Last Time: Poisson Processes have 3 equivalent descriptions.

[Not tested on non-homogeneous.

i) IID interarrival times $\{X_i\}_{i=1}^{\infty}$

$$X_i \sim Exp(\lambda),\ f_{X_i}(x) = \lambda e^{-\lambda x} \mathbb{1}\{x \geq 0\}$$

ii) Arrival Epochs $\{S_i\}_{i=1}^{\infty}$ $(S_{i+1} > S_i)$

$$S_1 = X_1,\quad S_2 = X_1 + X_2,\ \ldots,\ S_n = \sum_{i=1}^{n} X_i$$

$$X_i = S_i - S_{i-1},\quad i \geq 2.$$

cts time

iii) Poisson counting process $\{N(t) : t > 0\}$

$N(t) :=$ # of arrivals in $(0, t]$

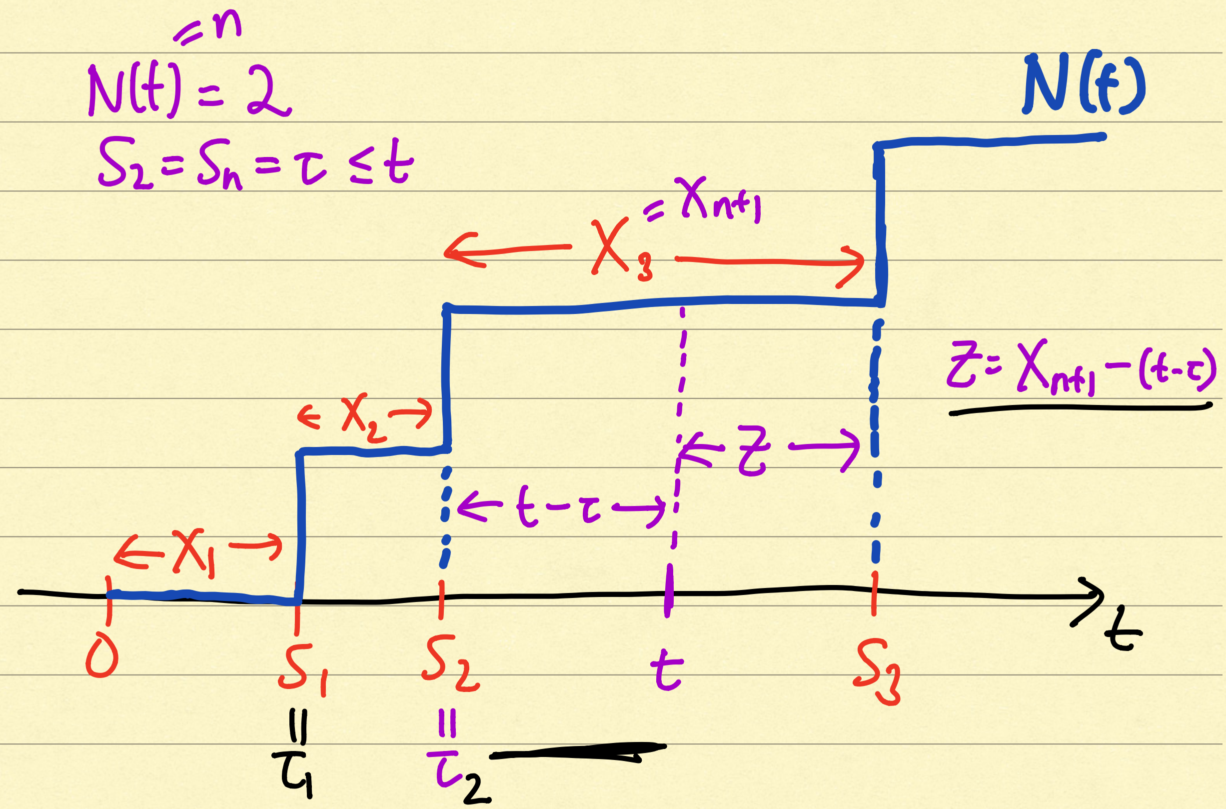
$N(t) = n \iff S_n \leq t < S_{n+1}$

<u>Thm</u>: For a Poisson process $Poi(\lambda)$, for any $t > 0$,

1) the length of the interval from t to the next arrival, denoted as $Z = Z_1$, is

ii) $\underline{\mathrm{Exp}(\lambda)}$.

Furthermore $Z \perp\!\!\!\perp N(t)$ &
$Z \perp\!\!\!\perp S_1, S_2, \dots, S_{N(t)}$
$Z \perp\!\!\!\perp \{N(\tau) : 0 < \tau \le t\}$.



Condition $Z = z_1$ on $N(t) = n$ & $S_n = \tau \le t$.

$$\begin{aligned}
&\Pr(Z > z \mid N(t) = n, S_n = \tau) \\
&= \Pr(X_{n+1} - (t-\tau) > z \mid N(t) = n, S_n = \tau) \\
&= \Pr(X_{n+1} > z + (t-\tau) \mid N(t) = n, S_n = \tau) \\
&\boxed{= \Pr(X_{n+1} > z + (t-\tau) \mid X_{n+1} > t-\tau, \cancel{S_n = \tau})}
\end{aligned}$$

$$\boxed{= Pr\left(X_{n+1} > z + \cancel{(t-\tau)} \mid X_{n+1} > \cancel{t-\tau}\right)}$$

$$\boxed{= Pr\left(X_{n+1} > z\right)} = e^{-\lambda z}$$

Given $S_n = \tau \leq t$, $\{N(t)=n\} = \{X_{n+1} > t-\tau\}$

$\{X_i\}_{i=1}^{\infty}$ are iid. & $S_n = \sum_{i=1}^{n} X_i$ $\left(X_{n+1} \perp\!\!\!\perp (X_1, \ldots X_n)\right)$

By memorylessness of X_{n+1}.

$$Pr(Z > z \mid N(t)=n, S_n = \tau) = e^{-\lambda z}.$$

Consider

$$Pr(Z > z \mid N(t)=n, \underbrace{S_1}_{X_1} = \tau_1, \ldots, \underbrace{S_n}_{X_1+\cdots+X_n} = \tau_n) = e^{-\lambda z}.$$

But $\{N(t)=n, S_1 = \tau_1, \ldots, S_n = \tau_n\}$ describes the entire process up to & incl. time t

i.e., $\{N(\tau): 0 < \tau \leq t\}$.

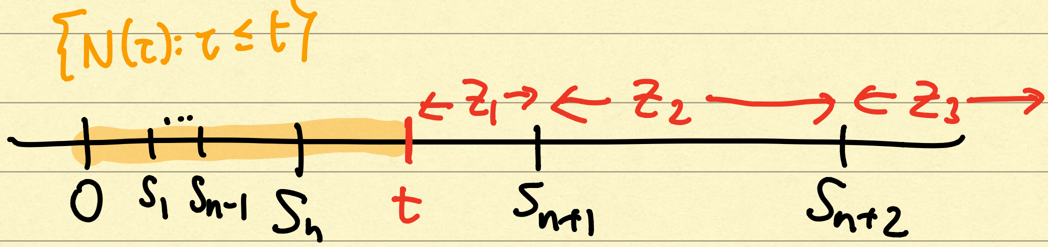
$$\boxed{Pr\left(Z > z \mid \underbrace{\{N(\tau): 0 < \tau \leq t\}}_{\{N(\tau):\, \tau \leq t\}}\right) = e^{-\lambda z}}$$

$\Rightarrow$ By integrating or averaging over $\{N(\tau): \tau \leq t\}$ we get

$$Pr(Z > z) = e^{-\lambda z} \quad (i)$$

$$Z_1 \perp\!\!\!\perp \{N(\tau) : \tau \leq t\} \quad \text{(ii).}$$

Stationary Inc. property (SIP)



Z_m: Time from $(m-1)^{st}$ arrival after t to m^{th} arrival

Z_1 = time from t to S_{n+1}

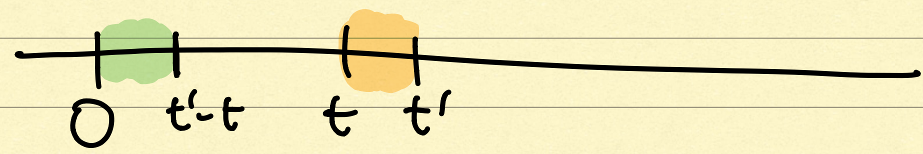
Fact: $X_{n+m} = Z_m$, $m \geq 2$.

Conditioned on $N(t) = n$, $S_n = \tau$, $Z_1, \ldots, Z_n, \ldots$ IID. $\sim Exp(\lambda)$ Exp(λ) rvs.

By conditioning $S_1, \ldots, S_n$ (as was done above) we see that $\{Z_i\}_{i=1}^{\infty}$ are unconditionally IID.

Poisson process starting at t is a probabilistic replica of the process starting at 0.

$t' > t \geq 0$ $\quad N(t') - N(t) \stackrel{d}{=} N(t'-t)$



Def: A counting process $\{N(t): t > 0\}$ has the SIP if

$$N(t') - N(t) \overset{d}{=} N(t'-t)$$
$$\forall \; 0 \leq t < t'$$

$$X \overset{d}{=} Y \Leftrightarrow \Pr(X \leq z) = \Pr(Y \leq z) \quad \forall z.$$

Thm: The Poisson counting has the SIP.

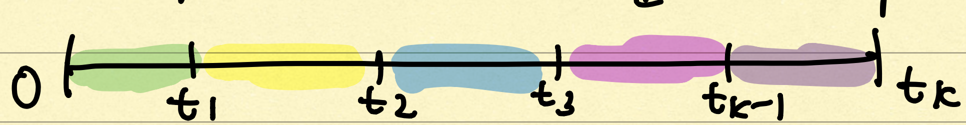
Notation: $\tilde{N}(t, t') := N(t') - N(t)$
Denotes the # of arrivals $(t, t']$

Def: A counting $\{N(t) : t > 0\}$ has the independent increment prop. (IIP) if $\forall k$

$\forall \; 0 < t_1 < t_2 < \cdots < t_k$

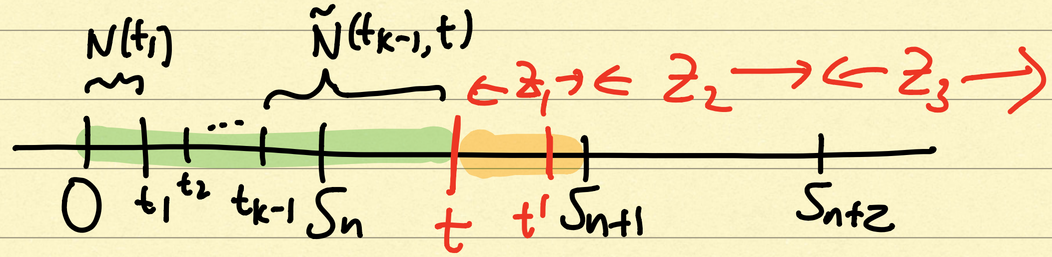
$N(t_1)$, $\tilde{N}(t_1, t_2)$, $\tilde{N}(t_2, t_3)$, ..., $\tilde{N}(t_{k-1}, t_k)$
are independent.

← are indep



Thm: The Poisson has the IIP.

Pf:



$Z_1 \perp\!\!\!\perp \{N(\tau) : 0 < \tau \leq t\}$

$Z_1 \perp\!\!\!\perp (N(t_1), \tilde{N}(t_1,t_2), \dots, \tilde{N}(t_{k-1}, t))$

But $\underline{Z_1, Z_2, Z_3, \dots}$ are IID.

$\underline{\tilde{N}(t,t')} \perp\!\!\!\perp (N(t_1), \tilde{N}(t_1,t_2), \dots, \tilde{N}(t_{k-1}, t))$

Rename $t = t_k$, $t' = t_{k+1}$

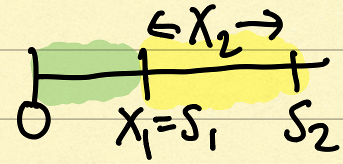
$\tilde{N}(t_k,t_{k+1}) \perp\!\!\!\perp (N(t_1), \tilde{N}(t_1,t_2), \dots, \tilde{N}(t_{k-1}, t_k))$

This is true $\forall k$ & $\forall$ times $t_1 < t_2 < \dots < t_{k+1}$ the Poisson counting proc. $\{N(t) : t > 0\}$ has the IIP.

Distribution of S_n : n^{th} arrival epoch.

$\underline{\underline{S_2}} = X_1 + X_2 \quad \{X_i\}_{i=1}^{\infty}$ IID $\mathrm{Exp}(\lambda)$

$X_1 = S_1 \quad , \quad S_2 = X_1 + X_2.$



$$f_{X_1 S_2}(x_1, s_2) = f_{X_1}(x_1) f_{S_2 | X_1}(s_2 | x_1)$$

$$= \lambda e^{-\lambda x_1} f_{X_2}(s_2 - x_1)$$

$s_2 \geq x_1$

$$= \lambda \cancel{e^{-\lambda x_1}} \lambda e^{-\lambda(s_2 \cancel{- x_1})}$$

$$\Rightarrow f_{X_1 S_2}(x_1, s_2) = \lambda^2 e^{-\lambda s_2}, \quad 0 \leq x_1 \leq s_2 .$$

Rmk: This density doesn't depend on x_1.
$\Rightarrow$ X_1 is <u>unif over $[0, s_2]$</u> if $S_2 = s_2$.
(Will discuss more abt this when we discuss conditional arrival densities)

$$f_{S_2}(s_2) = \int_{x_1=0}^{s_2} f_{X_1 S_2}(x_1, s_2) \, dx_1$$

$$\| $$

$$X_1 + X_2 \qquad = \int_0^{s_2} \lambda^2 e^{-\lambda s_2} dx_1 = \lambda^2 s_2 e^{-\lambda s_2}$$

$s_2 \in [0, \infty)$

This is the Erlang density of order 2 with rate λ. Erlang$(2,\lambda)$

Rmk: Erlang$(1,\lambda)$ = Exp(λ)

Rmk: Check the above via convolution & MGFs.

Thm: $f_{S_n}(s_n) = \dfrac{\lambda^n s_n^{n-1} \exp(-\lambda s_n)}{(n-1)!}$, $s_n \geq 0$

n^{th} arrival epoch

Rmk: This is true for $n=1$ & $n=2$

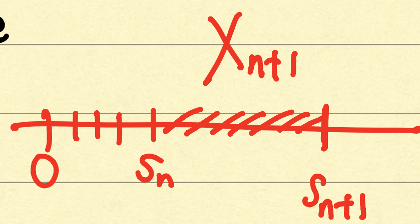
$\lambda e^{-\lambda s_1}$, $s_1 \geq 0$ Exp(λ) $\qquad$ $\lambda^2 s_2 e^{-\lambda s_2}$, $s_2 \geq 0$. Erlang$(2,\lambda)$

Sub-Claim: Joint density

$$f_{S_1 \cdots S_n}(s_1, \cdots, s_n) = \begin{cases} \lambda^n \exp(-\lambda s_n) & 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \\ 0 & \text{else} \end{cases}$$

Induction hypothesis

$f_{S_1 \cdots S_n S_{n+1}}(s_1, \cdots, s_n, s_{n+1})$



$$= f_{S_1 \dots S_n}(s_1, \dots, s_n) \boxed{f_{S_{n+1} | S_1 \dots S_n}(s_{n+1} | s_1, \dots, s_n)}$$

$$\overset{IH}{=} \lambda^n e^{-\lambda s_n} f_{X_{n+1}}(s_{n+1} - s_n)$$

$$= \underline{\lambda^n \cancel{e^{-\lambda s_n}}} \lambda e^{-\lambda(s_{n+1} - \cancel{s_n})} = \underline{\lambda^{n+1} e^{-\lambda s_{n+1}}}$$

$$0 \le s_1 \le s_2 \le \cdots \le s_{n-1} \le s_n \le s_{n+1}$$

$n = 3$

<u>Thm</u>: $f_{S_n}(s_n) = \dfrac{\lambda^n s_n^{n-1} \exp(-\lambda s_n)}{(n-1)!}, \quad s_n \ge 0$

Pf: Integrate out s_1, s_2 from $f_{S_1 S_2 S_3}$

$$f_{S_3}(s_3) = \int_{\substack{(s_1, s_2) \\ 0 < s_1 < s_2 < s_3}} f_{S_1 S_2 S_3}(s_1, s_2, s_3) \, ds_1 \, ds_2$$

Sub-Claim

$$= \underbrace{\int_0^{s_3} \underbrace{\left(\int_0^{s_2} \lambda^3 \exp(-\lambda s_3) \, ds_1 \right)}_{\because\ 0 < s_1 < s_2} ds_2}_{\because\ 0 < s_2 < s_3}$$

$$= \int_0^{s_3} \lambda^3 s_2 \exp(-\lambda s_3)\, ds_2$$

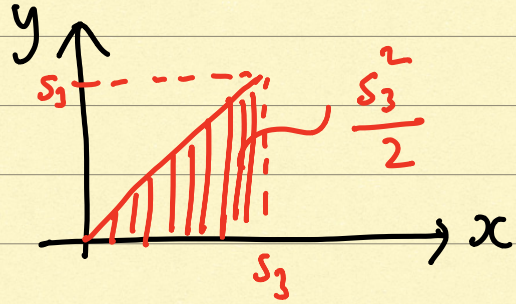
$$= \frac{s_3^2}{2} \lambda^3 \exp(-\lambda s_3), \quad s_3 \geq 0$$

Erlang density of order 3 w/ rate λ.

For $n=4$, see lec5.pdf.

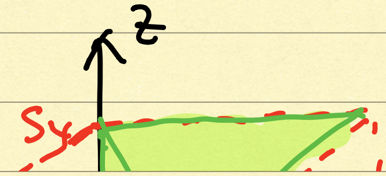
Rmk: $\exists$ factor $\frac{s_n^{n-1}}{(n-1)!}$ $\because$ the "volume" of the region $(s_1, \ldots, s_{n-1})$ s.t. $0 < s_1 < \cdots < s_{n-1} < s_n$ is exactly $\frac{s_n^{n-1}}{(n-1)!}$ $\nearrow$ $s_3^2/2$

Eg: $n=3$, Region $\{(x,y) : 0 \leq x \leq y \leq s_3\}$



$n=4$, Region $\{(x,y,z) : 0 \leq x \leq y \leq z \leq s_4\}$

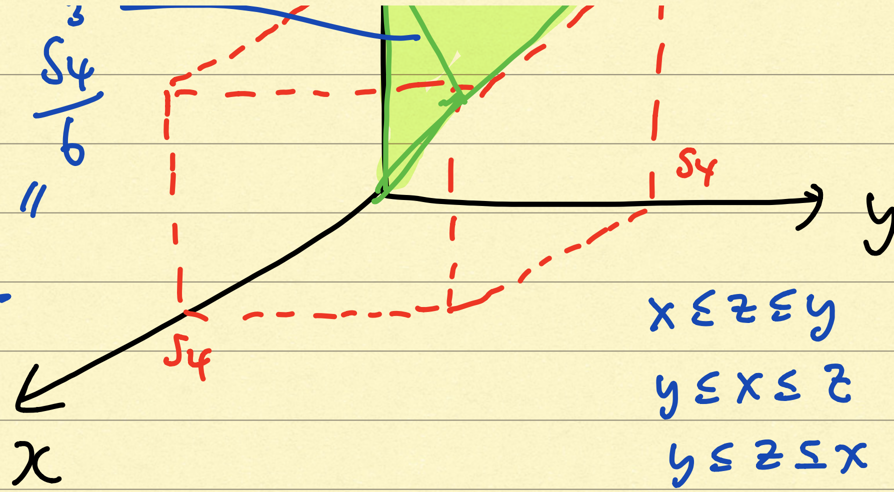
Volume



$= \frac{S_4}{6}$

$\| $

$\frac{S_4^3}{3!}$



$x \le z \le y$
$y \le x \le z$
$y \le z \le x$
$z \le x \le y$
$z \le y \le x$

PMF of $N(t)$: # of arrival in $(0,t]$.

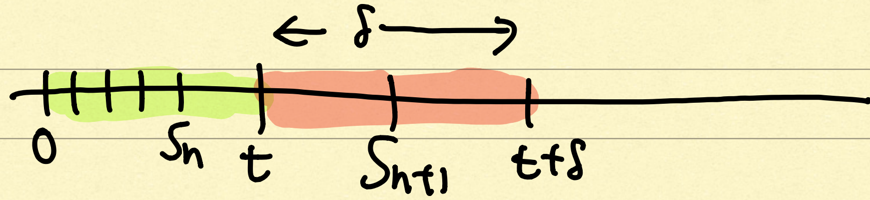
Thm: Poisson process with rate λ.
PMF of $N(t)$ is a Poisson rv with mean/rate λt, i.e.,

$$P_{N(t)}(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots$$

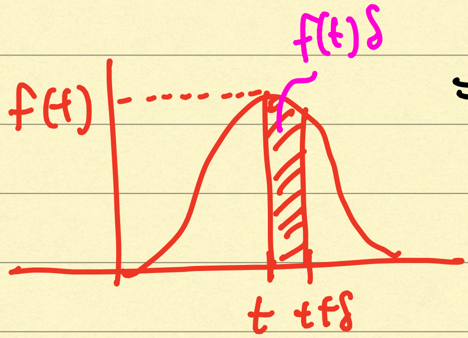
Pf 1: $g(\delta) = o(\delta)$ iff $\lim_{\delta \downarrow 0} \left| \frac{g(\delta)}{\delta} \right| = 0.$

e.g. $g(\delta) = \delta^2 \in o(\delta)$
$g(\delta) = \delta \notin o(\delta)$

Consider $\{t < S_{n+1} \le t+\delta\}$ for $\delta > 0$ small



$$\Pr(t < S_{n+1} \le t+\delta) = \int_t^{t+\delta} f_{S_{n+1}}(\tau)\,d\tau$$



$$= f_{S_{n+1}}(t)(\delta + o(\delta))$$

$$\Pr(t < S_{n+1} \le t+\delta) = \Pr(n \text{ arrivals} \in (0,t] \;\&\; 1 \text{ arrival } (t, t+\delta])$$

$$= \Pr(n \text{ arrivals in } (0,t]) \Pr(1 \text{ arrival in } (t, t+\delta])$$

$$= p_{N(t)}(n) \int_0^\delta f_X(\tau)\,d\tau$$

$$= p_{N(t)}(n) \int_0^\delta \lambda e^{-\lambda\tau}\,d\tau = \left[-e^{-\lambda\tau}\right]_0^\delta$$

$$= 1 - e^{-\lambda\delta}$$

$$= p_{N(t)}(n)(\lambda\delta + o(\delta)) \qquad = \lambda\delta + o(\delta)$$

$$p_{N(t)}(n)(\lambda\delta + o(\delta)) = f_{S_{n+1}}(t)(\delta + o(\delta))$$

$$p_{N(t)}(n)\,(\lambda \cancel{\delta}) = \frac{\lambda^{n+1} t^n e^{-\lambda t}}{n!} \cancel{\delta}$$

$$p_{N(t)}(n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, \ldots$$

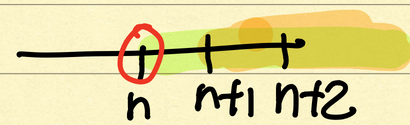
Pf 2: Recall $\{N(t) \geq n\} = \{S_n \leq t\}$

Fact: $n \geq 1$

$$\int_0^b x^{n-1} e^{-ax}\,dx = \frac{(n-1)!}{a^n}\left[1 - e^{-ab} \sum_{i=0}^{n-1} \frac{(ab)^i}{i!}\right]$$

$$\Pr(N(t) \geq n) = \Pr(S_n \leq t)$$

$$= \int_0^t \frac{\lambda^n s^{n-1} e^{-\lambda s}}{(n-1)!}\,ds.$$



$$= \frac{\lambda^n}{(n-1)!} \int_0^t s^{n-1} e^{-\lambda s}\,ds$$

$$= \frac{\cancel{\lambda^n}}{\cancel{(n-1)!}} \frac{\cancel{(n-1)!}}{\cancel{\lambda^n}} \left[1 - e^{-\lambda t} \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}\right].$$

$$\Pr(N(t) \geq n+1) = 1 - e^{-\lambda t} \sum_{i=0}^{n} \frac{(\lambda t)^i}{i!}$$

$$\Pr(N(t) = n) = \Pr(N(t) \geq n) - \Pr(N(t) \geq n+1)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Conclusion: $N(t) \sim \text{Poi}(\lambda t)$

$\mathbb{E}[N(t)] = \lambda t.$

Alternative def. of Poisson processes

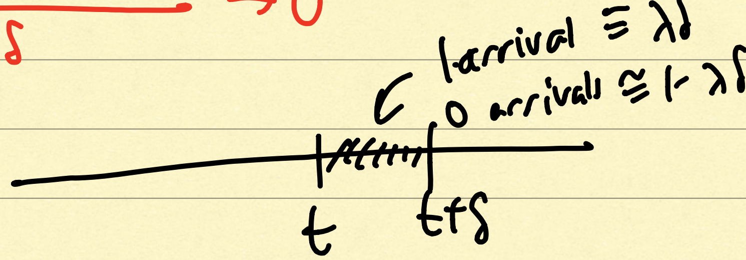
Def: A Poisson counting process $\{N(t): t \geq 0\}$ is a counting process s.t. $N(t) \sim \text{Poi}(\lambda t)$ and $\{N(t): t \geq 0\}$ has the SIP & IIP.

$$\tilde{N}(t, t+\delta) \overset{d}{=} N(\delta) \quad \text{SIP.}$$

Condition (*)

$$\left.\begin{array}{l} \Pr(\underbrace{\tilde{N}(t, t+\delta) = 0}_{= a(\delta)}) = 1 - \lambda\delta + o(\delta) \\ \Pr(\tilde{N}(t, t+\delta) = 1) = \lambda\delta + o(\delta) \\ \Pr(\tilde{N}(t, t+\delta) \geq 2) = o(\delta) \end{array}\right\} (*)$$

$$(*) \Rightarrow \frac{|a(\delta) - (1 - \lambda\delta)|}{\delta} \to 0 \quad \text{as } \delta \to 0^+$$



Def: A Poisson counting process is a counting process that satisfies (*) & satisfies SIP & IIP.

Poisson process as a limit of shrinking Bernoulli processes.

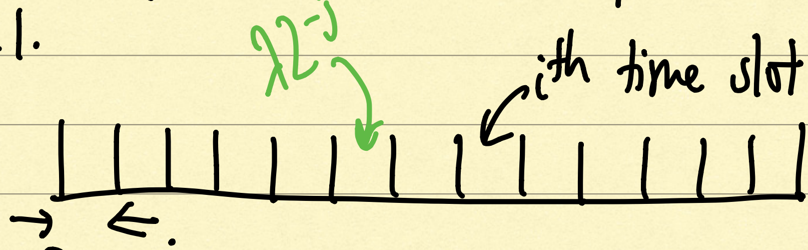
Bernoulli $\{Y_i\}_{i=1}^{\infty}$ $\quad Y_i \in \{0,1\}$

$\Pr(Y_i = 1) = p$, $\Pr(Y_i = 0) = 1-p$

$Y_i = 1 \Rightarrow$ arrival in time slot i.

$Y_i = 0 \Rightarrow$ no arrival in time slot i.

Consider a sequence of Bernoulli processes indexed by $j \geq 1$.



$\delta = 2^{-j}$ Each time slot is of size $\delta = 2^{-j}$.

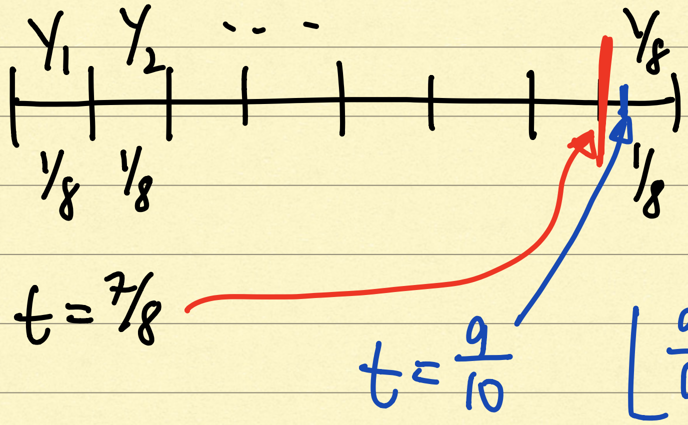
For the j^{th} Bernoulli process, let $p = \lambda 2^{-j} = \lambda\delta$.

The arrival rate is constant across all $j \geq 1$.

j^{th} Bernoulli counting process

$$N_j(t) = \sum_{i=1}^{\lfloor t 2^j \rfloor} Y_i$$

$j=3$



$$\left\lfloor \frac{9}{10} \times 2^3 \right\rfloor = \left\lfloor \frac{72}{10} \right\rfloor = 7$$

$$P_{N_j(t)}(n) = \binom{\lfloor t2^j \rfloor}{n} p^n (1-p)^{\lfloor t2^j \rfloor - n}$$

$p = \lambda\delta = \lambda 2^{-j}$. $\quad n = 0, 1, \ldots, \lfloor t\,2^j \rfloor$.

Thm: Consider the sequence of shrinking Bernoulli processes with arrival prob $\lambda 2^{-j} = p$ & slot size 2^{-j}.

Then fix t & fix n.

$$\lim_{j \to \infty} P_{N_j(t)}(n) = P_{N(t)}(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Pf: Similar to a problem in HW4.

Combining & Splitting of Poisson processes.

$\{N_1(t): t>0\}$ & $\{N_2(t): t>0\}$ independent.

rate $\downarrow$ λ_1 $\qquad$ $\downarrow$ λ_2.

Consider the sum process $N(t) = N_1(t) + N_2(t)$

Thm: $\{N(t): t>0\}$ is a Poisson process with rate $\underbrace{\lambda_1+\lambda_2}=\lambda$

$$
\begin{aligned}
&\Pr(\tilde{N}(t, t+\delta)=0) \\
&= \Pr(\tilde{N}_1(t, t+\delta)=0 \text{ and } \tilde{N}_2(t, t+\delta)=0) \\
&= \Pr(\tilde{N}_1(t, t+\delta)=0)\Pr(\tilde{N}_2(t, t+\delta)=0) \\
&= (1-\lambda_1\delta + o(\delta))(1-\lambda_2\delta + o(\delta)) \\
&= 1 - \underbrace{(\lambda_1+\lambda_2)}_{\lambda}\delta + o(\delta)
\end{aligned}
$$

$A \cap B = \phi$

$$
\begin{aligned}
&\Pr(\tilde{N}(t, t+\delta)=1) \\
&= \Pr(\underbrace{\{\tilde{N}_1(t, t+\delta)=0 \;\&\; \tilde{N}_2(t, t+\delta)=1\}}_{A} \text{ or } \\
&\qquad \underbrace{\{\tilde{N}_1(t, t+\delta)=1 \;\&\; \tilde{N}_2(t, t+\delta)=0\}}_{B}) \\
&= \Pr(\{\tilde{N}_1(t, t+\delta)=0 \;\&\; \tilde{N}_2(t, t+\delta)=1\}) \\
&\quad + \Pr(\{\tilde{N}_1(t, t+\delta)=1 \;\&\; \tilde{N}_2(t, t+\delta)=0\})
\end{aligned}
$$

$$= Pr(\tilde{N}_1(t,t+\delta)=0)\, Pr(\tilde{N}_2(t,t+\delta)=1)$$

$$+\; Pr(\tilde{N}_1(t,t+\delta)=1)\, Pr(\tilde{N}_2(t,t+\delta)=0)$$

$$= \lambda_1\delta\,(1-\lambda_2\delta) + \lambda_2\delta\,(1-\lambda_1\delta)$$

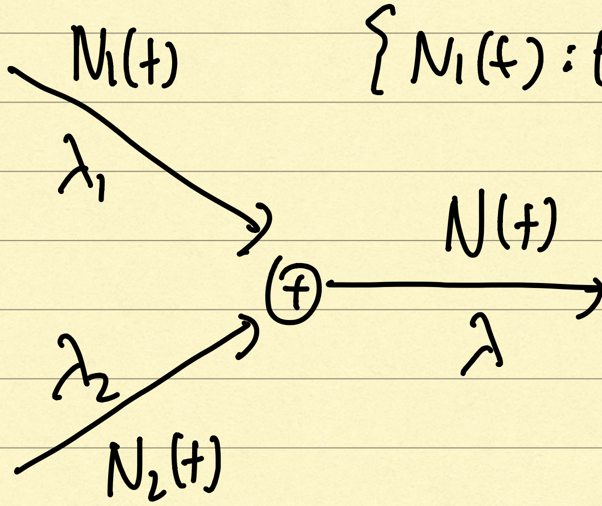
$$= \underset{\lambda}{\underset{\shortparallel}{(\lambda_1+\lambda_2)}}\,\delta + o(\delta)$$

$$Pr(\tilde{N}(t,t+\delta)\geq 2) = 1 - Pr(\tilde{N}(t,t+\delta)=0 \text{ or } 1)$$
$$= 1-(1-\lambda\delta + \lambda\delta + o(\delta)) = o(\delta).$$

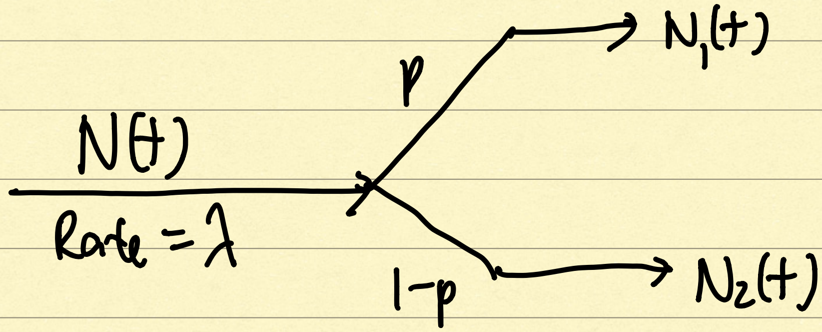
Also need to verify SIP & IIP.

$\Rightarrow$ $N(t)$ is a Poisson counting process with $\lambda = \lambda_1+\lambda_2$.

$\{N_1(t): t>0\} \perp\!\!\!\perp \{N_2(t): t\geq 0\}$



Splitting of Poisson Process



Split a Poisson process into 2 counting process $N_1(t)$ & $N_2(t)$

Whenever there's an arrival, assigned as type-I and type-II w.p. p & $1-p$ resp.

Thm: Resulting processes $N_1(t)$ & $N_2(t)$ are Poisson processes with rates λp & $(1-p)\lambda$ resp.

Moreover $\{N_1(t) : t > 0\} \perp\!\!\!\perp \{N_2(t) : t > 0\}$.

Pf: $\{N(t) = m+k\}$, $m, k \in \mathbb{N} \cup \{0\}$.

$$Pr(N_1(t) = m, N_2(t) = k \mid N(t) = m+k)$$

$$= \binom{m+k}{m} p^m (1-p)^k$$

But Bayes rule,

$$Pr(N_1(t)=m, N_2(t)=k \mid N(t)=m+k)$$

$$= \frac{Pr(N_1(t)=m, N_2(t)=k, N(t)=m+k)}{Pr(N(t)=m+k)}$$

$$= \frac{Pr(N_1(t)=m, N_2(t)=k, \cancel{N(t)=m+k})}{Pr(N(t)=m+k)}$$

$$\Rightarrow Pr(N_1(t)=m, N_2(t)=k)$$

$$= Pr(N(t)=m+k)\left(\binom{m+k}{m} p^m (1-p)^k\right)$$

$$= \frac{e^{-\lambda t}(\lambda t)^{m+k}}{\cancel{(m+k)}!} \frac{\cancel{(m+k)}!}{m!\,k!} p^m (1-p)^k$$

$$= \frac{e^{-\lambda t}(\lambda p t)^m (\lambda(1-p)t)^k}{m!\,k!}$$

$$= \frac{(\lambda p t)^m e^{-\lambda p t}}{m!} \cdot \frac{(\lambda(1-p)t)^k e^{-\lambda(1-p)t}}{k!}$$

$$= Pr(N_1(t)=m, N_2(t)=k)$$

$N_1(t) \sim \text{Poi}(\lambda p t)$ & $N_1(t) \perp\!\!\!\perp N_2(t)$

$N_2(t) \sim \text{Poi}(\lambda(1-p)t)$

Also need to show $\{N_1(t) : t > 0\} \perp\!\!\!\perp \{N_2(t) : t > 0\}$
This can be formally done by using SIP &
IIP of N_1 & N_2.