ELSEVIER

Applied Numerical Mathematics 29 (1999) 361–377

APPLIED NUMERICAL MATHEMATICS

On parametric semidefinite programming [☆]

D. Goldfarb [a,*], K. Scheinberg [b,1]

[a] *Columbia University, Department of Industrial Engineering and Operations Research, 500 W. 120th Street, Mudd Building, New York, NY 10027-6699, USA*

[b] *IBM T.J. Watson Research Center, Yorktown Heights, NY, USA*

Received 27 March 1997; received in revised form 6 April 1998

Abstract

In this paper we consider a semidefinite programming (SDP) problem in which the objective function depends linearly on a scalar parameter. We study the properties of the optimal objective function value as a function of that parameter and extend the concept of the optimal partition and its range in linear programming to SDP. We also consider an approach to sensitivity analysis in SDP and the extension of our results to an SDP problem with a parametric right-hand side. © 1999 Elsevier Science B.V. and IMACS. All rights reserved.

Keywords: Parametric semidefinite programming; Optimal partition; Sensitivity analysis

1. Introduction

In this paper we study certain properties of parametric semidefinite programming (SDP) problems. We consider an SDP problem whose objective function depends linearly on a scalar parameter. All results in the paper can be easily extended to the case where the right-hand side of an SDP is changed parametrically. It is also possible to generalize the results to the case of multiple parameters.

Semidefinite programming has attracted the attention of researchers in mathematical programming in the last few years. Most of this attention has been paid to theory and practical implementations of interior point methods for SDP. There has been relatively little work on issues of postoptimal analysis or parametric behavior in SDP. In several papers Bonnans, Cominetti and Shapiro [5,6,15] derive second order optimality conditions for SDP problems and study sensitivity analysis. Under the assumption that these conditions hold they obtain an expansion of a solution of an SDP. Their methods

☆ Research supported in part by NSF Grants DMS 91-06195, DMS 94-14438 and DMS 95-27124 and DOE Grant DE-FG02-92ER25126.

* Corresponding author. E-mail: gold@ieor.columbia.edu.

1 The doctoral research of this author was supported in part by an IBM Cooperative Fellowship.

PII: S0168-9274(98)00102-0

are based on a general theory of perturbation of smooth nonlinear problems and second order optimality conditions [8].

In this paper we exploit the special structure of an SDP problem and the fact that it is a natural extension of a linear programming problem to develop parametric and sensitivity analysis for SDP. In particular, we extend some of the properties of parametric linear and quadratic programming problems to the case of SDP. The classical approach to parametric linear programming is the parametric simplex method. This approach is based on the concept of an optimal basis. For every value of the parameter there is an optimal basis associated with it. Information about the local behavior of the optimal objective function value and the optimal solutions as a function of the parameter, then depends on the choice of the optimal basis for each value of the parameter. However, as shown in [12], in cases of primal and dual degeneracy such information may not be sufficient and may even be misleading. In [1,12] an alternative approach to parametric linear programming and sensitivity analysis was proposed that is based on the concept of an optimal partition. The optimal partition corresponds to a pair of primal–dual strictly complementary optimal solutions and is uniquely defined (unlike the optimal basis). It has the advantage that it contains the information needed to define the local behavior of the optimal solution and the optimal objective function value of a parametric linear program. The optimal partition approach was extended to quadratic programming problems in [3].

In this paper we extend this approach to SDP problems. We choose it not only because it is more powerful than the optimal basis approach, but also because there is no well-defined and established analogue of an optimal basis in SDP.

On the other hand we are able to define an analogue of the optimal partition which is readily available (in theory) once an optimal solution is found by an interior point algorithm. We show that sensitivity analysis can be performed using knowledge of the optimal partition, analogously to the way it is done in linear programming. We also show that the minimum ratio test used in sensitivity analysis in linear programming corresponds to finding the minimum eigenvalue of a matrix in the case of SDP. We also indicate where linear programming techniques fail to extend to SDP problems.

An example of a parametric SDP problem arises in certain engineering applications in which one has an SDP problem where one wants to find the trade-off curve between two different objective functions [16]. The trade-off curve is defined by the the value of the weighted sum of two objective functions as a function of a parameter, which is the ratio of these weights.

An example of the application of sensitivity analysis in SDP within the context of combinatorial optimization can be found in [11]. There sensitivity analysis is used for fixing variables in a branch-and-bound framework which uses SDP relaxations. The approach suggested in [11] is similar to the approach described in Section 4 for computing a sensitivity range involving the computation of the maximum eigenvalue of some matrix.

The paper is organized as follows. In the next section we introduce the concept of the optimal partition. In Section 3, we introduce the parametric SDP problem and discuss the behavior of the optimal objective function value and the optimal partition with respect to the parameter. In Section 4 we formulate the problems that one has to solve to find the range of the parameter for which the optimal solution is unchanged. We also show how to reduce these problems to the problem of finding the minimum eigenvalue of a matrix. The last section of the paper, Section 5, contains some concluding remarks.

2. Optimal partition

Let S^n denote the space of real symmetric $n \times n$ matrices. The primal semidefinite programming problem is

$$
\begin{aligned}
\text{(P)} \quad & \min \quad C \bullet X \\
& \text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, \ldots, m, \\
& \qquad\quad X \succeq 0, \ X \in S^n,
\end{aligned}
$$

where $C \in S^n$, $A_i \in S^n$, $i = 1, \ldots, m$, and $b \in \mathbb{R}^m$.

The problem dual to (P) is

$$
\begin{aligned}
\text{(D)} \quad & \max \quad \sum_{i=1}^{m} y_i b_i \\
& \text{s.t.} \quad \sum_{i=1}^{m} y_i A_i + Z = C, \\
& \qquad\quad Z \succeq 0, \ Z \in S^n.
\end{aligned}
$$

If the primal and the dual problems have optimal solutions and the duality gap is zero then the Karush–Kuhn–Tucker conditions for this primal–dual pair of problems are

$$
\begin{aligned}
& Z \bullet X = 0, \\
& A_i \bullet X = b_i, \quad i = 1, \ldots, m, \\
& \sum_{i=1}^{m} y_i A_i + Z = C, \\
& X, Z \succeq 0.
\end{aligned}
$$

For an optimal primal–dual solution (X, y, Z), it follows from $X \bullet Z = 0$ that $XZ = ZX = 0$. Thus $X = Q \Lambda Q^{\mathrm{T}}$ and $Z = Q \Omega Q^{\mathrm{T}}$, where Q is a matrix whose columns form a common orthonormal basis of eigenvectors of X and Z, $QQ^{\mathrm{T}} = I_n$, and Λ and Ω are diagonal matrices with the eigenvalues of X and Z, respectively, on their diagonals (see, e.g., [17]).

From complementary slackness, the common basis of eigenvectors of X and Z can be partitioned in the following way: $Q = [Q_{\mathrm{P}}, Q_{\mathrm{N}}, Q_{\mathrm{D}}]$, $X = Q_{\mathrm{P}} \Lambda_{\mathrm{P}} Q_{\mathrm{P}}^{\mathrm{T}}$, $\Lambda_{\mathrm{P}} \succ 0$, and $Z = Q_{\mathrm{D}} \Omega_{\mathrm{D}} Q_{\mathrm{D}}^{\mathrm{T}}$, $\Omega_{\mathrm{D}} \succ 0$. If we let

$$
\operatorname{rank}(X) = r, \qquad \operatorname{rank}(Z) = s,
$$

then $r + s \leqslant n$. If $r + s = n$ and, hence, $Q_{\mathrm{N}} = \emptyset$, then the primal–dual solution is said to be strictly complementary.

By $\boldsymbol{O}_{\mathrm{P}}$ ($\boldsymbol{O}_{\mathrm{D}}$) we denote the primal (dual) optimal face, i.e., the set of primal (dual) optimal solutions (possibly a single point). Suppose now that a partition $Q = [Q_{\mathrm{P}}, Q_{\mathrm{N}}, Q_{\mathrm{D}}]$ corresponding to optimal solutions $\overline{X} \in \operatorname{ri} \boldsymbol{O}_{\mathrm{P}}$ and $(\overline{y}, \overline{Z}) \in \operatorname{ri} \boldsymbol{O}_{\mathrm{D}}$ is given. The following simple fact is shown in [9]. $X \in \boldsymbol{O}_{\mathrm{P}}$ if and only if $X = Q_{\mathrm{P}} U Q_{\mathrm{P}}^{\mathrm{T}}$, where U is a solution to

$$
A_i \bullet Q_{\mathrm{P}} U Q_{\mathrm{P}}^{\mathrm{T}} = b_i, \quad i = 1, \ldots, m, \qquad U \succeq 0, \quad U \in S^r. \tag{1}
$$

Similarly, $(y, Z) \in \boldsymbol{O}_{\rm D}$ if and only if $Z = Q_{\rm D} V Q_{\rm D}^{\rm T}$, where V is a solution to

$$\sum_{i=1}^{m} y_i A_i + Q_{\rm D} V Q_{\rm D}^{\rm T} = C, \qquad V \succeq 0, \quad V \in \boldsymbol{S}^s. \tag{2}$$

Clearly, this means that given a primal (dual) solution X ((y, Z)) in the relative interior of the primal (dual) optimal face, one can completely describe this optimal face. We call such primal–dual pairs of solutions X and (y, Z) maximally complementary solutions as in [3]. Moreover, the set of solutions to (1) and (2) is independent of the particular choice of the bases $Q_{\rm P}$ and $Q_{\rm D}$ as long as the subspaces spanned by these bases remain the same; that is as long as $\boldsymbol{R}_{\rm P} \equiv \mathrm{span}(Q_{\rm P})$ and $\boldsymbol{R}_{\rm D} \equiv \mathrm{span}(Q_{\rm D})$ stay fixed. Hence, $\boldsymbol{R}_{\rm P}$ and $\boldsymbol{R}_{\rm D}$ are invariant with respect to the choice of $\overline{X} \in \mathrm{ri}\, \boldsymbol{O}_{\rm P}$ and $(\overline{y}, \overline{Z}) \in \mathrm{ri}\, \boldsymbol{O}_{\rm D}$, and for every pair of primal–dual SDP problems there is a triple associated with it, $(\boldsymbol{R}_{\rm P}, \boldsymbol{R}_{\rm N}, \boldsymbol{R}_{\rm D})$, where $\boldsymbol{R}_{\rm P}$ and $\boldsymbol{R}_{\rm D}$ are as defined above and $\boldsymbol{R}_{\rm N} = [\boldsymbol{R}_{\rm P} \oplus \boldsymbol{R}_{\rm D}]^{\perp}$. We refer to this triple as the *optimal partition*.

The optimal partition has been defined for parametric linear programming in [1,12]. For a standard form linear programming problem in which x denotes the primal variables and z denotes the dual slacks, the optimal partition is a partition of the indices $I = \{1, \ldots, n\}$ into two sets B and N so that $B = \{i \in I\colon \exists x \in \boldsymbol{O}_{\rm P},\ x_i > 0\}$ and $N = \{i \in I\colon \exists z \in \boldsymbol{O}_{\rm D},\ z_i > 0\}$. Clearly, from the properties of linear programming, $I = B \cup N$ and $B \cap N = \emptyset$. The concept of the optimal partition was also used in [3, 4,10], for quadratic programming problems and in [4,10] for linear complementarity problems. In this context the definition of the optimal partition is analogous to the linear programming case except that I is partitioned into *three* subsets of indices due to the possible absence of a strictly complementary solution. In the case of semidefinite programming we partition the space $\mathbb{R}^n$ rather then the set of indices I. Let us now give a formal definition of the optimal partition.

Definition 2.1. Given a primal–dual pair of SDPs (P) and (D), the optimal partition for these problems is $\pi = [\boldsymbol{R}_{\rm P}, \boldsymbol{R}_{\rm N}, \boldsymbol{R}_{\rm D}]$ if

(1) $\forall X \in \boldsymbol{O}_{\rm P}$, $\mathrm{span}(X) \subseteq \boldsymbol{R}_{\rm P}$ and $\exists X \in \boldsymbol{O}_{\rm P}$ such that $\mathrm{span}(X) = \boldsymbol{R}_{\rm P}$;
(2) $\forall (y, Z) \in \boldsymbol{O}_{\rm D}$, $\mathrm{span}(Z) \subseteq \boldsymbol{R}_{\rm D}$ and $\exists (y, Z) \in \boldsymbol{O}_{\rm D}$ such that $\mathrm{span}(Z) = \boldsymbol{R}_{\rm D}$;
(3) $\boldsymbol{R}_{\rm N} = [\boldsymbol{R}_{\rm P} \oplus \boldsymbol{R}_{\rm D}]^{\perp}$.

Let us consider the following example.

Example 1. Let $n = 3$ and $m = 3$,

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$b^{\rm T} = (1, 0, 0).$$

The feasible region of an SDP corresponding to the matrices A_1, A_2 and A_3 and the vector b is 3-dimensional (see Fig. 1).

Consider three different objective function matrices $C = C_i$, $i = 1, 2, 3$, and their respective pairs of primal–dual problems, where

$$C_1 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & -2 \\ 2 & -2 & 3 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \qquad C_3 = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

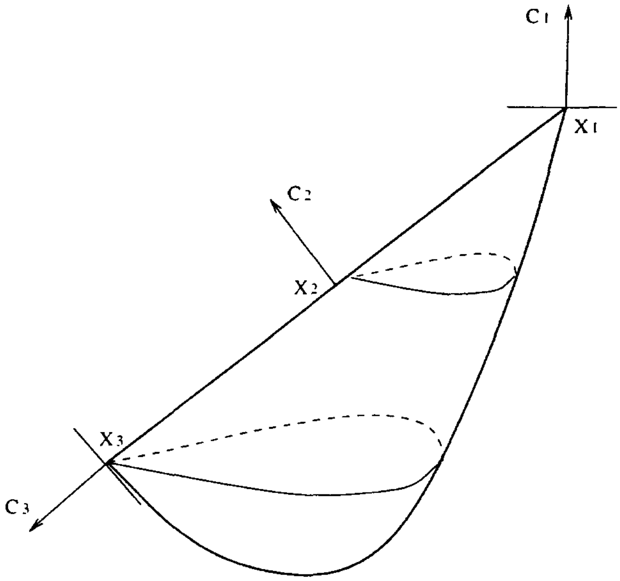

Fig. 1. Examples of unique, multiple and not strictly complementary solutions.

(1) If $C = C_1$, then the primal and dual optimal solutions are

$$X_1 = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, \qquad y = (-6,0,0)^{\mathrm{T}}, \qquad Z_1 = \begin{bmatrix} 8 & 2 & 2 \\ 2 & 3 & -2 \\ 2 & -2 & 3 \end{bmatrix}.$$

Moreover, $\operatorname{rank}(X_1) = 1$, $\operatorname{rank}(Z_1) = 2$, and the solutions are unique and satisfy strict complementarity. The optimal partition is

$$\boldsymbol{R}_{\mathrm{P}} = \operatorname{span}\left[\left(-\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)^{\mathrm{T}}\right],$$

$$\boldsymbol{R}_{\mathrm{D}} = \operatorname{span}\left[\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^{\mathrm{T}},\left(\frac{4}{3\sqrt{2}},\frac{1}{3\sqrt{2}},\frac{1}{3\sqrt{2}}\right)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{N}} = \emptyset.$$

(2) If $C = C_2$, then the primal–dual solutions are

$$X_2 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix}, \qquad y = (1,1,1)^{\mathrm{T}}, \qquad Z_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

and $\operatorname{rank}(X_2) = 2$ and $\operatorname{rank}(Z_2) = 1$. The primal solution X_2 is not unique; it is, in fact, the center of the optimal face described by

$$X_\theta = \theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1-\theta) \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, \qquad 0 \leqslant \theta \leqslant 1,$$

(see Fig. 1). The optimal partition is

$$\boldsymbol{R}_{\mathrm{P}} = \operatorname{span}\left[(1,0,0)^{\mathrm{T}},\left(-\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{D}} = \operatorname{span}\left[\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{N}} = \emptyset.$$

(3) If $C = C_3$, then, we have

$$X_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad y = (-1,-1,-1)^{\mathrm{T}}, \qquad Z_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$\operatorname{rank}(X_3) = 1$, $\operatorname{rank}(Z_3) = 1$ and the primal–dual pair does not satisfy complementary slackness, even though the primal and dual solutions are unique. In this case the optimal partition is

$$\boldsymbol{R}_{\mathrm{P}} = \operatorname{span}\left[(1,0,0)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{D}} = \operatorname{span}\left[\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}\right],$$

$$\boldsymbol{R}_{\mathrm{N}} = \operatorname{span}\left[\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}\right].$$

Cases (1) and (2) are similar to analogous cases in linear programming. The last case is somewhat similar to the case of multiple primal solutions in linear programming. However, case (3) has a unique solution due to the nonlinearity of the boundary of the feasible region.

3. Parametric objective function

In this section we consider the following primal SDP problem in which the objective function coefficients depend linearly on a scalar parameter γ:

$$\begin{array}{lll} (\mathrm{P}_\gamma) & \min & (C + \gamma\overline{C}) \bullet X \\ & \text{s.t.} & A_i \bullet X = b_i, \quad i = 1, \ldots, m, \\ & & X \succeq 0, \ X \in S^n. \end{array}$$

The dual problem then is

$$\begin{array}{lll} (\mathrm{D}_\gamma) & \max & \sum_{i=1}^{m} y_i b_i \\ & \text{s.t.} & \sum_{i=1}^{m} y_i A_i + Z = C + \gamma\overline{C}, \\ & & Z \succeq 0. \end{array}$$

The feasible region of the primal problem does not depend on γ. If the primal problem is infeasible, then (P_γ) is infeasible for all γ. Let us assume that the primal problem is strictly feasible; therefore, the Slater condition holds for all primal–dual pairs (P_γ)–(D_γ), which guarantees a zero duality gap at optimality (see, e.g., [13]). (If it is not strictly feasible, we consider the projection of the problem on the appropriate face of the cone of semidefinite matrices, such that the projected problem is strictly feasible.)

Let Γ be the set of values of γ for which (P_γ) has a bounded solution. It is easy to show that Γ is a closed (possibly unbounded) interval. From duality theory the dual problem is feasible and has a bounded solution for any $\gamma \in \Gamma$. We need the primal and the dual problems to have nonempty bounded optimal sets. This property is necessary to guarantee the convergence of an interior point method to a maximally complementary pair of primal–dual solutions. The dual problem has a nonempty bounded optimal set for all $\gamma \in \Gamma$ since the primal problem is strictly feasible [9].

Lemma 3.1. *The optimal set of the primal problem is either empty or unbounded for all $\gamma \in \Gamma$ or nonempty and bounded for any $\gamma \in \operatorname{int} \Gamma$.*

Proof. Let us assume that for some $\overline{\gamma} \in \operatorname{int} \Gamma$ (P_γ) has an empty or unbounded optimal face. Thus, there exists a nonzero direction $D \succeq 0$ such that $A_i \bullet D = 0$, $i = 1, \ldots, m$, and $(C + \overline{\gamma}\overline{C}) \bullet D = 0$. (This fact follows trivially in the case of an unbounded optimal set, and in the case of an empty optimal set it can be easily proved by choosing an infinite sequence of directions D_k, which converges to a direction D with desired properties.) Assume now that $\overline{C} \bullet D < 0$. Since $\overline{\gamma} \in \operatorname{int} \Gamma$, there exists an $\varepsilon > 0$ such that $\overline{\gamma} + \varepsilon \in \operatorname{int} \Gamma$ and $(C + (\overline{\gamma} + \varepsilon)\overline{C}) \bullet D < 0$. Consequently, $(\mathrm{P}_{\overline{\gamma}+\varepsilon})$ is unbounded, which contradicts the definition of Γ. Similarly we can dismiss the case $\overline{C} \bullet D > 0$. Thus $\overline{C} \bullet D = 0$ has to hold, which implies that $C \bullet D = 0$ and hence $(C \bullet D + \gamma\overline{C}) \bullet D = 0$ and (P_γ) has an empty or unbounded optimal face for any $\gamma \in \Gamma$. □

Henceforth, we make the following assumption, that guarantees that the optimal set of (P_γ) is nonempty and bounded for any $\gamma \in \operatorname{int} \Gamma$:

Assumption 3.2. For all $\gamma \in \operatorname{int} \Gamma$ (P_γ) and (D_γ) are strictly feasible.

Otherwise, it is possible to project the primal problem (independently of γ) onto a smaller subspace so that this property holds on $\operatorname{int} \Gamma$. Notice that by doing so we do not lose any information on the original problem, since the projection is independent of γ.

Let F_p denote the feasible set of (P_γ). Let

$$C(\gamma) = C + \gamma\overline{C},$$
$$X(\gamma) \in \mathcal{X}(\gamma) = \{X_\gamma\colon\ X_\gamma = \operatorname{argmin}\{C(\gamma) \bullet X\colon\ X \in F_p\}\}$$

be some optimal solution (for example, we can choose the analytic center of the optimal face, see, e.g., [9]) and

$$\phi(\gamma) = C(\gamma) \bullet X(\gamma).$$

We are interested in the properties of the function $\phi(\gamma)$. Recall that for parametric linear programming the analogous function is piecewise linear and concave. In [3] it is shown that for quadratic programming $\phi(\gamma)$ is concave and piecewise quadratic. In [14] it is shown that $\phi(\gamma)$ is concave and piecewise differentiable for the case of general convex programming. A general formula for the subdifferential (i.e., the interval between the values of the right and left derivatives in this case) of $\phi(\gamma)$ is also derived in [14]. Since an SDP problem is a special case of a convex programming problem, $\phi(\gamma)$ is concave in the SDP case and the formula in [14] for the subdifferential applies. However, we want to specialize that formula to our case and provide a simple proof for completeness. Consider for some $\gamma \in \operatorname{int} \Gamma$, $X(\gamma + \mathrm{d}\gamma)$ as $\mathrm{d}\gamma \to +0$. Since the points $X(\gamma + \mathrm{d}\gamma)$ for $\mathrm{d}\gamma < \sigma$, where σ is some positive number, lie in a compact set, $X(\gamma + \mathrm{d}\gamma)$ has a limit point as $\mathrm{d}\gamma \to +0$. Let $X^+(\gamma)$ be such a limit point. Analogously, define $X^-(\gamma)$ as a limit point of $X(\gamma + \mathrm{d}\gamma)$ as $\mathrm{d}\gamma \to -0$. We should point out that due to the possible multiplicity of the solutions in $\mathcal{X}(\gamma + \mathrm{d}\gamma)$ and in $\mathcal{X}(\gamma)$, it may happen that $X(\gamma + \mathrm{d}\gamma) \not\to X^+(\gamma)$ as $\mathrm{d}\gamma \to +0$, since there may be multiple limit points. However, we shall assume that $X(\gamma + \mathrm{d}\gamma) \to X^+(\gamma)$, since if this is not the case, we can choose an appropriate sequence that converges.

Lemma 3.3. *For any $\gamma \in \operatorname{int} \Gamma$*

$$\lim_{\mathrm{d}\gamma \to +0} \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X^+(\gamma)]}{\mathrm{d}\gamma} = 0$$

and

$$\lim_{\mathrm{d}\gamma \to -0} \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X^-(\gamma)]}{\mathrm{d}\gamma} = 0.$$

Proof. It is easy to see that $X^+(\gamma)$ is an optimal solution of (P_γ). Let us assume that

$$\liminf_{\mathrm{d}\gamma \to +0} \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X^+(\gamma)]}{\mathrm{d}\gamma} \leqslant \varepsilon < 0$$

(including the case $\liminf_{\mathrm{d}\gamma \to +0}(\cdot) = -\infty$). Then there exists a sequence $\{\mathrm{d}\gamma_k\} \to +0$ such that

$$[C + \gamma \overline{C}] \bullet X(\gamma + \mathrm{d}\gamma_k) \leqslant [C + \gamma \overline{C}] \bullet X^+(\gamma) + \varepsilon \mathrm{d}\gamma_k + \mathrm{o}(\mathrm{d}\gamma_k) < [C + \gamma \overline{C}] \bullet X^+(\gamma)$$

for $\mathrm{d}\gamma_k$ sufficiently small which contradicts the fact that $X^+(\gamma)$ is an optimal solution to (P_γ).

Similarly, assume that

$$\limsup_{\mathrm{d}\gamma \to +0} \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X^+(\gamma)]}{\mathrm{d}\gamma} \geqslant \varepsilon > 0$$

(including the case $\limsup_{\mathrm{d}\gamma \to +0}(\cdot) = \infty$). Then there exists a sequence $\{\mathrm{d}\gamma_k\} \to 0$ such that

$$\begin{aligned}
&[C + \gamma \overline{C} + \mathrm{d}\gamma_k \overline{C}] \bullet X(\gamma + \mathrm{d}\gamma_k) \\
&\quad \geqslant [C + \gamma \overline{C} + \mathrm{d}\gamma_k \overline{C}] \bullet X^+(\gamma) + \varepsilon \mathrm{d}\gamma_k + \mathrm{d}\gamma_k \overline{C} \bullet \big(X(\gamma + \mathrm{d}\gamma_k) - X^+(\gamma)\big) + \mathrm{o}(\mathrm{d}\gamma_k).
\end{aligned}$$

Since $X(\gamma + \mathrm{d}\gamma_k) - X^+(\gamma) \to 0$ as $\mathrm{d}\gamma_k \to +0$, it follows that for $\mathrm{d}\gamma_k$ sufficiently small,

$$[C + \gamma \overline{C} + \mathrm{d}\gamma_k \overline{C}] \bullet X(\gamma + \mathrm{d}\gamma_k) > [C + \gamma \overline{C} + \mathrm{d}\gamma_k \overline{C}] \bullet X^+(\gamma),$$

which contradicts the fact that $X(\gamma + \mathrm{d}\gamma_k)$ is an optimal solution of $(\mathrm{P}_{\gamma + \mathrm{d}\gamma_k})$.

Consequently,

$$\liminf_{\mathrm{d}\gamma \to +0} \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X(\gamma)]}{\mathrm{d}\gamma} = \limsup_{\mathrm{d}\gamma \to +0} \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X(\gamma)]}{\mathrm{d}\gamma} = 0$$

and it follows that the first limit in the statement of the lemma holds. The second limit is proved in an analogous fashion. □

Let us now consider

$$\lim_{\mathrm{d}\gamma \to 0} \frac{\phi(\gamma + \mathrm{d}\gamma) - \phi(\gamma)}{\mathrm{d}\gamma}:$$

$$\begin{aligned}
\frac{\phi(\gamma + \mathrm{d}\gamma) - \phi(\gamma)}{\mathrm{d}\gamma} &= \frac{[C + (\gamma + \mathrm{d}\gamma)\overline{C}] \bullet X(\gamma + \mathrm{d}\gamma) - [C + \gamma \overline{C}] \bullet X(\gamma)}{\mathrm{d}\gamma} \\
&= \overline{C} \bullet X(\gamma + \mathrm{d}\gamma) + \frac{[C + \gamma \overline{C}] \bullet [X(\gamma + \mathrm{d}\gamma) - X(\gamma)]}{\mathrm{d}\gamma}.
\end{aligned}$$

From Lemma 3.3 and from the fact that $C(\gamma) \bullet X(\gamma) = C(\gamma) \bullet X^+(\gamma) = C(\gamma) \bullet X^-(\gamma)$ we have

$$\lim_{\mathrm{d}\gamma \to +0} \frac{\phi(\gamma + \mathrm{d}\gamma) - \phi(\gamma)}{\mathrm{d}\gamma} = \overline{C} \bullet X^+(\gamma)$$

and

$$\lim_{\mathrm{d}\gamma \to -0} \frac{\phi(\gamma + \mathrm{d}\gamma) - \phi(\gamma)}{\mathrm{d}\gamma} = \overline{C} \bullet X^-(\gamma).$$

From this we conclude that $\phi(\gamma)$ is continuously differentiable at γ if and only if $\overline{C} \bullet X^+(\gamma) = \overline{C} \bullet X^-(\gamma)$. Below we show that this happens only when either the primal solution to (P_γ) is unique or any feasible direction of the optimal face is orthogonal to $\overline{C}$.

We need the following property.

Property 3.4. *For any $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 < \gamma_2$, and for any optimal solution $X(\gamma_1)$ of (P_{γ_1}), and any optimal solution $X(\gamma_2)$ of (P_{γ_2}), $\overline{C} \bullet X(\gamma_1) \geqslant \overline{C} \bullet X(\gamma_2)$.*

Proof. From the optimality of $X(\gamma_1)$ and $X(\gamma_2)$,

$$[C + \gamma_1 \overline{C}] \bullet X(\gamma_1) \leqslant [C + \gamma_1 \overline{C}] \bullet X(\gamma_2)$$

and

$$[C + \gamma_2 \overline{C}] \bullet X(\gamma_1) \geqslant [C + \gamma_2 \overline{C}] \bullet X(\gamma_2).$$

Subtracting the first inequality from the second and dividing both sides by $\gamma_2 - \gamma_1 > 0$ we obtain the desired result. □

Let $\mathcal{O}_\mathrm{P}(\gamma)$ denote the optimal face of (P_γ). From the above property and from the continuity of $\overline{C} \bullet X(\gamma + \mathrm{d}\gamma)$ as $\mathrm{d}\gamma \to 0$ it follows that for any $X(\gamma) \in \mathcal{O}_\mathrm{P}(\gamma)$, $\overline{C} \bullet X^-(\gamma) \leqslant \overline{C} \bullet X(\gamma) \leqslant \overline{C} \bullet X^+(\gamma)$. If $\overline{C} \bullet X^+(\gamma) \neq \overline{C} \bullet X^-(\gamma)$, then $\phi(\gamma)$ has a right and a left derivative and subdifferential $\partial\phi(\gamma) = \{\alpha \colon \alpha = \overline{C} \bullet X,\ X \in \mathcal{O}_\mathrm{P}(\gamma)\}$. This obviously follows from the facts above. Let $X(\gamma)$ be an optimal solution with maximal range (e.g., the central solution) and let Q_P be the matrix corresponding to an orthonormal basis for that range. We therefore obtain the following lemma.

Lemma 3.5. *The solutions of the following problems produce the right and left derivatives of $\phi(\gamma)$:*

$$\frac{\mathrm{d}\phi}{\mathrm{d}\gamma_+} = \min\{\overline{C} \bullet Q_\mathrm{P} U Q_\mathrm{P}^\mathrm{T} \mid A_i \bullet Q_\mathrm{P} U Q_\mathrm{P}^\mathrm{T} = b_i,\ i = 1, \ldots, m,\ U \succeq 0\} \tag{3}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}\gamma_-} = \max\{\overline{C} \bullet Q_\mathrm{P} U Q_\mathrm{P}^\mathrm{T} \mid A_i \bullet Q_\mathrm{P} U Q_\mathrm{P}^\mathrm{T} = b_i,\ i = 1, \ldots, m,\ U \succeq 0\}. \tag{4}$$

For related results on nonlinear semidefinite programming problems see [15].

Remark. Since the dual of a semidefinite program is also a semidefinite program and perturbing the objective function matrix of the primal problem is equivalent to perturbing the right-hand side matrix of the dual problem, all results of this section extend to the case of a problem with a parametric right-hand side.

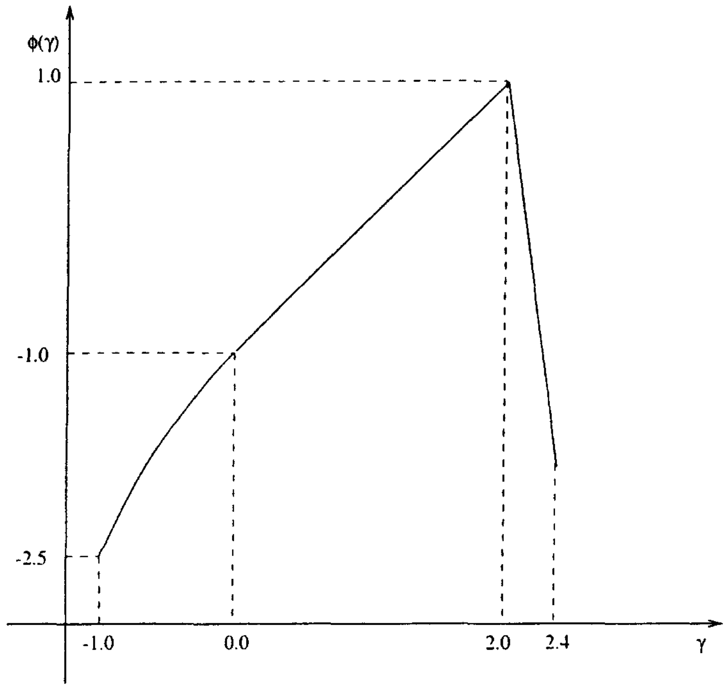

Fig. 2. Shape of $\phi(\gamma)$.

Let us return to the concept of the optimal partition. For a given $\gamma \in \operatorname{int} \Gamma$, let us denote the optimal partition corresponding to (P_γ) and (D_γ) by $\pi(\gamma) = [\boldsymbol{R}_\mathrm{P}(\gamma), \boldsymbol{R}_\mathrm{N}(\gamma), \boldsymbol{R}_\mathrm{D}(\gamma)]$. In the next section we show that the set of values of γ for which the optimal partition stays the same is either a single point or an interval. This is true for linear programming and quadratic programming and it is one of the basic properties of the optimal partition.

Example 2. Consider an SDP problem whose set of feasible solutions is defined as in Example 1, and whose objective function is $(C + \gamma \overline{C}) \bullet X$, where

$$C = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \overline{C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Then $C + \gamma \overline{C}$ equals C_3 of Example 1 for $\gamma = 0$, it equals C_2 for $\gamma = 2$ and it equals C_1 for $\gamma = 3$.

Let us vary γ in the interval $[-1.0, 2.4]$ (see Fig. 2).

(i) In the interval $[-1, 0]$, $X(\gamma)$ and $\pi(\gamma)$ vary continuously, and $\phi(\gamma)$ is a concave nonlinear function.

(ii) For $\gamma = 0$, from Example 1 we know that

$$X(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the optimal partition is

$$\boldsymbol{R}_{\mathrm{P}} = \mathrm{span}\left[(1,0,0)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{D}} = \mathrm{span}\left[\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}\right],$$
$$\boldsymbol{R}_{\mathrm{N}} = \mathrm{span}\left[\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}\right].$$

(iii) In the interval $(0, 1)$, $X(\gamma)$ is constant and equals $X(0)$, the optimal partition

$$\boldsymbol{R}_{\mathrm{P}} = \mathrm{span}\left[(1,0,0)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{D}} = \mathrm{span}\left[\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{\mathrm{T}}, \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}\right], \qquad \boldsymbol{R}_{\mathrm{N}} = \emptyset$$

is constant and $\phi(\gamma)$ is linear. Notice that $\forall \gamma \in (0, 1)$, $X(0) = X(\gamma)$ but $\pi(0) \neq \pi(\gamma)$.

(iv) At $\gamma = 2$ from Example 1 the optimal solution is not unique and the optimal face is defined by

$$\mathcal{X}(2) = \left\{X\colon X = \theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1-\theta) \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix},\ 0 \leqslant \theta \leqslant 1\right\}.$$

The optimal partition $\pi(2)$ is different from $\pi(\gamma)\ \forall \gamma \neq 2$. At $\gamma = 2$, $\phi(\gamma)$ has a break point (see Fig. 2). The right and left derivatives of $\phi(\gamma)$ at $\gamma = 2$ are

$$\frac{\mathrm{d}\phi}{\mathrm{d}\gamma_+} = \min\{\overline{C} \bullet X \mid X \in \mathcal{X}(2)\} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} = -7$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}\gamma_-} = \max\{\overline{C} \bullet X \mid X \in \mathcal{X}(2)\} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1.$$

(v) In the interval $[2, 2.4]$, $X(\gamma)$ is again constant and equals

$$\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix},$$

and $\phi(\gamma)$ is linear. The optimal partition is constant on $(2, 2.4]$ and is the same as the optimal partition in case 1 of Example 1.

4. Sensitivity analysis

In this section we study the problem of finding the range of γ for which some given primal solution remains optimal. In other words, we study the sensitivity analysis of primal optimal solutions. Such a range is, clearly, an interval which is possibly unbounded, and $\phi(\gamma)$ is linear on this interval. We show that the problem of finding such an interval in the case of primal nondegeneracy is in fact an extension of the linear programming minimum ratio test.

First, we consider the set of values of γ on which the *optimal partition* stays constant. Suppose, we have solved the problem (P_γ) for $\overline{\gamma} \in \operatorname{int} \Gamma$. Let us find the range of γ which we define as $\Gamma(\overline{\gamma}) = \{\gamma\colon \pi(\gamma) = \pi(\overline{\gamma})\}$. In the next lemma we show that $\Gamma(\overline{\gamma})$ is either an open interval or the point $\{\overline{\gamma}\}$. In case $\Gamma(\overline{\gamma})$ is an interval we describe how to find its endpoints.

Lemma 4.1. *Let $\pi(\overline{\gamma}) = (\boldsymbol{R}_{\mathrm{P}}(\overline{\gamma}), \boldsymbol{R}_{\mathrm{N}}(\overline{\gamma}), \boldsymbol{R}_{\mathrm{D}}(\overline{\gamma}))$ be the optimal partition corresponding to $\overline{\gamma}$, and let Q_{P} and Q_{D} be orthonormal bases of $\boldsymbol{R}_{\mathrm{P}}(\overline{\gamma})$ and $\boldsymbol{R}_{\mathrm{D}}(\overline{\gamma})$, respectively. The set of all values of γ, for which the optimal partition equals $\pi(\overline{\gamma})$, is either a single point $\{\overline{\gamma}\}$ or an open interval (a, b), where the endpoints a and b of the interval can be obtained by solving the following problems*:

$$a: \left\{\inf \gamma \;\Big|\; \sum_{i=1}^{n} A_i y_i + Q_{\mathrm{D}} V Q_{\mathrm{D}}^{\mathrm{T}} = C + \gamma \overline{C},\ V \succ 0\right\}, \tag{5}$$

$$b: \left\{\sup \gamma \;\Big|\; \sum_{i=1}^{n} A_i y_i + Q_{\mathrm{D}} V Q_{\mathrm{D}}^{\mathrm{T}} = C + \gamma \overline{C},\ V \succ 0\right\}. \tag{6}$$

Proof. Let a and b be defined by (5) and (6). We need to show that for any $\gamma \in (a, b)$, $\pi(\gamma) = \pi(\overline{\gamma})$ and vice versa. From the definition of the optimal partition it follows that for any γ, such that $\pi(\gamma) = \pi(\overline{\gamma})$, there exists V satisfying

$$\sum_{i=1}^{n} A_i y_i + Q_{\mathrm{D}} V Q_{\mathrm{D}}^{\mathrm{T}} = C + \gamma \overline{C}, \qquad V \succ 0; \tag{7}$$

hence, it follows from (5) and (6) that $a \leqslant \gamma \leqslant b$.

Suppose now that there exists (γ, y, V) with $\gamma \neq \overline{\gamma}$ that satisfies (7). Consider $Z = Q_{\mathrm{D}} V Q_{\mathrm{D}}^{\mathrm{T}}$. Z is feasible for (D_γ) from (7) and it is complementary to $X(\overline{\gamma})$ since $\mathcal{R}(Z) = \boldsymbol{R}_{\mathrm{D}}(\overline{\gamma})$. Since $X(\overline{\gamma})$ is feasible for (P_γ) we conclude that $(X(\overline{\gamma}), y, Z)$ is an optimal primal–dual solution for γ. Thus, we can conclude that

$$\boldsymbol{R}_{\mathrm{P}}(\gamma) \supseteq \boldsymbol{R}_{\mathrm{P}}(\overline{\gamma}) \quad \text{and} \quad \boldsymbol{R}_{\mathrm{D}}(\gamma) \supseteq \boldsymbol{R}_{\mathrm{D}}(\overline{\gamma})$$

(recall that $\pi(\gamma) = (\boldsymbol{R}_{\mathrm{P}}(\gamma), \boldsymbol{R}_{\mathrm{N}}(\gamma), \boldsymbol{R}_{\mathrm{D}}(\gamma))$ is defined by a maximally complementary primal–dual solution). Let us now show that the partition $\pi(\gamma) = \pi(\overline{\gamma})$.

Assume there is a primal solution to (P_γ) with a range larger than the range of $X(\overline{\gamma})$. This solution must be complementary to Z. Thus, it is complementary to $Z(\overline{\gamma})$ and, since it is feasible for $(\mathrm{P}_{\overline{\gamma}})$, we obtain a contradiction to $X(\overline{\gamma})$ being a maximally complementary solution for $\overline{\gamma}$. Thus $\boldsymbol{R}_{\mathrm{P}}(\gamma) = \boldsymbol{R}_{\mathrm{P}}(\overline{\gamma})$.

Now, assume that there is a dual solution (y', Z') to (D_γ), such that the range of Z' is larger than the range of Z. Since both Z' and Z are feasible for (D_γ), we must have

$$\sum_{i=1}^{n} A_i [y_i' - y_i] + Z' - Z = 0. \tag{8}$$

If we multiply the above equation by an arbitrary $\varepsilon > 0$ and add it to the feasibility equation for $(\mathrm{D}_{\overline{\gamma}})$,

$$\sum_{i=1}^{n} A_i y_i(\overline{\gamma}) + Z(\overline{\gamma}) = C + \overline{\gamma} \overline{C},$$

we obtain

$$\sum_{i=1}^{n} A_i [y_i(\overline{\gamma}) + \varepsilon(y_i' - y_i)] + Z(\overline{\gamma}) + \varepsilon(Z' - Z) = C + \overline{\gamma} \overline{C}.$$

Since $\mathcal{R}(Z(\overline{\gamma})) = \mathcal{R}(Z) \subset \mathcal{R}(Z')$ and $Z, Z(\overline{\gamma}), Z' \succeq 0$, there exists a small enough ε, such that $\overline{Z} = Z(\overline{\gamma}) + \varepsilon(Z' - Z) \succeq 0$ and $\mathcal{R}(\overline{Z}) \supset \mathcal{R}(Z(\overline{\gamma}))$. This contradicts the fact that $Z(\overline{\gamma})$ is a solution with the maximal range.

We have shown that any γ satisfies (7) if and only if the partition $\pi(\gamma)$ equals $\pi(\overline{\gamma})$. It is easy to see that the set of γ feasible for (7) is either a point or an open interval with endpoints a and b defined by (5) and (6). This completes the proof of the lemma. □

Next, we study the problem of finding the interval of values of γ for which an optimal primal solution remains optimal. This problem is not exactly the same as (5)–(6) but, clearly, is closely related to it. Since we do not require the partition to stay the same, but merely want complementary slackness to hold, we allow the dual solution to have any range in $\boldsymbol{R}_{\mathrm{N}} \oplus \boldsymbol{R}_{\mathrm{D}}$. For brevity of notation let Q_{ND} denote now a matrix whose columns form an orthonormal basis $\boldsymbol{R}_{\mathrm{N}} \oplus \boldsymbol{R}_{\mathrm{D}}$ and let $\operatorname{rank}(Q_{\mathrm{ND}}) = s$. The following problems then define the endpoints of the closed interval $[a, b]$:

$$a = \min\left\{\gamma \;\middle|\; \sum_{i=1}^{n} A_i y_i + Q_{\mathrm{ND}} V Q_{\mathrm{ND}}^{\mathrm{T}} = C + \gamma\overline{C},\ V \succeq 0,\ V \in \boldsymbol{S}^s\right\}, \tag{9}$$

$$b = \max\left\{\gamma \;\middle|\; \sum_{i=1}^{n} A_i y_i + Q_{\mathrm{ND}} V Q_{\mathrm{ND}}^{\mathrm{T}} = C + \gamma\overline{C},\ V \succeq 0,\ V \in \boldsymbol{S}^s\right\}. \tag{10}$$

In linear programming there is a finite set of such closed intervals, each of which corresponds to a particular optimal solution. In contrast, in semidefinite programming there can be whole regions of values of γ in which the optimal solution (and optimal partition) changes continuously. For example, the interval $-0.5 \leqslant \gamma \leqslant 0$ in Fig. 2 is such a region. Each point in this region corresponds to the case where the interval $[a, b]$ collapses to a single point (i.e., $a = b$).

Problems (5) and (6) are semidefinite programming problems. Let us try to reduce them to simpler problems. Sensitivity analysis in linear programming under an assumption of nondegeneracy reduces to a minimum ratio test. However, as shown in [12], in the degenerate case the minimum ratio test produces incorrect results. One then has to solve a linear programming problem to obtain the correct range of γ. The situation is similar in our case.

Assume (without loss of generality) that $\gamma = 0 \in \operatorname{int} \Gamma$. We want to find $a \leqslant 0$ and $b \geqslant 0$, such that $X(\gamma) = X(0)$ for all $\gamma \in [a, b]$. Consider the problem of finding a. For b the analysis is analogous.

There are two possible cases.

Case 1. There does not exist $\alpha \in \mathbb{R}^m$ such that $\overline{C} = \sum \alpha_i A_i + Q_{\mathrm{ND}} \Omega Q_{\mathrm{ND}}^{\mathrm{T}}$, $\Omega \in \boldsymbol{S}^s$. It is easy to see that $a = 0$, since a small perturbation of the original dual problem causes a change in the dual range.

Case 2. There exists $\alpha \in \mathbb{R}^m$ such that $\overline{C} = \sum \alpha_i A_i + Q_{\mathrm{ND}} \Omega Q_{\mathrm{ND}}^{\mathrm{T}}$, $\Omega \in \boldsymbol{S}^s$. We can rewrite the constraints to our problem as

$$\sum_{i=1}^{n} A_i \overline{y}_i + Q_{\mathrm{ND}} V Q_{\mathrm{ND}}^{\mathrm{T}} = C + \gamma Q_{\mathrm{ND}} \Omega Q_{\mathrm{ND}}^{\mathrm{T}}, \quad \overline{y} = y + \gamma\alpha.$$

Let $Q = [Q_{\mathrm{P}}, Q_{\mathrm{ND}}]$, where Q_{P} and Q_{ND} are orthonormal bases of $\boldsymbol{R}_{\mathrm{P}}$ and $\boldsymbol{R}_{\mathrm{D}} \oplus \boldsymbol{R}_{\mathrm{N}}$, respectively. By premultiplying the above system by Q^{T} and postmultiplying it by Q and considering the upper-left, upper-right (same as lower-left) and lower-right parts of the system separately, we obtain

$$\sum \overline{y}_i Q_{\mathrm{P}}^{\mathrm{T}} A_i Q_{\mathrm{P}} = Q_{\mathrm{P}}^{\mathrm{T}} C Q_{\mathrm{P}}, \tag{11}$$

$$\sum \overline{y}_i Q_{\mathrm{P}}^{\mathrm{T}} A_i Q_{\mathrm{ND}} = Q_{\mathrm{P}}^{\mathrm{T}} C Q_{\mathrm{ND}}, \tag{12}$$

$$\sum \overline{y}_i Q_{\mathrm{ND}}^{\mathrm{T}} A_i Q_{\mathrm{ND}} + V = Q_{\mathrm{ND}}^{\mathrm{T}} C Q_{\mathrm{ND}} + \gamma\Omega. \tag{13}$$

Let $A(\bar{y}) = -\sum \bar{y}_i Q_{\mathrm{ND}}^{\mathrm{T}} A_i Q_{\mathrm{ND}} + Q_{\mathrm{ND}}^{\mathrm{T}} C Q_{\mathrm{ND}}$, and let $\overline{Y}$ be the set of solutions to Eqs. (11) and (12). Then our problem reduces to

$$a = \min\{\gamma \mid A(\bar{y}) + \gamma \Omega \succeq 0,\ \bar{y} \in \overline{Y}\}. \tag{14}$$

We now need an assumption of primal nondegeneracy as defined in [2].

Definition 4.2. Let $\mathcal{M}_r = \{X \in S^n\colon \operatorname{rank}(X) = r\}$ and, for a given $X \in \mathcal{M}_r$, let $\mathcal{T}_X$ be the tangent subspace to $\mathcal{M}_r$ at X; i.e.,

$$\mathcal{T}_X = \left\{ Q \begin{bmatrix} U & V \\ V^{\mathrm{T}} & 0 \end{bmatrix} Q^{\mathrm{T}}\colon U \in S^{r\times r},\ V \in \mathbb{R}^{r\times n-r} \right\},$$

where Q is the orthogonal matrix of the eigenvectors of X. Let $\mathcal{N} = \{X \in S^{n\times n}\colon A_i \bullet X = 0,\ i = 1,\dots,m\}$.

We say that a solution X to problem (P) is primal nondegenerate if

$$\mathcal{T}_X + \mathcal{N} = S^n.$$

Assume that the unperturbed problem is primal nondegenerate. By the property of primal degeneracy proved in [2] the matrices

$$\begin{bmatrix} Q_{\mathrm{P}}^{\mathrm{T}} A_i Q_{\mathrm{P}} \\ Q_{\mathrm{P}}^{\mathrm{T}} A_i Q_{\mathrm{ND}} \end{bmatrix}, \quad i = 1,\dots,m,$$

are linearly independent. Hence, Eqs. (11) and (12) have a unique solution $\bar{y}$, so $A(\bar{y})$ does not depend on $\bar{y}$. $A(\bar{y}) = W_{\mathrm{ND}}$, where $W_{\mathrm{ND}} \succeq 0$ can be chosen to be the diagonal matrix of eigenvalues of the dual solution $Z(0)$ of the unperturbed problem.

If at $\bar{\gamma} = 0$ the solution is strictly complementary, then $W_{\mathrm{ND}} \succ 0$. Hence, $W_{\mathrm{ND}} + \gamma\Omega \succeq 0$ holds for some $\gamma < 0$ and (14) is equivalent to

$$a = \min\{\gamma \mid I + \gamma(W_{\mathrm{ND}}^{-1}\Omega) \succeq 0\}.$$

If $W_{\mathrm{ND}}^{-1}\Omega \preceq 0$, then for any $\gamma < 0$, $W_{\mathrm{ND}} + \gamma\Omega \succeq 0$, which mean that $\gamma_{\min} = -\infty$. If $(W_{\mathrm{ND}}^{-1}\Omega)$ has at least one positive eigenvalue, then it is easy to see that

$$\gamma_{\min} = -1/\lambda_{\max}(W_{\mathrm{ND}}^{-1}\Omega),$$

where $\lambda_{\max}(W_{\mathrm{ND}}^{-1}\Omega)$ is the largest (positive) eigenvalue of $W_{\mathrm{ND}}^{-1}\Omega$. This computation is analogous to a minimum ratio test in linear programming.

Let us consider the case when the solution of the unperturbed problem is not strictly complementary; i.e., $W_{\mathrm{ND}} \succeq 0$ and $W_{\mathrm{ND}} \not\succ 0$. Recall that this can happen even assuming nondegeneracy. In this case three subcases are possible:

(i) $\exists\varepsilon < 0$: $W_{\mathrm{ND}} + \varepsilon\Omega \succ 0$,
(ii) $\exists\varepsilon > 0$: $W_{\mathrm{ND}} + \varepsilon\Omega \succ 0$,
(iii) $\forall\varepsilon \in \mathbb{R}$: $W_{\mathrm{ND}} + \varepsilon\Omega \not\succ 0$.

If (iii) holds then $[a, b] = \{0\}$; i.e., the solution changes for any $\gamma \neq 0$. This happens because Ω is indefinite on the nullspace of the dual solution W_{ND}, so by adding or subtracting small multiples of Ω we obtain a matrix with negative eigenvalues.

If (i) holds then Ω is negative semidefinite on the nullspace of $W_{\rm ND}$ and we can only add negative multiples of Ω. So for small $\gamma < 0$, $W_{\rm ND} + \gamma\Omega \succ 0$ and hence $a < 0$ and $b = 0$. Similarly, if (ii) holds, $a = 0$ and $b > 0$.

Assuming that (i) holds, we have

$$a = \min\{\gamma \mid W_{\rm ND} + \varepsilon\Omega + (\gamma - \varepsilon)\Omega \succeq 0\}.$$

Since $W_{\rm ND} + \varepsilon\Omega \succ 0$, we can rewrite this problem as

$$a = \min\{\gamma \mid I + (\gamma - \varepsilon)[W_{\rm ND} + \varepsilon\Omega]^{-1}\Omega \succeq 0\},$$

whose solution is

$$\gamma_{\min} = \begin{cases} -\infty, & \text{if } [W_{\rm ND} + \varepsilon\Omega]^{-1}\Omega \preceq 0, \\ -1/\lambda_{\max}\left([W_{\rm ND} + \varepsilon\Omega]^{-1}\Omega\right) + \varepsilon, & \text{otherwise.} \end{cases}$$

When the solution of the unperturbed problem is primal degenerate, $A(\bar{y})$ is a linear operator applied to $\bar{y}$. Thus we have an SDP problem to solve in order to find a and one to find b. Since $\bar{y}$ has to satisfy (11) and (12), it is reasonable to expect that the degrees of freedom of $\bar{y}$ is small. In linear programming in the case of primal degeneracy, one has to solve a linear programming problem (rather than performing a simple minimum-ratio test) to obtain the correct bounds on the parameter (e.g., see [12]).

Remark 1. Let us consider a dual problem with a parametric objective function vector $b + \gamma\bar{b}$ for some $\bar{b} \in \mathbb{R}^m$:

$$\begin{aligned} (\mathrm{D}'_\gamma) \quad & \max \quad \sum_{i=1}^{m} y_i(b_i + \gamma\bar{b}_i) \\ & \text{s.t.} \quad \sum_{i=1}^{m} y_i A_i + Z = C, \\ & \qquad\quad Z \succeq 0. \end{aligned}$$

This is equivalent to considering the following primal problem with a parametric right-hand side:

$$\begin{aligned} (\mathrm{P}'_\gamma) \quad & \min \quad C \bullet X, \\ & \text{s.t.} \quad A_i \bullet X = b_i + \gamma\bar{b}_i, \quad i = 1, \ldots, m, \\ & \qquad\quad X \succeq 0, \quad X \in S^n. \end{aligned}$$

Since (D'_γ) is an SDP problem with the parametric objective matrix, all the results of the paper apply to it. Suppose we have solved the pair of problems (D'_γ) and (P'_γ) for $\gamma = 0$. As before, assume that $X(0)$ and $Z(0)$ are a maximally complementary pair of primal–dual optimal solutions. We call the problem of finding the interval $[a, b]$ of γ on which $Z(0)$ stays optimal for (D'_γ)—dual sensitivity analysis. We want to reduce this problem to one with a form similar to (14).

By similar arguments to those used above we can show that a and b can be found by solving the problems

$$a = \min\{\gamma \mid A_i \bullet Q_{\rm PN} U Q_{\rm PN}^{\rm T} = b_i + \gamma\bar{b}_i,\ i = 1, \ldots, m,\ U \succeq 0,\ U \in S^r\},$$
$$b = \max\{\gamma \mid A_i \bullet Q_{\rm PN} U Q_{\rm PN}^{\rm T} = b_i + \gamma\bar{b}_i,\ i = 1, \ldots, m,\ U \succeq 0,\ U \in S^r\}.$$

Here Q_{PN} denotes a matrix whose columns form an orthonormal basis in $\boldsymbol{R}_{\mathrm{P}} \oplus \boldsymbol{R}_{\mathrm{N}}$ and $\mathrm{rank}(Q_{\mathrm{PN}}) = r$.

Let us consider the first problem (the second one is, clearly, similar). As before, we have two cases:

Case 1. $\bar{b}_i = Q_{\mathrm{PN}}^{\mathrm{T}} A_i Q_{\mathrm{PN}} \bullet V$, $i = 1, \ldots, m$, for some $V \in \boldsymbol{S}^r$.

Case 2. $\nexists V \in \boldsymbol{S}^r$ such that $\bar{b}_i = Q_{\mathrm{PN}}^{\mathrm{T}} A_i Q_{\mathrm{PN}} \bullet V$, $i = 1, \ldots, m$.

In the second case, it is easy to see that $a = 0$. In the first case, if the dual solution $Z(0)$ is dual nondegenerate it follows that the matrices $Q_{\mathrm{PN}}^{\mathrm{T}} A_i Q_{\mathrm{PN}}$ are linearly independent [2]; thus the solution V to $\bar{b}_i = Q_{\mathrm{PN}}^{\mathrm{T}} A_i Q_{\mathrm{PN}} \bullet V$, $i = 1, \ldots, m$, is unique. Then, to find a, one needs to solve the problem $\{\min \gamma\colon\ \Lambda_{\mathrm{PN}} + \gamma V \succeq 0\}$. This can be done in the same way that problem (14) is solved in the case of primal nondegeneracy (see the discussion following Definition 4.2). In the case of dual degeneracy one has to solve an SDP problem, as in primal sensitivity analysis under primal degeneracy.

Remark 2. It is easy to see that if an SDP problem is reduced to an LP problem (by restricting all matrices to be diagonal) then our analysis reduces (under appropriate nondegeneracy assumptions) to standard linear programming sensitivity analysis; i.e., minimum ratio tests which guarantee nonnegativity of the reduced costs in the primal case and the nonnegativity of the primal basic variables in the dual case.

5. Conclusions

We have established a connection between properties of parametric linear programming problems and those of parametric SDP problems. Unfortunately, the nature of a parametric SDP is far more complicated due to regions of nonlinearity of $\phi(\gamma)$. As far as we know there is no simple way of describing $\phi(\gamma)$ completely. The results in Section 4 show that one can efficiently check if a given value of γ falls into an interval where $\phi(\gamma)$ is linear, and if this is the case, find the interval of linearity. Lemma 3.5 shows that one can also efficiently find the right and left derivatives of $\phi(\gamma)$.

References

[1] I. Adler and R.D.C. Monteiro, A geometric view of parametric linear programming, *Algorithmica* 8 (1992) 161–176.

[2] F. Alizadeh, J.-P.A. Haeberly and M.L. Overton, Complementarity and nondegeneracy in semidefinite programming, *Math. Programming* 77 (1997) 111–128.

[3] A.B. Berkelaar, B. Jansen, K. Roos and T. Terlaky, Sensitivity analysis in (degenerate) quadratic programming, Technical Report 96-26, Delft University of Technology (1996); also: *Management Sci.*, submitted.

[4] A.B. Berkelaar, B. Jansen, K. Roos and T. Terlaky, Basis- and tripartition identification for quadratic programming and linear complementarity problems. From an interior point solution to an optimal basis and vice versa, Technical Report 96-31, Delft University of Technology (1996); also: *Math. Programming*, submitted.

[5] J.F. Bonnans, R. Cominetti and A. Shapiro, Second order optimality conditions based on parabolic second order tangent sets, also: *SIAM J. Optim.*, to appear.

[6] J.F. Bonnans, R. Cominetti and A. Shapiro, Sensitivity analysis of optimization problems under second order regular constraints, Rapport de Recherche INRIA 2989 (1996).

[7] J.F. Bonnans, A.D. Ioffe and A. Shapiro, Expansion of exact and approximate solutions in nonlinear programming, in: W. Oettli and D. Pallaschke, eds., *Advances in Optimization*, Lecture Notes in Economics and Mathematical Systems, Vol. 382 (Springer, Berlin, 1992) 103–117.

[8] J.F. Bonnans and A. Shapiro, Optimization problems with perturbations, a guided tour, *SIAM Rev.* 40 (1998) 228–264.
[9] D. Goldfarb and K. Scheinberg, Interior point trajectories in semidefinite programming, *SIAM J. Optim.* 8 (1998) 871–886.
[10] O. Güler and Y. Ye, Convergence behavior of some interior point algorithms, *Math. Programming* 60 (1993) 215–228.
[11] C. Helmberg, Fixing variables in semidefinite relaxations, ZIB Preprint SC-96-43, Konrad Zuse Zentrum für Informationstechnik Berlin, Dahlem, Germany (1996).
[12] B. Jansen, K. Roos and T. Terlaky, An interior point method approach to postoptimal and parametric analysis in linear programming, in: *Proceedings of the Workshop Interior Point Methods*, Budapest, Hungary, January 1993.
[13] Yu.E. Nesterov and A.S. Nemirovskii, *Interior Point Methods in Convex Programming: Theory and Application* (SIAM, Philadelphia, PA, 1994).
[14] R.T. Rockafellar, *Conjugate Duality and Optimization*, Regional Conference Series in Applied Mathematics (SIAM, Philadelphia, PA, 1974).
[15] A. Shapiro, First and second order analysis of nonlinear semidefinite programs, *Math. Programming, Ser. B* 77 (1997) 301–320.
[16] L. Vandenberghe, Private communication.
[17] L. Vandenberghe and S. Boyd, Semidefinite programming, *SIAM Rev.* 38 (1995) 49–95.