Date – 18.07.22

Complex variable, fourier and Laplace Transform

Course Code: CSER – 2205

Complex number

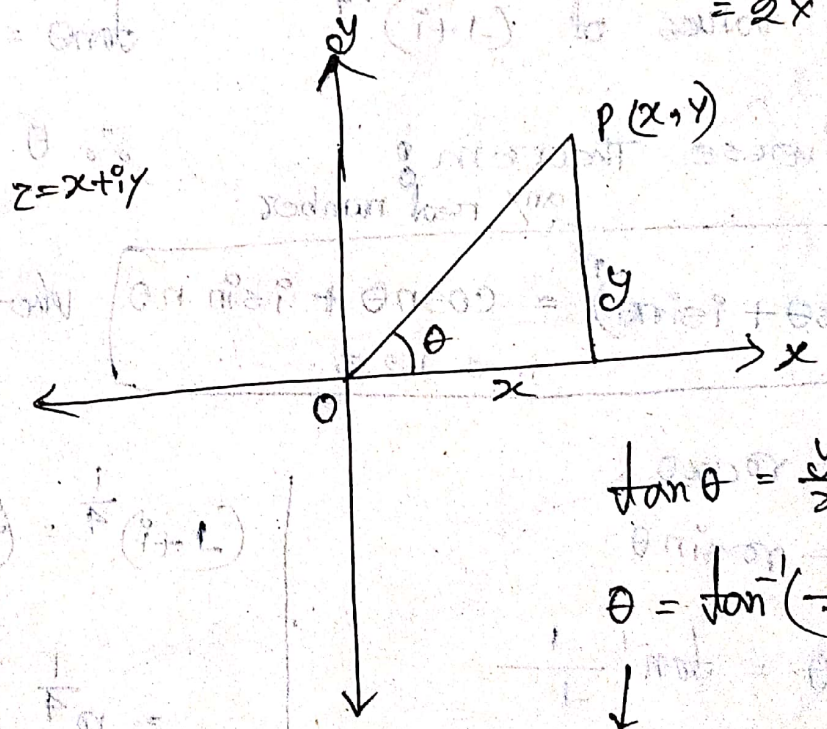
$z = x + iy$. $\quad i = \sqrt{-1}$ $\quad \bar{z} = x + iy = z$.

$|z| = \sqrt{x^2 + y^2}$ $\quad |\bar{z}| = \sqrt{x^2 + y^2}$ $\quad z + \bar{z} = 2x$.

$\bar{z} = x - iy$ $\quad = 2 \times$ Real part

$|z| = |\bar{z}|$ $\quad = 2 \times R(z)$.

$z - \bar{z} = 2iy$



$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

↓

amplitude of z.

1. Sum's outline series (complex variable)

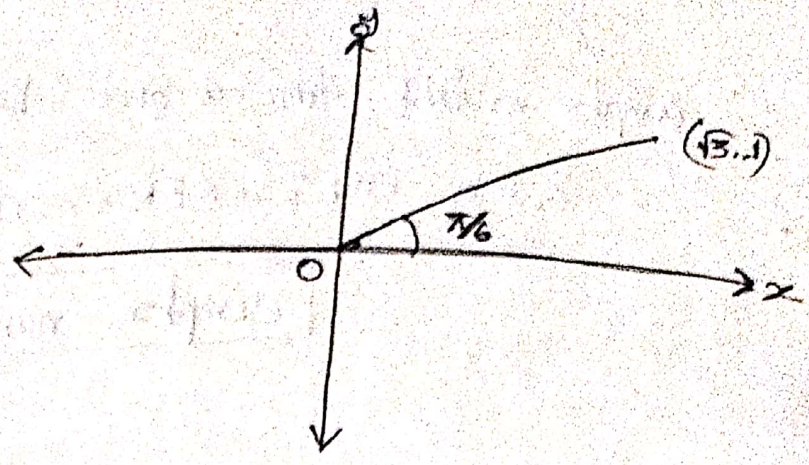
Q.

$z = \sqrt{3} + i$

$|z| = \sqrt{(\sqrt{3})^2 + (1)^2}$

$= \sqrt{4}$

$= 2$



$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$z = x + iy$

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

$\therefore \theta = \tan^{-1}\frac{y}{x}$

Find all values of $(1+i)^{1/4}$.

De-Moivre's Theorem:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad n = 0, 1, 2, 3 \ldots\ldots}$$

any real number.

where n is any real number.

$1 = r\cos\theta$

$1 = r\sin\theta$

$\therefore \theta = \tan^{-1}\frac{1}{1}$

$= \frac{\pi}{4}$

$r^2 = 1 + 1$

$\therefore r = \sqrt{2}$

$(1+i)^{\frac{1}{4}} = (r\cos\theta + ir\sin\theta)^{\frac{1}{4}}$

$= r^{\frac{1}{4}}(\cos\theta + i\sin\theta)^{1/4}$

$= r^{\frac{1}{4}}\left\{\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)\right\}^{1/4}$

~~op~~

$$= (2)^{\frac{1}{4}} \left\{ \cos\left(\frac{2\pi n+\theta}{4}\right) + i\sin\left(\frac{2\pi n+\theta}{4}\right) \right\}$$

$$n = 0, 1, 2, 3.$$

$$= (2)^{\frac{1}{8}} \left\{ \cos\left(2n\pi + \frac{\pi}{4}\right)\frac{1}{4} + i\sin\left(2n\pi + \frac{\pi}{4}\right)\cdot\frac{1}{4} \right\}$$

$$= 2^{\frac{1}{8}} \left\{ \cos(8n+1)\frac{\pi}{16} + i\sin(8n+1)\frac{\pi}{16} \right\}.$$

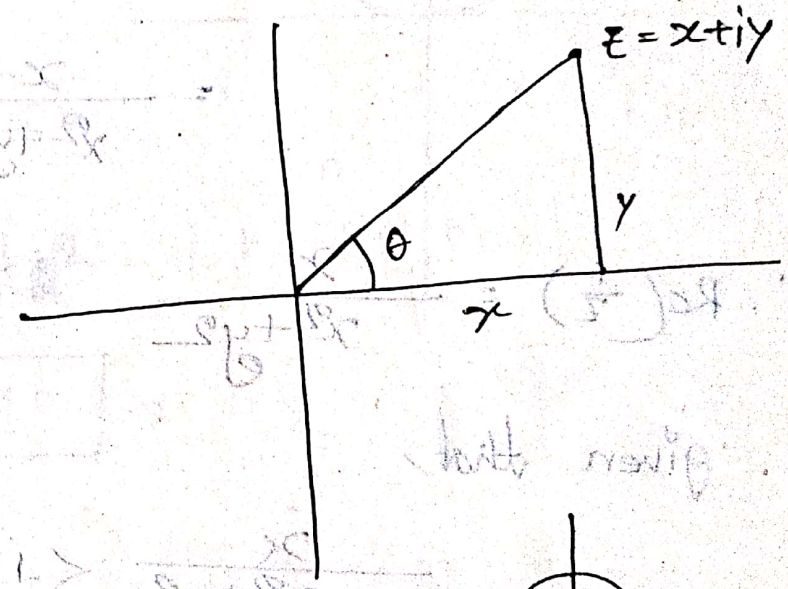
26-07-2022
Monday

$z = x + iy \qquad \theta = \tan^{-1}\frac{y}{x}$

$\sin \xrightarrow{\text{even}} \sin \xrightarrow{\text{odd}} \cos$

$\cos \xrightarrow{\text{even}} \cos \xrightarrow{\text{odd}} \sin$

$\tan \xrightarrow{\text{even}} \tan \xrightarrow{\text{odd}} \cot$

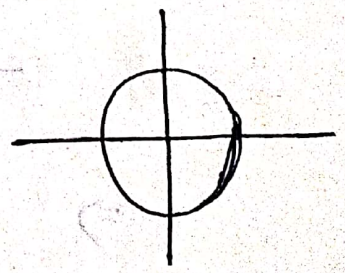


$(\sqrt{3} - i)^{\frac{1}{5}}$

we know,

$$x = r\cos\theta$$

$$y = r\sin\theta$$



$$\therefore r^2 = x^2 + y^2,$$

$$r = \sqrt{x^2+y^2}$$

$\sqrt{3} = r\cos\theta$

$-1 = r\sin\theta$

$\therefore r^2 = 3 + 1$

$\Rightarrow r^2 = 4$

$\therefore r = 2$

$\frac{1}{2} \to 0, 1$

$\frac{1}{5} \to 0, 1, 2, 3, 4$

$\boxed{\text{Page} - 61}$

$Re\left(\frac{1}{z}\right) < 1$

$z = x + iy$

$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$

$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$\therefore Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$

given that,

$\frac{x}{x^2+y^2} < 1$

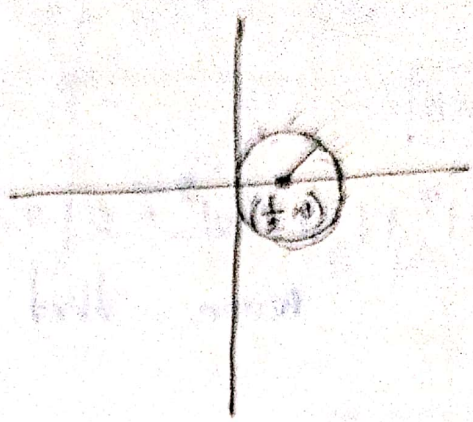
$\Rightarrow x^2+y^2 > x$

$\Rightarrow x^2+y^2-x > 0$

$\Rightarrow x^2 - 2x\cdot\frac{1}{2} + \frac{1}{4} + y^2 > \frac{1}{4}$

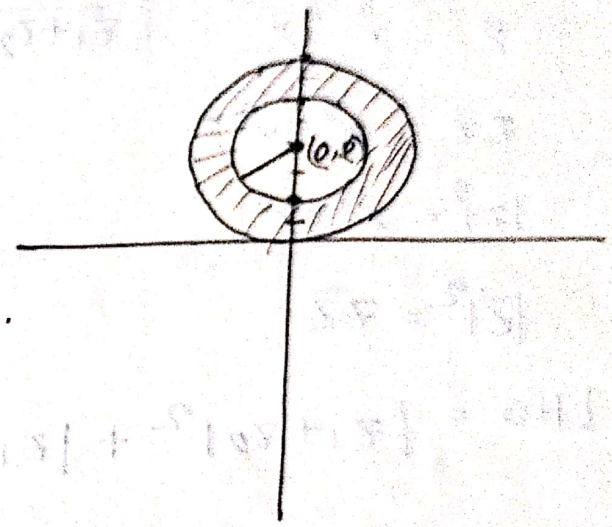
$\Rightarrow \left(x-\frac{1}{2}\right)^2 + y^2 > \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\therefore \left(x-\frac{1}{2}\right)^2+y^2 > \left(\frac{1}{2}\right)^2$



$\boxed{\text{Ex} - 40}$

$1 < |z-2i| < 2$



$\Rightarrow 1 < |x+iy-2i| < 2$

$\Rightarrow 1 < \sqrt{x^2+(y-2)^2} < 2$

$\Rightarrow 1 < x^2+(y-2)^2 < 2^2$

$\boxed{\text{Ex}-2}$ $\boxed{\text{Page}-44}$

$|z-i| = |z+i|$

$\Rightarrow |x+iy-i| = |x+iy+i|$

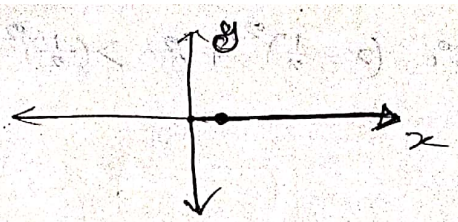
$= \sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$

$\Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2$

$\Rightarrow y^2-2y+1-y^2-2y-1=0$

$\Rightarrow -4y = 0$

$\therefore y = 0$



$\boxed{\text{Ex} - 30; \text{Page} - 38}$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

Prove that.

$z = x + iy$

$\bar{z} = x - iy$ $\qquad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$z\bar{z} = x^2 + y^2$

$|z|^2 = x^2 + y^2$

$|z|^2 = z\bar{z}$

$$\text{LHS} = |z_1 + z_2|^2 + |z_1 - z_2|^2$$

$$= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$$

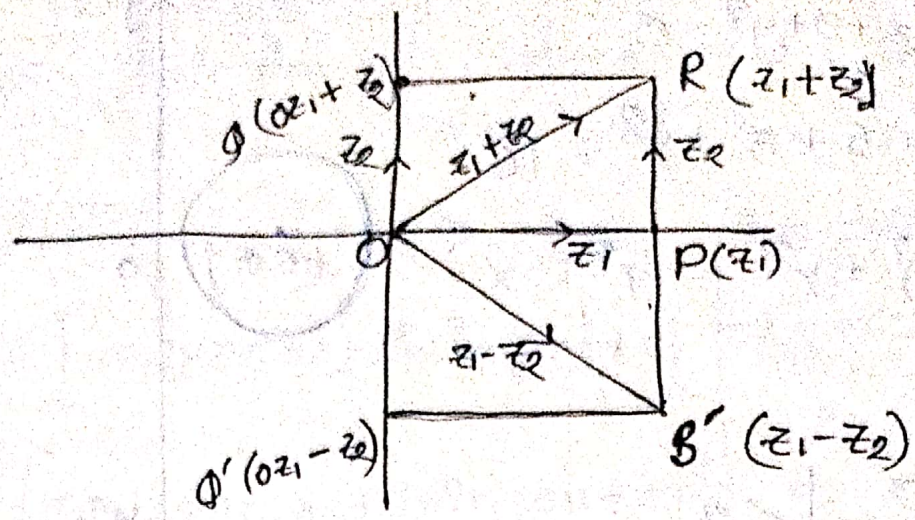
$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{\bar{z}_1 z_2} + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_2\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$$

$$= 2|z_1|^2 + 2|z_2|^2$$

$= Q$



Page - 27
Chapter - 1

28-07-2022
Thursday

$\left|\frac{z-3}{z+3}\right| = 2$ represent graphically.

$\Rightarrow |z-3| = 2|z+3|$

$\Rightarrow |x+iy-3| = 2|x+iy+3|$

$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$

$\Rightarrow (x-3)^2+y^2 = 4\left[(x+3)^2+y^2\right]$

$\Rightarrow x^2 - 6x + 9 + y^2 = 4\left[(x^2+6x+9)+y^2\right]$

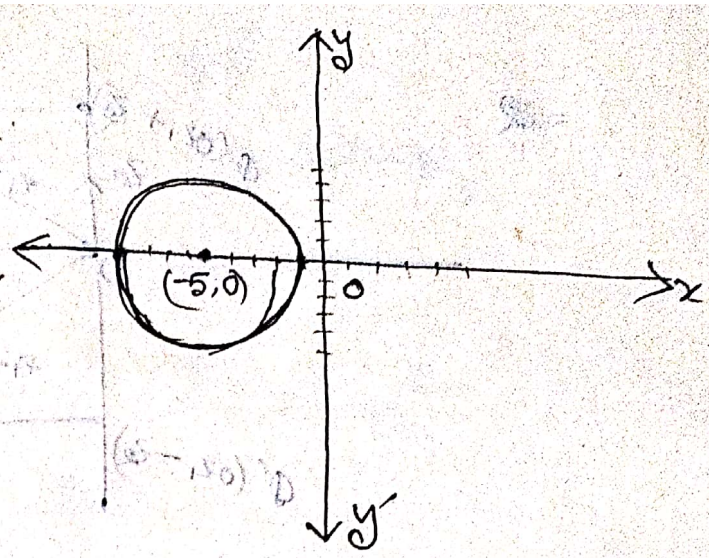
$\Rightarrow 3x^2 + 30x + 3y^2 + 27 = 0$

$\Rightarrow x^2 + 10x + y^2 + 9 = 0$

$\Rightarrow x^2 + 10x + 9 + y^2 = 0$

$\Rightarrow x^2 + 2 \cdot x \cdot 5 + 5^2 + y^2 = 16$

$\Rightarrow (x+5)^2 + y^2 = 4^2$



When $\left| \dfrac{z-3}{z+3} \right| < 2$

$\Rightarrow |z-3| < 2\,|z+3|$

$\Rightarrow |x+iy-3| < 2\,|x+iy+3|$

$\Rightarrow \sqrt{(x-3)^2 + y^2} < 2\sqrt{(x+3)^2 + y^2}$

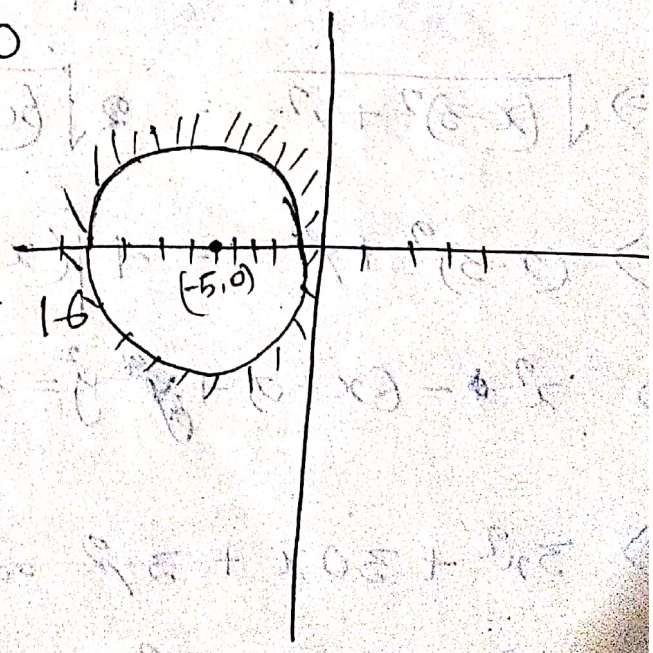
$\Rightarrow x^2 - 6x + 9 + y^2 < 4\,(x^2 + 6x + y^2)$

$\Rightarrow 3x^2 + 30x + 3y^2 + 27 > 0$

$\Rightarrow x^2 + 10x + y^2 + 9 > 0$

$\Rightarrow x^2 + 2 \cdot x \cdot 5 + 5^2 + y^2 > 16$

$\Rightarrow (x+5)^2 + y^2 > 4^2$



Page - 120 ; Ex - 1.28(a). (b)

$z^5 = -32$. find the all values of z and locate the values of z.

$\Rightarrow z^5 = 32(\cos\pi + i\sin\pi)$

$\Rightarrow z^5 = 32\left[\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\right]$

$\Rightarrow z^5 = 2^5\left[\cos(2n+1)\pi + i\sin(2n+1)\pi\right]$

$1 = \cos 0^\circ + i\sin 0^\circ$

$-1 = \cos\pi + i\sin\pi$

$= \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$

$\Rightarrow z = 2\left\{\cos(2n+1)\pi + i\sin(2n+1)\pi\right\}^{\frac{1}{5}}$

where $n = 0, 1, 2, 3, 4$.

$\Rightarrow z = 2\left\{\cos(2n+1)\frac{\pi}{5} + i\sin(2n+1)\frac{\pi}{5}\right\}$

when $n=0$;

$z_1 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

when $n=1$;

$z_2 = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

when $n=2$;

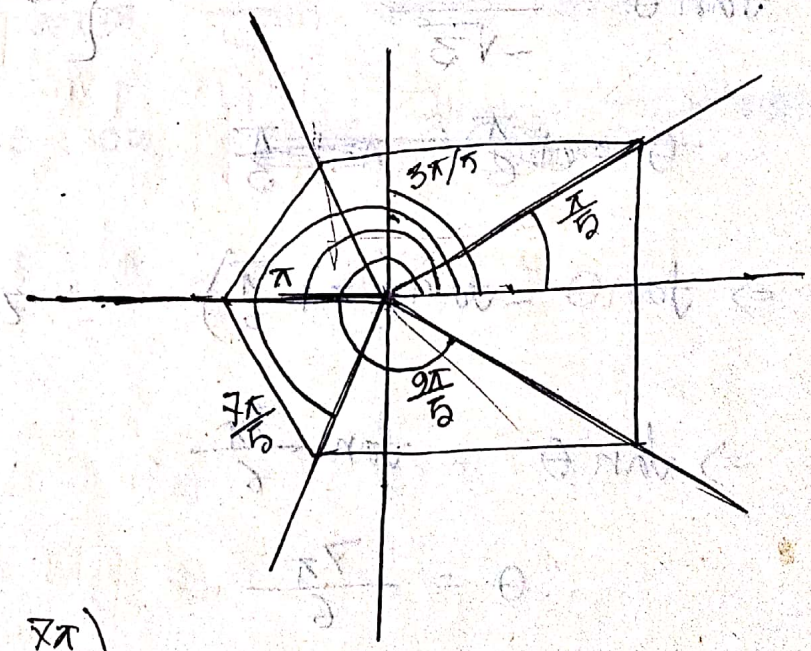
$z_3 = 2(\cos\pi + i\sin\pi)$

when $n=3$;

$z_4 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$

when $n=4$;

$z_5 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$



Ex 1.29 ; (b)

$$(-2\sqrt{3} - 2i)^{\frac{1}{4}}$$

$$-2\sqrt{3} = r\cos\theta$$

$$-2 = r\sin\theta$$

$$r^2 = (-2\sqrt{3})^2 + (-2)^2$$

$$= 4\times 3 + 4$$

$$= 12 + 4$$

$$= 16$$

$$\therefore r = 4$$

$$\tan\theta = \frac{-1}{-\sqrt{3}}$$

$$\therefore \theta = \frac{\pi}{6}$$

$$\Rightarrow \tan\theta = \tan\left(\pi + \frac{\pi}{6}\right)$$

$$\Rightarrow \tan\theta = \tan\frac{7\pi}{6}$$

$$\therefore \theta = \frac{7\pi}{6}.$$

$$\therefore (-2\sqrt{3} - 2i)^{\frac{1}{4}} = (r\cos\theta + ir\sin\theta)^{\frac{1}{4}}$$

$$= \{4(\cos\theta + i\sin\theta)\}^{\frac{1}{4}}$$

$$= \sqrt{2}\,(\cos\theta + i\sin\theta)^{\frac{1}{4}}$$

$$= \sqrt{2}\left\{\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)\right\}^{\frac{1}{4}}$$

$$= \sqrt{2}\left\{\cos\left(2n\pi + \frac{7\pi}{6}\right) + i\sin\left(2n\pi + \frac{7\pi}{6}\right)\right\}^{\frac{1}{4}}$$

$$= \sqrt{2}\left\{\cos(12n+7)\frac{\pi}{6} + i\sin(12n+7)\frac{\pi}{6}\right\}^{\frac{1}{4}}$$

$$= \sqrt{2}\left\{\cos(12n+7)\frac{\pi}{24} + i\sin(12n+7)\frac{\pi}{24}\right\}$$

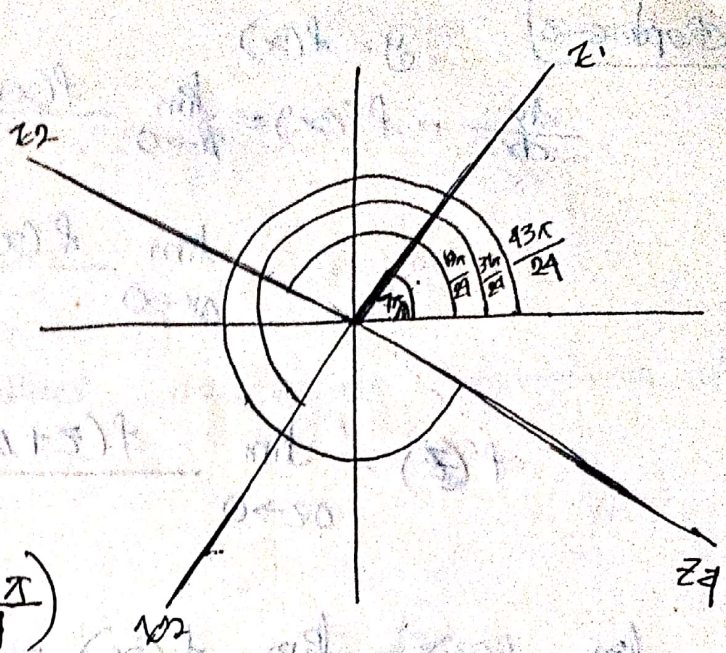
$$n = 0, 1, 2, 3.$$

$$\therefore z = \sqrt{2}\left\{\cos(12n+7)\frac{\pi}{24} + i\sin(12n+7)\frac{\pi}{24}\right\}$$

$$n = 0, 1, 2, 3.$$

$n = 0$;

$$z_1 = 2^{\frac{1}{2}}\left(\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right)$$



$n = 1$;

$$z_2 = 2^{\frac{1}{2}}\left(\cos\frac{19\pi}{24} + i\sin\frac{19\pi}{24}\right)$$

$n = 2$;

$$z_3 = 2^{\frac{1}{2}}\left(\cos\frac{31\pi}{24} + i\sin\frac{31\pi}{24}\right)$$

$n = 3$;

$$z_4 = 2^{\frac{1}{2}}\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right)$$

শুধু চিত্র হলে যোগ করতে হবে না।

find all values or locate থাকলে যোগ করতে হবে।

Chapter-3

1-08-2022
Monday

$y = f(x)$

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a) = L \text{ (for continuous function)}$$

$$L.H.D = R.H.D = L \rightarrow \text{(differentiable function)}$$

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

Analytic function (~~[illegible]~~ point is differentiable).

→ Analytic in a region
→ Analytic at a point

} Page - 2.39.

we know,

$$w = f(z) = u + iv.$$

$$= f(z) = u(x,y) + iv(x,y).$$

$$u(x,y) = x^2 + 3xy + y^2$$

$$u_x = 2x + 3y$$

$$u_{xx} = 2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \therefore \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ is called conjugate function of each other.

Harmonic function :

↳ satifies laplaces equation

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \end{aligned}\right\} \text{Laplaces equatio}$$

↳ Harmonic function

singular point

↳ যে point a function টি analyticity fail হবে, তার মানে যে point এ function টি differentiable না। (Page ৩.৫)

$$f(z) \longrightarrow z_0$$

isolated singularity

$\forall \epsilon > 0 \; ; \; \exists \; \delta > 0$

$|x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$

$\lim_{x \to a} f(x) = l$

$|x + iy - z_0| = \delta$

$x = a$

$z = z_0$

$\Rightarrow \sqrt{(x - z_0)^2 + y^2} = \delta$

$f(z) \quad \delta > 0$

$\Rightarrow (x - z_0)^2 + y^2 = \delta^2$

$|z - z_0| = \delta$

$|z| = a$

$z_0 = (x_0, y_0)$

isolated singularity

এই point ছাড়া অন্য কোন point নাই circle এর ভেতরে।

Pole:

$f(z) = \frac{1}{(z-2)^{-1}}$

$f(z) = \frac{1}{(z-3)}$ $\quad z = 3$ is a pole of order 1

$= \frac{1}{(z-3)^5}$ $\quad z = 3$ is a pole of order 5

$$f(z) = \frac{1}{(z-3)^5 \cdot (z-1)^2 (z-3)^3}$$

Branch point:

$f(x) = (x-1)^{1/2} \rightarrow$ multiple value function

↳ power যখন fraction হয়।

Removal singularity:

$z \rightarrow z_0$ $\quad$ z_0 is a isolated point

$$\lim_{z \to z_0} f(z)$$

essential singularity:

যা pole ও removal singularity হবে না

Isolated singularity, pole, removal singularity হবে না

$$f(z) = \frac{1}{\omega} \rightarrow \infty$$

↳ singularities and infinite.

৩.১০ Page - ৩.10 (৩.১ ; ৩.২ ; ৩.৩) H.W.

Page - 3.-10

3.1

$w = f(z) = z^3 - 2z$ at the point

a) $z = z_0$;

b) $z = -1$

here $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^3 - 2(z_0+\Delta z) - z_0^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^3} + 3z_0^2\Delta z + 3z_0\Delta z^2 + \Delta z^3 - \cancel{2z_0} - 2\Delta z - \cancel{z_0^3} + \cancel{2z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(3z_0^2 + 3z_0\Delta z + \Delta z^2 - 2\right)$$

$$= 3z_0^2 - 2.$$

$\boxed{3.2}$

$$\frac{d}{dz}\bar{z} \quad f(z) = \bar{z}$$

$z = x+iy$ $\qquad \Delta z \to 0$

$\Delta z = \Delta x + i\Delta y$ $\qquad \Delta x \to 0$, $\Delta y \to 0$

$$\frac{d}{dz}\{f(z)\} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta x + \Delta y \to 0} \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$\Delta x \to 0$, $\Delta y \to 0$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x+\Delta x) - i(y+\Delta y) - x + iy}{\Delta x + i\Delta y}$$

$\Delta y = 0$; $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{x + \Delta x - \cancel{iy} - \cancel{x} + \cancel{iy}}{\Delta x} = 1.$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 1 = 1.$$

$\Delta x = 0$; $\Delta y \to 0$

$$\lim_{\Delta y \to 0} \frac{\cancel{x} - \cancel{iy} - i\Delta y - \cancel{x} + \cancel{iy}}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} -1$$

$$= -1.$$

4-08-2022
Thursday

State and prove Cauchy-Riemann equation.

for necessary

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \quad \text{①}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$z = x + iy$

$\Delta z = \Delta x + i\Delta y$

$w = f(z) = u + iv$

$= u(x,y) + iv(x,y)$

$\cancel{= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}}$

Case-1 - $\Delta y = 0$, $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - \{u(x,y) + iv(x,y)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{-v(x,y) + v(x+\Delta x, y)}{\Delta x} \right]$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \therefore f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Case-2 ; $\Delta x = 0$; $\Delta y \to 0$

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - \{u(x,y) - iv(x,y)\}}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \right]$$

$$\cancel{\therefore f'(z) = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

equating the corresponding values (real and imaginary parts) we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

For sufficient condition,

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= u(x+\Delta x, y+\Delta y) + u(x, y+\Delta y) - u(x, y+\Delta y) - u(x,y)$$

$$= \left\{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\right\} + \left\{u(x, y+\Delta y) - u(x,y)\right\}$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y$$

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y$$

$\Delta x \to 0 \Rightarrow \varepsilon_1 \to 0$

$\Delta y \to 0 \to \eta_1 \to 0$

similarly

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

$$\Delta x \to 0 \to \varepsilon_2 \to 0$$

$$\Delta y \to 0 \to \eta_2 \to 0$$

we know

$$w = u + iv$$

$$\therefore \Delta w = \Delta u + i\Delta v$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \cancel{\varepsilon_1 \Delta x + \eta_1 \Delta y} + i\frac{\partial v}{\partial x}\Delta x + i\frac{\partial v}{\partial y}\Delta y$$

$$+ \underbrace{(\varepsilon_1 + i\varepsilon_2)}_{\varepsilon}\Delta x + \underbrace{(\eta_1 + i\eta_2)}_{\eta}\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{-\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(i^2\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y$$

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z + \varepsilon\Delta x + \eta\Delta y.$$

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

8-08-2020
Monday

Page 3.13

3.7 Prove that, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. —— (1)

harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

D. P w.r.to to x,

$$\therefore \frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - \left[-e^{-x}x\sin y + e^{-x}\sin y\right] - ye^{-x}\cos y$$

$$= -e^{-x}\sin y + e^{-x}x\sin y - e^{-x}\sin y - ye^{-x}\cos y$$

$$= xe^{-x}\sin y - 2e^{-x}\sin y - ye^{-x}\cos y \quad \text{—— (II)}$$

D. P.w.r.to y.

$$\frac{\partial u}{\partial y} = e^{-x}\left(x\cos y + y\sin y - \cos y\right)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}\left[-x\sin y + y\cos y + \sin y + \sin y\right]$$

$$= e^{-x}\left[2\sin y + y\cos y - x\sin y\right]$$

$$= 2e^{-x}\sin y + ye^{-x}\cos y - xe^{-x}\sin y \quad \text{——— (III)}$$

adding (II) and (III) we get,

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, u is harmonic.

b) Find v such that $f(z) = u + iv$ is analytic.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad u = e^{-x}(x\sin y - y\cos y) \quad \text{——— (1)}$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = e^{-x}\left(x\cos y + y\sin y - \cos y\right)$$

$$\therefore \frac{\partial v}{\partial x} = -e^{-x}\left(x\cos y + y\sin y - \cos y\right) \quad \text{——— (2)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$$

(2)

Integrating ② w.r.to y, (x is constant)

$$\cancel{v = -e^{-x}\cos y + xe^{-x}\cos y + y\sin y + \cos y}$$

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\left[y\sin y - \int 1.\sin y\, dy\right]$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + ye^{-x}\sin y + e^{-x}\cos y + \underbrace{f(x)}_{\to c}$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + \underbrace{f(x)}_{\to c}.$$

D.w.r.to x;

$$\frac{\partial v}{\partial x} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + f'(x) \quad \text{—— ④}$$

From ③ and ④ we get,

$$f'(x) = 0$$

$$\therefore f(x) = c$$

$\boxed{\text{Page - 3.37}}$

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$f'(z) = u_1(z,0) - iu_2(x,0)$$

$$u_1 = \frac{\partial u}{\partial x}$$

$$u_2 = \frac{\partial u}{\partial y}$$

$$u_1(x,y) = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + y e^{-x}\cos y$$

$$u_2(x,y) = \frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y.$$

$$f'(x) = 0 - i\left(xe^{-x} - e^{-x}\right)$$

$$\therefore f'(z) = \cancel{-i(ze^{-z}} - ize^{-z} + ie^{-z}$$

$$\therefore f(z) = -i\left\{\left[-ze^{-z} - \int 1 \cdot e^{-z}\,dz\right] + e^{-z}\right\}$$

$$\cancel{= -ize^{-z} + i\int e^{-z}dz - ie^{-z}}$$

$$= \cancel{ize^{-z} - ie^{-z}} = -i\left[-ze^{-z} + \cancel{e^{-z}} - \cancel{e^{-z}}\right]$$

$$= ize^{-z}$$

$$w = f(z) = u + iv$$

$$\Rightarrow ize^{-z} = u + iv$$

$$\Rightarrow \cancel{ize^{-z}} = u + iv = i(x+iy)\cdot e^{-(x+iy)}$$

$$\Rightarrow u + iv = i(x+iy)\,e^{-x}\cdot e^{-iy}$$

$$\Rightarrow u + iv = i(x+iy)\cdot e^{-x}\cdot(\cos y - i\sin y)$$

$$\Rightarrow u + iv = e^{-x}\left[i(x+iy)(\cos y - i\sin y)\right]$$

$\Rightarrow u+iv = e^{-x}\left[ix\cos y + x\sin y - y\cos y + iy\sin y\right]$

$= i\left[xe^{-x}\cos y + ye^{-x}\sin y\right] + e^{-x}x\sin y - e^{-x}y\cos y$

$\therefore v = xe^{-x}\cos y + ye^{-x}\sin y$

22-08-22
Monday

$\oint_C$ → closed integration
→ anticlock wise ঘুরে

1. $\oint_C f(z)dz = 0$ → point বাইরে থাকলে

point টা বাইরে থাকলে zero 0

2. $\oint_C f(z)dz = \oint_{C_1} f(z)dz$

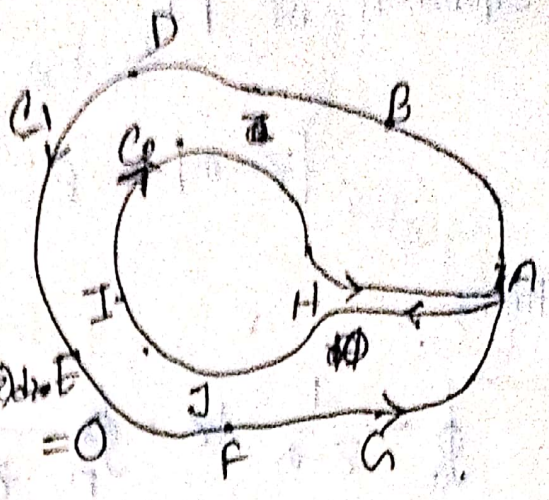
Page – 4.18

ex – 4.16

Cauchy – Gourcat theorem for multiply – connected regions.

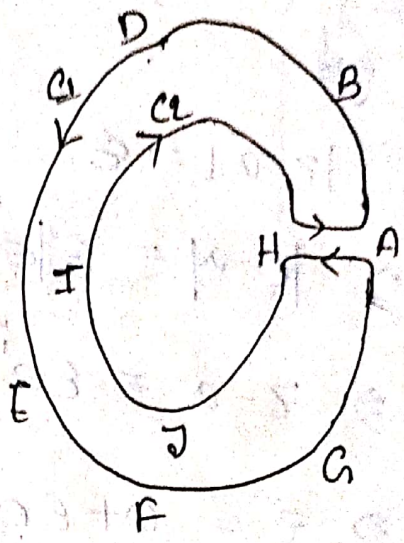
Solution

$$\int_{ABDEFGAHIJA} f(z)\,dz = 0$$



$$\int_{ABDEFGA} f(z)\,dz + \int_{AH} f(z)\,dz + \int_{HIJA} f(z)\,dz + \int_{JA} f(z)\,dz = 0$$

$$\Rightarrow \int_{ABDEFGA} f(z)\,dz + \int_{HIJA} f(z)\,dz = 0$$



$$\Rightarrow \oint_C f(z)\,dz = 0$$

Page – 4.20

Ex – 4.2-1

Given that,

$$\oint_C \frac{dz}{z-a}$$ and ~~z = a is~~

a) $z = a$ outside C

b) $z = a$ inside C

for,
$z = a$
$\therefore$ function is singular not analytic

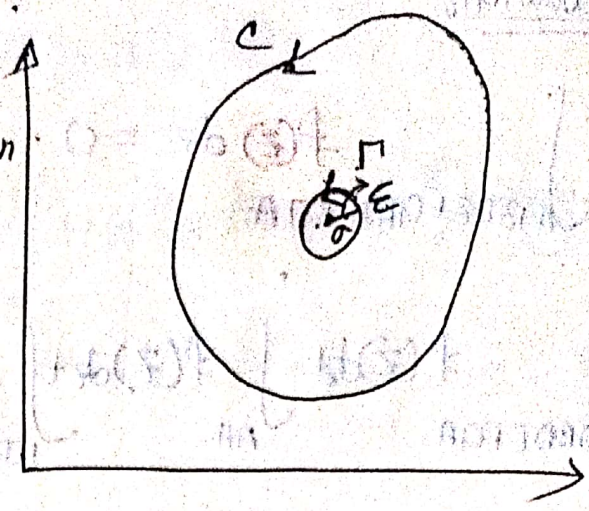
a) Ans: if a is outside C ~~then $f(z) = \frac{1}{z-a}$ is~~ then by cauchy's theorem we get,

$$\oint_C \frac{dz}{z-a} = 0$$

equation of Γ,

$$|z-a| = \varepsilon$$

$|z|=a$ → circle equation



Here,

$$\oint_C \frac{1}{z-a}\,dz = \oint_\Gamma \frac{dz}{z-a}$$

we get -

$$|z-a| = \varepsilon$$

$$\Rightarrow |z-a| = \varepsilon|e^{i\theta}|$$

$$\Rightarrow z-a = \varepsilon e^{i\theta} \quad ; \quad 0 \le \theta \le 2\pi$$

$$\Rightarrow z = a + \varepsilon e^{i\theta}$$

$$\Rightarrow dz = i\varepsilon e^{i\theta}\,d\theta \; ; \; 0 \to 2\pi$$

$|z|=a$

$\Rightarrow |x+iy| = a$

$\Rightarrow \sqrt{x^2+y^2} = a$

$\therefore x^2+y^2 = a^2$

$e^{i\theta} = \cos\theta + i\sin\theta$ (euler's theorem)

$\therefore |e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta}$

$= 1$

$$\oint \frac{1}{z-a}\,dz = \oint_\Gamma \frac{dz}{z-a}$$

$$= \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\,d\theta}{a+\varepsilon e^{i\theta}-a}$$

$$= \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\,d\theta}{\varepsilon e^{i\theta}\,d\theta}$$

$$= i\int_0^{2\pi} d\theta = i\,[\theta]_0^{2\pi}$$

when $z=a$ is inside C. Let Γ be a circle of radious ε with center at $z=a$. so that Γ is inside C.

$= i\left[2\pi - 0\right]$

$= 2\pi i$

volume $\rightarrow \iiint_V$ → triple integral.

Green's Theorem:

$\nabla \varphi \longrightarrow$ gradient

$\nabla \cdot V \longrightarrow$ divergense

$\nabla \times V \longrightarrow$ curl

$R \rightarrow$ double integral

$\rightarrow$ region

$$\oint_C B(z, \bar{z})\, dz = 2i \iint_R \frac{\partial B}{\partial \bar{z}}\, dx\, dy.$$

Formula:

$$\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$\boxed{\text{Page} - 4.10}$

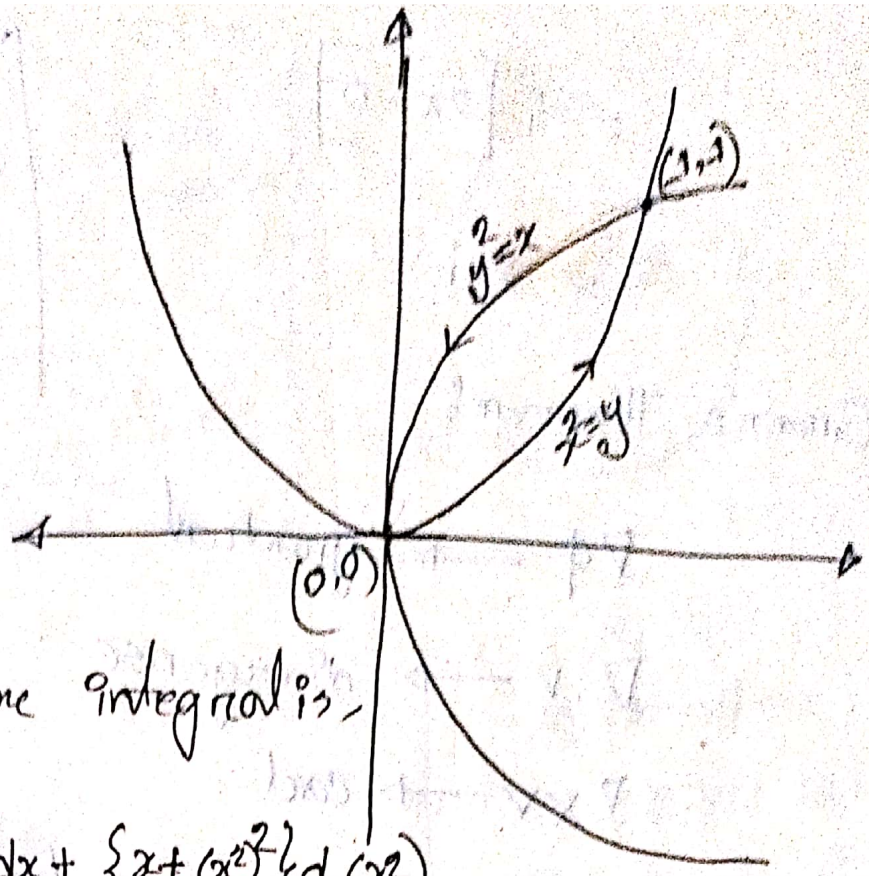
$\boxed{\text{Ex} - 4.5}$

$$\cancel{\oint_C P dx + Q dy = \int}$$

$$\oint_C \underbrace{(2xy - x^2)}_{P} dx + \underbrace{(x^2 + y^2)}_{Q} dy.$$

Curves are $y = x^2$ and $y^2 = x / x = y^2$

Q)



when $y=x^2$ the line integral is,

$$\int_{x=0}^{1} \left[\{2x\cdot(x^2)-x^2\}\,dx + \{x+(x^2)^2\}\,d(x^2)\right]$$

$$= \int_{0}^{1} (2x^3 - x^2)\,dx + (x+x^4)\,2x\,dx$$

$$= \int_{0}^{1} (2x^3 - x^2 + 2x^2 + 2x^5)\,dx$$

$$= \int_{0}^{1} (2x^3 + x^2 + 2x^5)\,dx$$

$$= \left[2\cdot\frac{x^4}{4} + \frac{x^3}{3} + 2\cdot\frac{x^6}{6}\right]_0^1$$

$$= \frac{2}{4} + \frac{1}{3} + \frac{2}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3}$$

$$= \frac{3+2+2}{6} = \frac{7}{6}.$$

when $x = y^2$ the line integral is,

$$\int_{y=1}^{0} (2y^2 \cdot y - y^4)\, dy^2 + (y^2 + y^2)\, dy$$

$$= \int_{1}^{0} (2y^3 - y^4)\, 2y\, dy + 2y^2 dy$$

$$= \int_{1}^{0} \left[4y^4 - 2y^5 + 2y^2 \right] dy$$

$$= \left[4 \cdot \frac{y^5}{5} - 2 \times \frac{y^6}{6} + 2 \cdot \frac{y^3}{3} \right]_1^0$$

$$= -\frac{4}{5} + \frac{\cancel{2}1}{\cancel{6}3} - \frac{2}{3} = \frac{-12+5-10}{15} = \frac{-22+5}{15} = \frac{-17}{15}$$

$\therefore$ The the required integral $= \frac{7}{6} - \frac{17}{15}$

$$= \frac{35-34}{30} = \frac{1}{30}$$

On the other hand,

$$\oint_C \underbrace{(2xy - x^2)}_{P} dx + \underbrace{(x+y^2)}_{Q} dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

$$= \iint_R \left[\frac{\partial}{\partial x} (x+y^2) - \frac{\partial}{\partial y} (2xy - x^2) \right] dx\, dy$$

$$= \iint_R (1-2x)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1-2x)\,dy\,dx$$

$$= \int_0^1 \left[y - 2xy\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^2 + 2x^3\right) dx$$

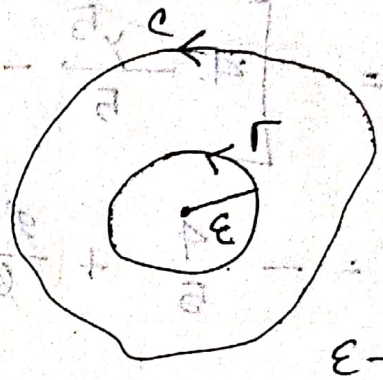
$$= \frac{1}{30}.$$

Page - 5.3

24-08-2022
Thursday

Ex - 5.1

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz$$



$\varepsilon \to 0$

$$\therefore \oint_C \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \times i\varepsilon e^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(a+\varepsilon e^{i\theta})\,d\theta$$

$|z-a| = \varepsilon\,|e^{i\theta}|$

$\Rightarrow z-a = \varepsilon e^{i\theta}$

$\therefore z = a+\varepsilon e^{i\theta}$

$\therefore dz = i\varepsilon e^{i\theta}\,d\theta$

$$\lim_{\varepsilon\to 0} \frac{f(z)}{z-a}\,dz = \lim_{\varepsilon\to 0} \int_0^{2\pi} i\,f(a+\varepsilon e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} f(a)\,d\theta$$

$$= i f(a)\,[\theta]_0^{2\pi}$$

$$\therefore \oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

$$\therefore f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

$$\therefore f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$\boxed{5.2\text{-Ex}}$

we get,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

$$\therefore \Rightarrow f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(a+h)}\,dz$$

$$\therefore \frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{1}{\cancel{h}}\,\frac{\cancel{z}-\cancel{a}-\cancel{z}+a+h}{(z-a-h)(z-a)}\,f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)}.$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

$$= \frac{1}{2\pi i}$$

$$\lim_{h\to 0} f'(a) = \lim_{h\to 0} \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-}$$

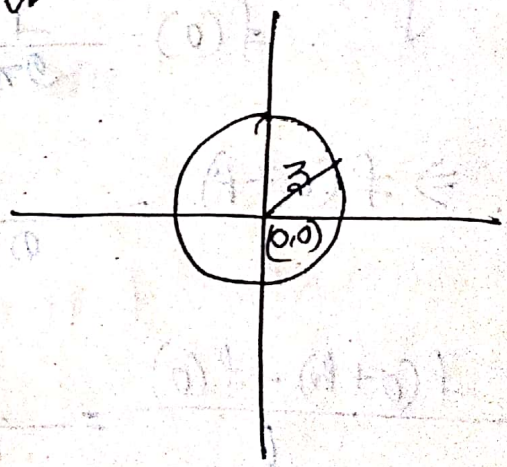
$$= \frac{f(z)}{(z-a)^2}\,dz$$

$\boxed{5.39}$ $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ i) $|z|=3$ ii) $|z|=1$.

i) when $|z|=3$ is ~~inside the~~

$\Rightarrow \sqrt{x^2+y^2}=3$

$\therefore x^2+y^2=3^2$



here, $z-2=0$

$\therefore z=2$.

the point $z=2$ is i

29-08-2022
Monday

$$\Rightarrow \oint_C \frac{dz}{z-a} = 0 \; ; \; z = a \text{ outside}$$

$$\oint_C \frac{dz}{z-a} = 2\pi i \; ; \; z = a \text{ inside.}$$

Page - 5.5

a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$. circle $|z| = 3$.

here,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)(1-2)} + \frac{1}{(2-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

Now,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

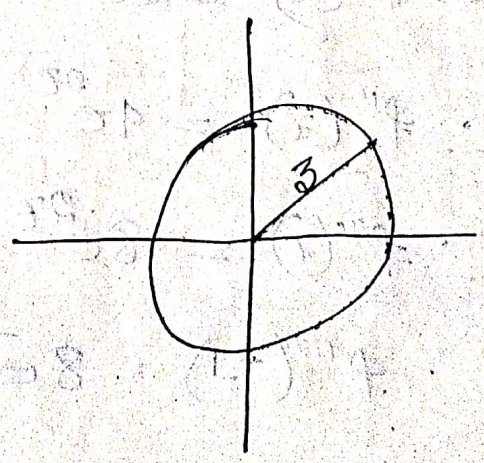
Here,

$f(z) = \sin \pi z^2 + \cos \pi z^2$

$|z| = 3$

$\Rightarrow z = 2$ and $z = 1$.

inside c.



$f(2) = \sin \pi 2^2 + \cos \pi 2^2 = 1$

$f(-1) = \sin \pi 1^2 + \cos \pi 1^2 = -1$

so,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$= 2\pi i - (-2\pi i)$

$= 4\pi i$

b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$

here, $a = -1$.

$f(z) = e^{2z}$.

$\therefore f'(z) = 2e^{2z}$

$\therefore f''(z) = 4e^{2z}$

$\therefore f'''(z) = 8e^{2z}$

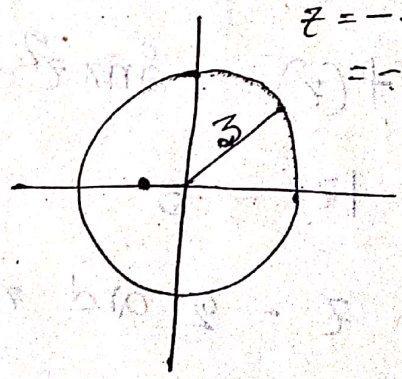
$f'''(-1) = 8e^{-2}$.

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-a)^2}$$

$$f^n(a) = \frac{\lfloor n}{2\pi i} \oint_C \frac{f(z)dz}{(z-a)^{n+1}}$$

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{\lfloor n} f^n(a)$$

$z = -1,$
$= -1 + 0i$



$$\therefore \oint_C \frac{e^{2z}}{(z+1)^{3+1}}\,dz = \frac{2\pi i}{\lfloor 3}\, f'''(-1)$$

$$= \frac{2\pi i}{3\times 2\times 1}\times 8e^{-2}$$

$$= \frac{2\pi i}{6}\times 8e^{-2}$$

$$= \frac{8\pi i e^{-2}}{3}$$

Page 5.8

57

evaluate $\oint_C \frac{3z^2+z}{z^2-1}\,dz$ | $C \longrightarrow |z-1| = 1$.

$$\Rightarrow \oint_C \frac{3z^2+z}{(z-1)(z-1)}\,dz$$

$$\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) = \frac{z+1-(z-1)}{(z+1)(z-1)}$$

$$= \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}$$

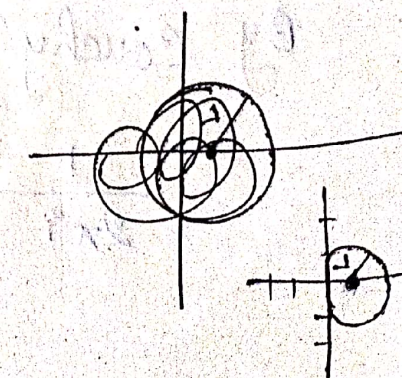
$|z-1| = 1$

$\Rightarrow |x-1+iy| = 1$

$\Rightarrow \sqrt{(x-1)^2+y^2} = 1$

$\therefore (x-1)^2+y^2 = 1$

The point $x=1$ inside contour C and point $x=-1$ outside contour C.



$$= \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}dz - \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}dz$$

$$\frac{1}{2}\oint_C \frac{3z^2+z}{z-1}dz = \frac{1}{2}\times 2\pi i f(1) = \pi i f(1) = \pi i\left[3\times 1^2+1\right] = 4\pi i$$

$$\frac{1}{2}\oint_C \frac{3z^2+z}{z+1}dz = 0$$

Page 5.22

Ex 5.39

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$$

a) $|z| = 3$

b) $|z| = 1$

here,

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$$

~~here~~ $\cancel{f(z) = e^z}$

a) Hence $f(z) = e^z$, the point $z=2$ inside the $|z| = 3$.

By Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz = f(2) = e^2.$$

b) the point $z=2$ outside $|z|=1$.

By cauchy's theorem,

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0.$$

$\boxed{\text{Q.41}}$

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz$$

a) $|z-1| = 4$

b) $|z-2| + |z+2| = 6$

a) the point $z=\pi i$ inside

$|z-1| = 4$;

$e^{ix} = \cos x + i\sin x$

so by cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$f(z) = e^{3z}$

$$\Rightarrow \oint_C \frac{e^{3z}}{z-\pi i}\,dz = 2\pi i\, f(\pi i)$$

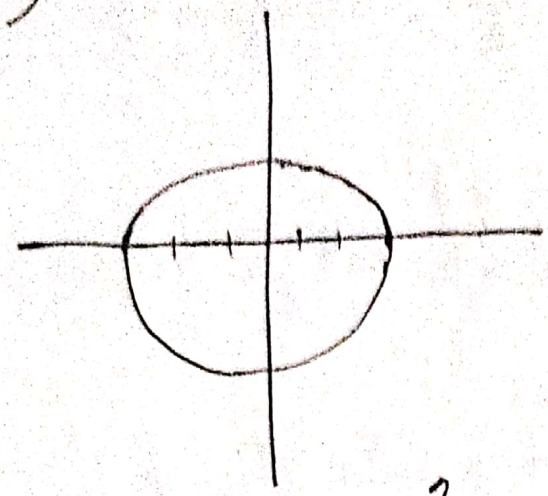
$$= 2\pi i \cdot e^{3\pi i}$$

$$= 2\pi i\,(\cos 3\pi + i\sin 3\pi)$$

$$= 2\pi i\,(-1+0)$$

$$= -2\pi i.$$

b)



$$\therefore \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$$

which is the equation of ellipse where major axis is 3 and minor axis is $\sqrt{5}$. So the point $z = \pi i$ outside C.

Hence by Cauchy's theorem we get,

$$\oint_C \frac{e^{3z}}{z-\pi i}\, dz = 0.$$

$$|z-2| + |z+2| = 6$$

$$\Rightarrow |x-2+iy| + |x+2+iy| = 6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} = 6 - \sqrt{(x+2)^2+y^2}$$

$$\Rightarrow (x-2)^2 + y^2 = 36 - 12\sqrt{(x+2)^2+y^2} + (x+2)^2 + y^2$$

$$\Rightarrow x^2 - 4x + 4 - x^2 - 4x - 4 = 36 - 12\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow -8x = 36 - 12\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow 12\sqrt{(x+2)^2+y^2} = 8x + 36$$

$$\Rightarrow 3\sqrt{(x+2)^2+y^2} = 9 + 2x$$

$$\Rightarrow 9\left(\cancel{(x+2)^2} + x^2 + 4x + 4 + y^2\right) = 81 + 36x + 4x^2$$

$$\Rightarrow 9x^2 + 36x + 36 + 9y^2 - 4x^2 - 36x - 81 = 0$$

$$\Rightarrow 5x^2 + 9y^2 = 45$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1.$$

Chapter-6

Page-6.17

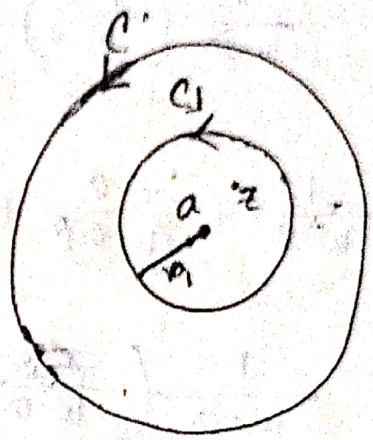
04-09-2022
Thursday

Theorem - 6.22

$f(z) \longrightarrow$ inside circle; (center = a)

$$f(z) = f(a) + f'(a)\cdot(z-a) + \frac{(z-a)^2}{\lfloor 2}f''(a) + \frac{(z-a)^3}{\lfloor 3}f'''(a) + \cdots\cdots$$

From cauchy's integral theorem,



$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \qquad z \to w,\ a \to z$$

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z)}\,dw \quad \text{--- (1)}$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left\{1-\frac{z-a}{w-a}\right\}} \qquad (1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\left(\frac{z-a}{w-a}\right)^3+\cdots\cdots + \left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^{n+\text{too}}+\left(\frac{z-a}{w-a}\right)^{n+1}+\cdots\cdots\right]$$

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \frac{(z-a)^n}{(w-a)^n} \times \frac{1}{w-a}$$

$$\frac{1}{w-a} \times \frac{(z-a)^n}{(w-a)^n} \left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots \right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} +$$

$$\frac{1}{w-a} \left(\frac{z-a}{w-a}\right)^n \left[1 - \frac{z-a}{w-a} \right]^{-1}$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} +$$

$$\frac{1}{w-a} \left(\frac{z-a}{w-a}\right)^n \frac{1}{1 - \frac{z-a}{w-a}}$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} +$$

$$\frac{1}{w-a} \left(\frac{z-a}{w-a}\right)^n \times \frac{1}{\frac{w-a-z+a}{w-a}}$$

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} +$$

$$\cdots + \left(\frac{z-a}{w-a}\right)^n \times \frac{1}{w-z}$$

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw +$$

$$+ \cdots\cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n$$

where $U_n = \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\times\frac{1}{w-z}\,dw$

$$\Rightarrow f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)}{\lfloor 2}f''(a) + \cdots + \frac{(z-a)^{n-1}}{\lfloor n-1}f^{n-1}(a) + U_n$$

$n \to \infty$, $|U_n| \to 0$

$$|U_n| = \left|\oint_C \frac{1}{2\pi i}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw\right|$$

$$\Rightarrow |U_n| < \frac{1}{2\pi}\times 2\pi r_1 \frac{M\cdot\gamma^n}{|w-z|}$$

$$\Rightarrow |U_n| < \frac{\gamma^n M r_1}{|w-z|}$$

here,

$$|w-z| = |(w-a)-(z-a)|$$

$$= |r_1 - (z-a)|$$

$\Rightarrow \frac{1}{|w-z|} \times$

$|i| = \sqrt{0^2+1^2}$
$= 1$.

$|f(w)| <$ constant
কোনো func কে mod নিলে তা কোনো constant অপেক্ষা ছোট তা বোঝায়.

$\oint_{C_1} dw = 2\pi r_1$

$\left|\frac{z-a}{w-a}\right| = \gamma < 1$.

$|w-z| \geq |r_1| - |z-a|$

r_1 = length

$$\therefore \frac{1}{|w-z|} < \frac{1}{r_1 - |z-a|}$$

$$|U_n| < \frac{\gamma^n M r_1}{r_1 - |z-a|}$$

$$\lim_{n\to\infty} |U_n| < 0$$

(proof)

Page – 6.19

Ex – 6.23

a) $f(z) = \ln(1+z)$

$z = 0$

$f(0) = \ln 1 = 0$

$f'(z) = \frac{1}{1+z}$ $\therefore f'(0) = 1$

$f''(z) = -1 \cdot (1+z)^{-2}$ $\therefore f''(0) = -1$

$f'''(z) = (-1)(-2)(1+z)^{-3}$ $\therefore f'''(0) = \lfloor 2$

$= \frac{\lfloor 2}{(1+z)^3}$

- - - - - - - - - - -

$$f^{n+1}(z) = \frac{(-1)^n \lfloor n}{(1+z)^{n+1}} \qquad \therefore f^{n+1}(0) = \frac{(-1)^n \lfloor n}{(1)^{n+1}} = (-1)^n \lfloor n$$

$$f(z) = \ln(1+z) = f(0) + f'(0)z + z^2 \frac{f''(0)}{2!} + z^3 \frac{f'''(0)}{\lfloor 3} + \cdots$$

$$= 0 + z - \frac{z^2}{\lfloor 2} + \frac{2z^3}{\lfloor 3} - \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

2nd method:

$$\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

integrating both side we get,

$$\therefore \int_0^z \frac{1}{1+z}\,dz = \int_0^z (1 - z + z^2 - z^3 + \cdots)$$

$$\Rightarrow \left[\ln(1+z)\right]_0^z = \left[\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right)\right]_0^z$$

$$\therefore \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

b) convergence → যদি finite মান আসে test = $\frac{U_{n+1}}{U_n}$

divergence → " infinite " "

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$U_n = \frac{(-1)^{n-1} \cdot z^n}{n}$$

$\lim_{n\to\infty} |U_n|$

$$\frac{U_{n+1}}{U_n} = \frac{\frac{(-1)^n z^{n+1}}{n+1}}{\frac{(-1)^{n-1} z^n}{n}}$$

$$= \frac{n \cancel{(-1)^n} \cancel{z^n} \cdot z}{(n+1) \cancel{(-1)^n} \cdot (-1)^{-1} \cancel{z^n}}$$

$$= \frac{nz}{-(n+1)}$$

$$\therefore \left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{nz}{(n+1)}\right|$$

$$\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{nz}{(n+1)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\cancel{n}z}{\cancel{n}(1+\frac{1}{n})}\right| = \lim_{n\to\infty} \left|\frac{z}{1+\frac{1}{n}}\right|$$

$$= |z| < 1.$$

c) $\ln\left(\frac{1+z}{1-z}\right)$ when $z=0$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

$$\ln(1+z) - \ln(1-z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

$$\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \frac{2 \cdot z^{2n+1}}{2n+1}$$

$\boxed{\text{Page-6.24}}$

$\boxed{\text{Ex-6.27}}$ $f(z) = \frac{1}{(z+1)(z+3)}$ expand the function.

a) $1 < |z| < 3$.

$$f(z) = \frac{1}{(z+1)(-1+3)} + \frac{1}{(-3+1)(z+3)}$$

$$= \frac{1}{2(z+1)} + \frac{1}{-2(z+3)}$$

$$= \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2} \times \left(\frac{1}{z+1}\right) - \frac{1}{2} \times \left(\frac{1}{z+3}\right)$$

Here,

$1 < |z| < 3$ $\qquad$ $|z| < 3$

$|z| > 1$ $\qquad$ $\therefore \frac{1}{|z|} > 3$

$\frac{1}{|z|} < 1$

$$\cancel{\frac{1}{2(z+1)} = \frac{1}{2z\left(1+\frac{1}{z}\right)}}$$

$$f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2} \times \frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

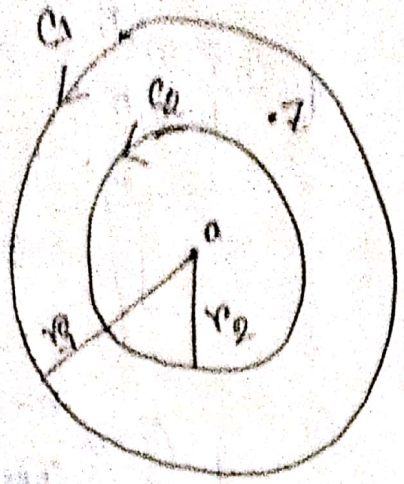
$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} \cdots$$

$$= \cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} + \cdots$$

Laurent's Theorem:

8-09-2022
Thursday



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)\,dz}{z-z_0} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)\,dz}{z-z_0}$$

centre $= a$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \quad \text{when } r_1 > r_2.$$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{(w-a)^{n+1}}, \quad a_{-n} = \oint_{C_2} \frac{1}{2\pi i} \cdot \frac{f(w)\,dw}{(w-a)^{-n+1}}$$

$n = 0 \ldots\ldots$ $\qquad$ $n = 1 \ldots\ldots$

Proof:

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z)}\,dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-z)}\,dw \qquad (1)$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3 + \cdots + \right.$$

$$\left(\frac{z-a}{w-a}\right)^n + \left(\frac{z-a}{w-a}\right)^{n+1} + \left(\frac{z-a}{w-a}\right)^{n+2} + \cdots\cdots \Bigg]$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{(w-a)^2} + \frac{(z-a)^3}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^n}\right.$$

$$\left.\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\cdots\right]\right]$$

$$= \frac{1}{w-a} + \frac{(z-a)}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \frac{(z-a)^3}{(w-a)^4} + \cdots\cdots + \frac{(z-a)^{n-1}}{(w-a)^n} +$$

$$\frac{(z-a)^n}{(w-a)^{n+1}} + \frac{1}{w-a}\left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$\therefore \frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} +$$

$$\frac{(z-a)^n}{(w-a)} \times \frac{1}{w-z}$$

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw = \oint_{C_1} \frac{1}{2\pi i}\,\frac{f(w)\,dw}{(w-a)} + \frac{(z-a)}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{(w-a)^2} +$$

$$\frac{(z-a)^2}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{(w-a)^n}$$

$$+ \frac{(z-a)^n}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{(w-a)^n(w-z)}$$

$$= a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots + (z-a)^{n-1} a_{n-1} + \psi$$

~~where a_0~~

$$\therefore \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{w-z} = a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots + (z-a)^{n-1} a_{n-1}$$

$$+ \psi_n$$

here,

$$\frac{-1}{w-z} = \frac{1}{z-w} = \frac{1}{(z-a)-(w-a)}$$

$$= \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)} = \frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$\cancel{= \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \frac{w-a}{z-a} + \frac{w-a}{z-a} + \cdots + \frac{(w-a)^n}{(z-a)^n}\right.}$$

$$= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3} + \cdots + \frac{(w-a)^{n-1}}{(z-a)^n}$$

$$+ \frac{(w-a)^n}{(z-a)^{n+1}}$$

$$\Rightarrow \frac{-1}{2\pi i}\oint_{C_2} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)\,dw}{z-a} + \frac{(w-a)}{2\pi i}\oint_{C_2} \frac{f(w)\,dw}{(z-a)^2}$$

$$+ \frac{(w-a)^2}{2\pi i}\oint_{C_2} \frac{f(w)\,dw}{(z-a)^3} + \cdots + \frac{(w-a)^{n-1}}{2\pi i}\oint_{C_2} \frac{f(w)\,dw}{(z-a)^n}$$

$$+ \psi_n$$

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dz = \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}$$

V_n

$$\therefore f(z) = \Big[a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{n+1}(z-a)^{n-1}$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + U_n + V_n\Big]$$

Now,

U_n

$$|V_n| = \left|\frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right|$$

$$\Rightarrow |V_n| \le \frac{1}{2\pi} \times 2\pi r_2 \times \frac{M\gamma^n}{|z-w|}$$

$$\Rightarrow |V_n| \le \frac{M\gamma^n r_2}{|z-w|}$$

$$|z-w| = |(z-a)-(w-a)|$$

$$= |(z-a)-r_2|$$

$$\Rightarrow |V_n| \le \frac{M\gamma^n r_2}{|(z-a)-r_2|}$$

$$\Rightarrow |V_n| \le \frac{M\gamma^n r_2}{|z-a|-r_2}$$

$\lim_{n\to\infty} \leq 0$

P- 6.22

12-09-2022
Monday

Ex- 6.26

$$\frac{e^{2z}}{(z-1)^3} \; ; \; z=1 \quad \text{find Lorenge series.}$$

let,
$\therefore u = z-1$
$\therefore z = u+1$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+u)}}{u^3} = \frac{e^2 \cdot e^{2u}}{u^3}$$

$$= \frac{e^2}{u^3}\left[1 + \frac{2u}{\lfloor 1} + \frac{(2u)^2}{\lfloor 2} + \frac{(2u)^3}{\lfloor 3} + \frac{(2u)^4}{\lfloor 4} + \cdots\right]$$

$$= \frac{e^2}{u^3}\left[1 + \frac{2u}{1} + \frac{2u^2}{1} + \frac{4u^3}{3} + \frac{2u^4}{3} + \cdots\right]$$

$$= \frac{e^2}{u^3}\left[\frac{1}{u^3} + \frac{2}{u^2} + \frac{2}{u} + \frac{4}{3} + \frac{2u}{3} + \cdots\right]$$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$\boxed{6.26\ (b)}$

$(z-3)\sin\frac{1}{z+2}$; $z=-2$.

let,

$z+2=u$

$\therefore z=u-2$

$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u}$$

$$= (u-5)\left[\frac{1}{u} - \frac{1}{\lfloor 3\, u^3} + \frac{1}{\lfloor 5\, u^5} - \frac{1}{\lfloor 7\, u^7} + \cdots\right]$$

$$= \frac{u-5}{u} - \frac{u-5}{6u^3} + \frac{u-5}{120u^5} - \frac{u-5}{5040u^7} + \cdots$$

$$= 1 - \frac{5}{u} - \frac{1}{6u^2} + \frac{5}{6u^3} + \frac{1}{120u^4} - \frac{1}{24u^5} - \frac{1}{5040u^6} + \frac{1}{1008u^7} + \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^2} + \frac{1}{120(z+2)^4} - \frac{1}{24(z+2)^5} - \frac{1}{5040(z+2)^6} + \frac{1}{1008(z+2)^7} + \cdots$$

$x=-2$ the series is essential singularity and convergence.

P-6.24

Ex-6.27

b) $f(z) = \dfrac{1}{(z+1)(z+3)}$

$= \dfrac{1}{u(u+2)}$

$= \dfrac{1}{2u\left(1+\frac{u}{2}\right)}$

$= \dfrac{1}{2u}\left(1+\dfrac{u}{2}\right)^{-1}$

$= \dfrac{1}{2u}\left(1 - \dfrac{u}{2} + \dfrac{u^2}{4} - \dfrac{u^3}{8} + \dfrac{u^4}{16} - \cdots\right)$

$= \dfrac{1}{2u} - \dfrac{1}{4} + \dfrac{u}{8} - \dfrac{u^2}{16} + \dfrac{u^3}{32} - \cdots$

$= \dfrac{1}{2(z+1)} - \dfrac{1}{4} + \dfrac{z+1}{8} - \dfrac{(z+1)^2}{16} + \dfrac{(z+1)^3}{32} + \cdots$

$0 < |z+1| < 2$

$u = z+1$

$0 < |u| < 2$

$\therefore z = u-1$

$0 < |u| < 2$

$u < 2$

$\dfrac{u}{2} < 1$

Chapter-7

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বৃহস্পতিবার

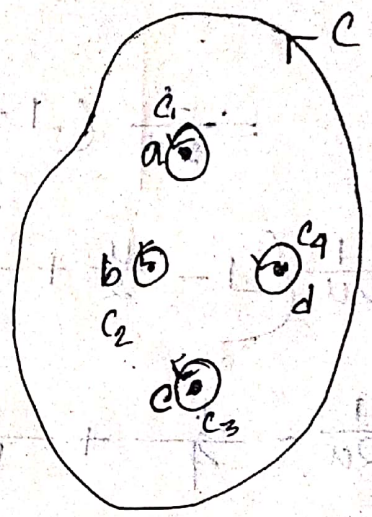
Page-7.7

Theorem – 7.2

$$\oint_C f(z) = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1}$$



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots$$

$$a_{-1} = \oint_{C_1} \frac{1}{2\pi i} f(z)\,dz$$

$$\therefore \oint_{C_1} f(z)\,dz = 2\pi i\, a_{-1}$$

$$\oint_{C_2} f(z)\,dz = 2\pi i\, b_{-1}$$

$$\oint_{C_3} f(z)\,dz = 2\pi i\, c_{-1}$$

$$\oint_{C_3} f(z)\,dz = 2\pi i\, d_{-1}$$

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1} + 2\pi i\, b_{-1} + 2\pi i\, c_{-1} + 2\pi i\, d_{-1} + \cdots$$

$$= 2\pi i\left(a_{-1} + b_{-1} + c_{-1} + d_{-1} + \cdots\right)$$

$(z-a)^m$ $z=a$ is a pole of order m.

$\boxed{7.3}$

$$a_{-1} = \lim_{z \to a} \frac{1}{\lfloor m-1} \times \frac{d^{m-1}}{dz^{m-1}} \left\{(z-a)^m f(z)\right\}$$

$\boxed{\text{Page} - 7.7}$

$\boxed{\text{Theorem} - 7.3}$

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots$$

$$\Rightarrow (z-a)^m f(z) = a_m + a_{m+1}(z-a) + \cdots + a_{-2}(z-a)^{m-2} + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + a_2(z-a)^{m+2} + a_3(z-a)^{m+3} + \cdots$$

result 0

$$\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z) = \lfloor m-1 \; a_{-1} + a_0(z-a)\, m(m-1)\cdots 2 + a_1(z-a)^2 (m+1)\, m(m-1)\cdots 3 + \cdots$$

$$\lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\} = a_{-1}\,\lfloor m-1$$

$$\therefore\ a_{-1} = \lim_{z\to a} \frac{1}{\lfloor m-1} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

Page 7.8 → Math 7.4 (a)

19-09-22
Monday

Page - 7.8

Ex - 7.4 (a)

$$f(z) = \frac{z^2-2z}{(z+1)^2 (z^2+4)}$$

$z = -1, -1$
↳ double pole

$z = 2i, -2i$
↳ single pole

$$(z+1)^2 (z^2+4) = 0$$

$$\therefore z = -1, -1, 2i, -2i$$

Hence $z=-1$ the double pole and $z=2i$ and $z=-2i$ are single pole.

Residue at $z=-1$ is a_{-1}

$$\therefore a_{-1} = \lim_{z\to -1} \frac{1}{\lfloor 1} \times \frac{d}{dz} \cancel{(z+1)^2} \times \frac{z^2-2z}{\cancel{(z+1)^2}(z^2+4)}$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)$$

$$a_{-1} = \lim_{z \to -1} \frac{(z^2+4)\cdot(2z-2) - (z^2-2z)\cdot 2z}{(z^2+4)^2}$$

$$= \lim_{z \to -1} \frac{\quad}{(z^2+4)^2}$$

$$= \frac{(1+4)(-2-2) - (1+2)\cdot 2\cdot -1}{(1+4)^2}$$

$$= \frac{-20+6}{25}$$

$$= \frac{-14}{25}.$$

$z^2+4=0$

$\Rightarrow z^2 = -4i^2$

$\Rightarrow z = \pm 2i$

$\Rightarrow (z+2i)(z-2i)=0$

Residue at $z=2i$ is b_{-1}

$$b_{-1} = \cancel{\lim_{z \to 2i} \left\{ (z-2i) \frac{(z^2-2z)}{(z+1)^2 (z+2i)(z+4)} \right\}}$$

$$= \lim_{z \to 2i} \left\{ (z-2i) \frac{z^2-2z}{(z+1)^2 (z^2+4)} \right\}$$

$$= \lim_{z \to 2i} \frac{\cancel{(z-2i)}\cdot(z^2-2z)}{(z+1)^2 \cancel{(z-2i)} (z+2i)}$$

$$= \frac{4i^2 - 2\cdot 2i}{(2i+1)^2 \cancel{(4i^2+4)} (2i+2i)}$$

$$= \frac{4i^2 - 4i}{(4i^2 + 4i + 1) \cdot 4i}$$

$$= \frac{-4 - 4i}{(-4 + 4i + 1) \cdot 4i}$$

$$= \frac{-\cancel{4}(1+i)}{(-3+4i) \cdot \cancel{4}i}$$

$$= \frac{-(1+i)}{(4i-3)i}$$

$m = \text{pole}$

$z = (-1+i)$

$z = -1 - i \rightarrow \{z - (-1-i)\}$

$\{z - (-i+i)\}$

$$= \frac{-(1+i)}{4i^2 - 3i} = \frac{-(1+i)}{-4-3i} = \frac{1+i}{4+3i}$$

$$= \frac{(1+i)(4-3i)}{(4+3i)(4-3i)} = \frac{(1+i)(4-3i)}{4^2 - (3i)^2} = \frac{4 - 3i + 4i + 3}{16 + 9}$$

$$= \frac{7+i}{25}$$

Residue at $z = -2i$ is C_{-1}

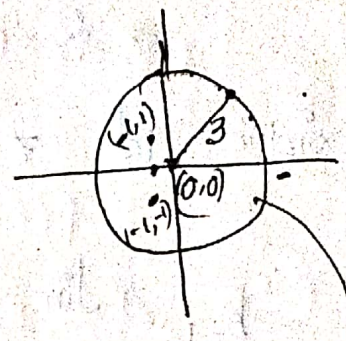
$$C_{-1} = \lim_{z \to -2i} \left\{ (z+2i) \frac{(z^2 - 2z)}{(z+1)^2 (z^2+4)} \right\}$$

$$= \frac{7-i}{25}$$

Page-7.10

Ex - 7.6

Evaluate $\frac{-1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$; $|z|=3$.



$z^2(z^2+2z+2)=0$

$\therefore z=0$ is double pole
$(0,0)$

$z^2+2z+2=0$

$\therefore z=\frac{-2\pm\sqrt{4-4\times1\times2}}{2}=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}$

$=\frac{-2\pm2i}{2}=-1\pm i$

$\therefore z=-1+i$ and $z=-1-i$ are simple pole.
$(-1,1)$ $(-1,-1)$

All the poles are inside $|z|=3$.

residue at $z=0$ is a_{-1}

$a_{-1}=\lim_{z\to0}\frac{1}{1!}\frac{d}{dz}(z-0)^2\cdot\frac{e^{zt}}{z^2(z^2+2z+2)}$

$=\lim_{z\to0}\frac{d}{dz}\frac{e^{zt}}{z^2+2z+2}$

$=\lim_{z\to0}\frac{(z^2+2z+2)\,t\cdot e^{zt}-e^{zt}\cdot(2z+2)}{(z^2+2z+2)^2}$

$=\frac{2t-1\cdot2}{2^2}$

$$= \frac{2(t-1)}{4}$$

$$= \frac{t-1}{2}.$$

residue at $z = -1+i$ is b_{-1},

$$b_{-1} = \lim_{z \to -1+i} \left[\left\{ z-(-1+i) \right\} \cdot \frac{e^{zt}}{z^2(z^2+2z+2)} \right] \text{ from } \frac{0}{0}$$

$$= \cancel{\lim_{z \to -1+i}} \frac{\cancel{te^{zt}}}{\cancel{z^2(2z+2)+2z(z^2+2z+2)}}$$

$$= \lim_{z \to -1+i} \left\{ \cancel{\{z-(-1+i)\}} \frac{e^{zt}}{z^2 \cancel{\{z-(-1+i)\}} \{z-(-1-i)\}} \right.$$

$$= \lim_{z \to -1+i} \frac{e^{zt}}{z^2 \{z-(-1-i)\}}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2 \{\cancel{-1}+\cancel{i}-\cancel{1}+\cancel{i}\}} = \frac{e^{-t+it}}{(i-1)^2 \cdot 2i}$$

$$= \frac{e^{-t} \cdot e^{it}}{(i^2-2i+1) \cdot 2i} = \frac{e^{-t} \cdot e^{it}}{-2i \cdot 2i}$$

$$\therefore b_{-1} = \frac{(e^{-1+i})^t}{4}$$

~~Need~~ residue at $z=-1-i$ is c_{-1},

$$c_{-1} = \lim_{z \to -1-i} e^{-i}$$

$$c_{-1} = \frac{e^{(-1-i)t}}{4}$$

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \oint 2\pi i\left(a_{-1}+b_{-1}+c_{-1}\right)$$

$$\therefore \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = a_{-1}+b_{-1}+c_{-1}$$

$$= \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2} + \frac{1}{4}\left\{e^{-t}\cdot e^{it} + e^{-t}\cdot e^{-it}\right\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\left\{e^{it} + e^{-it}\right\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\left\{\frac{e^{it}+e^{-it}}{2}\right\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\cos t$$

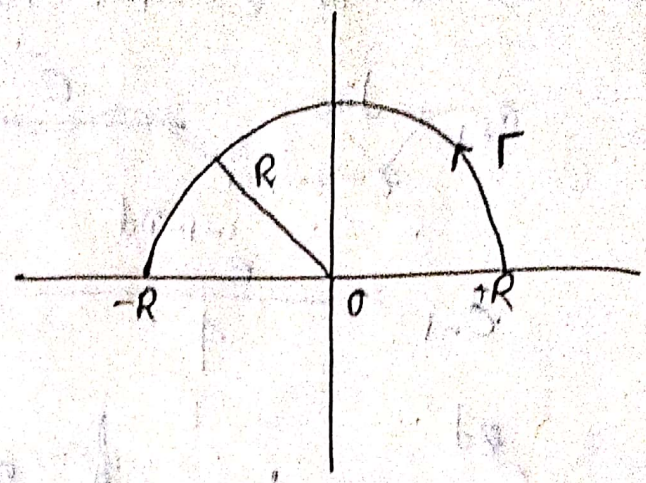
$$\boxed{\begin{aligned} \sin x &= \frac{e^{ix}-e^{-ix}}{2i} \\ \cos x &= \frac{e^{ix}+e^{-ix}}{2} \end{aligned}}$$

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

P-7.11

Ex-7.7



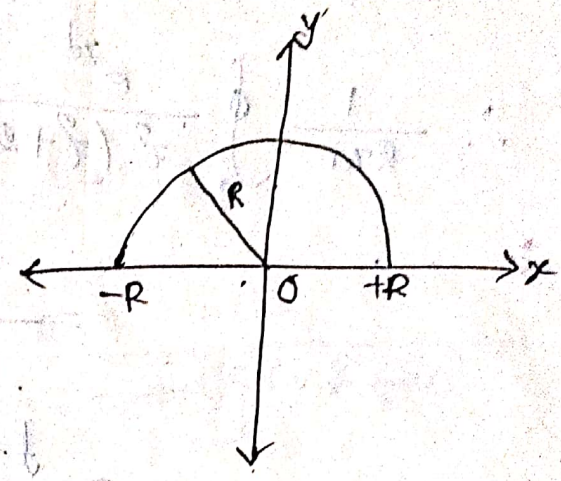
Id

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0$$

$$\boxed{\lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0}$$

Ex-7.9

$$\int_0^{\infty} \frac{dz}{x^6+1}$$



Id,

$$\oint_C \frac{dz}{z^6+1}$$

here

$$z^6+1=0 \Rightarrow z^6=-1 \Rightarrow z^6 = \cos\pi + i\sin\pi$$

$$\Rightarrow z^6 = \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$$

$$\Rightarrow z^6 = \cos(2n+1)\pi + i\sin(2n+1)\pi$$

$$\Rightarrow z = \left\{\cos(2n+1)\pi + i\sin(2n+1)\pi\right\}^{\frac{1}{6}}$$

$$\Rightarrow z = \cos(2n+1)\frac{\pi}{6} + i\sin(2n+1)\frac{\pi}{6} \text{ where}$$

$$n = 0, 1, 2 \cdots 5$$

$\Rightarrow z = e^{i(2n+1)\frac{\pi}{6}}$

~~when n = 0~~

$\therefore z_1 = e^{i\pi/6}$

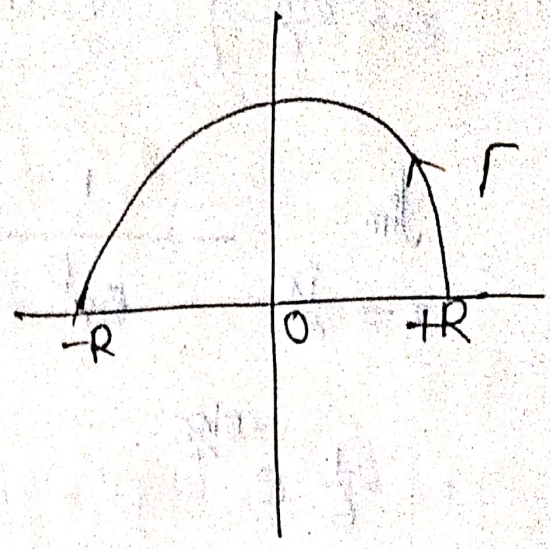
$z_2 = e^{i\pi/2}$

$z_3 = e^{\frac{i5\pi}{6}}$

$z_4 = e^{\frac{7i\pi}{6}}$

$z_5 = e^{\frac{9i\pi}{6}}$

$z_6 = e^{\frac{i11\pi}{6}}$



The point z_1, z_2, z_3 is insid Γ and z_4, z_5, z_6 is outside Γ. z_1, z_2, z_3 are simple pole.

Residue at $z_1 = e^{i\pi/6}$ is a_{-1}

$$a_{-1} = \lim_{z \to \frac{i\pi}{6}} \left\{ \frac{z - e^{i\pi/6}}{z^6+1} \right\}; \text{ form } \frac{0}{0}$$

$$= \lim_{z \to \frac{i\pi}{6}} \frac{1}{6z^5}$$

$$= \frac{1}{6\left(\frac{i\pi}{6}\right)^5} = \cancel{\frac{1}{6 \times \frac{\pi^5}{}}}$$

$$= \frac{1}{6} e^{-5i\pi/6}$$

$$b_{-1} = \lim_{z \to -\frac{5\pi i}{6}} \left\{ \frac{z - e^{-5\pi i/6}}{z^6+1} \right\} \quad \text{form } \frac{0}{0}$$

$$= \lim_{z \to -\frac{5\pi i}{6}} \frac{1}{5z^5}$$

$$= \frac{1}{6} e^{-5\pi i/2}$$

$$c_{-1} = \frac{1}{6} e^{-25\pi i/6}$$

$$\oint_C \frac{dz}{z^6+1} = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} \right)$$

$$= 2\pi i \left(\frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right)$$

$$\therefore \oint_C \frac{dz}{z^6+1} = \frac{\pi i}{3} \left(e^{-5\pi i/6} + e^{-5\pi i/2} + e^{-25\pi i/6} \right).$$

$$\Rightarrow \int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{\pi i}{3} \left[\cos\frac{5\pi}{6} - i \sin\frac{5\pi}{6} + \cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2} + \cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6} \right]$$

$$= \frac{\pi i}{3} \left[-\cancel{\frac{\sqrt{3}}{2}} - \frac{1}{2} i + 0 - i + \cancel{\frac{\sqrt{3}}{2}} - \frac{1}{2} i \right]$$

$$= \frac{-2\pi i^2}{3} = \frac{2\pi}{3}$$

$$\therefore \int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} + \lim_{R\to\infty} \int_{\Gamma} \frac{dz}{z^6+1} = \lim_{R\to\infty} \frac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$$

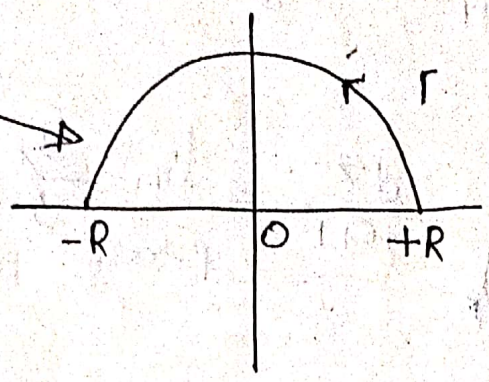
$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3} \quad \text{(Ans)}$$

$\boxed{\text{Ex-7.10, 7.11}}$

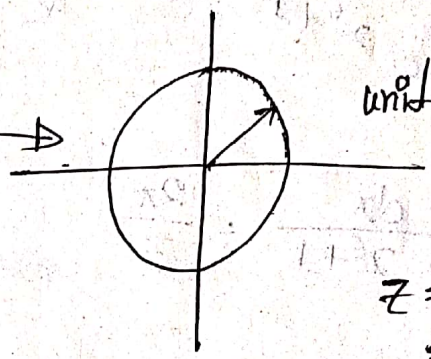
26-09-22
Monday

Algebraic function:



semi-circle.

trigonometric



unit circle

$$z = e^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Page - 7.15

Ex - 7.14

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$$

$$= \int_0^{2\pi} \frac{d\theta}{3 - 2\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$$e^{i\theta} = z$$

$$\Rightarrow i e^{i\theta} d\theta = dz$$

$$\therefore d\theta = \frac{dz}{i e^{i\theta}}$$

$$= \frac{dz}{iz}$$

$$= \int_0^{2\pi} d$$

$$= \oint_C \frac{\frac{dz}{iz}}{3 - (z + z^{-1}) + \frac{1}{2i}(z - z^{-1})}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\therefore \quad = \frac{z + z^{-1}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \frac{z - z^{-1}}{2i}$$

$$= \oint_C \frac{dz}{\cancel{iz}\left[\frac{6i - 2zi - 2iz^{-1} + z - z^{-1}}{\cancel{2i}}\right]}$$

$$= \oint_C \frac{2dz}{z\left[6i - 2iz - \frac{2i}{z} + z - \frac{1}{z}\right]}$$

$$= \oint_C \frac{2dz}{z\left[\frac{6iz - 2iz^2 - 2i + z^2 - 1}{z}\right]}$$

$$= \oint_C \frac{2dz}{z^2(-1 - 2i) + 6iz - 1 - 2i}$$

here,

$$z^2(1 - 2i) + 6iz - 1 - 2i = 0$$

$$\therefore z = \frac{-6i \pm \sqrt{(6i)^2 + 4(1+2i)(1-2i)}}{2(1-2i)}$$

$$= \frac{-6i \pm \sqrt{-36 + 4(1+4)}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)}$$

$$= \frac{-3i \pm 2i}{1-2i}$$

(+)ve,

$$z = \frac{-3i+2i}{1-2i}$$

$$= \frac{-i}{1-2i}$$

$$= \frac{-i(1+2i)}{1+4}$$

$$= \frac{-i(1+2i)}{5}$$

$$= \frac{2-i}{5}$$

(−)ve,

$$z = \frac{-3i-2i}{1-2i}$$

$$= \frac{-5i}{(1-2i)}$$

$$= \frac{-5i(1+2i)}{1+4}$$

$$= \frac{-\cancel{5}i(1+2i)}{\cancel{5}}$$

$$= 2-i$$

Recedue at $z = \frac{2-i}{5}$

$$a_{-1} = \lim_{z \to \frac{2-i}{5}} \left[\left(z - \frac{2-i}{5}\right) \frac{2}{(1-2i)z^2 + 6iz - 1 - 2i} \right]$$

form $\frac{0}{0}$

$$= \lim_{z \to \frac{2-i}{5}} \left[\frac{2}{2z(1-2i) + 6i} \right]$$

$$= \frac{\cancel{2}}{\cancel{2}\left[\left(\frac{2-i}{5}\right)(1-2i) + \cancel{5}i\right]}$$

$$= \frac{2}{\frac{1}{5}\left[(2-i)(1-2i) + 5 \cdot 15i\right]}$$

$$= \frac{5}{\cancel{2} - 4i - i - \cancel{2} + 5 \cdot 15i}$$

$$= \frac{5}{-5i + 5 \cdot 15i}$$

$$= \frac{5}{5 \cdot 10i}$$

$$= \frac{1}{2i}$$

P-7.10 P-7.11

H.W → 7.16, 7.17, 7.18, 7.19

22-09-22
Thursday

P-7.20

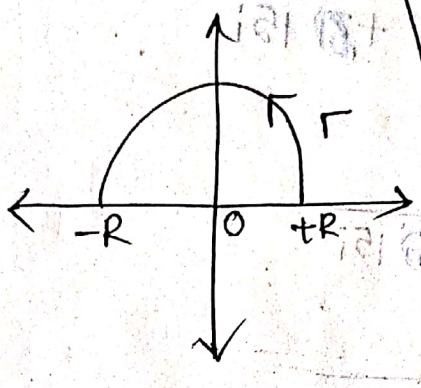
Ex-7.22

Algebric/ trigonometric → ~~[illegible]~~

show that, $\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}$, $m>0$

consider,

$$\oint_C \frac{e^{imz}}{z^2+1}\,dz$$



$e^{ix} = \cos x + i\sin x$

$\cos x$ = Real part of e^{ix}

$= \text{Real}(e^{ix})$.

~~where~~ e^{is}

here,

$z^2+1 = 0$

$\therefore z = \pm i$

residue at $z=i$ is,

$$a_{-1} = \lim_{z\to i}\left[(z-i)\frac{e^{imz}}{z^2+1}\right]$$

$$= \lim_{z\to i}\frac{\cancel{(z-i)}\,(e^{imz})}{(z+i)\,\cancel{(z-i)}}$$

$$= \frac{e^{-m}}{2i}$$

$$\oint_C \frac{e^{miz}}{z^2+1}\,dz = 2\pi i\, a_{-1}$$

$$= 2\pi i \times \frac{e^{-m}}{2i}$$

$$= \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{+R} \frac{e^{mix}}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{+R} \frac{\cos mx + i \sin mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + \lim_{R\to\infty} \int_{-R}^{R} \frac{i \sin mx}{x^2+1}\,dx + \lim_{R\to\infty} \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin mx}{x^2+1}\,dx + 0 = \pi e^{-m}$$

equating real part we get,

$$\int_{-\infty}^{+\infty} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

$$\Rightarrow 2\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

$$\therefore \int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m}$$

P – 7.24	P – 7.28	P – 7.26
Ex – 7.27	Ex 7.31	Ex – 7.29

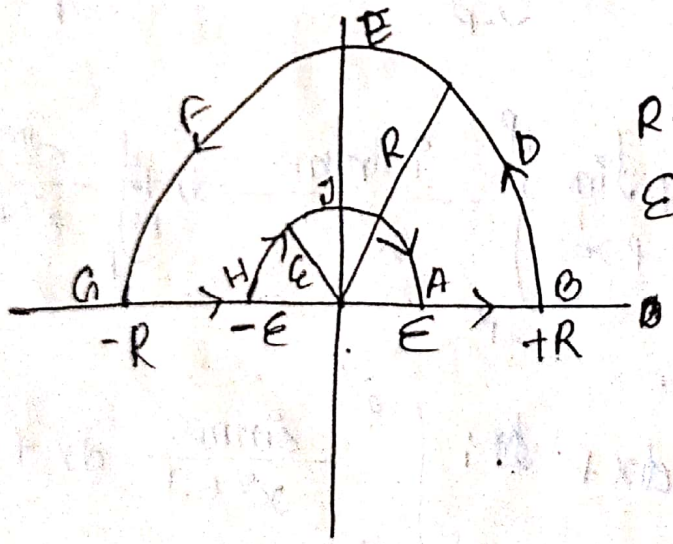
ব্যাখ্যাসহকারী

$\boxed{\text{P-7.25}}$

$\boxed{\text{Ex-7.27}}$

ব্যাখ্যাসহকারী

Prove that, $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.



$R \to \infty$

$\varepsilon \to 0$

$e^{ix} = \cos t + i\sin x$

$e^{iz} = \cos z + i\sin z$

$z = x + iy$

$z = 0$

$x = 0$, $y = 0$ $(0,0)$

$x = -x$

$dx = -dx$

$$\oint \frac{e^{iz}}{z}\,dz = 0$$

$$\Rightarrow \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{BCEFG} \frac{e^{iz}}{z}\,dz = 0$$

$$\Rightarrow -\int_{\varepsilon}^{R} \frac{e^{-ix}}{x}(-dx) + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

$$\Rightarrow \int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

$$\Rightarrow 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx - \int_{HJA} \frac{e^{iz}}{z}\,dz - \int_{BDEFG} \frac{e^{iz}}{z}\,dz$$

Let, $R \to \infty$, $\varepsilon \to 0$ then, $\int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$

$z = \varepsilon e^{i\theta}$ $\qquad \theta = \pi$ to 0

$\therefore dz = i\varepsilon e^{i\theta}\,d\theta$

Now

$$2i\int_{0}^{\infty} \frac{\sin x}{x}\,dx = -\lim_{\varepsilon \to 0}\int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\cancel{\varepsilon} e^{i\theta}} \cdot i\cancel{\varepsilon} e^{i\theta}\,d\theta$$

$$= -\lim_{\varepsilon \to 0} i\int_{\pi}^{0} e^{i\varepsilon e^{i\theta}}\,d\theta$$

$$= -i\int_{\pi}^{0} e^{0}\,d\theta = -i\int_{\pi}^{0} d\theta$$

$$\Rightarrow 2i\int_{0}^{\infty} \frac{\sin x}{x}\,dx = -i\left[\theta\right]_{\pi}^{0} = -i(0-\pi) = \pi i$$

$$\therefore \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

$\boxed{P-7.26}$

$\boxed{Ex-7.29}$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}; \quad 0<p<1.$$

$1+z=0$

$\therefore z=-1$

$$a_1 = \lim_{z\to -1} (z+1)\frac{z^{p-1}}{1+z}$$

$$= (-1)^{p-1}$$

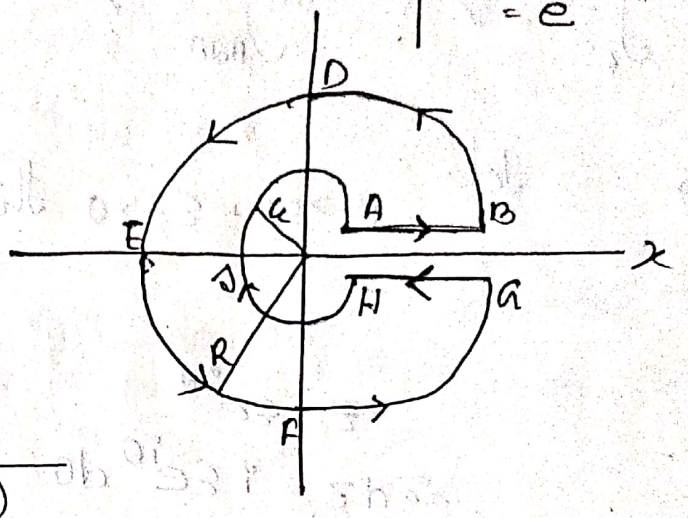
$$= (\cos\pi + i\sin\pi)^{p-1}$$

$$= (e^{i\pi})^{p-1}$$

$$= e^{(p-1)i\pi}$$

here,

$$\oint_C \frac{z^{p-1}}{1+z}\,dz$$



here,

$1+z=0 \quad \therefore z=-1$

$$a_1 = \lim_{z\to -1} (1+z)\frac{z^{p-1}}{(1+z)}$$

$$= \lim_{z\to -1} z^{p-1} = (-1)^{p-1} = (\cos\pi + i\sin\pi)^{p-1}$$

$$= (e^{i\pi})^{p-1} = e^{(p-1)i\pi}$$

$z = Re^{i\theta}$

$\therefore dz = iRe^{i\theta}d\theta$

$R\to\infty$

$\varepsilon\to 0$

Now,

$$\oint_C \frac{z^{p-1}}{1+z}\,dz = e^{(p-1)i\pi}$$

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)i\pi}$$

$$\int_{\varepsilon}^{R} \frac{x^{p-1}}{1+x}\,dx + \int_{BDEFG} \frac{z^{p-1}}{1+z}\,dz + \int_{R}^{\varepsilon} \frac{x}{1+x}\,dx + \int_{HJA} \frac{z^{p-1}}{z+1}\,dz = 2\pi i e^{(p-1)i\pi}$$

$$\Rightarrow \int_{\varepsilon}^{R} \frac{x^{p-1}}{1+x}\,dx + \int_{0}^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} \times Rie^{i\theta}\,d\theta + \int_{R}^{\varepsilon} \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx$$

$$+ \int_{2\pi}^{0} \frac{(Re^{i\theta})^{p-1} \cdot Rie^{i\theta}\,d\theta}{1+\varepsilon e^{i\theta}} = 2\pi i e^{(p-1)i\pi}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^{p-1}}{1+x}\,dx + \int_{\infty}^{0} \frac{x^{p-1} \cdot e^{2\pi i(p-1)}}{1+x} = 2\pi i e^{(p-1)i\pi}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^{p-1}}{1+x}\,dx - \int_{0}^{\infty} \frac{x^{p-1} \cdot e^{(p-1)2\pi i}}{1+x} = 2\pi i e^{(p-1)i\pi}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x^{p-1}}{1+x}\,dx \cdot \left(1 - e^{(p-1)2\pi i}\right) = 2\pi i e^{(p-1)\pi i}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i\, e^{(p-1)i\pi}}{1-e^{2\pi i(p-1)}}$$

$$\boxed{e^{-2\pi i} = 1}$$

$$\boxed{e^{\pi i} = -1}$$

$$\Rightarrow \quad = \frac{2\pi i \cdot e^{(p-1)\pi i} \cdot e^{-(p-1)\pi i}}{e^{-(p-1)\pi i} - e^{-(p-1)\pi i} \cdot e^{(p-1)2\pi i}}$$

$$= \frac{2\pi i}{e^{-p\pi i} \cdot e^{\pi i} - e^{-p\pi i} \cdot e^{\pi i} \cdot e^{2p\pi i} \cdot e^{-2\pi i}}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i}{-e^{-p\pi i} + e^{p\pi i}}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\dfrac{e^{p\pi i} - e^{-p\pi i}}{2i}}$$

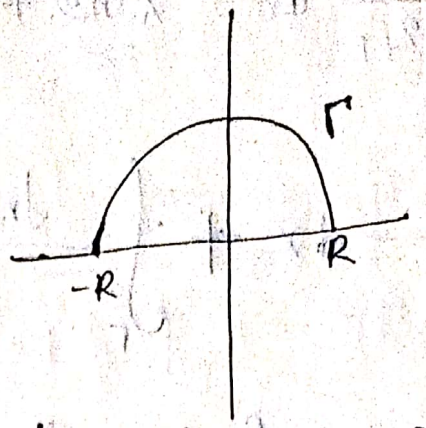
$$\therefore \int_0^\infty \frac{x^{p-1}\,dx}{1+x} = \frac{\pi}{\sin p\pi}$$

Ex-7.31

17-10-22
Monday

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$$

$$\oint_C \frac{\ln(z+i)}{z^2+1}\,dz$$



z^2+1

$= (z+i)(z-i)$

$= z^2 - i^2$

$= z^2+1$

$= x^2+1$

Residue of $z=i$ is

$z^2+1=0 \therefore z=\pm i$

$z=+i \ (0,1)$

$z=-i \ (0,-1)$

$z=-i$ outside contour.

$\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\Rightarrow e^{i\frac{\pi}{2}} = i$

$$a_{-1} = \left[\lim_{z\to i} \cancel{(z-i)} \cdot \frac{\ln(z+i)}{(z+i)\cancel{(z-i)}}\right]$$

$$= \lim_{z\to i} \frac{\ln(z+i)}{(z+i)}$$

$$= \frac{\ln 2i}{2i}$$

We know, $\oint_C \frac{\ln(z+i)}{z^2+1}\,dz = 2\pi i\, a_{-1}$

$$= \cancel{2}\pi \times \frac{\ln 2i}{\cancel{2i}}$$

$$= \pi \ln 2i$$

$$= \pi(\ln 2 + \ln i)$$

$$= \pi \ln 2 + \pi \ln e^{i\pi/2}$$

$$= \pi \ln 2 + \pi \times \frac{i\pi}{2}$$

$$= \pi \ln 2 + \frac{i\pi^2}{2}$$

here -

$$\oint_C \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \int_{-R}^{+R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}.$$

$$\Rightarrow \int_{-R}^{0} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \int_{R}^{0} \frac{\ln(i-x)}{x^2+1}\,(-dx) + \int_{0}^{R} \frac{\ln(i+x)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}.$$

$$\Rightarrow \int_{0}^{R} \frac{\ln(i-x)+\ln(i+x)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}.$$

$$\Rightarrow \int_{0}^{R} \frac{\ln(i-x)(i+x)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}.$$

$$\Rightarrow \int_{0}^{R} \frac{\ln(i^2-x^2)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \int_{0}^{R} \frac{\ln(-1)(1+x^2)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}.$$

$$\Rightarrow \int_0^R \frac{\ln(x^2+1)+\ln(-1)}{x^2+1}\,dx + \int_\Gamma \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \int_0^R \frac{\ln(x^2+1)}{x^2+1}\,dx + \int_0^R \frac{\ln e^{\pi i}}{x^2+1}\,dx + \int_\Gamma \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \int_0^R \frac{\ln(x^2+1)}{x^2+1}\,dx + \int_0^R \frac{\pi i\,dx}{x^2+1} + \int_\Gamma \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \lim_{R\to\infty} \int_0^R \frac{\ln(x^2+1)}{x^2+1}\,dx + i\lim_{R\to\infty}\int_0^R \frac{\pi\,dx}{x^2+1} + \lim_{R\to\infty}\int_\Gamma \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{i\pi^2}{2}$$

$$\Rightarrow \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx + i\int_0^\infty \frac{\pi\,dx}{x^2+1} = \pi\ln 2 + \frac{i\pi^2}{2}$$

equating the real part we get,

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi\ln 2$$

fourier series:

Odd function $\rightarrow f(-x) = -f(x) \Rightarrow \sin x, x, x^3 \ldots$

Even function $\rightarrow f(-x) = f(x) \Rightarrow \cos x, x^2, x^4 \ldots$

periodic function $\rightarrow f(T+x) = f(x)$.

$f(x) = \sin x$

$f(2\pi + x) = \sin(2\pi + x) = \sin x = f(x)$.

period

$$\int_{-\pi}^{\pi} \sin x \, dx = 0$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

$a_0, a_n, b_n \rightarrow$ fourier co-efficient.

$$\boxed{a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx}$$

$$\boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx}$$

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx}$$

If the function is even then $\boxed{b_n = 0}$

27-10-22
Thursday

$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$\int_{-\pi}^{\pi} \sin nx \, dx = 0$

$\int_{-\pi}^{\pi} \cos nx \, dx = 0$

$f(x+\pi) = f(x)$

$-\pi \le x \le \pi \longrightarrow 2\pi$

161 – 249 Page

Find the fourier series expansion of the function $f(x) = x^2$ in the interval $-\pi \le x \le \pi$ and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(1) $f(x) = x^2 ; -\pi \le x \le \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{6\pi} \left[\pi^3 - (-\pi)^3 \right]$$

$$= \frac{1}{6\pi} \left[\pi^3 + \pi^3 \right] = \frac{1}{6\pi} \times 2\pi^3$$

$$= \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \cancel{dx} \cos nx \, dx$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\left\{\left[x^2 \frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin nx}{n} dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2}{n} x \sin nx \, dx$$

$$= \frac{1}{n\pi}\left[x^2 \sin nx\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{n\pi} \times 0 - \frac{2}{n\pi}\left[-\frac{x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n\pi}\int_{-\pi}^{\pi} -\frac{\cos nx}{n} dx$$

$$= \frac{2}{n^2\pi}\left[\pi \cos n\pi + \pi \cos n\pi\right] + 0$$

$$= \frac{2}{n^2\pi} \times 2\pi \cos n\pi$$

$$= \frac{4}{n^2} \times (-1)^n$$

$f(x) = x^2$

$\therefore f(-x) = (-x)^2 = x^2$

$\therefore f(x)$ is an even function.

So, $b_n = 0$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4 \times (-1)^n}{n^2} \cos nx + 0 \right)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} + \cancel{\frac{4(-1)^n}{n^2}} \; 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \right).$$

Here, ~~let~~ $x = 0$, then $f(0) = 0$

$$\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2}} \qquad 0 = \frac{\pi^2}{3} - 4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \right)$$

$$\Rightarrow 4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \right) = \frac{\pi^2}{3}.$$

$$\Rightarrow \cancel{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

$$\Rightarrow 4\left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right)\right] = \frac{\pi^2}{3}$$

$$\Rightarrow 4\left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) - \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)\right] = \frac{\pi^2}{3}$$

$$\Rightarrow 4\left[\frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)\right] = \frac{\pi^2}{3}$$

$$\Rightarrow 2 \times \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

31-10-22
Monday

Even function : $b_n = 0 \longrightarrow$ Cosine Series

Odd function : $a_0 = 0$; $a_n = 0 \longrightarrow$ sine Series.

Find the fourier expansion of the function

(7) $f(x) = \begin{cases} 0 \; ; & -\pi < x \leq 0 \\ x \; ; & 0 < x \leq \pi \end{cases}$

Hence evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

We know, $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{0} f(x)dx + \frac{1}{2\pi} \int_{0}^{\pi} f(x)dx$$

$$= 0 + \frac{1}{2\pi} \int_{0}^{\pi} x\, dx$$

$$= \frac{1}{2\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{1}{4\pi} \left[x^2 \right]_0^{\pi} = \frac{1}{4\pi} \left[\pi^2 \right]$$

$$= \frac{\pi}{4}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi} 1\cdot\frac{\sin nx}{n}\,dx$$

$$= \frac{1}{n\pi}\times 0 + \frac{1}{n\pi}\cdot\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$\boxed{\begin{aligned}\sin n\pi &= 0\\ \cos n\pi &= (-1)^n\end{aligned}}$$

$$= \frac{1}{n^2\pi}\times\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi} \frac{\cos nx}{n}\,dx$$

$$= \frac{-1}{n\pi}\left[\pi\cos n\pi - 0\right] + 0$$

$$= -\frac{1}{n\cancel{\pi}} \times \cancel{\pi}\cos n\pi$$

$$= -\frac{1}{n} \times (-1)^n$$

$$= \frac{1}{n} \times (-1)^{n+1}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\left[(-1)^n - 1\right]\cos nx + \frac{1}{n}(-1)^{n+1}\sin nx\right\}$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left\{\frac{-2}{1^2}\cos x + 0 + \frac{-2}{3^2}\cos 3x + 0 + \frac{-2}{5^2}\cos 5x + 0 + \cdots\right\} + \left\{\frac{\sin x}{1} + \frac{-\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\} + \left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$$

Ans

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 0$$

Putting $x = 0$ in (*), $f(0) = 0$

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + 0$$

$$\Rightarrow \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi}{4}$$

$$\Rightarrow \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Half range fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

P.217 – [17]

17 $f(x) = x(\pi - x)$ ~~prove that~~ $0 \le x \le \pi$.

$f(x) = \pi x - x^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\,dx = \frac{2}{\pi}\left[\pi \frac{x^2}{2}\right]_0^{\pi} - \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \left[x^2\right]_0^{\pi} - \frac{2}{3\pi}\left[x^3\right]_0^{\pi}$$

$$= \pi^2 - \frac{2}{3\pi} \times \pi^3$$

$$= \pi^2 - \frac{2}{3}\pi^2 = \frac{3\pi^2 - 2\pi^2}{3}$$

$$= \frac{\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \cos nx\,dx.$$

$$= \frac{2}{\pi}\left[(\pi x - x^2) \cdot \frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} (\pi - 2x)\frac{\sin nx}{n}\,dx.$$

$$= 0 - \frac{2}{n\pi}\left[(\pi-2x)\cdot\frac{-\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}(0-2)\cdot\frac{-\cos nx}{n}\,dx$$

$$= \frac{2}{n^2\pi}\left[(\pi-2\pi)\cos n\pi - (\pi-0)\cdot\cos 0\right] + 0$$

$$= \frac{2}{n^2\pi}\left[-\pi(-1)^n - \pi\right]$$

$$= \frac{2}{n^2}\left[(-1)\cdot\{(-1)^n + 1\}\right]$$

$$= \frac{-2}{n^2}\left[(-1)^n + 1\right]$$

$$b_n = \frac{2}{n}\int_0^{\pi} f(x)\sin nx\,dx = 0.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$\Rightarrow x(\pi-x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{2}{n^2}\left[(-1)^n+1\right]\cos nx$$

$$= \frac{\pi^2}{6} - \left[0 + 2\cdot\frac{2}{2^2}\cos 2x + 0 + 2\cdot\frac{2}{4^2}\cos 4x + 0 + 2\cdot\frac{2}{6^2}\cos 6x + \cdots\right]$$

$$= \frac{\pi^2}{6} - \left[\frac{\cos 2\cdot 1x}{1^2} + \frac{\cos 2\cdot 2x}{2^2} + \frac{\cos 2\cdot 3x}{3^2} + \cdots\right]$$

$$= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^2}$$

Page: 187 – Ex(5)

189 – Ex(6(a))

192 – Ex(7)

195 – Ex(8)

198 – Ex(9)

200 – Ex(10)

203 – Ex(11)

207 – Ex(13)

210 – Ex(14)

212 – Ex(15)

214 – Ex(16)

217 – Ex(17)

220 – Ex(19)

complex variable → 5

fourier analysis → 1

vectors → 1

laplace → 1

8

10-11-22
Thursday

$\underline{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\underline{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\hat{i}.\hat{i} = 1$

$\hat{j}.\hat{j} = 1$

$\hat{k}.\hat{k} = 1$

$\underline{A}.\underline{B} = a_1b_1 + a_2b_2 + a_3b_3$

$|\underline{A}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$$\underline{A} \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$\underline{a}, \underline{b}, \underline{c}$

box product

$$\underline{a}.(\underline{b} \times \underline{c}) = [\underline{a}\ \underline{b}\ \underline{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$[\underline{a}\ \underline{b}\ \underline{c}] = 0 \longrightarrow$ co-planar. (সমতল)

$$\underline{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} = \sum i \frac{\partial}{\partial x}$$

$$\underline{\nabla}\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= \sum i \frac{\partial \phi}{\partial x}$$

$\nabla\phi \rightarrow$ gradient of $\phi \rightarrow$ grad Grade ϕ

$$\nabla \cdot V = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

divergence of V

$$= v_1 i + v_2 j + v_3 k$$

$$= \sum \frac{\partial v_1}{\partial x}$$

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

curl V

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + j\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \sum i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)$$

page — 623 [formula — 12 ति]

14-11-2022
Monday

Solinoidal → $\nabla \cdot \underline{V} = 0$

Irrotational → $\nabla \times \underline{V} = 0$

Directional derivatives = $\nabla \phi \cdot \underline{a}$

$\underline{a}$ = unit vector

$$\hat{a} = \frac{\underline{a}}{|\underline{a}|}$$

$$\therefore |\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Find the directional derivative of $\phi(x,y,z) = 4xz^3 - 3x^2y^2z$ at the point $(2,-1,2)$ in the direction of $2i - 3j + 6k$

$\phi(x,y,z) = 4xz^3 - 3x^2y^2z$ point $(2,-1,2)$ $(2i - 3j + 6k)$ direction

Here $\underline{A} = 2i - 3j + 6k$

$$\nabla \phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(4xz^3 - 3x^2y^2z\right)$$

$$= i\left(4z^3 - 6xy^2z\right) + j\,x - 6x^2yz + k\left(12xz^2 - 3x^2y^2\right)$$

At the point $(2,-1,2)$

$$\nabla \phi = i(32 - 24) + j\,48 + k(96 - 12)$$

$$= 8i + 48j + 84k$$

Unit vector, $\underline{a}$ of the direction.

$$\underline{a} = \frac{2i - 3j + 6k}{\sqrt{2^2 + 3^2 + 6^2}}$$

$$= \frac{2i - 3j + 6k}{7}$$

$$= \frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} k$$

directional derivative of ϕ is

$$\underline{\nabla} \phi \cdot \underline{a}$$

$$= (8i + 48j + 84k) \cdot \left(\frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} k\right)$$

$$= \frac{8 \times 2}{7} + 48 \times \frac{-3}{7} + 84 \times \frac{6}{7}$$

$$= \frac{16}{7} - \frac{144}{7} + \frac{504}{7}$$

$$= \frac{16 - 144}{7} + \frac{504}{7}$$

$$= \cancel{\frac{-128}{7} + \frac{504}{7}} = \frac{520 - 144}{7}$$

$$= \frac{376}{7}$$

$\underline{A}$ $\underline{B}$, θ

$$\cos\theta = \frac{A \cdot B}{|A| \cdot |B|}$$

P-638 Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

$\phi(x,y,z) = x^2+y^2+z^2-9$ $\quad (2,-1,2)$

$\psi(x,y,z) = x^2+y^2-3-z$

Let,

$$A = \nabla\cdot\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$B = \nabla\cdot\psi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Let θ be the angle between the surface.

$$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi|\,|\nabla\psi|}$$

$$= \frac{(2x\hat{i}+2y\hat{j}+2z\hat{k}) \cdot (2x\hat{i}+2y\hat{j}-\hat{k})}{\sqrt{4x^2+4y^2+4z^2} \cdot \sqrt{4x^2+4y^2+1^2}}$$

$$= \frac{4x^2+4y^2-2z}{2\sqrt{(4x^2+4y^2+1^2)(x^2+y^2+z^2)}}$$

At the point $(2,-1,2)$

$$\cos\theta = \frac{16 + 4 - 4}{2\sqrt{(16+4+1)\cdot(4+1+4)}}$$

$$= \frac{16}{2\times\sqrt{9\times 21}}$$

$$= \frac{\cancel{16}^{\cancel{8}^{4}}}{\cancel{2}\times 3\times\cancel{6}_{3}} = \frac{\cancel{16}^{8}}{\cancel{2}\times 3\sqrt{21}}$$

$$= \frac{\cancel{4}}{\cancel{9}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$(x, y, z) \longrightarrow x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\underline{r}|$

Show that $r^n \cdot \underline{r}$ is irrotational.

$\underline{\nabla} \times (r^n \cdot \underline{r}) = 0$

$r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^n = (x^2 + y^2 + z^2)^{\frac{n}{2}}$

P-623; No-5

P-623; No-4

$$\boxed{\underline{\nabla} \times (\phi \underline{V}) = (\underline{\nabla}\phi) \times \underline{V} + \phi (\underline{\nabla} \times \underline{V})}$$

$$\therefore \underline{\nabla} \times (r^n \cdot \underline{r}) = (\underline{\nabla} r^n) \times \underline{r} + r^n (\underline{\nabla} \times \underline{r})$$

$$\underline{\nabla} \times \underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + k(0-0)$$

$$= 0$$

$$\underline{\nabla} r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right)^{\frac{n}{2}}$$

$$= \left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} \times n\left[\hat{i}x + \hat{j}y + \hat{k}z\right]$$

$$= n\left(x^2+y^2+z^2\right)^{\frac{n-2}{2}} \cdot \underline{r}$$

$$= n\, r^{n-2} \cdot \underline{r}$$

$$\underline{\nabla} \times (r^n \cdot \underline{r}) = n \cdot r^{n-2} \cdot \underline{r} \times \underline{r} + 0$$

$$= n \cdot r^{n-2} \cdot 0$$

$$= 0$$

Hence $r^n \cdot \underline{r}$ is an irrotational.

Show that $r^n \cdot \underline{r}$ is solinoidal if $n = -3$

$$\underline{\nabla} \cdot (r^n \cdot \underline{r}) = 0$$

$$\boxed{\underline{\nabla}\cdot(\phi \underline{u}) = (\underline{\nabla}\phi)\cdot u + \phi\cdot(\underline{\nabla}\cdot u)}$$

$$\underline{\nabla}\cdot(r^n \cdot \underline{r}) = (\underline{\nabla} r^n)\cdot \underline{r} + r^n(\underline{\nabla}\cdot\underline{r})$$

$$\underline{\nabla} r^n = n r^{n-2} \cdot \underline{r}$$

$$\underline{\nabla}\cdot\underline{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 1+1+1$$

$$= 3.$$

$$\underline{\nabla}\cdot(r^n \cdot \underline{r}) = n r^{n-2} \cdot \underline{r}\cdot\underline{r} + r^n \cdot 3$$

$$= n r^{n-2} \cdot r^2 + 3r^n$$

$$= n r^n + 3r^n$$

$$= r^n (n+3).$$

when, $n = -3$ then $\underline{\nabla}\cdot(r^n \underline{r}) = 0$

Hence when $n = -3$, $r^n \cdot \underline{r}$ is solinoidal

625 - Page

1) line integral
2) surface integral
3) volume integral

page-661
theorem

$$\int_C F(R)\,dR = \int_C \left(f\hat{i} + g\hat{j}\right) \cdot \left(\hat{i}\,dx + \hat{j}\,dy\right)$$

$$= \int_C \left(f\,dx + g\,dy\right)$$

$$\iint_S \underline{F} \cdot d\underline{S} =$$

surface

$$\iint_S \underline{F} \cdot \underline{n}\,dS = \iint_R \underline{F} \cdot n \cdot \frac{dx\,dy}{|\underline{n} \cdot k|}$$

↳ regional integral

Page - 663

Ex - 32

evaluate $\iint_R \sqrt{x^2+y^2}\, dx\, dy$ over the region R in the x, y plane bounded by $x^2+y^2=36$.

Here,

$$\iint_R \sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_{x=-6}^{6} \int_{y=-\sqrt{36-x^2}}^{y=\sqrt{36-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$$= 4 \int_0^6 \int_0^{\sqrt{36-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$$= 4 \int_0^6 \left[\frac{y\sqrt{x^2+y^2}}{2} + \frac{x^2}{2} \log\left(y+\sqrt{x^2+y^2}\right) \right]_0^{\sqrt{36-x^2}} dx$$

$x^2+y^2=36$

$\Rightarrow y^2 = 36-x^2$

$\therefore y = \pm\sqrt{36-x^2}$

$x = \pm\sqrt{36-y^2}$

$\int \sqrt{a^2+x^2}\, dx$

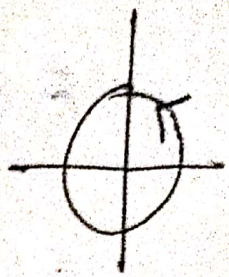
$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \log\left(x+\sqrt{a^2+x^2}\right)$

Page - 695

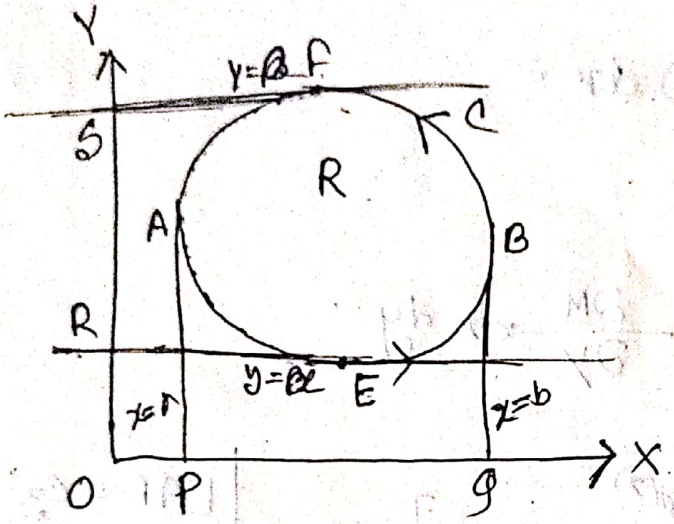
17-11-2022
Thursday

Green's theorem:

$\frac{\partial N}{\partial x}$



$$\oint_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$AFB = C_2$

$AEB = C_1$

$C = C_1 + C_2$

$y = Y_1(x)$

$y = Y_2(x)$

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{x=b}\left[\int_{y=Y_1(x)}^{y=Y_2(x)} \frac{\partial M}{\partial y}\,dy\right] dx$$

$$= \int_a^b \Big[M(x,y)\Big]_{Y_1(x)}^{Y_2(x)} dx$$

$$= \int_a^b \Big[M(x, Y_2(x)) - M(x, Y_1(x))\Big] dx$$

$$= -\int_a^b M(x, Y_1(x))\, dx + \int_a^b M(x, Y_2(x))\, dx$$

$$= -\left[\int_a^b M(x, Y_1(x))\, dx + \int_b^a M(x, Y_2(x))\, dx\right]$$

$$= -\oint_C M(x, Y)\, dx$$

$$\oint_C M\, dx = -\iint_R \frac{\partial M}{\partial y}\, dx\, dy$$

$$\iint_R \frac{\partial N}{\partial x}\, dx\, dy = \int_{y=\alpha}^{y=\beta}\left[\int_{x=X_1(x)}^{X_2(x)} \frac{\partial N}{\partial x}\, dx\right] dy$$

$$= \int_\alpha^\beta \left[N(x, y)\right]_{X_1}^{X_2} dy$$

$$= \int_\alpha^\beta \left[N(X_2, Y) - N(X_1, Y)\right] dy$$

$EAF = C_3$

$EBF = C_4$

$C_1 = C_3 + C_4$

y = fixed

x = variable

$$= \int_{\alpha}^{\beta} N(x_2, y)\, dy + \int_{\beta}^{\alpha} N(x_1, y)\, dy$$

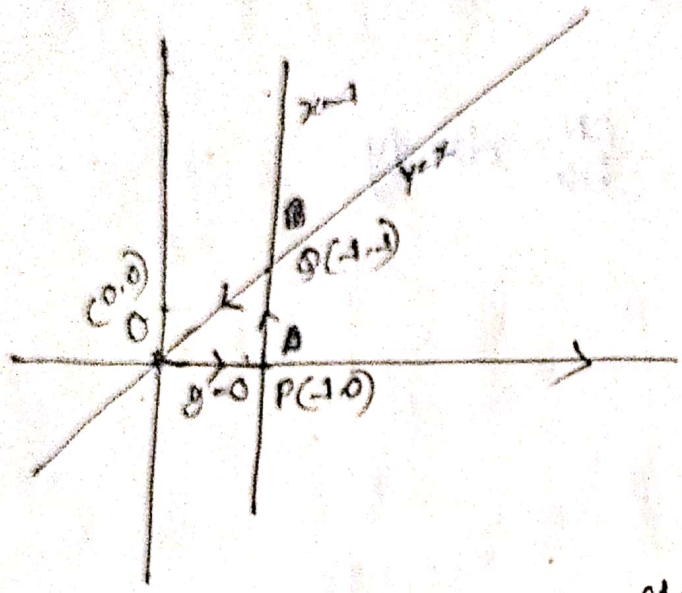
$$= \oint_C N(x, y)\, dy$$

$$\therefore \oint_C N(x, y)\, dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy$$

Page - 709
Ex - 48

Compute $\oint[(xy-x^2)\,dx + x^2y\,dy]$ over the triangle bounded by the lines $y=0, x=1, y=x$ and verify Green's Theorem.

Compute $\oint_C ((xy-x^2)\,dx + x^2y\,dy)$



$y=0$
$x=1$
$y=x$

$$\int_0^1 (0-x^2)\,dx + 0.dy = \int_0^1 \left[-\frac{x^3}{3}\right]_0^1$$

$$= -\frac{1}{3}[1-0] = -\frac{1}{3}$$

$$\int_{y=0}^{1} \left[(1.y-1^2).0 + 1^2.y\,dy\right] = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\int_{y=1}^{0} \left[(y.y-y^2)\,dy + y^2.y\,dy\right] = \int_1^0 y^3\,dy = \left[\frac{y^4}{4}\right]_1^0$$

$$= -\frac{1}{4}$$

$$L = -\frac{1}{3} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{-4+6-3}{12}$$

$$= \frac{-1}{12}$$

here, $N = x^2y$ $\quad M = xy - x^2$

$$\therefore \frac{\partial N}{\partial x} = 2xy \qquad \therefore \frac{\partial M}{\partial y} = x$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} (2xy - x)\,dy\,dx$$

$x = \text{const}$
$y = \text{variable}$

$$= -\frac{1}{12}$$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

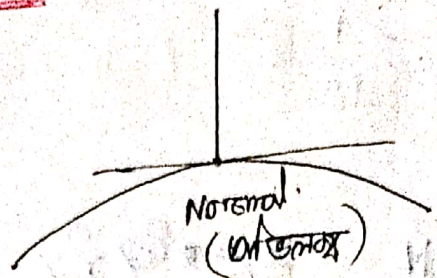
21-11-22
Monday

divergence theorem:

P-715

relation between surface and volume integral.

$V \Rightarrow \iiint$ $\quad$ $S \Rightarrow \iint$



↑ → positive
↓ - negative

Math term: State and prove Gausse's divergence theorem.

$$\iiint_V \nabla \cdot \underline{F}\, dV = \iint_S \underline{F} \cdot \hat{n}\, ds.$$

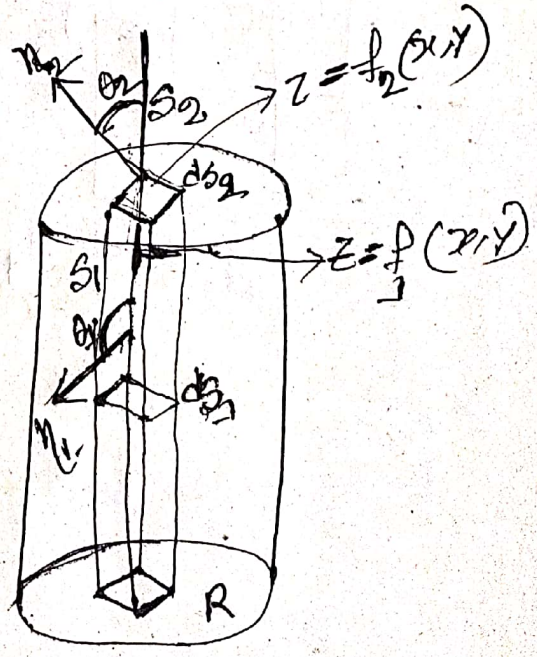
$\hat{n}$ = unit normal
↳ is out ward normal

$$\bar{F} = f_1(x,y,z)\,\hat{i} + f_2(x,y,z)\,\hat{j} + f_3(x,y,z)\,\hat{k}$$

Here,
$$\iint_S \underline{F} \cdot \hat{n}\, ds = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

$z_1 = f_1(x,y,z)$

$z_2 = f_2(x,y,z)$



$$\iiint_V \frac{\partial f_3}{\partial z}\, dx\, dy\, dz$$

$$= \iiint_V \frac{\partial f_3}{\partial z}\, dz\, dy\, dx$$

$$= \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial f_3}{\partial z}\, dz \right] dy\, dx$$

$$= \iint_R \left[f_3(x,y,z) \right]_{f_1(x,y)}^{f_2(x,y)} dy\, dx$$

$$= \iint_R \left[f_3(x,y,f_2) - f_3(x,y,f_1) \right] dy\, dx$$

$$\iint_R f_3(x,y,f_2)\, dy\, dx = \iint_{S_2} f_3 \cos\theta_2\, ds_2$$

$$\# \iint_{S_1} f_3 \cos\theta_1\, ds_1$$

for S_2 surface,

$dy\,dx = \cos\theta_2\, ds_2$

$\therefore \hat{k}\cdot\hat{n}_2\, ds_2 = |k||n_2| \cos\theta$

$\therefore \hat{k}\cdot\hat{n}_2\, ds_2 = \cos\theta_2\, ds_2$

for S_1 surface,

$dy\,dx = -\cos\theta_1\, ds_1$

$$\iint_S f_3\, \hat{k}\cdot\hat{n}\, ds = \iiint_V \frac{\partial f_3}{\partial z}\, dz\, dy\, dx \quad \text{—— (I)}$$

$$\iint_S f_1\, \hat{i}\cdot\hat{n}\, ds = \iiint_V \frac{\partial f_1}{\partial x}\, dx\, dy\, dz \quad \text{—— (II)}$$

$$\iint_S f_2\, \hat{j}\cdot\hat{n}\, ds = \iiint_V \frac{\partial f_2}{\partial y}\, dy\, dx\, dz \quad \text{—— (III)}$$

(I) + (II) + (III),

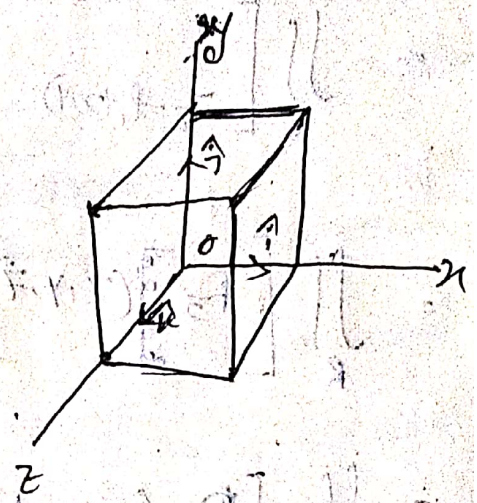
$$\iiint_V \underline{\nabla}\cdot\underline{F}\, dV = \iint_S \underline{F}\cdot\hat{n}\, ds$$

P - 720

Ex - 52 verify divergence theorem.

$$\underline{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} \; ; \; 0 \le x \le a, \; 0 \le y \le b, \; 0 \le z \le c$$

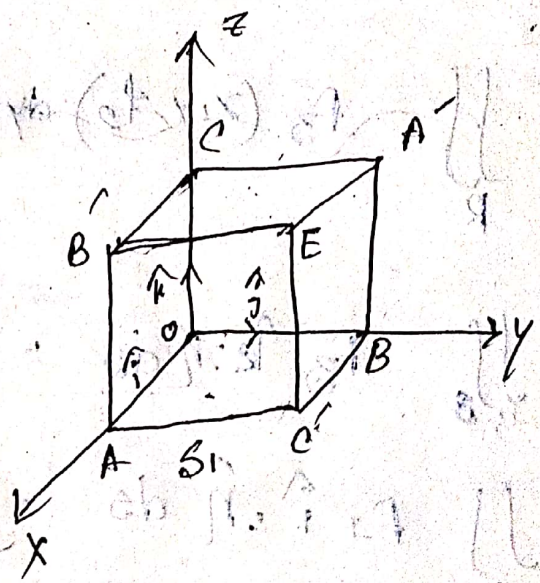
$$\iiint_V \underline{\nabla} \cdot \underline{F} \, dv = \iint_S \underline{F} \cdot \hat{n} \, ds$$



$$\underline{\nabla} \cdot \underline{F} = 2x + 2y + 2z = 2(x+y+z)$$

$$\iiint_V \underline{\nabla} \cdot \underline{F} \, dv = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} 2(x+y+z) \, dz \, dy \, dx$$

$$= abc(a+b+c)$$



$$\iint_{S_1} \underline{F} \cdot \hat{n} \, ds_1 \; ; \; n = -k$$

$$= \iint_{S_1} \underline{F} \cdot (-\hat{k}) \, ds_1$$

$$= -\iint_{S_1} (z^2 - xy) \, ds_1$$

$$= \int_{y=0}^{b} \int_{x=0}^{a} xy \, dx \, dy = \frac{a^2 b^2}{4}$$

example — 53 - 55

21-11-22
Thursday

Laplace Transformation:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \;;\; s>0$$

$$L(1) = \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$\therefore L\{1\} = -\frac{1}{s}(0-1) = \frac{1}{s}. \quad \therefore \boxed{L\{1\} = \frac{1}{s}}$$

$$L\{t\} = \int_0^\infty e^{-st}\, t\,dt$$

$$= \left[t \times \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\,dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$\therefore L\{t\} = \frac{1}{s} \times \frac{1}{s} = \frac{1}{s^2} \quad \therefore \boxed{L\{t\} = \frac{1}{s^2}}$$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[e^{-st} \cdot \frac{-\cos at}{a} \right]_0^\infty - \int_0^\infty -se^{-st} \cdot \frac{\cos at}{a} dt$$

$$= \left[0 + \frac{1}{a} \right]$$

$$= \left[\sin at \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty a \cos at \times \frac{e^{-st}}{-s} dt$$

$$= (0 - 0) + \frac{a}{s} \int_0^\infty \cos at \, e^{-st} \, dt$$

$$= \frac{a}{s} \left\{ \left[\cos at \times \frac{e^{-st}}{-s} \right]_0^\infty - \frac{a}{s} \int_0^\infty -a \sin at \, \frac{e^{-st}}{-s} \right.$$

$$= \frac{a}{s} \left[0 + \frac{1}{s} \right] - \frac{a^2}{s^2} \int_0^\infty \sin at \, e^{-st} \, dt$$

$$= \frac{a}{s^2} - \frac{a^2}{s^2} L\{(\sin at)\}$$

$$\Rightarrow \left(1 + \frac{a^2}{s^2}\right) L\{(\sin at)\} = \frac{a}{s^2}$$

$$\Rightarrow \frac{s^2+a^2}{s^2} L\{\sin at\} = \frac{a}{s^2}$$

$$\Rightarrow \boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= 0 + \frac{1}{s-a}$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

$$L\{f(t)\} = f(s)$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L\{f(t)\}$$

$$= -f(0) + s\,f(s) = s f(s) - f(0)$$

$$\boxed{L\{f'(t)\} = s f(s) - f(0)}$$

$$L\{f''(t)\} = \int_0^\infty e^{-st} \cdot f''(t)\,dt$$

$$= \left[e^{-st} \cdot f'(t)\right]_0^\infty - \int_0^\infty -s e^{-st} \cdot f'(t)\,dt$$

$$= 0 - f'(0) + s\int_0^\infty e^{-st} \cdot f'(t)\,dt$$

$$= -f'(0) + s\left[s f(s) - f(0)\right]$$

$$= -f'(0) + s^2 f(s) - s f(0)$$

$$\therefore \boxed{L\{f''(t)\} = s^2 f(s) - s f(0) - f'(0)}$$

1. $L\{f'(t)\} = s f(s) - f(0)$

2. $L\{f''(t)\} = s^2 f(s) - s f(0) - f'(0)$

3. $L\{f'''(t)\} = s^3 f(s) - s^2 f(0) - s f'(0) - f''(0)$

$$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$$

$$\cdots - s f^{(n-2)}(0) - f^{n-1}(0)$$

$$L\{f'''(t)\} = \int_0^\infty e^{-st} f'''(t)\,dt$$

$$= \left[e^{-st} f''(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f''(t)\,dt$$

$$= -f''(0) + s\int_0^\infty e^{-st} f''(t)\,dt$$

$$= -f''(0) + s\left[s^2 f(s) - sf(0) - f'(0)\right]$$

$$= -f''(0) + s^3 f(s) - s^2 f(0) - s f'(0)$$

$$\therefore L\{f'''(t)\} = s^3 f(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$$

$$\cdots - s f^{n-2}(0) - f^{n-1}(0)$$

$L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s)$

$$= (-1)^n f^n(s)$$

$L\{t f(t)\} = -f'(s)$

$L\{t^2 f(t)\} = f''(s)$

28-11-22
Monday

$L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

$$= \int_0^\infty e^{-st}\, e^{at} \sin bt \, dt$$

$$= \int_0^\infty e^{-t(s-a)} \sin bt \, dt$$

$$= \sin bt \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty - \int_0^\infty b\cos bt \cdot \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty dt$$

~~$L\{\int^n (f(t))\} =$~~

Chapte-2

Reverse Laplace Transform

$$L\{F(t)\} = f(s)$$

$$F(t) = L^{-1}\{f(s)\}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$\therefore \{e^{at}\} = L^{-1}\left\{\frac{1}{s-a}\right\}$$

$$L\{F_1(t)+F_2(t)\} = L\{F_1(t)\} + L\{F_2(t)\}$$

example:

$$L^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$$

$$= 4L^{-1}\left\{\frac{1}{s-2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + \frac{5}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

$$\boxed{\Gamma(n+1) = n\Gamma(n) = \lfloor n}$$

$$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$st = x$

$dt = \frac{dx}{s}$

$\therefore t = \frac{x}{s}$

$$L\{t^n F(t)\} = (-1)^n f^n(s)$$

$$L\{t^n\} = \int_0^\infty \underset{u}{e^{-st}} \cdot \underset{v}{t^n}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty t^n - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\, dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\, dt$$

$$= \frac{n}{s}\int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^{n-1} \frac{dx}{s}$$

$$= \frac{n}{s^2 \cdot s^{n-1}} \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$= \frac{n}{s^{n+1}} \cdot \cancel{\Gamma(n+1)}\ \Gamma n$$

(n) $\boxed{\Gamma(n+1) = \lfloor n}$

$$\therefore L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\# \quad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$$

$$L\left\{\frac{t^n}{\Gamma(n+1)}\right\} = \frac{1}{\Gamma(n+1)} L\{t^n\}$$

$$= \frac{1}{\cancel{\Gamma(n+1)}} \times \frac{\cancel{\Gamma(n+1)}}{s^{n+1}}$$

$$= \frac{1}{s^{n+1}}$$

$$\therefore L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$$

3018

$$L^{-1}\left\{s^{5}\right\} = \frac{t^{4}}{\Gamma 5}$$

$$= \frac{t^{4}}{\lfloor 4}$$

$$= \frac{t^{4}}{24}.$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$L^{-1}\left\{\frac{5s+4}{s^{3}} - \frac{2s-18}{s^{2}+9} + \frac{24-30\sqrt{s}}{s^{4}}\right\}$$

$$= L^{-1}\left\{\frac{5s+4}{s^{3}}\right\} - L^{-1}\left\{\frac{2s-18}{s^{2}+3^{2}}\right\} + L^{-1}\left\{\frac{24-30\sqrt{s}}{s^{4}}\right\}$$

$$= 5L^{-1}\left\{\frac{1}{s^{2}}\right\} + 4L^{-1}\left\{\frac{1}{s^{3}}\right\} - 2L^{-1}\left\{\frac{s}{s^{2}+3^{2}}\right\} + 6L^{-1}\left\{\frac{3}{s^{2}+3^{2}}\right\}$$

$$+ 24L^{-1}\left\{\frac{1}{s^{4}}\right\} - 30L^{-1}\left\{\frac{1}{s^{7/2}}\right\}$$

$$= 5t + \frac{4}{\lfloor 2}L^{-1}\left\{\frac{\lfloor 2}{s^{2+1}}\right\} - 2\cos 3t + 6\sin 3t + \frac{4}{\lfloor 3}L^{-1}\left\{\frac{\lfloor 3}{s^{3+1}}\right\}$$

$$- \frac{30}{\lfloor 5/2}L^{-1}\left\{\frac{\lfloor 5/2}{s^{5/2+1}}\right\}$$

or

$$\frac{30}{\lfloor 7/2}L^{-1}\left\{\frac{\lfloor 7/2}{s^{5/2+1}}\right\}$$

$= 5t + 2t^2 - 2\cos 3t + 6\sin 3t + 4t^3 - 16\, t^{5/2}\sqrt{\pi}$

$\boxed{\text{H.W}}$

Page-19, ex-7.

~~comp~~ ~~ex~~

Page-54, ex-17, 18.

Page - 58, ex-25.

Page - 59, ex-26.

Page - 60, ex-27, 28

Page -, ex -

Page - 82

Date - 05-12-22
Monday

$Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$

$\Rightarrow L\{Y'' - 3Y' + 2Y\} = L\{4e^{2t}\}$

$\Rightarrow L\{Y''\} - 3L\{Y'\} + 2L\{Y\} = 4L\{e^{2t}\}$

$\Rightarrow s^2 y(s) - sY(0) - Y'(0) - 3sy(s) + 3Y(0) + 2y(s) = 4 \times \frac{1}{s-2}$

$\Rightarrow y(s)(s^2 - 3s + 2) + 3s - 5 - 9 = \frac{4}{s-2}$

$\Rightarrow y(s)(s-1)(s-2) = \frac{4}{s-2} - 3s + 14$

$\Rightarrow y(s) = \frac{4}{(s-1)(s-2)(s-2)} - \frac{3s}{(s-1)(s-2)} + \frac{14}{(s-1)(s-2)}$

$\Rightarrow y(s) = \frac{4 - 3s(s-2) + 14(s-2)}{(s-1)(s-2)^2}$

$= \frac{4 - 3s^2 + 6s + 14s - 28}{(s-1)(s-2)^2}$

$= \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$

$$= \frac{-7}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$L\{f(t)\} = f(s)$$

$$\therefore f(t) = L^{-1}\{f(s)\}$$

$$\therefore y(t) = L^{-1}\left\{\frac{-7}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}\right\}$$

$$= -7e^{t} + 1e^{2t} + 1te^{2t}$$

Page 82 - 1, 2, 3, 4

Page 83 - 6, 7, 9.