6.4 Gaussian channels

3 most studied chanels:
- BSC
- BEC
- AWGN

AWGN: $Y = X + Z$

Y: chanel output; X: chanel input; Z: additive noise, Gaussian with mean 0, variance σ^2

AWGN: additive white Gaussian noise

$$w_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

$\Rightarrow$ can send an arbitrary amount of information in a single use of the channel!

$x_1 = 0, \quad x_2 = 2\cdot(3\sigma), \quad x_3 = 12\sigma, \quad \dots$

$\Rightarrow$ error $\leq 0.5\,\%$

can make this arbitrarily small by increasing distance

In practice, x is e.g. amplitude of an EM field

$\Rightarrow$ we have a power constraint:

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P \leftarrow \text{power}$$

6.4.1

Def. Differential entropy for X supported on S with prob. density p_X:

$$h(X) = -\int p_X(x) \log p_X(x)$$

$$h(X) = -\int_S p_X(x) \log p_X(x)$$

Examples:

1) $p_X(x) = \frac{1}{a}$ in $x \in [0, a]$

$$h(X) = \log(a) \qquad \text{(note: negative if } a < 1 \text{ !!)}$$

2) Gaussian distribution, mean μ, variance σ^2

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$h(X) = -\int_{-\infty}^{\infty} p_X(x) \left(\log \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{(x-\mu)^2}{2\sigma^2} \log e \right) dx$$

$$= -\frac{1}{2} \log \frac{1}{2\pi\sigma^2} + \frac{\log e}{2\sigma^2} \int_{-\infty}^{\infty} (x-\mu)^2 p_X(x)\, dx$$

$$= \frac{1}{2}\left(\log e + \log 2\pi\sigma^2 \right) = \underline{\frac{1}{2} \log (2\pi\sigma^2 e)}$$

Conditional entropy: $h(X|Y) = -\int_S p_{XY}(x,y) \log p_{X|Y}(x|y)\, dx dy$

Mutual information: $I(X:Y) = \int p_{XY}(x,y) \log \frac{p_{XY}(x,y)}{p_X(x) p_Y(y)} dx dy$

operationaly meaningful quantity

Lemma: $\boxed{I(X:Y) = \lim_{\Delta \to 0} I(X^\Delta : Y^\Delta)}$

where X^Δ

$$\int^{(r+1)\Delta} \qquad \text{for } r \in \mathbb{Z}$$

$$P_{x\Delta}(r) = \int_{r\Delta} p_x(x)\,dx \qquad \text{for } r \in \mathbb{Z}$$

$$\Rightarrow I(X:Y) \geq 0$$

6.4.2 Channel coding theorem for the AWGN channel

Theorem: The capacity of the AWGN channel W with noise variance σ^2 and power contraint P is

$$C(W) = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

signal-to-noise ratio! (SNR)

Lemma: $$\max_{\substack{P_X \\ \mathbb{E}[X^2] \leq P}} I(X:Y) = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

Proof: $$I(X:Y) = h(Y) - h(Y|X)$$
$$= h(X+Z) - \frac{1}{2}\log(2\pi\sigma^2 e)$$

we evaluate

$$\mathbb{E}[Y^2] = \mathbb{E}[X^2 + \underbrace{2XZ}_{} + Z^2] \leq P + \sigma^2$$

$$\mathbb{E}[XZ] = \mathbb{E}[X]\cdot\underbrace{\mathbb{E}[Z]}_{0} = 0!$$

$$\Rightarrow I(X:Y) \leq \frac{1}{2}\log\frac{2\pi\sigma^2(P+\sigma^2)e}{2\pi\sigma^2 e} = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

The other direction follows by choosing p_x as Gaussian with variance P and mean 0.

For any p_Y

Lemma: $h(Y)_p \leq h(Y)_\phi$ where ϕ is Gaussian with variance $\mathbb{E}[Y^2]$ and mean 0.

Proof: $D(p_Y \| \phi_Y) = -h(Y)_p + \int p_Y(y) \underbrace{\log \phi_Y(y)}_{\text{quadratic in } y} dy$

$$= -h(Y)_p + \int p_Y(y) \left[\left(-\frac{y^2}{2\mathbb{E}[Y^2]} \right) + \frac{1}{2} \log \frac{1}{2\pi\sigma^2} \right]$$

$$= -h(Y)_p + \int \phi_Y(y) \left[\quad \cdots \quad \right]$$

$$= -h(Y)_p + h(Y)_\phi \implies h(Y)_\phi \geq h(Y)_p. \quad \square$$

Converse: we maximized over sequences of channel inputs $x^n = x_1 \dots x_n$

here we can restrict to $\frac{1}{n}\sum_{i=1}^{n} x_i \leq P$

$$\to \mathbb{E}[X^2]_{p^{x^n}} = \sum_x P^{x^n}(x)\, x^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P$$

$i=1$

Achievability:

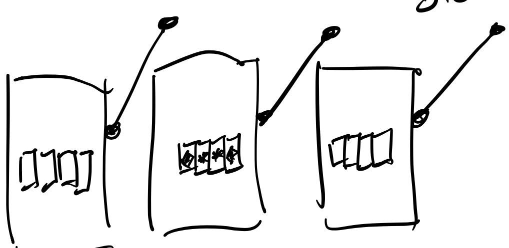
if p_X s.t. $\mathbb{E}[X^2] \le P - \varepsilon$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 \longrightarrow \mathbb{E}[X^2] \le P-\varepsilon$$

$$P\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 \le P\right] \longrightarrow 1$$

Chapter 7 : Multi-armed stochastic bandits

slot machine = one-armed bandit



many slot machines
= multi-armed bandit

select bandit $a \in A$
get reward X according to
some distribution p_a
that is unknown to you!

exploration: try all arms until you have sufficient statistics to estimate p_a

exploitation: you want to play the arms that give high reward (in expectation)

Definitions: • A bandit is determined by $\nu = \{p_a\}_{a\in A}$

- Algorithm/policy: $\pi_t(a_t \mid a_1, x_1, a_2, x_2, \dots, a_{t-1}, x_{t-1})$
  For round t

- The joint distribution
  $$p(a_1, x_1, \dots a_n, x_n) = \prod_{i=1}^{n} \pi_t(a_t \mid \dots)\, p_{a_t}(x_t)$$

- The expected reward of a is $\mu_a = \int p_a(x) \cdot x = \mathbb{E}[X]_{p_a}$

- The maximal expected reward: $\mu^* = \max_{a \in A} \mu_a$

- The expected regret for a policy π is
  $$R_n(\pi, \nu) = n\mu^* - \mathbb{E}\Big[\sum_{t=1}^{n} X_t\Big]_P$$

- The worst-case regret:
  $$R_n^*(\pi, \mathcal{P}) = \sup_{\nu \in \mathcal{P}} R_n(\pi, \nu)$$

- The minimax regret:
  $$R_n^*(\mathcal{P}) = \inf_{\pi} \max_{\nu \in \mathcal{P}} R_n(\pi, \nu)$$

Two types of bounds:

a) "achievability": Find π such that R_n^*
$$R_n^*(\mathcal{P}) \leq R_n^*(\pi, \mathcal{P}) \text{ is small}$$

b) "converse": find lower bound on $R_n^*(\mathcal{P})$
e.g. for each policy, find a bandit $\nu \in \mathcal{P}$ such that $R_n(\pi, \nu)$ is large.

[We will focus on b)]

7.2 Lower bound on minimax regret

<u>Theorem</u>: Let $K \geq 1$, $n \geq K-1$ and let $\mathcal{P}$ be a class of bandits with K arms with rewards distributed according to $\mathcal{N}(\cdot\,; v, 1)$ for $v \in [0,1]$.

unknown means

Then $R_n^*(\mathcal{P}) \geq \frac{1}{27}\sqrt{(k-1)n}$.

7.2.1 Decomposing the regret

<u>Lemma</u>: Define $\Delta_a = \mu^* - \mu_a \geq 0$

Define $T_a(n) = \sum_{t=1}^{n} \mathbb{1}\{A_t = a\}$

$$\boxed{R_n(\pi, v) = \sum_{a \in A} \Delta_a \mathbb{E}[T_a(n)]}$$

<u>Proof</u>: $R_n(\pi, v) = n\mu^* - \mathbb{E}\left[\sum_{t=1}^{n} X_t\right]$

$$= \underbrace{\sum_{a \in A} \mathbb{E}[T_n(a)]}_{n}\mu^* - \mathbb{E}\left[\sum_{t=1}^{n} \sum_{a \in A} \mathbb{1}\{A_t = a\} X_t\right]$$

$$= \sum_{a \in A} \left(\mathbb{E}[T_n(a)]\mu^* - \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\{A_t = a\}\right.\right.$$

$a \in A$ (

$\mathbb{E}[\underbrace{X_t \mid A_t = a}_{\mu_a}]$ /

$$= \sum_{a \in A} \mathbb{E}[T_n(a)] \cdot (\mu^* - \mu_a)$$

$\square$