Reading Sec 2.3.2 & 2.5.

Last Time: Combining & Splitting

$\{N_1(t): t>0\}$ Rate-λ_1 PP.

$\{N_2(t): t>0\}$ Rate-λ_2 PP

$$N_1 \perp\!\!\!\perp N_2$$

$N(t) = N_1(t) + N_2(t)$

Thm: $\{N(t): t>0\}$ is a PP of rate $\lambda_1 + \lambda_2$.

Splitting

$\{N(t): t>0\}$ PP w/ rate λ.

If there's an arrival, classified as type-I (resp. type-II) w.p. p (resp. $1-p$).

Type-I arrivals $N_1(t)$

Type-II arrivals $N_2(t)$

Thm: $N_1(t)$ is a PP with rate λp

$N_2(t)$ is a PP with rate $\lambda(1-p)$

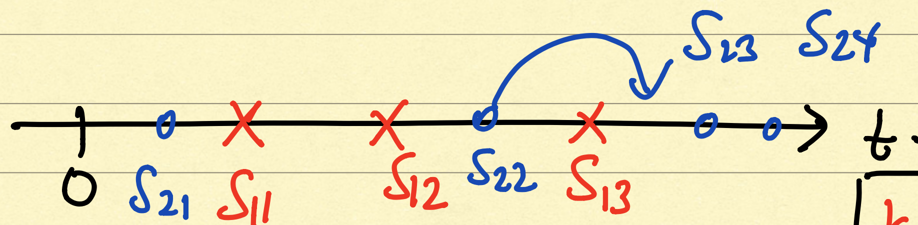
$$N_1 \perp\!\!\!\perp N_2.$$

Eg: Let S_{1k} be the epoch of the k^{th} arrival of $\underline{PP1}$ (rate λ_1) ← $N_1(t)$

Let S_{2j} be the epoch of the j^{th} arrival of PP2 (rate λ_2) ← $N_2(t)$

Qn: $Pr(S_{1k} < S_{2j}) = Pr(S_{13} < S_{24})$

Prob that the k^{th} arrival epoch of 1st process occurs before j^{th} arrival of 2nd process.



$\boxed{k=3 \quad x}$

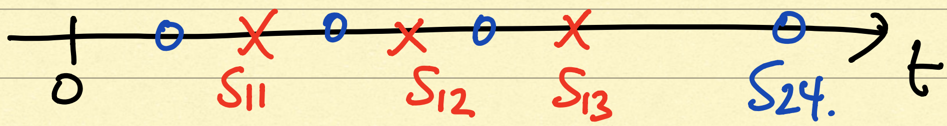
$j=4. \quad o$

Consider a combined/merged process with rate $\lambda_1+\lambda_2$

$Pr(S_{13} < S_{24})$ What do we need for $S_{13} < S_{24}$?

We need, out of the first $\boxed{6}$ arrivals of the combined process, 3 or more switched to the first.

$3 = k$ $\qquad\qquad$ $k+j-1 = 3+4-1 = 6$



$S_{24} > S_{13}$.

$$p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Pr(exactly k are switched to 1st proc)

$$= \binom{k+j-1}{k} p^k (1-p)^{j-1}$$

Pr(exactly $i \ge k$ are switched to 1st proc in the first $k+j-1$ arr)

$$= \binom{k+j-1}{i} p^i (1-p)^{k+j-1-i}$$

In general

$Pr(S_{1k} < S_{2j}) = Pr(\ge k$ are switched to first process out of $k+j-1$ arrivals)

$$= \sum_{i=k}^{k+j-1} Pr(\text{ exactly } i \text{ are switched to 1st proc})$$

$$= \sum_{i=k}^{k+j-1} \binom{k+j-1}{i} p^i (1-p)^{k+j-1-i} \quad ////.$$

Eg: M/M/1 Queue.

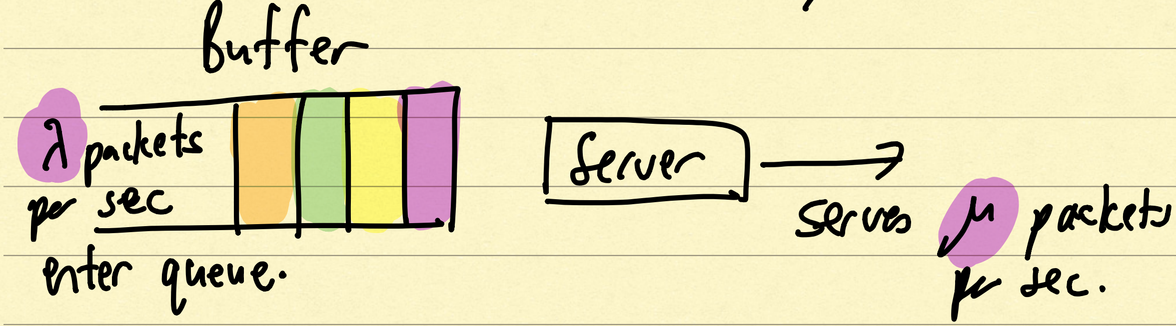
First character (M): memoryless arrival process.
Exponential interarrival times
Poisson process.

Second character (M): Memoryless service times.

Third character (1): Number of servers.

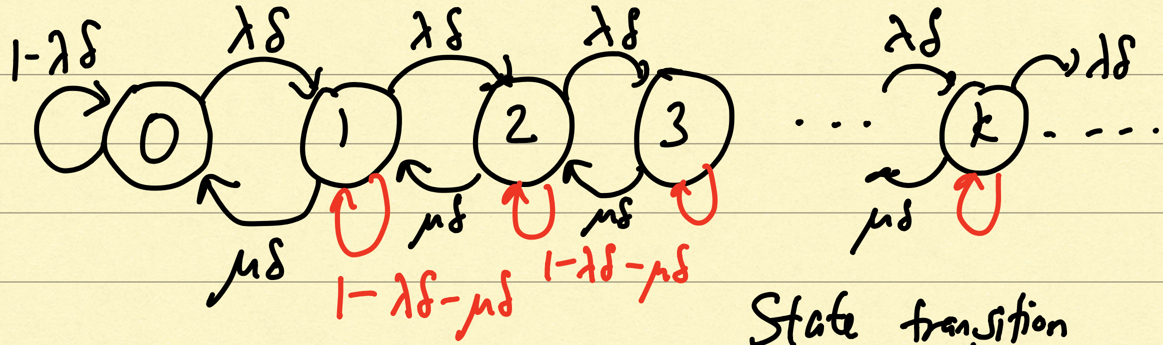
Queuing system with arrival of customers $PP(\lambda)$

Single server queue that serves customers with a service time dist^n $Exp(\mu)$.



Let state k denote that there are k customer in buffer / queue.

δ : infinitesimally small time int.



State transition diagram.

$P(i,j)$ = Prob of transitioning from state i to j.
$i,j \in \mathbb{N} \cup \{0\}$.

$P(0,0) = 1-\lambda\delta$, $P(j,j+1) = \lambda\delta$

current future

$P(j,j-1) = \mu\delta$, $P(j,j) = 1-(\lambda+\mu)\delta$.

Rmk: "birth-death process"

At "equilibrium", $\lambda P(k) = \mu P(k+1)$, $k \in \mathbb{N}$.
steady state $\underline{P(k)}$

prob. in state $k \in \mathbb{N} \cup \{0\}$ if the chain is run for a long time.

$$P(k) = \left(\frac{\lambda}{\mu}\right)^k P(0) = \rho^k P(0), \quad k \geq 0.$$

$$\sum_{k=0}^{\infty} P(k) = 1 \Rightarrow \sum_{k\geq 0} \left(\frac{\lambda}{\mu}\right)^k P(0) = 1$$

$$\Rightarrow \sum_{k\geq 0} \rho^k P(0) = 1 \quad \boxed{\rho<1}$$

$$P(0) = 1-\rho \quad \Leftarrow \quad P(0)\frac{1}{1-\rho} = 1$$

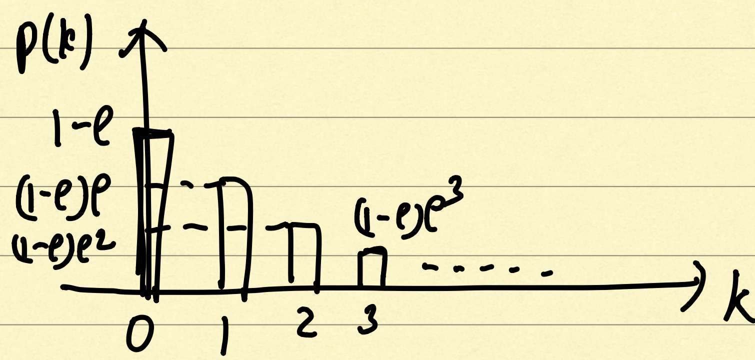
$$P(k) = (1-\rho)\rho^k$$

Average queue size $= EX = \sum_{k\geq 0} k\,P(k)$

$$= \sum_{k\geq 0} (1-\rho)\, k \rho^k = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

If $\rho<1 \Rightarrow \lambda<\mu \Rightarrow$ arrival rate < service rate

Intuitively all packets will be served before they accumulate.



Eg: PP(λ) Fix $t > 0$ $N(t) \sim \text{Poi}(\lambda t)$.

$\uparrow$ deterministic Thm $\Rightarrow$

T: Exp(ν) indep. of PP(λ).

$$f_T(t) = \nu e^{-\nu t} \mathbb{1}\{t \geq 0\}$$

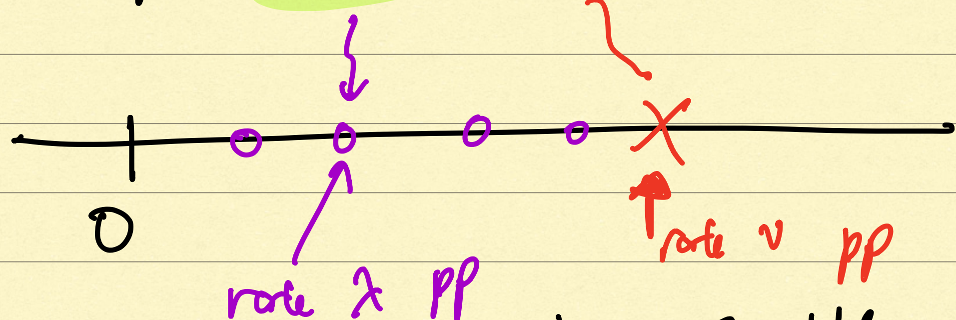
Question: What's the dist^n $N(T)$?

$N(T) = \#$ of arrivals in $(0, T]$.

Answer: View T as the first arrival of an indep. PP of rate ν.

Merge the new ν process w/ the rate λ PP.

Each arrival of the merged PP is original (new) one w.p. $\lambda/\lambda+\nu$ ($\nu/\lambda+\nu$).



inclusive of old & new procs.

K = # of arrivals until the first "success"

success ≡ arrival of new process.

$$p_K(k) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^{k-1}, \quad k=1,2,\dots$$

$N(T) = K-1$. (# of arrival of $PP(\lambda)$).

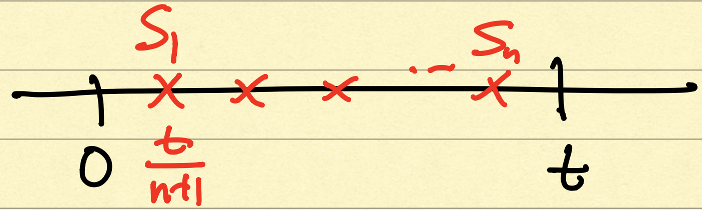
$$p_{N(T)}(n) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^{n}, \quad n=0,1,2\dots$$

Conditional Arrival Densities.

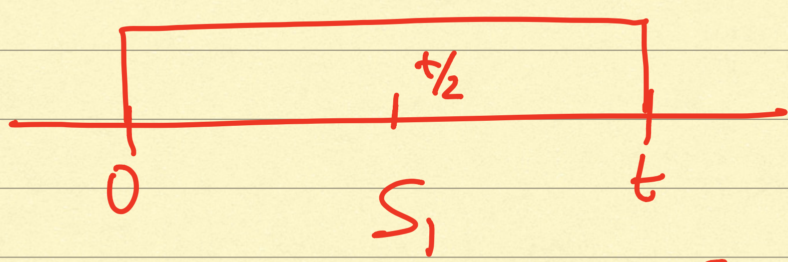
Question: Conditioned on event $\{N(t)=n\}$, there are n arrivals in $(0,t]$.

What's the dist^n of $(S_1, S_2, \dots, S_n)$?

$n \in \mathbb{N}$



$n=1$



$= S^n$

$S^{(n)} = (S_1, \dots, S_n)$

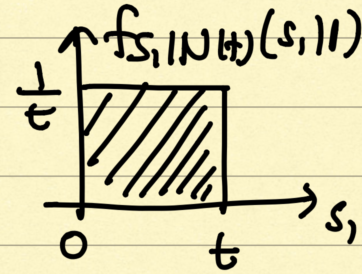
unif dist. on $(0,t]$

<u>Thm</u>: Let $f_{S^n|N(t)}(s^n|n)$ be the joint density of $S^n = (S_1, \ldots, S_n)$ given $N(t)=n$.

This density is constant in the region $0<s_1<s_2<\ldots<t$

$$f_{S^n|N(t)}(\underset{\substack{\parallel \\ (s_1, \ldots, s_n)}}{s^n}|n) = \frac{n!}{t^n}, \quad 0<s_1<s_2<\cdots<s_n<t$$

Eg: $f_{S_1|N(t)}(s_1|1) = \frac{1}{t}, \quad 0<s_1<t$



Pf: Joint dens. of S^{n+1}

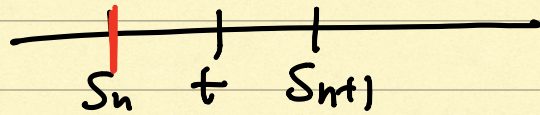
$$f_{S_1\ldots S_{n+1}}(s_1, \ldots, s_{n+1}) = \lambda^{n+1} \exp(-\lambda s_{n+1})$$

$$0 \le s_1 \le s_2 \ldots \le s_n \le s_{n+1}.$$

Bayes rule.

$$f_{S^{n+1}|N(t)}(s^{n+1}|n)\underbrace{P_{N(t)}(n)}_{\Rightarrow \text{ Poisson dist.}} = \overset{1}{\overset{\parallel}{P_{N(t)|S^{n+1}}(n|s^{n+1})}} f_{S^{n+1}}(s^{n+1})$$

Note that $N(t)=n$ iff $t \le S_n$, $S_{n+1} > t$.



$(s_1, \dots, s_n, s_{n+1})$

$$p_{N(t)|S^{n+1}}(n \mid s^{n+1}) = 1 \text{ if } s_n \le t,\ s_{n+1} > t.$$
and 0 otherwise.

Restrict our attn. $N(t)=n$, $S_n \le t$, $S_{n+1} > t$.

$$f_{S^{n+1}|N(t)}(s^{n+1} \mid n) = \frac{f_{S^{n+1}}(s^{n+1})}{p_{N(t)}(n)}$$

$f_{S^n|N(t)}$

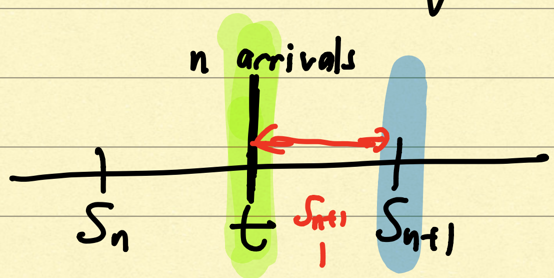
$$= \frac{\lambda^{n+1}\exp(-\lambda s_{n+1})}{(\lambda t)^n \exp(-\lambda t)/n!}$$

joint dist^{n} of S^{n+1}

Poisson dist^{n}.

$$= \frac{n!\,\lambda \exp(-\lambda(s_{n+1}-t))}{t^n}.$$

$$f_{S^{n+1}|N(t)}(s^{n+1} \mid n) = f_{S^n|N(t)}(s^n \mid n)\,\underbrace{f_{S^{n+1}|S^n,N(t)}(s_{n+1} \mid s^n, n)}$$



t

$$f_{S^{n+1}|N(t)}(s^{n+1}|n) = f_{S^n|N(t)}(s^n|n)\,\lambda\exp(-\lambda(s_{n+1}-t))$$

$$\frac{\dfrac{n!\,\cancel{\lambda}\,\cancel{\exp(-\lambda(s_{n+1}-t))}}{t^n}}{\cancel{\lambda\exp(-\lambda(s_{n+1}-t))}} = f_{S^n|N(t)}(s^n|n)$$

$$// \quad \frac{n!}{t^n}$$

in the region $0<s_1<s_2<\cdots<s_n<t$

Recall: $f_{S^n|N(t)}(s^n|n) = \dfrac{n!}{t^n}$, $\quad 0<s_1<s_2<\cdots<s_n<t.$

Arrival epochs $S_1, S_2, \ldots, S_n$

0 otherwise.

How about the joint density of the interarrival times $X^n = (X_1, X_2, \ldots, X_n)$?

$$\boxed{\begin{array}{l} X_i > 0 \\ \sum_{i=1}^{n} X_i < t. \end{array}}$$

There is a 1-1 transformation between S^n & X^n.

$S_1 = X_1, \quad X_i = S_i - S_{i-1}, \; i \in \{2, \dots, n\}.$

$S_1 = X_1, \quad S_n = \sum_{i=1}^{n} X_i.$

$\det(\cdot) = 1.$

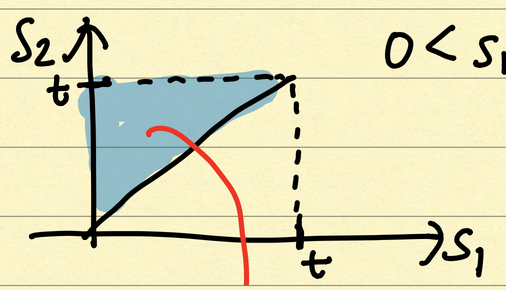
$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 \cdots & 0 \\ 1 & 1 & \ddots & 0 \\ 1 & 1 & 1 & \ddots \\ 1 & 1 & 1 \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ X_n \end{bmatrix}$$

$n=2 \qquad \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$\det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1.$

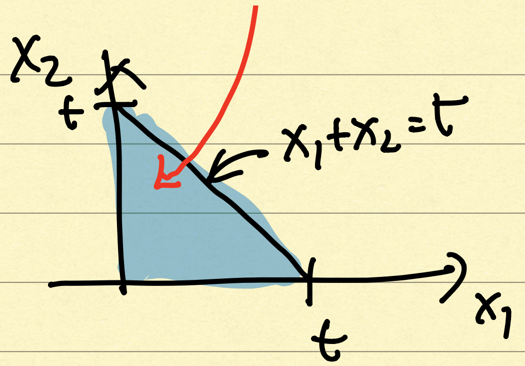
Hence the value of the density for interarrival times is unchanged

$$\underbrace{f_{X^n | N(t)}}_{(X_1, \dots, X_n)} (\underbrace{x^n}_{(x_1, \dots, x_n)} | n) = \frac{n!}{t^n}, \qquad X_i > 0, \; i \in [n], \quad \sum_{i=1}^{n} X_i < t$$



$0 < s_1 < s_2 < t.$

$f_{S^2 | N(t)} = 2 \quad (s^2 | 2)$



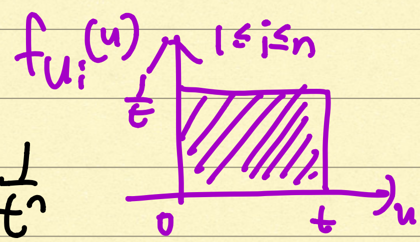
$$f_{X^2|N(t)=2}(x^2|2)$$

Rmk: $f_{X^n|N(t)}(x^n|n)$ is symm. in $(x_1,\dots,x_n)$

Connection to order statistics.

Let $U^n=(U_1,\dots,U_n)$ i.i.d. $(0,t]$ & unif.

For any $u^n \in (0,t]^n$,



$$f_{U^n}(u^n)=\left(\frac{1}{t}\right)^n=\frac{1}{t^n}$$

$\frac{n!}{t^n}$ $\qquad 0<u_i\le t,\ 1\le i\le n.$

$\parallel$

Fact: Both $f_{S^n|N(t)}(\cdot|n)$ & $f_{U^n}(\cdot)$ [as well as $f_{X^n|N(t)}(\cdot|n)$] are unif over the space in which they are positive.

But $f_{U^n}(u^n)=1/t^n$ for $u_i\in(0,t]$ $\forall i$ is $n!$ smaller than $\underline{f_{S^n|N(t)}(\cdot|n)}$ " $\frac{n!}{t^n}$

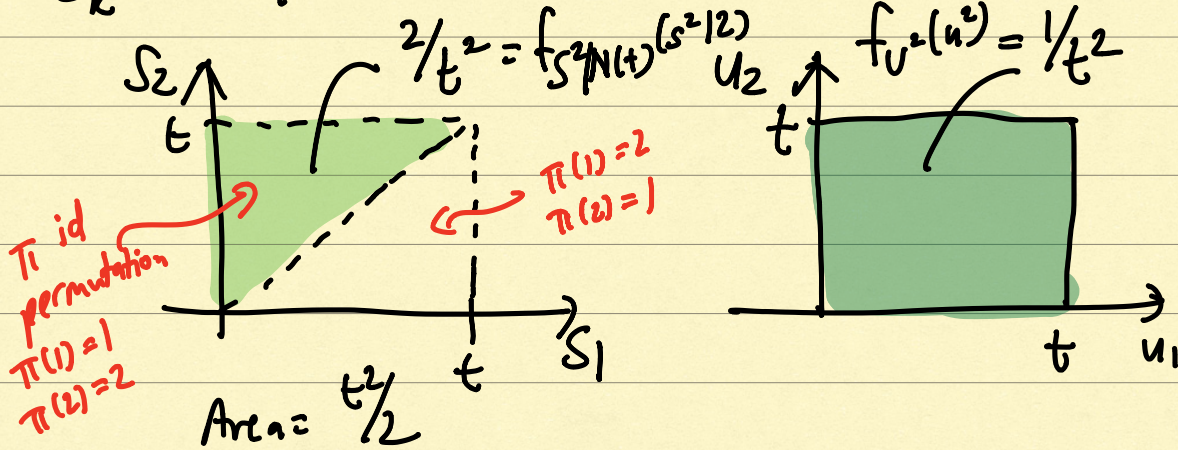
To see this, let $S^n = (S_1, \ldots, S_n)$ be defined as the order statistics of the iid $\{U_i\}$ process

$$S_1 = \min\{U_1, \ldots, U_n\}.$$

$$S_2 = \min[\{U_1, \ldots, U_n\} \setminus \{S_1\}]$$

$$\vdots$$

$$S_k = \min\{\{U_1, \ldots, U_n\} \setminus \{S_1, \ldots, S_{k-1}\}\}$$



Note that the n-cube (cube of length t in n dim) can be partitioned into $n!$ regions.

$\forall$ permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, there is a region $0 < U_{\pi(1)} < U_{\pi(2)} < \cdots < U_{\pi(n)}$

By symmetry each subregion indexed by π has the same volume.

Hence the conditional density $f_{S^n|N(t)}(s^n|n)$ is $n!$ times of $f_{U^n}(u^n)$

When we condition on $\{N(t)=n\}$, $S^n = (S_1, \dots, S_n)$ is distributed the same way as the order statistic of $U_i \sim \text{Unif}[0,t]$ iid

$n=3$ $\quad\boxed{u_1 < u_2 < u_3}\quad$ $u_3 < u_2 < u_1$

$|\Pi| = 3!$ $\quad u_1 < u_3 < u_2 \quad$ $u_3 < u_1 < u_2$

$u_2 < u_1 < u_3$

$u_2 < u_3 < u_1$

Another explanation $\quad f_{S^n|N(t)}(s^n|n) = \frac{n!}{t^n}$,

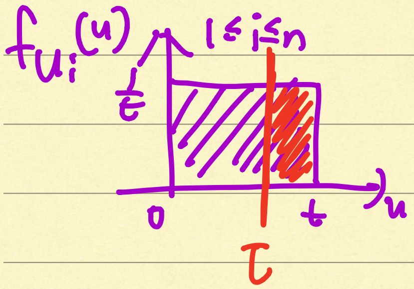
$$0 < s_1 < s_2 < \cdots < s_n < t.$$

What's the density of S_1 given $N(t)=n$?

$$S_1 \stackrel{\parallel}{=} \min\{U_1, \dots, U_n\}.$$

$\Pr(S_1 > \tau \mid N(t) = n) = \Pr(\min\{U_1, \dots, U_n\} > \tau)$

dist is the same $\min\{U_1, \dots, U_n\}$ $\quad = \Pr\left(\bigcap_{i=1}^{n} \{U_i > \tau\}\right)$



$$= \prod_{i=1}^{n} P(U_i > \tau)$$

$$= \prod_{i=1}^{n} \left(\frac{t-\tau}{t}\right) = \left(\frac{t-\tau}{t}\right)^n.$$

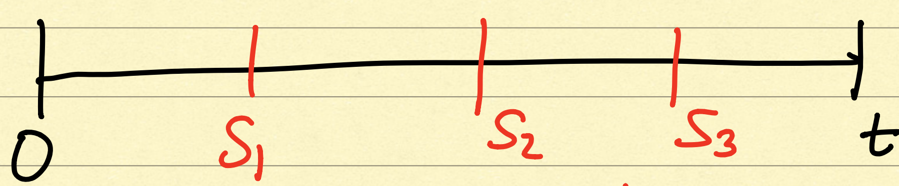
$$\mathbb{E}[S_1 \mid N(t)=n] = \int_0^t P(S_1 > \tau \mid N(t)=n)\, d\tau$$

$$= \int_0^t \left(\frac{t-\tau}{t}\right)^n d\tau$$

$$= \frac{1}{t^n}\left[-\frac{(t-\tau)^{n+1}}{n+1}\right]_0^t$$

$$= \frac{t}{n+1} \quad ///$$

$$\mathbb{E}[S_3 \mid N(t)=3] = \frac{3t}{4}.$$



$$\mathbb{E}[S_1 \mid N(t)=3] = \frac{t}{4}.$$

$$\mathbb{E}[S_1 \mid N(t)=n] = \frac{t}{n+1}$$

$$S_1 = X_1 \qquad \mathbb{E}[X_1 \mid N(t)=n] = \frac{t}{n+1}$$

$S_k = \sum_{i=1}^{k} X_i$ k^{th} arrival epoch.

$$\mathbb{E}[S_k | N(t)=n]$$
$$= \mathbb{E}\left[\sum_{i=1}^{k} X_i \,\Big|\, N(t)=n\right]$$
$$= \sum_{i=1}^{k} \mathbb{E}[X_i | N(t)=n] = \frac{kt}{n+1}$$

$\mathbb{E}[X_i | N(t)=n] = \frac{t}{n+1}$ $\because$ X_i's given $N(t)=n$ are symmetric & identically distributed.

$$f_{X^n | N(t)}(x^n | n) = \frac{n!}{t^n}, \quad X_i > 0,\ i \in [n], \quad \sum_{i=1}^{n} X_i < t$$

Ex 2.18 Gallager book.

a) Consider a Poisson process $N(t)$ with random rate Λ

$$f_\Lambda(\lambda) = \begin{cases} \alpha e^{-\alpha\lambda}, & \lambda \geq 0 \\ 0, & \text{else} \end{cases}$$

$$\Pr(N(t)=n \mid \Lambda=\lambda) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n=0,1,\ldots$$

b) $$\Pr(N(t)=n) = \int_0^\infty \Pr(N(t)=n \mid \Lambda=\lambda) f_\Lambda(\lambda)\, d\lambda$$

$$= \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \alpha e^{-\alpha\lambda}\, d\lambda.$$

$$= \frac{\alpha t^n}{(\alpha+t)^n} \int_0^\infty \frac{((\alpha+t)\lambda)^n e^{-(\alpha+t)\lambda}}{n!}\, d\lambda$$

$$= \frac{\alpha t^n}{(\alpha+t)^{n+1}} \underbrace{\left(\int_0^\infty \boxed{\frac{z^n e^{-z}}{n!}}\, dz\right)}_{=1}$$

$z = (\alpha+t)\lambda$

$dz = d\lambda\,(\alpha+t)$

Erlang of order $n+1$ with rate 1.

Erlang dist. of order n with rate λ

$$f_{S_n}(t) = \frac{\cancel{\lambda^n}\, t^{n-1} e^{-\cancel{\lambda} t}}{(n-1)!}, \quad t \geq 0.$$

$$= \frac{\alpha}{\alpha+t}\left(\frac{t}{\alpha+t}\right)^n, \quad n=0,1,2,\ldots$$

The above is too cumbersome.

Pf 2: Smarter method.

Observe that sol^{n} is $p(1-p)^n$, $p = \frac{\alpha}{\alpha+t}$ $\quad n=0,1,\dots$

Note that $\Pr(N_\lambda(t)=n)$ for a PP of fixed deterministic rate λ is a function of only (λt).

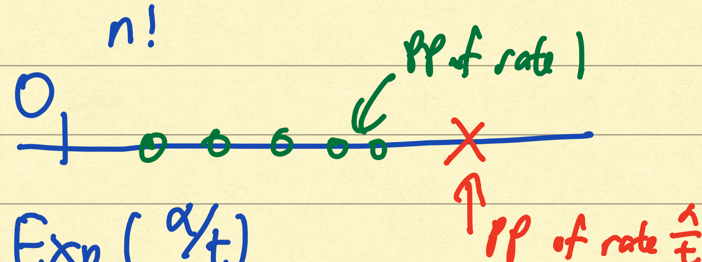
$$\Pr(N_\lambda(t)=n) = \frac{e^{-(\lambda t)}(\lambda t)^n}{n!} \qquad n=0,1,\dots$$

$$N_\lambda(t) \overset{d}{=} N_1(\lambda t)$$

$$\Pr(N_\lambda(t)=n) = \frac{e^{-(\lambda t)}(\lambda t)^n}{n!}$$

$$\Pr(N_1(\lambda t)=n) = \frac{e^{-(\lambda t)}(\lambda t)^n}{n!}$$

$\Lambda \sim \mathrm{Exp}(\alpha)$



Claim: $\Lambda t \sim \mathrm{Exp}(\alpha/t)$

$N(\Lambda t)$: represents the # of arrivals of a PP of rate 1 before the first arrival of an indep. PP w/ rate α/t.

$$\Pr(N(\Lambda t)=n) = \left(\frac{1}{1+\frac{\alpha}{t}}\right)^n \left(\frac{\frac{\alpha}{t}}{1+\frac{\alpha}{t}}\right)^1, \quad n=0,1,\dots$$

$$= \left(\frac{t}{\alpha+t}\right)^n \left(\frac{\alpha}{\alpha+t}\right)^1, \quad n=0,1,\ldots$$

c) $f_\Lambda(\lambda \mid N(t)=n)$

$$= \frac{\Pr(N(t)=n \mid \Lambda=\lambda)\, f_\Lambda(\lambda)}{\Pr(N(t)=n)}$$

$$= \frac{\frac{(\lambda t)^n e^{-\lambda t}}{n!}\, \alpha e^{-\alpha t}}{\frac{\alpha t^n}{(\alpha+t)^{n+1}}}$$

$$= \frac{\lambda^n e^{-\lambda(\alpha+t)} (\alpha+t)^{n+1}}{n!} \qquad \lambda \geq 0.$$

Erlang order $n+1$ rate $\alpha+t$.