9-th batch

1) b) we have,

$$\left.\begin{aligned} x+2y-3z &= 4 \\ 3x-y+5z &= 2 \\ 4x+y+(a^2-14)z &= a+2 \end{aligned}\right\} \text{—— (1)}$$

Let us represents 3 linear equations of system (1) by L_1, L_2, L_3 and L_3. Reduce it to echelon form.

$\underline{x+y+z=1}$

$$x+2y-3z=4$$
$$3x-y+5z=2$$
$$4x+y+(a^2-14)z=a+2$$

$$\left.\begin{aligned} & x+2y-3z=4 \\ L_2' \to 3L_1-L_2 \quad & 0+7y-14z=10 \\ L_3' \to 4L_1-L_3 \quad & 0+7y-(12+a^2-14)z = 16-a-2 \end{aligned}\right\} \to (2)$$

$$x+2y-3z=4$$
$$7y-14z=10$$

$L_3' \to L_2-L_3$ $\quad \cancel{0+(a^2-2-14)z} = (10-14+a)$

$\cancel{-(14+a^2-2)} = (a-4)$

$(a^2-16) = a-4$

$$x+2y-3z=4$$
$$7y-14z=10$$
$$(a+4)(a-4)z = a-4$$
$$\to (a+4)z=0$$

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So, the linear equations have a ~~ki~~ unique solution, If coefficient of z is non-zero

So that, $\boxed{\begin{aligned}(a+4) &\neq 0 \\ \therefore a &\neq -4\end{aligned}}$ or $\begin{aligned}a^2-16 &\neq 0 \\ a &\neq \pm 4\end{aligned}$

~~So, if a is not equal -4 then the for an~~

then the linear equations will be unique for any real value of a without (± 4)

(ii) The linear equation have more ~~that~~ than one value if the third equation is vanished, it is possible if, $a=4$

$$(a^2-16)z = a-4$$

$$0=0$$

(iii) The linear equation ~~has~~ will be inconsistent

If $a=-4$,

So that it will

$$0=-8$$

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1) c) Vector space and subspace definition.

Vector space: Let V be a non empty set over the scalar field F. V is said to be a vector space, if operation of addition is $u,v \in V \Rightarrow u+v \in V$ and operation of scalar multiplication $u \in V, \alpha \in F = \alpha u \in V$ defined on V and satisfied the following conditions.

(i) Addition is commutative:

$$u+v = v+u \;;\; u,v \in V$$

(ii) Addition is associative:

$$u+(v+w) = (u+v)+w \;;\; u,v,w \in V$$

(iii) Existence of zero vector:

$$u+0 = 0+u = u \;;\; u \in V$$

(iv) Existence of negative:-

$$u \in V, \; \exists -u \in V$$

Such that,

$$u+(-u) = (-u)+u = 0$$

(v) For any scalar $\alpha \in F$ and $u,v \in V$

then $\alpha(u+v) = \alpha u + \alpha v$

(vi) For any scalars $\alpha, \beta \in F$ and $u \in V$

Then $(\alpha+\beta)u = \alpha u + \beta u$

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7) For any scalars $\alpha, \beta \in F$ and $u \in V$

Then $(\alpha\beta)u = \alpha(\beta u)$

8) Unit scalar $1 \in F$, $1u = u$; $u \in V$

Subspace:- Let V be a vector space over the scalar field F and W be a non empty subset of V. W is called subspace of V, if it itself a vector space with respect to operation of addition and operation of scalar multiplication defined on V.

⇒ Not found :-

Ans no → 5

5) a) Basis:- Let V be a vector space over the scalar field F and $v_1, v_2, \dots v_n \in V$ then $\{v_1, v_2, \dots v_n\}$ called basis of V if and only if

(i) $\{v_1, v_2, \dots v_n\}$ is linearly independent

(ii) $\{v_1, v_2, \dots v_n\}$ spans of V

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<u>Dimension</u>:- The maximum number of linear independent set of vectors contained in it is called dimension of a vector space.

<u>Theorem 24, page 279</u>:-

Let $\dim S = s$
$\dim T = t$
$\dim (S \cap T) = r$

Let $\{u_1, u_2, \ldots u_r\}$ be a basis of $S \cap T$

Since, $S \cap T$ is a subset of S, also $S \cap T$ is a subset of T.

So, we extend the basis of $S \cap T$ to a basis of S and basis of T

So, basis of S is $\{u_1, u_2, \ldots u_r, v_1, v_2, \ldots v_{s-r}\}$

and basis of T is $\{u_1, u_2, \ldots u_r, w_1, w_2, \ldots w_{t-r}\}$

Let, $A = \{u_1, u_2, \ldots u_r, v_1, v_2, \ldots v_{s-r}, w_1, w_2, \ldots w_{t-r}\}$

The set A has exactly $r + s - r + t - r$
$= s + t - r$ elements

The theorem will complete if we can show that the set A is a basis of $S+T$.

Since $\{u_1, u_2, \ldots u_r, v_1, v_2, \ldots v_{s-r}\}$ is a basis of S

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So, $\{u_1, u_2, \dots u_r, v_1, v_2, \dots v_{s-r}\}$ generates of

S.

Similarly $\{u_1, u_2 \dots u_r, w_1, w_2, \dots w_{t-r}\}$ generates T

So, A is the union of S and T generates S+T

Now we have only to show that A is linear independent.

Suppose,

$\cancel{\alpha_1 u_1 + \alpha_2 u_2 + \dots \beta_1 v_1 + \beta_2 v_2}$

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_r u_r + \beta_1 v_1 + \beta_2 v_2 + \dots \beta_{s-r} v_{s-r} + \gamma_1 w_1 + \gamma_2 w_2 + \dots \gamma_{t-r} w_{t-r} = 0 \quad \text{----- (1)}$$

$\alpha_1, \alpha_2, \dots \alpha_r, \beta_1, \beta_2 \dots \beta_{s-r}, \gamma_1, \gamma_2, \dots \gamma_{t-r}$ be any scalars.

Let

$$V = \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_r u_r + \beta_1 v_1 + \beta_2 v_2 + \dots \beta_{s-r} v_{s-r} \longrightarrow (2)$$

For the basis of T.

$$V = \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_r u_r + \gamma_1 w_1 + \gamma_2 w_2 + \dots \gamma_{t-r} w_{t-r} \longrightarrow (3)$$

From (i) and (ii) ~~and (iii)~~ we get

~~$v+\gamma_1$~~

From (i) $\Rightarrow$ $v+\gamma_1 w_1+\gamma_2 w_2+\cdots\gamma_{t-r}w_{t-r}=0$

$\Rightarrow v=-\gamma_1 w_1-\gamma_2 w_2-\cdots\gamma_{t-r}w_{t-r} \longrightarrow (3)$

From (2) $\Rightarrow$ $v\in S$

From (3) $\Rightarrow$ $v\in T$

Since $\{u_1,u_2,\ldots u_r\}$ is a basis of $S\cap T$

There exist scalars $\delta_1,\delta_2,\ldots\delta_r$ such that

$v=\delta_1 u_1+\delta_2 u_2+\cdots\delta_r u_r \longrightarrow (4)$

From (3) and (4)

(3) $\Rightarrow$ $\delta_1 u_1+\delta_2 u_2+\cdots\delta_r u_r+\gamma_1 w_1+\gamma_2 w_2+\cdots\gamma_{t-r}w_{t-r}=0$

Since, $\{u_1,u_2,\ldots u_r,w_1,w_2,\ldots w_{t-r}\}$ is a basis of T.

So, they are linearly independent

and. $\delta_1=\delta_2=\cdots=\delta_r=\gamma_1=\gamma_2=\cdots=\gamma_{t-r}=0$

(1) $\Rightarrow$ $\alpha_1 u_1+\alpha_2 u_2+\cdots\alpha_r u_r+\beta_1 v_1+\beta_2 v_2+\cdots\beta_{s-r}v_{s-r}=0$

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Since, $\{u_1, u_2 \cdots u_r, v_1, v_2, \cdots v_{s-r}\}$ is a basis of S so they are linearly independent

$$\alpha_1 = \alpha_2 = \cdots = \alpha_r = \beta_1 = \beta_2 = \cdots = \beta_{s-r} = 0$$

Thus all scalars are zero

Hence the set A is linearly independent and so basis of S+T.

$\therefore \dim(S+T) = \dim S + \dim T$ (proved)

5(b) $S = \{(x,y,z,t) : \cancel{y=} y+z+t=0\}$

$$T = \left\{(x,y,z,t) : \begin{array}{l} x+y=0 \\ z-2t=0 \end{array}\right\}$$

For S,

$\cancel{so}$ $y+z+t=0$

we have only one equation for 4 variables has $(4-1)=3$ free variable, where x, z, t are free variables.

Let,

(i) $x=1,\ z=0,\ t=0,\ \therefore y=0$

(ii) $x=0,\ z=1,\ t=0,\ \therefore y=-1$

(iii) $x=0,\ z=0,\ t=1,\ \therefore y=-1$

$\therefore$ Basis of $S = \{(1,0,0,0), (0,-1,1,0), (0,-1,0,1)\}$

Dim of $S = 3$

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(ii) For T, $x+y=0$
$z-2t=0$

we have two equation for 4 variables, has $(4-2)=2$ free variable, where ~~x~~, y and t are free variables.

Let,
(i) $y=1$, $t=0$; $z=0$, $x=-1$

(ii) $y=0$, $t=1$ $\therefore$ $z=2$; $x=0$

$\therefore$ Basis of $T=\{(-1,1,0,0),(0,0,2,1)\}$

Dim of $T=2$

(iii) For $S\cap T$

$$\left.\begin{aligned} x+y+0z+0t &= 0 \\ y+z+t &= 0 \\ z-2t &= 0 \end{aligned}\right\}$$

~~Here we to~~ which is echelon form.

we have three equation for 4 variable has $(4-3)=1$ free variable which is t.

Let, $t=1$ then, $z=2$, $y=-3$, $x=3$

So Basis of $(S\cap T)=\{(3,-3,2,1)\}$

Dim of $(S\cap T)=1$ Ans

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4) a) Linearly dependance :- Let V be a vector space over the scalar field F and $v_1, v_2, \ldots v_n \in V$. The vectors $v_1, v_2, \ldots v_n$ are said to be linearly dependance if there exists scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in F$ not All zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$

where $\alpha_i \neq 0$ at least one i.

Linearly independent :- Let V be a vector space over the scalar field F and $v_1, v_2, \ldots v_n \in V$. The vectors $v_1, v_2, \ldots v_n$ are said to be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in F$ ~~not~~ al All zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots \alpha_n v_n = 0$

where $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

proof :- Suppose the vector v_k can be written as a linear combination of the vectors $v_1, v_2, \ldots v_{k-1}$

So, $v_k = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_{k-1} v_{k-1}$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_{k-1} v_{k-1} - v_k = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_{k-1} v_{k-1} + (-1) v_k + 0 \cdot v_{k+1} + \ldots 0 \cdot v_n = 0$

This shows that the vectors $v_1, v_2, \ldots v_n$ are linearly dependent.

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Conversely, Suppose $v_1, v_2 \cdots v_n$ are linearly dependent.

We have to say that the vectors of one of the vector v_k can be written as a linear combination of the preceding vectors.

We know,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$$

$\Rightarrow$ where the scalars are not zero

So, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{k-1} v_{k-1} + \alpha_k v_k + 0 v_{k+1} + \cdots 0 \cdot v_n = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{k-1} v_{k-1} + \alpha_k v_k = 0$

If $k=1$, then,

$$\alpha_1 v_1 = 0$$

Since vectors are linearly dependent

$v_1 = 0$, which is contradiction, so,

$k > 1$

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{k-1} v_{k-1} + \alpha_k v_k = 0$

$\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 - \cdots \alpha_{k-1} v_{k-1} = 0$

$\Rightarrow v_k = -\frac{\alpha_1}{\alpha_k} v_1 - \frac{\alpha_2}{\alpha_k} v_2 - \cdots \frac{\alpha_{k-1}}{\alpha_k} v_{k-1}$

$\Rightarrow v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_{k-1} v_{k-1}$

$\Rightarrow v_k$ written as a linear combination of the vectors $v_1, v_2, \cdots v_{k-1}$.

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4.b) Set a linear combination of the given polynomials u, v and w equal to the (zero) polynomial using the unknown scalars x, y and z: that is $xu + yv + zw = 0$

Thus $x(t^3 - 2t^2 + 5t + 1) + y(t^3 - t^2 + 8t + 2) + z(t^3 - 4t^2 + 9t + 5)$

$\Rightarrow xt^3 - 2xt^2 + 5xt + x + yt^3 - yt^2 + 8yt + 2y + zt^3 - 4zt^2 + 9zt + 5z = 0$

$\Rightarrow (x + y + z)t^3 + (-2x - y - 4z)t^2 + (5x + 8y + 9z)t + (x + 2y + 5z)1 = 0$

Setting the coefficients of the powers of t each equal to 0(zero),

we get the following homogenius linear system.

$$x + y + z = 0$$
$$-2x - y - 4z = 0$$
$$5x + 8y + 9z = 0$$
$$x + 2y + 5z = 0$$

or

$$\left.\begin{aligned} x + y + z &= 0 \\ 2x + y + 4z &= 0 \\ 5x + 8y + 9z &= 0 \\ x + 2y + 5z &= 0 \end{aligned}\right\} \longrightarrow (1)$$

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Let us represent four ~~eq~~ linear equations of system ① by L_1, L_2, L_3, L_4. And reduce it by echelon form.

$$\left.\begin{array}{ll} & x + y + z = 0 \\ L_2' \to 2L_1 - L_2 & y - 2z = 0 \\ L_3' \to 5L_1 - L_3 & -3y - 4z = 0 \\ L_4''' \to L_1 - L_4 & -y - 4z = 0 \end{array}\right\} \longrightarrow ②$$

Again, ~~x~~

$$\left.\begin{array}{ll} & x + y + z = 0 \\ & y - 2z = 0 \\ L_3' \to 3L_2 + L_3 & -10z = 0 \\ L_4' \to L_2 + L_4 & -6z = 0 \end{array}\right\} \longrightarrow ④$$

~~eqn~~ from equation L_3 and L_4 we get $z = 0$. ~~st~~ so the equation,

$$\left.\begin{array}{l} x + y + z = 0 \\ y - 2z = 0 \\ z = 0 \end{array}\right\}$$

which is echelon form, Hence $z=0$, $y=0$, $x=0$

So, $x = y = z = 0$

So the given polynomials are linearly independent.

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Linear combination:- Let V be a vector space over the scalar field F and $v_1, v_2, \dots v_n \in V$. Any vector $v \in V$ is called linear combination of $v_1, v_2, \dots v_n$
if there exist scalars $\alpha_1, \alpha_2, \dots \alpha_n \in F$
such that ~~$\alpha_1 x_1 +$~~

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n .$$

6) a) Kernel of T:- Let $T: U \to V$ be a linear transformation, Then kernel of T is a subset of U consisting of all $T(x) = 0$ where $x \in U$ and $0 \in V$

Range of T:- Let $T: U \to V$ be a linear transformation, then range of T is a subset of V consist of all $v \in V$,

$$T(u) = \cancel{v}v \quad u \in U$$

Theorem → 7.4:- Let $T: U \to V$ is a linear transformation, ~~then $\ker T$~~ we have to show that $\ker T$ is a subspace of U.

For this purpose, we have to show that $\ker T$ is closed vector addition ~~to~~ and

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scalar multiplication.

Since. $T(0) = 0$

$\therefore 0 \in \ker T$

i.e. $\ker T \neq \phi$

Let $x, y \in \ker T$

$T(x) = 0$; $T(y) = 0$

Thus $T(x+y) = T(x) + T(y) = 0 + 0 = 0$

$\Rightarrow x+y \in \ker T$

Again, Let $\alpha \in F$, $x \in \ker T$

$\therefore T(\alpha x) = \alpha T(x) = \alpha . 0 = 0$

$\therefore \alpha x \in \ker T$.

Hence $\ker T$ is a subspace of ~~V~~ U.

~~To~~ Now we have to show that Range of T is a subspace of V.

Let $v_1, v_2 \in$ Range of T. ~~There exist vectors v_1, v_2~~ then we have to prove that $v_1 + v_2 \in R(T)$ and $\alpha v_1 \in R(T)$ for any scalar α ; that is we must find vectors $u, u' \in U$ such that $T(u) = v_1 + v_2$ $T(u') = \alpha v_1$

Since $v_1, v_2 \in R(T)$, There exist u_1, u_2 in U such that $T(u_1) = v_1$, and $T(u_2) = v_2$. Let $u = u_1 + u_2$ and $u' = \alpha u_1$

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Then $T(u) = T(u_1+u_2) = T(u_1)+T(u_2) = v_1+v_2$

$T(u') = T(\alpha u_1) = \alpha T(u_1) = \alpha v_1$

which complete the proof.

6) b) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

Standard basis of $\mathbb{R}^3$ are $\{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

$T(0,0,1) = (-1,1,-2)$

Let us form a matrix whose rows are the vectors of Img T and reduce to echelon form.

$$\therefore \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

or,

$$\begin{array}{l} \\ r_2' \to 2r_1 - r_2 \\ r_3' \to r_1 + r_2 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ r_3' \to r_2 + r_3 \end{array}$$

which is echelon form

Echelon form has two non-zero rows

$\therefore$ Basis of Img of $T = \{(1,0,1), (0,1,-1)\}$

$\therefore \dim(\text{Img}\,T) = 2$

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For ker T,

$T(x, y, z) = 0$

$(x+2y-z,\ y+z,\ x+y-2z) = (0,0,0)$

Equating corresponding components

$$\left.\begin{aligned} x+2y-z &= 0 \\ y+z &= 0 \\ x+y-2z &= 0 \end{aligned}\right\} \rightarrow (1)$$

Let us represents 3 linear equation of system 1

Reduce it to echelon form.

$$\left.\begin{aligned} x+2y-z &= 0 \\ y+z &= 0 \\ L_3' \to L_3 - L_1 \qquad -y-z &= 0 \end{aligned}\right\} \rightarrow (2)$$

$$\left.\begin{aligned} x+2y-z &= 0 \\ \cancel{y+z} &\cancel{= 0} \\ y+z &= 0 \\ L_3' \to L_3 + L_2 \qquad 0 &= 0 \end{aligned}\right\} \rightarrow (3)$$

which is echelon form

Echelon form has two equation for 3 variables

$\therefore (3-2) = 1$ has a free variable

Where z is a free variable.

Let, $z = 1$, $y = -1$, $x = 3$

Basis of ker T $= \{(3, -1, 1)\}$

Dim of ker T $= 1$