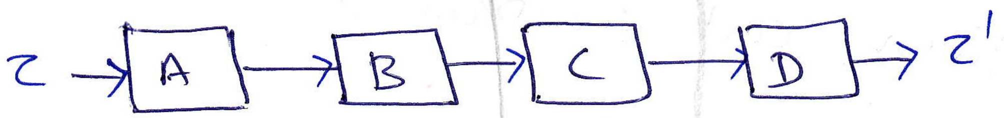
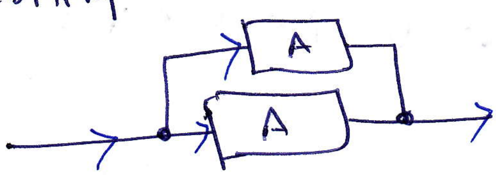
Dynamic Programming

Example

Reliability Maximization under Cost constraints



- Series configuration (H/W units in a pipeline)
- $R_S = R_A \cdot R_B \cdot R_C \cdot R_D$.
- Low-level redundancy adding to improve reliability



- increases overall reliability

Data Given:

# of units	A		B		C		D	
	R_A	C_A	R_B	C_B	R_C	C_C	R_D	C_D
1	0.6	100	0.7	80	0.7	75	0.8	60
2	0.7	130	0.75	100	0.8	80	0.85	70
3	0.8	150	0.8	120	0.9	85	0.88	75
4	0.9	170	0.85	140	0.95	90	0.9	80

DP-2

Cost Budget : $400

* How many units of A, B, C & D to inject into the system so that the overall reliability could be maximized under the given cost constraint?

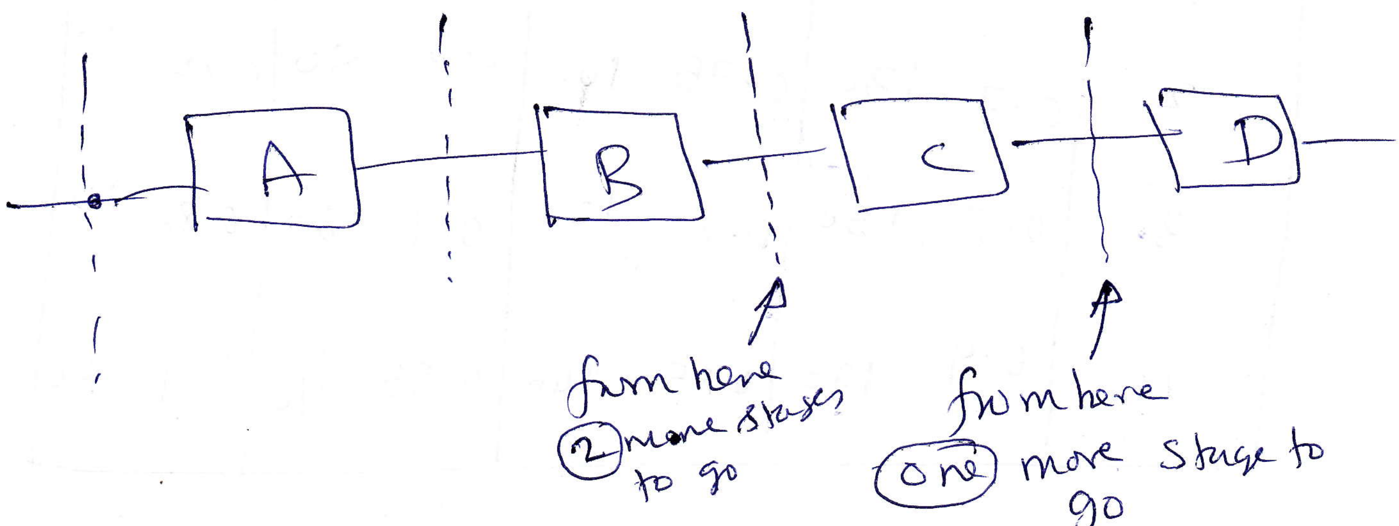
Solution approach

Stages : Each component

State : amount of resource ($s) available at each stage (resource availability)

Decision var : # of units of each component

Objective : maximization of Reliability



(DP-3)

Solution Steps

- Use backward recursion for solving this problem

With one more stage to go, i.e, $n=1$

$\boxed{n=1}$

X_4 → # of units of D that we are going to have

s	X_4	$f_1^*(\cancel{X_5})$	X_4^*
60-69	1	0.8	1
70-74	2	0.85	2
75-79	3	0.88	3
80-145	4	0.9	4

- Start with a minimum amount required to buy at least one unit of D, which is 60. So, start with \$60.

- The max we can have at this stage is when we had just 1 unit of A, 1 unit of B & 1 unit of C.

$$\Rightarrow 400 - (100 + 80 + 75) = 145$$

(DP-4)

- So, resource can vary anywhere between 60 to 145.

$f_1(s, x_4) = R_{x_4}$ is the relation

$$\boxed{f_1^*(s) = \max R_{x_4} \quad s.t\ C_{x_4} \leq s}$$

Optimal value

Example: suppose if we have $s = (70-74)$, say 73, we can buy 2 units so long as the $ required (which is 70 in this case) is less than what is available with us.

$$\left\{\begin{array}{l} \text{Best value} = x_4^* \\ \text{Best R value} = f_1^*(s) \end{array}\right.$$

(DP-5)

With 2 more stages to go,

$\boxed{n=2}$

s	x_3 = 1	2	3	4	$f_2^*(s)$	x_3^*
135–139	0.7× 0.8 = 0.56	—	—	—	0.56	1
140–144	0.56	0.64	—	—	0.64	2
145–149	0.595	0.64	0.72	—	0.72	3
150–154	0.616	0.68	0.72	0.76	0.76	4
155–159	0.63	0.704	0.765	0.76	0.765	3

(see page DP-7)

- What is the minimum resource I need at this stage? need to buy at least 1 of C & 1 of D ⇒ 75 + 60 = 135 is our starting value;
- Max value will be when I bought 1 of A & 1 of B ⇒ (400 − (100 + 80)) = 220

Let us complete the table above.

(DP-6) $X_3 = 1$, $s = 135$

If we have $s = 135$ & when $X_3 = 1$,

$X_3 = 1$ gives me a reliability of 0.7 &

we spent 75 on this component C &

left with \$60. $\Rightarrow$ this is the

resource we carry to the next stage

$\Rightarrow$ refer to our previous table with

value of $s = 60$. This gives,

$R = 0.8$ & $X_4 = 1$

$\Rightarrow$ The net reliability for $X_3 = 1$ with

$s = 135$ is $0.7 \times 0.8 = 0.56$

* Suppose if we want to buy 2 of C we

need to spend \$80 & balance = \$55,

which is insufficient to buy even 1 unit of

D. $\Rightarrow$ $X_3 = 2, 3, 4$ are infeasible in

this example.

This will happen for the range 135 – 139.

(verify!)

When s assumes $\underline{140}$, we have two alteratives;

1. 1 unit of C & 1 unit of D
2. 2 units of C & 1 unit of D

$\underbrace{0.7}_{1C} \times \underbrace{0.8}_{1D}$

$\underbrace{0.8}_{\text{for } 2C} \times \underbrace{0.8}_{1D}$

Thus, (from DP-5)

s	X_3 1	2	3	4	$f_2^*(s)$	X_3^*
160-164	0.63	0.72	0.792	0.8075	0.8075	4
165-169	0.63	0.72	0.81	0.836	0.836	4
170-220	0.63	0.72	0.81	0.855	0.855	4

recursive relation ?

$$\boxed{f_2(s, X_3) = R_{X_3} * f_1^*(s - C_{X_3})}$$

Check: Let $S = 160$ & say $X_3 = 4$ (we decided to buy 4) then,

for 4 C we spend \$90.

$\Rightarrow$ balance $= 160 - 90 = \$70$

$\Rightarrow$ reliability (max) possible

$= \underbrace{0.95}_{4C} \times \underbrace{0.85}_{2D}$ from table on Page DP-3

$= 0.8075$

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* Continue like the above & verify the following tables

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best value:

$$f_2^*(s) = \max f_2(s, X_3)$$

s.t.,

$$C_{X_3} \leq s$$

(DP-9)

$\boxed{n=3}$ (decision for buying B, C, D)

s	x_2 = 1	2	3	4	$f_3^*(s)$	x_2^*
215 – 219	0.392	—	—	—	0.392	1
220 – 224	0.448	—	—	—	0.448	1
225 – 229	0.504	—	—	—	0.504	1
230 – 234	0.532	—	—	—	0.532	1
235 – 239	0.5355	0.42	—	—	0.5355	1
240 – 244	0.56525	0.48	—	—	0.56525	1
245 – 249	0.5852	0.54	—	—	0.5852	1
250 – 254	0.5985	0.57			0.5985	1
⋮		⋮			⋮	⋮
300	0.5985	0.64125	0.684	0.68637	0.68637	4

(DP-10)

- recursion ?

$$\boxed{f_3(s, x_2) = R_{x_2} * f_2^*(s - C_{x_2})}$$

best value :

$$f_3^*(s) = \max f_3(s, x_2)$$

$$s.t.; \quad C_{x_2} \le s$$

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for $n = 4$ (four more stages to go)

available resource = \$400

s	x_1 = 1	2	3	4	$f_1^*(s)$	x_1^*
400	0.6 * 0.686375 = 0.4118	0.7 * 0.64125 = 0.44887	0.8 * 0.5985 = 0.4788	0.9 * 0.532 = 0.4788	0.4788	3 & 4

- <u>recursion</u> ?

$$f_1(s, x_1) = R_{x_1} * f_2^*(s - c_{x_1})$$

In the last computation we had
$x_1^* = 3$ & $x_1^* = 4$ as choices.

Let us get the final solution.

$\boxed{\text{with } x_1^* = 3}$

$\Rightarrow$ we start with \$150 & bal = \$250

- Look @ table (DP-9) for 250, we have $x_2^* = 1$.
- With $x_2^* = 1$, we spent \$80 & bal is $250 - 80 = 170$. Look @ table on (DP-7); for 170 we have $x_3^* = 4$

$\Rightarrow$ we spent \$90 $\Rightarrow$ bal $= 170 - 90 = 80$

From table (DP-3) for \$80 we have $x_4^* = 4$

Thus, the final solution is:

$x_1^* = 3$, $x_2^* = 1$, $x_3^* = 4$, $x_4^* = 4$
(A) (B) (C) (D)

Other solution: $x_1^* = 4$, $x_2^* = 4$, $x_3^* = 4$, $x_4^* = 1$
