Last Time: Poisson Processes have 3 equivalent descriptions.

i) IID Interarrival times $\{X_i\}_{i=1}^{\infty}$ where

$$X_i \sim \text{Exp}(\lambda),$$

ii) Arrival Epochs $\{S_i\}_{i=1}^{\infty}$ $(S_{i+1} > S_i)$

$S_1 = X_1$, $S_2 = X_1 + X_2$

$$S_n = \sum_{i=1}^{n} X_i.$$

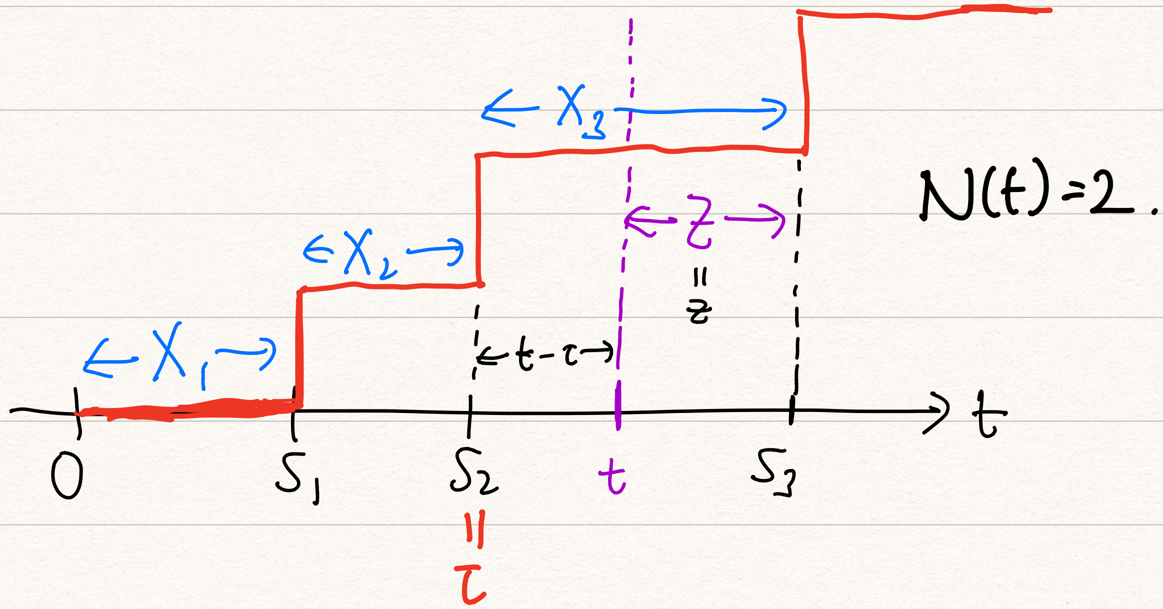
$$X_i = S_i - S_{i-1}, \quad i \geq 2.$$

iii) Poisson Counting Process. $\{N(t) : t > 0\}$

$N(t) = $ # of arrivals up to and including time t

$N(t) = n \iff S_n \leq t < S_{n+1}.$

Thm: For a Poisson process with rate λ, for any given $t>0$, the length of the interval from t to the next arrival, denoted as Z_1, is $\text{Exp}(\lambda)$



Furthermore $Z_1 \perp\!\!\!\perp N(t)$ &

$$Z_1 \perp\!\!\!\perp S_1, S_2, \ldots, S_{N(t)}$$

$$Z_1 \perp\!\!\!\perp \{N(\tau): 0<\tau\leq t\}.$$

Pf: We condition the comp. CDF of

Z on $N(t)=n,\ S_n=\tau$ (for some $\tau\le t$).
"Z_1

$$\Pr(Z>z \mid N(t)=n, S_n=\tau)$$
$$= \Pr(X_{n+1}-(t-\tau)>z \mid N(t)=n, S_n=\tau)$$
$$= \Pr(X_{n+1}>z+(t-\tau) \mid N(t)=n, S_n=\tau).$$
$$= \boxed{\Pr(X_{n+1}>z+(t-\tau) \mid X_{n+1}>t-\tau, S_n=\tau)}$$
$$= \boxed{\Pr(X_{n+1}>z+(t-\tau) \mid X_{n+1}>t-\tau)}$$
$$= \boxed{\Pr(X_{n+1}>z)} \boxed{= e^{-\lambda z}}$$

exponential interarrivals.

Follows because given $S_n=\tau\le t$,
$\{N(t)=n\}=\{X_{n+1}>t-\tau\}$. (Pls verify)

Follows because S_n depends on $X_1,\dots,X_n$
& $X_{n+1} \perp\!\!\!\perp (X_1,\dots,X_n)$

Follows because X_{n+1} has the memoryless prop.

To complete the proof, consider

$$\Pr(Z > z \mid N(t)=n, S_1=\tau_1, S_2=\tau_2, \dots, S_n=\tau_n)$$
$$= e^{-\lambda z}$$

But $\{N(t)=n, S_1=\tau_1, S_2=\tau_2, \dots, S_n=\tau_n\}$ describes the entire process up to & incl. time t, i.e, $\{N(\tau): 0<\tau\leq t\}$

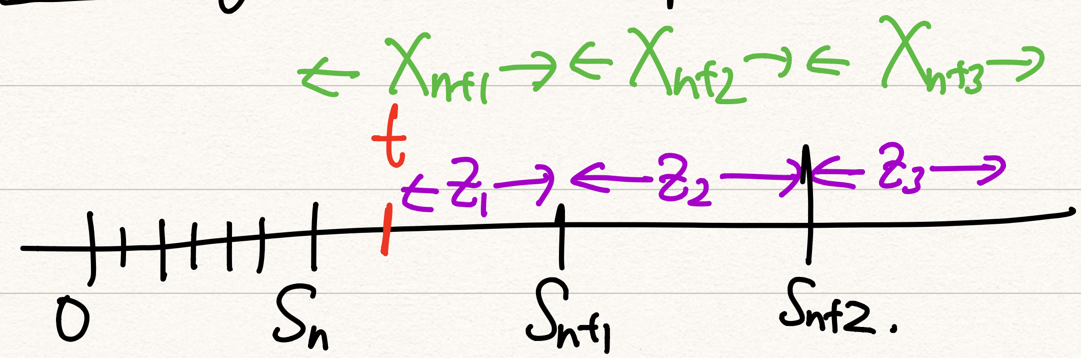
$$\Rightarrow \boxed{\Pr(Z > z \mid \{N(\tau): 0<\tau\leq t\}) = e^{-\lambda z}}$$

$\Rightarrow$ By integrating or averaging over $\{N(\tau): 0<\tau\leq t\}$, we get that

$$\boxed{\Pr(Z > z) = e^{-\lambda z}}$$

imply that $Z_1 \perp\!\!\!\perp \{N(\tau): 0<\tau\leq t\}$.

Stationary Increment Property.



Z_m: Time from $(m-1)^{st}$ arrival after t to m^{th} arrival $\Rightarrow$ $X_{n+m} = Z_m$.

Conditioned on $N(t) = n$, $S_n = \tau$, $Z_1, Z_2, Z_3, \cdots$ are IID. $Exp(\lambda)$ random variables.

By conditioning on all $S_1, \ldots, S_n$ (as was done above), we see that $Z_1, Z_2, \cdots$ are unconditionally IID.

Poisson process starting at t is a probabilistic replica of the process starting at 0. More precisely $N(t') - N(t)$ has the same distribution as $N(t'-t)$ for all $0 \leq t < t'$.

Def: A counting process $\{N(t) : t \geq 0\}$ has the stationary increments property if
$$N(t') - N(t) \overset{d}{=} N(t'-t) \quad \forall\, 0 \leq t < t'.$$

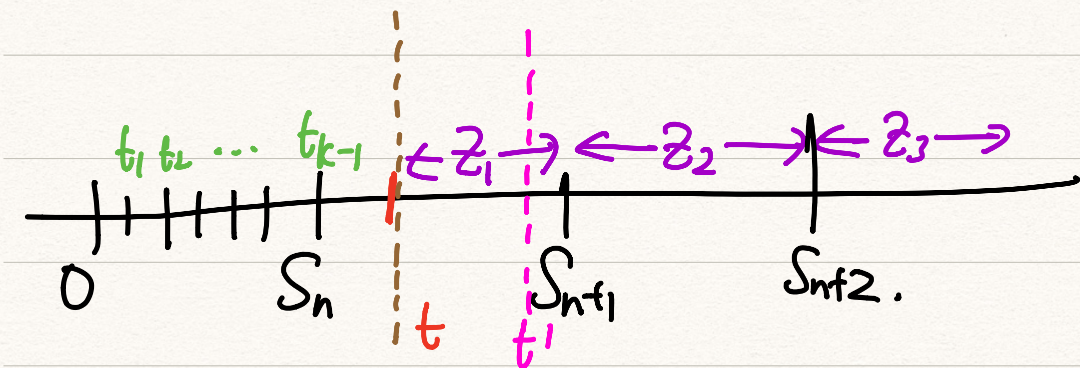
Thm:
The Poisson Counting process has the SIP.

For convenience $\tilde{N}(t, t') = N(t') - N(t)$.
is the # of arrivals in $(t, t']$.

Def: A counting process $\{N(t) : t \geq 0\}$ has

the independent increments property if
$\forall\ 0 < t_1 < t_2 < \cdots < t_k$,
$N(t_1), \tilde{N}(t_1, t_2), \tilde{N}(t_2, t_3), \ldots, \tilde{N}(t_{k-1}, t_k)$
are independent.

Thm: The Poisson process has the IIP.



$Z_1 \perp\!\!\!\perp \{N(\tau) : \tau \le t\}$.

$\Rightarrow Z_1 \perp\!\!\!\perp \left(N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t)\right)$.

But $Z_1, Z_2, Z_3, \ldots$ are IID.
$\Rightarrow \tilde{N}(t, t') \perp\!\!\!\perp \left(N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t)\right)$.

Rename $t = t_k$, $t' = t_{k+1}$

$\Rightarrow$

$$\tilde{N}(t_k, t_{k+1}) \perp\!\!\!\perp \left(N(t_1), \tilde{N}(t_1, t_2), \ldots, \tilde{N}(t_{k-1}, t_k) \right)$$

Since this is true $\forall k$ & $\forall t_1, \ldots, t_{k+1}$, we see that the Poisson counting process has the IIP.

<u>Distribution of $S_n = \sum_{i=1}^{n} X_i$.</u>

Most straightforward way is to get f_{S_n} by convolving $\mathrm{Exp}(\lambda)$ n times. This is too cumbersome.

$(X_1, S_2) \overset{d}{=} (S_1, S_2)$.

$$f_{X_1 S_2}(x_1, s_2) = f_{X_1}(x_1) f_{S_2|X_1}(s_2|x_1)$$

$$= \lambda e^{-\lambda x_1} \lambda e^{-\lambda(s_2 - x_1)}, \quad 0 \le x_1 \le s_2.$$

$$\Rightarrow f_{X_1 S_2}(x_1, s_2) = \lambda^2 e^{-\lambda s_2} \quad 0 \le x_1 \le s_2.$$

This joint density does not depend on x_1.

$\Rightarrow$ X_1 is unif. on $[0, s_2]$ given $S_2 = s_2$

(Will talk more about this when we discuss conditional arrival densities.)

$$f_{S_2}(s_2) = \int_{x_1=0}^{s_2} f_{X_1 S_2}(x_1, s_2)\, dx_1$$

$$= \int_0^{s_2} \lambda^2 e^{-\lambda s_2}\, dx_1$$

$$= \lambda^2 s_2 e^{-\lambda s_2}, \quad s_2 \geq 0.$$

This is the Erlang density with rate λ & order 2.

Hw: Check by convolution & transforms that $f_{S_2} = f_{X_1} * f_{X_1}$.

Claim: $$f_{S_n}(s_n) = \frac{\lambda^n s_n^{n-1} \exp(-\lambda s_n)}{(n-1)!} \quad -(*)$$

$s_n \geq 0.$

This is true for $\underbrace{n=1}_{Exp(\lambda)}$ & $\underbrace{n=2}_{Erlang(\lambda, 2)}$.

Pf:

Sub-Claim: Joint density

$$f_{S_1,\dots,S_n}(s_1,\dots,s_n) = \lambda^n \exp(-\lambda s_n)$$

$$0 \leq s_1 \leq s_2 \leq \dots \leq s_n .$$

$$\text{——} (*)$$

$$f_{S_1 \dots S_n S_{n+1}}(s_1,\dots,s_n,s_{n+1})$$

$$= f_{S_1,\dots,S_n}(s_1,\dots,s_n)\, f_{S_{n+1}|S_1,\dots,S_n}(s_{n+1}|s_1,\dots,s_n)$$

$$= \lambda^n \exp(-\lambda s_n)\, \lambda \exp(-\lambda(s_{n+1}-s_n)).$$

$$= \lambda^{n+1} \exp(-\lambda s_{n+1})$$

Now, we want f_{S_n} from $(*)$. We should "integrate out" $s_1,\dots,s_{n-1}$, but we must do this very carefully.

Let $n=3$.

$$f_{S_3}(s_3) = \int_{\substack{(s_1,s_2):\\ 0<s_1<s_2<s_3}} f_{S_1S_2S_3}(s_1,s_2,s_3)\,ds_1\,ds_2.$$

$$= \underbrace{\int_0^{s_3} \underbrace{\int_0^{s_2} \lambda^3 \exp(-\lambda s_3)\,ds_1}_{\because\ 0<s_1<s_2}\,ds_2.}_{\because\ 0<s_2<s_3.}$$

$$= \int_0^{s_3} \lambda^3 s_2 \exp(-\lambda s_3)\,ds_2.$$

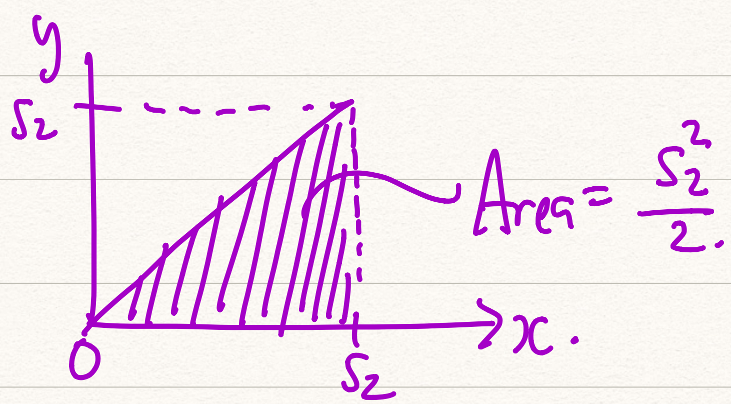
$$= \frac{s_3^2}{2}\lambda^3 \exp(-\lambda s_3), \quad s_3 \geq 0.$$

This checks out for $n=3$.

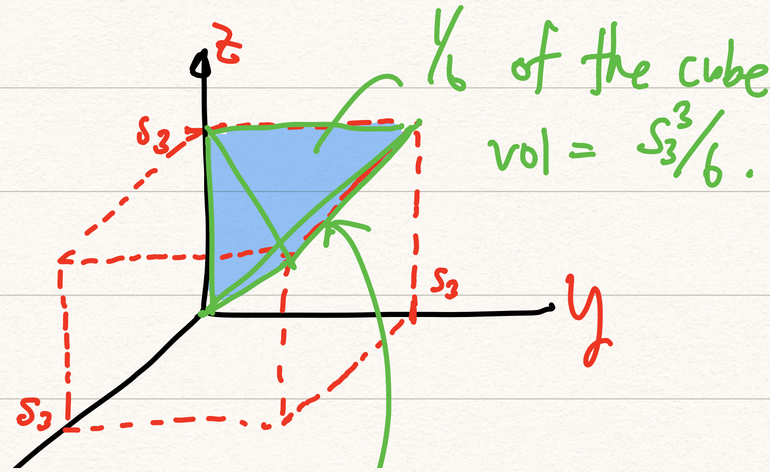
For $n=4$, see my supplementary notes.

Rmk: We have the factor $\dfrac{s_n^{n-1}}{(n-1)!}$ because the "volume" of the region $(s_1, \dots, s_{n-1})$ s.t. $0 < s_1 < s_2 < \dots < s_{n-1} < s_n$ is precisely $s_n^{n-1}/(n-1)!$

Eg: $n=2$. Region is $\{(x,y) : 0 \le x \le y \le s_2\}$



$n=3$



x

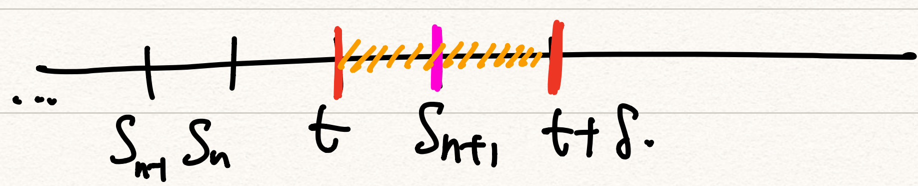
$\{(x,y,z): 0 \le x < y < z < s_3\}$

PMF of $N(t)$.

<u>Thm</u>: Poisson process of rate λ
PMF of $N(t)$, i.e., # of arrivals in $(0,t]$ is a Poisson rv with rate λt.

$$P_{N(t)}(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \dots$$

Pf: Consider $\{t < S_{n+1} \le t+\delta\}$ for small δ



$$\Pr(t < S_{n+1} \le t+\delta)$$

$$= \int_t^{t+\delta} f_{S_{n+1}}(\tau)\, d\tau$$

$$\boxed{\begin{array}{c} g(\delta) = o(\delta) \\ \Rightarrow \lim_{\delta \downarrow 0} \left| \dfrac{g(\delta)}{\delta} \right| = 0. \end{array}}$$

$$\cong f_{S_{n+1}}(t)\big(\delta + o(\delta)\big)$$

$$\Pr(t < S_{n+1} \le t+\delta)$$

$= \Pr(n$ arrivals in $(0,t]$ & 1 arrival in $(t, t+\delta])$

$$= p_{N(t)}(n) \cdot \int_0^{\delta} \underbrace{f_{X_{n+1}}(\tau)}_{\lambda e^{-\lambda\tau}}\, d\tau.$$

$$= p_{N(t)}(n)\big(\lambda\delta + o(\delta)\big)$$

$$\left[-e^{-\lambda\tau}\right]_0^{\delta} = \lambda\delta.$$

$$\Rightarrow p_{N(t)}(n)\big(\lambda\delta + o(\delta)\big) = f_{S_{n+1}}(t)\big(\delta + o(\delta)\big)$$

$$\lambda^{n+1} t^n e^{-\lambda t}$$

$$\Rightarrow \quad P_{N(t)}(n) = \frac{\lambda \quad t \quad e}{n!} \cdot \frac{1}{\lambda}$$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, \cdots$$

i.e., $N(t) \sim \mathrm{Poi}(\lambda t)$.

Alternative def$^{\underline{n}}$s of Poisson process.

Def: A Poisson counting process $\{N(t) : t \geq 0\}$ is a counting process s.t. $N(t) \sim \mathrm{Poi}(\lambda t)$ & has the SIP & IIP.

$$\tilde{N}(t, t+\delta) \stackrel{d}{=} N(\delta).$$

$$\left\{ \Pr(\tilde{N}(t, t+\delta) = 0) = e^{-\lambda\delta} \cong 1 - \lambda\delta + o(\delta). \right.$$

$$(*)\begin{cases} \Pr(\tilde{N}(t,t+\delta)=1) = 1-e^{-\lambda\delta} \cong \lambda\delta + o(\delta) \\ \Pr(\tilde{N}(t,t+\delta) \geq 2) = o(\delta). \end{cases}$$

Def: A Poisson counting process is a counting process that satisfies $(*)$ & has the SIP & IIP.

<u>Poisson process as a limit of shrinking Bernoulli processes.</u>

Bernoulli process $\{Y_i\}_{i=1}^{\infty}$

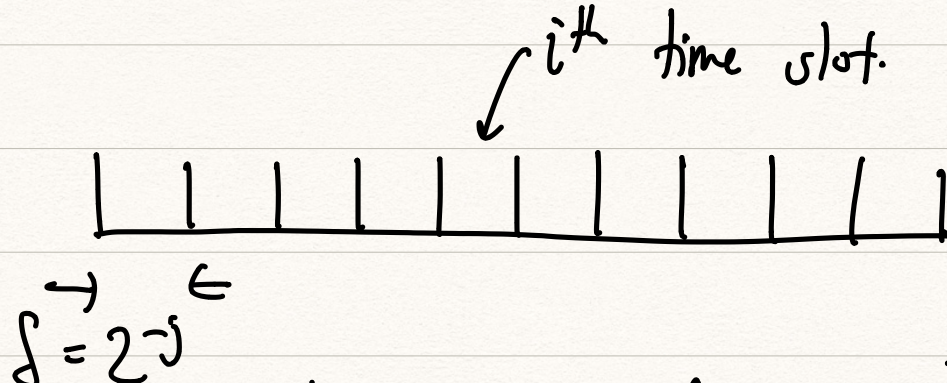
$Y_i \in \{0,1\}$ $\quad\quad P_Y(1)=p,\ P_Y(0)=1-p.$

$Y_i = 1 \Rightarrow$ arrival in time slot i.

$Y_i = 0 \Rightarrow$ no arrival in time slot j.

Consider a <u>sequence</u> of Bernoulli

processes indexed by $j \geq 1$.



Each time slot is of size 2^{-j}.

For the j^{th} Bernoulli process, let the arrival rate be $\lambda 2^{-j} = \lambda\delta$

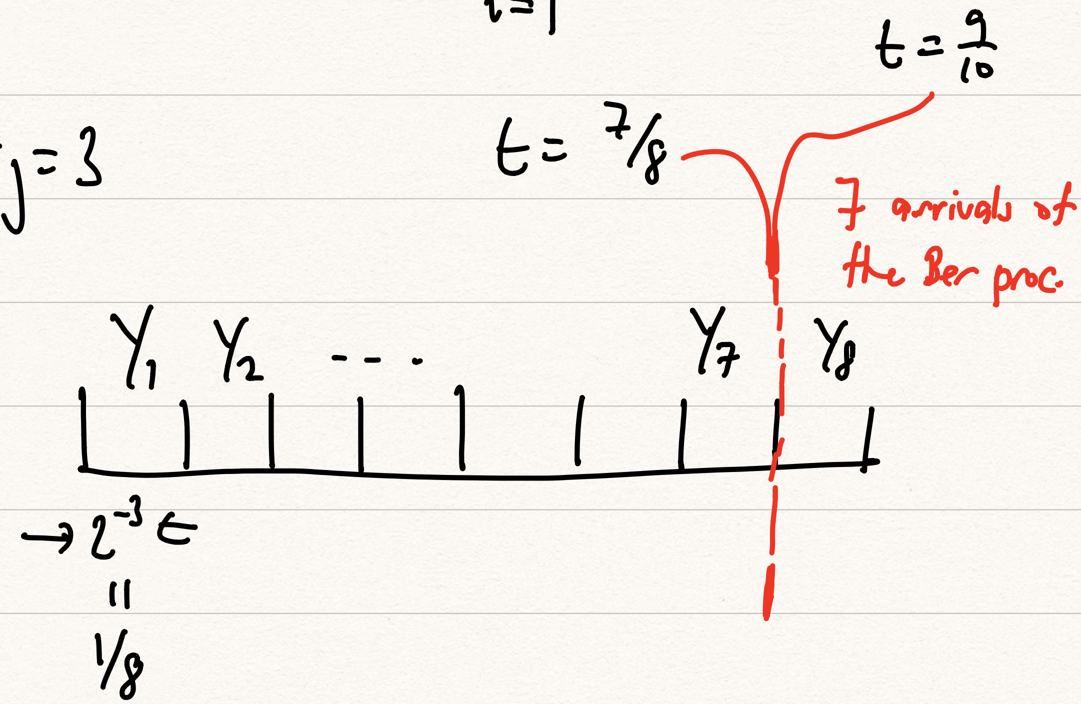
Probability of 1 arrival in a time slot $= \lambda\delta$

Prob. of no arrival in any time slot $= 1 - \lambda\delta$.

Bernoulli counting process

$$N_j(t) = \sum_{i=1}^{\lfloor t2^j \rfloor} Y_i$$

$j = 3$



$$P_{N_j(t)}(n) = \binom{\lfloor t2^j \rfloor}{n} p^n (1-p)^{\lfloor t2^j \rfloor - n}$$

$$p = \lambda 2^{-j}$$

<u>Thm</u>: Consider the sequence of shrinking Bernoulli processes with arrival probabilities

$\lambda 2^{-j} = p$ & time-slot size 2^{-j}. Then for every fixed time t & fixed # of arrivals n, the counting PMF $p_{N_j}(n)$ approaches the Poisson dist^n (of rate λ) with increasing j.

$$\lim_{j \to \infty} p_{N_j(t)}(n) = p_{N(t)}(n), \quad n = 0, 1, \ldots$$

Pf: Similar to a problem in HW 4.

Combining & Splitting Poisson processes.

$\{N_1(t) : t > 0\}$ & $\{N_2(t) : t > 0\}$ are two independent Poisson processes with rates λ_1 & λ_2 resp.

Consider the sum process $N(t) = N_1(t) + N_2(t)$.

Fact: $N(t)$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$.

Pf 1: $Pr(\tilde{N}(t, t+\delta) = 0)$

$= Pr(\tilde{N}_1(t, t+\delta) = 0 \ \& \ \tilde{N}_2(t, t+\delta) = 0)$

$= (1 - \lambda_1 \delta)(1 - \lambda_2 \delta)$

$= 1 - (\lambda_1 + \lambda_2)\delta + \lambda_1 \lambda_2 \delta^2$

$= 1 - (\lambda_1 + \lambda_2)\delta + o(\delta).$

$Pr(\tilde{N}(t, t+\delta) = 1)$

$= Pr(N_1(t, t+\delta) = 1 \ \& \ N_2(t, t+\delta) = 0)$

$\quad + Pr(N_1(t, t+\delta) = 0 \ \& \ N_1(t, t+\delta) = 1)$

$= \lambda_1 \delta (1 - \lambda_2 \delta) + \lambda_2 \delta (1 - \lambda_1 \delta)$

$= (\lambda_1 + \lambda_2)\delta - 2\lambda_1 \lambda_2 \delta^2$

$= (\lambda_1 + \lambda_2)\delta + o(\delta)$

$Pr(\tilde{N}(t, t+\delta] \geq 2) = o(\delta)$

Also need to verify that $N(t)$ possesses the IIP & SIP.

$\Rightarrow$ $N(t)$ is a Poisson counting process. with rate $\lambda = \lambda_1 + \lambda_2$.

Pf 2: $N(t) = N_1(t) + N_2(t)$

$$N_i(t) \sim Poi(\lambda_i t).$$

$\Rightarrow$ $N(t) \sim Poi((\lambda_1 + \lambda_2) t)$ by MGF or convolutions.

Pf 3: X_1: first inter-arrival time for $N(t)$

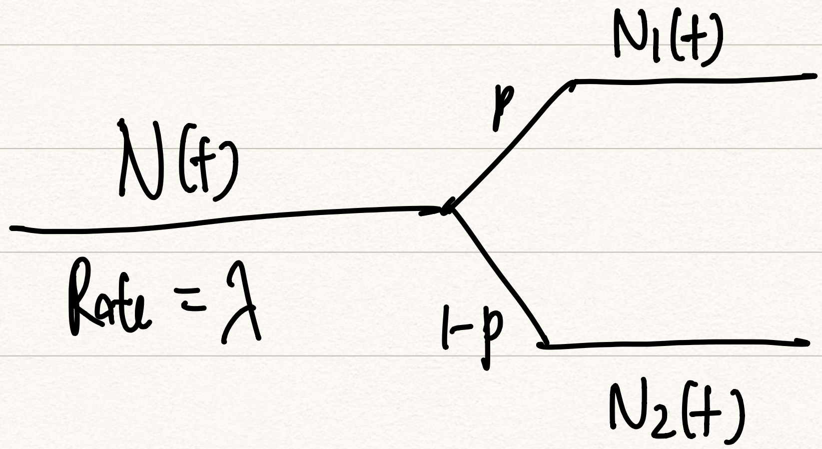
X_{11} : first inter-arrival time for $N_1(t)$.

X_{21} : first inter-arrival time for $N_2(t)$.

$$
\begin{aligned}
Pr(X_1 > t) &= Pr(X_1 > t \;\&\; X_2 > t) \\
&= Pr(X_1 > t)\, Pr(X_2 > t) \\
&= \left(\int_t^\infty \lambda_1 e^{-\lambda_1 s}\, ds \right) \cdot \left(\qquad \right) \\
&= \left[-e^{-\lambda_1 s} \right]_t^\infty \cdot \left(\qquad \right) \\
&= e^{-\lambda_1 t} e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2) t}.
\end{aligned}
$$

Use memoryless property of X_{1i} & X_{2j} to deduce that each subsequent inter-arrival X_k can be analyzed in the same way.

Splitting of a Poisson process



Consider splitting a Poisson process into 2 counting processes $N_1(t)$ & $N_2(t)$.

A particular arrival is assigned as type I with prob. p & type-II with prob $1-p$.

Claim: Resulting processes $N_1(t)$ & $N_2(t)$ are Poisson processes with rates $\lambda_1 = \lambda p$

& $\lambda_2 = (1-p)\lambda$ & furthermore they are independent.

Pf: Consider the event $\{N(t) = m+k\}$. $m, k \in \mathbb{N} \cup \{0\}$.

$$Pr(N_1(t) = m, N_2(t) = k \mid N(t) = m+k)$$
$$= \binom{m+k}{k} p^m (1-p)^k$$

But by Bayes rule,

$$Pr(N_1(t) = m, N_2(t) = k \mid N(t) = m+k)$$

$$= \frac{Pr(N_1(t) = m, N_2(t) = k)}{Pr(N(t) = m+k)}$$

$$= \binom{m+k}{k} p^m (1-p)^k$$

$\Rightarrow \Pr(N_1(t)=m, N_2(t)=k)$

$$= \binom{m+k}{k} p^m (1-p)^k \Pr(N(t)=m+k)$$

$$= \frac{\cancel{(m+k)!}}{m!\,k!} p^m (1-p)^k \frac{e^{-\lambda t} (\lambda t)^{m+k}}{\cancel{(m+k)!}}$$

$$= \frac{(p\lambda t)^m ((1-p)\lambda t)^k e^{-\lambda p t} e^{-\lambda(1-p)t}}{m!\,k!}$$

$$= \frac{(p\lambda t)^m e^{-\lambda p t}}{m!} \frac{((1-p)\lambda t)^k e^{-\lambda(1-p)t}}{k!}$$

$\Rightarrow N_1(t) \perp\!\!\!\perp N_2(t)$

$N_1(t) \sim \mathrm{Poi}(p\lambda t)$

$N_2(t) \sim \mathrm{Poi}((1-p)\lambda t)$.

Need to also show $\{N_1(t): t \geq 0\}$
indep. of $\{N_2(t): t \geq 0\}$.
This can be done formally by invoking
IIP for the two processes $\{N_i(t): t \geq 0\}$
$i = 1, 2$.