Math - IV (Lecture 1)

17/07/2022

Complex Number

$$z = \underbrace{x}_{\text{real}} + \underbrace{iy}_{\text{imaginary}}$$

$$i = \sqrt{-1}$$

$$|z| = \sqrt{x^2 + y^2}$$

$\bar{z} = x - iy$

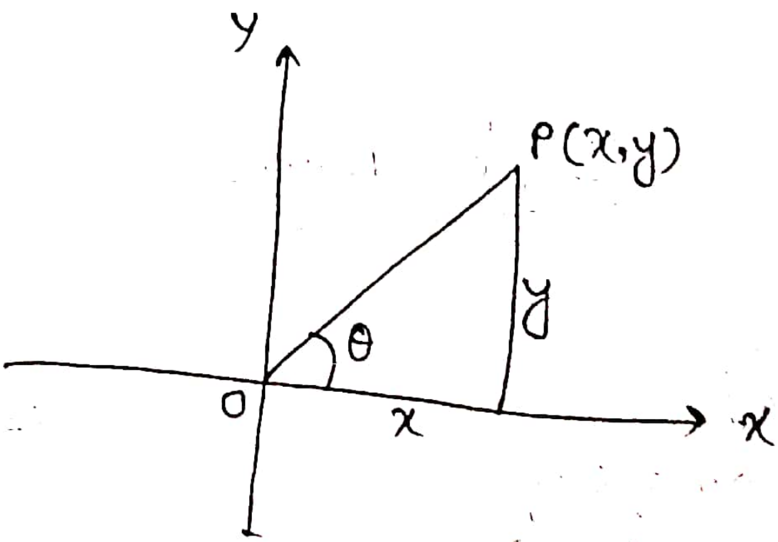
$|\bar{z}| = \sqrt{x^2 + y^2}$

$\therefore |z| = |\bar{z}|$

$\bar{\bar{z}} = x + iy = z$

$\tan 0^\circ = 0$, $\tan \underbrace{30^\circ}_{\pi/6} = \frac{1}{\sqrt{3}}$, $\tan \underbrace{45^\circ}_{\pi/4} = 1$,

$\tan \underbrace{60^\circ}_{\frac{\pi}{3}} = \sqrt{3}$, $\tan \underbrace{90^\circ}_{\frac{\pi}{2}} =$ undefined



$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

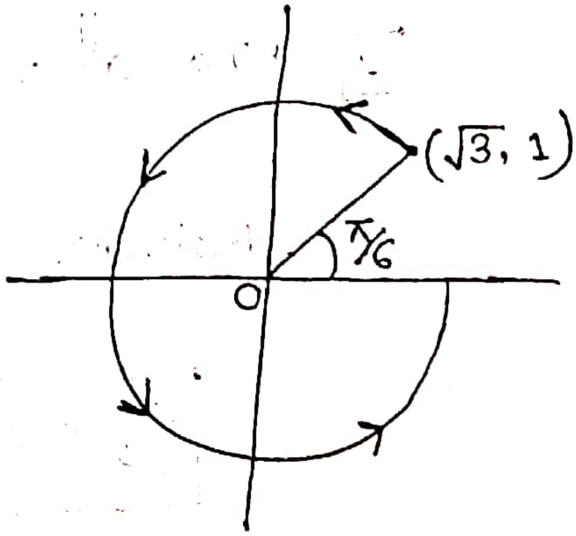
↳ amplitude of z

$$\Rightarrow z + \bar{z} = 2x$$

$$= 2 . Re(z)$$

$$\Rightarrow z - \bar{z} = 2 . i . y$$

$$= 2 . i . In(z)$$



<u>Ex:</u>

$$z = \sqrt{3} + i$$

$$|z| = \sqrt{3+1} = 2$$

$$Arg(z) \ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^{\circ} \text{ or } \frac{\pi}{6}$$

Note:

$z = x + iy$

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Find all values of $(1+i)$

De-Moiverse theorem:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

n = positive or negative (any real number or fraction)

$1 = r\cos\theta$ ...... (i)

$\sqrt{4} = \pm 2$

$1 = r\sin\theta$ .... (ii)

$\theta = \tan^{-1}\frac{1}{1}$

$= \frac{\pi}{4}$

from (i) & (ii), $\frac{r\sin\theta}{r\cos\theta} = 1$

$r^2 = 1+1 = \sqrt{2}$

$(1+i)^{1/4} = (r\cos\theta + ir\sin\theta)^{1/4}$

$= r^{1/4}(\cos\theta + i\sin\theta)^{1/4}$

$= r^{1/4}\{\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)\}^{1/4}$

$= (\sqrt{2})^{1/4}\{\cos\frac{(2n\pi+\theta)}{4}$ $\cancel{+ i\sin(2n\pi+\frac{\pi}{4})\frac{1}{4}}$

$+ i\sin\frac{(2n\pi+\theta)}{4}\}$, $[\because n = 0, 1, 2, 3]$

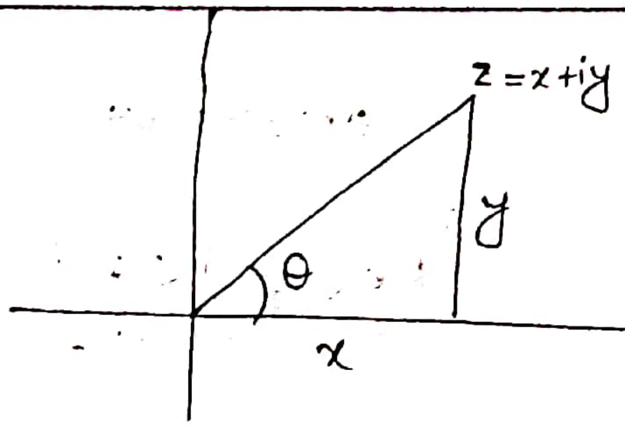
$$= 2^{1/8}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right)\frac{1}{4} + i\sin\left(2n\pi + \frac{\pi}{4}\right)\cdot\frac{1}{4}\right\}$$

$$= 2^{1/8}\left[\cos\,(8n+1)\cdot\frac{\pi}{16} + i\sin\,(8n+1)\frac{\pi}{16}\right]$$

* $n = 0, 1, 2, 3$ বসানোর পরে 4 টা answer আসবে।

Math- IV (Lec - 2)

$$\boxed{z = x+iy}$$



$\sin \xrightarrow{\text{even}} \sin \xrightarrow{\text{odd}} \cos$

$\cos \longrightarrow \cos \longrightarrow \sin$

$\tan \longrightarrow \tan \longrightarrow \cot$

$\theta = \tan^{-1} \frac{y}{x}$

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

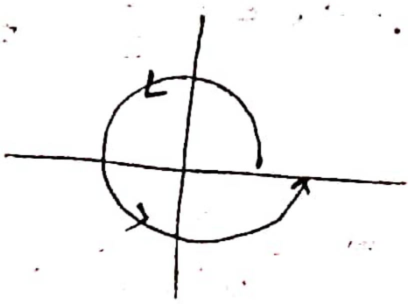
* $(\sqrt{3} - i)^{1/5}$

$\sqrt{3} = r\cos\theta$

$-1 = r\sin\theta$

$r^2 = 4$

$\therefore r = 2$



$$\cancel{\cos\theta + i\sin\theta}$$

$$\boxed{Re\left(\frac{1}{z}\right) < 1}$$

$$z = x + iy$$

$$\therefore \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right)$$

$$\therefore Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$$

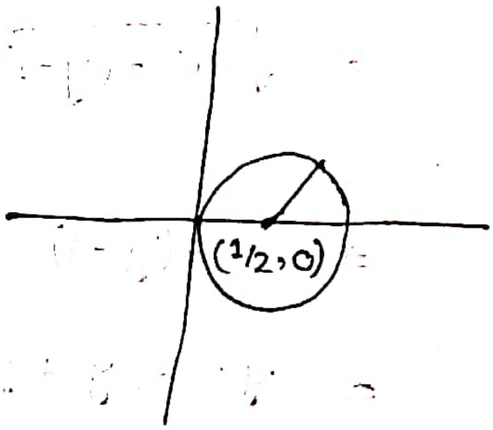
$$\Rightarrow \frac{x}{x^2+y^2} < 1$$

$$\Rightarrow x^2+y^2 > x$$

$$\Rightarrow x^2+y^2-x > 0$$

$$\Rightarrow x^2 - 2x \cdot \frac{1}{2} + \frac{1}{4} + \cancel{\quad} y^2 > \frac{1}{4}$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 > \left(\frac{1}{2}\right)^2$$



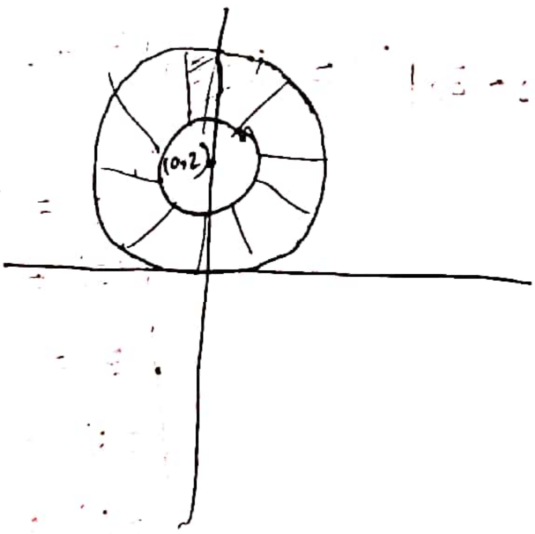
Ex-40

Figure it : $1 < |z - 2i| < 2$

$$\Rightarrow 1 < |x + iy - 2i| < 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y-2)^2} < 2$$

$$\Rightarrow 1 < x^2 + (y-2)^2 < 2^2$$



(*) $|z - i| = |z + \cancel{a}i|$

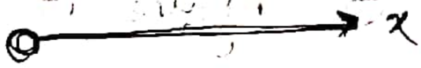
$\Rightarrow |x + iy - i| = |x + iy + \cancel{a}i|$

$\Rightarrow \sqrt{x^2 + (y-1)^2} = \sqrt{x^2 + (y+1)^2}$

$\Rightarrow x^2 + (y-1)^2 = x^2 + (y+1)^2$

$\Rightarrow y^2 - 2y + 1 = y^2 + 2y + 1$

$\Rightarrow 4y = 0$



$\Rightarrow \boxed{y = 0}$ (Ans)

(*) Ex-30
Pg - 38

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

$z = x + iy$

$|z| = x - iy$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$|z|^2 = x^2 + y^2$

$$\Rightarrow (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)}$$

$$\Rightarrow (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

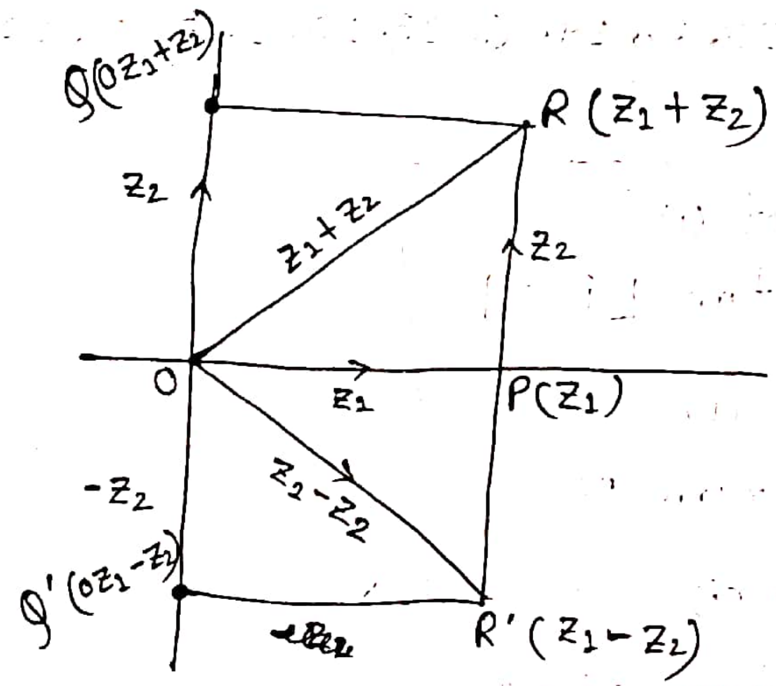
$$\Rightarrow z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_1\bar{z}_2$$

$$- z_2\bar{z}_1 + z_2\bar{z}_2$$

$$\Rightarrow |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$$

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

$$\Rightarrow 2|z_1|^2 + 2|z_2|^2$$



~~Algoritl~~

28/07/2022

Math - IV (Lec 3)

Pg: 1.27

* $$\left|\frac{z-3}{z+3}\right| = 2$$

$$\Rightarrow |z-3| = 2|z+3|$$

$$\Rightarrow |x+iy-3| = 2|x+iy+3|$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$$

$$\Rightarrow (x-3)^2+y^2 = 4\{(x+3)^2+y^2\}$$

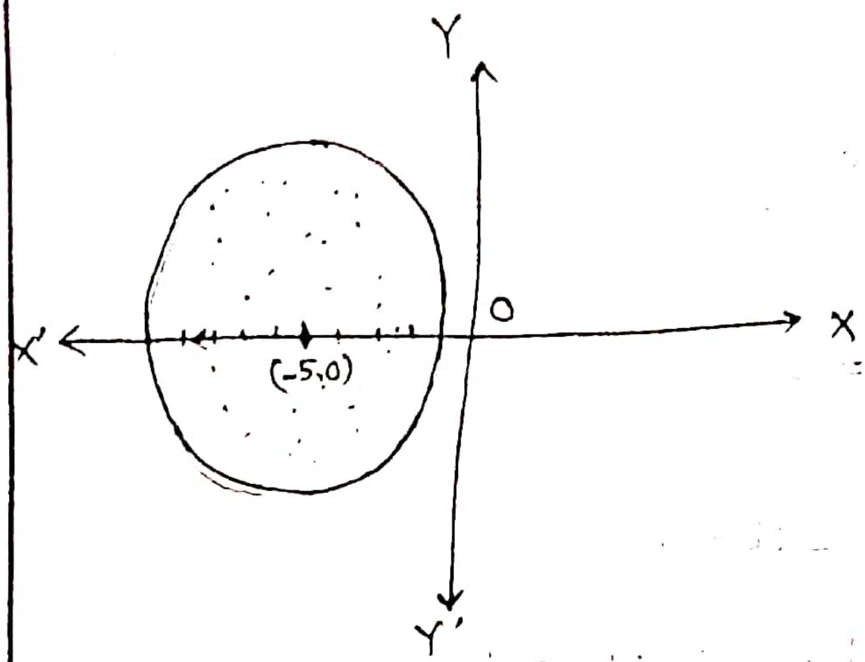
$$\Rightarrow x^2-6x+9+y^2 = 4(x^2+6x+9+y^2)$$

$$\Rightarrow 3x^2+30x+3y^2+27=0$$

$$\Rightarrow x^2+10x+9+y^2=0$$

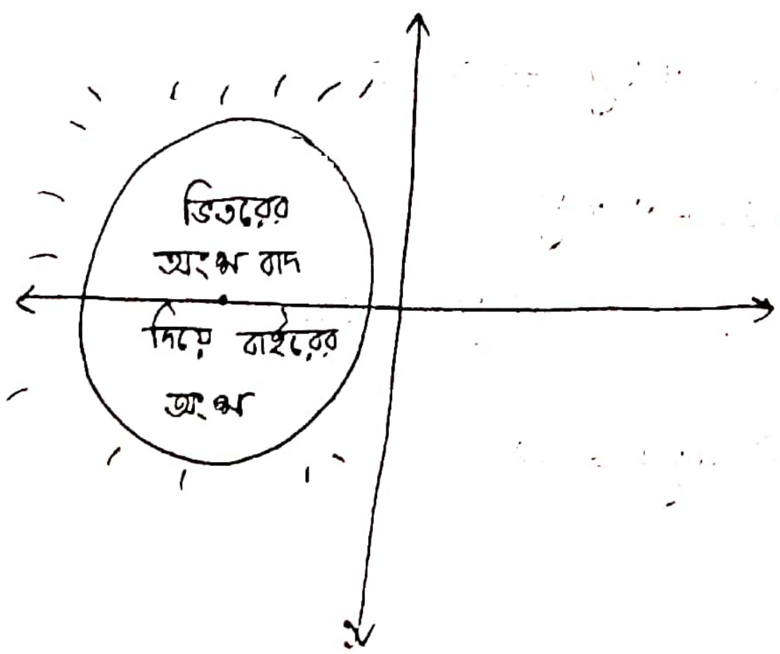
$$\Rightarrow x^2+2x\cdot5+5^2+y^2=16$$

$$\Rightarrow (x+5)^2+y^2=4^2$$



$$\circledast \quad \left| \frac{Z-3}{Z+3} \right| < 2$$

$$\Rightarrow (x+5)^2 + y^2 > 4^2$$



Eg-1.29 (1.28, a, b)

$z^5 = -32$

$\Rightarrow z^5 = 32(\cos\pi + i\sin\pi)$

$\Rightarrow z^5 = 32\{\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\}$.

$1 = \cos 0 + \sin 0$

$\Rightarrow -1 = \cos\pi + \sin\pi$

$\Rightarrow -1 = \cos(2n\pi+\pi) + \sin(2n\pi+\pi)$

$\Rightarrow z^5 = [2^5\{\cos(2n+1)\pi + i\sin(2n+1)\pi\}]$

$\Rightarrow z = 2\{\cos(2n+1)\pi + i\sin(2n+1)\pi\}^{1/5}$, $n = 0, 1, 2, 3, 4$

De-Moiverse Ther^m

$= 2\{\cos(2n+1)\frac{\pi}{5} + i\sin(2n+1)\frac{\pi}{5}\}$

$\boxed{n=0,1,2,3,4}$

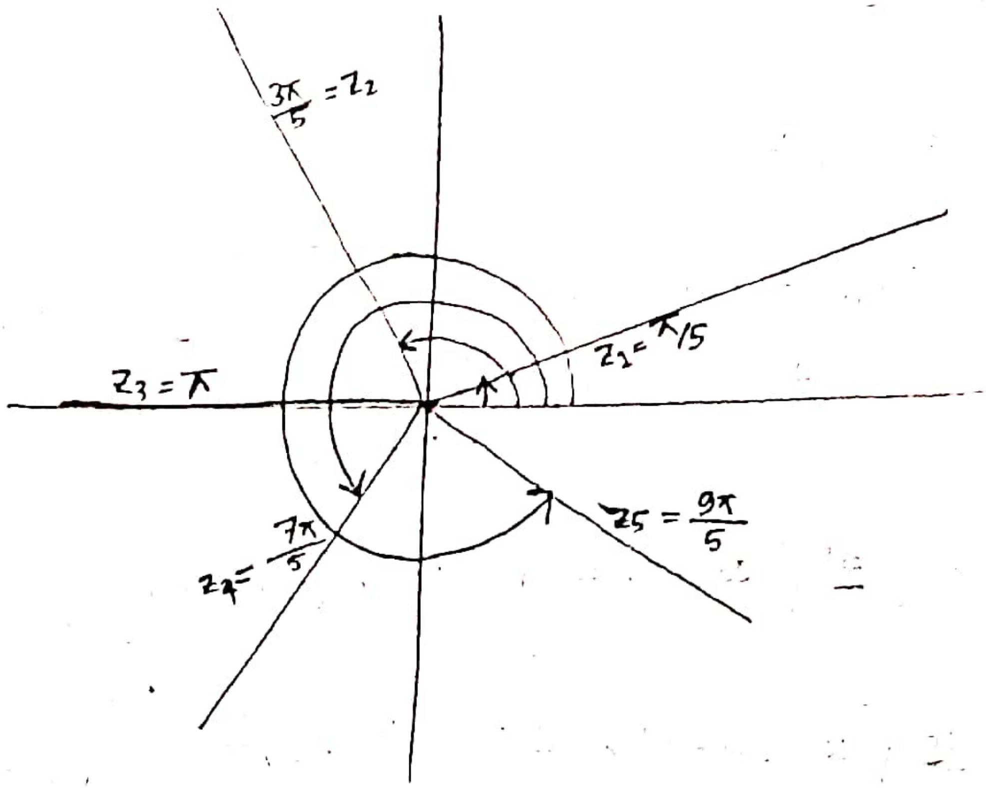
$z_1 = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$

$z_2 = 2(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5})$

$z_3 = 2(\cos\pi + i\sin\pi)$

$z_4 = 2(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5})$

$z_5 = 2(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5})$



1.29 (b)

$$(-2\sqrt{3} - 2i)^{1/4}$$

$$-2\sqrt{3} = r\cos\theta$$

$$-2 = r\sin\theta$$

$$r^2 = (-2\sqrt{3})^2 + (-2)^2$$

$$= 4 \cdot 3 + 4$$

$$= 16$$

$\therefore r = 4,$

$$\tan\theta = \left(\frac{-1}{-\sqrt{3}}\right) = \tan\left(\pi + \frac{\pi}{6}\right)$$

$$\therefore \theta = \frac{7\pi}{6}$$

$$\therefore z = (-2\sqrt{3} - 2i)^{1/4} = (r\cos\theta + r\sin\theta)^{1/4}$$

$$= \{4(\cos\theta + i\sin\theta)\}^{1/4}$$

$$= (2^2)^{1/4}\{\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)\}^{1/4}$$

$$= 2^{1/2}\{\cos(2n\pi + \frac{7\pi}{6}) + i\sin(2n\pi + \frac{7\pi}{6})\}^{1/4}$$

$$= 2^{1/2}\{\cos(12n+7)\frac{\pi}{6} + i\sin(12n+7)\frac{\pi}{6}\}^{1/4}$$

$$= 2^{1/2}\{\cos(12n+7)\frac{\pi}{24} + i\sin(12n+7)\frac{\pi}{24}\}$$

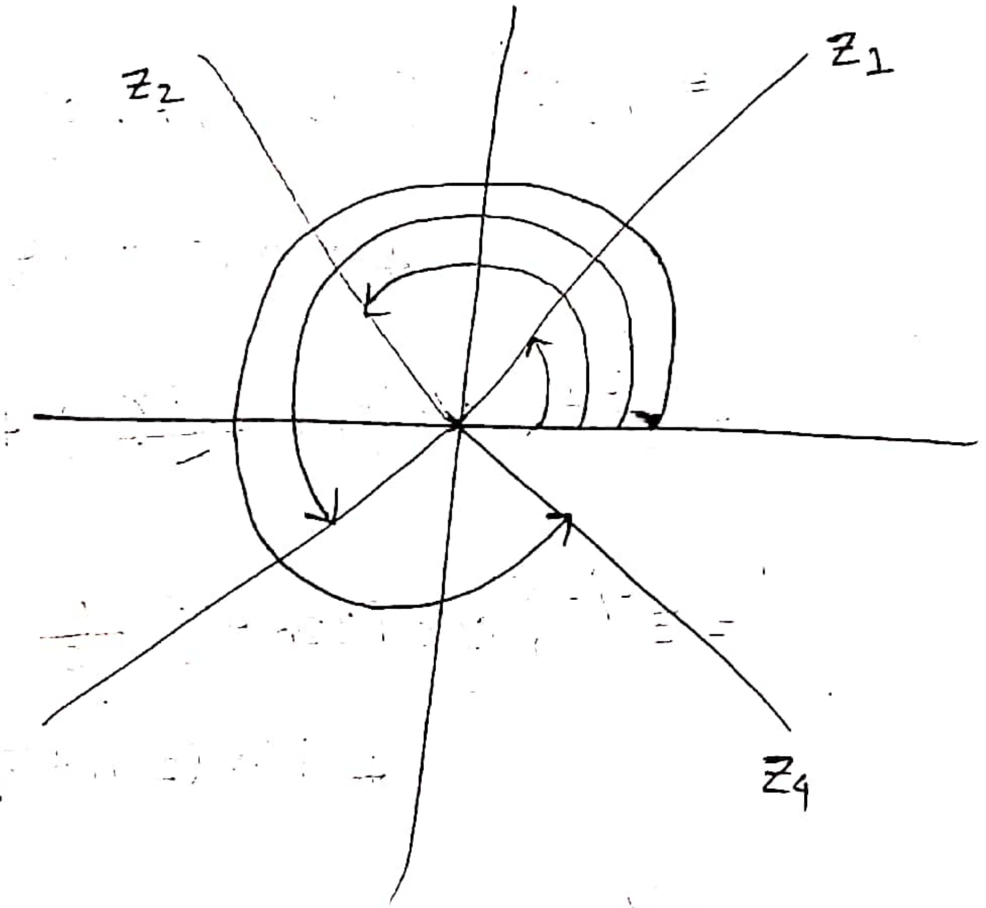
$$[n = 0, 1, 2, 3]$$

$n = 0$, $z_1 = 2^{1/2}\left(\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right)$

$n = 1$, $z_2 = 2^{1/2}\left(\cos\frac{19\pi}{24} + i\sin\frac{19\pi}{24}\right)$

$n = 2$, $z_3 = 2^{1/2}\left(\cos\frac{31\pi}{24} + i\sin\frac{31\pi}{24}\right)$

$n = 3, \quad z_4 = 2^{1/2}\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right).$



Math-IV (Lec. 4)

01/08/2022

C-3

$$y = f'(x)$$

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Differentiable

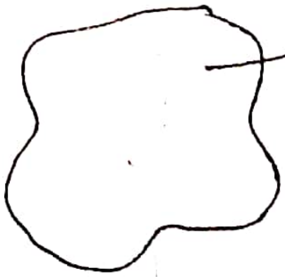
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a) = L$$

(Continuous)

L. H. Derivative = R.H.D = L (Differentiable)

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{f(h)} = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{f(h)}$$

Analytic Region

1. 

Region is differentiable

2. At a point (Differentiable)

\# $w = f(z) = u + iv$

$= f(z) = u(x,y) + v(x,y)$

$u(x,y) = x^2 + 3xy + y^2$

$u_x = 2x + 3y$

$u_{xx} = 2$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

$\boxed{\text{Condition}}$

(Analytic function)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Harmonic function

$$\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Laplas Equation

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \end{aligned}\right\} \text{Harmonic function}$$

Singular Point

$$f(z) \longrightarrow z_0$$

Isolated Singularity

Delta depends on epsilon

$z = z_0$

$f(z) \quad \delta > 0$

$|z - z_0| = \delta$

$\forall \ \varepsilon > 0, \ \exists \ \delta > 0$

$|x - a| < \delta$

$\Rightarrow |f(x) - l| < \varepsilon$

$|x + iy - z_0| = \delta$

$\sqrt{(x - z_0)^2 + y^2} = \delta$

Circle

$z_0 = x_0 + iy_0$

$|x + iy - z_0| = \delta$

$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \delta$

$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = \delta^2$

$\Rightarrow (x_0, y_0) \longrightarrow \delta$

* $|x - a| < \delta$ (not circle it's a distance) $|z| = a$

$|z + 5|$

$|x + 5 + iy|$ → circle

Pole

$$f(x) = \frac{1}{(x-2)^1} \longrightarrow \infty$$

$x=2$

poll of order 1 (single pole)

$\boxed{\text{double pole , triple pole}}$

$$f(z) = \frac{1}{(z-3)^5}$$

poll of order 5

$$f(z) = \frac{1}{(z-3)^5 (z-1)^2 (z-3)^3}$$

5

double pole

triple pole

power যদি ভগ্নাংশ আকারে থাকে (branch point)

$f(x) = (x-1)^{1/3}$

↳ Multiple values of function

$\boxed{z \to z_0}$ Isolated point

$\lim_{z \to z_0} f(z)$

Essential singularity

Singularity

\# $f(z) = f_w \to \infty$

↳ $w = f(z)$

Pg: 3.10

H.T 3.1, 3.2, 3.3

Pg: 3.10

3.1

$$w = f(z) = z^3 - 2z$$

a) $z = z_0$, b) $z = -1$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - z_0^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2 \Delta z + 3 . z_0 \Delta z^2 + \Delta z^3 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(3z_0^2 + 3z_0 \Delta z - 2 + \Delta z^2\right)$$

$$= 3z_0^2 - 2$$

3.2

$\frac{d}{dz}(\bar{z})$.

$f(z) = \bar{z}$

$z = x+iy$

$\Delta z \to 0$

$\Delta x \to 0$

$\Delta y \to 0$

$z = x+iy$

$\Delta z = \Delta x + i\Delta y$

$\frac{d}{dz}(f(z)) = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$

$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$

$z = x+iy$

$\bar{z} = x-iy$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x+\Delta x) - i(y+\Delta y) - (x - iy)}{\Delta x + i\Delta y}$$

$\Delta y = 0, \quad \Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{x + \Delta x - iy - x + iy}{\Delta x}$$

$$= \lim_{\Delta x} \frac{\Delta x}{\Delta x}$$

$= 1$

Again, $\Delta x = 0, \quad \Delta y \to 0$

$$\lim_{\Delta y \to 0} \frac{x - iy - i\Delta y - x + iy}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

∴ উক্ত point এ derivative না, যেহেতু not equal.

Math - IV (Lec 5) 01/08/2022

State & prove Cauchy-Riemann Equation

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

$z = x + iy$

$\Delta z = \Delta x + i\Delta y$

$w = f(z) = u + iv$

$= u(x,y) + v(x,y)$

$f(z) = u(x,y) + iv(x,y)$

$f(z + \Delta z) =$

Starting

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iu(x+\Delta x, y+\Delta y) - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y}$$

...... (i)

Case - I: $\Delta y = 0, \Delta x \to 0$

$$\therefore \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - \{u(x,y) + iv(x,y)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Case - II

$\Delta y \to 0 ; \Delta x = 0$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - \{u(x,y) + iv(x,y)\}}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + i \frac{v(x,y+\Delta y) - v(x,y)}{i\Delta y} \right]$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = f'(z)$$

$$\therefore \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

Continuous

$$\frac{\partial u}{\partial x} = \boxed{\frac{\partial u}{\partial y} = x + 3y = f_1(x, y)}$$

$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

$$= u(x+\Delta x, y+\Delta y) + u(x, y+\Delta y) -$$

$$- u(x, y+\Delta y) - u(x,y)$$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\}$$

$$+ \{u(x, y+\Delta y) - u(x,y)\}$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y$$

$$= \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + \varepsilon_1 \Delta x$$
$$+ \eta_1 \Delta y$$

$\Delta x \longrightarrow 0 \longrightarrow \varepsilon_1 \longrightarrow 0$

$\Delta y \longrightarrow 0 \longrightarrow \eta_1 \longrightarrow 0$

similarly, v এর জন্য করতে হবে।

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y} \cdot \Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

$w = u + iv$

$\Delta w = \Delta u + i\Delta v$

$$= \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + i\frac{\partial v}{\partial x} \cdot \Delta x$$
$$+ i\frac{\partial v}{\partial y} \cdot \Delta y + (\varepsilon_1 + i\varepsilon_2)\Delta x$$
$$+ (\eta_1 + i\eta_2)\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$

$$+ \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y$$

$$+ \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(i^2\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y$$

$$+ \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon \Delta x + \eta \Delta y$$

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z + \varepsilon \Delta x + \eta \Delta y$$

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Math - IV (Lec 6) 08.08.2022

Pg - 3.13

(3.7) Prove that, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. ----(i)

D.p.w.r to x

partially

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

Harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y + xe^{-x}\sin y - e^{-x}\sin y - ye^{-x}\cos y$$

$$= xe^{-x}\sin y - 2e^{-x}\sin y - ye^{-x}\cos y \quad \text{----(ii)}$$

D. p. w. r. to y,

$$\frac{\partial u}{\partial y} = e^{-x}\left(x\cos y + y\sin y - \cos y\right)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}\left(-x\sin y + y\cos y + \sin y + \sin y\right)$$

$$= 2e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \ldots (iii)$$

Adding (ii) & (iii), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{(harmonic)}$$

Cauchy - Riemann Equ^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{-x}\sin y - xe^{x}\sin y + ye^{-x}\cos y \quad \ldots (i)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -xe^{-x}\cos y - ye^{-x}\sin y + e^{-x}\cos y \quad \ldots ($$

Integrating (ii) with respect to y (x is constant)

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\left[y\sin y - \int 1.\sin y\,dy\right]$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + ye^{-x}\sin y + e^{-x}\cos y$$

$$+F(x) \hookrightarrow \text{I.C}$$

$$y\int\cos y - \int 1.\sin dy$$
$$= y\sin y + \cos y$$

$$v = xe^{-x}\cos y + ye^{-x}\sin y + F(x)$$

$$\frac{\partial v}{\partial x} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + F'(x) \quad \text{....(iv)}$$

From (iii) & (iv) $F'(x) = 0$

$$\therefore F(x) = c$$

$$v = xe^{-x}\cos y + ye^{-x}\sin y + c$$

Pg. 3.37

$$f'(x) = u_1(x,0) - i u_2(x,0)$$

$$f'(z) = u_1(z,0) - i u_2(z,0)$$

$$u_1 = \frac{\partial u}{\partial x}$$

(মিলনে - Thomson Method)

$$u_2 = \frac{\partial u}{\partial y}$$

$$u_1(x,y) = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$u_2(x,y) = \frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$f'(x) = 0 - i(xe^{-x} - e^{-x})$$

$$f'(z) = -ize^{-z} + ie^{-z}$$

$$\therefore f(z) = -i\left\{-ze^{-z} + \int 1.e^{-z}\,dz\right\} - ie^{-z}$$

$$= ize^{-z} + ie^{-z} - ie^{-z}$$

$$= ize^{-z}$$

$w = f(z) = u + iv$

$ize^{-z} = u + iv$

$\Rightarrow u + iv = i(x + iy) \cdot e^{-(x+iy)}$

$\Rightarrow u + iv = i(x + iy)\, e^{-x} \cdot e^{-iy}$

$\Rightarrow u + iv = i(x + iy)\, e^{-x} (\cos y - i \sin y)$

$= e^{-x} [(ix - y)(\cos y - i \sin y)]$

$\Rightarrow u + iv = e^{-x} [ix \cos y + x \sin y - y \cos y + iy \sin y]$

$= i[x e^{-x} \cos y + y e^{-x} \sin y] + e^{-x} x \sin y - e^{-x} y \cos y$

Equating imaginary part,

$v = x e^{-x} \cos y + y e^{-x} \sin y$ (Ans)

Math- IV (Lec 7)

22/08/2022

Pg: 3.11
(3.9)

Pg: 3.15
(3.10)

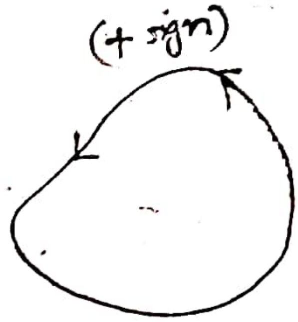
Pg: 3.16
(3.11)

Pg: 3.18
(3.14)

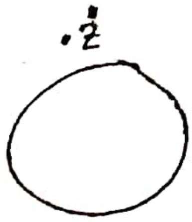
Pg: 3.20
(3.18)

Pg: 38 , Pg: 39
Cauchy Riemann

$\oint_C$ → closed integration



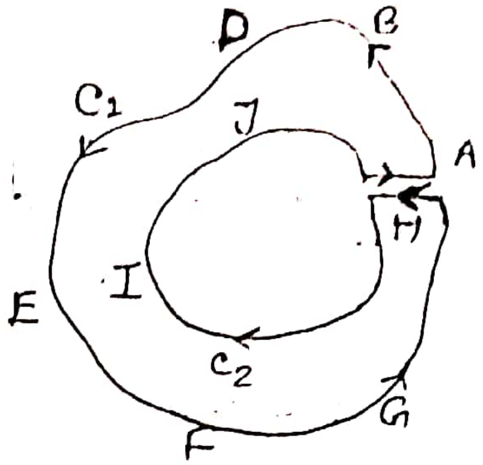
$\oint f(z)dz = 0$



$\oint_C f(z)dz = \oint_{C_1} f(z)dz$

Pg: 4.18

(4.16)



$$\int_{ABDEIGAHIJA} f(z)dz = 0$$

$$\int_{ABDEFGA} f(z) + \int_{AH} f(z)dz + \int_{HIJA} f(z)\,dz + \int_{HA} f(z)dz = 0$$

opposite

$$\Rightarrow \int_{ABDEFGA} f(z)dz + \int_{HIJH} f(z)dz = 0$$

$$\Rightarrow \oint_{C} f(z)\,dz = 0$$

~~Pg 4.22~~

Γ

Pg. 4.21

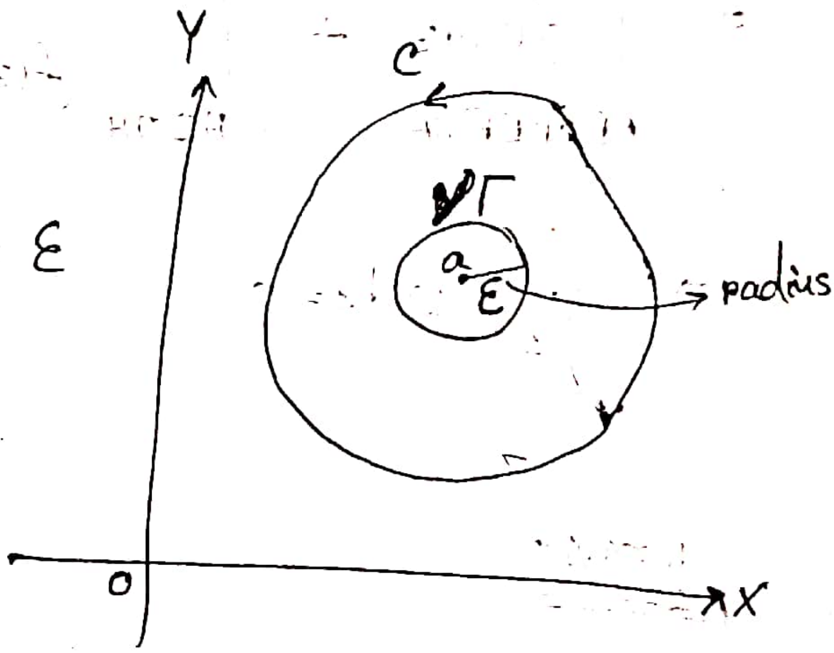
\# $\oint_c \left(\frac{1}{z-a}\right) dz$ $\qquad f(z) = \frac{1}{z-a}$

$z - a = 0$

$\boxed{z = a}$ → singular point

a) $z = a$ outside c

b) $z = a$ inside c



Equation of Γ $|z-a| = \varepsilon$

$$\oint_c \frac{1}{z-a}\,dz = \oint_\Gamma \frac{dz}{z-a}$$

$$|z-a| = \varepsilon$$

$$|z-a| = \varepsilon |e^{i\theta}|$$

$$z-a = \varepsilon e^{i\theta}$$

$$z = a + \varepsilon e^{i\theta}$$

$$dz = i\varepsilon e^{i\theta} d\theta$$

$$|z-a| = \varepsilon |e^{i\theta}|$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\oint_C \frac{1}{z-a}\,dz = \oint_\Gamma \frac{dz}{z-a}$$

$$= \int_0^{2\pi} \frac{\varepsilon i e^{i\theta}}{\varepsilon e^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

Green's Theorem

$\nabla \cdot \phi \quad \nabla \cdot v \quad \nabla \times v$

Green's Theorem

$$\oint_C p\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

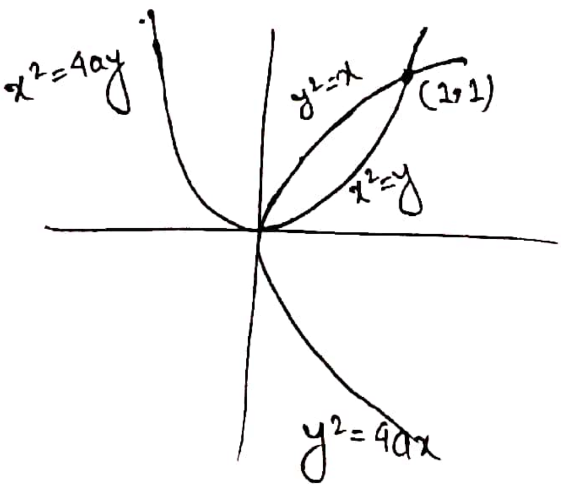
$\iint$
↓
double integration
= region

Pg: 4.10

4.5

$$\oint_C \underbrace{(2xy - x^2)}_{P}\,dx + \underbrace{(x+y^2)}_{Q}\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$\boxed{\begin{array}{l} y = x^2 \\ x = y^2 \end{array}}$$

$$\int_{x=0}^{1}\left[(2x\cdot x^2 - x^2)\,dx + (x+x^4)\,d(x^2)\right]$$

$x = y^2$

$x^2 = y$

$x = x^4$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$$= \int_0^1 \left[(2x^3 - x^2)\,dx + (x+x^4)\,2x\,dx\right]$$

$$= \left[\int_0^1 (2x^3 - x^2) + (2x^2 + 2x^5)\right]dx$$

$$= \int_0^1 (2x^3 - x^2 + 2x^2 + 2x^5)\,dx$$

$$= \left[2\frac{x^4}{4} + \frac{x^3}{3} + 2\frac{x^6}{6}\right]_0^1$$

$$= \frac{2}{4} + \frac{1}{3} + \frac{2}{6}$$

$$= \frac{7}{6}$$

$$\int_{y=1}^{0}\left[(2\cdot y^2\cdot y - y^4)\,d(y^2) + (y^2 + y^2)\,dy\right]$$

$$= \int_{y=1}^{0}\left[(2y^3 - y^4)\,2y\,dy + 2y^2\,dy\right] = \cancel{\frac{17}{30}}$$

$$\Rightarrow \frac{1}{30} \qquad \cancel{\frac{1}{30}}$$

Now,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x+y^2) - \frac{\partial}{\partial y}(2xy - x^2) \right] dx\,dy$$

$$= \iint_R (1-2x)\,dx\,dy$$

$y = x^2$
$y = \sqrt{x}$

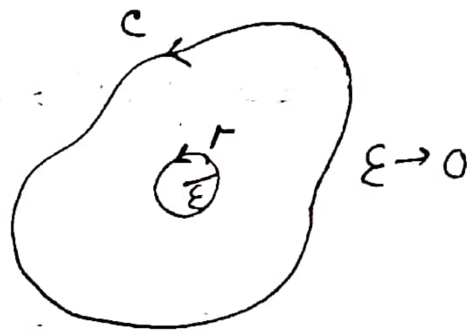
$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1-2x)\,dy\,dx$$

Pg: 5.3 (Math-IV) Lec 8 24.08.2022

5.1.

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$

$$\oint_c \frac{f(z)}{z-a} dz = \oint_\Gamma \frac{f(z)}{z-a} dz$$



$$= \int_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} . i\varepsilon e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} f(a+\varepsilon e^{i\theta}) d\theta$$

$|z-a| = \varepsilon |e^{i\theta}|, 0 \le \theta < 2\pi$

$z-a = \varepsilon e^{i\theta}$

$z = a + \varepsilon e^{i\theta}$

$dz = i\varepsilon e^{i\theta} d\theta$

$$\lim_{\varepsilon \to 0} \frac{f(z)}{z-a} dz = \lim_{\varepsilon \to 0} \int_0^{2\pi} f(a+\varepsilon e^{i\theta}) d\theta$$

$$= i \int_0^{2\pi} f(a) d\theta$$

$$= i f(a) [\theta]_0^{2\pi}$$

$$\therefore \oint_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

$$\# \quad f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}\,dz$$

$$f''(a) = \frac{\underline{|n}}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$* \quad f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

$$f(a+h) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-(a+h)}\,dz$$

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_c \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{1}{h}\,\frac{z-a-z+a+h}{(z-a-h)(z-a)}\,f(z)\,dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i} \oint_C \frac{f(z)}{(z-a-h)(z-a)^2}}$$

ল.আ.শূ.

$$\lim_{h \to 0} f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\,dz$$

Math - IV (Lec 9).

$\boxed{29.08.2022}$

$$\# \oint_C \frac{dz}{z-a} = 0 \quad ; \quad z = a \text{ outside}$$

$$\oint_C \frac{dz}{z-a} = 2\pi i \quad ; \quad z = a \quad \text{inside}$$

P-5.5

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz \qquad \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{(z-1)(1-2)} + \frac{1}{(2-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$$

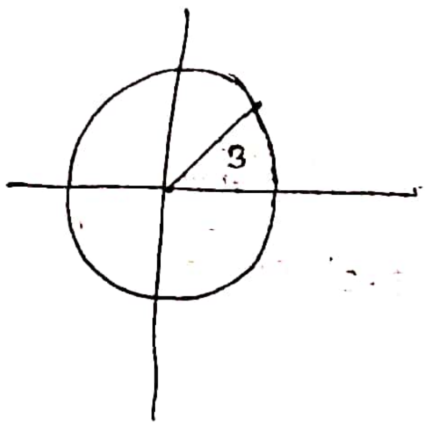
$$= \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} - \oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} dz$$

circle $|z| = 3$

Here, $f(z) = \sin\pi z^2 + \cos\pi z^2$

$|z| = 3 \quad \therefore z = 2$ and $z = 1$

inside c



$|z| = |x + iy| = 3$

$\sqrt{x^2 + y^2} = 3$

$x^2 + y^2 = 3^2$

$f(2) = \sin \pi 2^2 + \cos \pi 2^2 = 1$

$f(1) = \sin \pi 1^2 + \cos \pi 1^2 = -1$

$\therefore = 2\pi i - (-2\pi i)$

$= 4\pi i$

#(b) $\oint_c \frac{e^{2z}}{(z+1)^4} dz$

Here, $a = -1$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

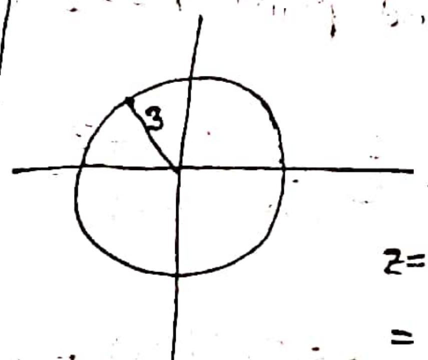
$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^2} dz$

$f^n(a) = \frac{\lfloor n}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz$

$\oint_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{\lfloor n} f^n(a)$



$z = -1$

$= -1 + 0i$

$\therefore \oint_c \frac{e^{2z}}{(z+1)^{3+1}} dz = \frac{2\pi i}{\lfloor 3} \cdot f'''(-1)$

$\Rightarrow \oint_c \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{6} \times 8e^{-2} = \frac{8}{3}\pi i e^{-2}$

Pg: 57

(57) $\oint_C \frac{3z^2+z}{z^2-1}\,dz$ $\quad C \to |z-1|=1$

$= \oint_C \frac{3z^2+z}{(z-1)(z+1)}\,dz$

$\frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right) = \frac{z+1-(z-1)}{(z-1)(z+1)}$

$= \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz$

$-\frac{1}{2}\oint_C \frac{3z^2+z}{z+1}\,dz$

$|z-1|=1$

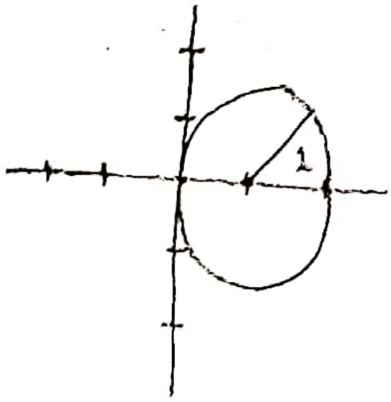
$|x-1+iy|=1$

$\sqrt{(x-1)^2+y^2}=1$

$(x-1)^2+y^2=1^2$

The point $x=1$ inside control C

and point $x=-1$ outside C



$\therefore \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz$

$= \frac{1}{2}\cdot 2\pi i f(1) = \pi i\,(3\cdot 1^2+1)$

$= 4\pi i$ (Ans)

$-1 = -1+0i$

outside

$\Rightarrow \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}\,dz = 0$

$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$

$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

Pg: 5.22

5.39 $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ (a) $|z|=3$ (b) $|z|=1$

(a) Here, $f(z)=e^z$

the point $z=2$ inside the $|z|=3$

By cauchy's integral formula

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = f(2) = e^2$$

(b) The point $z=2$ outside $|z|=1$

By cauchy's theorem $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0$

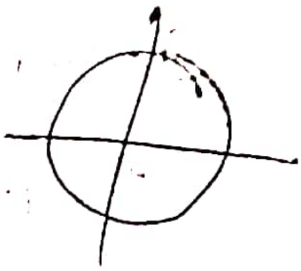
5.41 $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ (a) $|z-1|=4$

$\pi = 3.142 < 4$

(a) The point $z=\pi i$ inside $|z-1|=4$, So, cauchy's integral formula.

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$f(z) = e^{3z}$



$$\Rightarrow \oint_C \frac{e^{3z}}{z-\pi i}\,dz = 2\pi i f(\pi i)$$

$$= 2\pi i\, e^{3\pi i}$$

$$= 2\pi i\,(\cos 3\pi + i\sin 3\pi)$$

$$= -2\pi i$$

(b) $|z-2| + |z+2| = 6$

$$\Rightarrow |x-2+iy| + |x+2+iy| = 6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} = 6 - \sqrt{(x+2)^2+y^2}$$

$$\Rightarrow (x-2)^2+y^2 = 36 - 12\sqrt{(x+2)^2+y^2} + (x+2)^2 + y^2$$

$$\Rightarrow x^2 - 4x + 4 = 36 - 12\sqrt{(x+2)^2 + y^2} + x^2 + 4x + 4$$

(b) $\displaystyle\oint_C \frac{e^{3z}}{z - \pi i}\, dz$

$$\Rightarrow 12\sqrt{(x+2)^2 + y^2} = 36 + 8x$$

$$\Rightarrow 3\sqrt{(x+2)^2 + y^2} = 9 + 2x$$

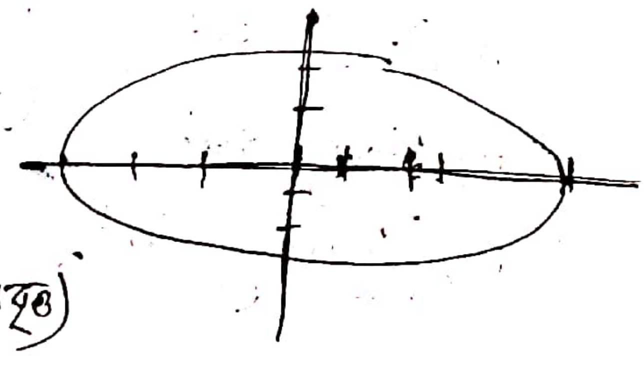
$$\Rightarrow 9(x^2 + 4x + 4 + y^2) = 81 + 36x + 4x^2$$

$$\Rightarrow 9x^2 + 36x + 36 + 9y^2 - 36x - 4x^2 = 81$$

$$\Rightarrow 5x^2 + 9y^2 = 45$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$$

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$$ (উপবৃত্ত)



which is equation of the eellipse whose major axis is 3. and minor axis is $\sqrt{5}$.

So, the point $z = \pi i$ outside c.

Hence, by cauchy's theorem, we get

$$\oint_c \frac{e^{3z}}{z - \pi i} \, dz = 0$$

Math- IV, Lec - 10

01.09.2022

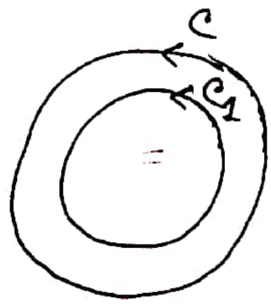
Pg: 6.17

Theorem : 6.22

Taylor's Theorem

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \frac{(z-a)^3}{\lfloor 3} f'''(a)$$

Proof:



$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$

$$f(z) = \frac{1}{2\pi i} \oint_{c_1} \frac{f(w)}{w-z} dw \quad \cdots\cdots (i)$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left\{1 - \frac{z-a}{w-a}\right\}}$$

$$= \frac{1}{w-a}\left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \frac{(z-a)^2}{(w-a)^2} + \cdots\cdots + \left(\frac{z-a}{w-a}\right)^{n-1} + \right.$$

$$+ \left(\frac{z-a}{w-a}\right)^n + \left(\frac{z-a}{w-a}\right)^{n+1} + \dots \Big]$$

$$\Rightarrow \frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \dots\dots$$

$$+ \frac{(z-a)^{n-3}}{(w-a)^n} + \frac{(z-a)^n}{(w-a)} \left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2\right]$$

$$= \dots\dots + \frac{1}{w-a}\left(\frac{z-a}{w-a}\right)^n \left[1 - \frac{z-a}{w-a}\right]^{-1}$$

$$= \dots\dots + \frac{1}{w-a}\left(\frac{z-a}{w-a}\right)^n \frac{1}{1 - \frac{z-a}{w-a}}$$

$$= \dots\dots + \frac{1}{w-a}\left(\frac{z-a}{w-a}\right)^n \frac{1}{\frac{(w-a)-(z-a)}{w-a}}$$

$$\therefore \frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} +$$

$$+ \dots\dots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \cdot \frac{1}{w-z}$$

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw$$

$$+\frac{(z-a)}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw$$

$$+\ldots\ldots+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw$$

$$+\ldots\ldots+U_n \,/\, \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw$$

$$\Rightarrow f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a)$$

$$+\ldots\ldots+\frac{(z-a)^{n-1}}{\lfloor n-a}f^{n-1}(a) + U_n$$

Math-IV, Lec-11

05.09.2022

$\lim_{n\to\infty} |U_n| \longrightarrow 0$

$$|U_n| = \left|\frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw\right|$$

$$|U_n| = \frac{1}{2\pi}$$

$$|U_n| < \frac{1}{2\pi}\cdot 2\pi r_1 \cdot \frac{M\cdot \gamma^n}{|w-z|}$$

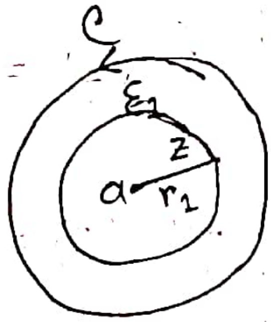
$$|U_n| < \frac{\gamma^n M r_1}{|w-z|}$$

$$|w-z| = |(w-a)-(z-a)|$$

$$= |r_1 - (z-a)|$$

$$|w-z| \geq |r_1| - |z-a|$$

$$\frac{1}{|w-z|} \le \frac{1}{r_1 - |(z-a)|}$$



$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|f(w)| < M$

$|z_1 z_2| = |z_1||z_2|$

$\oint_{C_1} dw$

$$\left|\frac{z-a}{w-a}\right| = \gamma < 1$$

$$|U_n| < \frac{r^n M r_1}{r_1 - |z-a|}$$

$$\lim_{n\to\infty} |U_n| < 0$$

$f(z) = \ln(1+z)$ ([illegible])

$$f'(z) = \frac{1}{1+z} \qquad \therefore f(0) = 0$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad \therefore f''(0) = 1$$

$$f'''(z) = \frac{(-1)(-1)\cdot 2}{(1+z)^3} \qquad \therefore f''(0) = -1$$

$$= \frac{(-1)^2 \lfloor\underline{2}}{(1+z)^3} \qquad \therefore f'''(0) = \lfloor\underline{2}$$

$$f^{n+1}(z) = \frac{(-1)^n \lfloor\underline{n}}{(1+z)^{n+1}} \qquad \therefore f^{n+1}(0) = (-1)^n \lfloor\underline{n}$$

$$f(z) = \ln(1+z) = f(0) + z f'(0) + \frac{z^2}{\lfloor 2} f''(0) + \frac{z^3}{\lfloor 3} f'''(0) + \cdots$$

$$= 0 + z - \frac{z^2}{\lfloor 2} + \frac{2z^3}{\lfloor 3} - \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

Another method,

$$\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

$$\int_0^z \frac{dz}{1+z} = \int_0^z (1 - z + z^2 - z^3 + \cdots) dz$$

$$\Rightarrow \left[\ln(1+z)\right]_0^z = \left[z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\right]_0^z$$

$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\cdots$

২। $$u_n = \frac{(-1)^{n-1}}{n} z^n$$

নিয়ম

নির্ণয়

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{nz}{n\left(1+\frac{1}{n}\right)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{z}{1+\frac{1}{n}}\right|$$

$$= |z| < 1$$

$$\frac{U_{n+1}}{U_n} = \frac{(-1)^n z^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} \cdot z^n}$$

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{n z^{n+1}}{(n+1) z^n}\right|$$

$$= \left|\frac{nz}{n+1}\right|$$

$\ln\left(\frac{1+z}{1-z}\right)$ $\quad z = 0$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

$$\ln(1+z) - \ln(1-z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4}$$
$$+ z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$$

$$\Rightarrow \ln\left(\frac{1+z}{1-z}\right) = 2z + \frac{2z^3}{3} + \frac{2z^5}{5} + \cdots$$

$$\Rightarrow \ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} 2 \cdot \frac{z^{2n+1}}{2n+1}$$

Pg: 6.24

\# $f(z) = \dfrac{1}{(z+1)(z+3)}$ $\quad 1 < z < 3$

$$= \frac{1}{(z+1)(-1+3)} + \frac{1}{(-3+1)(z+3)}$$

$$= \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{3+z}\right]$$

$$= \frac{1}{2}\left(\frac{1}{1+z}\right) - \frac{1}{2}\left(\frac{1}{3+z}\right)$$

$1 < |z|$

$$= \frac{1}{2z} \cdot \frac{1}{\left(1+\frac{1}{z}\right)} - \frac{1}{2} \cdot \frac{1}{3\left(1+\frac{z}{3}\right)}$$

$\frac{1}{|z|} < 1$

$|z| < 3$

$\frac{|z|}{3} < 1$

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\cdot\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

$$= \cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

Math-IV (Lec 12)

08/09/2022

Page. 6.20

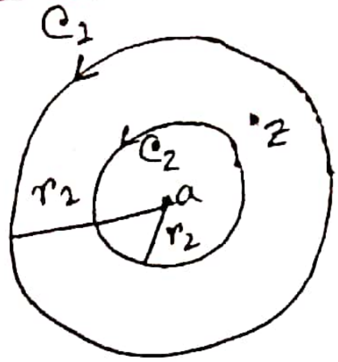
formula 5.19

6.25

Laurent's Theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(w)}{(w-a)^{n+1}}, \quad n = 0, \ldots$$



$r_1 > r_2$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)\,dw}{(w-a)^{-n+1}}, \quad n = 1 \ldots\ldots$$

Proof:

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z}\,dw \quad \text{----(i)}$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a} \cdot \left(1+\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2$$

$$+ \left(\frac{z-a}{w-a}\right)^3 + \cdots\cdots + \left(\frac{z-a}{w-a}\right)^n$$

$$+ \left(\frac{z-a}{w-a}\right)^{n+1} + \left(\frac{z-a}{w-a}\right)^{n+1} + \cdots\cdots$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3\right.$$

$$\left. + \cdots\cdots + \left(\frac{z-a}{w-a}\right)^n \left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\right]\right]$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots\cdots + \frac{(z-a)^{n-1}}{(w-a)^n}$$

$$+ \frac{(z-a)^n}{(w-a)^{n+1}} \left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$\Rightarrow \frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots$$

$$\cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n$$

$$\boxed{\frac{1}{\left(1 - \frac{z-a}{w-a}\right)(w-a)}} \rightarrow \frac{1}{w-z}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw$$

$$+ (z-a)\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw$$

$$+ \frac{(z-a)^2}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^3}\,dw + \dots\dots$$

$$\dots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)^n}$$

$$+ \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)^n(w-b)}$$

$$= a_0 + (z-a)a_1 + (z-a)^2 a_2 + \dots\dots$$

where $a_0 =$

$a_1 =$

$a_2 =$

$a_{n+1} =$

$U_n =$

$$\Rightarrow -\frac{1}{w-z} = \frac{1}{z-w} = \frac{1}{(z-a)-(w-a)}$$

$$= \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}$$

$$= \frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$\Rightarrow -\frac{1}{w-z} = \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3}$$

$$+ \dots \dots + \frac{(w-a)^{n-1}}{(z-a)^n} + \left(\frac{w-a}{z-a}\right)^n \frac{1}{z-w}$$

$$\Rightarrow -\frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(z-a)}\,dw$$

$$+ \frac{w-a}{2\pi i}\oint_{C_2} \frac{f(w)}{(z-a)^2}\,dw$$

$$+ \frac{(w-a)^2}{2\pi i}\oint_{C_2} \frac{f(w)}{(z-a)^3}\,dw + \dots$$

$$\dots + \frac{(w-a)^{n-1}}{2\pi i}\oint_{C_2} \frac{f(w)\,dw}{(z-a)^n} + V_n$$

$$= \frac{\cancel{a_{00}}\, a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\cdots + \frac{a_{-n}}{(z-a)^n}$$

$$+ V_n$$

$$f(z) = [a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{n-1}(z-a)^{n-1}]$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\cdots + \frac{a_{-n}}{(z-a)^n}$$

$$+ U_n + V_n$$

$$|V_n| = \left| \frac{1}{2\pi i} \oint_{C_2} \left(\frac{w-a}{z-a}\right)^n \frac{f(w)\,dw}{z-w} \right|$$

$$\leq \frac{1}{2\pi} \cdot \frac{M \cdot \gamma^n}{|z-w|\, r_2} \cdot 2\pi r_2 \longrightarrow \left(\frac{w-a}{z-a}\right)^n$$

$$r < 1$$

$$|z-w| = |(z-a) - (w-a)| = |z-a-r_2|$$

$$\lim_{n\to\infty} \leq 0$$

$$V_n <$$

Math - IV (Lec 13)

12/09/2022

Pg: 6.22

Ex: 2.66

$$\frac{e^{2z}}{(z-1)^3} \quad ; \quad z = 1$$

$$u = z - 1$$

$$z = u + 1$$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+u)}}{u^3} = \frac{e^2 \cdot e^{2u}}{u^3}$$

$$= \frac{e^2}{u^3} \left\{ 1 + \frac{2u}{\lfloor 1} + \frac{(2u)^2}{\lfloor 2} + \frac{(2u)^3}{\lfloor 3} + \frac{(2u)^4}{\lfloor 4} + \cdots\cdots \right\}$$

$$= \frac{e^2}{u^3} \left\{ 1 + \frac{2u}{1} + \frac{2u^2}{1} + \frac{4u^3}{3} + \frac{2u^4}{3} + \cdots\cdots \right\}$$

$$= e^2 \cdot \left\{ \frac{1}{u^3} + \frac{2}{u^2} + \frac{2}{u} + \frac{4}{3} + \frac{2}{3} u + \dots \dots \right\}$$

$$\Rightarrow \frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \dots \dots$$

6.26(b)

$(z-3)\sin\left(\frac{1}{z+2}\right)$; $z=-2$

$z+2 = u$

$z = u-2$

$$\therefore (z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u}$$

$$= (u-5)\left(\frac{1}{u} - \frac{1}{\lfloor 3 \, u^3} + \frac{1}{\lfloor 5 \, u^5} - \frac{1}{\lfloor 7 \, u^7} + \dots \dots\right)$$

$$= \frac{u-5}{u} - \frac{u-5}{6u^3} + \frac{u-5}{120u^5} - \frac{u-5}{5040u^7} + \dots \dots$$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots$$

$$= 1 - \frac{5}{u} - \frac{1}{6u^2} + \frac{5}{6u^3} + \frac{1}{120u^4} - \frac{1}{24u^5}$$

$$- \frac{1}{5040u^6} + \frac{5}{5040u^7} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3}$$

$$+ \frac{1}{120(z+2)^4} - \frac{1}{24(z+2)^5} + \cdots$$

Pg: 6.24

Ex: 6.27

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$0 < |z+1| < 2$

$z+1 = u$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)}$$

$0 < |u| < 2$

$z = u-1$

$$= \frac{1}{2u\left(1+\frac{u}{2}\right)}$$

$0 < |u| < 2$

$u < 2$

$\frac{u}{2} < 1$

$\cancel{2u}$

$$= \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8}+\frac{u^4}{16}-\cdots\right)$$

$$= \frac{1}{2u}-\frac{1}{4}+\frac{u}{8}-\frac{u^2}{16}+\frac{u^3}{32}-\cdots$$

Lec 19 (Math IV) 15.09.2022

~~Cha~~ Page: 7.7

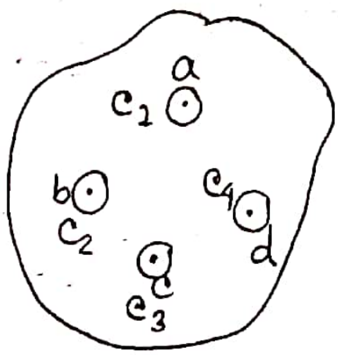
Theorem: 7.2

$$\oint_C f(z) = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + \dots\right)$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i . a_{-1}$$

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \dots$$



$$\boxed{a_{-1} = \frac{1}{2}\pi i \oint_{C_1} f(z)\,dz}$$

$$\oint_{C_1} f(z)\,dz = 2\pi i a_{-1}$$

$$\oint_{C_2} f(z)\,dz = 2\pi i b_{-1}$$

$$\oint_{C_3} f(z)\,dz = 2\pi i\, c_{-1}$$

$$= 2\pi i a_{-1} + 2\pi i b_{-1} + 2\pi i c_{-1} + \ldots$$

$$= 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + \ldots\right)$$

$$= 2\pi i$$

$$\# \; a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$* \; f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \ldots$$

$$+ \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)^1} + a_0$$

$$+ a_1 (z-a) + a_2 (z-a)^2 + a_3 (z-a)^3$$

$$+ \ldots$$

$$\Rightarrow (z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \dots$$

$$\dots + a_{-2}(z-a)^{m-2} + a_{-1}(z-a)^{m-1}$$

$$+ a_0 (z-a)^m + a_1 (z-a)^{m+1}$$

$$+ a_2 (z-a)^{m+2} + a_3 (z-a)^{m+3} + \dots$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\} = \lfloor m-1 \, . \, a_{-1} \cancel{+ a_0 (z-a)^{m+1}}$$

$$+ a_0 (z-a) \, m(m-1) \dots 2$$

$$+ a_1 (z-a)^2 (m+1) m (m-1) \dots 3$$

$$+ \dots$$

$y = x^5$

$y_1 = 5x^4$

$y_2 = 20x^3$

$y_3 = 60x^2$

$y_4 = 120x$

$y_5 = 5.4.3.2.1.$

$= \lfloor 5$

$y_6 = 0$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$= a_{-1} \lfloor m-1$$

$$\therefore a_{-1} = \lim_{z \to a} \frac{1}{\lfloor m-1} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

Math-IV (Lec 15) | 19/09/2022

Pg: 78

Ex: 7.4

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$$

$$a_{-1} = \lim_{z \to a} \frac{1}{\lfloor m-1} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$(z+1)^2 (z^2+4) = 0$$

$$z = -1, -1, 2i, -2i$$

Here, $z = -1$ is a double pole and

$z = 2i$ and $z = -2i$ are simple pole.

Residue at $z = -1$ is a_{-1}

$$\therefore a_{-1} = \lim_{z \to -1} \frac{1}{\lfloor 1} \cdot \frac{d}{dz} \left\{ (z+1)^2 \cdot \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right\}$$

$$= \lim_{z \to -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2+4} \right)$$

$$a_{-1} = \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2 - 2z) 2z}{(z^2+4)^2}$$

$$= \frac{(1+4)(-2-2)-(1+2)(-2)}{(1+4)^2}$$

$$= \frac{-14}{25}$$

Residue at $z = 2i$ is

$$\therefore b_{-1} = \lim_{z \to 2i} \left\{ (z-2i) \frac{z^2-2z}{(z+1)^2(z^2+4)} \right\}$$

$$= \lim_{z \to 2i} \frac{(z-2i)(z^2-2z)}{(z+1)^2 (z-2i)(z+2i)}$$

$$= \frac{4i^2 - 2.2i}{(2i+1)^2 \ (2i+2i)}$$

$$b_{-1} = \frac{-4-4i}{(4i^2+4i+1)4i} = \frac{-(1+i)}{(-4+4i+1)i} = \frac{-(1+i)}{(4i-3)i}$$

$$= \frac{-(1+i)}{4i^2-3i} = \frac{-(1+i)}{-4-3i} = \frac{1+i}{4+3i}$$

$$= \frac{(1+i)(4-3i)}{4^2 - (3i)^2}$$

$$= \frac{4 - 3i + 4i - 3i^2}{16 + 9}$$

$$= \frac{7+i}{25}$$

Residue at $z = -2i$ is c_{-1}

$$\therefore c_{-1} = \lim_{z \to -2i} \left\{ (z+2i) \frac{(z^2 - 2z)}{(z+1)^2 (z^2+4)} \right\}$$

$$= \frac{7-i}{25}$$

Pg: 7.10

Ex: 7.6

Evaluate $\frac{1}{2\pi i} \oint_c \frac{e^{zt}}{z^2 (z^2 + 2z + 2)} dz \Big| \; |z| = 3$

$$\therefore z^2 (z^2 + 2z + 2) = 0$$

$z = 0$ is double pole.

$$z^2 + 2z + 2 = 0$$

$$\therefore z = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

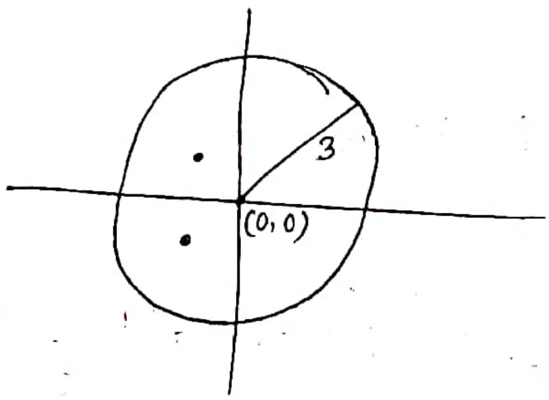
$$= \frac{-2 \pm 2i}{\quad}$$

$$= -1 \pm i$$

$\therefore$ $z = -1 + i$, $z = -1 - i$. are simple pole.

$(-1, 1)$, $(-1, -1)$, $(0, 0)$

All the poles are inside $|z| = 3$



Residue at $z = 0$ is a_{-1}

$$\therefore a_{-1} = \lim_{z \to 0} \frac{1}{\lfloor 1} \cdot \frac{d}{dz} \left\{ (z-0)^2 \frac{e^{zt}}{z^2(z^2+2z+2)} \right\}$$

$$= \lim_{z \to 0} \frac{d}{dz} \cdot \frac{e^{zt}}{z^2+2z+2}$$

$$= \lim_{z \to 0} \cancel{\frac{d}{dz}} \frac{(z^2+2z+2)e^{zt}.t - e^{zt}(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{2t-2}{4}$$

$$= \frac{t-1}{2}$$

Residue at $z = -1+i$ is b_{-1}

$$\therefore b_{-1} = \lim_{z \to -1+i} \left[\{z-(-1+i)\} \frac{e^{zt}}{z^2(z^2+2z+2)} \right]$$

$\boxed{\cancel{\text{(form } \tfrac{0}{0}) \text{ L'hospital}}}$

$$= \lim_{z \to -1+i} \frac{t e^{zt}}{z^2(2z+2) + (z^2+2z+2) \cdot 2z}$$

$$= \lim_{z \to -1+i} \frac{\{z-(-1+i)\} e^{zt}}{z^2 [\{z-(-1+i)\}\{z-(-1-i)\}]}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2 (-1+i+1+i)}$$

$$= \frac{e^{-t} \cdot e^{it}}{(1-2i+i^2) 2i}$$

$$= \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1-i$ is C_{-1}

$$C_{-1} = \frac{e^{(-1-i)t}}{4}$$

$$\therefore \oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1})$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$$

$$= \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$e^{ix} = \cos x + i\sin x$$

$$= \frac{t-1}{2} + \frac{1}{4}\left\{e^{-t}\cdot e^{it} + e^{-t}\cdot e^{-it}\right\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\left\{\frac{e^{it}+e^{-it}}{2}\right\}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\cdot\cos t$$

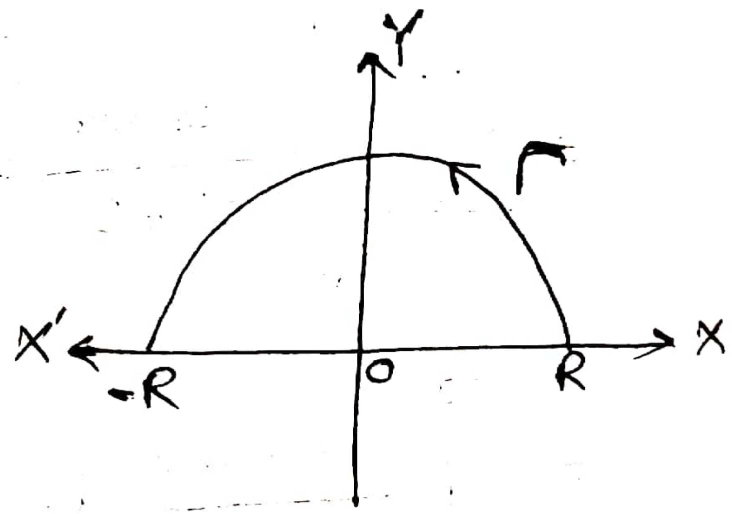
Math (IV) – Lec 16

22/09/2022

Pg: 7.11

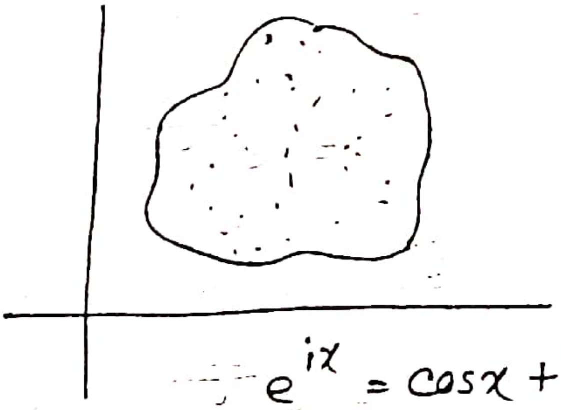
Ex: 7.7

$$\lim_{R \to \infty} \int_{\Gamma} F(z)\, dz = 0$$



$$\int_0^\infty \frac{dx}{x^6+1}$$

$$\oint_C \frac{dz}{z^6+1}$$



$e^{ix} = \cos x + i\sin x$

$z^6 + 1 = 0$

$\Rightarrow z^6 = -1 = \cos\pi + i\sin\pi$

$\Rightarrow z^6 = \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$

$\Rightarrow z^6 = \cos(2n+1)\pi + i\sin(2n+1)\pi$

$\therefore z = \{\cos(2n+1)\pi + i\sin(2n+1)\pi\}^{1/6}$

$= \cos(2n+1)\frac{\pi}{6} + i\sin(2n+1)\frac{\pi}{6}$ $(n = 0, 1, 2, \ldots, 5)$

$z_1 = e^{i(2n+1)\frac{\pi}{6}}$

$n=0$ হলে,

$z_1 = e^{\frac{i\pi}{6}}$ $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ ✓

$z_2 = e^{\frac{i\pi}{2}}$ $(0, 1)$ ✓

$z_3 = e^{\frac{i6\pi}{6}}$ ✓

$z_4 = e^{\frac{i7\pi}{6}}$ ✗

$z_5 = e^{\frac{i9\pi}{6}}$ ✗

$z_6 = e^{\frac{i11\pi}{6}}$ ✗

Resedue at $z_1 = e^{\frac{i\pi}{6}}$ is

$$a_{-1} = \lim_{z \to e^{\frac{i\pi}{6}}} \left\{ \frac{z - e^{\frac{i\pi}{6}}}{z^6+1} \right\} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{z \to e^{\frac{i\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6e^{\frac{5\pi i}{6}}}$$

$$= \frac{1}{6} e^{-\frac{5\pi i}{6}}$$

~~Rese~~ Residue at $z_2 = e^{\frac{i\pi}{2}}$ is

$$b_{-1} = \lim_{z \to e^{\frac{i\pi}{2}}} \left\{ \frac{z - e^{\frac{i\pi}{2}}}{z^6+1} \right\} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{z \to e^{\frac{i\pi}{2}}} \frac{1}{6z^5}$$

$$= \frac{1}{6} e^{-\frac{5\pi i}{2}}$$

$$c_{-1} = \frac{1}{6} e^{-\frac{25\pi i}{6}}$$

$$\therefore \oint_C \frac{dz}{z^6+1} = 2\pi i \left(a_{-1} + b_{-1} + c_{-1}\right)$$

$$= \frac{2\pi i}{6} \left[e^{-\frac{5\pi i}{6}} + e^{-\frac{5\pi i}{2}} + e^{-\frac{25\pi i}{6}} \right]$$

$$\Rightarrow \int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{i\pi}{3} \left[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2} + \cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6} \right]$$

$$= \frac{i\pi}{3} \left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i + 0 - i + \frac{\sqrt{3}}{2} - \frac{1}{2}i \right]$$

$$= \frac{-2\pi i^2}{3} = \frac{2\pi}{3}$$

$z = x + iy$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} + \lim_{R\to\infty} \int_{\Gamma} \frac{dz}{z^6+1}$$

$$= \lim_{R\to\infty} \frac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

(Ans)

$$\int_{-a}^{a} f(x)\,dx$$

$= 2\int_{0}^{a} f(x)\,dx$ → $f(x)$ is even

$= 0$ → $f(x)$ is odd

$\boxed{\text{Ex: } 7.9,\ 7.10,\ 7.11}$

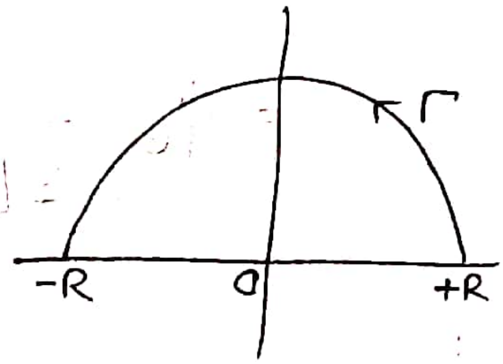
Math - IV (Lec 17) 26/09/2022

$\boxed{\text{Ex: } 7.10,\ 7.11}$

Algebraic Function

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$



Pg. 7.15

Ex: 7.14

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$$

$$= \int_0^{2\pi} \frac{d\theta}{3 - 2\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) + \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$= \oint_C \frac{\frac{dz}{iz}}{3 - (z + z^{-1}) + \frac{1}{2i}(z - z^{-1}}$$

$$e^{i\theta} = z$$

$$e^{i\theta} d\theta = dz$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{i}}{2i}$$

$$= \oint_c \frac{dz}{iz\left[\frac{6i - 6iz - 2iz^{-1} + z - z^{-1}}{2i}\right]}$$

$$= \oint_c \frac{2dz}{z\left[6i - 2iz - \frac{2i}{z} + z - \frac{1}{z}\right]}$$

$$= \oint_c \frac{2dz}{z\left[\frac{6iz - 2iz^2 - 2i + z^2 - 1}{z}\right]}$$

$$= \oint_c \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i}$$

$$= (1-2i)z^2 + 6iz - 1 - 2i$$

$$= 0$$

$$\therefore z = -6i \pm \frac{\sqrt{(6i)^2 + 4.(1-2i)(1+2i)}}{2(1-2i)}$$

$$= -6i \pm \frac{\sqrt{-36 + 4\ (1+4)}}{2(1-2i)}$$

$$\therefore z = \frac{-6i \pm 9i}{2(1-2i)} = \frac{-6i + 9i}{2(1-2i)} \quad (+)\text{ve value}$$

$$= \frac{i}{1-2i}$$

$$= \frac{i(1+2i)}{1+4}$$

$$= \frac{2-i}{5}$$

(-) for

$$z = \frac{-10i}{2(1-2i)} = \frac{5i(1+2i)}{5}$$

[Resedue হবে না, outside center এই]

$$= 2-i$$

Residue at $z = \frac{2-i}{5}$

$$a_{-1} = \lim_{z \to \frac{2-i}{5}} \left[\left(z - \frac{2i}{5}\right) \frac{2}{(1-2i)z^2 + 6iz - 1 - 2i} \right]$$

$$= \lim_{z \to \frac{2-i}{5}} \left[\frac{2}{2(1-2i)z+6i} \right]$$

$$= \frac{2}{2(1-2i)\left(\frac{2-i}{5}\right)+6i}$$

$$= \frac{1}{\frac{1}{5}\left(2-i-4i+2i^2\right)+3i}$$

$$= \frac{5}{-5i+15i} = \frac{5}{10i} = \frac{1}{2i}$$

Math- IV (Lec 18)

Pg: 7.20

29.09.2022

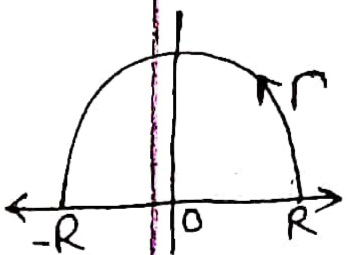
P - 7.16.
Ex - 7.15
P - 7.17
Ex - 7.16, 7.17

Ex- 7.22

$$\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m} \; ; \quad m > 0$$

$e^{ix} = \cos x + i \sin x$

$\cos x = \text{Real}\,(e^{ix})$



$$\oint_C \frac{e^{imz}}{z^2+1}\,dz \qquad z^2+1=0 \quad \therefore z = \pm i$$

$$a_{-1} = \lim_{z\to i} \left[(z-i) \frac{e^{imz}}{z^2+1} \right]$$

$$a_{-1} = \lim_{z\to a} \frac{1}{\lfloor m-1} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

$$= \lim_{z\to i} \frac{(z-i)e^{imz}}{(z+i)(z-i)} = \frac{1}{2i} e^{-m}$$

$$\therefore \oint_C \frac{e^{imz}}{z^2+1}\,dz = 2\pi i\, R_{-1}$$

$$= 2\pi i \frac{e^{-m}}{2i}$$

$$= \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{imx}}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{R} \frac{\cos mx + i\sin mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + \lim_{R\to\infty} \int_{-R}^{R} \frac{i\sin mx}{x^2+1}\,dx + \lim_{z\to\infty} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin mx}{x^2+1}\,dx + 0 = \pi e^{-m}$$

Equating real part.

$$\therefore \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\, dx = \pi e^{-m}$$

$$2\int_{0}^{\infty} \frac{\cos mx}{x^2+1}\, dx = \pi e^{-m}$$

$$\therefore \int_{0}^{\infty} \frac{\cos mx}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}$$

Pg - 7.24	Pg - 7.28	Pg : 7.26
Ex - 7.27	Ex - 7.31	Ex : 7.29