CSER 2207_8: Numerical Analysis-I

Lecture-2
Solution of equation in single variable

Dr. Mostak Ahmed
Associate Professor
Department of Mathematics, JnU

References

1. **Introduction to Numerical Analysis- S. S. Sastray.**
2. **Numerical Analysis- Burden & J.D. Faires.**
3. Numerical Methods & Calculus- S.S Kuo.
4. Numerical Method –E. Balagurusamy.
5. Numerical Analysis-Timothy Sauer.

Bisection method [2]

The first technique, based on the Intermediate Value Theorem, is called the **Bisection**, or **Binary-search, method**.

Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. The Intermediate Value Theorem implies that a number p exists in (a, b) with $f(p) = 0$. Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving (or bisecting) of subintervals of $[a, b]$ and, at each step, locating the half containing p.

To begin, set $a_1 = a$ and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

- If $f(p_1) = 0$, then $p = p_1$, and we are done.
- If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
  - If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
  - If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then reapply the process to the interval $[a_2, b_2]$. This produces the method described in Algorithm 2.1. (See Figure 2.1.)

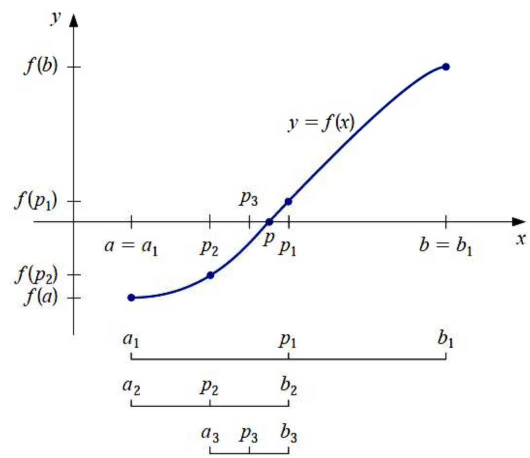
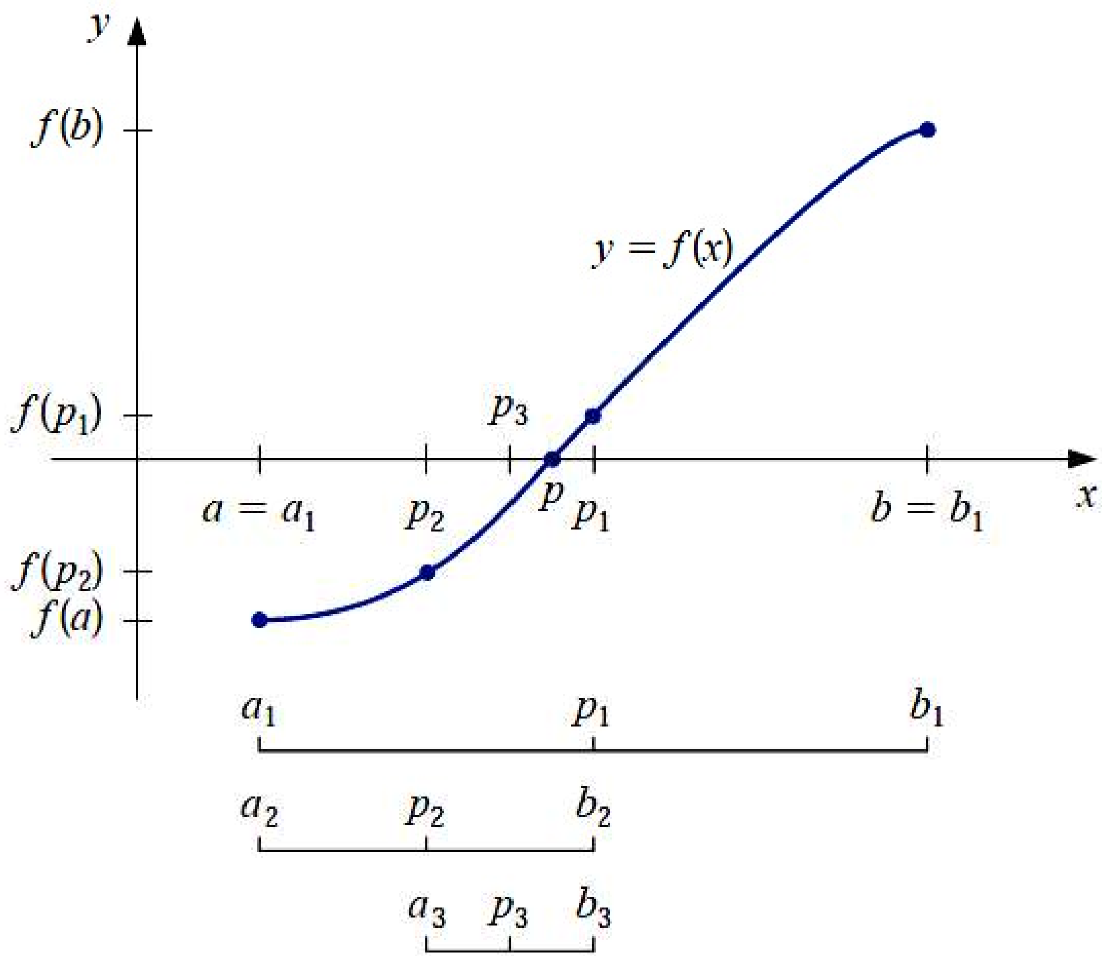


Figure 2.1

Figure 2.1



Algorithm [2]

Bisection

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$;
$FA = f(a)$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = a + (b - a)/2$; (*Compute* p_i.)
$FP = f(p)$.

Step 4 If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT (p); (*Procedure completed successfully.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 If $FA \cdot FP > 0$ then set $a = p$; (*Compute* a_i, b_i.)
$FA = FP$
else set $b = p$. (*FA is unchanged.*)

Step 7 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Example [1]

Example 2.1 Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Since $f(1)$ is negative and $f(2)$ positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2$. Then

$$f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8},$$ which is positive.

Hence the root lies between 1 and 1.5 and we obtain

$$x_1 = \frac{1 + 1.5}{2} = 1.25$$

We find $f(x_1) = -19/64$, which is negative. We therefore conclude that the root lies between 1.25 and 1.5. If follows that

$$x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$$x_3 = 1.3125, \qquad x_4 = 1.34375, \qquad x_5 = 1.328125, \text{etc.}$$

Example [1]

Example 2.2 Find a real root of the equation $x^3 - 2x - 5 = 0$.

Let $f(x) = x^3 - 2x - 5$. Then

$$f(2) = -1 \quad \text{and} \quad f(3) = 16.$$

Hence a root lies between 2 and 3 and we take

$$x_0 = \frac{2+3}{2} = 2.5$$

Since $f(x_0) = 5.6250$, we choose [2, 2.5] as the new interval. Then

$$x_1 = \frac{2+2.5}{2} = 2.25 \quad \text{and} \quad f(x_1) = 1.890625$$

Proceeding in this way, the following table is obtained.

n	a	b	x	$f(x)$
1	2	3	2.5	5.6250
2	2	2.5	2.25	1.8906
3	2	2.25	2.125	0.3457
4	2	2.125	2.0625	−0.3513
5	2.0625	2.125	2.09375	−0.0089
6	2.09375	2.125	2.10938	0.1668
7	2.09375	2.10938	2.10156	0.07856
8	2.09375	2.10156	2.09766	0.03471
9	2.09375	2.09766	2.09570	0.01286
10	2.09375	2.09570	2.09473	0.00195
11	2.09375	2.09473	2.09424	−0.0035
12	2.09424	2.09473		

Example

Find a root of the equation $x^3 - 3x - 5 = 0$ by bisection method.

Solution. Let $f(x) = x^3 - 3x - 5$. Then we observe that $f(2) = -3$ and $f(3) = 13$. Thus, a root of the given equation lies between 2 and 3. Let $x_0 = 2.5$. Then

$$f(2.5) = (2.5)^3 - 3(2.5) - 5 = 3.125\,(+\text{ve}).$$

Thus, the root lies between 2.0 and 2.5. Then

$$x_1 = \frac{2+2.5}{2} = 2.25.$$

We note that $f(2.25) = -0.359375\,(-\text{ve})$. Therefore, the root lies between 2.25 and 2.5. Then we take

$$x_2 = \frac{2.25+2.5}{2} = 2.375$$

and observe that $f(2.375) = 1.2715\,(+\text{ve})$. Hence, the root lies between 2.25 and 2.375. Therefore, we take

$$x_3 = \frac{2.25+2.375}{2} = 2.3125.$$

Now $f(2.3125) = 0.4289\,(+\text{ve})$. Hence, a root lies between 2.25 and 2.3125. We take

$$x_4 = \frac{2.25+2.3125}{2} = 2.28125.$$

Now

$$f(2.28125) = 0.0281\,(+\text{ve}).$$

We observe that the root lies very near to 2.28125. Let us try 2.280. Then

$$f(2.280) = 0.0124.$$

Thus, the root is 2.280 approximately.

Thank You