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MULTIPARAMETRIC LINEAR PROGRAMMING*

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The multiparametric linear programming (MLP) problem for the right-hand sides (RHS) is to maximize $z = c^T x$ subject to $Ax = b(\lambda)$, $x \geqq 0$, where $b(\lambda)$ can be expressed in the form

$$b(\lambda) = b^* + F\lambda,$$

where F is a matrix of constant coefficients, and λ is a vector-parameter.

The multiparametric linear programming (MLP) problem for the prices or objective function coefficients (OFC) is to maximize $z = c^T(\nu)x$ subject to $Ax = b$, $x \geqq 0$, where $c(\nu)$ can be expressed in the form $c(\nu) = c^* + H\nu$, and where H is a matrix of constant coefficients, and ν a vector-parameter.

Let B_i be an optimal basis to the MLP-RHS problem and R_i be a region assigned to B_i such that for all $\lambda \in R_i$ the basis B_i is optimal. Let K denote a region such that $K = \bigcup_i R_i$, provided that the R_i for various i do not overlap.

The purpose of this paper is to present an effective method for finding all regions R_i that cover K and do not overlap. This method uses an algorithm that finds all nodes of a finite connected graph. An analogous method is presented for the MLP-OFC problem.

1. Introduction

In a linear model of an economic system, the interdependences among the entire system, its environment and the subsystems are fixed. By the introduction of parameters into the linear model or in the optimal solution obtained by solving the appropriate linear programming problem, the interdependences can be varied. Thus, parametric programming combined with other analytic methods, such as parametric sensitivity analysis and use of several objective functions, etc., can become a serious tool for solving management problems.

These tools can also be used to simulate the stochastic nature of some elements in the model, to make decisions, to measure the effects of inaccuracies in information flow, and for decisions under risk and uncertainty, etc.

As more parameters are introduced into the model, the model naturally becomes more intractable. Consequently, most of the papers dealing with multiparametric linear programming are of a theoretical nature rather than of a practical interest (see, e.g., [1], [2], [4], [5], [20], [21], [23], [28]). The computational procedure for a scalar parameter has been worked out completely.[1] As far as we know, the computational aspects of the multiparametric linear programming problem have not yet been completely resolved. In the literature one can find actual solutions for at most two parameters, in which case a geometric approach is frequently used (see, e.g., [3]).

This paper considers the computational aspects of the multiparametric linear programming problem. §2 introduces basic notations, theorems and definitions, and presents a solution method for the multiparametric linear programming problem when parameters are found only in the right-hand sides of the linear programming model.

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[1] See, for example, W. Orchard-Hays who describes in *Advanced Linear Programming Computing Techniques*, McGraw-Hill, 1968, a computational algorithm for a scalar parameter in the right-hand side or in the prices.

The solution method is based upon an algorithm that finds all nodes of a connected graph. §3 does likewise, but briefly, when parameters are found only in the prices or objective function coefficients. §4 considers a so-called "Constrained Multiparametric Linear Programming Problem", and, finally, §5 presents a numerical example of the method given in §2.

2. The Multiparametrization of the Right-Hand Sides

2.1. *Notation*

The multiparametric linear programming (MLP) problem for the right-hand sides (RHS) is to maximize

$$z = c^T x \tag{2.1}$$

subject to

$$Ax = b^* + F\lambda, \tag{2.2}$$

and

$$x \geqq 0, \tag{2.3}$$

with respect to $x = (x_1, \cdots, x_n)^T$, where[2] $A = (a_{ij})$ is an (m, n) matrix, $c, x, 0 \in E^n$, $b \in E^m$ are column vectors, $\lambda = (\lambda_1, \cdots, \lambda_s)^T$ is a variable (column) vector-parameter, $F = (f_{ik})$ is an (m, s) matrix. The elements of A, F, b and c are constant. Let the rank of matrix A be m.

We shall denote

$$b(\lambda) = b^* + F\lambda. \tag{2.4}$$

Let $I = \{i: i = 1, \cdots, m\}$, $J = \{j: j = 1, \cdots, n\}$. Let $\rho = [j_1, \cdots, j_m], j \in J$, be an m-tuple consisting of the subscripts of basic variables. The m-tuple ρ will be called "the index of the basis". Let ${}^\rho B$ denote the (regular) matrix corresponding to ρ, and let ${}^\rho B^{-1}$ denote the matrix inverse of ${}^\rho B$. Further, let $J_1^{(\rho)} \subset J$, and $J_2^{(\rho)} \subset J$ be the sets of subscripts of the basic and the nonbasic variables, respectively, that correspond to the basis with the index ρ. Evidently, $J_1^{(\rho)} \cup J_2^{(\rho)} = J$, $J_1^{(\rho)} \cap J_2^{(\rho)} = \varnothing$.

The simplex tableau corresponding to the basis ${}^\rho B$ can be written in the following form[3]

$$z + {}^\rho c^T x - {}^\rho p^T \lambda = z^{(\rho)}, \tag{2.5}$$

$${}^\rho A x - {}^\rho F \lambda = {}^\rho b^*, \tag{2.6}$$

where

$${}^\rho A = {}^\rho B^{-1} A, \quad {}^\rho F = {}^\rho B^{-1} F, \quad {}^\rho b^* = {}^\rho B^{-1} b^*, \tag{2.7}$$

$$\begin{gathered} {}^\rho c^T = c_B{}^{T\rho} A - c^T, \quad {}^\rho p^T = c_B{}^{T\rho} F, \quad z^{(\rho)} = c_B{}^{T\rho} b^*, \\ c_B{}^T = (c_{j_1}, \cdots, c_{j_m}), \quad [j_1, \cdots, j_m] = \rho. \end{gathered} \tag{2.8}$$

From

$${}^\rho B^{-1} b(\lambda) = {}^\rho b(\lambda) \tag{2.9}$$

[2] It is possible to add the constraints $G\lambda \leqq g$ to problems (2.1)–(2.3). The treatment of such a "constrained MLP-RHS" problem is briefly described in §4.

[3] The left superscript ρ indicates only that the relevant vector or matrix has been transformed into the basis with the index ρ. The value of z corresponding to ${}^\rho B$ is $z^{(\rho)}$.

and from (2.4) we get

$$ {}^{\rho}b(\lambda) = {}^{\rho}B^{-1}b^{*} + {}^{\rho}B^{-1}F\lambda = {}^{\rho}b^{*} + {}^{\rho}F\lambda, \tag{2.10} $$

and denote

$$ {}^{\rho}b(\lambda) = ({}^{\rho}b_1(\lambda), \cdots, {}^{\rho}b_m(\lambda))^T. \tag{2.11} $$

If we want to distinguish different bases, we shall use $\rho_1, \rho_2, \cdots, \rho_k, \cdots$, and we shall denote the corresponding terms by B_k, B_k^{-1}, ${}^kA$, ${}^kF$, ${}^kb^*$, $J_2^{(k)}$, etc.

Finally, denote by $z_{\max}^{(\rho)}(\lambda)$ (or, $z_{\max}^{(k)}(\lambda)$) the maximal value of the objective function related to the basis with the index ρ (or, ρ_k), and depending on λ, and by $z_{\max}(\lambda)$ the function expressing the dependence of the value of the objective function on λ over all relevant bases.

2.2. *Basic Theorems and Definitions*

In this part we shall introduce the basic theorems and definitions that are needed to solve problem (2.1)–(2.3). The proofs of theorems found in other works are not repeated here.

Definition 2.1. If there exists a vector-parameter λ such that the problem (2.1)–(2.3) has a finite optimal solution with respect to this λ, then such a λ will be said to be a feasible vector-parameter. Let K denote the set of all feasible vector-parameters.

THEOREM 2.1. *Assume that, for a fixed $\lambda^0 \in E^s$, there exists a finite optimum to the problem* (2.1)–(2.3). *Then, for every $\lambda \in E^s$, the problem* (2.1)–(2.3) *either has a finite optimal solution, or has no feasible solution.*

PROOF. The dual problem to (2.1)–(2.3) is to minimize

$$ w = u^T b(\lambda) \quad \text{subject to} \quad A^T u \geqq c. \tag{2.12} $$

From the duality theorem it follows that if for $\lambda = \lambda^0$ there exists a finite optimum to (2.1)–(2.3), then (2.12) has for $\lambda = \lambda^0$ a feasible solution u^0. Since A and c are constants (not depending on λ), the solution u^0 is feasible for all $\lambda \in E^s$. Again from the duality theorem it follows that the problem (2.1)–(2.3) does not have an unbounded solution for any $\lambda \in E^s$, and, consequently, for any $\lambda \in K$. Q.E.D.

Definition 2.2. The basis ${}^{\rho}B$ is said to be an optimal basis, if, and only if, there exists $\lambda \in K$ such that ${}^{\rho}B$ is optimal (primal and dual feasible) with respect to λ.

THEOREM 2.2. *K is a convex polyhedron in E^s.*

The proof is found in [3], [1].

Let ${}^{\rho}B$ be an optimal basis in the sense of Definition 2.2. Since the dual solution does not depend on λ, it remains feasible for all $\lambda \in E^s$ (see also the proof of Theorem 2.1). The basis ${}^{\rho}B$ remains primal feasible, if

$$ {}^{\rho}B^{-1}(b^* + F\lambda) \geqq 0, $$

that is, if

$$ -{}^{\rho}F\lambda \leqq {}^{\rho}b^*. \tag{2.13} $$

The condition (2.13) determines uniquely a convex polyhedron R_ρ such that for all $\lambda \in R_\rho$ the basis ${}^{\rho}B$ is optimal,[4] and for $\lambda \in K - R_\rho$ the basis ${}^{\rho}B$ is primal infeasible.

[4] For the various ρ, i.e., $\rho_1, \rho_2, \cdots, \rho_k, \cdots$, we shall write $R_1, R_2, \cdots, R_k, \cdots$.

THEOREM 2.3. *The function $z_{\max}(\lambda)$, defined over K, is a concave function.*

The proof is found in [3], [1]. In [3] is also proved that the function $z_{\max}(\lambda)$ is continuous.

Definition 2.3. Consider any two bases B_1 and B_2 with the indices ρ_1 and ρ_2, respectively. These two bases are said to be neighboring bases (or, neighbors), if and only if

(1) there exists $\lambda^* \in K$ such that B_1 and B_2 are both optimal bases to (2.1)–(2.3) for λ^*, and

(2) it is possible to pass from B_1 to B_2 (and vice versa) by one step of the dual simplex algorithm.

Note 1. Suppose that there exists $\lambda^* \in K$ satisfying Definition 2.3. Then by setting $\lambda = \lambda^*$ in (2.6), it follows from (2.13) regarding (2.10) that ${}^1b_i^* + \sum_{k=1}^{s} {}^1f_{ik}\lambda_k^* = 0$ for at least one $i \in I$, say for $i = r$. If there exists ${}^1a_{rj} < 0$ for at least one $j \in J_2^{(1)}$, it is possible to pass to a neighboring optimal basis using the dual simplex algorithm. From the simplex pivoting step it is evident that

$$ {}^1b_r^* + \sum_{k=1}^{s} {}^1f_{rk}\lambda_k^* = {}^2b_r^* + \sum_{k=1}^{s} {}^2f_{rk}\lambda_k^* = 0, $$

and, finally, $z_{\max}^{(1)}(\lambda^*) = z_{\max}^{(2)}(\lambda^*)$.

Note 2. Suppose that there is a dual degeneracy in B_1. This degeneracy clearly does not depend on λ. Suppose this degeneracy occurs for $j = s$, $s \in J_2^{(1)}$. If there exists ${}^1a_{is} > 0$ for at least one $i \in I$, it is possible to pass to another optimal basis, say B_3, by one step of the *primal* simplex algorithm. The basis B_3 is *not* to be regarded as a neighbor of basis B_1 in the sense of Definition 2.3.

Definition 2.4. Consider two optimal bases B_1 and B_2, and let R_1 and R_2, respectively, be the corresponding feasible regions of λ uniquely defined by (2.13). Then, the regions R_1, R_2 are said to be neighbors, if and only if B_1 and B_2 are neighbors.

Note. If R_1 and R_2 are neighbors, then clearly

$$ R_1 \cap R_2 \neq \emptyset. $$

THEOREM 2.4. *Two neighbors R_1 and R_2 lie in opposite half spaces of E^s.*

PROOF. Let B_1 and B_2 be the optimal bases relating to R_1 and R_2, respectively. The simplex tableau corresponding to these bases is of the form (2.5), (2.6). Assume that the numbering of the rows in the tableaux corresponding to each of the given bases is the same in the sense that it is possible to pass from one to another without renumbering them.

Let ${}^1a_{rk}$ be the pivot element in the simplex tableau for the index ρ_1 used to generate the basis B_2. By assumption, ${}^1a_{rk} < 0$. The rth row of the simplex tableau for index ρ_2 is obtained by dividing the corresponding row of the former simplex tableau by ${}^1a_{rk}$. This implies that

$$ \frac{{}^1b_r(\lambda)}{{}^1a_{rk}} = {}^2b_r(\lambda) \quad \text{for all} \quad \lambda \in E^s, $$

and since ${}^1a_{rk} < 0$, the two terms must have opposite signs. Since

$$ {}^1b_r(\lambda) \geqq 0 \quad \text{for} \quad \lambda \in R_1, $$

and

$$ {}^2b_r(\lambda) \geqq 0 \quad \text{for} \quad \lambda \in R_2, $$

the theorem is proved. Q.E.D.

Definition 2.5. The region R_ρ is said to have a neighbor along its ith face if it is possible to pass to that neighbor by excluding the ith basic variable from the basis.

Note. Assume that the numbering of the basic variables in (2.6) related to ρ_k is the same as the numbering of the rows in (2.6). The elements of the ith row of $-{}^kF$ can be regarded as the coefficients of the corresponding equation of a hyperplane representing the "ith face" of R_k, i.e., $-{}^kf_{i1}\lambda_1 - \cdots - {}^kf_{is}\lambda_s = {}^kb_i$. If we wish to exclude the ith basic variable from the basis using the dual step as required by Definition 2.3, there must exist $\lambda^0 \in R_k$ such that the ith element of ${}^kb(\lambda^0) = {}^kb^* + {}^kF\lambda^0$ vanishes, and, simultaneously, there must exist at least one negative element in the ith row of ${}^kA$.

Thus, the region R_ρ has a neighbor along its ith face ($i \in I$), if

(1′) there exists $\lambda^0 \in R$ such that ${}^\rho b_i(\lambda^0) = 0$, and

(2′) ${}^ka_{ij} < 0$ for at least one $j \in J_2^{(\rho)}$.

Up to now we have introduced preliminary notations, theorems and definitions. We shall now turn to the solution method for the MLP-RHS problem. A solution to the MLP-RHS problem will be shown to be related to the nodes of a connected (undirected) subgraph G_0 of a (undirected) graph $G = (S, \Gamma)$ with node set S and arc set Γ, that is generated by the MLP-RHS problem.

We shall now define graph G and describe a connected subgraph G_0 of graph G.

Definition 2.6. Graph $G = (S, \Gamma)$ is said to be generated by a given MLP-RHS problem, if and only if

1. The set S of nodes is a subset of the m-tuples $\rho = [j_1, \cdots, j_m]$ of integers j_i, $1 \leqq j \leqq n$, $i = 1, \cdots, m$, such that $\rho \in S$ if and only if ${}^\rho B$ is an optimal basis to the problem (2.1)–(2.3), and

2. For $\rho_1, \rho_2 \in S$, the index $\rho_2 \in \Gamma(\rho_1)$ (and, at the same time, $\rho_1 \in \Gamma(\rho_2)$), if and only if the bases B_1 and B_2 are neighbors. Let nodes ρ_1, ρ_2 be called neighboring nodes.

Note. From Definition 2.6 it is obvious that each node $\rho \in S$ corresponds uniquely to an optimal basis ${}^\rho B$ (and vice versa), and that $\Gamma(\rho)$ is the set of nodes joined with the node ρ by an arc, i.e., $\Gamma(\rho)$ is the set of all neighbors to ρ.

Definition 2.7. Two nodes $\rho_1, \rho_2 \in S$ are said to have the distance Δ, $0 < \Delta \leqq m$, between them, if exactly Δ elements of ρ_2 are different from elements of ρ_1.

THEOREM 2.5. *Let $\lambda^1, \lambda^2 \in K$ be two arbitrary feasible vector-parameters and $\rho_1 \in S$ be given such that $\lambda^1 \in R_1$. Then there exists a path $\{\rho_1, \cdots, \rho_k\}$ in the graph $G = (S, \Gamma)$ such that $\lambda^2 \in R_k$.*

PROOF. Express the line segment joining λ^1, λ^2 in parametric form

$$\lambda(t) = \lambda^1 + t(\lambda^2 - \lambda^1), \qquad 0 \leqq t \leqq 1. \tag{2.14}$$

Evidently, $\lambda(t) \in K$ for all $t \in [0, 1]$, because K is a convex set. Setting $\lambda = \lambda(t)$ in (2.1)–(2.3), we obtain a parametric problem with scalar parameter t. Perform the parametrization procedure of this problem for $0 \leqq t \leqq 1$ starting at basis B_1 and following the dual simplex algorithm. The resulting sequence of optimal bases corresponds to the path $\{\rho_1, \cdots, \rho_k\}$ in graph G. Q.E.D.

Theorem 2.5 enables us to assign a connected graph to a given MLP-RHS problem. However, we should realize that the graph $G = (S, \Gamma)$ need not be connected. If G is not connected, then in some of the optimal bases ${}^\rho B$ there obviously occur dual degeneracies. As mentioned in Note 2 to Definition 2.3, if in B_1 there is a dual degeneracy, it is possible to pass from B_1 to another optimal basis B_3 in one step of the primal simplex algorithm. But, in this case, basis B_3 is not a neighbor to basis B_1, ρ_3 is not a neighbor to ρ_1, and R_1 and R_3 will overlap.[5]

If, in this case, we tried to find *all* existing optimal bases (or, all existing feasible regions) depending on λ, we would encounter serious computational difficulties. Moreover, as mentioned above, the solution method requires a connected graph. Therefore, to determine efficiently the set K of all feasible vector-parameters, we shall cover K by nonoverlapping (neighboring) regions R_ρ (see Theorems 2.4 and 2.5, Definitions 2.3, 2.4 and 2.7). Then the corresponding subgraph G_0 of graph G will surely be connected.

Hence, if there exists a dual degeneracy in some of the optimal bases, it is sufficient to find an arbitrary component $G_0 = (S_0, \Gamma_0)$ of the (entire) graph $G = (S, \Gamma)$. Namely, any component G_0 of G determines a covering of K as required above, i.e.,

$$\bigcup_{\rho \in S_0} R_\rho = K. \tag{2.15}$$

This fact follows immediately from Theorem 2.5.

By Theorem 2.5 it is also obvious that $R_3 \subset \bigcup_{\rho\in S_0} R_\rho$, i.e., $R_3 \subset K$, where R_3 is a region corresponding to basis B_3 obtained from a primal step.

Now we may conclude that if we determine an arbitrary optimal basis ${}^\rho B$ to the problem (2.1)–(2.3) we have, in fact, found a node $\rho \in S_0$. With this node a component G_0 of G is uniquely determined. From (2.15) it follows that if one finds all the nodes $\rho \in S_0$, then the covering of K is obtained, i.e., all feasible λ to the given MLP-RHS problem stated in (2.1)–(2.3), or all nonoverlapping feasible regions R_ρ are found. Obviously, if there is no dual degeneracy in any of the optimal bases ${}^\rho B$, then $G = G_0$.

We may now approach the MLP-RHS problem in the following way:

(1) Determine a region $K \subset E^s$ of vector-parameters λ such that the MLP-RHS problem stated in (2.1)–(2.3) has a finite optimal solution for all $\lambda \in K$ and has no feasible solution for $\lambda \in E^s - K$.

(2) Find all nonempty regions R_ρ such that $\bigcup_{\rho\in S_0} R_\rho = K$ and such that for every $\rho \in S_0$ the basis ${}^\rho B$ is an optimal basis to (2.1)–(2.3).

(3) Determine the functions ${}^\rho b(\lambda)$ and $z_{\max}^{(\rho)}(\lambda)$ defined over R_ρ, and consequently the functions $x(\lambda)$ and $z_{\max}(\lambda)$ defined over K.

As shown, by Definition 2.6, Theorem 2.5 and by relation (2.15), it is generally possible to generate a connected graph (a component) for each MLP-RHS problem. Hence, by a successive exhaustion of all nodes $\rho \in S_0$ of this component G_0, one obtains the solution of the MLP-RHS problem.

Furthermore, having found an arbitrary node $\rho \in S_0$, the relevant simplex tableau (2.5), (2.6) is generated. From this tableau the region R_ρ and the desired functions ${}^\rho b(\lambda)$ and $z_{\max}^{(\rho)}(\lambda)$ defined over R_ρ can be derived easily.

2.3. *The Solution Method*

This part presents an algorithm that finds all nodes $\rho \in S_0$ of a connected graph, and describes how to use this algorithm for solving the MLP-RHS problem. This algorithm is described and proved in [18], and we shall refer to it as The Algorithm.

For The Algorithm the following assumptions must be satisfied:

1. The graph must be finite and connected.

[5] This can be easily proved by showing that R_1 and R_3 do not lie in opposite half spaces of E^s (see Theorem 2.4).

2. There must be a way to compute an initial node ρ_0, and

3. There must be a procedure that determines all nodes that are neighbors to a given node.

Then The Algorithm may be stated as follows:

Step A. Compute node $\rho_0 \in S_0$.

Step B. Construct the sets (lists)

$$V_0 = \{\rho_0\}, \quad W_0 = \{\Gamma(\rho_0)\}.$$

Step C. Assume that the kth step of The Algorithm that finds the node ρ_{k-1}, and the lists V_{k-1}, W_{k-1} has already been performed.

Step C1. If $W_{k-1} = \emptyset$, then $V_{k-1} = S_0$, and the procedure is finished.

Step C2. If $W_{k-1} \neq \emptyset$, select a node $\rho_k \in W_{k-1}$ with a distance Δ from ρ_{k-1} that is as small as possible. Specifically, if $M_k = W_{k-1} \cap \Gamma(\rho_{k-1}) \neq \emptyset$, select $\rho_k \in M_k$.

Step C3. Construct the lists $V_k = V_{k-1} \cup \{\rho_k\}$, $W_k = W_{k-1} \cup \Gamma(\rho_k) - V_k$, and return to Step C1.

From the previous parts it is obvious that the graph G_0 generated by a given MLP-RHS problem is finite and connected. Thus, the first assumption for using The Algorithm is satisfied.

Now we shall describe a procedure, called Phase 1, that computes the initial node $\rho_0 \in S_0$ and another procedure, called Phase 2, that determines all neighbors to a given node. By this description it will be shown that in the case of a MLP-RHS problem the remaining assumptions for The Algorithm are also satisfied.

Phase 1. (1°) Find a feasible solution (x^0, λ^0) to the system

$$Ax - F\lambda = b^*, \tag{2.16}$$

$$x \geqq 0, \tag{2.17}$$

where the elements of λ are unrestricted. If there does not exist any solution to (2.16), (2.17), then $K = \emptyset$.

(2°) Suppose that there exists a feasible solution (x^0, λ^0) to (2.16), (2.17). Then put $\lambda = \lambda^0$ into the initial problem (2.1)–(2.3) and solve

$$\text{maximize } z = c^T x \tag{2.18}$$

subject to

$$Ax = b^* + F\lambda^0, \qquad x \geqq 0. \tag{2.19}$$

By Theorem 2.1, region $K = \emptyset$ if the latter problem has no finite solution. If the problem (2.18), (2.19) has a finite solution, then in sense of Definition 2.2, there exists an optimal basis B_0 to problem (2.18), (2.19), that is simultaneously an optimal basis to the problem (2.1)–(2.3). Thus, $\lambda^0 \in R_0$, $R_0 \neq \emptyset$, $R_0 \subset K$, hence, $K \neq \emptyset$. Consequently, ρ_0 is the initial node of S_0 that has been found.

Phase 2. Suppose that the kth step of The Algorithm has generated the sets V_{k-1} and W_{k-1}. List V_{k-1} consists of all optimal bases or nodes $\rho_0, \cdots, \rho_{k-1}$ such that to each of them we have an appropriate simplex tableau (2.5), (2.6). It is easy to derive the conditions determining the feasible region R_ρ and the expressions of the functions ${}^\rho b(\lambda)$ and $z_{\max}^{(\rho)}(\lambda)$ defined over R_ρ from these tableaux. In other words, V_{k-1} consists of nodes such that we already know their neighbors; these neighbors are listed either in V_{k-1} or in W_{k-1}.

List W_{k-1} consists of those nodes ρ, that are indices of optimal bases, but whose simplex tableaux have not yet been generated. Hence, we cannot yet describe the regions R_ρ or the desired functions of λ. In other words, W_{k-1} consists of those nodes, whose neighbors have not yet been explored, consequently, we do not know yet if among these neighbors there exist indices of optimal bases that have not yet been listed.

According to Step C2, suppose $W_{k-1} \neq \emptyset$. After selecting a convenient node $\rho_k \in W_{k-1}$ (with distance Δ from ρ_{k-1} as small as possible) we pass from B_{k-1} to B_k, and obtain the tableaux (2.5), (2.6) related to ρ_k.

In order to generate W_k, we add to W_{k-1} all the nodes in $\Gamma(\rho_k)$ except those nodes in $\Gamma(\rho_k)$ which are already listed in W_{k-1} or V_{k-1}. Let Q_k denote the set of nodes to be added to W_{k-1} in order to generate W_k. Thereafter we move the selected node ρ_k from W_{k-1} to V_{k-1} in order to generate V_k.

The only remaining problem is to construct the set Q_k. This can be done in the following way:

($1^{\circ\circ}$) Let $P_k \subset I$ be the set of rows in (2.6) related to ρ_k, or the set of indices of faces of R_k such that condition ($2'$) is satisfied.

($2^{\circ\circ}$) Form the system

$$\begin{aligned} \xi_i &= {}^k b_i^* - \textstyle\sum_{j=1}^{s} (-{}^k f_{ij})\lambda_j, \qquad i = 1, \cdots, m, \\ \xi_i &\geqq 0 \quad \text{all} \quad i \in I, \end{aligned} \tag{2.20}$$

by transforming (2.13) into a set of equations using "slack" variables ξ_i. Find successively the $\min_{i \in P_k} \xi_i$ subject to (2.20). In this way we determine those $i \in P_k$ for which condition ($1'$) is satisfied.

Note. The method of determining $\min \xi_i$ is shown in the numerical example. In connection with a modification of the simplex algorithm for unrestricted variables, in [22] there is a detailed description of the method of finding $\min \xi_i$ (see, also, [11]).

($3^{\circ\circ}$) Let $\bar{P}_k \subset P_k$ be the set of $i \in I$ such that both conditions ($1'$) and ($2'$) are satisfied.

($4^{\circ\circ}$) According to the dual simplex rules, determine the possible pivot elements for all $i \in \bar{P}_k$.

($5^{\circ\circ}$) Without actually performing the appropriate dual steps, determine successively the indices of neighboring optimal bases, i.e., $\Gamma(\rho_k)$, by using the pivot elements found in ($4^{\circ\circ}$).

($6^{\circ\circ}$) Select from $\Gamma(\rho_k)$ all nodes already listed in W_{k-1} or in V_{k-1}. The remainder of $\Gamma(\rho_k)$ is exactly Q_k, i.e., $W_k = W_{k-1} \cup Q_k - \{\rho_k\}$.

Due to a possible nonunique determination of the pivot element in the row, there may exist two or more neighbors along the same face of R_ρ.

Now we can summarize the procedure.

Find the initial optimal basis B_0, i.e., node $\rho_0 \in S_0$, according to (1°) and (2°). This concludes Phase 1.

Phase 2 starts with the optimal simplex tableau for ρ_0. In this tableau find all rows (i.e., the set P_0) such that there is at least one negative element (in the part ${}^0A$ of the tableau). Determine the rows for which the corresponding "slack" variables ξ_i can equal zero (i.e., minimizing ξ_i for $i \in P_0$). The found subset $\bar{P}_0$ of P_0 denotes the faces of R_0 where there exists a neighbor. By an obvious exchange of the appropriate variables according to the dual simplex rules we obtain those nodes that are neighbors to ρ_0.

Now we can construct the lists $V_0 = \{\rho_0\}$, and $W_0 = \Gamma(\rho_0) = Q_0$. Choosing any node $\rho_k \in W_0$, the distance Δ between ρ_k and ρ_0 equals one. Having chosen say $\rho_1 \in W_0$, and after passing from B_0 to B_1, repeat the procedure of finding $\bar{P}_1$ and define $\Gamma(\rho_1)$. To generate W_1 we construct Q_1, i.e., find all nodes from $\Gamma(\rho_1)$ that are not yet listed in V_0 and W_0. Add the obtained set Q_1 of nodes to W_0 and take out ρ_1. Add node ρ_1 to V_0 in order to generate V_1, which evidently becomes $V_1 = \{\rho_0, \rho_1\}$.

Repeat the whole procedure until in a kth step the list $W_k = \emptyset$, i.e., $V_k = S_0$.

3. The Multiparametrization of Prices

3.1. *Basic Theorems and Definitions*

The procedure used for the parametrization of the right-hand sides can with certain modifications also be used for the parametrization of prices (the coefficients of the objective function). We describe this process very briefly, using the ideas of §2.

The MLP problem for the prices or for the objective function coefficients (OFC) is to maximize

$$w = c^* + H\nu \tag{3.1}$$

subject to

$$Ax = b, \tag{3.2}$$

$$x \geqq 0. \tag{3.3}$$

Denote

$$c(\nu) = c^* + H\nu, \tag{3.4}$$

where $\nu \in E^s$ is a vector-parameter, A is a constant (m, n) matrix, H is a constant (n, s) matrix, $b \in E^m$ and $c^* \in E^n$ are constant vectors.

The MLP-OFC problem can be rewritten in the equivalent form

$$\text{maximize } w = \nu^T y + z \tag{3.5}$$

subject to

$$Ax = b, \tag{3.6}$$

$$y - H^T x = 0, \tag{3.7}$$

$$z - c^{*T} x = 0, \tag{3.8}$$

$$x \geqq 0, \tag{3.9}$$

where $y \in E^s$ is a variable vector, z is a scalar variable.

Since the elements of y are unbounded, they always remain in the basis. Therefore, every basis can be fully characterized by the subscripts of its basic variables x_i.

As in §2, let $\rho = [j_1, \cdots, j_m]$ be the index of the basis. The system (3.6)–(3.9) that is transformed into ${}^\rho B$ is

$${}^\rho A x = {}^\rho b, \tag{3.10}$$

$$y + {}^\rho H^T x = {}^\rho q, \tag{3.11}$$

$$z + {}^\rho c^{*T} x = z^{(\rho)}. \tag{3.12}$$

Let H_B be the part of H that corresponds to the basic variables in ${}^\rho B$. Then we can write

$${}^\rho H^T = H_B{}^T\, {}^\rho A - H^T, \qquad {}^\rho c^{*T} = c_B^{*T}\, {}^\rho A - c^{*T}, \tag{3.13}$$

(3.14) $${}^{\rho}q = H_B{}^T\,{}^{\rho}b, \qquad z^{(\rho)} = c_B^{*T}\,{}^{\rho}b.$$

For a fixed ν^0 we obtain the representation of ${}^{\rho}c(\nu^0)$ in ${}^{\rho}B$ by the linear combination of equations (3.11), (3.12)

(3.15) $${}^{\rho}w = -({}^{\rho}H\nu^0 + {}^{\rho}c^*)^T x + \nu^{0T}\,{}^{\rho}q + z^{(\rho)}.$$

Thus,

(3.16) $${}^{\rho}c(\nu^0) = {}^{\rho}H\nu^0 + {}^{\rho}c^*.$$

The primal solution does not depend on ν. For maintaining ${}^{\rho}B$ optimal the dual feasibility condition must be satisfied, i.e.,

(3.17) $$-{}^{\rho}H\nu \leqq {}^{\rho}c^*.$$

Relation (3.17) defines uniquely the region $R_\rho \subset E^s$ corresponding to the basis ${}^{\rho}B$. Basis ${}^{\rho}B$ is an optimal basis to the MLP-OFC problem, if and only if $R_\rho \neq \varnothing$ (cf. Definition 2.2).

We shall now outline the analogous basic theorems and definitions for the MLP-OFC problem.

A feasible vector-parameter is defined in a way similar to Definition 2.1.

Region K is the set of all feasible vector-parameters. Thus, K is the union of all R_ρ, that do not overlap, corresponding to the optimal bases ${}^{\rho}B$.

THEOREM 3.1. *The MLP-OFC problem either has a feasible solution for each* $\nu \in E^s$, *or it has no feasible solution.*

Theorem 2.2 holds in the defined form.

As an analogy to Theorem 2.3 we have

THEOREM 3.2. *Function* $z_{\max}(\nu)$, *defined over* K, *is a convex (and continuous) function.*

Definition 3.1. Take two optimal bases B_1 and B_2 with indices ρ_1 and ρ_2, respectively. These two bases are said to be neighbors, if and only if

(1) there is $\nu^* \in K$ such that B_1 and B_2 are both optimal bases for ν^*, and

(2) it is possible to pass from B_1 to B_2 (and vice versa) by one step of the primal simplex algorithm.

Definitions 2.4, 2.6 and 2.7, and Theorems 2.4 and 2.5 remain valid.

As an analogy to Definition 2.5 we have

Definition 3.2. The set R_ρ defined by (3.17) has a neighbor along its jth face, $j \in J_2^{(\rho)}$, if it is possible to pass to that neighbor by introducing the jth nonbasic variable into the basis, i.e., if the following conditions are satisfied:

(1″) There exists $\nu^0 \in R_\rho$ such that ${}^{\rho}c_j(\nu^0) = 0$, and

(2″) ${}^{\rho}a_{ij} > 0$ for at least one $i \in I$.

Note. The set J of subscripts may always be renumbered so that

$$J_2^{(\rho)} = \{1, \cdots, n - m\}.$$

3.2. *Solving the MLP-OFC Problem*

To find all nodes of the graph $G_0 = (S_0, \Gamma_0)$ The Algorithm is used again. We shall show how to determine a feasible basis and how to find its neighbors.

Phase 1. (1*) Find a feasible solution (u^0, ν^0) to the system

(3.18) $$A^T u - H\nu \geqq c^*,$$

where $u \in E^m$ and $\nu \in E^s$ are variable vectors. This system can also be solved dually by finding a dual feasible solution to the system

$$Ax = b, \quad y - H^T x = 0, \quad z - c^{*T} x = 0.$$

If the system (3.18) has no solution, then there exists no dual feasible solution to the MLP-OFC problem, i.e., $K = \varnothing$.

(2^{**}) Put ν^0 (obtained by solving (3.18)) into (3.1)–(3.3), and solve

$$\text{maximize } w = c^T(\nu^0)x \tag{3.19}$$

subject to

$$Ax = b, \qquad x \geqq 0. \tag{3.20}$$

If there exists a primal feasible solution to (3.19), (3.20), there obviously also exists an optimal basis B_0 to the MLP-OFC problem. At the same time, B_0 is an optimal basis subject to all $\nu \in R_0$. The region R_0 is not empty, since $\nu^0 \in R_0$. Thus, ρ_0 is a node in set $S_0 \subseteq S$.

The set Q_k can be determined similarly as in §2 using the system of equations

$$\xi = {}^\rho c^{*N} - (-{}^\rho H^N)\nu, \tag{3.21}$$

$$\xi \geqq 0, \tag{3.22}$$

where $\xi \in E^{n-m}$, ${}^\rho H^N$, and ${}^\rho c^{*N}$ are the respective parts of ${}^\rho H$ and ${}^\rho c^*$ that correspond to the nonbasic variables in ${}^\rho B$. The remaining parts of matrix ${}^\rho H$ and of vector ${}^\rho c^*$ are null. The set Q_k contains the unlisted neighbors along those faces, for which actually $\xi_i = 0$ subject to (3.21), (3.22).

4. The Constrained MLP problem

The general problem of the MLP-RHS problem can be written in the form

$$\text{maximize } z = c^T x \tag{4.1}$$

subject to

$$Ax = b(\lambda), \tag{4.2}$$

$$G\lambda \leqq g, \tag{4.3}$$

$$x \geqq 0, \tag{4.4}$$

where (4.3) is a system of constraints for the vector-parameter λ, G is an (m, s) matrix of constant coefficients, and $g \in E^m$ is a constant vector. As the practical applications of MLP demonstrate, the prevailing part of constraints (4.3) can be written in the form

$$\underline{\lambda}_i \leqq \lambda_i \leqq \bar{\lambda}_i, \tag{4.5}$$

where $\underline{\lambda}_i$ and $\bar{\lambda}_i$ are the lower and the upper limits, respectively, of the element λ_i of the vector λ.

The problem (4.1)–(4.4) might be called the constrained multiparametric linear programming problem (CMLP) and can be solved by almost the same procedure that was described in §2. Since the additional constraints (4.3) are linear, the λ corresponding to finite optimal solutions again form a convex polyhedron. Let $M = \{\lambda \mid G\lambda \leqq g\}$, and let $K \subset M$ be the set of feasible vector-parameters $\lambda \in M$, and let ${}^\rho B$ be the optimal basis corresponding to the nonempty region R_ρ defined by

$$-{}^{\rho}F\lambda \leqq {}^{\rho}b^*, \tag{4.6}$$

$$G\lambda \leqq g. \tag{4.7}$$

Then all the assertions, on which the method in §§1 and 2 is based, are satisfied. In particular, Theorem 2.5, and consequently, relation (2.15) are true, and therefore The Algorithm may be used. The applications of this algorithm to the problem (4.1)–(4.4) differ from the previous one only in that the auxiliary systems (2.14), (2.15) and (2.19) are enlarged to include (4.7).

In a similar way we may proceed in the case of a constrained MLP-OFC problem.

5. Numerical Example for the MLP-RHS Problem

In this section we shall find the covering of K by nonoverlapping regions R_ρ for a specific MLP-RHS problem.

Consider the problem: Maximize $z = 3x_1 + 2x_2$, subject to

$$x_1 \leqq 10 + \lambda_1 + 2\lambda_2,$$
$$x_2 \leqq 2 - \lambda_1 + \lambda_2,$$
$$x_1 + x_2 \leqq 20 - \lambda_2,$$
$$-x_1 + x_2 \geqq 4 - \lambda_3,$$
$$x_1 + 2x_2 \leqq 12 + \lambda_1 - \lambda_3,$$
$$x_1 \geqq 0, \quad x_2 \geqq 0.$$

Phase 1. After adding slack variables x_3, x_4, x_5, x_6, x_7, and an artificial variable p we construct the initial simplex tableau (Table 1). After one primal simplex step, we generate Table 2, from which it is evident that there is a feasible vector-parameter $\lambda^0 = (0, 0, 2)^T$. Replacing vector $b(\lambda)$ by $b(\lambda^0)$ in the initial problem, we obtain a particular problem, whose optimal solution is given in Table I.

Phase 2. In Phase 1 we found that $\rho_0 = [1, 2, 3, 5, 7]$. The region R_0 corresponding to B_0 is defined by (see Table I)

$$-2\lambda_1 - \lambda_2 + \lambda_3 \leqq 12,$$
$$\lambda_1 - \lambda_2 \leqq 2,$$
$$-2\lambda_1 + 3\lambda_2 + \lambda_3 \leqq 20,$$
$$\lambda_1 - \lambda_2 - \lambda_3 \leqq -2,$$

TABLE 1

	1	2	6	$-F$			b^*
3	1	0	0	−1	−2	0	10
←4	0	$\boxed{1}$	0	1	−1	0	2
5	1	1	0	0	1	0	20
p	−1	1	−1	0	0	1	4
7	1	2	0	−1	0	1	12
z	−3	−2	0	0	0	0	0
$-p$	1	−1	1	0	0	−1	−4

TABLE 2

	1	4	6	$-{}^{p}F$			${}^{p}b^{*}$
3	1	0	0	−1	−2	0	10
→2	0	1	0	1	−1	0	2
5	1	−1	0	−1	2	0	18
←p	$\boxed{-1}$	−1	−1	−1	1	1	2
7	1	−2	0	−3	2	1	8
	−3	2	0	2	−2	0	4
	1	1	1	1	−1	−1	−2

TABLE I

ρ_0	4	6	$-{}^{0}F$			${}^{0}b^{*}$	${}^{0}b(\lambda^0)$
←3	−1	$\boxed{-1}$	−2	−1	1	12	10
2	1	0	1	−1	0	2	2
5	−2	−1	−2	3	1	20	18
→1	1	1	1	−1	−2	−2	0
7	−3	−1	−4	3	2	10	6
	5	3	5	−5	−3	−2	4

$$-4\lambda_1 + 3\lambda_2 + 2\lambda_3 \leqq 10,$$

and

$$z_{\max}^{(0)}(\lambda) = -2 - 5\lambda_1 + 5\lambda_2 + 3\lambda_3, \qquad x_3(\lambda) = 12 + 2\lambda_1 + \lambda_2 - \lambda_3,$$

$$x_2(\lambda) = 2 - \lambda_1 + \lambda_2, \qquad x_5(\lambda) = 20 + 2\lambda_1 - 3\lambda_2 - \lambda_3,$$

$$x_1(\lambda) = -2 - \lambda_1 + \lambda_2 + \lambda_3, \qquad x_7(\lambda) = 10 + 4\lambda_1 - 3\lambda_2 - 2\lambda_3,$$

where all functions are defined over R_0.

From Table I and condition (1′) we see that a neighbor to ρ_0 could exist only along the 1st, 3rd and 5th faces (rows in Table I), i.e., $P_0 = \{1, 3, 5\}$. To generate W_0, we need to know $\Gamma(\rho_0)$, i.e., all neighbors to ρ_0.

According to ($2^{\circ\circ}$) we have to check if $\xi_1 = \xi_3 = \xi_5 = 0$, i.e., if there exists λ^* such that

$$12 + 2\lambda_1^* + \lambda_2^* - \lambda_3^* = 0,$$

$$20 + 2\lambda_1^* - 3\lambda_2^* - \lambda_3^* = 0,$$

$$10 + 4\lambda_1^* - 3\lambda_2^* - 2\lambda_3^* = 0.$$

For this purpose we form the system (2.20) given in Table I.1. An artificial variable p is necessary in order to have a feasible solution in this tableau. To exclude p, we may choose between the 2nd or 3rd columns; in both cases the minimum of the ratios 20/3, 2/1, 10/3 or 12/1, 20/1, 2/1, 10/2 is the same. It does not matter, therefore, which of the variables λ_2, λ_3 we choose to replace p in the basis.

Table I.2 is generated from Table I.1 by using the pivot element in the box. If we want to minimize ξ_1, we may regard the first row as the transformed "objective

TABLE I.1

	λ_1	λ_2	λ_3	ξ_4	
ξ_1	-2	-1	1	0	12
ξ_2	1	-1	0	0	2
ξ_3	-2	3	1	0	20
$\leftarrow p$	-1	1	$\boxed{1}$	-1	2
ξ_5	-4	3	2	0	10

TABLE I.2

	λ_1	λ_2	ξ_4	
ξ_1	-1	-2	1	10
ξ_2	1	-1	0	2
ξ_3	-1	2	1	18
$\rightarrow\lambda_3$	-1	1	-1	2
$\leftarrow\xi_5$	$\boxed{-2}$	1	2	6

TABLE I.3

	ξ_5	λ_2	ξ_4	
$\leftarrow\xi_1$	$-\frac{1}{2}$	$\boxed{-\frac{5}{2}}$	0	7
ξ_2	$\frac{1}{2}$	$-\frac{1}{2}$	1	5
ξ_3	$-\frac{1}{2}$	$\frac{3}{2}$	0	15
λ_3	$-\frac{1}{2}$	$\frac{1}{2}$	-2	-1
$\rightarrow\lambda_1$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	-3

function coefficients". Furthermore, λ_i, $i = 1, 2, 3$, are unbounded variables, therefore we must introduce them into the basis and keep them there.

In the usual sense of the minimization procedure, the minimum of ξ_1 is apparently reached, since the coefficients in the first two columns of the first row are negative. We may, however, change the minimization criterion, since the λ_i are unrestricted. Then, in order to determine the pivot element, we must find min (10/1, 18/1, 6/2) or min (10/2, 2/1). In the second case, we would exclude ξ_2, however we are not interested in minimizing ξ_2 (see P_0). Therefore we use the first case and exclude ξ_5.

In this way, we obtain Table I.3. Only λ_2 remains nonbasic. Therefore, we find either min (7.2/5, 5.2/1), or min (15.2/3). Both cases are of interest, and therefore either may be chosen. Exchanging λ_2 with ξ_1, we obtain Table I.4.

We can now conclude that min ξ_1 = min $\xi_5 = 0$, but we must still consider min ξ_3. In the appropriate row we cannot choose a negative element in order to maintain all basic ξ_i feasible (nonnegative). We have, thus, only one possibility, i.e., to exchange ξ_1 and ξ_3. In this way Table I.5 is generated.

The procedure of minimizing ξ_i is described precisely in [22] and [11].

Thus, we have found $\bar{P}_0 = P_0 = \{1, 3, 5\}$. Returning to Table I, we determine the pivot elements in the three rows by the dual rules, and fictitiously exchanging the pertinent variables we obtain: $\rho_1 = [1, 2, 5, 6, 7]$, $\rho_2 = [1, 2, 3, 4, 7]$, $\rho_3 = [1, 2, 3, 4, 5]$, i.e., $Q_0 = \Gamma(\rho_0) = \{\rho_1, \rho_2, \rho_3\}$, $V_0 = \{\rho_0\}$, $W_0 = Q_0$.

Choosing $\rho_1 \in W_0$ (with distance $\Delta = 1$ from ρ_0) and performing the appropriate

TABLE I.4

	ξ_5	ξ_1	ξ_4	
$\rightarrow\lambda_2$	$\frac{1}{5}$	$-\frac{2}{5}$	0	$-\frac{14}{5}$
ξ_2	$\frac{3}{5}$	$-\frac{1}{5}$	1	$\frac{18}{5}$
$\leftarrow\xi_3$	$-\frac{4}{5}$	$\boxed{\frac{3}{5}}$	0	$\frac{96}{5}$
λ_3	$-\frac{3}{5}$	$\frac{1}{5}$	-2	$\frac{2}{5}$
λ_1	$-\frac{2}{5}$	$-\frac{1}{5}$	-1	$-\frac{22}{5}$

TABLE I.5

	ξ_5	ξ_3	ξ_4	
λ_2	$-\frac{1}{3}$	$\frac{2}{3}$	0	10
ξ_2	$\frac{1}{3}$	$\frac{1}{3}$	1	10
$\rightarrow\xi_1$	$-\frac{4}{3}$	$\frac{5}{3}$	0	32
λ_3	$-\frac{1}{3}$	$-\frac{1}{3}$	-2	-6
λ_1	$-\frac{2}{3}$	$\frac{1}{3}$	-1	2

TABLE II

ρ_1	4	3		$-{}^1F$		${}^1b^*$
$\rightarrow 6$	1	-1	2	1	-1	-12
2	1	0	1	-1	0	2
5	-1	-1	0	4	0	8
1	0	1	-1	-2	0	10
$\leftarrow 7$	$\boxed{-2}$	-1	-2	4	1	-2
	2	3	-1	-8	0	34

TABLE III

ρ_4	7	3		$-{}^4F$		${}^4b^*$
6	$\frac{1}{2}$	$-\frac{3}{2}$	1	3	$-\frac{1}{2}$	-13
$\leftarrow 2$	$\frac{1}{2}$	$\boxed{-\frac{1}{2}}$	0	1	$\frac{1}{2}$	1
5	$-\frac{1}{2}$	$-\frac{1}{2}$	1	2	$-\frac{1}{2}$	9
1	0	1	-1	-2	0	10
$\rightarrow 4$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-2	$-\frac{1}{2}$	1
	1	2	-3	-4	1	32

dual step, we obtain Table II. Here we have $P_1 = \{1, 3, 5\}$. As before, we solve the auxiliary problem. This computation is not illustrated. We find that $\bar{P}_1 = \{1, 5\}$, $\Gamma(\rho_1) = \{\rho_0, \rho_4\}$, where $\rho_4 = [1, 2, 4, 5, 6]$. Then $Q_1 = \{\rho_4\}$, and $V_1 = \{\rho_0, \rho_1\}$, $W_1 = \{\rho_2, \rho_3, \rho_4\}$.

Choosing $\rho_4 \in W_1$, and after having performed the dual step, we obtain Table III. Here $P_2 = \{1, 2, 3, 5\}$, and after carrying out the auxiliary computations for finding $\min \xi_i$, we find that $\bar{P}_2 = \{1, 2, 5\}$. Then, $\Gamma(\rho_4) = \{\rho_1, \rho_3, \rho_5\}$, where $\rho_5 = [1, 3, 4, 5, 6]$, $Q_2 = \{\rho_5\}$, and $V_2 = \{\rho_0, \rho_1, \rho_4\}$, $W_2 = \{\rho_2, \rho_3, \rho_5\}$.

And now very briefly: choosing $\rho_5 \in W_2$, we obtain Table IV. Here the new node is $\rho_6 = [1, 3, 4, 6, 7]$, and $V_3 = \{\rho_0, \rho_1, \rho_4, \rho_5\}$, $W_3 = \{\rho_2, \rho_3, \rho_6\}$.

TABLE IV

ρ_5	7	2		$-{}^5F$		${}^5b^*$
6	−1	−3	1	0	−2	−16
→3	−1	−2	0	−2	−1	−2
←5	$\boxed{-1}$	−1	1	1	−1	8
1	1	2	−1	0	1	12
4	0	1	1	−1	0	2
	3	4	−3	0	3	36

TABLE V

ρ_6	5	2		$-{}^6F$		${}^6b^*$
←6	−1	$\boxed{-2}$	0	−1	−1	−24
3	−1	−1	−1	−3	0	−10
→7	−1	1	−1	−1	1	−8
1	1	1	0	1	0	20
4	0	1	1	−1	0	2
	3	1	0	3	0	60

TABLE VI

ρ_2	5	6		$-{}^2F$		${}^2b^*$
→2	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	12
3	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	$-\frac{5}{2}$	$\frac{1}{2}$	2
←7	$\boxed{-\frac{3}{2}}$	$\frac{1}{2}$	−1	$-\frac{3}{2}$	$\frac{1}{2}$	−20
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	8
4	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{3}{2}$	$-\frac{1}{2}$	−10
	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{5}{2}$	$-\frac{1}{2}$	48

TABLE VII

ρ_3	7	6		$-{}^3F$		${}^3b^*$
2	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$
3	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	−2	$\frac{1}{3}$	$\frac{26}{3}$
→5	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{1}{3}$	$\frac{40}{3}$
1	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{4}{3}$
4	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	−1	$-\frac{2}{3}$	$-\frac{10}{3}$
	$\frac{5}{3}$	$\frac{4}{3}$	$-\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{44}{3}$

Choosing $\rho_6 \in W_3$ we obtain Table V. There are no new unlisted neighbors. Hence $V_4 = \{\rho_0, \rho_1, \rho_4, \rho_5, \rho_6\}$, and $W_4 = \{\rho_2, \rho_3\}$.

In the last two steps, choosing successively ρ_2 and ρ_3, we obtain no new nodes (Table VI and Table VII). Finally we have: $V_6 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$, $W_6 = \varnothing$ and $S_0 = V_6$.

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