Explicitly Parameterized Solutions of Parametric Cone Programs

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1 The Overall Idea

1.1 Parametric Cone Programs

Let us consider the following cone program in inequality form:

$$\begin{array}{rll} \Pi(\theta): & \underset{x}{\text{minimize}} & h(\theta)^\intercal x \\ & \text{subject to} & F(\theta)x + g(\theta) \preceq_K 0 \,, \end{array} \tag{1}$$

where the problem data depends on parameters θ. For the proper cone K, (a combination of) the nonnegative orthant, the second-order cone and the positive semidefinite cone are considered. We want to solve $\Pi(\theta)$ for all parameter values $\theta \in \Theta$, where we make the following *assumptions*:

A1. The problem $\Pi(\theta)$ is strictly feasible for all $\theta \in \Theta$.

A2. The problem data depend polynomially on θ. That is, $h(\theta)$, $F(\theta)$ and $g(\theta)$ are polynomial mappings.

The optimal value of $\Pi(\theta)$ is denoted $p^\star(\theta)$, and its optimal set $X^\star(\theta)$. In addition, we use $X^\star_\epsilon(\theta)$ to denote the ϵ-suboptimal set of $\Pi(\theta)$:

$$X^\star_\epsilon(\theta) = \{x \mid F(\theta)x + g(\theta) \preceq_K 0 \,,\; h(\theta)^\intercal x \leq p^\star(\theta) + \epsilon\} \,.$$

Note, $X^\star(\theta) = X^\star_0(\theta)$.

Relying of the fact that the considered cones K are self-dual, the dual cone program of $\Pi(\theta)$ amounts to

$$\begin{array}{rll} \Delta(\theta): & \underset{y}{\text{maximize}} & g(\theta)^\intercal y \\ & \text{subject to} & F(\theta)^\intercal y + h(\theta) = 0 \\ & & y \succeq_K 0 \,. \end{array} \tag{2}$$

Its optimal value and optimal set are denoted by respectively $d^\star(\theta)$ and $Y^\star(\theta)$, and on account of assumption A1:

$$p^\star(\theta) = d^\star(\theta) \,, \quad \forall \theta \in \Theta \,.$$

The ϵ-suboptimal set of $\Delta(\theta)$ is denoted by $Y^\star_\epsilon(\theta)$.

The common approach in parametric programming is to compute an explicit description of an optimal solution $x^\star(\theta)$ as a function of θ [??? refs]. This approach suffers from the following drawbacks

- limited complexity: one scalar parameter – multiple parameters, only $g(\theta)$ parameter dependent, while h and F parameter independent, ...
- computational cost: exponential (???) growth with the number of parameters
- non-uniqueness of $x^\star(\theta)$???

- ???

Given these drawbacks, recently attention has been devoted to the computation of approximate solutions [??? refs]. Yet, these approaches only mitigate the drawbacks above to limited extent.

To complete

- The last paragraph.
- Optimal set just $X_0^\star(\theta)$ instead of $X^\star(\theta)$. Do we want to use the suboptimal sets as in the elaborations below?

1.2 Explicitly parameterized solutions

As an alternative to the aforementioned approaches, we propose to construct approximate solutions of $\Pi(\theta)$ and $\Delta(\theta)$ of the form:

$$\hat{x}(\theta) \;=\; C_x b_x(\theta)\ , \tag{3}$$
$$\hat{y}(\theta) \;=\; C_y b_y(\theta)\ . \tag{4}$$

The entries of $b_x(\theta)$ and $b_y(\theta)$ correspond to (given) basis functions on Θ. For $\hat{x}(\theta)$ to qualify as an approximate solutions of $\Pi(\theta)$, is must be feasible. Hence, we enforce that for all $\theta \in \Theta$, there exists $\epsilon(\theta) < \infty$ such that

$$\hat{x}(\theta) \in X^\star_{\epsilon(\theta)}(\theta)\ .$$

To obtain the best possible approximation, we minimize $\int_\Theta \epsilon(\theta) d\theta$ while respecting the containment constraint above for all $\theta \in \Theta$. This amounts to the following semi-infinite cone program

$$\begin{aligned} \hat{\Pi} : \quad & \underset{C_x}{\text{minimize}} && \int_\Theta h(\theta)^\intercal C_x b_x(\theta) d\theta \\ & \text{subject to} && F(\theta) C_x b_x(\theta) + g(\theta) \preceq_K 0\ , \quad \forall \theta \in \Theta\ . \end{aligned} \tag{5}$$

Similarly for the dual:

$$\begin{aligned} \hat{\Delta} : \quad & \underset{C_x}{\text{maximize}} && \int_\Theta g(\theta)^\intercal C_y b_y(\theta) d\theta \\ & \text{subject to} && F(\theta)^\intercal C_y b_y(\theta) + h(\theta) = 0\ , \quad \forall \theta \in \Theta \\ & && C_y b_y(\theta) \succeq_K 0\ , \quad \forall \theta \in \Theta\ . \end{aligned} \tag{6}$$

To complete

- Note on parametric equality constraints: should be eliminated! Done by equalizing the coefficients (also guarantees the equality outside Θ, but no conservatism because e.g. 2 polynomials of degree n are equal if that are equal at $n + 1$ points). Is this always possible???
- Conversion of semi-infinite programs to finite-dimensional programs. + Discussion on the choice of the basis functions.

1.3 Discussion

- GP: Should we adopt a more general problem formulation, where we replace the constraint in (**??**) by

  $$\mathcal{F}(x, \theta) + F_0(\theta) \preceq_K 0$$

  with $\mathcal{F}(x, \theta)$ a mapping linear in x and polynomial in θ? The dual problem then involves $\mathcal{F}^{\text{adj}}(y, \theta)$.

- GP: The parameter dependencies $h(\theta)$, $F(\theta)$ and $g(\theta)$ can be extended to be piece-wise polynomial. What is the effect on the overall complexity (e.g. generally more knots and consequently, more constraints at the end) ???
- GP: I'm rather sure that the problem formulation can be extended to rational parameter dependencies $h(\theta)$, $F(\theta)$ and $g(\theta)$. Ways to convert the semi-infinite constraints to a finite number of generalized inequalities: nurbs, S-procedure, descriptor forms ...
- GP: I'm rather sure that, when considering rational parameter dependencies of the data, the solutions $\hat{x}(\theta)$ and $\hat{y}(\theta)$ should still be parameterized as a linear combination of basis functions. Hence, no general rational functions with a free numerator and a free denominator. What would be appropriate rational basis functions ???
- GP: Which forms of Θ are allowed (polyhedra, semi-algebraic sets ...) and how to derive such inner approximations of the set of parameter values for which $\Pi(\theta)$ is strictly feasible? First idea: solve the parametric phase-1 type program:

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & t \\ \text{subject to} & F(\theta)x + g(\theta) \preceq_K t\mathbf{1} \end{array}$$

2 Applications

- explicit MPC
- time-optimal point-to-point motion
- trade-off curves
- combined structure control design
  - Similar to LPV control for static parameter, but with different objective (ℓ_1 instead of ℓ_∞).
  - What is the effect of the choice of the objective function ??? Also, what is the value of the ℓ_1 objective for varying parameters (suppose for instance that the worst case ℓ_2 gain gets only slightly larger, but better performance over large parts of the paramter domain)???