Reading: Sec 2.1 & 2.2.1 (Memoryless Property).

Def: [Arrival Process]

An arrival process is a sequence of increasing rvs $0 < S_1 < S_2 < S_3 < \dots$

Rmk: $S_i < S_{i+1} \Leftrightarrow S_{i+1} - S_i$ is a positive rv.

$\Leftrightarrow \Pr(S_{i+1} > S_i) = 1.$

S_i: Arrival epochs.

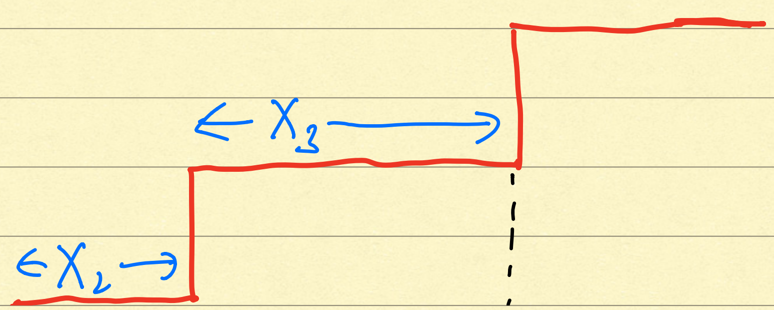
S_n is a rv & $\{S_n \le t\}$ is an event

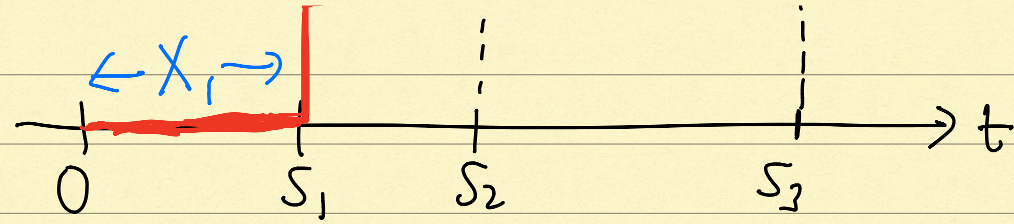
$\Updownarrow$

The n^{th} arrival occured before t.

Specify the collection of arrival epochs $\{S_i\}_{i=1}^{\infty}$ using two related stochastic processes.

indexed by $i \in \mathbb{N}$.
S_i's cts rv.





Interarrival times: $\left|\begin{array}{l} X_1 = S_1 \\ X_i = S_i - S_{i-1}, \quad i = 2, 3, \ldots \end{array}\right.$

$$S_n = \sum_{i=1}^{n} X_i$$

Indexed by $i \in \mathbb{N}$.
X_i's cts rv.

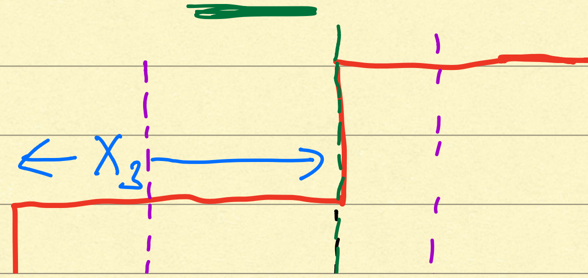
$\because$ $S_{i+1} - S_i$ is a positive rv. $\Rightarrow$ X_i is a pos. rv.
$\Pr(X_i \leq 0) = 0$.

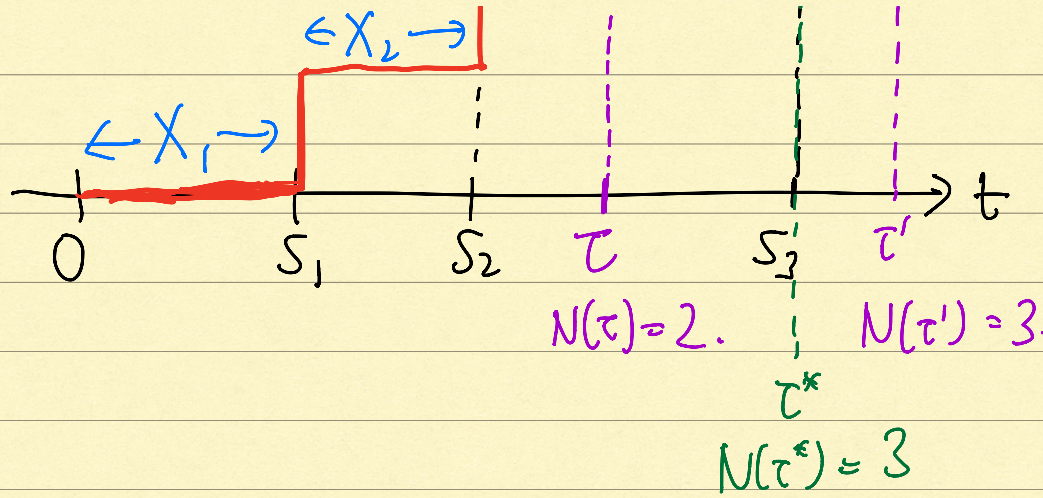
Rmk: Usually (for renewal processes) the X_i's are i.i.d. Hence, usually easier to specify an arrival process via $\{X_i\}_{i=1}^{\infty}$.

<u>Counting Process</u> $\{N(t) : t > 0\}$.

Indexed by $t > 0$ (uncountable)
Each $N(t)$ integer-valued

$N(t) = \#$ of arrivals in $(0, t]$.





Rmk: Clearly, $\forall\ \tau \geq t$, $N(\tau) \geq N(t)$

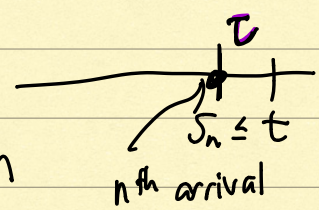
i.e., $N(\tau) - N(t)$ is a non-neg. rv.

$Pr(X \geq 0) = 1.$

Fact: $\forall n \geq 1,\ t > 0$

$$\{S_n \leq t\} = \{N(t) \geq n\}.$$

Pf: $(\subseteq)$ Suppose $S_n \leq t$. This means that the n^{th} arrival occured at some time $\tau \leq t$.

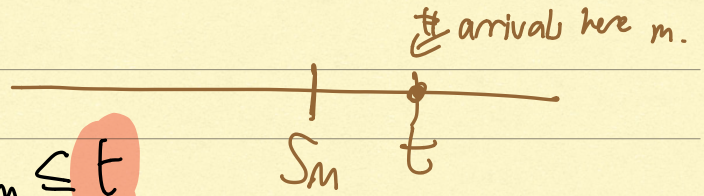


$\Rightarrow N(\tau) = n. \Rightarrow \underline{N(t) \geq N(\tau)} = n$

$\Rightarrow \{S_n \leq t\} \subseteq \{N(t) \geq n\}.$ —(1) ///

$(\supseteq)$ Now suppose $N(t) \geq n$. Implies that the number of arrivals up to and including time t is

Some $m \geq n$.



$$S_m \leq t \Rightarrow S_n \leq S_m \leq t$$

$$\Rightarrow \{N(t) \geq n\} \subseteq \{S_n \leq t\}. \quad -(2)$$

$$(1) \,\&\, (2) \Rightarrow \{S_n \leq t\} = \{N(t) \geq n\}.$$

Poisson process

Def: A renewal process is an arrival process in which the interarrival times $\{X_i\}_{i=1}^{\infty}$ are i.i.d.

Def: A Poisson process is a renewal process in which the interarrival times $\{X_i\}_{i=1}^{\infty}$ are exponentially distributed, i.e.,

$$f_{X_i}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

λ: rate of the process

$$EX_i = 1/\lambda, \qquad Var(X_i) = 1/\lambda^2$$

$\lambda t = \mathbb{E}[\,\#\text{ of arrivals in any interval of length } t].$

$\lambda = \frac{1}{t}\mathbb{E}[\,\#\text{ of arrivals in any interval of length } t].$

$= \mathbb{E}[\,\#\text{ of arrivals in a unit interval}] = \text{rate}$

Memoryless Property of the Poisson process

Def: A rv X possesses the memoryless property if X is a positive rv & $\forall\, t, x \geq 0$,

$$\Pr(X > t+x) = \Pr(X > x)\Pr(X > t).$$

$$\Rightarrow \quad 1 - F_X(t+x) = [1 - F_X(x)][1 - F_X(t)].$$

The following is more intuitive

Memorylessness

$$\boxed{\frac{\Pr(X > t+x)}{\Pr(X>t)}} = \Pr(X > t+x \mid X > t) = \Pr(X > x)$$

Bayes rule

$$\Pr(X > t+x \mid X>t) = \frac{\Pr(X > t+x \cap \cancel{X>t})}{\Pr(X>t)}$$

$t = 10$ mins

$x = 10$ mins.

$$\Pr(X > t+x \mid X > t) = \Pr(X > x) \quad -(*)$$

?

Bus has taken > 10 mins to arrive

Bus has taken > 20 mins to arrive

Rmk: X represents waiting time until arrival (*) means that if arrival has not occurred by time t, dist^{n} of remaining time is the same as original waiting dist^{n}.

Claim: $X \sim Exp(\lambda)$ is a memoryless rv. $(X > 0 \text{ a.s.})$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = \int_{-\infty}^{x} \lambda e^{-\lambda u} \mathbb{1}\{u \geq 0\}\, du$$

$$= \int_0^x \lambda e^{-\lambda u}\, du = \left[-e^{-\lambda u}\right]_0^x$$

$$= 1 - e^{-\lambda x}$$

$$1 - F_X(x) = e^{-\lambda x}.$$

$$\underset{\substack{\parallel \\ e^{-\lambda(t+x)}}}{1 - F_X(t+x)} = \underbrace{\underset{\substack{\parallel \\ e^{-\lambda x}}}{[1 - F_X(x)]}\underset{\substack{\parallel \\ e^{-\lambda t}}}{[1 - F_X(t)]}}_{e^{-\lambda(t+x)}}.$$

as desired

$\Rightarrow$ $Exp(\lambda)$ has the memoryless property.

Claim: If $X \geq 0$ a.s. & X memoryless, X must be $\text{Exp}(\lambda)$ for some $\lambda > 0$. (Ex 2.6)

Pf sketch:

$$h(x) := \log \Pr(X > x)$$

If X memoryless, then $\underline{h(t+x) = h(t) + h(x)}$ — (1)

$$\Downarrow$$

$$\because \quad \Pr(X > x+t) = \Pr(X > x)\Pr(X > t)$$

$$\Rightarrow \quad h(x+t) = h(x) + h(t)$$

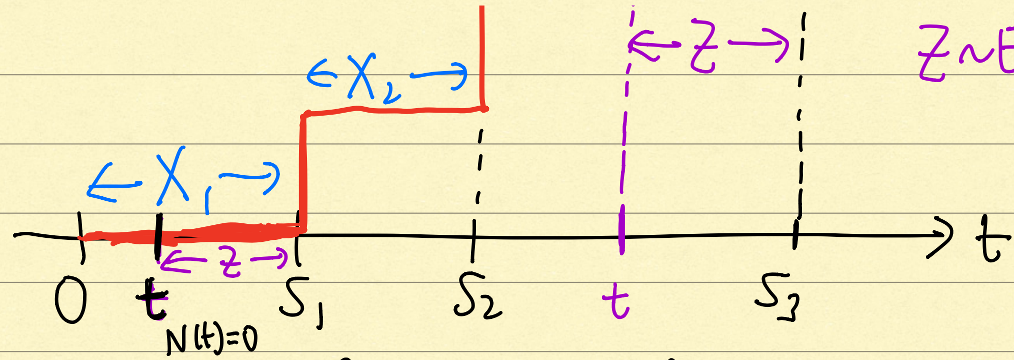
$h(\cdot)$ is non-increasing / monotone. — (2)

It suffices to show that under (1) & (2), h is linear; i.e., $\underline{h(x) = ax}$ for some $a \in \mathbb{R}$.

i) Show for $x \in \mathbb{Z}$.
ii) Show for $1/x \in \mathbb{Z}$.
iii) Show for $x \in \mathbb{Q}$. (rational)
iv) Use (2) "Some form of continuity" to show $\forall x \in \mathbb{R}$.

What's the distribution of the first arrival after a given deterministic time t?

$X_i \sim \text{Exp}(\lambda)$



$Z \sim \text{Exp}(\lambda)$.

<u>Thm</u>: For a Poisson process of rate λ, for any $t>0$, define Z to be the length from t to the next arrival. Then Z is $\text{Exp}(\lambda)$, i.e.,

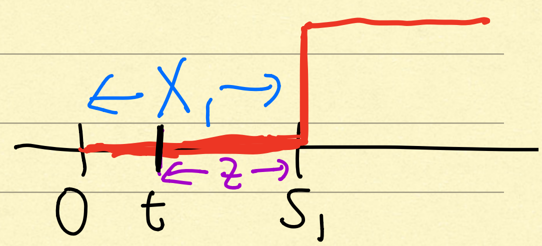
$$\Pr(Z > z) = e^{-\lambda z}$$

(CDF) $\quad \underset{\Pr(Z \le z)}{\underbrace{F_Z(z)}} = 1 - e^{-\lambda z}, \quad z \ge 0.$

Furthermore $Z \perp\!\!\!\perp N(t)$ & $Z \perp\!\!\!\perp$ all the $N(t)$ arrivals before t.

Pf: First (warm up) consider t s.t. $N(t) = 0$.
(i.e., no arrivals yet at t)

Given $N(t) = 0$, $\quad X_1 > t$
& $Z = \underline{\underline{X_1 - t}}$

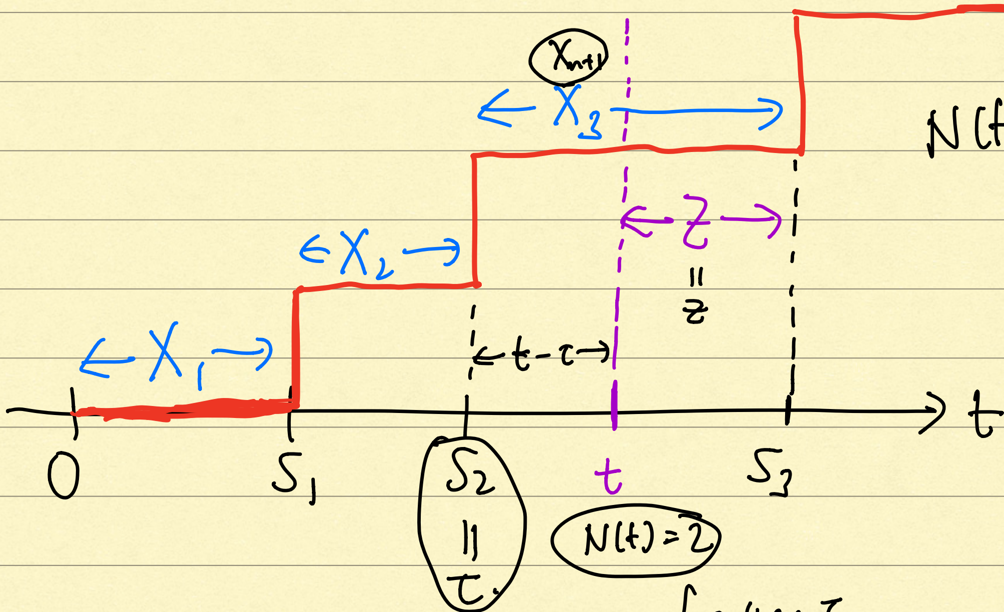


$N(t) = 0$.

$$\begin{aligned}
&\Pr(Z > z \mid N(t) = 0) \\
&= \Pr(X_1 - t > z \mid N(t) = 0) \\
&= \Pr(X_1 > t + z \mid N(t) = 0) \\
&= \Pr(X_1 > t + z \mid X_1 > t) \\
&= \underset{\uparrow \text{Exp}(\lambda)}{\Pr(X_1 > z)} = e^{-\lambda z}
\end{aligned}$$

General case $N(t)=n \quad (n \geq 1)$

$\Rightarrow \quad S_n = \tau \leq t$ for some τ.



$N(t) = \underline{\underline{2}} = n$

for some τ

Conditioned on $N(t)=n$, $S_n = \tau \leq t$, first arrival after t is the first arrival after S_n.

$$Z = z \iff X_{n+1} = z + (t-\tau)$$

some $\tau \leq t$

$$\underline{\Pr(Z > z \mid N(t)=n, S_n = \tau)}$$

$$= \Pr(X_{n+1} - (t-\tau) > z \mid N(t)=n, S_n=\tau)$$

$$= \Pr(X_{n+1} > z + t - \tau \mid \underline{N(t)=n}, \underline{S_n=\tau})$$

$$= \Pr(X_{n+1} > z + t - \tau \mid X_{n+1} > t-\tau, \cancel{S_n = \tau})$$

$$= \Pr(X_{n+1} > z + t - \tau \mid X_{n+1} > t - \tau)$$

$t - \tau$ $\not\perp$ depends on $\{X_i\}_{i=1}^{n}$

$\because S_n = \sum_{i=1}^{n} X_i$.

$$= \underline{\underline{e^{-\lambda z}}}$$

conditional

Z has cdf given by $1 - e^{-\lambda z}$ $z \geq 0$.

exponential.