Review of Markov chains & Stationary Distributions.

Def: A discrete-time, finite-state, ~~homogeneous~~ Markov chain is a sequence of rvs $\{X_n : n \geq 0\}$ in which each $X_n \in S$ (S finite set)

$$Pr(X_n \mid X_{n-1}, X_{n-2}, \dots, X_0) = Pr(X_n \mid X_{n-1}), \quad n \geq 1.$$

Def: i & $j \in S$ communicate if $i \to j$ & $j \to i$. We write $i \leftrightarrow j$.

Def: A class C of states in S is a non-empty subset of S s.t. $\forall i \in C$ and every other $j \neq i$ satisfies $j \in C$ iff $i \leftrightarrow j$ & $j \notin C$ iff $i \not\leftrightarrow j$.

Def: A recurrent state is state i that is accessible from all states that are accessible from i (if $i \to j$ then $j \to i$).

A transient state is one that not recurrent.

Def: Period of state $i = \gcd\{n : P_{ii}^n > 0\}$.

Thm: For a finite-state Markov chain

i) Either all states in a class are recurrent or all are transient

ii) All states in same class have the same period.

Def: An ergodic class is one that is recurrent & aperiodic (all states have period 1).

Thm: For an ergodic M-state Markov chain, $P_{ij}^{m} > 0 \ \forall i,j \in S$ all $m \geq (M-1)^2 + 1$.

The Matrix Representation

$[P]$: row stochastic, i.e., all rows sums are 1.

$$[P]_{ij} = \Pr(X_n = j \mid X_{n-1} = i) \quad \forall n \geq 1.$$

$\quad \overset{\shortparallel}{P_{ij}}$

$$\begin{aligned}
P_{ij}^{2} &= \Pr(X_2 = j \mid X_0 = i) \\
&= \sum_k \Pr(X_2 = j, X_1 = k \mid X_0 = i) \\
&= \sum_k \Pr(X_2 = j \mid X_1 = k, \cancel{X_0 = i}) \Pr(X_1 = k \mid X_0 = i) \\
&= \sum_k \Pr(X_2 = j \mid X_1 = k) \Pr(X_1 = k \mid X_0 = i) \\
&= \sum_k P_{kj} P_{ik} = [P^2]_{ij}.
\end{aligned}$$

Moral: The 2-step transition probabilites are given by the elements of $[P^2] = [P][P]$.

P_{ij}^n is the $(i,j)^{th}$ entry of $[P^n]$ ← n^{th} power $[P]$.
represents $Pr(X_n = j \mid X_0 = i)$

$$[P^{m+n}] = [P^m][P^n] \Rightarrow P_{ij}^{m+n} = \sum_k P_{ik}^m P_{kj}^n$$

Goal: Understand long term behavior $[P^n]$ as $n \to \infty$.

Note that if the "memory" of starting state i "dies out" as $n \to \infty$, $\boxed{P_{ij}^n}$ should become indep. of i as $n \to \infty$.

$$[P^n] \to \begin{pmatrix} P_{11}^\infty & \cdots & P_{1M}^\infty \\ \vdots & \ddots & \vdots \\ P_{M1}^\infty & \cdots & P_{MM}^\infty \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_M \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_M \end{pmatrix}$$

rank 1 matrix.

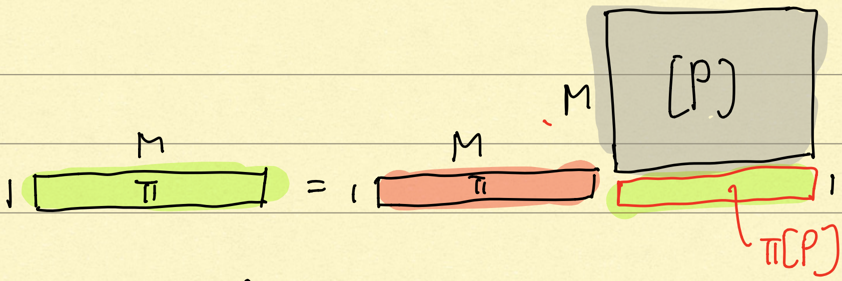
To find π (assuming it exists), look at

$$P_{ij}^{n+1} = \sum_k P_{ik}^n P_{ki}^1 \quad \text{(Chapman-Kolmogorov eqn)}$$

$$\pi_j = \sum_k \pi_k P_{ki}$$

$$\Rightarrow \quad \pi = \pi [P]$$

$\pi = (\pi_1, \cdots, \pi_M)$
(row vector.
M



Moral: To find the stationary distⁿ (if it exists), solve the left-eigenvector / eigenvalue problem

$$\pi = \pi [P].$$

i.e., find the left-eigenvector with eigenvalue 1.

Def: A steady-state vector for an M-state MC with $[P]$ is a row vector $\pi \in \mathbb{R}^{1 \times M}$ s.t.

$$\pi = \pi[P], \quad \sum_{i=1}^{M} \pi_i = 1, \quad \pi_i \geq 0 \quad \forall i \in [M].$$

$\pi_j = \sum_i P_{ij} \pi_i$ $\qquad$ $\{1, .., M\}$

Qn: What happens if π is the initial pmf of MC at time 0?

$$\pi_i = \Pr(X_0 = i), \quad i \in [M] = S.$$

Claim: The pmf is maintained forever ($\Pr(X_n = i) = \pi_i$)

$$\Pr(X_1 = j) = \sum_i \Pr(X_0 = i) \Pr(X_1 = j \mid X_0 = i)$$

$$= \sum_i \pi_i P_{ij} = \pi_j \qquad \checkmark$$

$$\pi = \pi[P] \Rightarrow \pi[P] = \pi[P]^2 \Rightarrow \pi = \pi[P]^2$$

$$\pi = \pi[P^n] \quad \forall n$$

Rmk: Note that if $[P^n]$ converges to $\begin{pmatrix} -\pi- \\ -\pi- \\ \vdots \\ -\pi- \end{pmatrix}$, then π satisfies $\pi = \pi[P]$.

However if $\pi = \pi[P]$, this doesn't necessarily mean that $[P^n]$ converges to $\begin{pmatrix} -\pi- \\ -\pi- \\ \vdots \\ -\pi- \end{pmatrix}$.

Questions:

1. When does $\pi = \pi[P]$ have a prob. vector solution?
   Ans: ALWAYS

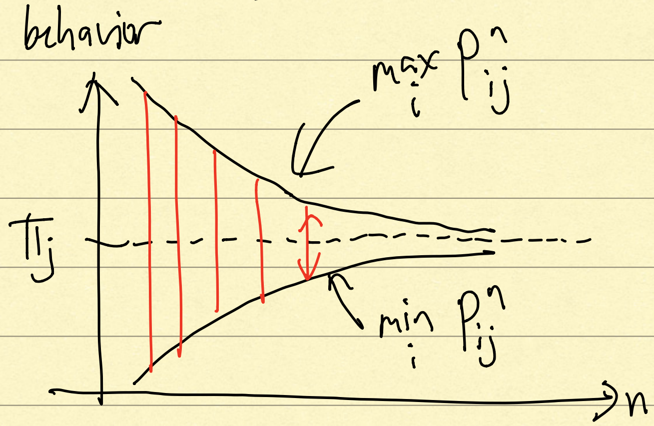
2. When is π satisfying $\pi = \pi[P]$ unique?
   Ans: Iff $[P]$ is TM of a <u>unichain</u>.

3. When does each row of $[P^n]$ converge to π satisfying $\pi = \pi[P]$
   Ans: <u>Ergodic</u> <u>unichain</u>

Def: Unichain $\equiv$ MC that contains a <u>single</u> recurrent class plus possibly some transient states

Def: Ergodic unichain $\equiv$ Unichain in which the recurrent class is ergodic.

For some MCs (ergodic unichains) we have the following behavior



Study the diff between largest & smallest elements of each col. of $[P^n]$.

<u>Lem:</u> $\forall$ state j & $\forall n \geq 1$

$$\max_i P_{ij}^{n+1} \leq \max_l P_{lj}^n \; ; \; \min_i P_{ij}^{n+1} \geq \min_l P_{lj}^n .$$

Pf: Fix any $i,j \in S$ & $n \in \mathbb{N}$,

$$P_{ij}^{n+1} = \sum_k P_{ik} P_{kj}^n \leq \sum_k P_{ik} \left(\max_l P_{lj}^n \right)$$

$$= \left(\max_l P_{lj}^n \right) \sum_k P_{ik} = \max_l P_{lj}^n$$

$$\max_i p_{ij}^{n+1} \leq \max_\ell p_{\ell j}^n \qquad ////.$$

Rmk: Max over all elements of fixed col j is non-increasing.

Assumption:

Consider the scenario in which $[P] > 0$ (i.e., $P_{ij} > 0$ $\forall\, i,j \in S'$).

Lem: Define $\alpha := \min_{i,j} P_{ij} > 0$. Then $\forall$ states j & $n \geq 1$,

$$\max_i P_{ij}^{n+1} - \min_i P_{ij}^{n+1} \leq (1-2\alpha)\left[\max_i P_{ij}^n - \min_i P_{ij}^n\right]$$

$$\Rightarrow \max_i P_{ij}^n - \min_i P_{ij}^n \leq (1-2\alpha)^n$$

$$\Rightarrow \pi_j = \lim_{n\to\infty} \max_i P_{ij}^n = \lim_{n\to\infty} \min_i P_{ij}^n \geq \alpha > 0.$$

Rmk: $\Rightarrow$ elements of P_{ij}^n for column j of $[P^n]$ achieve equality over i as $n \to \infty$.

Fix $i,j \in S, n \in \mathbb{N}$.

Pf: Let $\ell_{min} = \operatorname{argmin}_\ell P_{\ell j}^n$. Note that

$$\sum_{k \neq \ell_{min}} P_{ik} = 1 - P_{i\ell_{min}}.$$

$P_{\ell_{min} j}^n$

$\parallel$

$$P_{ij}^{n+1} = \sum_k P_{ik} P_{kj}^n = \left(\sum_{k \neq l_{min}} P_{ik} P_{kj}^n\right) + P_{il_{min}} \left(\min_l P_{lj}^n\right)$$

$$\leq \left(\sum_{k \neq l_{min}} P_{ik}\right)\left(\max_l P_{lj}^n\right) + P_{il_{min}}\left(\min_l P_{lj}^n\right)$$

$$= \left(\max_l P_{lj}^n\right)\left(1 - P_{il_{min}}\right) + P_{il_{min}}\left(\min_l P_{lj}^n\right)$$

$$= \max_l P_{lj}^n - \underbrace{P_{il_{min}}}_{\geq \alpha}\left(\underbrace{\max_l P_{lj}^n - \min_l P_{lj}^n}_{\geq 0}\right)$$

$$P_{ij}^{n+1} \leq \max_l P_{lj}^n - \alpha\left(\max_l P_{lj}^n - \min_l P_{lj}^n\right) \quad -(1)$$

$$\Rightarrow \max_i P_{ij}^{n+1} \leq \max_l P_{lj}^n - \alpha\left(\max_l P_{lj}^n - \min_l P_{lj}^n\right) \quad -(1')$$

Similarly, $\Rightarrow \min_i P_{ij}^{n+1} \geq \min_l P_{lj}^n - \alpha\left(\max_l P_{lj}^n - \min_l P_{lj}^n\right) \quad -(2')$

$$P_{ij}^{n+1} \geq \min_l P_{lj}^n + \alpha\left(\max_l P_{lj}^n - \min_l P_{lj}^n\right) \quad -(2)$$

(1') − (2')

$$\Rightarrow \max_i P_{ij}^{n+1} - \min_i P_{ij}^{n+1} \leq (1-2\alpha)\left(\max_l P_{lj}^n - \min_l P_{lj}^n\right)$$

$$gap(n+1) \leq (1-2\alpha)\, gap(n)$$

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Thm$^{(*)}$: $[P]$ transition matrix s.t. $[P] > 0$.

Then $\exists$ unique steady-state vector π which is $\pi > 0$ (i.e., $\pi_j > 0 \ \forall j \in S$) & $\pi = \pi[P]$, $\sum \pi_j = 1$.

Convergence is geometric in n.

<u>Thm</u>: If $[P]$ is ergodic, the same is true.

Pf: If $[P]$ is ergodic, then $[P^m] > 0$ for all $m \geq (M-1)^2 + 1$.
Apply previous result (*) to $[P^m]$.

Define the vector $\pi = (\pi_1, \ldots, \pi_M) > 0$ by

$$\pi_j = \lim_{n\to\infty} \max_{\ell} P^n_{\ell j} = \lim_{n\to\infty} \min_{\ell} P^n_{\ell j} > 0$$

$$[P^n] \longrightarrow \begin{pmatrix} - \pi - \\ \vdots \\ - \pi - \end{pmatrix} \qquad n \to \infty.$$

ergodic unichain.