

# **CSER 2207\_8: Numerical Analysis-I**

## **Lecture-5**

### **Solution of equation in single variable**

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# Newton's Method

## (Newton-Raphson Method)

Suppose that  $f \in C^2[a, b]$ . Let  $p_0 \in [a, b]$  be an approximation to  $p$  such that  $f'(p_0) \neq 0$  and  $|p - p_0|$  is “small.” Consider the first Taylor polynomial for  $f(x)$  expanded about  $p_0$  and evaluated at  $x = p$ .

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p)),$$

where  $\xi(p)$  lies between  $p$  and  $p_0$ . Since  $f(p) = 0$ , this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p)).$$

Newton's method is derived by assuming that since  $|p - p_0|$  is small, the term involving  $(p - p_0)^2$  is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0).$$

Solving for  $p$  gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

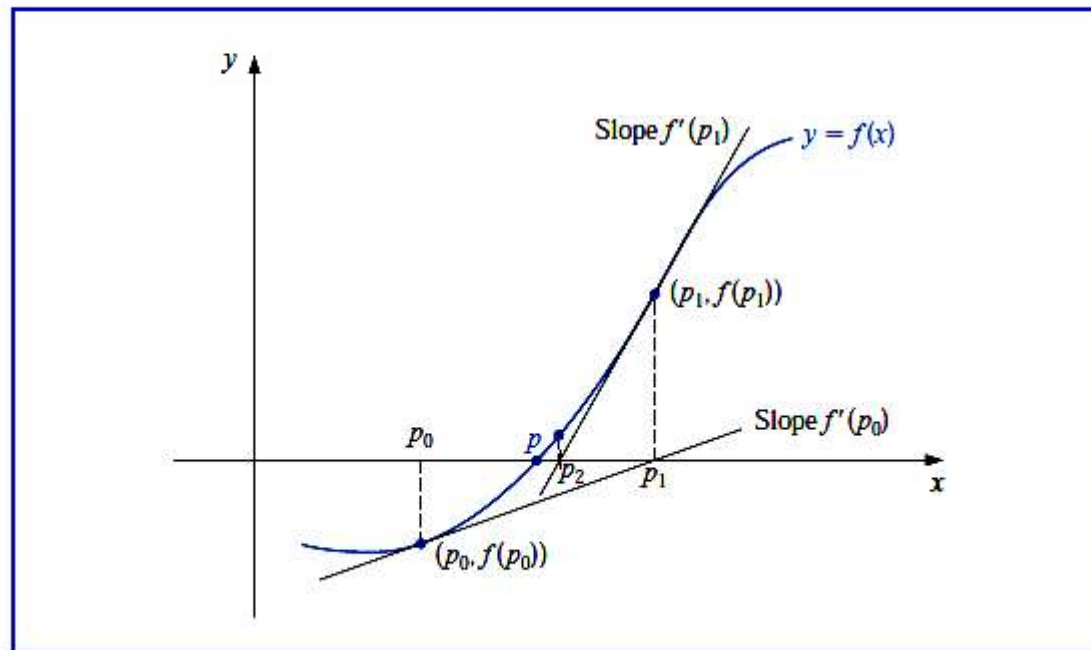
This sets the stage for Newton's method, which starts with an initial approximation  $p_0$  and generates the sequence  $\{p_n\}_{n=0}^{\infty}$ , by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \quad (2.7)$$

# Illustration

Figure 2.8 on page 68 illustrates how the approximations are obtained using successive tangents. (Also see Exercise 15.) Starting with the initial approximation  $p_0$ , the approximation  $p_1$  is the  $x$ -intercept of the tangent line to the graph of  $f$  at  $(p_0, f(p_0))$ . The approximation  $p_2$  is the  $x$ -intercept of the tangent line to the graph of  $f$  at  $(p_1, f(p_1))$  and so on. Algorithm 2.3 follows this procedure.

Figure 2.8



# Algorithm 2.3

## Newton's

To find a solution to  $f(x) = 0$  given an initial approximation  $p_0$ :

**INPUT** initial approximation  $p_0$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $i = 1$ .

**Step 2** While  $i \leq N_0$  do Steps 3–6.

**Step 3** Set  $p = p_0 - f(p_0)/f'(p_0)$ . (*Compute  $p_i$ .*)

**Step 4** If  $|p - p_0| < TOL$  then  
    **OUTPUT** ( $p$ ); (*The procedure was successful.*)  
    **STOP**.

**Step 5** Set  $i = i + 1$ .

**Step 6** Set  $p_0 = p$ . (*Update  $p_0$ .*)

**Step 7** **OUTPUT** ('The method failed after  $N_0$  iterations,  $N_0 =$ ',  $N_0$ );  
    (*The procedure was unsuccessful.*)  
    **STOP**.



# Example

**Example 2.11** Find a root of the equation  $x \sin x + \cos x = 0$ .

We have

$$f(x) = x \sin x + \cos x \quad \text{and} \quad f'(x) = x \cos x.$$

The iteration formula is therefore

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}.$$

With  $x_0 = \pi$ , the successive iterates are given below

$n$	$x_n$	$f(x_n)$	$x_{n+1}$
0	3.1416	-1.0	2.8233
1	2.8233	-0.0662	2.7986
2	2.7986	-0.0006	2.7984
3	2.7984	0.0	2.7984

# Example

**Example 2.12** Find a real root of the equation  $x = e^{-x}$ , using the Newton-Raphson method.

We write the equation in the form

$$f(x) = xe^x - 1 = 0 \quad (i)$$

Let  $x_0 = 1$ . Then

$$x_1 = 1 - \frac{e-1}{2e} = \frac{1}{2} \left( 1 + \frac{1}{e} \right) = 0.6839397$$

Now

$$f(x_1) = 0.3553424, \quad \text{and} \quad f'(x_1) = 3.337012,$$

so that

$$x_2 = 0.6839397 - \frac{0.3553424}{3.337012} = 0.5774545.$$

Proceeding in this way, we obtain

$$x_3 = 0.5672297 \quad \text{and} \quad x_4 = 0.5671433.$$



# Example

Find by Newton-Raphson method, the real root of the equation  $3x = \cos x + 1$ .

**Solution.** The given equation is

$$f(x) = 3x - \cos x - 1 = 0.$$

We have

$$f(0) = -2 \text{ (-ve)} \text{ and } f(1) = 3 - 0.5403 - 1 = 1.4597 \text{ (+ve)}.$$

Hence, one of the roots of  $f(x) = 0$  lies between 0 and 1. The values at 0 and 1 show that the root is nearer to 1. So let us take  $x = 0.6$ . Further,

$$f'(x) = 3 + \sin x.$$

Therefore, the Newton-Raphson formula gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \\ &= \frac{3x_n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}. \end{aligned}$$

Hence,

$$\begin{aligned} x_1 &= \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.6(0.5646) + 0.8253 + 1}{3 + 0.5646} = 0.6071, \\ x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{(0.6071)(0.5705) + 0.8213 + 1}{3 + 0.5705} = 0.6071. \end{aligned}$$

Hence the required root, correct to four decimal places, is 0.6071.

# Thank You