3.50. (a) Prove that the function $u = 2x(1-y)$ is harmonic. (b) Find a function v such that $f(z) = u + iv$ is analytic [i.e., find the conjugate function of u]. (c) Express $f(z)$ in terms of z.

3.51. Answer Problem 3.50 for the function $u = x^2 - y^2 - 2xy - 2x + 3y$.

3.52. Verify that the Cauchy–Riemann equations are satisfied for the functions (a) e^{z^2}, (b) $\cos 2z$, (c) $\sinh 4z$.

3.53. Determine which of the following functions u are harmonic. For each harmonic function, find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

(a) $3x^2y + 2x^2 - y^3 - 2y^2$, (b) $2xy + 3xy^2 - 2y^3$, (c) $xe^z \cos y - ye^z \sin y$, (d) $e^{-2xy}\sin(x^2 - y^2)$.

3.54. (a) Prove that $\psi = \ln[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point $(1, 2)$. (b) Find a function ϕ such that $\phi + i\psi$ is analytic. (c) Express $\phi + i\psi$ as a function of z.

3.55. Suppose $\text{Im}\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3 - 2i$, $f(1) = 6 - 5i$. Find $f(1+i)$.

$\boxed{3.50}$

(a) Prove that the function $u = 2x(1-y)$ is harmonic.

(b) Find a function v such that $f(z) = u + iv$ is analytic

[i.e. find the conjugate function of u]

(c) Express $f(z)$ in terms of z.

(a) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x - 2xy)$

$= 2 - 2y$

$\frac{\partial^2 u}{\partial x^2} = 0$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x - 2xy)$

$= 0 - 2x$

$\frac{\partial^2 u}{\partial y^2} = 0.$

Adding, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, thus u is harmonic

(b) $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$

$= \cancel{2x}\ 2-2y$ —— (1)

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$= +2x$ —— (2)

Integrating equation (1) with respect to y keeping x constant we get,

$v = \int (2-2y)\,dy$

$= 2\int dy - 2\int y\,dy$

$= 2y - \cancel{2} \times \frac{y^2}{\cancel{2}} + F(x)$

$= 2y - y^2 + F(x)$; where $F(x)$ is an arbitrary real function of x

D.w.r.to x,

$\frac{\partial v}{\partial x} = \cancel{-2x} + F'(x)$ —— (3)

From equation (2) and (3),

$F'(x) = 2x$

Integrating,

$F(x) = x^2$

$\therefore v = 2y - y^2 + x^2$

~~Integrating w.r.t. x.~~

~~$f(x) = 2 \cdot \frac{x^2}{2}$~~

~~$\therefore F(x) = x^2$~~

~~$\therefore v = 2y - y^2 + x^2$.~~

(c) $f(z) = u + iv$

$\Rightarrow 2x(1-y) + i(2y - y^2 + x^2)$

$\Rightarrow 2x - 2xy + i2y - iy^2 + ix^2$

$\Rightarrow 2x + 2xyi^2 + i2y + i^3y^2 + ix^2$

$\Rightarrow i\{x^2 + (iy)^2 + 2xiy\} + 2(x+iy)$

$\Rightarrow i(x+iy)^2 + 2(x+iy)$

$\Rightarrow iz^2 + 2z$. (Ans:)

3.52

(a) Verify that the Cauchy-Riemann equations are satisfied for the function e^{z^2}

Here,

$$f(z) = e^{z^2}$$

$$\Rightarrow u+iv = e^{(x+iy)^2} \quad ; [f(z) = u+iv \text{ and } z = x+iy]$$

$$\Rightarrow u+iv = e^{x^2+2ixy-y^2}$$

$$\Rightarrow u+iv = e^{x^2-y^2} \cdot e^{2ixy}$$

$$\Rightarrow u+iv = e^{x^2-y^2}(\cos 2xy + i\sin 2xy) \, ; [e^{i\theta} = \cos\theta + i\sin\theta]$$

$$\Rightarrow u+iv = e^{x^2-y^2}\cos 2xy + e^{x^2-y^2} \cdot \sin 2xy\, i.$$

So,

$$u = e^{x^2-y^2} \cdot \cos 2xy$$

$$v = e^{x^2-y^2} \cdot \sin 2xy$$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2} \cdot 2x \cdot \cos 2xy + e^{x^2-y^2}(-\sin(2xy)) \cdot 2y$$

$$= 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin(2xy)$$

$$\frac{\partial v}{\partial y} = -2ye^{x^2-y^2}\,\cancel{co}\,\sin 2xy + e^{x^2-y^2}\cos 2xy \cdot 2x$$

$$= 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Again,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(e^{x^2-y^2}\cos 2xy\right)$$

$$= -2y \cdot e^{x^2-y^2} \cdot \overset{\cos}{\cancel{\sin}}(2xy) + e^{x^2-y^2}(-\sin(2xy))\} 2x$$

$$= -2ye^{x^2-y^2} \cdot \cos 2xy - 2xe^{x^2-y^2} \sin 2xy.$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(e^{x^2-y^2} \cdot \sin 2xy\right)$$

$$= 2xe^{x^2-y^2} \sin 2xy + e^{x^2-y^2} \cdot \cos 2xy \cdot 2y$$

$$= 2ye^{x^2-y^2} \cdot \cos 2xy + 2xe^{x^2-y^2} \sin 2xy.$$

$$\therefore \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ Cauchy-Riemann equations are satisfied for the given function e^{z^2}.

(b)

$$f(z) = \cos 2z$$

$$\Rightarrow u+iv = \{\cos 2(x+iy)\}$$

$$\Rightarrow u+iv = \{\cos(2x+2iy)\}$$

$$\Rightarrow u+iv = \cos 2x \cdot \cosh 2y - i\sin 2x \sinh 2y$$

① $\cos(x+iy) = \cos x \cosh y - i\sin x \sinh y$

② $\cos(x-iy) = \cos x \cosh y + \sin x \sinh y$

$\frac{d}{dx}\cosh(2x) = -2\sin(2hx)$

$\frac{d}{dx}\sinh(2x) = 2\cosh(2x)$.

So,

$$u = \cos 2x \cosh 2y$$

$$v = -\sin 2x \cdot \sinh 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\cos 2x \cdot \cosh 2y)$$

$$= -2\sin 2x \cdot \cosh 2y$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-\sin 2x \sinh 2y)$$

$$= -\sin 2x \cdot 2\cosh 2y$$

$$= -2\sin 2x \cdot \cosh 2y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Again,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\cos 2x \cosh 2y)$$

$$= \cos 2x \cdot 2\sinh 2y$$

$$= 2\cos 2x \cdot \sinh 2y$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(-\sin 2x \cdot \sinh 2y)$$

$$= -\cos 2x \cdot 2 \cdot \sinh 2y$$

$$= -2\cos 2x \cdot \sinh 2y$$

$$\therefore \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

$\therefore$ Cauchy-Riemann equations are satisfied.

3.52 (b) $f(z) = \sinh 4z$

$\Rightarrow u+iv = \sinh\{4(x+iy)\}$

$\Rightarrow u+iv = -i \sin i(4x+4iy)$ [1]

$\Rightarrow u+iv = -i \sin(i4x - 4y)$

$\Rightarrow u+iv = -i\left[\sin i4x . \cos 4y - \cos i4x . \sin 4y\right]$ [2]

$\Rightarrow u+iv = -i\left[i \sinh 4x . \cos 4y - \cosh 4x \sin 4y\right]$ [3]

$\Rightarrow u+iv = -i^2 \sinh 4x \cos 4y + i \cosh 4x \sin 4y$

$\Rightarrow u+iv = \sinh 4x \cos 4y + i \cosh 4x \sin 4y$

① $\sinh z = -i \sin iz$

② $\sin(A-B) = \sin A \cos B - \cos A \sin B$

③ $\sin iz = \frac{\sinh z}{-i} = \frac{i \sinh z}{-i^2} = i \sinh z$

So,

$u = \sinh 4x \cos 4y$

$v = \cosh 4x \sin 4y$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sinh 4x \cos 4y)$$

$$= 4\cosh 4x . \cos 4y \; ; \left[\frac{d}{dx}(\sinh 4x) = 4\cosh 4x\right]$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(\cosh 4x . \sin 4y)$$

$$= \cosh 4x . 4\cos 4y \; ; \left[\frac{d}{dx}(\sin 4y) = 4\cos 4y\right]$$

$$= 4\cosh 4x . \cos 4y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\sinh 4x \cos 4y)$$

$$= \sinh 4x \cdot -4\sin 4y$$

$$= -4\sinh 4x \cdot \sin 4y$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(\cosh 4x \; \overset{\sin}{\cancel{\cos}}\, 4y)$$

$$= 4\sinh 4x \cdot \cancel{\cos}\, \sin 4y$$

$$\therefore \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

So Cauchy-Riemann equations are satisfied for the function $\sinh 4z$.

3.53 Determine which of the following functions u are harmonic. For each harmonic function, find the conjugate harmonic function v and express $u+iv$ as an analytic function of z.

(a) $3x^2y+2x^2-y^3-2y^2$

$\Rightarrow$ Given that,

$$u = 3x^2y+2x^2-y^3-2y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2y+2x^2-y^3-2y^2)$$

$$= 6xy+4x$$

$$\frac{\partial^2 u}{\partial x^2} = 6y+4$$

Again,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2y+2x^2-y^3-2y^2)$$

$$= 3x^2-3y^2-4y$$

$$\frac{\partial^2 u}{\partial y^2} = -6y-4$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$\therefore$ u is a harmonic function.

Now from Cauchy-Reimann equation,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$= 6xy + 4x \text{ --- } (1)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$= 3y^2 - 3x^2 + 4y \text{ --- } (2)$$

Integrating (1) with respect to y keeping x constant,

$$v = 6x\frac{y^2}{2} + 4xy + F(x)$$

$$= 3xy^2 + 4xy + F(x) \text{; [where } F(x) \text{ is an arbitrary real function of } x]$$

D.w.r. to x,

$$\frac{\partial v}{\partial x} = 3y^2 + 4y + F'(x) \text{ --- } (3)$$

From (2) and (3),

$$F'(x) = -3x^2$$

Integrating,

$$F(x) = -3\frac{x^3}{3}$$

$$= -x^3$$

$\therefore V = 3xy^2 + 4xy - x^3$

$f(z) = u + iv$

$= 3x^2y + 2x^2 - y^3 - 2y^2 + i3xy^2 + i4xy - ix^3$

$= \{(ix)^3 - 3(ix)^2y + 3.(ix).y^2 - y^3\} + 2x^2 + i4xy - 2y^2$

$= (ix - y)^3 + 2\{x^2 + 2ixy + (iy)^2\}$

$= (ix - y)^3 + 2(x + iy)^2$

$= (-y + ix)^3 + 2z^2.$

(b) $2xy + 3xy^2 - 2y^3$

$\Rightarrow$ Given that,

$$u = 2xy + 3xy^2 - 2y^3$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2xy + 3xy^2 - 2y^3)$$

$$= 2y + 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2xy + 3xy^2 - 2y^3)$$

$$= 2x + 6xy - 6y^2$$

$$\frac{\partial^2 u}{\partial y^2} = 6x - 12y$$

As $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$, the given function is not harmonic

(c) $xe^x\cos y - ye^x\sin y$

$\Rightarrow$ Given that,

$$u = xe^x\cos y - ye^x\sin y$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xe^x\cos y - ye^x\sin y)$$

$$= e^x\cos y + xe^x\cos y - ye^x\sin y.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^x\cos y + xe^x\cos y - ye^x\sin y)$$

$$= e^x\cos y + e^x\cos y + xe^x\cos y - ye^x\sin y$$

$$= 2e^x\cos y + xe^x\cos y - ye^x\sin y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(xe^x\cos y - ye^x\sin y)$$

$$= -xe^x\sin y - e^x(y\cos y + \sin y).$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x\cos y - e^x(-y\sin y + \cos y + \cos y)$$

$$= -xe^x\cos y + ye^x\sin y - 2e^x\cos y$$

$$= -2e^x\cos y - xe^x\cos y + ye^x\sin y$$

here, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ so the function u is harmonic.

From Cauchy-Reimann equation,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$= e^x \cos y + x e^x \cos y - y e^x \sin y \quad \text{------ (1)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$= x e^x \sin y + e^x y \cos y + e^x \sin y \quad \text{------ (2)}$$

Integrating equation (1) w.r.t. y keeping x constant,

$$v = e^x \sin y + x e^x \sin y - e^x \left\{ y \int \sin y \, dy - \int \left(\frac{d}{dy}(y) \int \sin y \, dy \right) dy \right\}$$

$$= e^x \sin y + x e^x \sin y - e^x \left\{ -y \cos y - \int (-\cos y) \, dy \right\}$$

$$= e^x \sin y + x e^x \sin y - e^x \{ -y \cos y + \sin y \}$$

$$= e^x \sin y + x e^x \sin y + e^x y \cos y - e^x \sin y + F(x)$$

$$= x e^x \sin y + e^x y \cos y + F(x).$$

Differentiating w.r.t. x,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(xe^x \sin y + e^x y \cos y + F(x))$$

$$= e^x \sin y + xe^x \sin y + e^x y \cos y + F'(x) \quad \text{—— (3)}$$

From (2) and (3),

$$F'(x) = 0$$

Integrating, $F(x) = c$.

$$\therefore V = xe^x \sin y + e^x y \cos y + c \quad \text{(Ans.)}$$