“Ordinary Differential Equations” Math-III(Note)

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Chapter-1

Introduction

[Differential Equation]

$\frac{d}{dx}$ → differential operator

$$\boxed{\frac{dy}{dx} = Dy}$$

যেসকল equation এ differential operator থাকবে তাকে differential equation বলে.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 7y = 0$$

$$\text{or, } D^2 + 5D + 7y = 0$$

সর্বোচ্চ $\boxed{\text{order}}$ এর যে power থাকে সেটি হলো degree.

→ সর্বোচ্চ যতবার differentiate করা হয়.

$$\left(\frac{d^3y}{dx^3}\right)^{1} + 3\frac{d^2y}{dx^2} + y = x$$

→ Order

→ Degree

→ Differential Equation

- → Ordinary Differential Equation

$$\left(\frac{dy}{dx}\right)^2 + 2y^2 = 4\left(\frac{dy}{dx}\right) + 4x$$

- → Partial Differential Equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz.$$

Differential Equation:

Equations which involve differential coefficients are called the differential equations.

Ordinary Differential Equation:

Equations which involve a single independent variable are called ordinary differential equation.

$$\frac{dy}{dx}=\sqrt{\frac{1-x^2}{2-y^2}}\ ,\ \left(\frac{dy}{dx}\right)^2+2y^2=4\left(\frac{dy}{dx}\right)+4x.$$

Partial Differential Equation:

Equations which involve partial differential coefficients with respect to more than one independent variable are called partial differential equation.

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=kz\ ,\ \frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$

Page-3 এর Ex. 4:

Form the differential equation corresponding to the family of curves $y=c(x-c)^2$, where c is an arbitrary constant.

Here,

$$y=c(x-c)^2 \quad \ldots \ldots \ldots (1)$$

$$\Rightarrow \frac{dy}{dx}=2c(x-c) \quad \ldots \ldots \ldots (2)$$

Dividing the equation (1) by (2),

$$\frac{y}{\frac{dy}{dx}}=\frac{c(x-c)^2}{2c(x-c)}$$

$$\Rightarrow \frac{y}{\frac{dy}{dx}}=\frac{(x-c)}{2}$$

$$\Rightarrow 2y=(x-c)\frac{dy}{dx}$$

$$\Rightarrow (x-c)=\frac{2y}{\frac{dy}{dx}}$$

$$\therefore c=x-\frac{2y}{p} \; ; \left[\text{where } p=\frac{dy}{dx}\right]$$

Printing the value of c in (2) we get,

$$p=2\left(x-\frac{2y}{p}\right)\cdot\frac{2y}{p}$$

$$\Rightarrow p=2\cdot\frac{px-2y}{p}\cdot\frac{2y}{p}$$

$$\Rightarrow p^3 = 4y\,(px - 2y)$$

$$\therefore \left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right),$$

which is the required differential equation.

Page-3 এর Ex. 5:

Find the differential equation of all circles passing through the origin and having their centres on x-axis.

$$x^2+y^2+2gx+2fy+c=0$$

$\rightarrow f=0$ (because of having their centers on the x-axis.

$\therefore x^2+y^2+2gx+c=0$

$\rightarrow c=0$ (because all circles pass through the origin

$\therefore x^2+y^2+2gx=0$

Equation of circles passing through the origin and having their centres on the x-axis is,

$$x^2+y^2+2gx=0,$$

where g is an arbitrary constant.

Differentiating,

$$x+y\frac{dy}{dx}+g=0.$$

$$\Rightarrow g=-\left(x+y\frac{dy}{dx}\right).$$

Putting this value of g in the equation of circle we get,

$$x^2+y^2+2\{-(x+y\frac{dy}{dx})\}x=0.$$

$$\Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx}=0$$

$$\therefore y^2=x^2+2xy\frac{dy}{dx},$$

which is the required differential equation.

Another way:

$$x^2+y^2+2gx=0$$

$$\Rightarrow 2x+2y\frac{dy}{dx}+2g=0.$$

$$\Rightarrow 2g=-2x-2y\frac{dy}{dx}$$

$$\Rightarrow 2g=-2(x+y\frac{dy}{dx})$$

$$\therefore g=-(x+y\frac{dy}{dx})$$

Now putting the vallue of g in the equation (1),

$$x^2+y^2+2\{-(x+y\frac{dy}{dx})\}x=0$$

$$\Rightarrow x^2+y^2-2x(x+y\frac{dy}{dx})=0$$

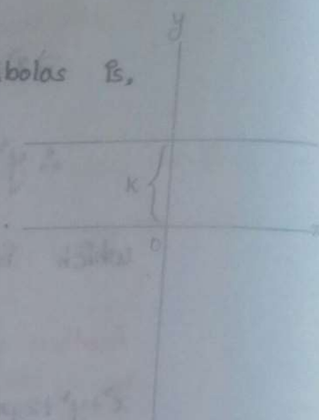
$$\Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx}=0$$

$$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}=0$$

$$\therefore y^2=x^2+2xy\frac{dy}{dx}.$$

Page-4, [illegible] Ex. 7:

Form the differential equation that represents all parabolas each of which has a latus rectum 4a and whose axes are parallel to x-axis.



Equation of the family of such parabolas is,

$$(y-k)^2 = 4a(x-h) \ldots \ldots \ldots (i)$$

where h and k are arbitrary constant.

Differentiating,

$$2(y-k)\frac{dy}{dx} = 4a$$

$$\Rightarrow (y-k)\frac{dy}{dx} = 2a$$

Differentiating again,

$$(y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right) \times \left(\frac{dy}{dx}\right) = 0 ; \left[\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)\right]$$

$$\Rightarrow (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \frac{2a}{\frac{dy}{dx}} \times \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad ; \left[(y-k) = \frac{2a}{\frac{dy}{dx}}\right]$$

$$\Rightarrow \frac{2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3}{\frac{dy}{dx}} = 0$$

$$\therefore 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0 ;$$

which is the required differential equation.

Chapter-2

"Equations of First Order and First Degree"

Differential equation of the first order and first degree:

A differential equation of the type,

$$M + N\frac{dy}{dx} = 0,$$

where M and N are functions of x and y or constant is called a differential equation of first order and first degree.

$$\left\{\begin{matrix} M(x,y) \rightarrow f(\ )\,dx \\ N(x,y) \rightarrow f(\ )\,dy \end{matrix}\right\}$$

Solution of the differential equation when variables are separable:

Page-6 এর Ex-1:

Solve $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

Separating the variables the equation becomes,

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Integrating we get,

$$\int \frac{dy}{1+x^2} = \int \frac{dx}{1+y^2} + c$$

$$\Rightarrow \tan^{-1} x = \tan^{-1} y + c \quad ;\left[\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$\Rightarrow \tan^{-1}x - \tan^{-1}y = c$

$\Rightarrow \tan^{-1}\dfrac{x-y}{1+xy} = c \quad ;\left[\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}\right]$

$\Rightarrow x-y = (1+xy)\tan c$

$\therefore x = y + (1+xy)\tan c$

which is the solution.

Another way:

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$$\Rightarrow \int\frac{dy}{1+y^2} = \int\frac{dx}{1+x^2} + A$$

$$\Rightarrow \tan^{-1}y = \tan^{-1}x + A$$

$$\Rightarrow \tan^{-1}y - \tan^{-1}x = A$$

$$\Rightarrow \tan^{-1}\frac{y-x}{1+xy} = A$$

$\Rightarrow \tan^{-1}\dfrac{y-x}{1+xy} = \tan^{-1}c \quad ;[\text{say } A = \tan^{-1}c]$

$$\Rightarrow \frac{y-x}{1+xy} = c$$

$\therefore y-x = (1+xy)c$, which is the solution.

Page-7 এর Ex-4:

Solve $(y-px)x=y$

Equation is,

$$(y-px)x=y$$

$$\text{or, } xy-px^2=y$$

$$\text{or, } px^2=xy-y$$

$$\text{or, } px^2=y(x-1)$$

i.e. $\dfrac{dy}{dx}=\dfrac{y(x-1)}{x^2}$; $\left[p=\dfrac{dy}{dx}\right]$

$$\text{or, } \frac{dy}{y}=\frac{x-1}{x^2}dx$$

$$\text{or, } \frac{dy}{y}=\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

Integrating,

$$\int\frac{dy}{y}=\int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$$\Rightarrow \log y=\log x+\frac{1}{x}+\log A$$

$$\Rightarrow y=xAe^{\frac{1}{x}}$$

$$\therefore \frac{y}{x}=Ae^{\frac{1}{x}}$$

[* Page-7 এর Ex-6(i) & 6(ii) → Same Rules]

Equations reducible to the form in which variables are separables:

Page-7 এর Ex.1:

Solve $\frac{dy}{dx} = (4x+y+1)^2$ ... (1)

Let, $4x+y+1 = v$

$\Rightarrow 4 + \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{dv}{dx} - 4$ ... ... (2)

From equation (1) and (2),

$\frac{dv}{dx} - 4 = v^2$

$\Rightarrow \frac{dv}{dx} = v^2 + 4$

$\Rightarrow \frac{dv}{v^2+4} = dx$

$\Rightarrow \int \frac{dv}{v^2+4} = \int dx$

$\Rightarrow \frac{1}{2} \tan^{-1} \frac{v}{2} = x + c$; $\left[\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$

$\therefore \frac{1}{2} \tan^{-1} \frac{4x+y+1}{2} = x + c$, which is the solution.

Page-8 এর Ex.2:

Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Here,

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y) \dots \dots \dots (1)$$

Let, $x+y=v$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1 \dots \dots \dots (1)$$

From equation (1) and (2),

$$\frac{dv}{dx} - 1 = \sin v + \cos v$$

$$\Rightarrow \frac{dv}{dx} = 1 + \sin v + \cos v$$

$$\left\{ \begin{array}{l} 1.\ 1+\cos\theta = 2\cos^2\frac{\theta}{2} \\ 2.\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \end{array} \right\}$$

$$\Rightarrow \frac{dv}{1+\sin v+\cos v} = dx$$

$$\Rightarrow \frac{dv}{1+\cos v+\sin v} = dx$$

$$\Rightarrow \frac{dv}{2\cos^2\frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}} = dx$$

$$\Rightarrow \frac{dv}{2\cos^2\frac{v}{2}\left(1+\tan\frac{v}{2}\right)} = dx$$

$$\Rightarrow \frac{1}{2}\sec^2\frac{v}{2} \times \frac{dv}{(1+\tan\frac{v}{2})} = dx$$

$$\Rightarrow \frac{\frac{1}{2}\sec^2\frac{v}{2}}{1+\tan\frac{v}{2}}\,dv = dx$$

$$\Rightarrow \int \frac{\frac{1}{2}\sec^2 v}{1+\tan\frac{v}{2}}\,dv = \int dx$$

$$\Rightarrow \ln\left|1+\tan\frac{v}{2}\right| = x+c$$

$\therefore \ln\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x+c$, is the required solution.

1. $\frac{d}{dx}(\tan x) = \sec^2 x$
2. $\int \frac{f'(x)}{f(x)}\,dx = \int \frac{d\{f(x)\}}{f(x)}$
   $= \ln|f(x)|+c$

Now, let, $1+\tan\frac{v}{2} = z$

$\Rightarrow \sec^2\frac{v}{2}\cdot\frac{1}{2}\,dv = dz$

$\Rightarrow \frac{1}{2}\sec^2\frac{v}{2}\,dv = dz$

Page-9 এর Ex-5:

Solve $\frac{x\,dx+y\,dy}{x\,dy-y\,dx} = \sqrt{\left(\frac{a^2-x^2-y^2}{x^2+y^2}\right)}$

Here we change to polar co-ordinates by putting,

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$x\,dx + y\,dy = r\,dr$

$\frac{y}{x} = \tan\theta$

$\therefore \frac{x\,dy-y\,dx}{x^2} = \sec^2\theta\,d\theta$

or, $x\,dy - y\,dx = r^2\,d\theta$

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$

$\therefore$ the equation becomes,

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\frac{a^2-(x^2+y^2)}{x^2+y^2}}$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \sqrt{\frac{a^2-r^2}{r^2}}$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sqrt{a^2-r^2}}{r}$$

$$\Rightarrow \frac{dr}{d\theta} = \sqrt{a^2-r^2}$$

Separating the variables,

$$\frac{dr}{\sqrt{a^2-r^2}} = d\theta$$

Integrating,

$$\sin^{-1}\left(\frac{r}{a}\right) = \theta + c \qquad \left[\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c\right]$$

$$\text{or, } r = a\sin(\theta+c)$$

$$\therefore \sqrt{x^2+y^2} = a\sin\left[\tan^{-1}\left(\frac{y}{x}\right)+c\right]$$

[Page-9 এর Ex-6 & 7 same rules]

☐ Homogeneous Differential Equation:

An equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous function of x and y of the same degree can be reduced to an equation in which variables are separable by putting, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

A function $f(x,y)$ is called homogeneous of degree n if $f(tx, ty) = t^n f(x,y)$, $\forall (t > 0)$

{$ax^2 + 2hxy + by^2 \rightarrow$ homogeneous হলে $y = vx$ ধরে solve করতে হবে}

Page-8 এর Ex-2:

Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Here,

$$x^2y\,dx - (x^3+y^3)\,dy = 0$$

$$\text{or, } x^2y\,dx = (x^3+y^3)\,dy$$

$$\text{or, } \frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad \cdots\cdots\cdots (1) \quad [\text{Homogeneous}]$$

Putting, $y = vx$

or, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, the equation (1) becomes,

$$\frac{x^2\cdot(vx)}{x^3+(vx)^3} = v + x\frac{dv}{dx}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{x^3 v}{x^3+v^3x^3} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-v-v^4}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \frac{1+v^3}{v^4}dv = -\frac{dx}{x}$$

$$\Rightarrow \left(v^{-4}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

$$\Rightarrow \int\left(v^{-4}+\frac{1}{v}\right)dv = -\int\frac{1}{x}dx \quad [\text{Integrating}]$$

$$\Rightarrow \frac{v^{-3}}{-3} + \log v = -\log x + \log c$$

$$\Rightarrow \frac{1}{v^3} - 3\log v = 3\log x - 3\log c.$$

$$\Rightarrow \frac{1}{v^3} = 3\log v + 3\log x - 3\log c$$

$$\Rightarrow \frac{1}{v^3} = \log v^3x^3 - \log c^3$$

$$\Rightarrow \frac{1}{v^3} = \log y^3 + \log A \;;\; [\text{As } y = vx \text{ and say } \log A = -\log c^3]$$

$$\Rightarrow \frac{x^3}{y^3} = \log Ay^3$$

$$\therefore x^3 = y^3 \log(Ay^3).$$ (Ans:)

Page-12 এর Ex-7:

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y = \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx}$$

or, $x\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}(x\,dy - y\,dx)$.

The equation is,

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y = \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx}$$

Putting, $y = vx$

or, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, the equation becomes,

$$(x\cos v + vx\sin v)vx = (vx\sin v - x\cos v)x\left(v + x\frac{dv}{dx}\right)$$

$$\Rightarrow v(\cos v + v\sin v) = (v\sin v - \cos v)\left(v + x\frac{dv}{dx}\right)$$

$$\Rightarrow v(\cos v + v\sin v) = (v\sin v - \cos v)v + (v\sin v - \cos v)x\frac{dv}{dx}$$

$$\Rightarrow v\cos v + v^2\sin v = v^2\sin v - v\cos v + x(v\sin v - \cos v)\frac{dv}{dx}$$

$$\Rightarrow v\cos v + v^2\sin v - v^2\sin v + v\cos v = x(v\sin v - \cos v)\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2v\cos v}{x(v\sin v - \cos v)}$$

$$\Rightarrow \left(\frac{v\sin v - \cos v}{v\cos v}\right) dv = \frac{2dx}{x}$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right) dv = 2\frac{dx}{x}$$

$$\Rightarrow \int\left(\tan v - \frac{1}{v}\right) dv = 2\int \frac{1}{x} dx \; ; \text{[Integrating]}$$

$$\Rightarrow \log \sec v - \log v = 2\log x + \log c$$

$$\Rightarrow \log \frac{\sec v}{v} = \log x^2 + \log c$$

$$\Rightarrow \log \frac{\sec v}{v} = \log cx^2$$

$$\Rightarrow \sec v = Cvx^2$$

$$\Rightarrow \sec \frac{y}{x} = c \cdot \frac{y}{x} \cdot x^2$$

$$\therefore Cxy = \sec \frac{y}{x}.$$

1. $\int \frac{1}{x} dx = \ln|x| + c$

2. $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$= \int \frac{-(-\sin x)}{\cos x} dx$

$= -\ln|\cos x| + c \; ; \left[\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c\right]$

$= \ln\left|\frac{1}{\sec x}\right| + c$

$= -[\ln 1 - \ln|\sec x|] + c$

$= 0 + \ln|\sec x| + c$

$= \ln|\sec x| + c$

Equation Reducible to Homogeneous Form:

An equation of the type $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$, when $\frac{a}{a'}\neq\frac{b}{b'}$ can be reduced to homogeneous form as follows:

Put $x=X+h$, $y=Y+k$; then $\frac{dy}{dx}=\frac{dY}{dX}$, where X, Y are new variables and h, k are arbitrary constants. The equation now becomes,

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$$

We choose the constants h and k in such a way that $ah+bk+c=0$, $a'h+b'k+c'=0$.

With this substitution the differential equation reduces to $\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y}$, which is a homogeneous equation in X, Y and can be solved by putting $Y=vX$ as earlier.

Page-15 এর Ex.-5:

Solve $(2x+y+3)\frac{dy}{dx} = x+2y+3$

The equation is,

$$(2x+y+3)\frac{dy}{dx} = x+2y+3$$

$$\text{or, } \frac{dy}{dx} = \frac{x+2y+3}{2x+y+3} \quad \cdots \cdots \cdots (1)$$

Put $x = X+h$, $y = Y+k$, where h and k are constants.

$$dx = dX, \quad dy = dY.$$

$$\therefore \frac{dy}{dx} = \frac{dY}{dX}$$

Now,

$$\frac{dY}{dX} = \frac{(X+h)+2(Y+k)+3}{2(X+h)+(Y+k)+3}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+Y+(2h+k+3)} \quad \cdots \cdots (2)$$

Choose h and k such that,

$$h+2k+3=0$$

Now, $\quad 2h+k+3=0$

$$\frac{h}{6-3} = \frac{k}{6-3} = \frac{1}{1-4}$$

$$\Rightarrow \frac{h}{3} = \frac{k}{3} = \frac{1}{-3}$$

$\therefore k=-1,\ h=-1$

Let,

$$Y = vX$$

$$\therefore \frac{dY}{dX} = v + X\frac{dv}{dX}$$

(From equation (2),

$$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{1+2v}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+2v-2v-v^2}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{2+v}{1-v^2}dv = \frac{dX}{X}$$

$$\Rightarrow \frac{2}{1-v^2}dv + \frac{v}{1-v^2}dv = \frac{dX}{X}$$

$$\Rightarrow \frac{2}{(1-v)(1+v)}dv - \frac{1}{2}\left(\frac{-2v\,dv}{1-v^2}\right) = \frac{dX}{X}$$

$$\Rightarrow \left[\frac{2}{(1-v)(1+1)} + \frac{2}{(1-(-1))\quad(1+v)}\right]dv - \frac{1}{2}\left(\frac{-2v}{1-v^2}\right)dv = \frac{dX}{X}$$

$$\Rightarrow \int \frac{-1}{1-v}\,dv + \int \frac{1}{1+v}\,dv - \frac{1}{2}\int \frac{-2v\,dv}{1-v^2} = \frac{dx}{x} \quad \text{[Integrating]}$$

$$\Rightarrow -\log(1-v) + \log(1+v) - \frac{1}{2}\log(1-v^2) = \log x + \log c$$

$$\Rightarrow \log\frac{1+v}{1-v} - \frac{1}{2}\log(1-v^2) = \log cx$$

$$\Rightarrow \log\frac{1+v}{1-v} - \log\sqrt{1-v^2} = \log cx$$

$$\Rightarrow \log\frac{1+v}{1-v} = \log cx + \log\sqrt{1-v^2}$$

$$\Rightarrow \log\frac{1+v}{1-v} = \log cx\sqrt{1-v^2}$$

$$\Rightarrow \frac{1+v}{1-v} = cx\sqrt{1-v^2}$$

$$\Rightarrow \frac{1+\frac{Y}{X}}{1-\frac{Y}{X}} = cX\sqrt{1-\frac{Y^2}{X^2}}$$

$$\Rightarrow \frac{X+Y}{X-Y} = cX\sqrt{\frac{X^2-Y^2}{X^2}}$$

$$\Rightarrow \frac{x-h+y-k}{x-h-y+k} = cX\cdot\frac{1}{X}\sqrt{X^2-Y^2}$$

$$\Rightarrow \frac{x+1+y+1}{x+1-y-1} = c\sqrt{(x-h)^2-(y-k)^2}$$

$$\therefore \frac{x+y+2}{x+y} = c\sqrt{(x+1)^2-(y+1)^2}$$

Another way:

$$\frac{dy}{dx}(2x+y+3)=x+2y+3$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+2y+3}{2x+y+3} \quad \dots\dots\dots (1)$$

$$\Rightarrow \frac{dY}{dX}=\frac{X+h+2Y+2k+3}{2X+2h+Y+k+3}$$

$$\Rightarrow \frac{dY}{dX}=\frac{X+2Y+h+2k+3}{2X+Y+2h+k+3} \quad \dots\dots\dots (2)$$

Choose h and k such that,

$$h+2k+3=0$$

$$2h+k+3=0$$

$$\therefore \frac{h}{6-3}=\frac{k}{6-3}=\frac{1}{1-4}$$

$$\Rightarrow \frac{h}{3}=\frac{k}{3}=\frac{1}{-3}$$

$\therefore h=-1, k=-1.$

$$\therefore \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1+2v-2v-v^2}{2+v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{2+v}{1-v^2}\,dv = \frac{dx}{x}$$

$$\Rightarrow \frac{2}{1-v^2}\,dv + \frac{v}{1-v^2}\,dv = \frac{dx}{x}$$

$$\Rightarrow 2\int \frac{1}{1-v^2}\,dv + \left(-\frac{1}{2}\right)\int \frac{-2v}{1-v^2} = \frac{dx}{x}$$

$$\Rightarrow 2\log(1-v^2) - \frac{1}{2}\log(1-v^2) = \log x + \log c$$

$$\Rightarrow 2\log\frac{1+v}{1-v} - \log\sqrt{1-v^2} = \log(cx)$$

$$\Rightarrow 2\log\frac{1+v}{1-v} = \log(cx) + \log\sqrt{1-v^2}$$

$$\Rightarrow 2\log\frac{1+v}{1-v} = \log\left(cx\sqrt{1-v^2}\right)$$

$$\Rightarrow 2\cdot\frac{1+\frac{y}{x}}{1-\frac{y}{x}} = cx\sqrt{1-\frac{y^2}{x^2}}$$

$$\Rightarrow 2\,\frac{x+y}{x-y} = c\sqrt{x^2-y^2}$$

$$\Rightarrow 2\,\frac{x-h+y-k}{x-h-y+k} = c\sqrt{(x-h)^2-(y-k)^2}$$

$$\Rightarrow 2\,\frac{x+y+2}{x-y} = c\sqrt{(x-1)^2-(y+1)^2}$$

Page-18:

Linear Differential Equations:

A differential equation of the form,

$$\frac{dy}{dx} + Py = Q,$$

where P, Q are functions of x or constants, is called the linear differential equation of the first order.

Integrating Factor:

$$I.F. = e^{\int P dx}$$

It's solution is,

$$y \times (I.F.) = \int [Q \times (I.F.)] dx + c.$$

Page-19 এর Ex-2(a):

Solve $x\frac{dy}{dx}+2y=x^2\log x$

The equation is,

$$x\frac{dy}{dx}+2y=x^2\log x$$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x}y=x\log x$$

which is the form $\frac{dy}{dx}+Py=Q$.

So, it is linear differential equation.

$$\therefore I.F.=e^{\int\frac{2}{x}dx}$$

$$=e^{2\log x}$$

$$=e^{\log x^2}$$

$$=x^2.$$

$\therefore$ It's solution is,

$$y.x^2=\int x^2.x\log x\,dx+c$$

$$\Rightarrow x^2y=\int \log x.x^3\,dx+c$$

$$\Rightarrow x^2y=\log x.\frac{x^4}{4}-\int\frac{1}{x}.\frac{x^4}{4}dx+c;\left[\int uv\,dx=u\int v\,dx-\int\left\{\frac{du}{dx}\int v\,dx\right\}dx\right]$$

$$\Rightarrow x^2y=\frac{x^4}{4}\log x-\frac{1}{4}\int x^3\,dx+c$$

$$\Rightarrow x^2y=\frac{x^4}{4}\log x-\frac{1}{4}.\frac{x^4}{4}+c$$

$$\Rightarrow x^2 y = \frac{x^4}{16}(4\log x - 1) + c$$

$$\therefore y = \frac{x^2}{16}(4\log x - 1) + cx^{-2}. \quad \text{(Ans:)}$$

Another process:

$$x\frac{dy}{dx} + 2y = x^2\log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

$$\Rightarrow x^2\frac{dy}{dx} + \frac{2x^2}{x}y = x^2 . x\log x$$

$$\Rightarrow x^2 dy + 2xy\,dx = x^3\log x\,dx$$

$$\Rightarrow d(x^2y) = x^3\log x\,dx$$

$$\Rightarrow \int d(x^2y) = \int x^3\log x\,dx$$

$$\Rightarrow x^2y = \log x . \frac{x^4}{4} - \int \frac{1}{x}\cdot\frac{x^4}{4}dx + c$$

$$\Rightarrow x^2y = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 dx + c$$

$$\Rightarrow x^2y = \frac{x^4}{4}\log x - \frac{1}{4}\cdot\frac{x^4}{4} + c$$

$$\Rightarrow x^2y = \frac{x^4}{16}(4\log x - 1) + c.$$

$$\therefore y = \frac{x^2}{16}(4\log x - 1) + cx^{-2} \quad \text{(Ans:)}$$

[Page-19 to 22 → same rules]

Page-22 এর Ex. 3:

Solve $(1+y^2)dx + (x-\tan^{-1}y)dy = 0$

The equation is.

$$(1+y^2)dx + (x-\tan^{-1}y)dy = 0$$

$$\Rightarrow (x-\tan^{-1}y)dy = -(1+y^2)dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1+y^2}{x-\tan^{-1}y}$$

$$\Rightarrow \frac{dy}{dx} + \frac{1+y^2}{x-\tan^{-1}y} = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{x-\tan^{-1}y}{1+y^2} = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is the form $\frac{dx}{dy} + Px = Q$

$$\therefore I.F. = e^{\int P dy}$$

$$= e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1}y}$$

Let,

$\tan^{-1}y = t$

$$\Rightarrow \frac{d}{dt}(\tan^{-1}y) = \frac{d}{dt}($$

$$\Rightarrow \frac{1}{1+y^2}\frac{dy}{dt} = 1$$

$$\Rightarrow \frac{1}{1+y^2}dy = dt$$

$\therefore$ It's solution is,

$$xe^{\tan^{-1}y} = \int e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}dy + c$$

$$\Rightarrow x.e^{\tan^{-1}y} = \int e^t . t\, dt + c$$

$$\Rightarrow x e^{\tan^{-1} y} = t e^{t} - \int e^{t}\, dt + c$$

$$\Rightarrow x e^{\tan^{-1} y} = t e^{t} - e^{t} + c$$

$$\Rightarrow x e^{\tan^{-1} y} = e^{t}(t-1) + c$$

$$\Rightarrow x e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

$$\therefore x = \tan^{-1} y - 1 + c e^{-\tan^{-1} y}$$

Ex Page -22:

Equations reducible to linear form:

$$\frac{dy}{dx} + Py = Qy^{n}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + y^{1-n}\cdot P = Q \quad (1)$$

Let,

$$y^{1-n} = z$$

$$\Rightarrow (1-n) y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\cdot\frac{dz}{dx}$$

Then (1) becomes,

$$\frac{1}{1-n}\cdot\frac{dz}{dx} + Pz = Q$$

$\Rightarrow \frac{dz}{dx} + P(1-n)z = (1-n)Q$, which is a linear equation in z and x

Page-23 এর Ex-1:

Solve $\frac{dy}{dx} = x^3y^3 - xy$

The equation is $\frac{dy}{dx} = x^3y^3 - xy$

Dividing by y^3,

$$y^{-3}\frac{dy}{dx} = x^3 - \frac{x}{y^2}$$

$$\Rightarrow y^{-3}\frac{dy}{dz} + \frac{x}{y^2} = x^3$$

$$\Rightarrow -\frac{1}{2}\frac{dz}{dx} + zx = x^3$$

$$\Rightarrow \frac{dz}{dx} - 2zx = -2x^3$$

let,

$$\frac{1}{y^2} = z$$

$$\Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$$

which is linear equation.

$$\therefore I.F. = e^{\int -2x\,dx}$$

$$= e^{-x^2}$$

It's solution is,

$$z e^{-x^2} = \int -2x^3 \cdot e^{-x^2}\,dx + c$$

$$\Rightarrow y^{-2} \cdot e^{-x^2} = \int x^2 \cdot e^{-x^2}(-2x)\,dx + c$$

$$\Rightarrow y^{-2} \cdot e^{-x^2} = -\int t e^t\,dt + c$$

$$\Rightarrow y^{-2} \cdot e^{-x^2} = -te^t + \int e^t\,dt + c$$

let,

$$-x^2 = t$$

$$\Rightarrow -2x\,dx = dt$$

$$\Rightarrow y^{-2}e^{-x^2} = -te^t + e^t + c$$

$$\Rightarrow y^{-2}e^{-x^2} = e^t(-t+1) + c$$

$$\Rightarrow y^{-2}e^{-x^2} = e^{-x^2}(x^2+1) + c$$

$$\therefore y^{-2} = (x^2+1) + c \quad \text{(Ans:)}$$

[Page - (31 - 35) → Same Rules]

Chapter: 03
Equations of First Order and First Degree

Exact Differential Equation:

$$M\,dx + N\,dy = 0$$

$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ হলে, exact differential equation হবে.

Necessary and Sufficient Condition:

$$M\,dx + N\,dy = 0$$

$$\Rightarrow M + N\frac{dx}{dy} = 0 \quad \ldots\ldots (1)$$

Let, $u = c$ be its primitive .... (2)

Differentiating (2) we get,

$$\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} = 0 \quad \ldots\ldots (3)$$

Comparing (1) and (3) we get,

$M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$ so that

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x\,\partial y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y}$$

Hence the condition is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition and it is proved.

if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then we show that $M+N\frac{\partial y}{\partial x}=0$

Let, $\int M\,dx = U$

$\therefore \frac{\partial U}{\partial x}=M$ so that,

$\frac{\partial^2 U}{\partial x\,\partial y}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ $\left[\text{as } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right]$

i.e.,

$\frac{\partial^2 U}{\partial x\,\partial y}=\frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{dU}{\partial y}\right)$

$\Rightarrow N=\frac{\partial U}{\partial y}+f(y)$; [Integrating]

$\int \frac{\partial N}{\partial x}\,dx=\int \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) f(x)+f(y)$

$\Rightarrow \int \partial N=\int \partial\left(\frac{\partial U}{\partial y}\right)+f(y)$

$\Rightarrow N=\frac{\partial U}{\partial y}+f(y)$

[$f(y)$ is a function of y free from x]

$\therefore M+N\frac{dy}{dx}=\frac{\partial U}{\partial x}+\left[\frac{\partial U}{\partial y}+f(y)\right]\frac{dy}{dx}$

$=\frac{d}{dx}\left[U+\int f(y)\frac{dy}{dx}\,dx\right]$

$=\frac{d}{dx}\left[U+F(y)\right]$

$=0.$

This shows that $M+N\frac{dy}{dx}=0$ is an exact equation.

Page- 37 এর Ex-1 :

$$(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0.$$

The equation,

$$(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0$$

$$\Rightarrow (y^4+4x^3y+3x)+(x^4+4xy^3+y+1)\frac{dx}{dy}=0.$$

Here,

$M=y^4+4x^3y+3x$ and $N=x^4+4xy^3+$

$\therefore \frac{\partial M}{\partial y}=4y^3+4x^3$ and $\frac{\partial N}{\partial x}=4x^3+4y^3.$

Since these are equal, the equation is exact.

To find solution of the differential equation, integrating M i.e. y^4+4x^3y+3x w.r.t. x, keeping y as constant, we get,

$$y^4x+\frac{4}{4}x^4y+\frac{3}{2}x^2$$

In x^4+4xy^3+y+1, terms free from x are $y+1$ whose integral with respect to y is $\frac{1}{2}y^2+y$

Therefore the general solution is,

$$y^4x+x^4y+\frac{3}{2}x^2+\frac{1}{2}y^2+y=c$$

Page-38 এর Ex-2:

Solve $x(x^2+y^2-a^2)dx+y(x^2-y^2-b^2)dy=0$

Comparing the equation with $Mdx+Ndy=0$ we get,

$M=x^3+xy^2-xa^2$, $N=x^2y-y^3-b^2y$

$\therefore \frac{\partial M}{\partial y}=2xy$ and $\frac{\partial N}{\partial x}=2xy$

Since these are equal, the equation is exact.

$\therefore$ It's solution,

$$\int M dx = c$$

$$\Rightarrow \int (x^3+xy^2-xa^2)\,dx=c$$

$$\Rightarrow \frac{x^4}{4}+\frac{x^2y^2}{2}-\frac{x^2a^2}{2}=c$$

$$\therefore x^4+2x^2y^2-2x^2a^2=c.$$

$$\int N\,dy = c$$

$$\Rightarrow \int (x^2y-y^3-b^2y)\,dy=c$$

$$\Rightarrow \frac{x^2y^2}{2}-\frac{y^4}{4}-\frac{b^2y^2}{2}=c$$

$$\therefore 2x^2y^2-y^4-2b^2y^2=c$$

$\therefore$ The general solution is, $x^4-y^4+2x^2y^2-2x^2a^2-2b^2y^2=c$

Page-39 এর Ex-5(b):

Solve $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

The given equation is,

$$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$$

$$\Rightarrow x\,dx + y\,dy + \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = 0$$

$$\Rightarrow \left(x - \frac{y}{x^2+y^2}\right)dx + \left(y + \frac{x}{x^2+y^2}\right)dy = 0.$$

Here, $M = x - \frac{y}{x^2+y^2}$ and $N = y + \frac{x}{x^2+y^2}$

$$\therefore \frac{\partial M}{\partial y} = -\frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = -\frac{(x^2-y^2)}{(x^2+y^2)^2} \quad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

and $\frac{\partial N}{\partial x} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} = -\frac{(x^2-y^2)}{(x^2+y^2)^2}$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Integrating M w.r.t. x regarding y as constant, we get,

$$\left[\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx\right]$$

$$\int \left(x - \frac{y}{x^2+y^2}\right)dx$$

$$= \int x\,dx - \int \frac{y}{x^2+y^2}dx$$

$$= \frac{x^2}{2} - y \int \frac{1}{x^2+y^2}\,dx - \int \left\{ \frac{d}{dx}(y) \int \frac{1}{x^2+y^2}\,dx \right\} dx$$

$$= \frac{x^2}{2} - y \cdot \frac{1}{y} \tan^{-1}\frac{x}{y} - 0 \quad \left[\frac{d}{dx}(c) = 0 \text{ and } \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \right]$$

$$= \frac{x^2}{2} - \tan^{-1}\frac{x}{y}$$

$$= x^2 - 2\tan^{-1}\frac{x}{y}$$

In N, term free from x is y whose integral is, $\frac{y^2}{2}$.

Hence, the solution is,

$$x^2 + y^2 - 2\tan^{-1}\frac{x}{y} = c.$$

(Ans:)

Page- 39 এর Ex. 7 :

Solve $[\cos x \tan y + \cos(x+y)]dx + [\sin x \sec^2 y + \cos(x+y)]dy = 0$

Here,

$$M = \cos x \tan y + \cos(x+y)$$

$$N = \sin x \sec^2 y + \cos(x+y).$$

Now,

$$\frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y) \qquad \left\{ \begin{array}{l} 1.\ \frac{d}{dx}(\tan x) = \sec^2 x \\ 2.\ \frac{d}{dx}(\cos x) = -\sin x \end{array} \right.$$

$$\frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$$

Since these are equal, the equation is exact.

Now integrating M, i.e. $\cos x \tan y + \cos(x+y)$ with respect to x keeping y as constant we get.

$$\sin x \tan y + \sin(x+y).$$

In N, there is no term free from x.

Hence the general solution is,

$$\sin x \tan y + \sin(x+y) = C.$$

Page-40 প্রশ্ন Ex-8:

Solve $(\cos x \tan y - \sin x \sec y)dx + (\sin x \sec^2 y + \cos x \tan^2 y \operatorname{cosec} y)dy =$

Here,

$M = \cos x \tan y - \sin x \sec y$

$N = \sin x \sec^2 y + \cos x \tan^2 y \operatorname{cosec} y$

1. $\frac{d}{dx}(\tan x) = \sec^2 x$
2. $\frac{d}{dx}(\sec x) = \sec x \tan x$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$

$\therefore \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin x \sec y \tan y$

$\frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin x \cdot \tan y \cdot \frac{\sin y}{\cos y} \cdot \frac{1}{\sin y}$

$= \cos x \sec^2 y - \sin x \tan y \cdot \sec y$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Integrating M with regard to x keeping y as constant we get,

$\sin x \tan y + \cos x \sec y$

In N there is no term free from x.

Hence the general solution is,

$\sin x \tan y + \cos x \sec y = C.$

Page-41:

Integrating factor by inspection:

Group of terms:

(a) $x\,dy - y\,dx$

I.F.	Exact Differential	I.F.	Exact Differential
(1) $\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2}$ $= d\left(\frac{y}{x}\right)$	(4) $\frac{1}{x^2+y^2}$	$\frac{x\,dy - y\,dx}{x^2+y^2}$ $= \frac{x\,dy - y\,dx}{x^2\left(1+\frac{y^2}{x^2}\right)}$ $= \frac{\frac{x\,dy - y\,dx}{x^2}}{1+\left(\frac{y}{x}\right)^2}$ $= d\left(\tan^{-1}\frac{y}{x}\right)$.
(2) $\frac{1}{y^2}$	$\frac{x\,dy - y\,dx}{y^2}$ $= -\frac{y\,dx - x\,dy}{y^2}$ $= d\left(-\frac{x}{y}\right)$.		
(3) $\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy}$ $= \frac{dy}{y} - \frac{dx}{x}$ $= d(\log y - \log x)$ $= d\left(\log\frac{y}{x}\right)$		

Group of term:

(b) $x\,dy + y\,dx$

(c) $x\,dx + y\,dy$

I.F.	Exact Differential	I.F.	Exact Differential
$\frac{1}{(xy)^n}$	$\frac{x\,dy + y\,dx}{xy}$ $= \frac{dy}{y} + \frac{dx}{x}$ $= d[\log y + \log x]$ $= d[\log(xy)]$ (for $n=1$)	$\frac{1}{(x^2+y^2)^n}$	$\frac{x\,dx + y\,dy}{(x^2+y^2)^n}$ $= \left[\frac{1}{2} \cdot \frac{2x\,dx + 2y\,dy}{x^2+y^2}\right]$ $= d\left[\frac{1}{2}\log(x^2+y^2)\right]$ (if $n=1$)

Page-42 এর Ex-3:

Solve $x\,dy - y\,dx - x(x^2-y^2)^{\frac{1}{2}}\,dx = 0$

The equation is,

$$x\,dy - y\,dx - x(x^2-y^2)^{\frac{1}{2}}\,dx = 0$$

$$\Rightarrow \frac{x\,dy - y\,dx}{(x^2-y^2)^{\frac{1}{2}}} - x\,dx = 0.$$

$$\Rightarrow \frac{\dfrac{x\,dy-y\,dx}{x^2}}{\dfrac{1}{x^2}\cdot x\left(1-\dfrac{y^2}{x^2}\right)^{\frac{1}{2}}} - x\,dx = 0.$$

$$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{\frac{1}{x}\left(1-\frac{y^2}{x^2}\right)^{\frac{1}{2}}} - x\,dx = 0$$

$$\Rightarrow \frac{x\,d\left(\frac{y}{x}\right)}{\sqrt{1-\frac{y^2}{x^2}}} - x\,dx = 0$$

$$\Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} - \int dx = 0$$

$$\Rightarrow \sin^{-1}\frac{y}{x} - x + c = 0 \qquad \left[\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c\right]$$

Page-43 এর Ex.5:

Solve $y\,dx - x\,dy + \log x\,dx = 0$

The equation is,

$$y\,dx - x\,dy + \log x\,dx = 0$$

$$\Rightarrow (y + \log x)\,dx = x\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \log x}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x} \quad \cdots \cdots \cdots (1)$$

$$\therefore y \cdot \frac{1}{x} = \int \frac{\log x}{x} \cdot \frac{1}{x}\,dx$$

$$\Rightarrow \frac{y}{x} = \int \frac{1}{x^2} \log x\,dx$$

$$\Rightarrow \frac{y}{x} = \log x\left(-\frac{1}{x}\right) - \int \frac{1}{x} \cdot \left(-\frac{1}{x}\right) dx; \left[\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx\right]$$

$$\Rightarrow \frac{y}{x} = \frac{-1}{x} \log x + \int \frac{1}{x^2}\,dx$$

$$\Rightarrow \frac{y}{x} = \frac{-1}{x} \log x - \frac{1}{x} - c$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{x}(1 + \log x) - c$$

$\therefore y + \log x + cx + 1 = 0$ is the solution.

Page-34 (Rules for finding the integrating factor):

When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (সমান হলে exact হবে),

Rule 1:

If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x only, then

Integrating Factor (I.F.) $= e^{\int f(x)\, dx}$.

Rule 2:

If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y alone, then

Integrating Factor (I.F.) $= e^{\int -g(y)\, dy}$.

Page-44 এর Ex. 1:
Solve $(x^2+y^2+x)dx + xy\, dy = 0$

The equation is,

$$(x^2+y^2+x)dx + xy\, dy = 0 \quad \ldots\ldots\ldots (1)$$

Here, $M = x^2+y^2+x$, $N = xy$.

$\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = y$.

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

However,

$$\frac{\partial M/\partial y - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x}, \text{ a function of } x \text{ alone.}$$

$$\therefore \text{I.F.} = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$= x.$$

Multiplying by I.F. ; the equation becomes,

$$(x^3+xy^2+x^2)dx + x^2y\,dy = 0 \quad \dots \dots \dots (2)$$

Here, $M_1 = x^3+xy^2+x^2$, $N_1 = x^2y$

$\frac{\partial M_1}{\partial y} = 2xy$ and $\frac{\partial N_1}{\partial x} = 2xy$.

So, equation (2) is an exact.

Integrating M_1 w.r.t. x, keeping y as constant, we get,

$$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3}$$

and in N_1 there is no term free from x.

Therefore the solution is,

$$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$$

$$\text{or, } 3x^4 + 6x^2y^2 + 4x^3 = c$$

{ Page - 44 to 47 Same Rules }

Page - 47 :

Rule - III :

If $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$,

Integrating Factor (I.F.) $= \dfrac{1}{Mx + Ny}$.

Rule IV :

If the equation can be written in the form,

$$y.f(xy)\,dx + x.g(xy)\,dy = 0, \quad f(xy) \neq g(xy),$$

then $\dfrac{1}{xy[f(xy) - g(xy)]} = \dfrac{1}{Mx - Ny}$ is an integrating factor.

Page - 48 এর Ex-2:

Solve $(x^4+y^4)dx - xy^3 dy = 0$

The equation is,

$$(x^4+y^4)dx - xy^3\,dy = 0 \ldots \quad \ldots \quad \ldots (1)$$ which is homogeneous.

Now,

$Mx + Ny$

$= (x^4+y^4)x - xy^3 \cdot y$

$= x^5 + xy^4 - xy^4$

$= x^5 \neq 0.$

$\therefore I.F. = \frac{1}{x^5}$

Multiplying by $\frac{1}{x^5}$ the equation becomes,

$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx - \frac{y^3}{x^3} = 0$, exact now.

Integrating $\frac{1}{x}+\frac{y^4}{x^5}$ with respect to x keeping y constant,

we get,

$\log x - \frac{y^4}{4x^4}$

Also $-\frac{y^3}{x^4}$ is not free from x.

Hence the complete solution is $= \log x - \frac{y^4}{4x^4} = C$

or, $4x^4 \log x - Cx^4 = y^4$

$\therefore y^4 = 4x^4 \log x - Cx^4$.

যদি $\frac{dy}{dx} = -\frac{dx}{dy}$
যদি $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$ } তবে orthogonal trajectories হয়

Chapter : 04

Trajectories

Page-54 এর Ex.1:

Find orthogonal trajectories of hyperbolas $xy = c^2$.

Family of hyperbolas is $xy = c^2$.

Differentiating with regard to x,

$$y + x\frac{dy}{dx} = 0$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, the differential equation of orthogonal trajectories is,

$$y - x\frac{dx}{dy} = 0$$

$$\text{or } x\,dx - y\,dy = 0.$$

Integrating,

$$\frac{x^2}{2} - \frac{y^2}{2} = c$$

$$\therefore x^2 - y^2 = c,$$

$y - x\frac{dx}{dy} = 0$

or, $y\,dy - x\,dx = 0$

or, $\frac{y^2}{2} - \frac{x^2}{2} = c_1$; [Integrating]

or, $y^2 - x^2 = -2c_1$

or, $x^2 - y^2 = c$; $[-2c_1 = c]$

This gives family of orthogonal trajectories of the hyperbolas $xy = c^2$.

Page - 55 अनु Ex. 2:

Show that the system of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, is self-orthogonal.

Equation of the curve,

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

Differentiating the curve w.r.t. x we get,

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\cdot p=0$$

$$\Rightarrow \frac{(b^2+\lambda)x+(a^2+\lambda)y.p}{(a^2+\lambda)(b^2+\lambda)}=0$$

$$\Rightarrow b^2x+\lambda x+a^2yp+\lambda yp=0$$

$$\Rightarrow (x+py)\lambda=-(b^2x+a^2yp)$$

$$\therefore \lambda=-\frac{b^2x+a^2yp}{x+py}$$

$$\therefore a^2+\lambda=a^2-\frac{b^2x+a^2yp}{x+py}$$

$$=\frac{a^2x+a^2py-b^2x-a^2yp}{x+py}$$

$$=\frac{(a^2-b^2)x}{x+py}$$

and $b^2+\lambda = b^2 - \frac{b^2x+a^2yp}{x+py}$

$$= \frac{b^2x+b^2py-b^2x-a^2yp}{x+py}$$

$$= \frac{(b^2-a^2)py}{x+py}$$

$$= -\frac{(a^2-b^2)py}{x+py}$$

Hence the differential equation of the given conics is,

$$\frac{x^2(x+py)}{(a^2-b^2)x} - \frac{y^2(x+yp)}{(a^2-b^2)yp} = 1$$

$$\Rightarrow \frac{x(x+py)}{(a^2-b^2)} - \frac{y(x+yp)}{(a^2-b^2)p} = 1$$

$$\Rightarrow x(x+py) - \frac{1}{p}y(x+py) = 1(a^2-b^2)$$

$$\Rightarrow x(x+py)\left(x-\frac{y}{p}\right) = a^2-b^2 \quad \ldots \ldots \ldots (1)$$

Now replacing p by $-\frac{1}{p}$, the differential equation of the orthogonal trajectories is,

$$\left(x-\frac{y}{p}\right)(x+yp) = a^2-b^2,$$

which is just same as(1). Thus the system of confocal conics is self-orthogonal.

Page-56 এর Ex-6:

Determine the 45° trajectories of the family of concentric circles $x^2+y^2=c^2$

Differentiating $x^2+y^2=c^2$, the differential equation of the family of circles is,

$$2x+2y\frac{dy}{dx}=0$$

$$\Rightarrow x+y\,p=0.$$

Now to find differential equation of the 45° trajectories, we shall replace p by,

$$\frac{p-\tan45^\circ}{1+p\tan45^\circ}=\frac{p-1}{1+p}$$

Hence the differential equation of the 45° trajectories

$$x+y\,\frac{p-1}{1+p}=0$$

$$\Rightarrow x(p+1)+y(p-1)=0$$

$$\Rightarrow px+py+x-y=0$$

$$\Rightarrow \frac{dy}{dx}(x+y)+x-y=0$$

This is a homogeneous equation.

$\therefore \left(v + x\frac{dv}{dx}\right)(x+vx) + (x - vx) = 0$

let, $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \left(v + x\frac{dv}{dx}\right)(1+v) + (1-v) = 0.$

$\Rightarrow v + x\frac{dv}{dx} = -\frac{1-v}{1+v}$

$\Rightarrow x\frac{dv}{dx} = -v - \frac{1-v}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{-v \quad v^2 - 1 + v}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{-(v^2+1)}{1+v}$

$\Rightarrow \frac{dx}{x\,dv} = -\frac{1+v}{v^2+1}$

$\Rightarrow \frac{1+v}{v^2+1}dv = -\frac{dx}{x}$

Integrating,

$\int\left(\frac{1}{v^2+1} + \frac{v}{v^2+1}\right)dv = -\int\frac{1}{x}dx$

$\Rightarrow \int\frac{1}{v^2+1}dv + \frac{1}{2}\int\frac{2v}{v^2+1}dv = -\log x + \log c$

$\Rightarrow \tan^{-1}v + \frac{1}{2}\log(v^2+1) = \log c - \log x$

$\Rightarrow \tan^{-1}\frac{y}{x} + \log x + \frac{1}{2}\log\left(\frac{y^2}{x^2}+1\right) = \log c$

$$\Rightarrow 2\log x+\log\left(\frac{x^2+y^2}{x^2}\right)=\log c+\log e^{-2\tan^{-1}\frac{y}{x}}$$

$$\Rightarrow \log x^2\cdot\left(\frac{x^2+y^2}{x^2}\right)=\log ce^{-2\tan^{-1}\frac{y}{x}}$$

$$\therefore x^2+y^2=ce^{-2\tan^{-1}\frac{y}{x}}.$$

Page-57 এর Ex.8:

Find the orthogonal trajectories of the cardioid $r=a(1-\cos\theta)$, where a is the parameter

The cardioid is,

$$r=a(1-\cos\theta)$$

Differentiating with respect to θ,

$$\therefore \frac{dr}{d\theta}=a(0+\sin\theta)=a\sin\theta$$

$$\text{i.e. } a=\frac{1}{\sin\theta}\cdot\frac{dr}{d\theta}$$

Hence differential equation of the family of cardioids is,

$$r=\frac{1}{\sin\theta}\cdot\frac{dr}{d\theta}(1-\cos\theta).$$

$$\therefore \frac{1}{r}\cdot\frac{dr}{d\theta}=\frac{\sin\theta}{(1-\cos\theta)}$$

Now replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, the differential equation of the orthogonal trajectories is,

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow -\frac{dr}{r} = \frac{1-\cos\theta}{\sin\theta}d\theta.$$

$$\Rightarrow -\frac{dr}{r} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\theta/_2\cos\theta/_2}d\theta.$$

$$\Rightarrow -\frac{dr}{r} = \tan\frac{\theta}{2}d\theta$$

$$\Rightarrow \log r = 2\log\cos\frac{\theta}{2} + \log 2c \quad [\text{Integrating}]$$

$$\Rightarrow r = 2c\cos^2\frac{\theta}{2}$$

$$\therefore r = c\ (1+\cos\theta). \quad ;\left[2\cos^2\frac{\theta}{2} = 1+\cos\theta\right]$$

Integrating,

$$\int \tan\frac{\theta}{2}d\theta = \int -\frac{dr}{r}$$

$$\Rightarrow -2\log(\cos\theta/_2) = -\log r + \log c$$

$$\Rightarrow 2\log(\cos\theta/_2) = \log r - \log c$$

$$\Rightarrow \log\cos^2\theta/_2 = \log r - \log c$$

$$\Rightarrow \log r = \log c + \log\cos^2\frac{\theta}{2}$$

$$\Rightarrow \log r = \log c\cdot\cos^2\frac{\theta}{2}$$

$$\Rightarrow r = c\cos^2\frac{\theta}{2}$$

$$\Rightarrow r = \frac{1}{2}c\ 2\cos^2\frac{\theta}{2}$$

$$\therefore r = c_1(1+\cos\theta).$$

Page-58 প্রশ্ন Ex. 11:

Find the orthogonal trajectories of $r^n \sin n\theta = a^n$

Taking log on both sides,

$$\log(r^n \sin n\theta) = \log a^n$$

$$\Rightarrow n\log r + \log \sin n\theta = n\log a$$

Differentiating with respect to θ,

$$\frac{n}{r}\cdot\frac{dr}{d\theta} + \frac{1}{\sin n\theta}\cdot\cos n\theta \cdot n = 0$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{\cos n\theta}{\sin n\theta} = 0.$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, the differential equation of orthogonal trajectories,

$$-\frac{r^2}{r}\cdot\frac{d\theta}{dr} = -\cot n\theta$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \tan n\theta$$

$$\Rightarrow \frac{dr}{r} = \tan n\theta \, d\theta$$

$$\Rightarrow \int \frac{1}{r} dr = \int \tan n\theta \, d\theta$$

$$\Rightarrow \log r = \frac{1}{n}\log \sec n\theta + \log c.$$

$$\Rightarrow n\log r = \log(\sec n\theta) + 2\log c$$

$\Rightarrow \log r^n = \log (\sin n\theta) . c^n$

$\Rightarrow r^n = c^n \sin n\theta$

$\therefore r^n \cos n\theta = c^n$

Chapter-5

Linear Differential Equations with Constant Coefficients

Linear Differential Equation:

A differential equation of the form,

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X$$

where $P_1, P_2, \ldots, P_n$ and X are functions of x or constants, is called a linear differential equation of nth order.

And if $P_1, P_2, \ldots, P_n$ are all constants (not functions of x) and X is some function of x, then the equation is a linear differential equation with constant coefficients.

Page-61 এর Ex.1:

Solve $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0$

The equation is, $(D^3 - 13D - 12)y = 0$

The auxiliary equation is, $D^3 - 13D - 12 = 0$

i.e. $(D+1)(D+3)(D-4) = 0$,

$D = -1, -3, 4.$

Hence, the complete solution is,

$$y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{4x}.$$

Page-62 এর Ex.1:

Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{9d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$

Let, $y = e^{mx}$ be the trial Sol^n of (1),

$$\therefore \frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2e^{mx}, \quad \frac{d^3y}{dx^3} = m^3e^{mx}, \quad \frac{d^4y}{dx^4} = m^4e^{mx}$$

The equation (1) becomes,

$$(m^4 - m^3 - 9m^2 - 11m - 4)e^{mx} = 0.$$

Auxiliary equation is,

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0; [e^{mx} \neq 0]$$

$$\Rightarrow m^4 + m^3 - 2m^3 - 2m^2 - 7m^2 - 7m - 4m - 4 = 0$$

$$\Rightarrow m^3(m+1) - 2m^2(m+1) - 7m(m+1) - 4(m+1)$$

$$\Rightarrow (m+1)(m^3 - 2m^2 - 7m - 4) = 0$$

$$\Rightarrow (m+1)(m^3 + m^2 - 3m^2 - 3m - 4m - 4) = 0$$

$$\Rightarrow (m+1)\{m^2(m+1) - 3m(m+1) - 4(m+1)\} = 0$$

$$\Rightarrow (m+1)(m+1)(m^2 - 3m - 4) = 0$$

$$\Rightarrow (m+1)(m+1)(m+1)(m-4) = 0.$$

$$\therefore m = -1, -1, -1, 4.$$

$\therefore$ General solution is, $y=(c_1+c_2x+c_3x)e^{-x}+c_4e^{4x}$.

Auxiliary equation having imaginary roots:

Let $m=\alpha\pm i\beta$ be the imaginary roots of an equation of second order (since imaginary roots occur in pairs).

Then its general solution is,

$$y=e^{\alpha x}\left[A\cos\beta x+B\sin\beta x\right]$$

Page-63 এর Ex.1:

Solve $(D^4+5D^2+6)y=0$

Auxiliary equation is, $m^4+5m^2+6=0$; $[e^{mx}\neq 0]$ .... (1)

let, $m^2=p$

$\therefore$ eq^n (1),

$$p^2+5p+6=0$$

$$\Rightarrow p^2+3p+2p+6=0$$

$$\Rightarrow p(p+3)+2(p+3)=0$$

$$\Rightarrow (p+3)(p+2)=0$$

$$\Rightarrow (m^2+3)(m^2+2)=0$$

$$\therefore m=\pm\sqrt{2}i\,,\ \pm\sqrt{3}i.$$

The general solution, $\left[e^{ix} = \cos x + i\sin x\right]$

$$y = c_1 e^{\pm\sqrt{2}ix} + c_2 e^{\pm\sqrt{3}ix}$$

$$= c_1\left[\cos(\pm\sqrt{2})x + i\sin(\pm\sqrt{2}x)\right] + c_2\left[\cos(\pm\sqrt{3})x + i\sin(\pm\sqrt{3})x\right]$$

$$= A\cos\sqrt{2}x + B\sin\sqrt{2}x + E\cos\sqrt{3}x + F\sin\sqrt{3}x.$$

Another way:

[$m = \pm i\alpha$ ~~হলে~~, general solution: $y = c_1 e^{i\alpha x} + c_2 e^{-i\alpha x}$]

$$y = c_1 e^{\sqrt{2}ix} + c_2 e^{-\sqrt{2}ix} + c_3 e^{\sqrt{3}ix} + c_4 e^{-\sqrt{3}ix}$$

$$= c_1(\cos\sqrt{2}x + i\sin\sqrt{2}x) + c_2(\cos\sqrt{2}x - i\sin\sqrt{2}x) + c_3(\cos\sqrt{3}x + i\sin\sqrt{3}x)$$

$$+ c_4(\cos\sqrt{3}x - i\sin\sqrt{3}x).$$

$$= (c_1 + c_2)\cos\sqrt{2}x + (c_1 i - c_2 i)\sin\sqrt{3}x + (c_3 + c_4)\cos\sqrt{3}x + (c_3 i - c_4 i)\sin\sqrt{3}x.$$

$$= A\cos\sqrt{2}x + B\sin\sqrt{2}x + E\cos\sqrt{3}x + F\sin\sqrt{3}x.$$

Synopsis of the forms of solution:

To solve an equation of the form:

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)y = 0$$

① Find the roots of the auxiliary equation,

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n = 0$$

② Put the general solution as follows:

Roots of Auxi. Equ^n.	Complete Solution
Case I All roots $m_1, m_2, \ldots, m_n$ are real and different	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
Case II $m_1 = m_2$ but other roots are real and different	$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
Case III ① $\alpha + i\beta$, a pair of imaginary roots	$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ or, $c_1 e^{\alpha x}\cos(\beta x + c_2)$ or, $c_1 e^{\alpha x}\sin(\beta x + c_2)$ $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$
② $(\alpha \pm i\beta)(\alpha + i\beta)$ repeated twice	$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$
*** $m = \pm\alpha i$	$y = A\cos\alpha x + B\sin\alpha x$

General solution of $(D^n + a_1 D^{n-1} + \dots + a_n)y = X$

Complementary Function $(C.F.) = y_c$

Particular Integral $(P.I.) = y_p$

The general solution, $y = y_p + y_c$

Page-69 এর Ex.1:

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

The equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$$

$$\Rightarrow (D^2 + D + 1)y = \sin x \dots\dots\dots (1)$$

Let $y = e^{mx}$ be the trial solution of $(D^2+D+1)y = 0 \dots\dots (2)$

$$Dy = me^{mx}$$

$$D^2y = m^2 e^{mx}$$

$\therefore$ Equ^n (2),

$$(m^2+m+1)e^{mx} = 0$$

$\therefore$ Auxiliary equation is,

$$m^2+m+1 = 0 \; ; e^{mx} \neq 0$$

$$\therefore m = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Complementary Function,

$$y_c = e^{-\frac{x}{2}} \left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right.$$

Particular Integration,

$$y_p = \frac{1}{D^2+D+1} \sin 2x$$

$$= \frac{1}{-(2)^2+D+1} \sin 2x$$

$$= \frac{1}{D-3} \sin 2x$$

$$= \frac{D+3}{D^2-9} \sin 2x$$

$$= \frac{D+3}{-(2)^2-9} \sin 2x$$

$$= -\frac{1}{13}(D+3)\sin 2x$$

$$= -\frac{1}{13}\left[D\sin 2x + 3\sin 2x\right]$$

$$= -\frac{1}{13}(2\cos 2x + 3\sin 2x).$$

The general solution,

$$y = y_e + y_p$$

$$= e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x).$$

(Ans.)

Page-69 এর Ex.2:

Solve $(D^2+1)^2 y = \cos 3x$

The equation,

$$(D^2+1)^2 y = \cos 3x \quad \cdots\cdots (1)$$

Let $y = e^{mx}$ be the trial solution of $(D^2+1)^2 y = 0 \cdots\cdots (2)$

$\therefore$ Auxiliary equation is,

$$(m^2+1)^2 = 0 \; ; [e^{mx} \neq 0]$$

$$\Rightarrow (m^2+1)(m^2+1) = 0$$

$$\Rightarrow m = \pm i, \pm i$$

$$\therefore m = i, i, -i, -i.$$

Complementary Function, $y_e = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x$

Particular Integration, $y_e = \frac{1}{(D^2+1)^2} \cos 3x$

$$= \frac{1}{(-3^2+1)^2} \cos 3x$$

$$= \frac{1}{(-8)^2} \cos 3x$$

$$= \frac{1}{64} \cos 3x$$

$\therefore$ The complete solution,

$$y = y_e + y_p$$

$$= (c_1 + c_2 x)\cos x + (c_3 + c_4)\sin x + \frac{\cos 3x}{64}.$$

Page-67 (Particular Integral when $X = e^{\alpha x}$):

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided that } f(a) \neq 0.$$

[illegible] Page-71 [illegible] Ex. 1:

Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$

The equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

$$\Rightarrow (D^2 - 3D + 2)y = e^x \quad \ldots \ldots \ldots (1)$$

Let, $y = e^{mx}$ be the trial solution of $(D^2 - 3D + 2)y = 0$

Auxiliary equation is,

$$m^2 - 3m + 2 = 0 \; ; [e^{mx} \neq 0]$$

$$\therefore m = 1, 2.$$

The complementary function is,

$$y_c = c_1 e^x + c_2 e^{2x}$$

The Particular Integration is,

$$y_p = \frac{1}{D^2 - 3D + 2} e^x$$

$$= x \cdot \frac{1}{2D - 3} e^x$$

$$= x \cdot \frac{1}{2 \cdot 1 - 3} e^x$$

$$= x \cdot \frac{1}{-1} \cdot e^x$$

$$= -xe^x$$

$\therefore$ The complete solution is,

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{2x} - xe^{x}.$$

(Ans:)

Page-72 ত্র Ex-4:

Solve, $2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = e^{x} + 1$

The equation,

$$2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = e^{x} + 1$$

$$\Rightarrow (2D^3 - 3D^2 + 1)y = e^{x} + 1 \ldots \ldots (1)$$

Let $y = e^{mx}$ be the trial solution of $(2D^3 - 3D^2 + 1)y = 0 \ldots (2)$

$\therefore$ Auxiliary equation of (2) is,

$$2m^3 - 3m^2 + 1 = 0 \ ; \ [e^{mx} \neq 0]$$

$$\Rightarrow 2m^2(m-1) - m(m-1) - 1(m-1) = 0$$

$$\Rightarrow (m-1)(2m^2 - m - 1) = 0$$

$$\Rightarrow (m-1)\{2m(m-1) + 1(m-1)\} = 0$$

$$\Rightarrow (m-1)(m-1)(2m+1) = 0$$

$$\therefore m = 1, 1, -\frac{1}{2}.$$

$\therefore$ The complementary function is,

$$y_c = (c_1 + c_2 x)e^x + c_3 e^{-\frac{x}{2}}$$

and the partial integral is,

$$y_p = \frac{1}{2D^3 - 3D^2 + 1}(e^x + 1)$$

$$= \frac{1}{2D^3 - 3D^2 + 1} e^x + \frac{1}{2D^3 - 3D^2 + 1} e^{0.x}$$

$$= x \frac{1}{6D^2 - 6D} e^x + \frac{1}{0 - 0 + 1} e^{0.x}$$

$$= x.x. \frac{1}{12D - 6} e^x + 1$$

$$= \frac{x^2.e^x}{12 - 6} + 1$$

$$= \frac{x^2 e^x}{6} + 1$$

$\therefore$ The general solution is, $y = y_c + y_p$

$$= (c_1 + c_2 x)e^x + c_3 e^{-\frac{x}{2}} + \frac{x^2 e^x}{6} + 1$$

{Page -75 ⇒ Ex-10 & 11 → Same Rules}

Page- 76

Ex.: Solve $(D^3+2D^2+D)y = e^{2x}+x^2+x$

The given equation,

$$(D^3+2D^2+D)y = e^{2x}+x^2+x \quad \ldots\ldots (1)$$

Let, $y=e^{mx}$ be the trial solution of $(D^3+2D^2+D)y=0 \ldots\ldots (2)$

$\therefore$ A.E. is, $m^3+2m^2+m=0 \; ; [e^{mx} \neq 0]$

$$\Rightarrow m(m^2+2m+1)=0$$

$$\Rightarrow m(m+1)^2=0$$

$$\therefore m = 0, -1, -1.$$

$$\therefore y_c = c_1 e^{0x} + (c_2+c_3x)e^{-x}$$

$$= c_1 + (c_2+c_3x)e^{-x}$$

and $y_p = \dfrac{1}{D^3+2D^2+D} e^{2x}+x^2+x$

$$= \frac{1}{D^3+2D^2+D} e^{2x} + \frac{1}{D^3+2D^2+D}(x^2+x)$$

$$= \frac{e^{2x}}{2^3+2.2^2+2} + \frac{1}{D(D+1)^2}(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}(1+D)^{-2}(x^2+x)$$

$$= \frac{1}{18}e^{2x} + \frac{1}{D}\left[1-2D+3D^2-4D^3+5D^4-\cdots\right](x^2+x)$$

$$= \frac{1}{18}e^{2x} + \frac{1}{D}\left[x^2+x-2(2x+1)+3(2+0)-0+0-\cdots\right]$$

$$= \frac{1}{18}e^{2x} + \frac{1}{D}(x^2+x-4x-2+6)$$

$$= \frac{1}{18}e^{2x} + \frac{1}{D}(x^2-3x+4)$$

$$= \frac{1}{18}e^{2x} + \left(\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right).$$

$\therefore$ The general solution is,

$$y = y_c + y_p$$

$$= c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18}e^{2x} + \left(\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right) \text{(Ans:)}$$

Ith Page-77 :

$$\frac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\frac{1}{f(D+a)}V$$, where V is function of x

ক্ষ Page-77 এর Ex.2:

Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = xe^x + e^x$

Let, $y = e^{mx}$

AE. is,

$$m^3 - 3m^2 + 3m - 1 = 0 \text{ , } [e^{mx} \neq 0]$$

$$\Rightarrow (m-1)^3 = 0$$

$$\therefore m = 1, 1, 1$$

$$\therefore y_c = (c_1 + c_2x + c_3x^2)e^x$$

$$y_p = \frac{1}{(D-1)^3}\cdot(xe^x + e^x)$$

$$= \frac{1}{(D-1)^3}\cdot e^x(x+1)$$

$$= e^x\frac{1}{(D+1-1)^3}\cdot(x+1)$$

$$= e^x\cdot\frac{1}{D^3}(x+1)$$

$$= e^x\cdot\frac{1}{D^2}\left(\frac{x^2}{2}+x\right)$$

$$= e^x\cdot\frac{1}{D}\left(\frac{x^3}{6}+\frac{x^2}{2}\right)$$

$$= e^x\left(\frac{x^4}{24}+\frac{x^3}{6}\right).$$

$\therefore$ The complete solution $= y_c + y_p$

$$= (c_1 + c_2 x + c_3 x^2) e^x + e^x \left(\frac{x^4}{24} + \frac{x^3}{6}\right).$$

Page-78 এর Ex. 4:

Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x$

Let, $y = e^{mx}$

$\therefore$ The equation, $(D^3 - 2D + 4) e^{mx} = e^x \cos x$

A.E. is, $D^3 - 2D + 4 = 0$; $[e^{mx} \neq 0]$

$\Rightarrow D^2(D+2) - 2D(D+2) + 2(D+2) = 0$

$\Rightarrow (D+2)(D^2 - 2D + 2) = 0.$

$\therefore D = -2,\ D = \frac{2 \pm \sqrt{2^2 - 4.1.2}}{2.1}$

$= \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i.$

$\therefore y_c = c_1 e^{-2x} + c_2 e^x \cos(x + c_3)$

$$y_p = \frac{1}{D^3 - 2D + 4} \cdot e^x \cos x$$

$$= \frac{e^x}{(D+1)^3 - 2(D+1) + 4} \cdot \cos x$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2 + 3D + 1 - 2D - 2 + 4} \cos x$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x.$$

$$= x e^x \cdot \frac{1}{3D^2 + 6D + 1} \cos x$$

$$= x e^x \cdot \frac{1}{3 \cdot (-1^2) + 6D + 1} \cos x$$

$$= x e^x \cdot \frac{1}{6D - 2} \cos x$$

$$= x e^x \cdot \frac{1}{2(3D - 1)} \cdot \cos x$$

$$= x e^x \cdot \frac{3D+1}{2(9D^2 - 1)} \cos x.$$

$$= x e^x \cdot \frac{3D+1}{2(9 \cdot -(1)^2 - 1)} \cos x$$

$$= -\frac{x e^x}{20} (3D+1) \cos x$$

$$= -\frac{x e^x}{20} (-3\sin x + \cos x)$$

$$= \frac{x e^x}{20} (3\sin x - \cos x)$$

Hence, The complete solution is,

$$y = y_c + y_p$$

$$= c_1 e^{-2x} + c_2 e^{x}\cos(x + c_3) + \frac{1}{20} x e^{x}(3\sin x - \cos x).$$

Page-81 এর Ex. 2:

Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

Let, $y = e^{mx}$

$\therefore$ A.E. is, $m^2 + 4 = 0$; $e^{mx} \neq 0$

$$\Rightarrow m = \pm 2i$$

$\therefore y_c = A\cos 2x + B\sin 2x$

and $y_p = \frac{1}{D^2+4} \cdot x\sin x$

$= \text{I.P. of } \frac{1}{D^2+4} \cdot x e^{ix}$

$= \text{I.P. of } e^{ix} \frac{1}{(D+i)^2+4} x$

$= \text{I.P. of } e^{ix} \frac{1}{D^2+2iD-1+4} x$

$= \text{I.P. of } e^{ix} \frac{1}{3+2iD+D^2} x$

Let,

$X = x\cos x$

$Y = x\sin x$

$X + iY = x\cos x + ix\sin x$

$= x(\cos x + i\sin x)$

$= x e^{ix}$

$$= \text{I.P. of } e^{ix} \frac{1}{3\left(1+\frac{2iD+D^2}{3}\right)} x$$

$$= \text{I.P. of } e^{ix} \frac{1}{3}\left(1+\frac{2iD+D^2}{3}\right)^{-1} x$$

$$= \text{I.P. of } \frac{e^{ix}}{3}\left[1-\frac{2iD+D^2}{3}+\left(\frac{2iD+D^2}{3}\right)^2-\cdots\right]x$$

$$= \text{I.P. of } \frac{e^{ix}}{3}\left[x-\frac{2i+0}{3}+0\cdots\right]$$

$$= \text{I.P. of } \frac{e^{ix}}{3}\left(x-\frac{2i}{3}\right)$$

$$= \text{I.P. of } \frac{1}{3}(\cos x+i\sin x)\left(x-\frac{2i}{3}\right)$$

$$= \text{I.P. of } \frac{1}{3}\left(x\cos x-\frac{2i}{3}\cos x+ix\sin x+\frac{2\sin x}{3}\right)$$

$$= \text{I.P. of } \frac{1}{3}\left[x\cos x+\frac{2\sin x}{3}+i\left(x\sin x-\frac{2}{3}\cos x\right)\right]$$

$$= \frac{1}{3}\left(x\sin x-\frac{2}{3}\cos x\right)$$

$$= \frac{1}{9}(3x\sin x-2\cos x)$$

The complete solution is,

$$y = y_c + y_p$$

$$= A\cos 2x + B\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$$

Page - 80 এর Ex.1:

Solve $\frac{d^4y}{dx^4} - y = x\sin x$

Auxiliary equation is,

$$D^4 - 1 = 0$$

$$\Rightarrow (D^2+1)(D^2-1) = 0$$

$$\therefore D = \pm i, \pm 1.$$

Hence,

$$y_c = c_1\cos x + c_2\sin x + c_3e^{x} + c_4e^{-x}$$

and $y_p = \frac{1}{D^4-1} \cdot x\sin x$

$$= \text{Imaginary Part of } \frac{1}{D^4-1} xe^{ix}$$

$$= \text{I.P. of } e^{ix}\frac{1}{(D+i)^4-1} x$$

$$= \text{I.P. of } e^{ix}\frac{1}{D^4+4iD^3-6D^2-4iD} x$$

$$= \text{I.P. of } -e^{ix}\frac{1}{4iD}\left[1-\frac{3}{2}iD - D^2 + \frac{1}{4}iD^3\right]^{-1} x \quad \left[\because \frac{1}{i} = -i\right]$$

$$= \text{I.P. of } -e^{ix}\frac{1}{4iD}\left[1+\frac{3}{2}iD\right] x$$

$$= \text{I.P. of } -e^{ix}\frac{1}{4iD}\left[x+\frac{3}{2}i\right]$$

$$= \text{I.P. of } \frac{i}{4}(\cos x + i\sin x)\left[\frac{1}{2}x^2 + \frac{3}{2}ix\right]$$

$$= \frac{1}{8}x^2\cos x - \frac{3}{2}x\sin x.$$

Hence, the complete solution is,

$$y = y_c + y_p$$

$$= C_1\cos x + C_2\sin x + C_3e^{x} + C_4e^{-x} + \frac{1}{8}x^2\cos x - \frac{3}{8}x\sin x.$$

(Ans:)

[Page-81 এর Ex.3(a), Page-82 এর Ex.3(b), Page-83 এর Ex.7]

[Page-85 এর Ex.5, Page-87 এর Ex.8, Page-88 এর Ex.11]

Page-78 এর Ex-3:

Solve $(D^3-7D-6)y = e^{2x}.x^2$

A.E. is, $D^3-7D-6=0$

$\Rightarrow D^2(D+1)-D(D+1)-6(D+1)=0$

$\Rightarrow (D+1)(D^2-D-6)=0.$

$\Rightarrow (D+1)(D-3)(D+2)=0.$

$\therefore D=-1, 3, -2$

$\therefore y_c = c_1e^{-x}+c_2e^{3x}+c_3e^{-2x}$

and $y_p = \frac{1}{D^3-7D-6} e^{2x}.x^2.$

$= e^{2x}. \frac{1}{(D+2)^3-7(D+2)-6} x^2.$

$= e^{2x} \frac{1}{D^3+6D^2+12D+8-7D-14-6} x^2$

$= e^{2x} \frac{1}{D^3+6D^2+5D-12} x^2.$

$= -\frac{e^{2x}}{12} \left(1-\frac{5}{12}D-\frac{1}{2}D^2-\frac{1}{12}D^3\right)^{-1} x^2$

$= -\frac{e^{2x}}{12} \left\{1-\left(\frac{5}{12}D+\frac{1}{2}D^2+\frac{1}{12}D^3\right)\right\}^{-1} x^2$

$$= -\frac{e^{2x}}{12}\left[1+\left(\frac{5}{12}D+\frac{1}{2}D^2+\frac{1}{12}D^3\right)+\left(\frac{5}{12}D+\frac{1}{2}D^2+\frac{1}{12}D^3\right)^2+\cdots\right]x^2$$

$$= -\frac{e^{2x}}{12}\left[1+\frac{5}{12}D+\frac{1}{2}D^2+\frac{25}{144}D^2\right]x^2$$

$$= -\frac{e^{2x}}{12}\left(x^3+\frac{5}{6}x+1+\frac{25}{72}\right).$$

$$= -\frac{e^{2x}}{12}\left(x^3+\frac{5}{6}x+\frac{97}{72}\right).$$

$\therefore$ The complete solution is.

$$y = y_c + y_p$$

$$= c_1e^{-x}+c_2e^{3x}+c_3e^{-2x}-\frac{e^{2x}}{12}\left(x^3+\frac{5}{6}x+\frac{97}{72}\right).$$

(Ans:)

Page-81 এর Ex. 3(a):

Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x$

The equation is,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x \quad \cdots\cdots (1)$$

Let $y = e^{mx}$ be the trial solution,

$$(D^2 - 4D + 4)y = 0 \quad \cdots\cdots (2)$$

$\therefore$ A.E. is,

$$m^2 - 4m + 4 = 0;\ [e^{mx} \neq 0]$$

$$\Rightarrow (m-2)^2 = 0$$

$$\therefore m = 2, 2.$$

$$\therefore y_c = (c_1 + c_2x)e^{2x}.$$

Let, $X = x^2\cos 2x$

$Y = x^2\sin 2x$

$X + iY = x^2\cos 2x + ix^2\sin 2x$

$= x^2e^{i2x}$.

and $y_p = \frac{1}{D^2 - 4D + 4} \cdot 3x^2 \cdot e^{2x} \cdot \sin 2x$.

$$= \frac{1}{(D-2)^2} \cdot 3e^{2x} \cdot x^2\sin 2x$$

$$= 3e^{2x}\left(\frac{1}{D+2-2}\right) \cdot x^2\sin 2x$$

$$= 3e^{2x} \cdot \frac{1}{D^2}(x^2\sin 2x)$$

$= I.P.$ of $3e^{2x} \cdot \frac{1}{D^2} x^2 . e^{2ix}$

$= I.P.$ of $3e^{2x} \frac{1}{(D+2i)^2} \cdot x^2$

$= I.P.$ of $3e^{2x} \frac{1}{(D+2i)^2} \cdot x^2$

$= I.P.$ of $3.e^{2x} . e^{2ix} \frac{1}{D^2+4iD-4} \cdot x^2$

$= I.P.$ of $3e^{2x} . e^{2ix} \frac{1}{-4\left(1-\frac{4iD+D^2}{4}\right)} x^2$

$= I.P.$ of $-\frac{3}{4} e^{2x} . e^{2ix} \left(1 - \frac{4iD+D^2}{4}\right)^{-1} x^2$

$= I.P.$ of $-\frac{3}{4} e^{2x} . e^{2ix} \left[1 + \frac{4iD+D^2}{4} + \left(\frac{4iD+D^2}{4}\right)^2 + \cdots\right] x^2$

$= I.P.$ of $-\frac{3}{4} e^{2x} . e^{2ix} \left[1 + iD + \frac{D^2}{4} + \frac{1}{16}(-16D^2 + 8iD^3 + D^4) - \cdots\right] x^2$

$= I.P.$ of $-\frac{3}{4} e^{2x} . e^{2ix} \left[x^2 + 2ix + \frac{2}{4} + \frac{2}{4} - 2 + 0\right]$

$= I.P.$ of $-\frac{3}{4} e^{2x} \quad (\cos 2x + i \sin 2x) \left(x^2 + 2ix - \frac{3}{2}\right)$

$= I.P.$ of $-\frac{3}{4} e^{2x} \left(x^2 \cos 2x + 2ix \cos 2x - \frac{3}{2}\cos 2x + i x^2 \sin x - 2x \sin x - \frac{3}{2} i \sin 2x\right)$

$$= -\frac{3}{4}e^{2x}\left(2x\cos 2x + x^2\sin 2x - \frac{3}{4}\sin 2x\right)$$

$$= -\frac{3}{8}e^{2x}\left[4\cos 2x + (2x^2-3)\sin 2x\right]$$

The complete solution is,

$$y = y_c + y_p$$

$$= (c_1 + c_1 x)e^{2x} - \frac{3}{8}e^{2x}\left[4\cos 2x + (2x^2-3)\sin 2x\right]$$

(Ans:)

{ Page - (92 - 93) उद् Ex - 21, 22, 23, 24 }.

Chapter : 06

Homogeneous Linear Equations :

An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = X, \cdots \cdots (i)$$

where $P_1, P_2, \ldots, P_n$ are constants and X is a function of x, is called the Homogeneous Linear Equation.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 5y = 10xe^x \text{ ——— } (1)$$

Here, equ^n (1) is an example of homogeneous linear equation.

Let, $x = e^z$

$\Rightarrow \log x = \log e^z$

$\Rightarrow z = \log x$

$\Rightarrow \frac{dz}{dx} = \frac{1}{x} \cdots\cdots (ii)$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \cdots \cdots \cdots (iii)$

Again,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right)$$

$$= \frac{x \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{dy}{dz} \cdot 1}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx} = \frac{x\frac{d^2y}{dz^2}\cdot\frac{1}{x} - \frac{dy}{dz}}{x^2}$$

Here,

$D = \frac{d}{dx}$

$D_1 = \frac{d}{dz}$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz} - \frac{dy}{dz}$$

$$\therefore x^2 D^2 y = D_1^2 y - D_1 y$$

$$= D_1(D_1 - 1)y.$$

So,

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y,$$

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\ldots(D-n+1)y.$$

Page-96 সূত্র Ex. 1:

Solve $x^2 \frac{d^2y}{dx^2} + y = 3x^2$

Let, $x = e^z$ and $\frac{d}{dz} \equiv D$, the equation becomes,

$$D(D-1)y + y = 3e^{2z}$$

$$\Rightarrow (D^2 - D + 1)y = 3e^{2z}.$$

The auxiliary equation is,

$$D^2 - D + 1 = 0$$

$$\therefore D = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$\therefore$ C.F. is, $y_c = c_1 e^{z/2} \cos\left(\frac{1}{2}\sqrt{3}\, z + c_2\right)$.

and P.I is, $y_p = \frac{3e^{2z}}{D^2 - D + 1}$

$$= \frac{3e^{2z}}{2^2 - 2 + 1}$$

$$= e^{2z}$$

Therefore the solution is,

$$y = c_1 e^{z/2} \cos\left(\frac{1}{2}\sqrt{3}z + c_2\right) + e^{2z}$$

$$= c_1 x^{1/2} \cos\left(\frac{1}{2}\sqrt{3}\log x + c_2\right) + x^2.$$

(Ans.)

Page-97 এর Ex.4(a):

Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

Let, $x = e^z$

$\Rightarrow z = \log x$

$x\frac{dy}{dx} = \frac{dy}{dz}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

The equation becomes,

$\frac{d^2y}{dz^2} - \frac{dy}{dz} - \frac{3dy}{dz} + 4y = 2e^{2z}$

$\Rightarrow \frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 2e^{2z}$.

$\Rightarrow (D_1^2 - 4D_1 + 4)y = 2e^{2z}; \left[D_1 = \frac{d}{dz}\right]$

$\Rightarrow (D_1 - 2)^2 y = 2e^{2z}$.

$\therefore$ A.E. is $(D_1 - 2)^2 = 0$

$\therefore D = 2, 2$

$\therefore$ C.F. is $y_c = (c_1 + c_2 z)e^{2z}$

$= (c_1 + c_2 \log_x)x^2$

let, $y = e^{mx}$ be the trial sol^n,

$(D_1 + 2)^2 y = 0$

$\therefore$ A.E. is,

$(m-2)^2 = 0 ; [e^{mz} \neq 0]$

$\therefore m = 2, 2$

$y_c = (c_1 + c_2 z)e^{2z}$

$= (c_1 + c_2 \log_x)x^2$

and P.I. is $y_p = \frac{1}{(D_1-2)^2} \cdot 2e^{2z}$

$$= 2z \frac{1}{2(D_1-2)} \cdot e^{2z}$$

$$= 2 \cdot z \cdot \frac{1}{(1-0)} \cdot e^{2z}$$

$$= z^2 e^{2z}$$

$$= (\log x)^2 \cdot x^2$$

$\therefore y = y_c + y_p$

$$= (c_1 + c_2 \log x) x^2 + (\log x)^2 x^2$$

[Page-98 এর Ex-6(a), Page-99 এর Ex-7, Page-100 এর Ex-9, 10]

Page- 98 এর Ex. 6(a):

Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^3 + 3x$

Putting, $x = e^z$ and $D = \frac{d}{dz}$, the equation becomes,

$$[D(D-1)(D-2) - D(D-1) + 2D - 2]y = e^{3z} + 3e^z$$

$$\Rightarrow [(D^2-D)(D-2) - (D^2-D) + 2D - 2]y = e^{3z} + 3e^z$$

$$\Rightarrow [D^3 - 2D^2 + 2D - D^2 - D^2 + D + 2D - 2]y = e^{3z} + 3e^z$$

$$\Rightarrow [D^3 - 4D^2 + 5D - 2]y = e^{3z} + 3e^z.$$

$\therefore$ A.E. is, $D^3 - 4D^2 + 5D - 2 = 0$

i.e. $(D-1)^2(D-2) = 0$.

$\therefore D = 1, 1, 2.$

$\therefore$ C.F. $= (c_1 + c_2 z)e^z + c_3 e^{2z}$.

$= (c_1 + c_2 \log x)x + c_3 x^2$

and P.I. $= \frac{e^{3z}}{(D-1)^2(D-2)} + 3\frac{e^z}{(D-1)^2(D-2)}$

$= \frac{e^{3z}}{(3-1)^2(3-2)} + 3z^2 \frac{e^z}{6D-8}$ [multiplying the second term by z^2 and differentiating the denominator twice w.r.t. D]

$= \frac{1}{4}e^{3z} - \frac{3}{2}z^2e^z.$

$= \frac{1}{4}x^3 - \frac{3}{2}(\log x)^2 . x.$

Therefore the general solution is,

$$y = (c_1 + c_2 \log x)x + c_3 x^2 + \frac{1}{4}x^3 - \frac{3}{2}(\log x)^2 . x.$$

Page-99 এর Ex. 7:

Solve $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

Dividing by x, the equation can be written as,

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{1}{x}$$

Now putting $x = e^z$ and $D \equiv \frac{d}{dz}$, this becomes.

$$[D(D-1)(D-2) + 2D(D-1) - D + 1]y = e^{-z}$$

$$\Rightarrow (D^2 - D)(D-2) + (2D^2 - 2D) - D + 1]y = e^{-z}$$

$$\Rightarrow [D^3 - 2D^2 - D^2 + 2D + 2D^2 - 2D - D + 1]y = e^{-z}$$

$$\Rightarrow [D^3 - D^2 - D + 1]y = e^{-z}.$$

The Auxiliary equation is,

$$D^3 - D^2 - D + 1 = 0.$$

$$\text{or, } (D-1)^2(D+1) = 0.$$

$\therefore$ The complementary function is,

$$y_c = (c_1 + c_2 z)e^z + c_3 e^{-z}$$

$$= (c_1 + c_2 \log x)x + c_3 x^{-1}$$

and the partial Integral is,

$$y_p = \frac{1}{(D-1)^2(D+1)} e^{-z}$$

$$= z \frac{1}{3D^2-2D-1} e^{-z}$$ [multiplying by z and differentiating the denominator w.r.t. D]

$$= z \frac{e^{-z}}{3(-1)^2-2(-1)-1}$$

$$= \frac{1}{4} z e^{-z}$$

$$= \frac{1}{4}(\log x) \cdot \frac{1}{x}.$$

Hence, the complete solution is,

$$y = y_c + y_p$$

$$= (C_1 + C_2 \log x)x + C_3 x^{-1} + \frac{1}{4}(\log x)\frac{1}{x}.$$ (Ans:)

LT Page-100 এর Ex.10:

Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

putting $x = e^z$ and $D \equiv \frac{d}{dz}$, the equation becomes,

$$[D(D-1) - D - 3]y = x^2 \log x.$$

Auxiliary equation is,

$$D^2 - 2D - 3 = 0$$

$$\Rightarrow (D-3)(D+1) = 0.$$

$$\therefore D = 3, -1.$$

$\therefore$ The complementary function $= y_c = c_1 e^{3z} + c_2 e^{-z}$

$$= c_1 x^3 + c_2 x^{-1}.$$

and partial Integration, $y_p = \frac{z e^{2z}}{D^2 - 2D - 3}$

$$= e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} z$$

$$= e^{2z} \frac{1}{D^2 + 2D - 3} z$$

$$= -\frac{e^{2z}}{3}\left[1 - \frac{2}{3}D - \frac{1}{3}D^2\right]^{-1} z$$

$$= -\frac{1}{3} e^{2z}\left(1 + \frac{2}{3}D + \cdots\right) z$$

$$= -\frac{1}{3} e^{2z}\left(z + \frac{2}{3}\right).$$

$$= -\frac{1}{3} x^2\left(\log x + \frac{2}{3}\right).$$

Hence, the complete solution is,

$$y = y_c + y_p$$

$$= c_1 x^3 + c_2 x^{-1} - \frac{1}{3} x^2\left(\log x + \frac{2}{3}\right) \text{(Ans.)}$$

Chapter:05 (Flash Back)

Linear Differential Equation with Constant Coefficients.

Page-74 এর Ex. 8:

Solve $\frac{d^6y}{dx^6}+y=\sin\frac{3}{2}x\sin\frac{1}{2}x$

Auxiliary equation is,

$$D^6+1=0.$$

$$\Rightarrow D^6+D^4-D^4-D^2+D^2+1=0$$

$$\Rightarrow D^4(D^2+1)-D^2(D^2+1)+1(D^2+1)=0$$

$$\Rightarrow (D^2+1)(D^4-D^2+1)=0$$

$$\Rightarrow (D^2+1)\left[(D^2+1)^2-3D^2\right]=0$$

$$\Rightarrow (D^2+1)(D^2-\sqrt{3}D+1)(D^2+\sqrt{3}D+1)=0.$$

When $D^2+1=0$, $D=\pm i$

When $D^2-\sqrt{3}D+1=0$, $D=\frac{\sqrt{3}\pm i}{2}$

When $D^2-\sqrt{3}D+1=0$, $D=\frac{\sqrt{3}\pm i}{2}$

Hence,

C.F. is, $y_c=C_1\cos x(x+C_2)+C_3e^{\sqrt{3}x}\cos\left(\frac{1}{2}x+c_4\right)+C_5e^{-\sqrt{3}x}\cos\left(\frac{1}{2}x\right.$

Now,

$$\sin\frac{3}{2}x\sin\frac{1}{2}x=\frac{1}{2}(\cos x-\cos 2x)$$

$\therefore$ P.I. $y_p = \frac{1}{2}\frac{\cos x}{D^6+1} - \frac{1}{2}\frac{\cos 2x}{D^6+1}$ (first term case of failure)

$= \frac{1}{2} x \frac{\cos x}{6D^5} - \frac{1}{2} \cdot \frac{\cos 2x}{(-4)^3+1}$

$= \frac{1}{2} x \frac{\cos x}{6(-1)^2 D} + \frac{1}{126}\cos 2x$

$= \frac{1}{12} x \sin x + \frac{1}{126}\cos 2x$; [as $\frac{1}{D}$ means integration]

Hence, the complete solution is,

$$y = y_c + y_p$$

বই Page-92 এর Ex. 21 :

Solve $\frac{d^2y}{dx^2}+a^2y=\sec ax$

A.E. is $D^2+a^2=0$

i.e., $D=\pm ai$

$\therefore$ C.F. is $y_c=(C_1\cos ax+C_2\sin ax)$

Now,

P.I. is $y_p=\frac{1}{D^2+a^2}\sec ax$

$=\frac{1}{(D+ai)(D-ai)}\sec ax$

$=\frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]e^{aix}\cdot\frac{e^{-aix}}{\cos ax}$

$=\left[\frac{1}{2ai}\cdot\frac{1}{D-ai}\cdot e^{aix}\cdot\frac{e^{-aix}}{\cos ax}\right]-\left[\frac{1}{2ai}\cdot\frac{1}{D+ai}\cdot e^{aix}\cdot\frac{e^{-aix}}{\cos ax}\right]$

$=\frac{e^{aix}}{2ai}\cdot\frac{1}{D}\cdot\frac{e^{-aix}}{\cos ax}-\frac{e^{-aix}}{2ai}\cdot\frac{1}{D}\cdot\frac{e^{aix}}{\cos ax}$

$=\frac{e^{aix}}{2ai}\int\frac{\cos ax-i\sin ax}{\cos ax}dx-\frac{e^{-aix}}{2ai}\int\frac{\cos ax+i\sin ax}{\cos ax}dx$

$=\frac{e^{aix}}{2ai}\int(1-i\tan ax)dx-\frac{e^{-aix}}{2ai}\int(1+i\tan ax)dx$

$$= \frac{e^{aix}}{2ai}\left[x + \frac{i}{a}\log\cos ax\right] - \frac{e^{-aix}}{2ai}\left[x - \frac{i}{a}\log\cos ax\right]$$

$$= \frac{1}{2ai}\left[x\left(e^{aix} - e^{-aix}\right)\right] + \frac{1}{2a^2}\log(\cos ax)\left[e^{aix} + e^{-aix}\right]$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\log(\cos ax)\cos ax.$$

Hence, the complete solution is,

$$y = y_c + y_p$$

$$= (c_1\cos ax + c_2\sin ax) + \frac{x}{a}\sin ax + \frac{1}{a^2}\log(\cos ax)\cos ax$$

Chapter: 07

Equations of the First Order But not of the First Degree

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Page-116 এর Ex.2:

Solve $p^3+2xp^2-y^2p^2-2xy^2p=0$

The equation can be written as,

$$p(p^2+2xp-y^2p-2xy^2)=0$$

$$\Rightarrow p\{p(p+2x)-y^2(p+2x)\}=0$$

$$\Rightarrow p(p+2x)(p-y^2)=0$$

The component equations are,

$$\frac{dy}{dx}=0,\ \frac{dy}{dx}+2x=0,\ \frac{dy}{dx}-y^2=0.$$

Solution of these component equations are,

$\frac{dy}{dx}=0$	$\frac{dy}{dx}+2x=0$	$\frac{dy}{dx}-y^2=0$
$\Rightarrow dy=0$	$\Rightarrow dy+2x\,dx=0$	$\Rightarrow \frac{dy}{y^2}-dx=0$
$\Rightarrow y=c$	$\Rightarrow y+x^2=c$	$\Rightarrow \frac{-1}{y}-x=c$
$\therefore y-c=0$	$\therefore y+x^2-c=0$	$\therefore 1+xy+cy=0.$

Therefore, the most general solution is,

$$(y-c)(y+x^2-c)(xy+yc+1)=0.$$

Page-117 এর Ex.5:

Solve $x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}+2y^2-x^2=0$

Writing p for $\frac{dy}{dx}$, the equation becomes

$$x^2p^2-2xyp+2y^2-x^2=0.$$

Solving for P,

$$P=\frac{2xy\pm\sqrt{[4x^2\cdot y^2-4x^2\cdot(2y^2-x^2)]}}{2x^2}$$

$$=\frac{2xy\pm\sqrt{4x^2y^2-8x^2y^2+4x^4}}{2x^2}$$

$$=\frac{2xy\pm\sqrt{4x^4-4x^2y^2}}{2x^2}$$

$$=\frac{2xy\pm 2x\sqrt{x^2-y^2}}{2x^2}$$

$$=\frac{y\pm\sqrt{x^2-y^2}}{x}$$

The component equations are

$$\frac{dy}{dx}=\frac{y\pm\sqrt{x^2-y^2}}{x}$$

These are homogeneous.

Let $y=vx$

$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2x^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx \pm x\sqrt{1-v^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v \pm \sqrt{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \pm\sqrt{1-v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x} \quad \text{and} \quad \frac{dv}{\sqrt{1-v^2}} = -\frac{dx}{x}$$

$$\Rightarrow \sin^{-1}v = \log x + \log c \qquad \Rightarrow \sin^{-1}v = \log x - \log c$$

$$\Rightarrow \sin^{-1}v = \log cx \qquad \Rightarrow \sin^{-1}v = -\log cx$$

$$\therefore \sin^{-1}v = \pm\log cx$$

$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \pm\log cx$, which form the required solution

Page-119 এর Ex. 10:

Solve $p^3-(x^2+xy+y^2)p^2+(x^3y+x^2y^2+xy^3)p-x^3y^3=0$

The equation on factorization is,

$$(p-x^2)(p-y^2)(p-xy)=0.$$

The component equations are,

$$\frac{dy}{dx}=x^2, \quad \frac{dy}{dx}=y^2, \quad \frac{dy}{dx}=xy$$

$$\Rightarrow y=\frac{x^3}{3}+c \quad \Rightarrow \frac{dy}{y^2}=dx \quad \Rightarrow \frac{dy}{y}=xdx$$

$$\Rightarrow 3y-x^3-3c=0 \quad \Rightarrow -\frac{1}{y}=x+c \quad \Rightarrow \log y=x+c$$

$$\Rightarrow 1+xy+cy=0 \quad \Rightarrow \log y-x-c=0.$$

Therefore the complete solution is,

$$(3y-x^3-3c)(1+xy+cy)(\log y-x-c)=0.$$

(Ans.)

Online Class 01

13/07/2020
Monday

Page-117 এর Ex.6:

Solve $x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0$

Solution:

We have,

$$x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0$$

$\Rightarrow x^2p^2+3xyp+2y^2=0$; [where $p=\frac{dy}{dx}$]

$\Rightarrow x^2p^2+2xyp+xyp+2y^2=0$

$\Rightarrow x^2p^2+xyp+2xyp+2y^2=0$

$\Rightarrow xp(xp+y)+2y(xp+y)=0$

$\Rightarrow (xp+y)(xp+2y)=0.$

∴ The component equations are,

$xp+y=0$ and $xp+2y=0$

$\Rightarrow x\frac{dy}{dx}+y=0$

$\Rightarrow \frac{dy}{y}+\frac{dx}{x}=0$

Integrating,

$\log y+\log x=\log c$

$\Rightarrow \log xy=\log c$

$\Rightarrow xy=c$

$\therefore xy-c=0$

$\Rightarrow x\frac{dy}{dx}+2y=0$

$\Rightarrow \frac{dy}{y}+2\frac{dx}{x}=0.$

Integrating,

$\log y+2\log x=\log c$

$\Rightarrow \log y+\log x^2=\log c$

$\Rightarrow \log x^2y=\log c$

$\Rightarrow x^2y=c$

$\therefore x^2y-c=0$

Hence, the solution is,

$$(xy-c)(x^2y-c)=0 \quad \text{(Ans:)}$$

Page-117 এর Ex.7(a):

Solve $yp^2+(x-y)p-x=0$

Solution:

Given that,

$$yp^2+(x-y)p-x=0$$

$$\Rightarrow yp^2+xp-yp-x=0$$

$$\Rightarrow p(yp+x)-1(yp+x)=0$$

$$\Rightarrow (yp+x)(p-1)=0$$

The component equations are,

$p-1=0$ and, $yp+x=0$

$\Rightarrow \frac{dy}{dx}=1$

Integrating,

$y=x+c$

$\therefore y-x-c=0$

$\Rightarrow y\frac{dy}{dx}+x=0$

$\Rightarrow ydy+xdx=0$

Integrating,

$y^2+x^2=c$

$\therefore x^2+y^2-c=0$.

$\therefore$ The solution is, $(y-x-c)(x^2+y^2-c)=0$

(Ans:).

◫ Page-118 এর Ex. 7(b):

Solve $xp^2+(y-x)p-y=0$

Sol^n:

Given that,

$$xp^2+(y-x)p-y=0$$

$$\Rightarrow xp^2+py-xp-y=0$$

$$\Rightarrow p(xp+y)-1(xp+y)=0$$

$$\Rightarrow (xp+y)(p-1)=0.$$

The component equation are,

$p-1=0$ and, $x\frac{dy}{dx}+y=0$

$\Rightarrow \frac{dy}{dx}=1$

$\Rightarrow \frac{dy}{y}+\frac{dx}{x}=0$

Integrating,

$y=x+c$

$\therefore y-x-c=0$

Integrating,

$\log y+\log x=\log c$

$\Rightarrow \log xy=\log c$

$\Rightarrow xy=c$

$\therefore xy-c=0.$

$\therefore$ The solution is, $(y-x-c)(xy-c)=0$. (Ans.)

Page-118 এর Ex. 8:

Solve $p^3 - p(x^2+xy+y^2) + xy(x+y) = 0$

Solution:

We have,

$$p^3 - p(x^2+xy+y^2) + xy(x+y) = 0$$

$$\Rightarrow p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$$

$$\Rightarrow p^2(p-x) - py(x-y) - y^2(p-x) = 0$$

[Vanishing method]

p এর স্থানে x বসালে 0 হবে সেজন্য (p−x) একটি উৎপাদক

$$\Rightarrow (p-x)(p^2 - py - y^2) = 0$$

$$\Rightarrow (p-x)\{x(p-y) + (p-y)(p+y)\} = 0$$

$$\Rightarrow (p-x)(p-y)(x+y+p) = 0$$

The component equations are,

$$p - x = 0, \quad p - y = 0, \quad p + x + y = 0.$$

Now,

$$\frac{dy}{dx} - x = 0$$

$$\Rightarrow dy - x\,dx = 0$$

Integrating,

$$y - \frac{x^2}{2} = c_1$$

$$\Rightarrow 2y - x^2 = 2c_1$$

$$\Rightarrow 2y - x^2 - 2c_1 = 0$$

$$\frac{dy}{dx} - y$$

$$\Rightarrow \frac{dy}{y} - dx = 0$$

Integrating,

$$\log y - x = \log c$$

$$\Rightarrow \log y = \log c + \log e^x$$

$$\Rightarrow \log y = \log ce^x$$

$$\Rightarrow y = ce^x$$

$$\therefore y - ce^x = 0$$

$$p + x + y = 0$$

$$\Rightarrow \frac{dy}{dx} + x + y = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -x$$

which is linear equation

$$\therefore \text{I.F.} = e^{\int dx}$$

$$= e^x$$

So, solution is,

$$ye^x = -\int x e^x dx$$

$$\Rightarrow ye^x = -xe^x + \int e^x dx$$

$$\Rightarrow ye^x = -xe^x + e^x + c$$

$$\Rightarrow ye^x = -e^x(x-1) + c$$

$$\Rightarrow y = -(x-1) + ce^{-x}$$

$$\Rightarrow y + x - 1 - ce^{-x} = 0.$$

$\therefore$ The required solution is,

$$(2y - x^2 - c)(y - ce^x)(x + y - 1 - ce^{-x}) = 0 \quad \text{(Ans.)}$$

Online Class 02

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14/07/2020

Tuesday

Page-119 এর Ex. 11:

Solve $xyp^2+(x^2+xy+y^2)p+x^2+xy=0$

Solution:

Given that,

$$xyp^2+(x^2+xy+y^2)p+x^2+xy=0$$

$$\Rightarrow xyp^2+x^2p+xyp+y^2p+x^2+xy=0$$

$$\Rightarrow xp(yp+x)+x(yp+x)+y(yp+x)=0$$

$$\Rightarrow (yp+x)(xp+x+y)=0.$$

The component equations are,

$yp+x=0$

$\Rightarrow y\frac{dy}{dx}+x=0$

$\Rightarrow ydy+xdx=0$

Integrating,

$\frac{y^2}{2}+\frac{x^2}{2}=c$

$\Rightarrow x^2+y^2=c$

$\therefore x^2+y^2-c=0$

$xp+x+y=0$

$\Rightarrow x\frac{dy}{dx}+x+y=0$

$\Rightarrow xdy+xdx+ydx=0.$

$\Rightarrow xdy+ydx+xdx=0$

$\Rightarrow d(xy)+xdx=0$

Integrating,

$xy+\frac{x^2}{2}=c$

$\Rightarrow 2xy+x^2=2c$

$\therefore 2xy+x^2-2c=0$

$\therefore$ The solution is, $(x^2+y^2-c)(2xy+x^2-\cancel{x})=0$

(Ans:)

[I] Clairaut's Equation: (Page-130)

$$y = px + f(p)$$

To solve $y = px + f(p)$.

Solution: Given that,

$$y = px + f(p) \quad \dots\dots (1)$$

D.W.R. to x,

$$\therefore \frac{dy}{dx} = p.1 + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\Rightarrow p = p + \{x + f'(p)\}\frac{dp}{dx}$$

$$\Rightarrow \{x + f'(p)\}\frac{dp}{dx} = 0.$$

$$\therefore \frac{dp}{dx} = 0, \text{ neglecting } x + f'(p) = 0$$

$$\Rightarrow dp = 0$$

$$\therefore p = c$$

Putting the value of p in (1) we get,

$$y = cx + f(c).$$

(Ans:).

Page-131 এর Ex.04:

Solve $\sin px \cos y = \cos px \sin y + p$

Solution:

Given that,

$$\sin px \cos y = \cos px \sin y + p$$

$$\Rightarrow \sin px \cos y - \cos px \sin y = p$$

$$\Rightarrow \sin(px - y) = p$$

$$\Rightarrow (px - y) = \sin^{-1}(p)$$

$$\Rightarrow y = px - \sin^{-1}(p) \ldots \ldots \ldots (1)$$

which is the form of $y = px + f(p)$

So, the equation (1) is a clairaut's equation,

putting $p = c$ in (1),

The solution is,

$$y = cx - \sin^{-1}(c).$$

(Ans:)

Page-132 প্রশ্ন Ex-06:

Solve $(y-px)^2 = 1+p^2$

Solution:

Given that,

$$(y-px)^2 = 1+p^2$$

$$\Rightarrow y-px = \pm\sqrt{1+p^2}$$

$$\Rightarrow y = px \pm \sqrt{1+p^2}$$

$$\therefore y = px + \sqrt{1+p^2} \quad \dots\dots (1)$$

and $y = px - \sqrt{1+p^2} \quad \dots\dots (2)$

Both the equation (1) and (2) are of the clairauts form.

Putting $p=c$,

The solutions are,

$$y = cx + \sqrt{1+c^2} \text{ and } y = cx - \sqrt{1+c^2}$$

$$\Rightarrow y - cx - \sqrt{1+c^2} = 0 \quad \Rightarrow y - cx + \sqrt{1+c^2} = 0$$

Therefore the primative solution is,

$$(y - cx - \sqrt{1+c^2})(y - cx + \sqrt{1+c^2}) = 0$$

$$\Rightarrow \{(y-cx) - \sqrt{1+c^2}\}\{(y-cx) + \sqrt{1+c^2}\} = 0$$

$$\Rightarrow (y-cx)^2 - (1+c^2) = 0$$

$$\therefore (y-cx)^2 = 1+c^2 \quad \text{(Ans:)}$$

Online Class 03

Monday

Page-133 এর Ex-11:

Reduce the equation $y^2(y-xp)=x^4p^2$, where $p=\frac{dy}{dx}$, to Clairaut's form by the substitution $x=\frac{1}{X}$, $y=\frac{1}{Y}$ and hence solve the equation

Solution:

Given that,

$$y^2(y-xp)=x^4p^2 \quad \cdots \cdots (1)$$

When, $x=\frac{1}{X}$ and $y=\frac{1}{Y}$

$\Rightarrow 1=-\frac{1}{X^2}\frac{dX}{dx}$ $\qquad \Rightarrow dy=-\frac{1}{Y^2}dY$

$\Rightarrow dx=-\frac{1}{X^2}dX$

$$\therefore p=\frac{dy}{dx}=\frac{X^2}{Y^2}\cdot\frac{dY}{dX}$$

Putting this value of p in given equation, we get,

$$(1)\Rightarrow \frac{1}{Y^2}\left(\frac{1}{Y}-\frac{1}{X}\cdot\frac{X^2}{Y^2}\frac{dY}{dX}\right)=\frac{1}{X^4}\cdot\frac{X^4}{Y^4}\left(\frac{dY}{dX}\right)^2$$

$$\Rightarrow \frac{1}{Y}-\frac{X}{Y^2}\frac{dY}{dX}=\frac{1}{Y^2}\left(\frac{dY}{dX}\right)^2$$

$$\Rightarrow \frac{Y-X\frac{dY}{dX}}{Y^2}=\frac{1}{Y^2}\left(\frac{dY}{dX}\right)^2$$

$$\Rightarrow Y-X\frac{dY}{dX}=\left(\frac{dY}{dX}\right)^2$$

$\Rightarrow Y = XP + P^2$; $\left[P = \frac{dY}{dX}\right]$

whic is Clairaut's form.

Putting $P = c$, so it's solution is,

$Y = Xc + c^2$

$= \frac{1}{y} = \frac{c}{x} + c^2$ (Ans:)

Online Class 04

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Method of Variation of Parameters

Tuesday

(Page-85(2))

Ex.1(a): Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+n^2y=\sec nx$.

Solution:

Given that,

$$\frac{d^2y}{dx^2}+n^2y=\sec nx \quad \cdots\cdots (1)$$

Let $y=e^{mx}$ be the trial solution of,

$$\frac{d^2y}{dx^2}+n^2y=0 \quad \cdots\cdots (2)$$

$\therefore \frac{dy}{dx}=me^{mx}$, $\frac{d^2y}{dx^2}=m^2e^{mx}$

$\therefore (2) \Rightarrow m^2e^{mx}+n^2e^{mx}=0$

$\Rightarrow (m^2+n^2)e^{mx}=0$

A.E. is, $m^2+n^2=0$, $e^{mx}\neq 0$

$\Rightarrow m^2=-n^2$

$\therefore m=\pm in$

The complementary function (C.F.) is,

$$y=c_1\cos nx+c_2\sin nx \quad \cdots\cdots (3)$$

where c_1 and c_2 are constants.

Let $y=A\cos nx+B\sin nx \quad \cdots\cdots (4)$

be the particular solution of (1) where A and B are functions of x.

Differentiating (4) with respect to x we get,

$$\frac{dy}{dx} = -An\sin nx + A_1\cos nx + Bn\cos nx + B_1\sin nx$$

$$= -An\sin nx + Bn\cos nx + (A_1\cos nx + B_1\sin nx)$$

Choose A and B such that,

$$A_1\cos nx + B_1\sin nx = 0 \quad \ldots\ldots (5)$$

$$\therefore \frac{dy}{dx} = -An\sin nx + Bn\cos nx$$

D.W.R. to x we get,

$$\frac{d^2y}{dx^2} = -An^2\cos nx - A_1 n\sin nx - Bn^2\sin nx + B_1 n\cos nx$$

$$= -n^2(A\cos nx + B\sin nx) - A_1 n\sin nx + B_1 n\cos nx$$

$$= -n^2 y - A_1 n\sin nx + B_1 n\cos nx$$

Putting the value of $\frac{d^2y}{dx^2}$ in (1) we get,

$$(1) \Rightarrow \frac{d^2y}{dx^2} + n^2 y = \sec nx$$

$$\Rightarrow -n^2 y - A_1 n\sin nx + B_1 n\cos nx + n^2 y = \sec nx$$

$$\Rightarrow -A_1 n\sin nx + B_1 n\cos nx = \sec nx \quad \ldots\ldots (6)$$

(5) $\times n\sin nx$ + (6) $\times \cos nx$, we get,

$A_1 n\sin nx\cos nx + B_1 n\sin^2 nx - A_1 n\sin nx\cos nx + B_1 n\cos^2 nx = \sec nx.\cos nx$

$\Rightarrow B_1 n\sin^2 nx + B_1 n\cos^2 nx = \sec nx.\frac{1}{\sec nx}$

$\Rightarrow B_1 n(\sin^2 nx + \cos^2 nx) = 1$

$\Rightarrow B_1 n = 1$

$\therefore B_1 = \frac{1}{n}$

$\Rightarrow \frac{dB}{dx} = \frac{1}{n}; \left[B_1 = \frac{dB}{dx}\right]$

$\Rightarrow dB = \frac{dx}{n}$

$\therefore B = \frac{x}{n} + c_2$

From (5),

$A_1\cos nx + B_1\sin nx = 0$

$\Rightarrow A_1\cos nx + \frac{1}{n}\sin nx = 0$

$\Rightarrow \frac{dA}{dx} = -\frac{1}{n}\tan nx$

$\Rightarrow dA = -\frac{1}{n}\tan nx\, dx$

$\therefore A = \frac{1}{n^2}\log(\cos nx) + c_1$

∴ The complete solution is,

$$y = A\cos nx + B\sin nx$$

$$= \left\{\frac{1}{n^2}\log(\cos nx) + c_1\right\}\cos nx + \left(\frac{x}{n} + c_2\right)\sin nx$$

$$= \frac{1}{n^2}\log(\cos nx).\cos nx + \frac{x}{n}\sin nx + c_1\cos nx + c_2\sin nx$$

$$= c_1\cos nx + c_2\sin nx + \frac{1}{n^2}\log(\cos nx) + \frac{x}{n}\sin nx$$

(Ans:)

Ex For Practise:

→ page-87 এর Ex. 1(b), 2

→ Page-88 এর Ex. 3(a)

→ Page-89 এর Ex. 4

→ Page-93 এর Ex. 10

Online Class 05

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Part-II এর Chapter 2

27/07/2020

Monday

"Linear Equations of Second Degree"

Page: 47

For $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$

(i) $y=e^x$ is a particular integral if, $1+P+Q=0$

(ii) $y=e^{-x}$ " " " if, $1-P+Q=0$

(iii) $y=e^{mx}$ " " " if, $m^2+mP+Q=0$

(iv) $y=x$ " " " if, $P+Qx=0$

(v) $y=x^2$ " " " if, $2+2Px+Qx^2=0$

(vi) $y=x^m$ " " " if, $m(m-1)+Pmx+Qx^2=0$.

Page-48 এর Ex.1:

Solve $\frac{d^2y}{dx^2}-\frac{3}{x}\frac{dy}{dx}+\frac{3}{x^2}y=2x-1$

Given that,

$$\frac{d^2y}{dx^2}-\frac{3}{x}\frac{dy}{dx}+\frac{3}{x^2}y=2x-1 \quad \text{-----} (1)$$

Here, $P=-\frac{3}{x}$, $Q=\frac{3}{x^2}$

Since $P+Qx=0$, therefore $y=x$ is a part of C.F.

Let $y=vx$,

$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore \frac{d^2y}{dx^2}=\frac{dv}{dx}+\frac{dv}{dx}+x\frac{d^2v}{dx^2}$

$=x\frac{d^2v}{dx^2}+2\frac{dv}{dx}$

The equation (1) becomes,

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}-\frac{3}{x}\left(x\frac{dv}{dx}+v\right)+\frac{3}{x^2}vx=2x-1$$

$$\Rightarrow x\frac{d^2v}{dx^2}+2\frac{dv}{dx}-3\frac{dv}{dx}-\frac{3v}{x}+\frac{3v}{x}=2x-1$$

$$\Rightarrow x\frac{d^2v}{dx^2}-\frac{dv}{dx}=2x-1$$

$$\Rightarrow \frac{d^2v}{dx^2}-\frac{1}{x}\frac{dv}{dx}=2-\frac{1}{x}$$

Now putting $\frac{dv}{dx}=P$, $\frac{d^2v}{dx^2}=\frac{dP}{dx}$ the equation becomes

$$\frac{dP}{dx}-\frac{1}{x}P=2-\frac{1}{x}$$

which is a linear equation.

$$\therefore \text{I.F.}=e^{\int\left(-\frac{1}{x}\right)dx}$$

$$=e^{-\ln x}$$

$$=\frac{1}{x}$$

$\therefore$ Solution of the equation is,

$$P.\frac{1}{x}=\int\left(2-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx+K_1$$

$$\Rightarrow P.\frac{1}{x}=2\ln x+\frac{1}{x}+K_1$$

$$\Rightarrow P=2x\ln x+1+K_1x$$

$$\Rightarrow\frac{dv}{dx}=2x\ln x+1+K_1x$$

$\Rightarrow dv = (2x\ln x + 1 + K_1 x)dx$

$\Rightarrow v = \int (2x\ln x + 1 + K_1 x)\,dx$

$\Rightarrow v = 2\left[\ln x \int x\,dx - \int \frac{d}{dx}(\ln x)\int x\,dx\right] + x + \frac{K_1}{2}x^2 + C_2$

$\Rightarrow v = 2\ln x \frac{x^2}{2} - 2\int \frac{1}{x}\cdot\frac{x^2}{2}\,dx + x + K_1\frac{x^2}{2} + C_2$

$\Rightarrow v = x^2\ln x - \frac{x^2}{2} + x + K_1\frac{x^2}{2} + C_2$

$\Rightarrow v = x^2\ln x + c_1 x^2 + x + C_2.$

$\therefore$ The complete solution is,

$y = vx$

$= x^3\ln x + x^2 + c_1x^3 + c_2 x$ (Ans.)

*For Practice:

$\rightarrow$ Page-49 এর Ex: 2, 3

$\rightarrow$ Page-50 এর Ex: 4(a)

$\rightarrow$ Page-51 এর Ex: 4(b), 5.

Page-53 Ex-8:

Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)y = e^x \sin x$

Given that,

$$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)y = e^x \sin x \quad \ldots\ldots (1)$$

Here,

$P = -\cot x$, $Q = -(1-\cot x)$

$\therefore 1+P+Q = 0$

So, $y = e^x$ is a part of C.F.

Putting $y = ve^x$

$$\frac{dy}{dx} = ve^x + e^x \frac{dv}{dx} = e^x\left(v + \frac{dv}{dx}\right)$$

$$\frac{d^2y}{dx^2} = ve^x + e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2} + e^x \frac{dv}{dx}$$

$$= e^x\left[\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right]$$

$\therefore$ The equation (1) becomes,

$$e^x\left[\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right] - \cot x\, e^x\left(v + \frac{dv}{dx}\right) - (1-\cot x)ve^x = e^x \sin x$$

$$\Rightarrow \frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v - v\cot x - \cot x \frac{dv}{dx} - v + v\cot x = \sin x$$

$$\Rightarrow \frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$$

Let $p = \frac{dv}{dx}$

$\therefore \frac{d^2v}{dx^2} = \frac{dp}{dx}$

$\therefore \frac{dp}{dx} + (2-\cot x)p = \sin x,$

which is a linear equation.

$\therefore I.F. = e^{\int(2-\cot x)dx}$

$= e^{2x-\log\sin x}$

$= e^{2x} \cdot e^{\log\frac{1}{\sin x}}$

$= \frac{e^{2x}}{\sin x}$

$\therefore$ Solution is,

$p \cdot \frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x}\, dx + c_1$

$\Rightarrow p \cdot \frac{e^{2x}}{\sin x} = \frac{1}{2}e^{2x} + c_1$

$\Rightarrow p = \frac{1}{2}\sin x + c_1 e^{-2x} \cdot \sin x$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$

$\Rightarrow dv = \left(\frac{1}{2}\sin x + c_1 e^{-2x}\sin x\right)dx$

Integrating we get,

$$\therefore v = -\frac{1}{2}\cos x + c_1 e^{-2x}\left(\frac{\cos x + 2\sin x}{2^2+1^2}\right) + c_2$$

$$= -\frac{1}{2}\cos x + \frac{c_1}{5}e^{-2x}(\cos x + 2\sin x) + c_2$$

$\therefore$ The complete solution is,

$$y = ve^x$$

$$= -\frac{1}{2}e^x\cos x + \frac{c_1}{5}e^{-x}(\cos x + 2\sin x) + c_2 e^x \quad \text{(Ans:)}$$

*For practise

→ page-52 এর Ex. 07

→ Page-53 এর Ex. 09

→ Page-54 এর Ex. 10

→ Page-55 এর Ex. 11

Online Class 06

720

04/08/2020

Tuesday

Reduction of Order

Q1 $y_1 = e^x$ is a solution of D.E. $y'' - 2y' + y = 0$. Find the another linear independent solution

Solution:

Given that,

$$y'' - 2y' + y = 0 \quad \dots \dots \dots (1)$$

Putting, $y = vy_1 = ve^x$ in eq^n (1) we get,

$$(ve^x)'' - 2(ve^x)' + ve^x = 0$$

$$\Rightarrow \{(ve^x)'\}' - 2(ve^x)' + ve^x = 0$$

$$\Rightarrow \{ve^x + v'e^x\}' - 2(ve^x + v'e^x) + ve^x = 0$$

$$\Rightarrow ve^x + v'e^x + v'e^x + v''e^x - 2ve^x - 2v'e^x + ve^x = 0$$

$$\Rightarrow v''e^x = 0$$

$$\Rightarrow v'' = 0; \ [e^x \neq 0]$$

$$\Rightarrow v' = c_1; [\text{Integrating}]$$

$$\Rightarrow v = c_1 x + c_2, [\text{Integrating}]$$

Let $c_1 = 1$ and $c_2 = 0$

Then $v = x$

So, $y = ve^x = xe^x$ is an another linear independent solution

Ex. Show that $y=x^2+1$ is a solution of the equation $(x^2-1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$. Hence solve the equation by reducing its order.

Solution:

Given that,

$$(x^2-1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0 \cdots \cdots \cdots (1)$$

Putting $y=x^2+1$ in L.H.S. of (1) we get,

$$(x^2-1)\frac{d^2}{dx^2}(x^2+1)-2x\frac{d}{dx}(x^2+1)+2(x^2+1)=0$$

$$\Rightarrow (x^2-1)\frac{d}{dx}(2x)-2x.2x+2x^2+2=0$$

$$\Rightarrow (x^2-1).2-4x^2+2x^2+2=0$$

$$\Rightarrow 2x^2-2-4x^2+2x^2+2=0$$

$$\Rightarrow 0$$

$$\Rightarrow R.H.S.$$

Hence $y=x^2+1$ is solution of equation (1)

let $y=v(x^2+1)$ is an another linear independent solution

$$\therefore \frac{dy}{dx}=(x^2+1)\frac{dv}{dx}+2vx$$

$$\frac{d^2y}{dx^2}=(x^2+1)\frac{d^2v}{dx^2}+2x\frac{dv}{dx}+2x\frac{dv}{dx}+2v$$

$$=(x^2+1)\frac{d^2v}{dx^2}+4x\frac{dv}{dx}+2v$$

Putting the value of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y in (1) we get,

$$(x^2-1)\left\{(x^2+1)\frac{d^2v}{dx^2}+4x\frac{dv}{dx}+2v\right\}-2x\left\{(x^2+1)\frac{dv}{dx}+2vx\right\}+2v(x^2+1)=0$$

$$\Rightarrow (x^4-1)\frac{d^2v}{dx^2}+4x(x^2-1)\frac{dv}{dx}+2v(x^2-1)-2x^3\frac{dv}{dx}-2x\frac{dv}{dx}-4vx^2+2vx^2+2v=0$$

$$\Rightarrow (x^4-1)\frac{d^2v}{dx^2}+(4x^3-4x)\frac{dv}{dx}+2vx^2-2v-2x^3\frac{dv}{dx}-2x\frac{dv}{dx}-2vx^2+2v=0$$

$$\Rightarrow (x^4-1)\frac{d^2v}{dx^2}+(4x^3-4x-2x^3-2x)\frac{dv}{dx}=0.$$

$$\Rightarrow (x^4-1)\frac{d^2v}{dx^2}+(2x^3-6x)\frac{dv}{dx}=0$$

Let, $p=\frac{dv}{dx}$

$\Rightarrow \frac{dp}{dx}=\frac{d^2v}{dx^2}$

$$\Rightarrow (x^4-1)\frac{dp}{dx}+(2x^3-6x)p=0$$

$$\Rightarrow \frac{dp}{p}+\frac{(x^2-3)2x}{x^4-1}dx=0$$

$$\Rightarrow \frac{dp}{p}+\frac{(x^2-3)2x\,dx}{(x^2)^2-1}=0$$

Let, $x^2=z$

$\Rightarrow 2x\,dx=dz$

$$\Rightarrow \frac{dp}{p}+\frac{(z-3)}{z^2-1}dz=0$$

$$\Rightarrow \frac{dp}{p}+\frac{z-3}{(z-1)(z+1)}dz=0$$

$$\Rightarrow \frac{dP}{P} + \left\{ \frac{-1-3}{(z+1)(-1-1)} + \frac{1-3}{(1+1)(z-1)} \right\} dz = 0$$

$$\Rightarrow \frac{dP}{P} + \left\{ \frac{2}{z+1} - \frac{1}{z-1} \right\} dz = 0$$

$$\Rightarrow \ln P + 2\ln(z+1) - \ln(z-1) = \ln c \; ; \text{[Integrating]}$$

$$\Rightarrow \ln P + \ln (z+1)^2 = \ln c + \ln(z-1)$$

$$\Rightarrow \ln P(z+1)^2 = \ln c(z-1)$$

$$\Rightarrow P(z+1)^2 = c(z-1)$$

$$\Rightarrow (x^2+1)^2 \frac{dv}{dx} = c(x^2-1)$$

$$\Rightarrow dv = \frac{x^2-1}{(x^2+1)^2} c\, dx$$

$$\Rightarrow dv = c \cdot \frac{x^2-1}{(x^2+1)^2} dx$$

$$\Rightarrow dv = c \cdot \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x+\frac{1}{x}\right)^2} dx$$

$$\Rightarrow \int dv = c \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2}$$

$$\Rightarrow v = -c \frac{1}{x+\frac{1}{x}} + c_1$$

$$\Rightarrow v = -c\,\frac{x}{x^2+1} + c_1$$

Let $c = 1$, $c_1 = 0$. Then,

$$v = -\frac{x}{x^2+1}$$

$$\therefore y = v(x^2+1) = -\frac{x}{x^2+1}(x^2+1) = -x$$

$\therefore y = a(x^2+1) - bx$ is another linear independent solution where a and b are constants.

Online Class 07

10/08/2020
Monday

Method of Undetermined Coefficients

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{x} \rightarrow \text{U.C. function}$$

(1) U.C. ফাংশন $= c$ (ধ্রুবক) হলে UC সেট $= \{1\}$

(2) ,, $= x$,, ,, ,, $= \{x, 1\}$

(3) ,, $= x^2$,, ,, ,, $= \{x^2, x, 1\}$

(4) ,, $= e^x$,, ,, ,, $= \{e^x\}$

(5) ,, $= \sin ax$,, ,, ,, $= \{\sin ax, \cos ax\}$

Ex. Solve $(D^2-1)y = e^{2x}$ by the method of undetermined cofficient

Solution:

Given that,

$$(D^2-1)y = e^{2x} \quad \dots \dots \dots (1)$$

Let $y = e^{mx}$ be the trial solution of $(D^2-1)y = 0$

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

$\therefore$ A.E is $m^2 - 1 = 0$; $(e^{mx} \neq 0)$

$\therefore m = \pm 1$

$\therefore$ C.F. is $y_c = c_1 e^{x} + c_2 e^{-x}$

Here, U.C. function $= e^{2x}$

$\therefore$ U.C. set $= \{e^{2x}\}$

Let P.I. by $y_p = Ae^{2x}$

Replacing y by y_p

$\therefore Dy = 2Ae^{2x}$

$D^2y = 4Ae^{2x}$

$\therefore$ The equation becomes,

$4Ae^{2x} - Ae^{2x} = e^{2x}$; $[e^{2x} \neq 0]$

$\Rightarrow 4A - A = 1$

$\Rightarrow 3A = 1$

$\therefore A = \frac{1}{3}$

$\therefore y = \frac{1}{3} e^{2x}$

$\therefore$ The general solution is, $y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} e^{2x}$

(Ans:)

Q) Solve $(D^2+16)y = 48\cos 4x$ by the method of undetermine coefficients

Solution:

Given that,

$$(D^2+16)y = 48\cos 4x \quad \dots \dots \dots (1)$$

Let $y = e^{mx}$ be the trial solution of $(D^2+16)y=0$

$$\therefore Dy = me^{mx}, \quad D^2y = m^2 e^{mx}$$

$\therefore$ A.E. is $m^2+16=0$; $e^{mx} \neq 0$

$$\Rightarrow m = \pm 4i$$

$\therefore$ C.F., $y_c = c_1\cos 4x + c_2 \sin 4x$

U.C. function $= 48\cos 4x$

$\therefore$ U.C. set $= \{\cos 4x, \sin 4x\}$

Here C.F. and U.C. function has a common element.

$\therefore$ U.C. set $= \{x\cos 4x, x\sin 4x\}$

Let P.I. be $y_p = Ax\cos 4x + Bx\sin 4x$.

Replace y by y_p,

$\therefore y = Ax\cos 4x + Bx\sin 4x$

$\therefore Dy = A\cos 4x - 4Ax\sin 4x + B\sin 4x + 4Bx\cos 4x$

$\therefore D^2y = -4A\sin 4x - 4A\sin 4x - 16Ax\cos 4x + 4B\cos 4x + 4B\cos 4x - 16Bx\sin 4x$

$= -8A\sin 4x - 16Ax\cos 4x + 8B\cos 4x - 16Bx\sin 4x$

Equation (1) becomes,

$-8A\sin 4x - 16Ax\cos 4x + 8B\cos 4x - 16Bx\sin 4x + 16Ax\cos 4x + 16Bx\sin 4x$

$= 48\cos 4x$

$\Rightarrow -8A\sin 4x + 8B\cos 4x = 48\cos 4x$

$\Rightarrow -8A = 0$ and $8B = 48$

$\therefore A = 0$ $\Rightarrow B = 6$

$\therefore y_p = 0 + 6x\sin 4x$

$= 6x\sin 4x$

$\therefore$ The complete solution is $= y_c + y_p$

$= c_1\cos 4x + c_2\sin 4x + 6x\sin 4x$

(Ans)

Online Class 08

17/08/2020
Monday

Example-08: Solve by the method of undetermined coefficients

$$(D^2-2D+1)y = x\sin x$$

Solution:

Given that,

$$(D^2-2D+1)y = x\sin x \quad \dots \dots \dots (1)$$

Let $y=e^{mx}$ be the trial solution of $(D^2-2D+1)y=0$

$\therefore Dy = me^{mx}$, $D^2y = m^2e^{mx}$

$\therefore$ A.E. is $m^2-2m+1=0$; $e^{mx} \neq 0$

$\Rightarrow (m^2-1)=0$

$\therefore m=1,-1$

$\therefore$ C.F. is $y_c = (c_1+c_2x)e^x$

U.C. Function $= x\sin x$

$\therefore$ U.C. set $= \{x\sin x,\ x\cos x,\ \sin x,\ \cos x\}$

$\therefore$ P.I. be y_p , Replacing y_p by y

$\therefore y = (A+Bx)\sin x + (C+Ex)\cos x$

$\therefore Dy = (A+Bx)\cos x + B\sin x - (C+Ex)\sin x + E\cos x$

$= (A+E+Bx)\cos x + (B-C-Ex)\sin x$

$D^2y = B\cos x - (A+E+Bx)\sin x - E\sin x + (B-C-Ex)\cos x$

$= (B+B-C-Ex)\cos x - (A+E+Bx+E)\sin x.$

Then equation (1) becomes,

$$\left[(B+B-C-Ex)\cos x - (A+E+Bx+E)\sin x - 2(A+E+Bx)\cos x - 2(B-C-Ex)\sin x + (A+Bx)\sin x + (C+Ex)\cos x\right] = x\sin x$$

$$\Rightarrow (2B-C-Ex-2A-2E-2Bx+C+Ex)\cos x + (-A-2E-Bx-2B+2C+2Ex+A+Bx)\sin x = x\sin x.$$

$$\Rightarrow (2B-C-2A-2E+C)\cos x + (-E-2B+E)\cos x + (-A-2E-2B+2C+A)\sin x + (-B+2E+B)x\sin x = x\sin x$$

Equating the corresponding components,

$2B-2A-2E=0$, $-2B=0 \therefore B=0$

$-2E-2B+2C=0$, $2E=1 \therefore E=\frac{1}{2}$

$2A = -2E+2B = -1+0=-1 \therefore A = -\frac{1}{2}$

$2C = 2E+2B = -\frac{1}{2}$

$$\therefore y_p = \frac{1}{2}\sin x + \left(-\frac{1}{2} + \frac{1}{2}x\right)\cos x$$

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^x + \frac{1}{2}\sin x + \left(-\frac{1}{2} + \frac{1}{2}x\right)\cos x$$

(Ans:)

Online Class 09

132

18/08/2020

Tuesday

[4] The population of a community is known to increase at a rate proportional to the number of people present at time t. If the population has doubled in 50 years. How long will it take to triple?

Solution:

Let, x be a population of a community at a time t.

$$\frac{dx}{dt} \propto x$$

$$\Rightarrow \frac{dx}{dt} = Kx \quad (K \text{ is a constant})$$

$$\Rightarrow \frac{dx}{x} = K dt$$

When $t=0$, then $x = x_0$

$t=50$, then $x = 2x_0$

$$\therefore \int_{x=x_0}^{2x_0} \frac{dx}{x} = K \int_{t=0}^{50} dt$$

$$\Rightarrow \left[\ln x\right]_{x_0}^{2x_0} = K\left[t\right]_0^{50}$$

$$\Rightarrow \ln 2x_0 - \ln x_0 = K[50-0]$$

$$\Rightarrow \ln \frac{2x_0}{x_0} = 50K$$

$$\Rightarrow 50K = \ln 2$$

$$\therefore K = \frac{\ln 2}{50}$$

Again when $t=0$, then $x=x_0$

and when $t=t_1$, then $x=3x_0$

$$\int_{x_0}^{3x_0} \frac{dx}{x} = x\int_0^t dt$$

$$\Rightarrow [\ln x]_{x_0}^{3x_0} = K[t]_0^t$$

$$\Rightarrow \ln 3x_0 - \ln x_0 = K[t-0]$$

$$\Rightarrow Kt = \ln\frac{3x_0}{x_0}$$

$$\Rightarrow t = \ln 3/K$$

$$\Rightarrow t = \frac{\ln 3}{\frac{\ln 2}{50}}$$

$$\Rightarrow t = \frac{50\ln 3}{\ln 2}$$

$\therefore t = 79$ years.

(Ans:)

Online Class 10

734

24/08/2020

Monday

Example: 05 (Page-200 of Ordinary Differential Equations)

In a certain bacteria culture, the rate of increase in the number of bacteria is proportional to the numbers present. If the number double in 4hr, how many will be present at 12 hr? If the number 400 in 3hr and 2000 in 10hr, then find the number initially present?

Let x be the number of bacteria after a time t. Then,

$$\frac{dx}{dt} \propto x$$

$$\therefore \frac{dx}{dt} = Kx \quad (K \text{ is constant})$$

$$\Rightarrow \frac{dx}{x} = K dt$$

Initially, $t=0$ then $x=n$

and $t=4$ then $x=2n$

$$\therefore \int_{n}^{2n} \frac{dx}{x} = K \int_{t=0}^{4} dt$$

$$\Rightarrow [\ln x]_{n}^{2n} = K[t]_{0}^{4}$$

$$\Rightarrow \ln 2n - \ln n = 4K$$

$$\Rightarrow \ln 2 = 4K$$

Again $t=0$ then $x=n$

and $t=12$ then $x=x$

$$\therefore \int_{x=n}^{x} \frac{dx}{x} = K\int_{t=0}^{12} dt$$

$$\Rightarrow \left[\ln x\right]_{n}^{x} = K\left[t\right]_{0}^{12}$$

$$\Rightarrow \ln x - \ln n = 12K$$

$$\Rightarrow \ln x = \ln n + 3.4.K$$

$$\Rightarrow \ln x = \ln n + 3.\ln 2$$

$$\Rightarrow \ln x = \ln n + \ln 2^3$$

$$\Rightarrow \ln x = \ln n + \ln 8$$

$$\Rightarrow \ln x = \ln 8n$$

$$\therefore x = 8n$$

$\therefore$ After 12 hours population will be $8n$.

Again, when $t=3$ then $x=400$

and when $t=10$ then $x=2000$

$$\therefore \int_{400}^{2000} \frac{dx}{x} = K\int_{3}^{10} dt$$

$$\Rightarrow \left[\ln x\right]_{400}^{2000} = K\left[t\right]_{3}^{10}$$

$$\Rightarrow \ln 2000 - \ln 400 = K\,[10-3]$$

$$\Rightarrow \ln \frac{2000}{400} = 7K$$

$$\Rightarrow \ln 5 = 7K$$

$$\therefore K = \ln 5^{\frac{1}{7}}$$

Initially $t=0$ then $x=n$

when $t=3$, then $x=400$

$$\therefore \int_{n}^{400} \frac{dx}{x} = K\int_{0}^{3} dt$$

$$\Rightarrow \left[\ln x\right]_{n}^{400} = K\left[t\right]_{0}^{3}$$

$$\Rightarrow \ln 400 - \ln n = 3K$$

$$\Rightarrow \ln n = \ln 400 - 3.\ln 5^{\frac{1}{7}}$$

$\Rightarrow \ln n = \ln \dfrac{400}{5^{\frac{3}{7}}}$

$\Rightarrow n = 200.68$

$\therefore n \simeq 200$

So, initially the number of bacteria was 200 (almost).

Online Class 12

138

অধ্যায়-২ 01/09/2020

(Existence and Uniqueness of Solution) Tuesday

Initial Value Problem & Boundary value problem

Page: 36

Example-2: Solve the initial value problem $y''=0$, $y(0)=1$, $y'(0)=1$

We have,

$$\frac{d^2y}{dx^2}=0$$

Integrating,

$$y'=\frac{dy}{dx}=c_1$$

Again, Integrating,

$$y=c_1x+c_2$$

when $x=0$, then $y(0)=c_2$

$$\Rightarrow 1=c_2$$

$$\therefore c_2=1$$

when $x=0$, then, $y'(0)=c_1$

$$\therefore c_1=1$$

$\therefore y=x+1$. (Ans:)

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Example: 6 // Solve the initial value problem:

$$x^2y'' + 4xy' + 2y = 0, \quad y(1) = 1, \quad y'(1) = 2$$

We have,

$$x^2y'' + 4xy' + 2y = 0$$

$$\Rightarrow \frac{d}{dx}(x^2y' + 2xy) = 0$$

$$\Rightarrow \frac{d^2}{dx^2}(x^2y) = 0$$

Integrating we get,

$$\frac{d}{dx}(x^2y) = c_1$$

$$\Rightarrow x^2y = c_1x + c_2$$

when $x = 1$ then, $y(1) = c_1 + c_2$

$$\Rightarrow 1 = c_1 + c_2$$

$$\therefore c_1 + c_2 = 1$$

when $x = 1$ then $\frac{dy}{dx} = 2$

$$\frac{d}{dx}(x^2y' + 2xy) = 0$$

$$\Rightarrow x^2y' + 2xy = c_1$$

$\Rightarrow 1^2.2+2.1.1=c_1$

$\therefore c_1=4$

$\therefore c_2=1-c_1$

$=-3.$

$\therefore x^2y=4x-3.$ (Ans:).

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Example 10// Show that the function $f(x,y)=xy^2$ satisfies the Lipschitz condition over the rectangular region R.

$$R=\{(x,y): 0\leq x\leq a, |y-y_0|\leq b\}$$

We have,

$$f(x,y)=xy^2$$

let (x,y_1) and $(x,y_2)\in R$

$$\therefore |f(x,y_1)-f(x,y_2)|=|xy_1^2-xy_2^2| \quad \ldots\ldots\ldots (1)$$

$$=|x(y_1^2-y_2^2)|$$

$$=|x|\,|(y_1+y_2)(y_1-y_2)|$$

$$\therefore |f(x,y_1)-f(x,y_2)|\leq a\,|y_1+y_2|\,|y_1-y_2| \quad \ldots\ldots\ldots (2)$$

Now,

$$|y_1+y_2| = |(y_1-y_0)+(y_2-y_0)+2y_0|$$

$$\Rightarrow |y_1+y_2| \leq |y_1-y_0| + |y_2-y_0| + 2|y_0|$$

$$\Rightarrow |y_1+y_2| \leq b+b+2|y_0|$$

$$\therefore |y_1+y_2| \leq 2b+2|y_0| \quad \ldots\ldots\ldots (3)$$

From equation (1), (2) and (3),

$$|f(x,y_1)-f(x,y_2)| \leq a\{2b+2|y_0|\}|y_1-y_2|$$

$$\Rightarrow |f(x,y_1)-f(x,y_2)| \leq K|y_1-y_2| \; ; \left[\text{where } K = 2a\{b+|y_0|\}\right]$$

So, the given function satisfies the lipchitzs condition.

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Example 20: State existence and uniqueness theorem and examine it for $\frac{dx}{dt} = \frac{\sqrt{x}}{t}$, $x(1)=0$

let, $R = \{(t, x) : (t - t_0) \in a, |x - x_0| \leq b\}$ be a rectangular area, where $f(t, x)$ and $\frac{\delta f}{\delta x}$ are continuous

$\therefore \frac{dx}{dt} = \frac{\sqrt{x}}{t}$, $x(1) = 0$

$\therefore f(t, x) = \frac{\sqrt{x}}{t}$, $x_0 = 0$, $t_0 = 1$

$\therefore \frac{\delta f}{\delta t} = \frac{1}{2t\sqrt{x}}$, let $a = 1$, $b = 1$

$\therefore R = \{(t, x) : |t - t_0| \leq 1, |x - 0| \leq 1\}$

In the rectangle R, $\frac{\delta f}{\delta x}$ is discontinuous, so, it has no unique solution.

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Example 22 // Show that there doesn't exist unique solution of the initial value problem $y' = y^{1/3}$, $y(0) = 0$.

Given that,

$$y' = y^{1/3}$$

$$\Rightarrow f(x, y) = y^{1/3}; \text{ where } y(0) = 0,\ y_0 = 0,\ x_0 = 0.$$

Suppose, $a = 1$, $b = 1$

$$\therefore R = \{(x, y) : |x - x_0| \leq a,\ |y - y_0| \leq b\}$$

$$= \{(x, y) : |x - 0| \leq 1,\ |y - 0| \leq 1\}$$

$$= \{(x, y) : |x| \leq 1, |y| \leq 1\}$$

$\therefore f(x, y)$ is continuous in the region R.

Let (x, y_1) and $(x, y_2) \in \mathbb{R}$, then,

$$|f(x, y_1) - f(x, y_2)| = |y_1^{1/3} - y_2^{1/3}|$$

$$\Rightarrow |f(x, y_1) - f(x, y_2)| = \frac{|(y_1^{1/3} - y_2^{1/3})(y_1^{2/3} + y_1^{1/3} y_2^{1/3} + y_2^{2/3})|}{|y_1^{2/3} + y_1^{1/3} y_2^{2/3} + y_2^{2/3}|}$$

$$\Rightarrow |f(x, y_1) - f(x, y_2)| = \frac{|y_1 - y_2|}{|y_1^{2/3} + y_1^{1/3} + y_2^{1/3} + y_2^{2/3}|}$$

$$\therefore |f(x, y_1) - f(x, y_2)| \leq A |y_1 - y_2|$$

So, the given function $f(x, y) = y^{1/3}$ does not satisfies the lipchitz condition. Hence, the given initial value problem has no unique solution.