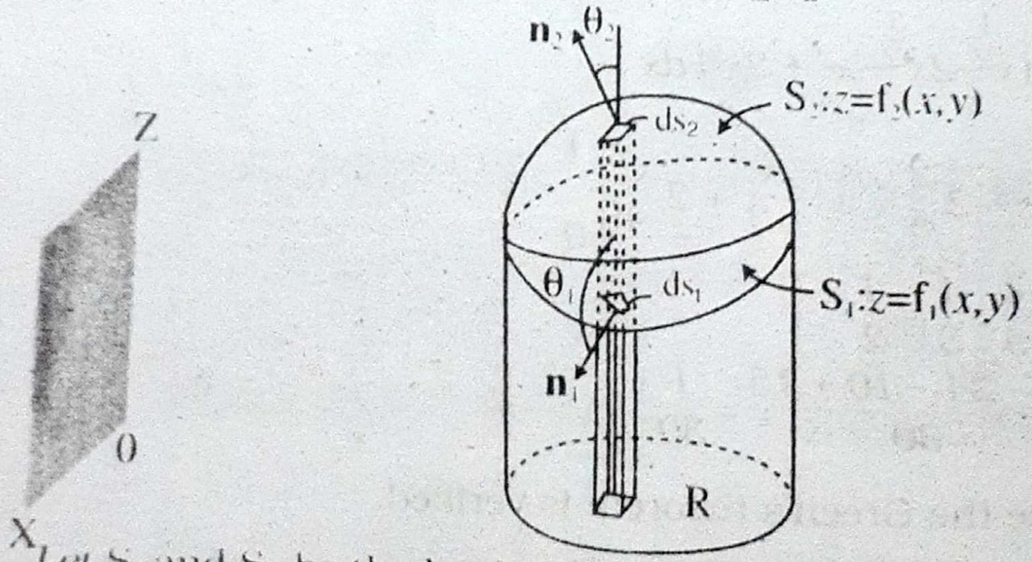
$$\text{that is, } \iiint_V \nabla.\mathbf{F} dV = \iint_S \mathbf{F.n}dS$$

where **n** is the positive (outward drawn) normal to S.

Proof : Taking **i, j, k** as unit vectors along the axes of x, y and z respectively, we have $\mathbf{F} = F_1(x, y, z)\ \mathbf{i} + F_2\ (x, y, z)\ \mathbf{j} + F_3\ (x, y, z)\ \mathbf{k}$ where F_1, F_2, F_3, are components of **F** along the axes of x, y and z respectively, Now we have to show that

$$\iint_S \mathbf{F}.\ \mathbf{n}dS = \iiint_V (\nabla.\mathbf{F})\ dV = \iiint_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F}{\delta z}\right) dxdydz\,.$$

Firstly, we have to consider the closed surface S which is such that any line parallel to z – axis cuts it in at most two points, say $P_1\ (x, y, z_1($ and $P_2\ (x, y, z_2)$, $z_1 \le z_2$



Let S_1 and S_2 be the lower and upper portions of the surface S corresponding to the points P_1 and P_2 and let the equations of S_1 and S_2 be $z = f_1\ (x, y)$ and $z = f_2\ (x, y)$ respectively. Let the projection of the surface S on the x–y plane be R.

Now consider the integral

$$\iiint\limits_V \frac{\delta F_3}{\delta z} dV = \iiint\limits_V \frac{\delta F_3}{\delta z} dz dy dx .$$

$$= \iint\limits_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\delta F_3}{\delta z} dz \right] dx dy$$

$$= \iint\limits_R [F_3(x, y, z)]_{f_1}^{f_2} dy dx$$

$$\iint\limits_R [F_3(x, y, f_2) - F_3(x, y, f_1)] dy dx$$

For the upper portion S_2, we have $dydx = \cos\theta_2 dS_2 = \mathbf{k}.\, \mathbf{n}_2 dS_2$ since the normal $\mathbf{n}_2$ to S_2 makes an acute angle θ_2 with $\mathbf{k}$.

Similarly, for the lower portion S_1, we have $dydx = -\cos\theta_1 dS_1 = -\mathbf{k}.\, \mathbf{n}_1 dS_1$, since the normal $\mathbf{n}_1$ to S_1 makes an obtuse angle θ_1 with $\mathbf{k}$.

Then $\displaystyle\iint\limits_R F_3(x, y, f_2)\, dy dx = \iint\limits_{S_2} F_3\, \mathbf{k}.\, \mathbf{n}_2\, dS_2$

and $\displaystyle\iint\limits_R F_3(x, y, f_1)\, dy dx = -\iint\limits_{S_1} F_3\, \mathbf{k}.\, \mathbf{n}_1\, dS_1$

Therefore, $\displaystyle\iint\limits_R F_3(x, y, f_2)\, dy dx - \iint\limits_R F_3(x, y, f_1)\, dy dx$

$$= \iint\limits_{S_2} F_3\, \mathbf{k}.\, \mathbf{n}\, 2dS_2 + \iint\limits_{S_1} F_3\, \mathbf{k}.\, \mathbf{n}_1\, dS_1 = \iint\limits_S F_3\, \mathbf{k}.\mathbf{n} dS.$$

Consequently , $\displaystyle\iiint\limits_V \frac{\delta F_3}{\delta z} dV = \iint\limits_S F_3\, \mathbf{k}.\, \mathbf{n} dS \quad (1)$

Similarly, by projecting the surface S on the yz and zx planes respectively, we get

$$\iiint_V \frac{\delta F_1}{\delta x} dV = \iint_S F_1 \, \mathbf{i}.\, \mathbf{n} dS \quad (2)$$

and $$\iiint_V \frac{\delta F_2}{\delta y} dV = \iint_S F_2 \mathbf{j}.\, \mathbf{n} dS \quad (3)$$

Adding equations (1), (2) & (3), we get

$$\iiint_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} \right) dV = \iint_S (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}).\, \mathbf{n} dS$$

Or, $$\iiint_V \left(\mathbf{i} \frac{\delta}{\delta x} + \mathbf{j} \frac{\delta}{\delta y} + \mathbf{k} \frac{\delta}{\delta z} \right). (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) dV = \iint_S \mathbf{F}.\, \mathbf{n} dS$$

Or, $$\iiint_V \nabla. \mathbf{F} d\mathbf{V} = \iint_S \mathbf{F}.\, \mathbf{n} dS,$$

Hence theorem is proved.

10. 22 Expression of Gauss's divergence theorem in rectangular form.

Let $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$.

Then div $\mathbf{F} = \nabla. \mathbf{F}$

$$= \left(\mathbf{i} \frac{\delta}{\delta x} + \mathbf{j} \frac{\delta}{\delta y} + \mathbf{k} \frac{\delta}{\delta z} \right). (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k})$$

$$= \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}.$$

The unit normal to S is $\mathbf{n} = n_1\,\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$.

The $\left.\begin{array}{l} n_1 = \mathbf{n}.\mathbf{i} = \cos\theta \\ n_2 = \mathbf{n}.\,\mathbf{j} = \cos\varphi \\ n_3 = \mathbf{n}.\mathbf{k} = \cos\Psi \end{array}\right\}$

where θ, φ, Ψ, are the angles which $\mathbf{n}$ makes with the positive direction of x, y and z axes or $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ direction respectively.

The quantities $\cos\theta$, $\cos\varphi$ and $\cos\Psi$ are the direction cosines of $\mathbf{n}$. The $\mathbf{F}.\ \mathbf{n} = (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}).\ (\cos\theta\mathbf{i} + \cos\phi\mathbf{j} + \cos\Psi\mathbf{k})$

$= F_1\cos\theta + F_2\cos\varphi + F_3\cos\Psi$.

Therefore, the Gauss's divergence theorem can be written

as $$\iiint_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dxdydz$$

$$= \iint_S (F_1\cos\theta + F_2\cos\varphi + F_3\cos\Psi)dS.$$

Example 51. Evaluate $\iint_S \mathbf{F}.\ \mathbf{n}dS$ where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Solution : By divergence theorem, we have

$$\iiint_V \nabla.\ \mathbf{F}dV = \iint_S \mathbf{F}.\ \mathbf{n}dS.$$

Therefore, $$\iint_S \mathbf{F}.\ \mathbf{n}dS = \iiint_V \nabla.\ \mathbf{F}dV$$

$$= \iiint_V \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\ (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k})\ dV$$

$$= \iiint_V \left[\frac{\delta}{\delta x}(4xz) + \frac{\delta}{\delta y}(-y^2) + \frac{\delta}{\delta z}(yz)\right] dV$$

$$= \iiint_V (4z - 2y + y)\, dV$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (4z - y)\, dz dy dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \left[\frac{4z^2}{2} - yz\right]_0^1 dy dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} (2 - y)\, dy dx$$

$$= \int_{x=0}^{1} \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_{x=0}^{1} \left(2 - \frac{1}{2}\right) dx = \int_{x=0}^{1} \frac{3}{2} dx = \frac{3}{2}\left[x\right]_0^1 = 3/2.$$

Hence $\iint_S \mathbf{F}.\, \mathbf{n} dS = 3/2.$

Example 52. Verify Gauss,s divergence theorem for $\mathbf{F} = (x^2 - yz)\, \mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\, k$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$

Verification : By divergence theorem,

we have $\iiint_V \nabla.\mathbf{F} dV = \iint_S \mathbf{F}.\, \mathbf{n} dS.$

$$\nabla \mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta y} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).\{(x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}\}$$

$$= \frac{\delta}{\delta x}(x^2 - yz) + \frac{\delta}{\delta y}(y^2 - zx) + \frac{\delta}{\delta z}(z^2 - xy)$$

$$= 2x + 2y + 2z = 2(x + y + z).$$

$$\therefore \iiint_V \nabla . F \, dV$$

$$= 2\int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} (x + y + z) \, dz dy dx$$

$$= 2\int_{x=0}^{a} \int_{y=0}^{b} \left[xz + yz + \frac{z^2}{2}\right]_0^c dy dx$$

$$= 2\int_{x=0}^{a} \int_{y=0}^{b} \left[cx + cy + \frac{c^2}{2}\right] dy dx$$

$$= 2\int_{x=0}^{a} \left[cxy + c\frac{y^2}{2} + \frac{c^2}{2}y\right]_0^b dx$$

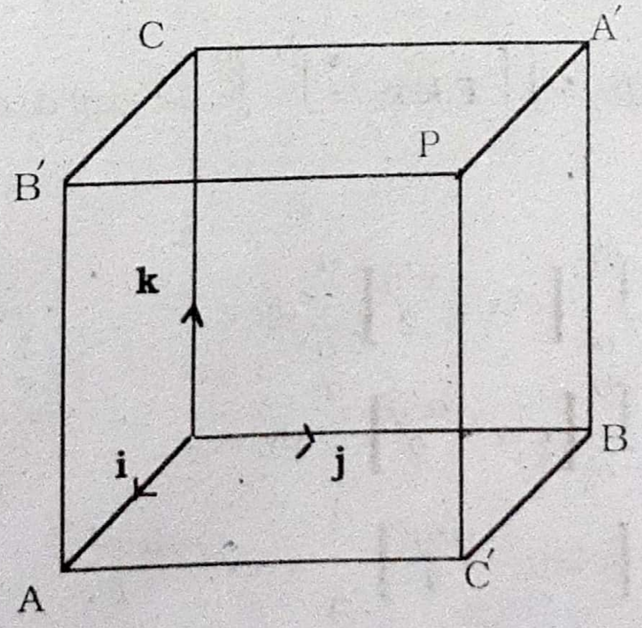
$$= 2\int_{0}^{a} \left[bcx + \frac{cb^2}{2} + \frac{c^2 b}{2}\right] dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{cb^2}{2}x + \frac{c^2 b}{2}x\right]_0^a$$

$$= [bca^2 + cb^2 a + c^2 ba]$$

$$= abc(a + b + c)$$

$$\therefore \iiint_V \nabla . \mathbf{F} dV = abc(a + b + c).$$



Also $$\iint_S \mathbf{F}.\mathbf{n}dS = \iint_{S_1} \mathbf{F}.\mathbf{n}S_1 + \iint_{S_2} \mathbf{F}.\mathbf{n}dS_2 + \iint_{S_3} \mathbf{F}.\mathbf{n}dS_3$$

$$+ \iint_{S_4} \mathbf{F}.\mathbf{n}dS_4 + \iint_{S_5} \mathbf{F}.\mathbf{n}dS_5 + \iint_{S_6} \mathbf{F}.\mathbf{n}dS_6 \qquad (2)$$

where S_1 is the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$ and S_6 the face $BA'PC'$.

Now $$\iint_{S_1} \mathbf{F}.\mathbf{n}dS_1 = \iint_{S_1} \mathbf{F}.(-\mathbf{k})S_1$$

$$= \iint_{S_1} \{(x^2-yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}\}.(-\mathbf{k})dS_1$$

{since $z = o$, $\mathbf{i}.\mathbf{k} = \mathbf{j}.\mathbf{k} = 0$ and $\mathbf{k}.\mathbf{k} = 1$}

$$= \int_0^b \int_0^a (o - xy)\, dxdy$$

$$= \int_o^b \left[y\frac{x^2}{2}\right]_o^a dy = \int_o^b \frac{a^2y}{2} dy = \frac{a^2}{2}\left[\frac{y^2}{2}\right]_o^b = \frac{a^2b^2}{4}.$$

$$\iint_{S_2} \mathbf{F}.\mathbf{n}dS_2 = \iint_{S_2} \mathbf{F}.\mathbf{k}dS_2 = \int_0^b \int_0^a (c^2-xy)\, dxdy$$

{Since $z = c$}

$$= \int_o^b \left[c^2x - \frac{x^2y}{2}\right]_o^a dy$$

$$= \int_o^b \left[c^2a - \frac{a^2y}{2}\right]_o^a dy$$

$$= \left[c^2ay - \frac{a^2y^2}{4}\right]_o^a = abc^2 - \frac{a^2b^2}{4}.$$

Similarly,

$$\iint_{S_3} \mathbf{F}.\mathbf{n}dS_3 = \iint_{S_3} \mathbf{F}.(-\mathbf{i})dS_3 = -\int_0^b \int_0^c (o - yz)\, dydz = \frac{b^2c^2}{4}$$

$$\iint_{S} \mathbf{F}.\mathbf{n}dS_4 = \iint_{S_4} \mathbf{F}.\mathbf{i}dS_4 = \int_0^b \int_0^c (a^2 - yz)\, dydz = a^2bc - \frac{b^2c^2}{4}$$

$$\iint_{S_5} \mathbf{F}.\mathbf{n}dS_5 = \iint_{S_5} F.(-j)dS_5 = -\int_0^c \int_0^a (o - zx)dzdx = \frac{c^2a^2}{4}$$

$$\iint_{S_6} \mathbf{F}.\mathbf{n}\, dS_6 = \iint_{S_6} \mathbf{F}.\mathbf{j}\, .dS_6 = \int_0^c \int_0^a (b^2 - zx)\, dzdx = ab^2c - \frac{c^2a^2}{4}$$

Substituting these values of the integrals in (2), we get

$$\iint_S \mathbf{F}.\mathbf{n}dS = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{c^2a^2}{4} + ab^2c - \frac{c^2a^2}{4},$$

$$= abc^2 + a^2bc + ab^2c$$

$$= abc(a + b + c) \quad (3)$$

Thus $\iint_S \mathbf{F}.\mathbf{n}dS = abc(a + b + c)$.

Hence from (1) and (3) we have

$$\iiint_V \nabla.\mathbf{F}dV = \iint_S \mathbf{F}.\mathbf{n}dS.$$

Therefore, the Gauss's divergence theorem is verified.

Example 53. Verify the divergence theorem for $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. **[D. U. S. 1986]**

Verification : Volume integral $= \iiint\limits_V \nabla.\mathbf{F}dV$

$$= \iiint\limits_V \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).(4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k})$$

$$= \iiint\limits_V \left[\frac{\delta}{\delta x}(4x) + \frac{\delta}{\delta y}(-2y^2) + \frac{\delta}{\delta z}(z^2)\right] dV$$

$$= \iiint\limits_V (4 - 4y + 2z)\, dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)dzdydx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 dydx$$

$$= \int_{x=-2}^{2} y \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dydx$$

$$= \int_{x=-2}^{2} \left[21y - 6y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} \left[42\sqrt{4-x^2} - 0\right] dx$$

$$= 42\int_{x=-2}^{2} \sqrt{2^2 - x^2}dx$$

$$= 42\left[\frac{x\sqrt{2^2-x^2}}{2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 42\left[0 + 2\left\{\sin^{-1}\frac{2}{2} - \sin^{-1}\frac{(-2)}{2}\right\}\right]$$

$$= 84\left[\sin^{-1} 1 + \sin^{-1} 1\right] = 84\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 84\pi$$

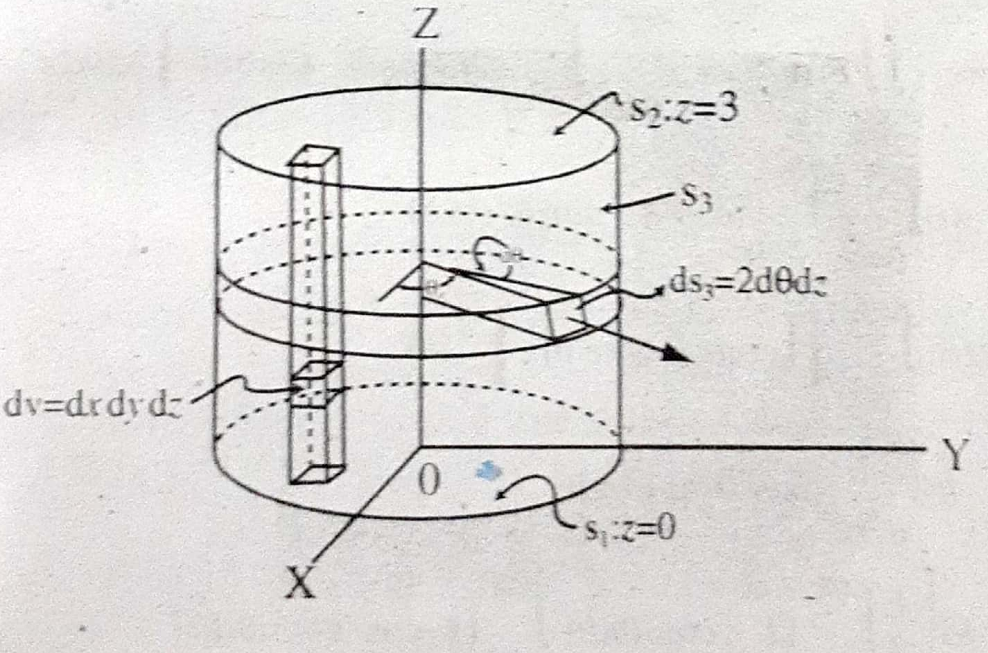
The surface S of the cylinder consists of a base $S_1 (z=0)$, the top $S_2 (z=3)$ and the convex portion $S_3 (x^2+y^2=4)$.

The surface integral

$$\iint_S \mathbf{F.n}dS = \iint_{S_1} \mathbf{F.n}dS_1 + \iint_{S_2} \mathbf{F.n}dS_2 + \iint_{S_3} \mathbf{F.n}dS_3$$

On S_1 $(z = o)$ $\mathbf{n} = -\mathbf{k}$, $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j}$ and $\mathbf{F.n} = 0$

so that $\iint_{S_1} \mathbf{F.n}dS_1 = 0.$



On $S_2 (z = 3)$, $\mathbf{n} = \mathbf{k}$, $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + 9\mathbf{k}$

And $\mathbf{F.n} = 9$ so that

$$\iint_{S_2} \mathbf{F.n}dS_2 = 9\iint_{S_2} dS_2 = 9 \times 4\pi = 26\pi$$

(Since area of $S_2 = 4\pi$)

On $S_3 (x^2 + y^2 = 4)$

A perpendicular to $x^2 + y^2 = 4$ has the direction

$$\nabla (x^2 + y^2) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2)$$

$$= \mathbf{i}\frac{\delta}{\delta x}(x^2 + y^2) + \mathbf{j}\frac{\delta}{\delta y}(x^2 + y^2) + 0$$

$$= 2x\mathbf{i} + 2y\mathbf{j}$$

Then unit normal is

$$\boldsymbol{n} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{4}} = \frac{x\mathbf{i} + y\mathbf{j}}{2}.$$

(since $x^2 + y^2 = 4$)

So $\mathbf{F}.\boldsymbol{n} = (4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}).\left(\frac{x\mathbf{i} + y\mathbf{j}}{2}\right) = 2x^2 - y^3$.

From the figure above $x = 2\cos\theta$, $y = 2\sin\theta$ $dS_3 = 2d\theta dz$.

Therefore, $$\iint_{S_3} \mathbf{F}.\boldsymbol{n}dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} 2(2\cos\theta)^2 - (2\sin\theta)^3\Big] 2d\theta dz$$

$$= 16\int_{\theta=0}^{2\pi}\int_{z=0}^{3}(\cos^2\theta - \sin^3\theta)\, dz d\theta$$

$$= 16\int_{\theta=0}^{2\pi}\Big[(\cos^2\theta - \sin^3\theta)z\Big]_0^3 d\theta$$

$$= 48\int_{\theta=0}^{2\pi}(\cos^2\theta - \sin^3\theta)d\theta$$

$$= 48\left[\frac{1}{2}\int_0^{2\pi}(1 + \cos 2\theta)d\theta - \int_0^{2\pi}(1 - \cos^2\theta)\sin\theta d\theta\right]$$

$$= 48\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} - 48\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{2\pi}$$

$$= 48\left[\frac{1}{2}.2\pi + 0\right] - 48 \times 0 = 48\pi.$$

$$\therefore \iint_{S_3} \mathbf{F}.\boldsymbol{n}dS_3 = 48\pi.$$

Therefore, the surface integral

$$\iint_S \mathbf{F}.\,\mathbf{n}dS = 0 + 36\pi + 48\pi = 84\pi.$$

So $\iiint_V \nabla.\,\mathbf{F}dV = \iint_S \mathbf{F}.\,\mathbf{n}dS.$

Hence the Gauss's divergence theorem is verified.

Example 54. Apply Gauss's divergence theorem to evaluate

$\iint_S (lx^2 + my^2 + nz^2)\, dS$ taken over the sphere

$(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$; l, m, n being the direction cosines of the external normal to the sphere.

Solution : By Gauss's divergence theorem, we have

$$\iint_V \nabla.\,\mathbf{F}dV = \iint_S \mathbf{F}.\mathbf{n}dS$$

The parametric equations of the sphere given by

$(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$ are $x = a + \rho\sin\theta\cos\varphi$, $y = b + \rho\sin\theta\sin\varphi$, $z = c + \rho\cos\theta$ and to cover the whole sphere r varies from 0 to ρ, θ varies from 0 to π and φ varies from 0 to 2π.

Now $\iint_S (lx^2 + my^2 + nz^2)dS$

$$= \iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}).\,(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})dS$$

$$= \iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}).\,\mathbf{n}dS$$

$\left\{ \begin{array}{l} \text{since } \mathbf{n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}, \text{ where} \\ l, m, n \text{ are direction cosines to } \mathbf{n} \end{array} \right.$

$$=\iiint_V \nabla.\mathbf{F}dV$$

$$=\iiint_V \left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right).(x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k})\,dV$$

$$=\iiint_V \left\{\frac{\delta}{\delta x}(x^2)+\frac{\delta}{\delta y}(y^2)+\frac{\delta}{\delta z}(z^2)\right\}dV$$

$$=2\iiint_V (x+y+z)\,dydxdz$$

[changing the equation to parametric form]

$$=2\int_{r=0}^{\rho}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\{(a+b+c)+\rho\,(\sin\theta\cos\varphi$$

$$+\sin\theta\sin\varphi+\cos\theta)\}\times r^2\sin\theta drd\theta d\varphi$$

$$=2\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\{(a+b+c)+\rho\,(\sin\theta\cos\varphi+\sin\theta\sin\varphi$$

$$+\cos\theta)\}\times\left[\frac{r^3}{3}\right]_0^{\rho}\sin\theta d\theta d\varphi$$

$$=\frac{2}{3}\rho^3\int_{\theta=0}^{\pi}\ [(a+b+c)\,\varphi]_0^{2\pi}+\rho\,[\sin\theta\sin\varphi]_0^{2\pi}$$

$$-\rho\,[\sin\theta\cos\varphi]_0^{2\pi}+[[\cos\theta.\varphi]_0^{2\pi}]\sin\theta d\theta.$$

$$=\frac{2}{3}\rho^3\int_0^{\pi}[(a+b+c)\,2\pi+o-o+\cos\theta.\,2\pi]\sin\theta d\theta$$

$$=\frac{2}{3}\rho^3\,[2\pi(a+b+c)\,(-\cos\theta)]_0^{\pi}+2\pi\left[\frac{\sin^2\theta}{2}\right]_0^{\pi}$$

$$=-\frac{2}{3}\rho^3\,2\pi\,(a+b+c)(\cos\pi-\cos 0)+2\pi\times 0$$

$$=-\frac{4}{3}\pi\rho^3\,(a+b+c)\,(-1-1)+0$$

$$=\frac{8}{3}\pi\rho^3\,(a+b+c).$$

Example 55. If $\vec{OA} = a\mathbf{i}$, $\vec{OB} = a\mathbf{j}, \vec{OC} = \vec{a}\mathbf{k}$ form three coterminous edges of a cube and S denotes the surface of the cube; evaluate $\iint_S \{(x^3-yz)\,\mathbf{i}-2x^2y\mathbf{j} + 2\mathbf{k}\}\,.\,\mathbf{n}dS$ by expressing it as a volume integral.

Also verify the result by direct evaluation of the surface integral.

Solutioin : By Gauss's divergence theorem we have

$$\iint_S \mathbf{F}.\,\mathbf{n}dS = \iiint_V \nabla.\,\mathbf{F}dV$$

Let $\mathbf{F} = (x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}$ then

$$\nabla.\,\mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta x}\right) \{(x^3 - yz)\,\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}$$

$$= \frac{\delta}{\delta x}(x^3 - yz) - \frac{\delta}{\delta y}(2x^2y) + \frac{\delta}{\delta z}(2)$$

$= 3x^2 - 2x^2 = x^2$. Also $dV = dxdydz$.

$$\therefore \iint_S \mathbf{F}.\,\mathbf{n}dS = \iiint_V \nabla.\,\mathbf{F}dS$$

$$= \iiint_V x^2\,dxdydz$$

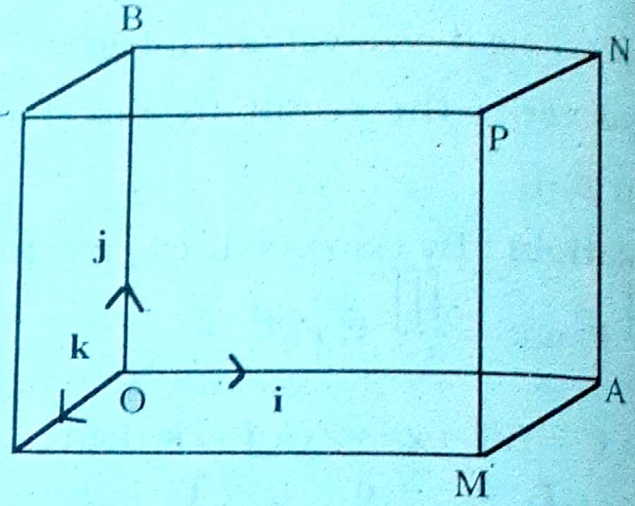
$$= \int_0^a\int_0^a\int_0^a = x^2\,dxdydz.$$

$$= \int_0^a\int_0^a \left[\frac{x^3}{3}\right]_0^a dydz$$

$$= \frac{a^3}{3}\int_0^a [y]_0^a\,dz = \frac{a^4}{3}\int_0^a dz = \frac{a^4}{3}\,[z]_0^a = \frac{1}{3}a^5.$$

Now we shall evaluate the surface integral directly. The surface of a cube consists of six faces.

Over the surface AOBN $\iint_S \mathbf{F}.\mathbf{n}dS = \iint_S \mathbf{F}.(-\mathbf{k})\, dS$



$$= \iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2k\}.(-k)dS$$

$$= \iint_S (-2)\, dS$$

$$= -2\int_0^a \int_0^a dxdy$$

$$= -2\int_0^a [x]_0^a dy$$

$$= -2a\int_0^a dy = -2a[y]_0^a = -2a^2.$$

Also over the surface PLCM

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iint_S \mathbf{F}.\,\mathbf{k}dS$$

$$= \iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}.\,\mathbf{k}dS$$

$$= 2\int_0^a \int_0^a dxdy$$

$$= 2\int_0^a [x]_0^a dy = 2a\int_0^a dy = 2a\,[y]_0^a = 2a^2.$$

Similarly, over the faces NPMA, BLCO, AOCM, NBLP, the corresponding surface integrals are respectively.

$$\iint_S \mathbf{F}.\,\mathbf{i}\,dS = \iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}.\,\mathbf{i}dS$$

$$= \iint_S (x^3 - yz)\,dS \text{ (since } x = a)$$

$$= \int_0^a \int_0^a (a^3 - yz)\,dydz$$

$$= \int_0^a \left[a^3y - \frac{y^2z}{2}\right]_0^a dz$$

$$= \int_0^a \left(a^4 - \frac{a^2z}{2}\right) dz$$

$$= \left[a^4z - \frac{a^2}{2}.\frac{z^2}{2}\right]_o^a = a^5 - \frac{1}{4}a^4.$$

$$\iint_S \mathbf{F}.(-\mathbf{i})dS = \iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}.\,(-\mathbf{i})\,dS$$

$$= -\iint_S (x^3 - yz)\,dS \text{ (since } x = 0)$$

$$= -\int_0^a \int_0^a (0 - yz)\,dydz$$

$$= \int_0^a \left[\frac{y^2z}{2}\right]_o^a dz = \frac{a^2}{2}\int_0^a zdz = \frac{a^2}{2}\left[\frac{z^2}{2}\right]_o^a = \frac{1}{4}a^4.$$

$$\iint_S \mathbf{F}.(-\mathbf{j})\, dS = \iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}.(-\mathbf{j})\, dS$$

$$= \iint_S 2x^2y dS = \int_0^a\int_0^a 2x^2y dx dz = 0. \text{ since } y = 0 \text{ in this face.}$$

$$\iint_S \mathbf{F}.\mathbf{j} dS = \iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}.\mathbf{j} dS$$

$$= -2\iint_S x^2y dS \text{ (since } y = a)$$

$$= -2a\int_0^a\int_0^a x^2 dx\, dz$$

$$= -2a\int_0^a \left[\frac{x^3}{3}\right]_0^a dz$$

$$= -\frac{2}{3}a^4\int_0^a dz = -\frac{2}{3}a^4\left[z\right]_0^a = -\frac{2}{3}a^5.$$

Adding we see that over the whole surface.

$$\iint_S \mathbf{F}.\mathbf{n}dS = -2a^2 + 2a^2 + a^5 - \frac{1}{4}a^4 + \frac{1}{4}a^4 + 0 - \frac{2}{3}a^5 = \frac{1}{3}a^5.$$

Hence the Gauss's divergence theorem is verified.

Example 56. Verify divergence theorem for the vector $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ taken over the cube $0 \le x, y, z \le 1$.

Solution : From divergence theorem we have

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iiint_V = \nabla.\mathbf{F}dV.$$

Consider the cube as shown in the figure of Example 52. Let us take the adjacent edges OA, OB, OC, as axes of reference with O as origin.

Given $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$\nabla.\mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta y} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) . (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})$$

$$= \frac{\delta}{\delta x}(x^2) + \frac{\delta}{\delta y}(y^2) + \frac{\delta}{\delta z}(z^2)$$

$$= 2x + 2y + 2z = 2(x + y + z).$$

Now the value of x, y and z vary from 0 to 1 and therefore for the volume of the cube, we have

$$\iiint_V \nabla.\mathbf{F}\, dV = 2\int_0^1\int_0^1\int_0^1 (x + y + z)dxdydz$$

$$= 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + yx + zx\right]_0^1 dydz$$

$$= 2\int_0^1\int_0^1 \left(\frac{1}{2} + y + z\right) dydz$$

$$= 2\int_0^1 \left[\frac{1}{2}y + \frac{y^2}{2} + zy\right]_0^1 dz.$$

$$= 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z\right] dz.$$

$$= 2\int_0^1 (1 + z)dz = 2\left[z + \frac{z^2}{2}\right]_0^1 = 2\left(1 + \frac{1}{2}\right) = 3.$$

Again for the surface of the cube, we have

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iint_{S_1} \mathbf{F}.\mathbf{n}dS_1 + \iint_{S_2} \mathbf{F}.\mathbf{n}dS_2 + \iint_{S_3} \mathbf{F}.\mathbf{n}dS_3$$

$$+ \iint_{S_4} \mathbf{F}.\mathbf{n}dS_4 + \iint_{S_5} \mathbf{F}.\mathbf{n}dS_5 + \iint_{S_6} \mathbf{F}.\mathbf{n}dS_6 \quad (1)$$

where S_1 is the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$, and S_6 the face $BA'PC'$. Then

$$\iint_{S_1} \mathbf{F}.\mathbf{n}dS_1 = \iint_{S_1} \mathbf{F}.(-\mathbf{k})\, dS_1$$

$$= \int_0^1\int_0^1 (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}).\,(-\mathbf{k})dS_1$$

$$= \int_0^1\int_0^1 (-\,z^2)dxdy = 0 \text{ since } z = 0.$$

$$\iint_{S_2} \mathbf{F}.\mathbf{n}dS_1 = \iint_{S_2} \mathbf{F}.\mathbf{k}dS_2 = \int_0^1\int_0^1 (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}).\ \mathbf{k}dS_2$$

$$= \int_0^1\int_0^1 dxdy \text{ (since } z = 1)$$

$$= \int_0^1\int_0^1 dxdy = \int_0^1 \Big[x\Big]_0^1 dy = \int_0^1 dy = \Big[y\Big]_0^1 = 1.$$

Similarly, $\displaystyle\iint_{S_3} \mathbf{F}.\mathbf{n}dS_3 = \iint_{S_3} \mathbf{F}.(-\mathbf{i})dS_3$

$$= \int_0^1\int_0^1 (-x^2)dydz = 0 \text{ (since } x = 0)$$

$$\iint_{S_4} \mathbf{F}.\mathbf{n}dS_4 = \iint_{S_4} \mathbf{F}.\mathbf{i}dS_4$$

$$\int_0^1 \int_0^1 x^2 dydz = \int_0^1 \int_0^1 dydz = 1 \ (since \ x = 1)$$

$$\iint_{S_5} \mathbf{F}.\mathbf{n}dS = \iint_{S_5} \mathbf{F}.(-\mathbf{j})dS_5$$

$$\int_0^1 \int_0^1 (-y^2)dzdx = 0 \ (since \ y = 0)$$

$$\iint_{S_6} \mathbf{F}.\mathbf{n}dS_6 = \iint_{S_6} \mathbf{F}.\mathbf{j}dS_6$$

$$\int_0^1 \int_0^1 y^2 dzdx \ (since \ y = 1)$$

$$\int_0^1 \int_0^1 dzdx = 1.$$

Substituting these values in (1), we have

$$\iint_S \mathbf{F}.\mathbf{n}dS = 0 + 1 + 0 + 1 + 0 + 1 = 3$$

Hence the Gauss's divergence theorem is verified.

Example 57. Use the divergence theroem to evaluate

$\iint_S \varphi.\ nd\sigma$ where $d\sigma$ is the surface element, $\varphi = y^2\mathbf{i} + x^2\mathbf{j} + e^z\mathbf{k}$

and S is the spherical surface $x^2 + y^2 + z^2 = 1,\ z \geq 0)$.

[D.U.P. 1985]

[Supplementary]

Slolution : By Gauss's divergence theorem we have

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iiint_V \nabla.\mathbf{F}dV.$$

$$\therefore \iint_S \varphi\text{-}\mathbf{n}d\sigma = \iiint_V \nabla.\varphi dV$$

$$= \iiint_V \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).(y^2\mathbf{i} + x^2\mathbf{j} + e^z\mathbf{k})dV$$

$$= \iiint_V \left\{\frac{\delta}{\delta x}(y^2) + \frac{\delta}{\delta y}(x^2) + \frac{\delta}{\delta z}(e^z)\right\} dV$$

$$= \iiint_V e^z dV = \int_0^1\int_0^1\int_0^1 e^z\, dzdydx$$

$$= \int_0^1\int_0^1 \Big[e^z\Big]_0^1 dydx = \int_0^1\int_0^1 (e-1)dydx$$

$$= \int_0^1 \Big[(e-1)y\Big]_0^1 dx = \int_0^1 (e-1)dx = \Big[(e-1)x\Big]_0^1 = e-1.$$

Example 58. If S is any closed surface enclosing a volume V and $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$, show that

$$\iint_S \mathbf{F}.\mathbf{n}dS = (a+b+c)V.$$

Proof : By Gauss's divergence theorem, we have

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iiint_V \nabla.\mathbf{F}dV$$

$$= \iiint_V \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).(ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k})dV$$

$$= \iiint_V \left\{\frac{\delta}{\delta x}(ax) + \frac{\delta}{\delta y}(by) + \frac{\delta}{\delta z}(cz)\right\} dV$$

$$= \iiint_V (a+b+c)dV$$

$$= (a+b+c)\iiint_V dV = (a+b+c)V.$$

Consider first $\iint_S [\nabla \times (F_1\mathbf{i})].\boldsymbol{n}dS$

Since $\nabla \times (F_1\mathbf{i}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ F_1 & 0 & 0 \end{vmatrix}$

$$= \frac{\delta F_1}{\delta z}\boldsymbol{j} - \frac{\delta F_1}{\delta y}\boldsymbol{k}.$$

$$\text{So } [\nabla \times (F_1 i)].\boldsymbol{n}dS = \left[\frac{\delta F_1}{\delta z}\boldsymbol{j} - \frac{\delta F_1}{\delta y}\boldsymbol{k}\right].\boldsymbol{n}dS$$

$$= \left[\frac{\delta F}{\delta z}\boldsymbol{n}.\boldsymbol{j} - \frac{\delta F_1}{\delta y}\boldsymbol{n}.\boldsymbol{k}\right]dS \quad (1)$$

If $z = f(x,y)$ is taken as the equation of S, then the position vector to any point of S is $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{i} + z\boldsymbol{k} = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k}$.

so that $\dfrac{\delta \boldsymbol{r}}{\delta y} = 0 + \boldsymbol{j} + \dfrac{\delta f}{\delta y}\boldsymbol{k} = \boldsymbol{j} + \dfrac{\delta f}{\delta y}\boldsymbol{k}$.

Now $\dfrac{\delta \boldsymbol{r}}{\delta y}$ is perpendicular to $\boldsymbol{n}$ as $\dfrac{\delta \boldsymbol{r}}{\delta y}$ is the tangent to the surface S.

$$\boldsymbol{n}.\frac{\delta \boldsymbol{r}}{\delta y} = \boldsymbol{n}.j + \boldsymbol{n}.\frac{\delta f}{\delta y}k = 0$$

$$\text{or, } \boldsymbol{n}.\boldsymbol{j} = -\frac{\delta f}{\delta y}\boldsymbol{n}.k = -\frac{\delta z}{\delta y}\boldsymbol{n}.k$$

$$\text{Therefore, } \boldsymbol{n}.\boldsymbol{j} = -\frac{\delta z}{\delta y}\boldsymbol{n}.k$$

Substituting this value in equation no (1), we get

$$[\nabla \times (F_1 i)].\boldsymbol{n}dS = \left(-\frac{\delta F_1}{\delta z}\frac{\delta z}{\delta y}\boldsymbol{n}.\boldsymbol{k} - \frac{\delta F_1}{\delta y}\boldsymbol{n}.\boldsymbol{k}\right)dS$$

$$= -\left(\frac{\delta F_1}{\delta y} + \frac{\delta F_1}{\delta z}\frac{\delta z}{\delta y}\right)\boldsymbol{n}.\boldsymbol{k}dS \quad (2)$$

Now on S, $F_1(x, y, z) = F_1(x, y, f(x,y)) = G(x, y)$

Hence $\frac{\delta F_1}{\delta y} + \frac{\delta F_1}{\delta z}, \frac{\delta z}{\delta y} = \frac{\delta G}{\delta y}$ (3)

Therefore, from equation no (2), we get

$$[\nabla \times (F_1\mathbf{i})].\ \mathbf{n}dS = -\frac{\delta G}{\delta y}\mathbf{n.k}dS = -\frac{\delta G}{\delta y}dxdy$$

Hence $\iint_S [\nabla \times (F_1\mathbf{i}).\ \mathbf{n}dS = \iint_R -\frac{\delta G}{\delta y}dxdy$ (4)

where R is the projection of S on the xy–plane.

By Green's theorem for the plane, we have

$$\iint_R -\frac{\delta G}{\delta y}\ dxdy = \oint_\Gamma Gdx$$

where Γ is the boundary of R.

Now since at each point (x, y) of the curve Γ the value of G is the same as the value of F_1 at each point (x, y, z) of C and dx is the. the same for both curves Γ and C, we conclude that

$\oint_\Gamma Gdx = \oint_C F_1dx$

that is, $\oint_C F_1dx = \iint_R -\frac{\delta G}{\delta y}\ dxdy$ (5)

Now from equations (4) and (5), we get

$\iint_S [\nabla \times (F_i\mathbf{i})].\mathbf{n}dS = \oint_C F_1dx$ (6)

Similarly, by projections on the yz ard zx planes, we have

$\iint_S [\nabla \times (F_2\mathbf{j})].\mathbf{n}dS = \oint_C F_2dx$ (7)

amd $\iint_S [\nabla \times (F_3\mathbf{k})].\mathbf{n}dS = \oint_C F_3dz$ (8)

Thus by adding (6), (7) and (8), we get

$$\iint_S [\nabla \times (F_i\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})]\ .\mathbf{n}dS = \oint_C (F_idx + F_2dy + F_3dz)$$

that is, $\iint_S [\nabla \times (\mathbf{F})].\mathbf{n}dS = \oint_C \mathbf{F}.d\mathbf{r}$

Hence the theorem is proved.

10.24. Expression of Stoke's theorem in rectangular form

By Stoke's theorem we have

$$\iint_S (\nabla\times\mathbf{F}).\mathbf{n}dS = \oint_C \mathbf{F}.d\mathbf{r}$$

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

Then $\nabla\times\mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\delta F_3}{\delta y} - \frac{\delta F_2}{\delta z}\right)\mathbf{i} + \left(\frac{\delta F_1}{\delta z} - \frac{\delta F_3}{\delta x}\right)\mathbf{j} + \left(\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y}\right)\mathbf{k}$$

Now the unit normal to S is $\mathbf{n} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$ where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of $\mathbf{n}$.

$$\text{So}(\nabla\times\mathbf{F}).\mathbf{n} = \left(\frac{\delta F_3}{\delta y} - \frac{\delta F_2}{\delta z}\right)\cos\alpha + \left(\frac{\delta F_1}{\delta z} - \frac{\delta F_3}{\delta x}\right)\cos\beta + \left(\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y}\right)\cos\gamma$$

Also $\mathbf{F}.d\mathbf{r} = (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) . (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$

$$= F_1dx + F_2dy + F_3dz.$$

Hence the Stoke's theorem becomes

$$\iint_S \left[\left(\frac{\delta F_3}{\delta y} - \frac{\delta F_2}{\delta z}\right)\cos\alpha + \left(\frac{\delta F_1}{\delta z} - \frac{\delta F_3}{\delta x}\right)\cos\beta + \left(\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y}\right)\cos\gamma\right]ds$$

$$= \oint_C (F_1dx + F_2dy + F_3dz).$$

which is the required rectangular form of Stoke's theorem.

10.25. Derivation of Green's theorem from Gauss's divergence theorem.

According to Gauss's divergence theorem we have

$$\iiint_V \nabla.\mathbf{F}dV = \iint_S \mathbf{F}.\mathbf{n}dS \quad (1)$$

$$\nabla.\mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right).(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

$$= \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}.$$

Take $\mathbf{F} = \varphi \nabla \psi = \varphi \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \psi$

or, $F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \mathbf{i}\left(\varphi\frac{\delta\psi}{\delta x}\right) + \mathbf{j}\left(\varphi\frac{\delta\psi}{\delta y}\right) + \mathbf{k}\left(\varphi\frac{\delta\psi}{\delta z}\right)$

$\therefore F_1 = \varphi\frac{\delta\psi}{\delta x}, F_2 = \varphi\frac{\delta\psi}{\delta y}, F_3 = \varphi\frac{\delta\psi}{\delta z}.$

Therefore, $\nabla.\mathbf{F} = \frac{\delta}{\delta x}\left(\varphi\frac{\delta\psi}{\delta x}\right) + \frac{\delta}{\delta y}\left(\varphi\frac{\delta\psi}{\delta y}\right) + \frac{\delta}{\delta y}\left(\varphi\frac{\delta\psi}{\delta z}\right)$

$$= \frac{\delta\varphi}{\delta x}\frac{\delta\psi}{\delta x} + \varphi\frac{\delta^2\psi}{\delta x^2} + \frac{\delta\varphi}{\delta y}\frac{\delta\psi}{\delta y} + \varphi\frac{\delta^2\psi}{\delta y^2} + \frac{\delta\phi}{\delta z}\frac{\delta\psi}{\delta z} + \varphi\frac{\delta^2\psi}{\delta z^2}.$$

$$= \varphi\left(\frac{\delta^2\psi}{\delta x^2} + \frac{\delta^2\psi}{\delta y^2} + \frac{\delta^2\psi}{\delta z^2}\right) + \frac{\delta\varphi}{\delta x}\frac{\delta\psi}{\delta x} + \frac{\delta\varphi}{\delta y}\frac{\delta\psi}{\delta y} + \frac{\delta\varphi}{\delta z}\frac{\delta\psi}{\delta z}$$

$$= \varphi\nabla^2\psi + \nabla\varphi.\nabla\psi.$$

Putting this value in equation no (1), we get

$$\iiint_V (\varphi\nabla^2\psi + \nabla\varphi.\nabla\psi)dV = \iint_S (\phi\nabla\psi).\mathbf{n}dS$$

or, $\iiint_V (\phi\nabla^2\psi + \nabla\phi.\nabla\psi)dV = \iint_S (\varphi\nabla\psi).d\mathbf{S}$ (2)

Now interchanging φ and ψ in equation (2),

we get $\iiint_V \psi\nabla^2\varphi + \nabla\psi.\nabla\varphi\, dV = \iint_S (\psi\nabla\phi).dS$ (3)

Now subtracting (3) from (2), we get

$$\iiint_V (\varphi\nabla^2\psi - \psi\nabla^2\varphi)dV = \iint_S (\varphi\nabla\psi - \psi\nabla\varphi).dS \quad (4)$$

which is Green's theorem in the plane.

10. 26 Derivation of Gauss's theorem from Green's theorem.

If we put $\varphi = 1$ and $\psi = \psi$ in equation no (4), then we have

$$\iiint_V (\nabla.\nabla\varphi)dV = \iint_{\mathbf{S}} \nabla\psi.\, d\mathbf{S}$$

Now if we have a vector field $\mathbf{F} = \nabla\varphi$

then $$\iiint_V (\nabla.\mathbf{F})dV = \iint_{\mathbf{S}} \mathbf{F}.\, d\mathbf{S} = \iint_S \mathbf{F}.\, \mathbf{n}dS.$$

which is Gauss's divergence therem.

10. 27 Derivation of Green's theorem in the plane from Stoke's theorem.

According to Stoke's theorem, we have

$$\oint_C \mathbf{F}.\, d\mathbf{R} = \iint_S (\nabla\times\mathbf{F}).\, \mathbf{n}dS = \iint_S (\nabla\times\mathbf{F}).\, d\mathbf{S}$$

If the curve is plotted in the x–y plane then

$\mathbf{n} = \mathbf{k}$ *and* $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ for this plane.

So $$\oint_C \mathbf{F}.\, \mathbf{dr} = \iint_S (\nabla\times\mathbf{F}).\, \mathbf{k}dS \quad (2)$$

Let $F_1 = M$ *and* $F_2 = N$, so that $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

Then $$\nabla\times\mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \frac{\delta}{\delta z}\mathbf{k}\right) \times (M\mathbf{i} + N\mathbf{j} + O\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ M & N & O \end{vmatrix}$$

$$= -\frac{\delta N}{\delta z}\mathbf{i} + \frac{\delta M}{\delta z}\mathbf{j} + \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)\mathbf{k}$$

Therefore, $(\nabla \times \mathbf{F}).\,\mathbf{k} = \dfrac{\delta N}{\delta x} - \dfrac{\delta M}{\delta y}$

Also $\mathbf{F}.\,d\mathbf{r} = (M\mathbf{i} + N\mathbf{j}).\,(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$

$$= Mdx + Ndy \text{ and } dS = dxdy.$$

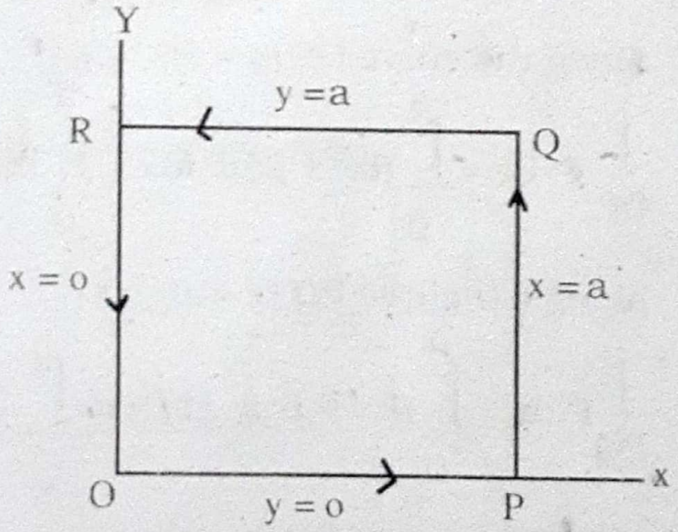
Then from equation no (2), we get

$$\oint_C (Mdx + Ndy) = \iint_S \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dxdy$$

which is Green's theorem in the plane.

Example 59. Verify Stoke's theorem for the function $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ integrated round the square in the plane z=0 whose sides are along the lines $x = 0,\ y = 0,\ x = a,\ y = a$

Verification : According to Stoke's theorem we have $\oint_C \mathbf{F}.\,d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}).\,\mathbf{n}dS$



$$\nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times (x^2\mathbf{i} + xy\mathbf{j} + 0\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= -\mathbf{i}\frac{\delta}{\delta z}(xy) + \mathbf{j}\frac{\delta}{\delta z}(x^2) + \mathbf{k}\left\{\frac{\delta}{\delta x}(xy) - \frac{\delta}{\delta y}(x^2)\right\}$$

$$= 0 + \mathbf{k}y = \mathbf{k}y.$$

Also in the plane $z = 0$, $\mathbf{n} = \mathbf{k}$ & $dS = dxdy$

$$\therefore \iint_S (\nabla \times \mathbf{F}).\, \mathbf{n} dS = \int_0^a \int_0^a \mathbf{k}y.\, \mathbf{k}dxdy$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^a dx$$

$$= \frac{1}{2}\int_0^a a^2 dx = \frac{1}{2}a^2.\left[x\right]_0^a = \frac{1}{2}a^3.$$

Again $\oint_C \mathbf{F}.\, d\mathbf{r} = \int_{OPQRO} \mathbf{F}.d\mathbf{r} = \int_{OP} \mathbf{F}.d\mathbf{r} + \int_{PQ} \mathbf{F}.d\mathbf{r} + \int_{QR} \mathbf{F}.d\mathbf{r} + \int_{RO} \mathbf{F}.\, d\mathbf{r}$

Along the curve OP $(y = 0)$

$$\int_{OP} \mathbf{F}.\, d\mathbf{r} = \int_0^a (\mathbf{i}x^2 + \mathbf{j}xy).\, \mathbf{i}dx = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{1}{3}a^3.$$

Along the curve PQ $(x = a)$

$$\int_{PQ} \mathbf{F}.d\mathbf{r} = \int_0^a (\mathbf{i}x^2 + \mathbf{j}xy).\, \mathbf{j}dy = a\int_0^a ydy = a.\left[\frac{y^2}{2}\right]_{0.}^a = \frac{1}{2}a^3.$$

Along the curve QR $(y = a)$

$$\int_{QR} \mathbf{F}.d\mathbf{r} = \int_a^0 (\mathbf{i}x^2 + \mathbf{j}xy).\, \mathbf{i}dx$$

$$= \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{1}{3}a^3.$$

Along the curve RO $(x = 0)$

$$\int_{RO} \mathbf{F}.\,d\mathbf{r} = \int_a^0 (\mathbf{i}x^2 + \mathbf{j}xy).\,\mathbf{j}dy = 0$$

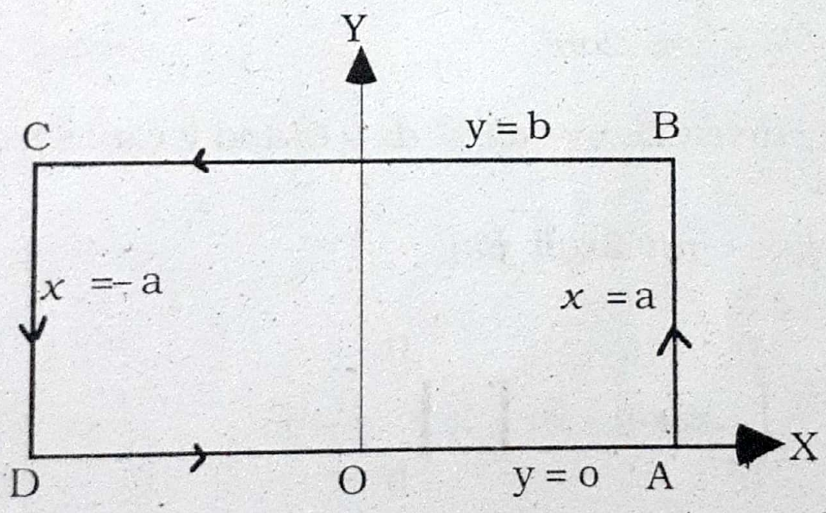
Therefore, $\oint_C \mathbf{F}.\,d\mathbf{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3.$

Hence the Stoke's theorem is verified.

Example 60. Verify Stoke's theorem for $\mathbf{F}=(x^2 + y^2)\mathbf{i}-2xyi$ taken round the rectangle bounded by the lines $x = \pm a$, $y=0$, $y = b$.

Verification : By Stoke's theorem we have

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}).\,\mathbf{n}dS$$



Let ABCD be thé given rectangle as shown in the adjacent figure.

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint \mathbf{F}.d\mathbf{r}, ABCDA)$$

$$= \int \mathbf{F}, AB).d\mathbf{r} + \int \mathbf{F}, BC).d\mathbf{r} + \int \mathbf{F}, CD).\,d\mathbf{r} + \int \mathbf{F}, DA).\,d\mathbf{r}.$$

Along AB, $x = a(i,\,e,\,dx = 0)$ and y varies from 0 to b.

$$\therefore \int_{AB} (\mathbf{F}.AB).\,d\mathbf{r} = \iint [(x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}].\,\mathbf{j}dy$$

$$= 0 - \int_0^b 2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2.$$

Similarly, along BC, $y = b$ (i. e. $dy = 0$ and x varies from a to – a

$$\therefore \int_{BC} \mathbf{F}.d\mathbf{r} = \int_{BC} [(x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}].\mathbf{i}dx$$

$$= \int_a^{-a} (x^2 + b^2)dx = \left[\frac{x^3}{3}\right] + \left[b^2x\right]_a^{-a}$$

$$= -\frac{2}{3}a^3 - 2ab^2.$$

Along the curve CD, $x = -a$(i. e. $dx = 0$) and y varies from b to 0.

$$\int_{CD} \mathbf{F}.d\mathbf{r} = [(x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}].\,\mathbf{j}dy$$

$$= \int_b^0 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_b^0 = -ab^2.$$

Along the curve DA, $y = 0$, (i. e. $dy = 0$) and x varies from–a to a.

$$\therefore \int_{DA} \mathbf{F}.d\mathbf{r} = \int_{DA} [(x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}].\mathbf{i}dy$$

$$= \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{3}a^3 + \frac{1}{3}a^3 = \frac{2}{3}a^3.$$

Thus

$$\oint_C \mathbf{F}.\,d\mathbf{r} = -ab^2 - \frac{2}{3}a^3 - 2ab^3 - ab^2 + \frac{2}{3}a^3 = -4ab^2.$$

Also since

$$\nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \cdot \{(x^2+y^2)\mathbf{i} - 2xy\mathbf{j} + 0\mathbf{k}\}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \mathbf{i}\frac{\delta}{\delta z}(2xy) + \mathbf{j}\frac{\delta}{\delta z}(x^2+y^2) + \mathbf{k}\left\{\frac{\delta}{\delta x}(-2xy) - \frac{\delta}{\delta y}(x^2+y^2)\right\}$$

$$= 0 + 0 - 2\mathbf{k}y - 2ky = -4\mathbf{k}y.$$

$$\therefore \iint_S (\nabla \times \mathbf{F}).\ \mathbf{n}dS = \int_0^b \int_{-a}^{a} (-4\mathbf{k}y).\ \mathbf{k}dxdy$$

$$= \int_0^b \int_{-a}^{a} (-4y)dxdy$$

$$= \int_0^b -4y\left[x\right]_{-a}^{a} dy$$

$$= -8a\int_0^b ydy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -8a.\frac{b^2}{2} = -4ab^2.$$

Hence the Stoke's theorem is verified.

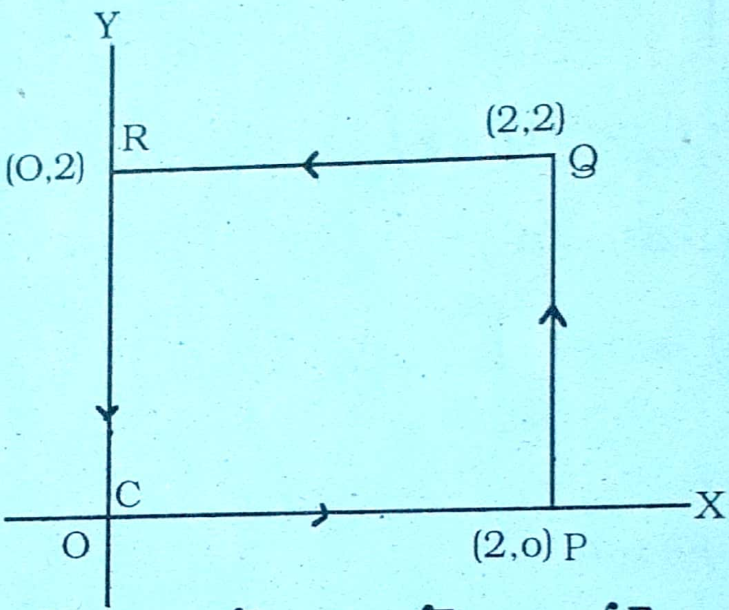
Example 61. Evaluate $\iint_S (\nabla \times \mathbf{F}).\ \mathbf{n}dS$ for

$\mathbf{F} = (y - z + 2)\mathbf{i} + (yz + 4)\mathbf{j} - xz\mathbf{k}$, where S is the surface of the cube $x = y = z = 0$ and $x = y = z = 2$. **[R. U. P. 1980]**

Solution : By Stoke's theorem, we have

$$\oint_C \mathbf{F}.d\mathbf{r} + \iint_S (\nabla \times \mathbf{F}).\ \mathbf{n}dS$$

Here the boundary C of the surface S is a square bounded by the lines. $x = 0$ x, $= 2$, $y = 0$, $y = 2$ in the xy–plane.



$$\text{So} \oint_C \mathbf{F}.\,d\mathbf{r} = \int_{OP} \mathbf{F}.d\mathbf{r} + \int_{PQ} \mathbf{F}.\,d\mathbf{r} + \int_{QR} \mathbf{F}.\,d\mathbf{r} + \int_{RO} \mathbf{F}.d\mathbf{r}.$$

Along OP, $y = 0$ and x varies from o to 2.

$$\therefore \int_{OP} \text{F. dr} = \int_{x=0}^{2} (2\mathbf{i} + 4\mathbf{j}).\ \mathbf{i}dx \quad \left\{ \begin{matrix} \text{since } dy = \theta \\ dz = 0 \end{matrix} \right\}$$

$$= \int_0^2 2dx = 2\left[\ x\ \right]_0^2 = 4.$$

Along PQ, $x = 2$ and y varies from 0 to 2, so that $dx = 0$ and $dz = 0$.

$$\therefore \int_{PQ} \mathbf{F}.d\mathbf{r} = \int_0^2 \{(y + 2)\mathbf{i} + 4\mathbf{j}\}.\ \mathbf{j}dy$$

$$= \int_0^2 4dy = 4\left[\ y\ \right]_0^2 = 8.$$

Along QR, $y = 2$ and x varies from 2 to 0

so that $dy = 0$, $dz = 0$.

$$\therefore \int_{QR} \mathbf{F}.d\mathbf{r} = \int_2^0 (4\mathbf{i} + 4\mathbf{j}).\mathbf{i}dx$$

$$= 4\int_2^0 dx = 4\Big[\ x\ \Big]_2^0 = -8.$$

Along RO, $x = 0$ and y varies from 2 to 0

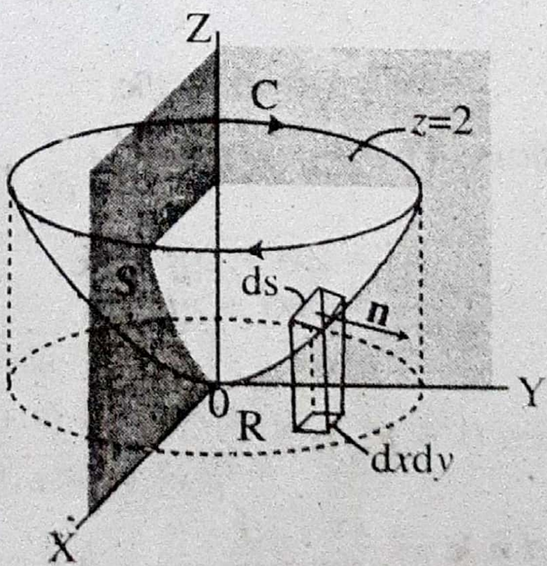
so that $dx = 0$, $dz = 0$.

$$\therefore \int_{RO} \mathbf{F}.d\mathbf{r} = \int_2^0 \{(y + 2)\mathbf{i} + 4\mathbf{j}\}.\mathbf{j}dy$$

$$= \int_2^0 4dy = 4\Big[\ y\ \Big]_2^0 = -8.$$

Thus $\oint_C F.d\mathbf{r} = 4 + 8 - 8 - 8 = -4.$

Example 62. Verify Stoke's theroem for $\mathbf{F} = 3y\mathbf{i} - xz\mathbf{j} + yz^2\mathbf{k}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by z =2 and C is its boundary.



Verification : The boundary C of S is a circle with equations $x^2 + y^2 = 4$, $z = 2$ and parametric equations $x = 2\cos\theta$, $y = 2\sin\theta$, $z = 2$, where $0 \le \theta \le 2\pi$.

Then $\oint_C \mathbf{F}.d\mathbf{r} = \oint_C (3ydx - xzdy + yz^2dz)$ $\{dz = 0\}$

$$= \int_{2\pi}^{0} 3(2\sin\theta)(-2\sin\theta)d\theta - 2\cos\theta\,(2)\,2\cos\theta\, d\theta + 0$$

$$= \int_{0}^{2\pi} (12\sin^2\theta d\theta + 8\cos^2\theta d\theta)$$

$$= 6\int_{0}^{2\pi} (1 - \cos 2\theta)d\theta + 4\int_{0}^{2\pi} (1 + \cos 2\theta)\, d\theta$$

$$= 6\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} + 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 6(2\pi - 0) + 4(2\pi + 0) = 20\pi.$$

Also $\nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times (3y\mathbf{i} - xz\mathbf{j} + yz^2\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\delta}{\delta y}(yz^2) - \frac{\delta}{\delta z}(-zx)\right\} + \mathbf{j}\left\{\frac{\delta}{\delta z}(3y) - \frac{\delta}{\delta x}(yz^2)\right\}$$
$$+ \mathbf{k}\left\{\frac{\delta}{\delta x}(-xz) - \frac{\delta}{\delta y}(3y)\right\}$$

$$= (z^2 + x)\mathbf{i} - (z + 3)\mathbf{k}.$$

Again, $\nabla(x^2 + y^2 - 2z) = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right)(x^2 + y^2 - 2z)$

$$= 2x\mathbf{i} + 2y\mathbf{j} - 2\mathbf{k}.$$

$$\therefore \quad \mathbf{n} = \frac{2(x\mathbf{i} + y\mathbf{j} - \mathbf{k})}{\sqrt{(2x)^2 + (2y)^2 + (-2)^2}} = \frac{x\mathbf{i} + y\mathbf{j} - \mathbf{k}}{\sqrt{x^2 + y^2 + 1}}.$$

Then $(\nabla \times \mathbf{F}).\,\mathbf{n} = \dfrac{(z^2 + x)x + (z + 3)}{\sqrt{x^2 + y^2 + 1}} = \dfrac{z^2x + x^2 + z + 3}{\sqrt{x^2 + y^2 + 1}}$

and $\mathbf{n}.\mathbf{k} = \left(\dfrac{x\mathbf{i} + y\mathbf{j} - k}{\sqrt{x^2 + y^2 + 1}}\right).\,k = -\dfrac{1}{\sqrt{x^2 + y^2 + 1}}.$

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Therefore. $\iint\limits_S (\nabla \times \mathbf{F}).\, \mathbf{n} dS = \iint\limits_R (\nabla \times \mathbf{F}).\, \mathbf{n} \dfrac{dxdy}{|\mathbf{n}.\mathbf{k}|}$

$$= \iint\limits_R \left\{ x\left(\frac{x^2+y^2}{2}\right)^2 + x^2 + \frac{x^2+y^2}{2} + 3 \right\} dxdy$$

Since $z = \dfrac{x^2+y^2}{2}$

Changing the equation to polar coordinates

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} \textbf{limits} \left\{\begin{array}{l} r = 0 \text{ to } 2 \\ \varphi = 0 \text{ to } 2\pi \end{array}\right.$$

Then we have $\iint\limits_S (\nabla \times \mathbf{F}).\, \mathbf{n} dS$

$$= \int_{r=0}^{2} \int_{\varphi=0}^{2\pi} \left\{ r\cos\varphi \left(\frac{r^4}{4}\right) + \frac{r^2}{2} + 3 \right\} r d\theta d\varphi$$

$$= \int_0^{2\pi} \left[\cos\varphi \left[\frac{r^7}{28}\right]_0^2 + \cos^2\varphi \left[\frac{r^4}{4}\right]_0^2 \right] d\varphi$$

$$= \frac{2^5}{7}\int_0^{2\pi} \cos\varphi d\varphi + \frac{2^3}{4}\int_0^{2\pi} (1+\cos 2\varphi)\, d\varphi$$

$$= 0 + 2\left[\varphi + \frac{\sin 2\varphi}{2}\right]_0^{2\pi} + 16\pi + 0,$$

$$= 4\pi + 0 + 16\pi = 20\pi.$$

Hence Stoke's theorem is verified.

Example 63. Verify Stoke's theorem for the vector

$\mathbf{F} = (x + y)\, \mathbf{i} + (2x - z)\, \mathbf{j} + (y + z)\, \mathbf{k}$ taken over the triangle ABC cut from the plane $3x + 2y + z = 6$ by the coordinate planes.

Verification : The given plane is $3x+2y+z=6$

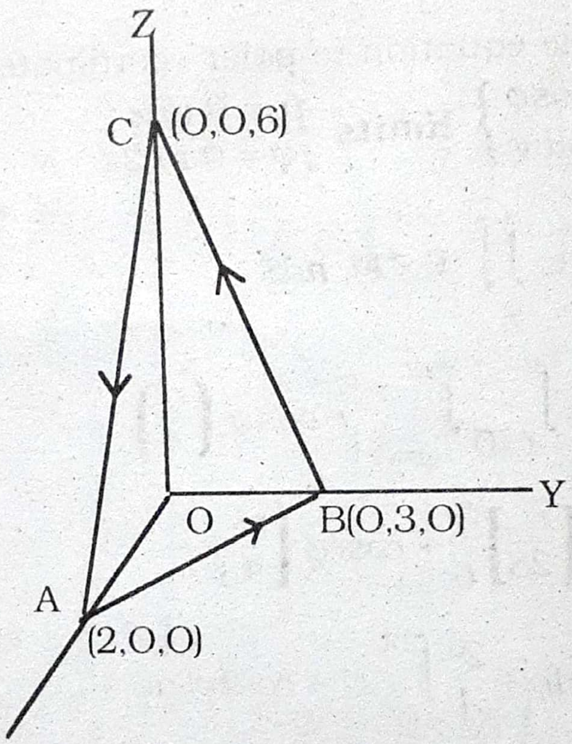
that is, $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$

The intercepts on the axes of the given plane are 2, 3, 6, respectively.

$\therefore$ for the point A, $y = z = 0$, $x = 2$;

for the point B, $z = x = 0$, $y = 3$;

for the point C, $x = y = 0$, $z = 6$



The given vector is $\mathbf{F} = (x+y)\mathbf{i} + (2x - z)\,\mathbf{j} + (y+z)\mathbf{k}$.

$$\nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times \{(x+y)\mathbf{i} + (2x-z)\mathbf{j} + (y+z)\mathbf{k}\}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\delta}{\delta y}(y+z) - \frac{\delta}{\delta z}(2x-z)\right\} + \mathbf{j}\left\{\frac{\delta}{\delta z}(x+y) - \frac{\delta}{\delta x}(y+z)\right\}$$

$$+ \left\{\frac{\delta}{\delta x}(2x-z) - \frac{\delta}{\delta y}(x+y)\right\}$$

$$= \mathbf{i} + \mathbf{i} + 0 + 2\mathbf{k} - \mathbf{k} = 2\mathbf{i} + \mathbf{k}.$$

$$\iint_S (\nabla \times \boldsymbol{F}).\ \boldsymbol{n}dS = \iint_{BOC} (\nabla \times \boldsymbol{F}).\ \boldsymbol{n}dS + \iint_{COA} \nabla \times \boldsymbol{F}).\ \boldsymbol{n}dS$$
$$+ \iint_{AOB} (\nabla \times \boldsymbol{F}.\ \boldsymbol{n}dS$$

For the Face BOC, $dS = \frac{1}{2} dydz;\ \mathbf{n} = \mathbf{i}$

$$\iint_S (\nabla \times \boldsymbol{F}).\ \boldsymbol{n}dS = \int_0^3 \int_0^6 (2\mathbf{i} + \mathbf{k}).\ \mathbf{i}\frac{1}{2} dydz$$

$$= \int_0^3 \int_0^6 dydz = \int_0^3 [z]_0^6\, dy$$

$$= 6\int_0^3 dy = 6\,[y]_0^3 = 18.$$

For the face CAO, $dS = \frac{1}{2} dzdx;\ \mathbf{n} = \mathbf{j}$

$$\iint_{CAO} (\nabla \times \boldsymbol{F}).\ \boldsymbol{n}dS = \int_{z=0}^{6} \int_{x=0}^{2} (2\mathbf{i} + \mathbf{k}).\ \mathbf{j}\frac{1}{2} dzdx = 0$$

For the face OAB, $dS = \frac{1}{2} dxdy;\ \mathbf{n} = \mathbf{k}$

$$\iint_{OAB} (\nabla \times \boldsymbol{F}).\ \boldsymbol{n}dS = \int_{x=0}^{2} \int_{y=0}^{3} (2\mathbf{i}+\mathbf{k}).\ \mathbf{k}\frac{1}{2} dxdy$$

$$= \frac{1}{2}\int_{x=0}^{2} \int_{y=o}^{3} dxdy$$

$$= \frac{1}{2}\int_0^2 [y]_0^3\, dx = \frac{3}{2}\int_0^2 dx = \frac{3}{2}\,[x]_0^2 = 3$$

Therefore, $\iint_S (\nabla \times \boldsymbol{F}),\ \boldsymbol{n}dS = 18 + 0 + 3 = 21.$

Again, $\int_{arcABCA} \mathbf{F}.d\mathbf{r} = \int_{arcBC} \mathbf{F}.\ d\mathbf{r} + \int_{arcCA} \mathbf{F}.\ d\mathbf{r} + \int_{arcAB} \mathbf{F}.\ d\mathbf{r}$

Now along BC, $x = 0$ so that $2y + z = 6$

and $\mathbf{F} = y\mathbf{i} - z\mathbf{j} + (y + z)\,\mathbf{k}$.

$$\therefore \int_{\text{arcBC}} \mathbf{F}.d\mathbf{r} = \int_{\text{arcBC}} \{y\hat{i} - z\hat{j} + (y+z)\mathbf{k}\}.\,(dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_{\text{arcBC}} \{-zdy + (y + z)dz\}$$

$$= \int_3^0 \{-(6-2y)\,dy + (y + 6 - 2y).\,-2dy\}$$

$$= \int_0^3 (6 - 2y + 12 - 2y)\,dy$$

$$= \int_0^3 (18 - 4y)dy$$

$$= [18y - 2y^2]_0^3 = 54 - 18 = 36.$$

Along CA, $y = 0$ so that $3x + z = 6$ and

$\mathbf{F} = x\mathbf{i} + (2x - z)\,\mathbf{j} + z\mathbf{k}$.

$$\therefore \int_{\text{arcCA}} \mathbf{F}.d\mathbf{r} = \int_{\text{arcCA}} \{x\mathbf{i} + (2x-z)\mathbf{j} + z\mathbf{k}\}.\,(dx\mathbf{i} + dz\mathbf{k})$$

$$= \int_{\text{arcCA}} (xdx + zdz)$$

$$= \int_0^2 \{xdx + (6 - 3x) - 3dx\}$$

$$= \int_0^2 (10x - 18)dx$$

$$= [5x^2 - 18x]_0^2 = 20 - 36 = -16.$$

Along AB, $z = 0$ so that $3x + 2y = 6$ and
$\mathbf{F} = (x + y)\,\mathbf{i} + 2x\mathbf{j} + y\mathbf{k}$.

$$\therefore \int_{arcAB} \mathbf{F}.d\mathbf{r} = \int_{arcAB} \{(x+y)\mathbf{i} + 2x\mathbf{j} + y\mathbf{k}\}.\,(dx\mathbf{i} + dy\mathbf{j})$$

$$= \int_{arcAB} (x+y)dx + 2xdy$$

$$= \int_0^2 \left[\left\{x + \frac{6-3x}{2}\right\}dx + 2x.\left(-\frac{3}{2}\right)dx\right]$$

$$= \frac{1}{2}\int_0^2 (6-7x)\,dx$$

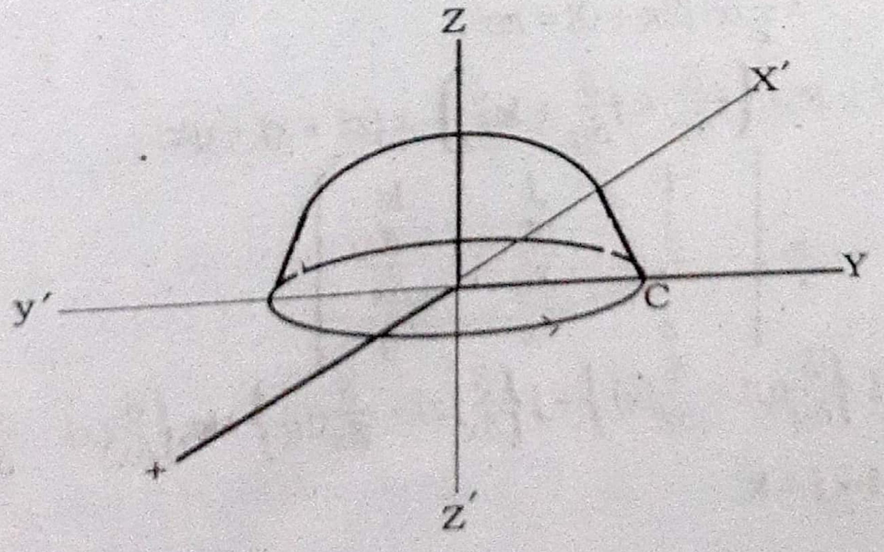
$$= \frac{1}{2}\left[6x - \frac{7}{2}x^2\right]_2^0 = \frac{1}{2}(-12 + 14) = 1.$$

Therefore, $\displaystyle\int_{arcABCA} = 36 - 16 + 1 = 21$

Hence the Stoke's theorem is ver ified.

Example 64. Verify Stoke's theorem for the vector $F = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ taken over the half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy–plane.

Verification :



For Stoke's theorem $\oint_C \mathbf{F} d\mathbf{r} = \iint_S (\nabla \times \boldsymbol{F}).\, \boldsymbol{n} dS$

The projection of the surface on the plane z = 0 (i.e xy – plane) is a circle $x^2+y^2=a^2$ having parametric equations $x=a\cos\theta$, $y = a\sin\theta$ where $0\le\theta\le2\pi$.
Let C be the boundary of the surface.

$$\text{Then } \oint_C \boldsymbol{F}\, d\boldsymbol{r} = \int_C (x\boldsymbol{j} + y\boldsymbol{k}).\,(dx\boldsymbol{i} + dy\boldsymbol{j})$$

$$= \int_C x dy$$

$$= \int_0^{2\pi} a\cos\theta\, d(a\sin\theta)$$

$$= \int_0^{2\pi} a\cos\theta\, a\cos\theta d\theta.$$

$$= \frac{1}{2} a^2. \int_0^{2\pi} (1+\cos2\theta)d\theta.$$

$$= \frac{1}{2} a^2 \left[\theta + \frac{1}{2}\sin2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2} a^2 [2\pi + 0] = \pi a^2.$$

$$\text{Again } \nabla\times\boldsymbol{F} = \left(\boldsymbol{i}\frac{\delta}{\delta x} + \boldsymbol{j}\frac{\delta}{\delta y} + \boldsymbol{k}\frac{\delta}{\delta z}\right) \times (z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k})$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ z & x & y \end{vmatrix}$$

$$= \boldsymbol{i}\left\{\frac{\delta}{\delta y}(y) - \frac{\delta}{\delta z}(x)\right\} + \boldsymbol{j}\left\{\frac{\delta}{\delta x}(z) - \frac{\delta}{\delta x}(y)\right\} + \boldsymbol{k}\left\{\frac{\delta}{\delta x}(x) - \frac{\delta}{\delta y}(z)\right\}$$

$$= \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}.$$

Since R is the projection of S on the xy-plane, $\mathbf{n}=\mathbf{k}$

$$\therefore \iint_S (\nabla \times \mathbf{F}).\mathbf{n}dS = \iint_S (\mathbf{i}+\mathbf{j}+\mathbf{k}).\ \mathbf{n}dS$$

$$= \iint_R (\mathbf{i}+\mathbf{j}+\mathbf{k}).\ \mathbf{k}dxdy$$

$$= \iint_R dxdy$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dxdy$$

$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} dxdy$$

$$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= 4\left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[0 + \frac{a^2}{2}\sin^{-1} 1 - 0\right]$$

$$= 2a^2.\frac{\pi}{2} = \pi a^2.$$

Thus the stoke's theorem is verfified.

Example 65. Verify Stoke's theorem for

$\mathbf{F}=(2x-y)\mathbf{i}-yz^2\mathbf{j}-y^2z\mathbf{k}$, where S is the upper half of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

The boundary C of S is the circle $x^2+y^2=1$.

Verification : Let $x = \cos\theta$, $y = \sin\theta$, and $z = 0$, $0 \le \theta \le 2\pi$ be the parametric equations of C. Then

$$\int_C \mathbf{F}.\, d\mathbf{r} = \int_C (2x-y)\mathbf{i}.\, (dx\mathbf{i}+dy\mathbf{j})$$

$$= \int_C (2x-y)dx$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)\, d(\cos\theta)$$

$$= \int_0^{2\pi} (\sin\theta - 2\cos\theta)\, \sin\theta d\theta.$$

$$= \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 2\sin^2\theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1-\cos 2\theta)d\theta. + \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} + 0$$

$$= \frac{1}{2}(2\pi - 0) + 0 = \pi.$$

Again $\nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\delta}{\delta x} + \mathbf{j}\frac{\delta}{\delta y} + \mathbf{k}\frac{\delta}{\delta z}\right) \times (2x-y)\,\mathbf{i}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2x-y & 0 & 0 \end{vmatrix}$$

$$= 0 + \mathbf{j}\left\{\frac{\delta}{\delta z}(2x-y)-0\right\} + \mathbf{k}\left\{0-\frac{\delta}{\delta y}(2x-y)\right\}$$

$$= 0 + 0 + \mathbf{k} = \mathbf{k}.$$

$$\therefore \quad \iint_S (\nabla \times \mathbf{F}).\, \mathbf{n}dS = \iint_S \mathbf{k}.\mathbf{n}dS = \iint_R k.kdxdy = \iint_R dxdy$$

Since $\mathbf{k}.\mathbf{n}dS = \mathbf{k}.\mathbf{k}\, dxdy$ and R is the projection of S on the xy — plane.

$$\text{Now } \iint_R dxdy = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dxdy$$

$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} dxdy$$

$$= 4\int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= 4\int_0^1 \sqrt{1-x^2}dx)$$

$$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= 4\left[0 + \frac{1}{2}\sin^{-1} 1\right] = 2.\frac{\pi}{2} = \pi.$$

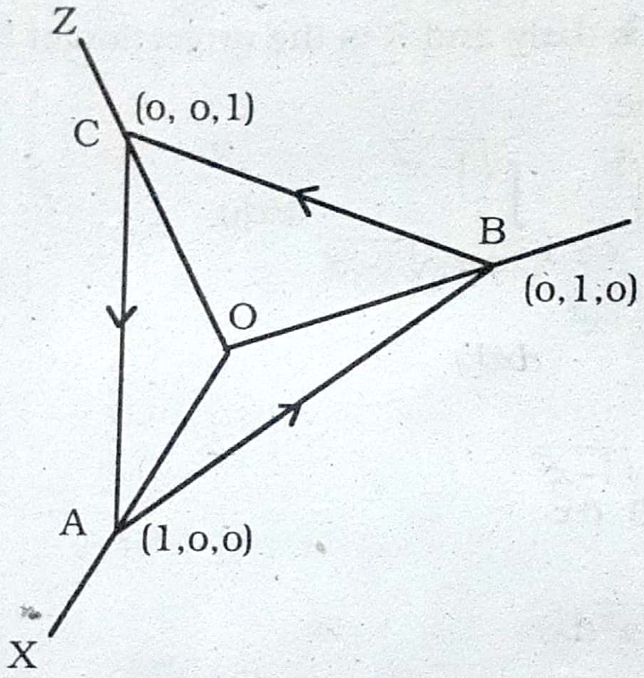
Thus $\iint_S (\nabla \times \mathbf{F}).\, \mathbf{n}dS = \pi$. So Stoke's theorem is verified.

Example 66. Verify Stoke's theorem for $\mathbf{F} = (2y + z)\mathbf{i} + (x-z)\mathbf{j} + (y - z)\,\mathbf{k}$ taken over the triangle ABC cut from the plane $x + y + z = 1$ by the coordinate planes.

[R.U.P. 1969]

Verification : The given plane is $\frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$.

Its interepts on the axes are 1, 1, 1 respectively. So the plane intersects the coordinate axes at $A(1,0,0)$, $B(0, 1, 0)$ and $C(0,0,1)$. The given vector is $\mathbf{F}=(2y+z)\mathbf{i}+(x-z)\mathbf{j}+(y-z)\mathbf{k}$.



$$\nabla\times\mathbf{F}=\left(\mathbf{i}\frac{\delta}{\delta x}+\mathbf{j}\frac{\delta}{\delta y}+\mathbf{k}\frac{\delta}{\delta z}\right)\{(2y+z)\,\mathbf{i}+(x-z)\,\mathbf{j}+(y-z)\mathbf{k}\}$$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{l} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2y+z & x-z & y-z \end{vmatrix}$$

$$=\mathbf{i}\left\{\frac{\delta}{\delta y}(y-z)-\frac{\delta}{\delta z}(x-z)\right\}+\mathbf{j}\left\{\frac{\delta}{\delta z}(2y+z)-\frac{\delta}{\delta x}(y-z)\right\}$$

$$+\mathbf{k}\left\{\frac{\delta}{\delta x}(x-z)-\frac{\delta}{\delta y}(2y+z)\right\}$$

$$=\mathbf{i}(1+1)+\mathbf{j}(1-0)+\mathbf{k}(1-2)$$

$$=2\mathbf{i}+\mathbf{j}-\mathbf{k}.$$

Now $\iint\limits_S (\nabla \times \mathbf{F}).\mathbf{n}dS = \iint\limits_{BOC} (\nabla \times \mathbf{F}).\mathbf{n}dS + \iint\limits_{COA} (\nabla \times \mathbf{F}).\mathbf{n}dS.$

$$+\iint\limits_{AOB} (\nabla \times F).ndS.$$

For the face BOC, $dS = \frac{1}{2}\, dydz$, $\mathbf{n} = \mathbf{i}$

$$\iint\limits_S (\nabla \times \mathbf{F}).\mathbf{n}S = \int_0^1 \int_0^1 (2\mathbf{i} + \mathbf{j} - \mathbf{k}).\, \mathbf{i}\frac{1}{2}dydz$$

$$= \int_0^1 \int_0^1 \mathrm{dydz} = 1.$$

$$\iint\limits_{COA} (\nabla \times \mathbf{F}).\, \mathbf{n}dS = \int_0^1 \int_0^1 (2\mathbf{i} + \mathbf{j} - \mathbf{k}).\, \mathbf{j}\frac{1}{2}dxdy$$

$$= \frac{1}{2}\int_0^1 \int_0^1 dzdx = \frac{1}{2}.$$

$$\iint\limits_{AOB} (\nabla \times \mathbf{F}).\mathbf{n}dS = \int_0^1 \int_0^1 (2\mathbf{i} + \mathbf{j} - \mathbf{k}).\, \mathbf{k}\frac{1}{2}dxdy$$

$$= -\frac{1}{2}\int_0^1 \int_0^1 dxdy = -\frac{1}{2}.$$

$$\therefore \quad \iint\limits_S (\nabla \times F).ndS = 1 + \frac{1}{2} - \frac{1}{2} = 1.$$

Again $\int\limits_{\text{arc } ABCA} \mathbf{F}.\mathrm{d}\mathbf{r} = \int\limits_{\text{arc } AB} \mathbf{F}.\, \mathrm{d}\mathbf{r} + \int\limits_{\text{arc } BC} \mathbf{F}.\, \mathrm{d}\mathbf{r} + \int\limits_{\text{arc } CA} \mathbf{F}.\, \mathrm{d}\mathbf{r}$

Along BC, $x = 0$ so that

$\mathbf{F} = (2y + z)\mathbf{i} - z\mathbf{j} + (y - z)\,\mathbf{k}$ and $y + z = 1.$

$$\therefore \int_{\text{arc BC}} \mathbf{F}.d\mathbf{r} = \int_{\text{arc BC}} \{2y + z)\,\mathbf{i} - z\mathbf{j} + (y-z)\,k\}.\,(dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_{\text{arc BC}} \{-zdy + (y - z)dz\}$$

$$= \int_1^0 [-(1 - y)dy + \{y - (1 - y)\}\, d(1-y)]$$

$$= \int_1^0 (y-1-2y + 1)dy$$

$$= -\int_1^0 ydy = -\left[\frac{y^2}{2}\right]_1^0 = \frac{1}{2}.$$

Along CA, $y = 0$, so that

$$\mathbf{F} = z\mathbf{i} + (x - z)\mathbf{j} - zk \text{ and } x + z = 1.$$

$$\therefore \int_{\text{arc CA}} \mathbf{F}.d\mathbf{r} = \int_{\text{arc CA}} \{z\mathbf{i} + (x-z)\mathbf{j} - zk\}.\,(dx\mathbf{i}, + dz\mathbf{k})$$

$$= \int_{\text{arc CA}} (zd\,x - zdz)$$

$$= \int_1^0 \{zd(1-z) - zdz\}$$

$$= \int_1^0 (-zdz - zdz)$$

$$= -2\int_1^0 zdz = -2\left[\frac{z^2}{2}\right]_1^0 = 1.$$

Along AB, $z = 0$ so that $\mathbf{F} = 2y\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and $x + y = 1$.

$$\therefore \int_{\text{arcAB}} \mathbf{F}.d\mathbf{r} \int_{\text{arcAB}} (2y\mathbf{i} + x\mathbf{j} + y\mathbf{k}).\,(dx\mathbf{i} + dy\mathbf{j})$$

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$$= \int_{arcAB} (2y dx + x dy).$$

$$= \int_1^0 \{2(1-x)dx + xd(1-x)\}$$

$$= \int_1^0 \{(2-2x)dx - xdx\}$$

$$= \int_1^0 (2-3x)dx$$

$$= 2\,[x]_1^0 - 3\left[\frac{x^2}{2}\right]_1^0$$

$$= -2 + \frac{3}{2} = -\frac{1}{2}.$$

Therefore, $\displaystyle\iint_{arc\ ABCA} \mathbf{F}.\, d\mathbf{r} = \frac{1}{2} + 1 - \frac{1}{2} = 1.$

Hence the Stoke's theorem is verified.

EXERCISES 4(D)

[GREEN'S THEOREM]

1. Evaluate by Green's theorem

(i) $\oint (e^x \sin y\,dx + e^x \cos y\,dy)$ around the rectangle with vertices $(0, 0)$, $1,0)$, $(1, \pi/2)$, $(0, \pi/2)$.

(ii) $\oint (2x^3 - y^3)\,dx + (x^3 + y^3)dy\}$ around the circle $x^2 + y^2 = 1$.

Answer : (i) 0 (ii) $\frac{3}{2}\pi$

2. Use Green's theorem to find the integral $\oint_C \{(2x-y)dx + (x+3y)dy\}$ where C is the ellipse with equation $b^2x^2 + a^2y^2 = a^2b^2$

Answer : $2\pi ab$.

3. Use Green's theorem to evaluate the integral

$$\oint_C \left\{\frac{ydx-(x-1)dy}{(x-1)^2+y^2}\right\} \text{ where } C \text{ is the graph of } x^2+y^2=4.$$

Answer : O.

4. Use Green's theorem to evaluate the integral

$$\oint_C \{xy^6dx+(3x^2y^5+6x)dy\}$$

where C is ellipse $x^2+4y^2=4$.

Answer : 12π.

5. Use Green's theroem to evaluate the line integral

$$\oint_C \{2\tan^{-1}\{y/x\}\,dx+\log(x^2+y^2)dy\}$$

where C is the circle $(x-2)^2+y^2=1$.

Answer : 0.

6. Use Green's theorem to find the integral

$\oint_C (e^x\sin y dx + e^x\cos y dy)$ where C is any regular closed curve passing through the points (0, 0) and (a, b).

Answer : $e^a \sin b$.

7. Use Green's theorem to evaluate the integral

$\oint_C \{(x^3-3y)dx+(x+\sin y)dy\}$ where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, $(o, 2)$.

Answer : 4.

8. Use Green's therem to evaluate the integral

$\oint_C (x-y)^3 (dx+dy)\}$ where C is the graph of $x^2+y^2=a^2$.

Answer : $3\pi a^4$.

9. Evaluate the line integral

$\oint_C \{(x^2 + xy)dx + (x^2 + y^2)dy\}$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

Answer : 0.

10. Verify Green's theorem for $\oint_C (x^2 y dx + xy^2 dy)$ where C is the boundary of the region R in the first quadrant bounded by the graphs of $y=x$ and $y^3 = x^2$.

11. Verify Green's theorem for

$\oint_C \{(3x^2 - 8y^2)dx + (4y - 6xy)dy\}$ where C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$.

[common value =3/2]

12. Verify Green's theorem in the plane for

$\oint_C \{(2x - y^3)dx + xydy\}$ where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

[common value =60π]

13. Verify Green's theorem in the plane for

$\oint_C \{(x^3 - x^2 y)dx + xy^2 dy\}$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

[common value = 120π]

14. Evaluate $\int_C \{2xy^3 - y^2\cos x)dx + (1 - 2y\sin x + 3x^2 y^2)\, dy\}$ along the parabola $2x = \pi y^2$ from $(0,0)$ to $(\pi/2, 1)$.

Answer : $\frac{\pi^2}{4}$

15. (i) Find the area bounded by the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ having parametric equations $x = a\cos^3\theta,\ y = a\sin^3\theta,$

$0 \le \theta \le 2\pi.$

(ii) Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ a > 0$ *and the* $x-$*axis.*

Answer : (i) $\frac{3}{8}\pi a^2$. (ii) $3\pi a^2$

(GAUSS'S DIVERGENCE THEOREM)

16. Evaluate $\iint_S \mathbf{F}.\mathbf{n}ds$ *where*

$\mathbf{F} = (2x + 3z)\mathbf{i} - (xz + y)\mathbf{j} + (y^2 + 2z)\mathbf{k}$ and S is the surface of the sphere having centre at $(3, -1, 2)$ *and radius* 3.

Answer : 108π.

17. Use Gauss's divergence therem to evaluate $\iint_S \mathbf{F}.\ \mathbf{n}dS$ where

$\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Answer : $\frac{12}{5}\pi a^5$.

18. Evaluate $\iint_S \mathbf{F}.\mathbf{n}dS$ *where*

$\mathbf{F} = (z^2 - x)\mathbf{i} - xy\mathbf{j} + 3z\mathbf{k}$ *and* S is the surface of the region bounded by $z = 4 - y^2,\ x = 0,\ x = 3$ and the $xy-$plane.

Answer : 16.

19. Evaluate $\iint_S \mathbf{F} . \mathbf{n} dS$, *where* $\mathbf{F} = 2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k}$ *and* S is the surface of the parallelopiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1$, *and* $z = 3$. **[D. U. P. 1981]**

Answer : 30.

20. Verify Gauss's divergence theorem for $\mathbf{F} = (2xy + z)\mathbf{i} + y^2\mathbf{j} - (x + 3y)\mathbf{k}$ taken over the region bounded by $2x + 2y + z = 6, x = 0, z = 0$. [common value = 27]

21. Verify Gauss's divergence theorem for the funnction $F = 2xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ taken over upper half of the sphere $x^2 + y^2 + z^2 = a^2$. [common value = $\frac{5}{4}\pi a^4$]

22. Verify Gauss's divergence theorem for the function $F = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ taken over the cube whose vertices are (o, o, o), (1, o, o), (1, 1, o), (o,1,o) (o, o, 1), (1, o, 1), (1, 1, 1) *and* (o, 1, 1).

[common *value* = $\frac{3}{2}$].

23. Vertfy Gauss's divergence theorem for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ over the tetrahedron with vertices (o, o, o), (1, o, o), (o, 2, o) *and* (o, o, 1). [common value = $\frac{4}{3}$]

24. By transforming to a triple integral evaluate

$$I = \iint_S x^2 (xdydz + ydzdx + zdxdy)$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(o \le z \le b)$ and the circular disks $z = o$ *and* $z = b$ $(x^2 + y^2 \le a^2)$.

Answer : $\frac{5}{4}\pi a^4 b$.

(STOKE'S THEOREM)

25. Compute $\oint_C \mathbf{F}.d\mathbf{r}$ around the circle $(x-1)^2 + y^2 = 1,\ z = 3,$ when $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2\mathbf{k}$. **[R. U. P. 1969]**

Answer : 2π

26. Evaluate, $\iint_S (\nabla \times \mathbf{F}).\ d\mathbf{S}$ where S is the srface of the paraboloid $x^2 + y^2 + z = 4$ above the xy – plane and $F = (x^2 + y + 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$.

Answere : -4π.

27. Evaluate $\iint_S (\nabla \times \mathbf{F})\ .\mathbf{n}dS$ where $F = (x-z)\mathbf{i} + (x^3 + yz)\mathbf{j} - 3xy^2\mathbf{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy – plane.

Answer : 12π.

28. If $\mathbf{F} = 2yz\mathbf{i} - (x + 3y - 2)\mathbf{j} - (x^2 + z)\mathbf{k}$, evaluate $\iint_S (\nabla \times \mathbf{F}).\mathbf{n}dS$ over the surface of interssction of the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ which is included in the first octant.

Answer : $-\dfrac{a^2}{12}(3\pi + 8a)$.

29. Apply Stoke's theomem to evaluate $\oint_C (ydx + zdy + xdx)$ where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

Answer :– $\dfrac{\pi a^2}{\sqrt{2}}$

30. If $F = y\mathbf{i} - (x - 2xz)\mathbf{j} - xy\mathbf{k}$, evaluate $\iint_S (\nabla \times \mathbf{F}).\, \mathbf{n} dS$,

where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy – plane.

Answer : 0.

31. Evaluate $\iint_S (\nabla \times \mathbf{F}).\, \mathbf{n} dS$ where S is the portion of the sphere $x^2 + y^2 + z^2 = 16$ below the plane $z = 2$, $\mathbf{n}$ is the outward normal and $\mathbf{F} = yz\mathbf{i} + 3xz\mathbf{j} + z^2\mathbf{k}$.

Answer : -48π.

32. Evaluate $\iint_S (\nabla \times \mathbf{F}).\, \mathbf{n} dS$ where S is the cylindrical surface $x^2 + y^2 = a^2$ between $z = 0$ and $z = b$; $\mathbf{n}$ is the outward normal and $\mathbf{F} = xz\mathbf{i} + (4 + z^2)\mathbf{j} + xe^{yz}\mathbf{k}$.

Answer : 0.

33. Verify Stoke's theorem for the vector field

$\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$ integrated round the rectangle in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$.

34. Verify Stoke's theorem for

$\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is its boundary.

(commnon value = 9π]

35. Verify Stoke's theorem for $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$, where S is the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane z = o and C is its boundary (the curve of intersectioin). [common value = 8π.]

10. 28. Transformation of coordinates

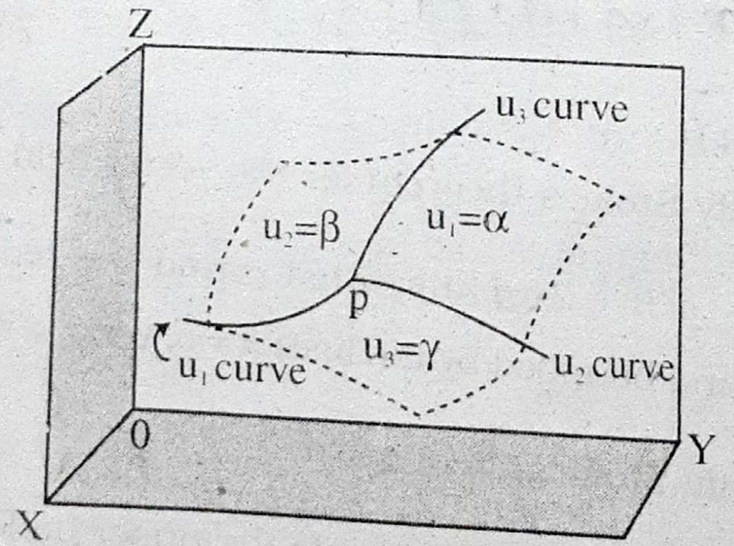
Consider the rectangular coordinates (x, y, z) of any point which can be expressed as functions of (u_1, u_2, u_3) so that

$x = x(u_1, u_2, u_3)$ $y = y(u_1, u_2, u_3)$, $z = z(u_1, u_2, u_3)$, (1)

Let the equations in (1) be solved for u_1, u_2, u_3 in terms of x, y, z; that is, $u_1 = u_1(x, y, z)$, $u_2 = u_2(x, y, z)$, $u_3 = u_3(x, y, z)$ (2)

We assume that the functions in (1) and (2) are single valued and have continuous derivatives so that the correspondence between (x, y, z) and (u_1, u_2, u_3) is unique. Then (u_1, u_2, u_3) are called **curvilinear coordinates** of (x, y, z). The sets of equations (1) or (2) define a **transformation of co-ordinates.**

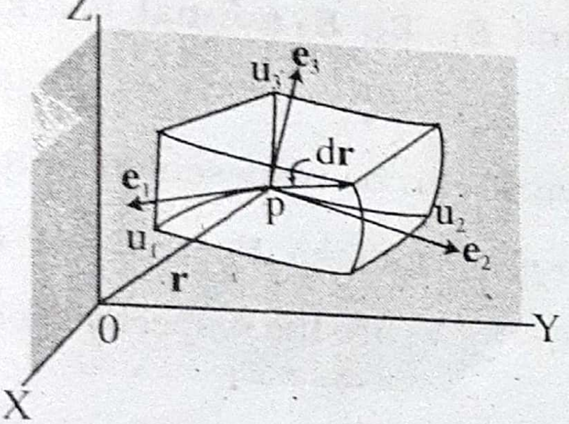
10. 29. Orthogonal Curvilinear coordinates.



The surfaces $u_1 = \alpha$, $u_2 = \beta$, $u_3 = \gamma$, where α, β, γ are constants are called **coordinate surfaces** and each pair of these surfaces intersect in curves **called coordinate curves or lines.**

In particular when the coordinate surfaces intersect at right angles, the three coordinate curves are also mutually **orthogonal** and u_1, u_2, u_3 are called the **orthogonal curvilinear coordinates.**

10.30 Unit vectors in curvilinear systems



Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P. Then (1) can be written as $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$. A tangent vector to the u_1 curve at P (for which u_2 and u_3 are constants) is $\dfrac{\delta r}{\delta u_1}$.

Then a unit tangent vector in this direction is

$$\mathbf{e}_1 = \frac{\delta \mathbf{r}}{\delta u_1} \Big/ \left|\frac{\delta \mathbf{r}}{\delta u_1}\right| \text{, so that } \frac{\delta \mathbf{r}}{\delta u_1} = h_1 e_1 \text{ where, } h_1 = \left|\frac{\delta \mathbf{r}}{\delta u_1}\right|$$

Similarly, if $\mathbf{e_2}$ and $\mathbf{e_3}$ are unit tangent vectors to u_2 and u_3 curves at P respectively then

$$\frac{\delta \mathbf{r}}{\delta u_2} = h_2\mathbf{e_2} \text{ and } \frac{\delta \mathbf{r}}{\delta u_3} = h_3\mathbf{e_3} \text{ where } h_2 = \left|\frac{\delta \mathbf{r}}{\delta u_2}\right| \text{ and } h_3 = \left|\frac{\delta \mathbf{r}}{\delta u_3}\right|.$$

The quantities h_1, h_2, h_3 are called **scale factors.**

Since ∇u_1 is a vector at P normal to the surface $u_1 = \alpha$, a unit vector in this direction is given by $\mathbf{E}_1 = \dfrac{\nabla u_1}{|\nabla u_1|}$.

Similarly, the unit vectors $\mathbf{E}_2 = \dfrac{\nabla u_2}{|\nabla u_2|}$ and $\mathbf{E}_3 = \dfrac{\nabla u_3}{|\nabla u_3|}$

at P are normal to the surfaces $u_2 = \beta\ annd . u_3 = \gamma$ respectively.

Thus at each point P of a curvilinear system there exist, in general, two sets of unit vectors. $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ **tangent** to the coordinate curves and $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ **normal** to the coordinate surfaces.

Therefore, a vector **V** can be respesented in terms of unit base vectors $V = v_1\boldsymbol{e_1} + v_2\boldsymbol{e_2} + v_3\boldsymbol{e_3} = V_1\boldsymbol{E_1} + V_2\mathbf{E_2} + V_3\mathbf{E_3}$

where v_1, v_2, v_3 *and* V_1, V_2, V_3 are the respective components of V in each system.

Now we assume that the system (u_1, u_2, u_3) is such that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right handed orthogonal system ie. $[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3] = 1$.

Hence the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ satisfy relations respectively similar to those for **i, j,** and **k**. Since the gradient of a scalar point function at a given point is perpendicular to the level surface of the function passing through that point, we have in orthogonal system.

$\boldsymbol{e}_1 = h_1\ \nabla u_1, \mathbf{e}_2 = h_2\ \nabla u_2; \mathbf{e}_3 = h_3\ \nabla u_3$ *where* $h_1\ h_2, h_3$ are multipliers and in general functions of the coordinates.

Also we have

$$\boldsymbol{e}_1 = \boldsymbol{e}_2 \times \boldsymbol{e}_3 = h_2\, h_3\, (\nabla u_2 \times \nabla u_3)$$

$$\boldsymbol{e}_2 = \boldsymbol{e}_3 \times \boldsymbol{e}_1 = h_3 h_1\, (\nabla u_3 \times \nabla u_1)$$

$$e_3 = e_1 \times e_2 = h_1 h_2 (\nabla u_1 \times \nabla u_2)$$

and $[\nabla u_1 \; \nabla u_2 \; \nabla u_3] = \left[\frac{e_1}{h_1} \; \frac{e_1}{h_2} \; \frac{e_3}{h_3}\right]$

$$= \frac{1}{h_1 h_2 h_3} [e_1 \; e_2 \; e_3] = \frac{1}{h_1 h_2 h_3}.$$

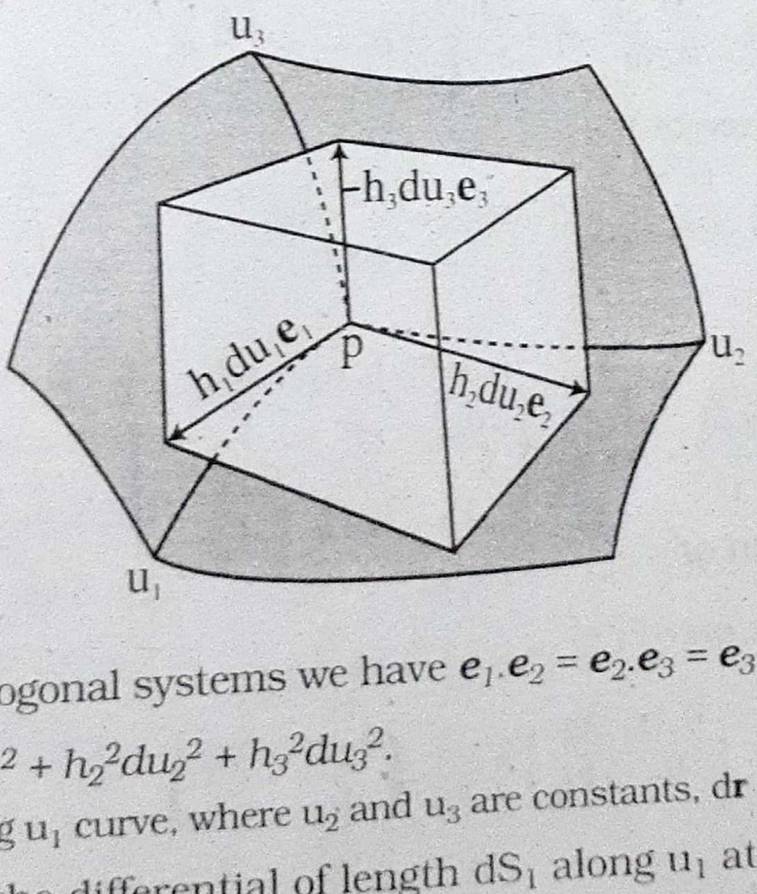
10. 31. Arc length and volume elements.

When $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ *is the position vector of a point P, then we have*

$$d\mathbf{r} = \frac{\delta \mathbf{r}}{\delta u_1} du_1 + \frac{\delta \mathbf{r}}{\delta u_2} du_2 + \frac{\delta \mathbf{r}}{\delta u_3} du_3$$

$$= h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$$

Therefore, the differential of arc length dS is determined from $dS^2 = d\mathbf{r}.d\mathbf{r}$.



For orthogonal systems we have $\mathbf{e}_1.\mathbf{e}_2 = \mathbf{e}_2.\mathbf{e}_3 = \mathbf{e}_3.\mathbf{e}_1 = 0$. and $dS^2 = h_1{}^2 du_1{}^2 + h_2{}^2 du_2{}^2 + h_3{}^2 du_3{}^2$.

Now along u_1 curve, where u_2 and u_3 are constants, $d\mathbf{r} = h_1 du_1 \mathbf{e}_1$. Therefore, the differential of length dS_1 along u_1 at P is $h_1 du_1$. Similarly, the differential arc lengths along u_2 and u_3 at P are $dS_2 = h_2 du_2$ and $dS_3 = h_3 du_3$ respectively.

Thus the volume element for an orthogonal curvilinear coordinate system is given by

$$d\bar{V} = (h_1\, du_1\, \mathbf{e}_1).(h_2 du_2 \mathbf{e}_2) \times (h_3 du_3 \mathbf{e}_3)$$

$$= h_1 h_2 h_3\, du_1 du_2 du_3\, |\,\mathbf{e}_1.e_2 \times \mathbf{e}_3\,|$$

$$= h_1 h_2 h_3\, du_2 du_3\, du_1 \text{ since } |\,\mathbf{e}_1.\mathbf{e}_2 \times \mathbf{e}_3\,| = 1.$$

10.32 Differential operators in terms of curvilinear coordinates.

[Gradient, Divergence, Curl and Laplacian in orthogonal curvilinear coordinates]

If φ is a scalar function and $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, then we have the following results :

(a) Gradient of φ

$$\nabla\varphi = grad\varphi = \frac{1}{h_1}\frac{\delta\varphi}{\delta u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\delta\varphi}{\delta u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\delta\varphi}{\varphi u_3}\mathbf{e}_3$$

(b) Divergence of F

$$\nabla.\mathbf{F} = div\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\delta}{\delta u_1}(h_2 h_3 F_1) + \frac{\delta}{\delta u_2}(h_3 h_1 F_2) + \frac{\delta}{\delta u_3}(h_1 h_2 F_3)\right]$$

(e) Curl of F

$$\nabla \times \mathbf{F} = curl\mathbf{F} = \frac{1}{h_1\, h_2\, h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\delta}{\delta u_1} & \dfrac{\delta}{\delta u_2} & \dfrac{\delta}{\delta u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

(d) Laplacian of φ

$$\nabla^2\varphi = \text{Laplacian of } \varphi$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\delta}{\delta u_1}\left(\frac{h_2 h_3}{h_1}\frac{\delta\varphi}{du_1}\right) + \frac{\delta}{\delta u_2}\left(\frac{h_3 h_1}{h_2}\frac{\delta\varphi}{\delta u_2}\right) + \frac{\delta}{\delta u_3}\left(\frac{h_1 h_2}{h_3}\frac{\delta\varphi}{\delta u_3}\right)\right]$$

Proof of (a) : Let $\nabla\varphi = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$ (1)

Then we have to determine α_1, α_2, *and* α_3.

$$\text{Since } d\mathbf{r} = \frac{\delta \mathbf{r}}{\delta u_1} du_1 + \frac{\delta \mathbf{r}}{\delta u_2} du_2 + \frac{\delta \mathbf{r}}{\delta u_3} du_3$$

$$= h_1\mathbf{e}_1 du_1 + h_2\mathbf{e}_2 du_2 + h_3\mathbf{e}_3 du_3.$$

Thus we have $d\varphi = \nabla\varphi.\, d\mathbf{r} = h_1\alpha_1 du_1 + h_2\alpha_2 du_2 + h_3\alpha_3 du_3$ (2)

but $d\varphi = \frac{\delta\varphi}{\delta u_1} du_1 + \frac{\delta\varphi}{\delta u_2} du_2 + \frac{\delta\varphi}{\delta u_3} du_3$ (3)

Equating corresponding coefficients of du_1, du_2 & du_3 *respectively from right hands sides of (2) and (3) we get*

$$h_1\alpha_1 = \frac{\delta\varphi}{\delta u_1},\ h_2\alpha_2 = \frac{\delta\varphi}{\delta u_2},\ h_3\alpha_3 = \frac{\delta\varphi}{\delta u_3}$$

$$\text{or, } \alpha_1 = \frac{1}{h_1}\frac{\delta\varphi}{\delta u_1},\ \alpha_2 = \frac{1}{h_2}\frac{\delta\varphi}{\delta u_2},\ \alpha_3 = \frac{1}{h_3}\frac{\delta\varphi}{\delta u_3}$$

Putting the values of α_1, α_2 *and* α_3 *in (1) we get*

$$\nabla\varphi = \frac{e_1}{h_1}\frac{\delta\varphi}{\delta u_1} + \frac{e_2}{h_2}\frac{\delta\varphi}{\delta u_2} + \frac{e_3}{h_3}\frac{\delta\varphi}{\delta u_3}.$$

This indicates the operator equivalence

$$\nabla = \frac{e_1}{h_1}\frac{\delta}{\delta u_1} + \frac{e_2}{h_2}\frac{\delta}{\delta u_2} + \frac{e_3}{h_3}\frac{\delta}{\delta u_3}$$

which reduces to the usual expression for the operator ∇ in rectangular coordinates.

Proof of (b) : *Let* $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$

$$\text{then } \nabla.\mathbf{F} = \nabla.(F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3)$$

$$= \nabla.(F_1\mathbf{e}_1) + \nabla.(F_2\mathbf{e}_2) \cdot \nabla.(F_3\mathbf{e}_3) \quad (1)$$

Since for an orthogonal system

$$\nabla u_1 = \frac{\mathbf{e}_1}{h_1},\ \nabla u_2 = \frac{\mathbf{e}_2}{h_2},\ \nabla u_3 = \frac{\mathbf{e}_3}{h_3},$$

$$\text{Then } \nabla u_2 \times \nabla u_3 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{h_2 h_3} = \frac{\mathbf{e}_1}{h_2 h_3}$$

$$\text{or, } \mathbf{e}_1 = h_2 h_3\, \nabla u_2 \times \nabla u_3$$

Similarly, $\mathbf{e}_2 = h_3 h_1 \nabla u_3 \times \nabla u_1$

$\mathbf{e}_3 = h_1 h_2 \nabla u_1 \times \nabla u_2$

Now $\nabla.(F_1\mathbf{e}_1) = \nabla.(F_1 h_2 h_3 \nabla u_2 \times \nabla u_3)$

$$= \nabla(F_1 h_2 h_3).\nabla u_2 \times \nabla u_3 + F_1 h_2 h_3 \nabla.(\nabla u_2 \times \nabla u_3)$$

$$= \nabla(F_1 h_2 h_3).\frac{\mathbf{e}_2 \times \mathbf{e}_3}{h_2 h_3} + 0 = \nabla(F_1 h_2 h_3).\frac{\mathbf{e}_1}{h_2 h_3}$$

$$= \left[\frac{\mathbf{e}_1}{h_1}\frac{\delta}{\delta u_1}(F_1 h_2 h_3) + \frac{\mathbf{e}_2}{h_2}\frac{\delta}{\delta u_2}(F_1 h_2 h_3) + \frac{\mathbf{e}_3}{h_3}\frac{\delta}{\delta u_3}(F_1 h_2 h_3)\right].\left(\frac{\mathbf{e}_1}{h_2 h_3}\right)$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\delta}{\delta u_1}(F_1 h_2 h_3) \quad (2)$$

Similarly, $\nabla.(F_2 e_2) = \dfrac{1}{h_1 h_2 h_3}\dfrac{\delta}{\delta u_2}.(F_2 h_3 h_1) \quad (3)$

$$\nabla.(F_3 e_3) = \frac{1}{h_1 h_2 h_3}\frac{\delta}{\delta u_3}(F_3 h_1 h_2) \quad (4)$$

Adding (2), (3) and (4), we get

$$\nabla.\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\delta}{\delta u_1}(F_1 h_2 h_3) + \frac{\delta}{\delta u_2}(F_2 h_3 h_1) + \frac{\delta}{\delta u_3}(F_3 h_1 h_2)\right].$$

Proof of (c) : Let $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$

then $\nabla \times \mathbf{F} = \nabla \times (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3)$

$$= \nabla \times (F_1\mathbf{e}_1) + \nabla \times (F_2\mathbf{e}_2) + \nabla \times (F_3\mathbf{e}_3)$$

since for an orthogonal system

$$\nabla u_1 = \frac{e_1}{h_1},\ \nabla u_2 = \frac{e_2}{h_2};\ \nabla u_3 = \frac{e_3}{h_3}$$

i, e, $\mathbf{e}_1 = h_1 \nabla u_1$, $\mathbf{e}_2 = h_2 \nabla u_2$, $\mathbf{e}_3 = h_3 \nabla u_3$.

Now $\nabla \times (F_1\mathbf{e}_1) = \nabla \times (F_1 h_1 \nabla u_1)$

$$= \nabla(F_1 h_1) \times \nabla u_1 + F_1 h_1 \nabla \times \nabla u_1$$

$$= \nabla(F_1 h_1) \times \nabla u_1 + 0$$

$$= \left[\frac{\mathbf{e}_1}{h_1}\frac{\delta}{\delta u_1}(F_1 h_1) + \frac{\mathbf{e}_2}{h_2}\frac{\delta}{\delta u_2}(F_1 h_1) + \frac{\mathbf{e}_3}{h_3}\frac{\delta}{\delta u_3}(F_1 h_1)\right] \times \nabla u_1$$

$$= \left[\frac{\mathbf{e}_1}{h_1}\frac{\delta}{\delta u_1}(F_1 h_1) + \frac{\mathbf{e}_2}{h_2}\frac{\delta}{\delta u_2}(F_1 h_1) + \frac{\mathbf{e}_3}{h_3}\frac{\delta}{\delta u_3}(F_1 h_1)\right] \times \frac{\mathbf{e}_1}{h_1}$$

$$= \frac{\mathbf{e}_1 \times \mathbf{e}_1}{h_1^2} \frac{\delta}{\delta u_1} (F_1 h_1) + \frac{\mathbf{e}_2 \times \mathbf{e}_1}{h_2 h_1} \frac{\delta}{\delta u_2} (F_1 h_1) + \frac{\mathbf{e}_3 \times \mathbf{e}_1}{h_3 h_1} \frac{\delta}{\delta u_3} (F_1 h_1)$$

$$= 0 - \frac{\mathbf{e}_3}{h_2 h_1} \frac{\delta}{\delta u_2} (F_1 h_1) + \frac{\mathbf{e}_2}{h_3 h_1} \frac{\delta}{\delta u_3} (F_1 h_1)$$

$$= \frac{\mathbf{e}_2}{h_3 h_1} \frac{\delta}{\delta u_3} (F_1 h_1) - \frac{\mathbf{e}_3}{h_2 h_1} \frac{\delta}{\delta u_2} (F_1 h_1) \quad (1)$$

Similarly, we have

$$\nabla \times (F_2 \mathbf{e}_2) = \frac{\mathbf{e}_3}{h_1 h_2} \frac{\delta}{\delta u_1} (F_2 h_2) - \frac{\mathbf{e}_1}{h_2 h_3} \frac{\delta}{\delta u_3} (F_2 h_2) \quad (2)$$

and $\nabla \times (F_3 \mathbf{e}_3) = \dfrac{\mathbf{e}_1}{h_2 h_3} \dfrac{\delta}{\delta u_2} (F_3 h_3) - \dfrac{\mathbf{e}_2}{h_3 h_1} \dfrac{\delta}{\delta u_1} (F_3 h_3) \quad (3)$

Adding (1), (2) and (3), we get

$$\nabla \times \mathbf{F} = \frac{\mathbf{e}_2}{h_3 h_1} \frac{\delta}{\delta u_3} (F_1 h_1) - \frac{\mathbf{e}_3}{h_1 h_2} \frac{\delta}{\delta u_2} (F_1 h_1)$$

$$+ \frac{\mathbf{e}_3}{h_1 h_2} \frac{\delta}{\delta u_1} (F_2 h_2) - \frac{\mathbf{e}_1}{h_2 h_3} \frac{\delta}{\delta u_3} (F_2 h_2)$$

$$+ \frac{\mathbf{e}_1}{h_2 h_3} \frac{\delta}{\delta u_2} (F_3 h_3) - \frac{\mathbf{e}_2}{h_3 h_1} \frac{\delta}{\delta u_1} (F_3 h_3)$$

$$= \frac{\mathbf{e}_1}{h_2 h_3} \left[\frac{\delta}{\delta u_2} (F_3 h_3) - \frac{\delta}{\delta u_3} (F_2 h_2) \right]$$

$$+ \frac{\mathbf{e}_2}{h_3 h_1} \left[\frac{\delta}{\delta u_3} (F_1 h_1) - \frac{\delta}{\delta u_1} (F_3 h_3) \right] + \frac{\mathbf{e}_3}{h_1 h_2} \left[\frac{\delta}{\delta u_1} (F_2 h_2) - \frac{\delta}{\delta u_2} (F_1 h_1) \right]$$

$$= \frac{h_1 \mathbf{e}_1}{h_1 h_2 h_3} \left[\frac{\delta}{\delta u_2} (F_3 h_3) - \frac{\delta}{\delta u_3} (F_2 h_2) \right]$$

$$+ \frac{h_2 \mathbf{e}_2}{h_1 h_2 h_3} \left[\frac{\delta}{\delta u_3} (F_1 h_1) - \frac{\delta}{\delta u_1} (F_3 h_3) \right]$$

$$+ \frac{h_3 \mathbf{e}_3}{h_1 h_2 h_3} \left[\frac{\delta}{\delta u_1} (F_2 h_2) - \frac{\delta}{\delta u_2} (F_1 h_1) \right]$$

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \dfrac{\delta}{\delta u_1} & \dfrac{\delta}{\delta u_2} & \dfrac{\delta}{\delta u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}$$

Hence

$$\nabla \times \mathbf{F} = \text{Curl}\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\delta}{\delta u_1} & \frac{\delta}{\delta u_2} & \frac{\delta}{\delta u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}$$

Proof of (d) : Let $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$

$$\nabla\phi = \frac{\mathbf{e}_1}{h_1}\frac{\delta\psi}{\delta u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\delta\psi}{\delta u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\delta\psi}{\delta u_2}$$

If $\mathbf{F} = \nabla\phi$, *then* $F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 = \dfrac{\mathbf{e}_1}{h_1}\dfrac{\delta\psi}{\delta u_1} + \dfrac{\mathbf{e}_2}{h_2}\dfrac{\delta\psi}{\delta u_2} + \dfrac{\mathbf{e}_3}{h_3}\dfrac{\delta\psi}{\delta u_3}$.

Equating the coefficients of $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$ *from both sides of the* above equation, we get

$$F_1 = \frac{1}{h_1}\frac{\delta\Psi}{\delta u_1},\ F_2 = \frac{1}{h_2}\frac{\delta\psi}{\delta u_2},\ F_3 = \frac{1}{h_3}\frac{\delta\psi}{\delta u_3}.$$

Now we know that

$$\nabla.\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\delta}{\delta u_1}(F_1h_2h_3) + \frac{\delta}{\delta u_2}(F_2h_3h_1) + \frac{\delta}{\delta u_3}(F_3h_1h_2)\right]$$

Putting the values of F_1, F_2. F_3, *we get*

$$\nabla.\nabla\phi = \nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\delta}{\delta u_1}\left(\frac{h_2h_3}{h_1}\frac{\delta\psi}{\delta u_1}\right) + \right.$$

$$\left.\frac{\delta}{\delta u_2}\left(\frac{h_3h_1}{h_2}\frac{\delta\psi}{\delta u_2}\right) + \frac{\delta}{\delta u_3}\left(\frac{h_1h_2}{h_3}\frac{\delta\psi}{\delta u_3}\right)\right].$$

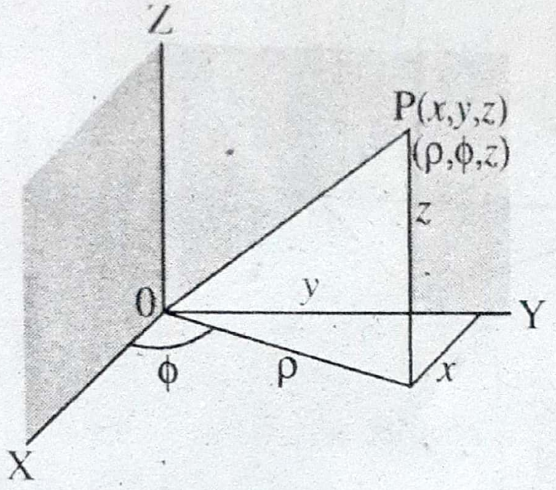
10. 33 . Special orthogonal coordinate systems.

(a) Cylindrical coordinates (ρ, φ, z)

Transformation equations :

$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z$

where $\rho \geq 0,\ 0 \leq \varphi \leq 2\pi,\ -\infty \angle z \angle \infty.$



Scale factors : $h_1 = 1,\ h_2 = \rho,\ h_3 = 1.$

Element of arc length : $dS^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$

Jacobian : $\dfrac{\delta(x,\ y,\ z)}{\delta(\rho,\ \varphi,\ z)} = \rho.$

Element of volume : $dV = \rho d\rho d\varphi dz$

Laplacian :

$$\nabla^2 \mathbf{U} = \frac{1}{\rho}\frac{\delta}{\delta\rho}\left(\rho\frac{\delta U}{\delta p}\right) + \frac{1}{\rho^2}\frac{\delta^2 U}{\delta\varphi^2} + \frac{\delta^2 U}{\delta z^2}$$

$$= \frac{\delta^2 U}{\delta\rho^2} + \frac{1}{\rho}\frac{\delta U}{\delta\rho} + \frac{1}{\rho^2}\frac{\delta^2 U}{\delta\varphi^2} + \frac{\delta^2 U}{\delta z^2}.$$

(b) Spherical Coordinates : (r, θ, ϕ)

Transformation equations :

$x = r\sin\theta\cos\varphi,$

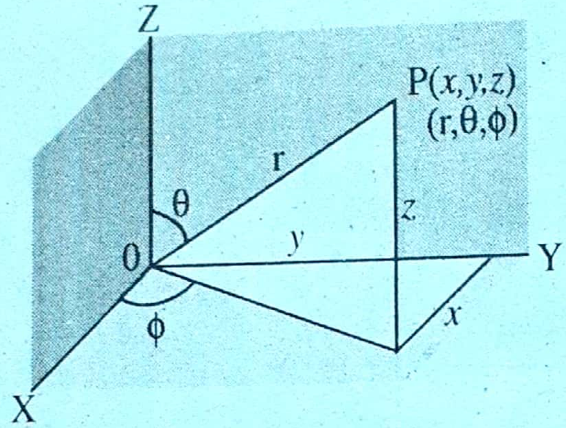
$y = r\sin\theta\sin\varphi,\ z = r\ \cos\theta.$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$.

Scale of factors : $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

Element of arc length :

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$



Jacobian : $\dfrac{\delta(x, y, z)}{\delta(r, \theta, \varphi)} = r^2 \sin\theta.$

Element of volume : $dV = r^2 \sin\theta \, dr d\theta d\varphi$

Laplacian :

$$\nabla^2 U = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta U}{\delta r}\right) + \frac{1}{r^2 \sin\theta}\frac{\delta}{\delta\theta}\left(\sin\theta \frac{\delta U}{\delta\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\delta^2 U}{\delta\varphi^2}.$$

Example 67. Express in cylidrical coordinates the quantities *(a)* $\nabla\varphi$ *(b)* $\nabla . \mathbf{F}$ *(c)* $\nabla \times \mathbf{F}$ *(d)* $\nabla^2\varphi$

Solution : For cylindrical coordinates (ρ, φ, z)

$u_1 = \rho$, $u_2 = \varphi$, $u_3 = z$; $\mathbf{e}_1 = \mathbf{e}_\rho$, $\mathbf{e}_2 = \mathbf{e}_\varphi$, $\mathbf{e}_3 = \mathbf{e}_z$

and $h_1 = h_\rho = 1$, $h_2 = h_\varphi = \rho$, $h_3 = h_z = 1$.

$$(a)\ \nabla\varphi = \frac{1}{h_1}\frac{\delta\varphi}{\delta u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\delta\varphi}{\delta u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\delta\varphi}{\delta u_3}\mathbf{e}_3$$

$$= \frac{1}{1}\frac{\delta\varphi}{\delta\rho}\mathbf{e}_\rho + \frac{\delta\varphi}{\delta\phi}\mathbf{e}_\phi + \frac{1}{1}\frac{\delta\varphi}{\delta z}\mathbf{e}_z$$

$$= \frac{\delta\varphi}{\delta\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\delta\varphi}{\delta\phi}\mathbf{e}_\phi + \frac{\delta\varphi}{\delta z}\mathbf{e}_z.$$

(b) $$\nabla.\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\delta}{\delta u_1}(h_2h_3F_1) + \frac{\delta}{\delta u_2}(h_3h_1F_2) + \frac{\delta}{\delta u_3}(h_1h_2F_3)\right]$$

$$= \frac{1}{(1)(\rho)(1)}\left[\frac{\delta}{\delta\rho}\left((\rho)(1)\,F_\rho\right) + \frac{\delta}{\delta\phi}\left((1)(1)F_\phi\right) + \frac{\delta}{\delta z}\left((1)(\rho)F_z\right)\right]$$

$$= \frac{1}{\rho}\left[\frac{\delta}{\delta\rho}(\rho F_\rho) + \frac{\delta F_\phi}{\delta\phi} + \frac{\delta}{\delta z}(\rho F_z)\right]$$

where $\mathbf{F} = F_\rho\,\mathbf{e}_1 + F_\phi\,\mathbf{e}_2 + F_z\mathbf{e}_3$, *i.e* $F_1 = F_\rho$, $F_2 = F_\phi$, $F_3 = F_z$.

(c) $$\nabla\times F = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_1 & h_3e_3 \\ \dfrac{\delta}{\delta u_1} & \dfrac{\delta}{\delta u_2} & \dfrac{\delta}{\delta u_3} \\ h_1\,F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\delta}{\delta\rho} & \dfrac{\delta}{\delta\phi} & \dfrac{\delta}{\delta z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\left(\frac{\delta F_z}{\delta\phi} - \frac{\delta}{\delta z}\rho F_\phi\right)\mathbf{e}_\rho + \left(\rho\frac{\delta F_\rho}{\delta z} - \rho\frac{\delta F_z}{\delta\rho}\right)\mathbf{e}_\phi + \left(\frac{\delta}{\delta\rho}(\rho F_\phi) - \frac{\delta F_\rho}{\delta\phi}\right)\mathbf{e}_z\right]$$

(d) $$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left[\frac{\delta}{\delta u_1}\left(\frac{h_2h_3}{h_1}\frac{\delta\varphi}{\delta u_1}\right) + \frac{\delta}{\delta u_2}\left(\frac{h_3h_1}{h_2}\frac{\delta\varphi}{\delta u_2}\right) + \frac{\delta}{\delta u_3}\left(\frac{h_1h_2}{h_3}\frac{\delta\varphi}{\delta u_3}\right)\right]$$

$$= \frac{1}{(1)(\rho)(1)}\left[\frac{\delta}{\delta\rho}\left(\frac{(\rho)(1)}{(1)}\frac{\delta\varphi}{\delta\rho}\right) + \frac{\delta}{\delta\emptyset}\left(\frac{(1)(1)}{\rho}\frac{\delta\varphi}{\delta\phi}\right) + \frac{\delta}{\delta z}\left(\frac{(1)\,(\rho)}{(1)}\frac{\delta\varphi}{\delta z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\delta}{\delta\rho}\left(\rho\frac{\delta\varphi}{\delta\rho}\right) + \frac{1}{\rho^2}\frac{\delta^2\varphi}{\delta\phi^2} + \frac{\delta^2\varphi}{\delta z^2}$$

Example 68. Express *(a)* $\nabla \times \mathbf{F}$ *and (b)* $\nabla^2\phi$ in spherical coordinates.

Solution : (a) Here $u_1 = r,\ u_2 = \theta,\ u_3 = \phi,\ e_1 = e_r,\ e_2 = e_\theta,\ e_3 = e_\phi;$

$h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_\phi = r\sin\theta$

$$\nabla \times \mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \dfrac{\delta}{\delta u_1} & \dfrac{\delta_2}{\delta u_2} & \dfrac{\delta}{\delta u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

$$= \frac{1}{(1)(r)(r\sin\theta)}\begin{vmatrix} e_r & re_\theta & r\sin\theta e_\phi \\ \dfrac{\delta}{\delta r} & \dfrac{\delta}{\delta\theta} & \dfrac{\delta}{\delta\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left\{\frac{\delta}{\delta\theta}(r\sin\theta F_\phi) - \frac{\delta}{\delta\phi}(rF_\theta)\right\}e_r + \left\{\frac{\delta F_r}{\delta\phi} - \frac{\delta}{\delta r}(r\sin\theta F_\phi)\right\}re_\theta\right.$$

$$\left. + \left\{\frac{\delta}{\delta r}(rF_\theta) - \frac{\delta F_r}{\delta\theta}\right\}r\sin\theta e_\phi\right]$$

(b) $$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\delta}{\delta u_1}\left(\frac{h_2h_3}{h_1}\frac{\delta\psi}{\delta u_1}\right) + \frac{\delta}{\delta u_2}\left(\frac{h_3h_1}{h_2}\frac{\delta\psi}{\delta u_2}\right)\right.$$

$$\left. + \frac{\delta}{\delta u_3}\left(\frac{h_1h_2}{h_3}\frac{\delta\psi}{\delta u_3}\right)\right]$$

$$= \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\delta}{\delta r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\delta\psi}{\delta r}\right) + \frac{\delta}{\delta\theta}\left(\frac{(r\sin\theta)(1)}{(r)}\frac{\delta\psi}{\delta\theta}\right)\right.$$

$$\left. + \frac{\delta}{\delta\phi}\left(\frac{(1)(r)}{(r\sin\theta)}\frac{\delta\psi}{\delta\phi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\delta}{\delta r}\left(r^2\frac{\delta\psi}{\delta r}\right) + \frac{\delta}{\delta\theta}\left(\sin\theta\frac{\delta\psi}{\delta\theta}\right) + \frac{1}{\sin\theta}\frac{\delta^2\psi}{\delta\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta\psi}{\delta r}\right) + \frac{1}{r^2\sin\theta}\frac{\delta}{\delta\theta}\left(\sin\theta\frac{\delta\psi}{\delta\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\delta^2\psi}{\delta\phi^2}.$$

Example 69. Prove that $\iiint\limits_V (x^2 + y^2 + z^2)dxdydz = \frac{4\pi a^5}{5}$ where V is a sphere having centre at the origin and radius equal to a.

Proof :

The given integral is equal to eight times the integral evaluated over that part of the sphere contained in the first octant. Then in rectangular coordinates the integral becomes

$$8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} (x^2 + y^2 + z^2)dzdydx.$$

Let us evaluate the integral by using spherical coordinates. In changing to spherical coordinates, the integrand $x^2 + y^2 + z^2$ is replaced by r^2 while the volume element dxdydz is replaced by the volume element $r^2 \sin\theta drd\theta d\phi$. To cover the required region in the first octant, fix θ and ϕ and integrate with respect to r from r = 0 to r = a, then keep ϕ constant and integrate with respect to θ from $\theta = 0$ to $\theta = \pi/2$; finally integrate with respect to ϕ from $\phi = 0$ to $\phi = \pi/2$. Then the above integral becomes

$$8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2(r^2 \sin\theta)drd\theta d\phi$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^4\sin\theta\, drd\theta d\phi$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\frac{r^5}{5}\sin\theta\right]_{r=0}^{a} d\theta d\phi$$

$$= \frac{8a^5}{5}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \sin\theta d\theta d\phi$$

$$= \frac{8a^5}{5}\int_{\phi=0}^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2} d\phi = \frac{8a^5}{5}\int_{\phi=0}^{\pi/2} d\phi = \frac{8a^5}{5}\frac{\pi}{2} = \frac{4\pi a^5}{5}.$$

Hence $\iiint_V (x^2 + y^2 + z^2)\, dxdydz = \dfrac{4\pi a^5}{5}$.

The End