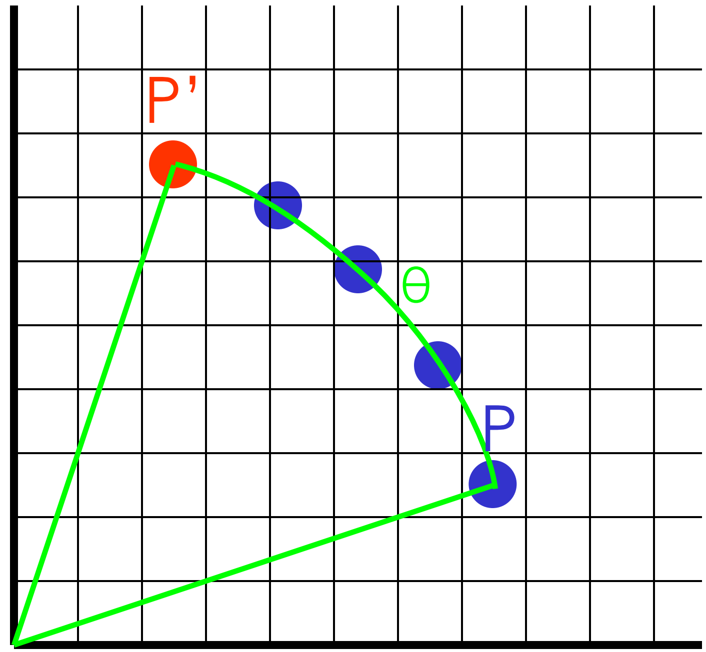
CSE- 4105

Lecture- 07

Transformation-II

Rotation

- A rotation repositions all points in an object along a circular path in the plane centered at the pivot point.
- First, we'll assume the pivot is at the origin.

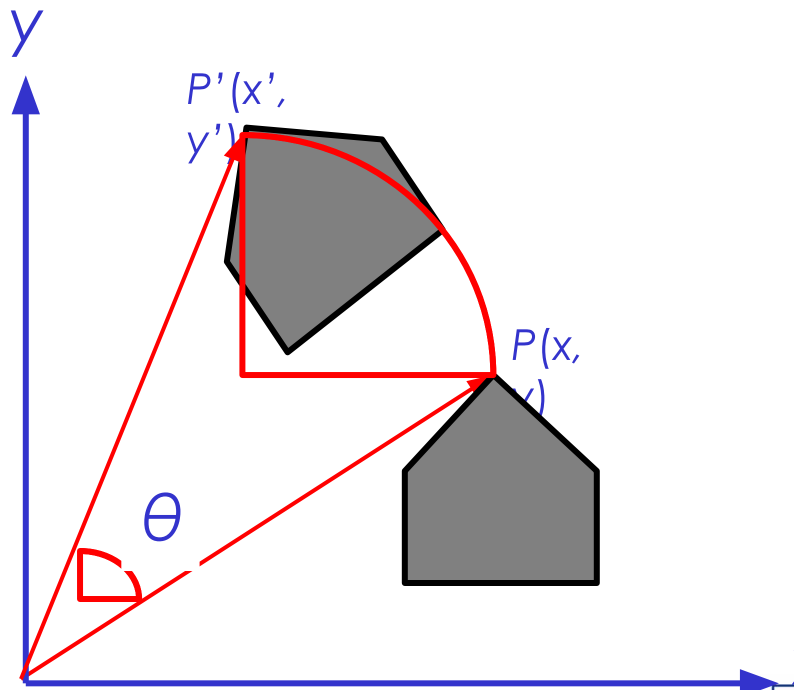


Rotation

- Rotate through an angle θ about the origin

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
- $P' = R(\theta) \cdot P$



Rotation

- Derivation of the rotation equation

$$x = r \cdot \cos\phi$$

$$y = r \cdot \sin\phi$$

$$x' = r \cdot \cos(\phi + \theta)$$

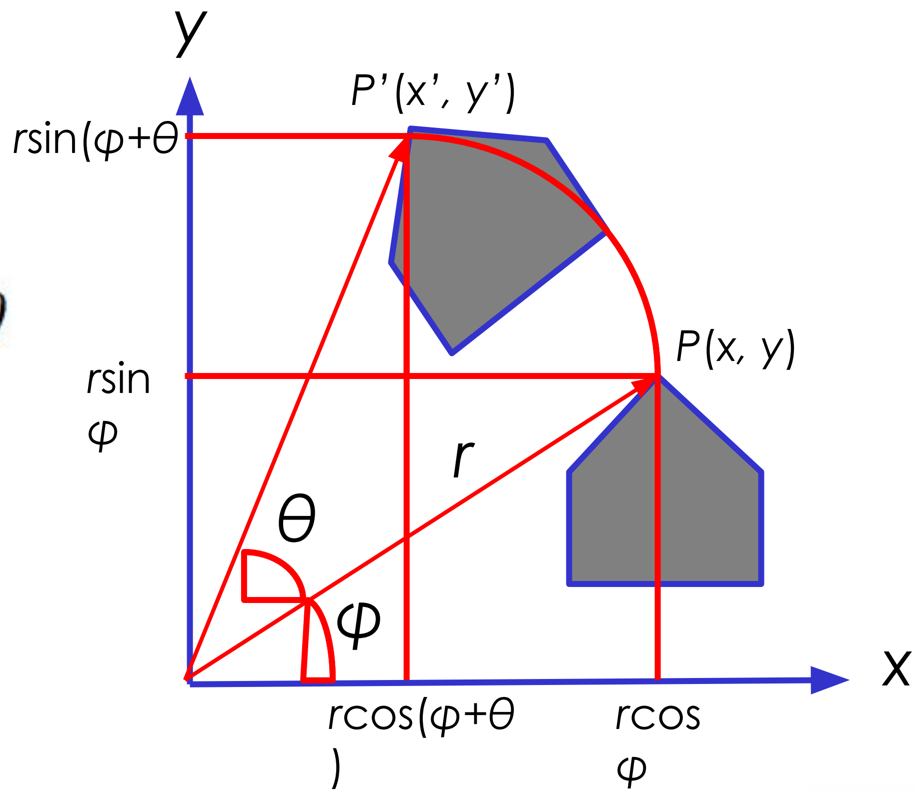
$$= r \cdot \cos\phi \cdot \cos\theta - r \cdot \sin\phi \cdot \sin\theta$$

$$y' = r \cdot \sin(\phi + \theta)$$

$$= r \cdot \cos\phi \cdot \sin\theta + r \cdot \sin\phi \cdot \cos\theta$$

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$

$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$



Rotation

- $$x' = x.\cos\theta - y.\sin\theta$$
  $$y' = x.\sin\theta + y.\cos\theta$$
- $$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
- Rewriting in matrix form gives us :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

- $$P' = R \cdot P$$

Rotation

- Example
  - Find the transformed point, P', caused by rotating P= (5, 1) about the origin through an angle of 90°.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cdot\cos\theta - y\cdot\sin\theta \\ x\cdot\sin\theta + y\cdot\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot\cos 90 - 1\cdot\sin 90 \\ 5\cdot\sin 90 + 1\cdot\cos 90 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 0 - 1\cdot 1 \\ 5\cdot 1 + 1\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Transformations.

- Translation.
  - $P'=T+P$
- Scale
  - $P'=S \cdot P$
- Rotation
  - $P'=R \cdot P$
- We would like all transformations to be multiplications

Homogeneous Coordinates

- A point (x, y) can be re-written in **homogeneous coordinates** as (x_h, y_h, h)
- The **homogeneous parameter** h is a non-zero value such that:

$$x = \frac{x_h}{h} \qquad\qquad y = \frac{y_h}{h}$$

- We can then write any point (x, y) as (hx, hy, h)
- We can conveniently choose $h = 1$ so that (x, y) becomes $(x, y, 1)$

Homogenous Coordinates

- Mathematicians commonly use homogeneous coordinates as they allow scaling factors to be removed from equations
- We will see in a moment that all of the transformations we discussed previously can be represented as 3*3 matrices
- Using homogeneous coordinates allows us use matrix multiplication to calculate transformations – extremely efficient!

Translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{aligned} x' &= x + d_x \\ y' &= y + d_y \end{aligned} \longrightarrow \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1*x+0*y+dx*1 \\ 0*x+1*y+dy*1 \\ 0*x+0*y+1*1 \end{bmatrix} = \begin{bmatrix} x+dx \\ y+dy \\ 1 \end{bmatrix}$$

Scaling

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Sx & 0 \\ 0 & Sy \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{aligned} x' &= xS_x \\ y' &= yS_y \end{aligned} \longrightarrow \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} Sx \times x \\ Sy \times y \\ 1 \end{bmatrix}$$

Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$x' = x.\cos\theta - y.\sin\theta$$
$$y' = x.\sin\theta + y.\cos\theta$$

$$\longrightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta \times x - \sin\theta \times y \\ \sin\theta \times x + \cos\theta \times y \\ 1 \end{bmatrix}$$

At a glance...

- Translation: $$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$
- Scaling $$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$
- Rotation $$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Inverse Transformations

- Transformations can easily be reversed using inverse transformations

$$T^{-1} = \begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining Transformations

- A number of transformations can be combined into one matrix to make things easy
  - Allowed by the fact that we use homogenous coordinates
- Imagine rotating a polygon around a point other than the origin
  - Transform to centre point to origin
  - Rotate around origin
  - Transform back to centre point

Combining Transformations (cont…)

- The three transformation matrices are combined as follows

$$\begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$v' = T(-dx,-dy)R(\theta)T(dx,dy)v$$

REMEMBER: Matrix multiplication is not commutative so order matters

Summary

- 2D Transformations
    - Translation
    - Scaling
    - Rotation
- Homogeneous coordinates
- Matrix multiplications
- Combining transformations

Thank You