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Source: *Journal of the Operations Research Society of America*, Vol. 2, No. 3 (Aug., 1954), pp. 316-319
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/166644
Accessed: 14/05/2014 08:49

PARAMETRIC OBJECTIVE FUNCTION (PART 1)

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(Received February 23, 1954)

In the linear programming problem where there is a linear function to be optimized (called the objective function), it is desirable to study the behaviour of solutions when the (cost) coefficients in the objective function are parametrized. The problem is then to find the set of x_j $(j=1,2,\cdots,n)$ which minimizes the linear form $\sum (c_j+\lambda c_j')\, x_j$ and satisfy the constraints $x_j \geq 0$ and $\sum a_{ij}x_j=a_{i0}$ $(i=1,2,\cdots,m)$ where c_j, c_j', a_{ij}, and a_{i0} are constants, with at least one $c_j \neq 0$, and λ a parameter. Using the simplex method, a computational procedure is described which enables one, given a feasible solution, to determine the values of the parameter (if any) for which the solution minimizes the objective function; and given a minimum feasible solution how one can generate by the simplex algorithm all minimum feasible solutions and the corresponding values of the parameter. To a minimum solution there corresponds one interval of values of λ (non-degeneracy assumed). The process indicates how to obtain a new solution and the corresponding interval of values of λ which is contiguous to the previous one both having only one point in common.

THE LINEAR-PROGRAMMING problem[1] which is being extensively used in a variety of ways in industrial and economic research,[2] has an interesting aspect which is to be investigated in this paper.

The problem briefly stated is as follows: Find values of the x_j $(j=1, 2, \cdots, n)$ which minimize the linear form

$$\sum_{j=1}^{n} c_j x_j, \qquad \text{(objective function)} \quad (1)$$

and satisfy

$$x_j \geq 0, \quad (2)$$

$$\sum_{j=1}^{\infty} a_{ij}x_j = b_i \qquad (i=1,2,\cdots,m) \quad (3)$$

where c_j, a_{ij}, and b_i are constants. The c_j are known as the cost coefficients. They are studied in detail here.

Let us first define a *feasible solution* to the above problem as a set x_j which satisfies (2) and (3) without necessarily yielding a minimum. If in fact (1) is minimized, we then have a *minimum feasible solution*.

It is important, as will be clear later, to seek solutions to the problem not only for fixed c_j, but also when one or more among them are allowed to vary. In other words, we shall show how solutions are obtained when some of the c_j are linear functions of a parameter λ.

There are several methods used to solve the problem. From a practical standpoint the simplex process[3] is favored here.

Now, when all the c_j's are fixed, the simplex process starts out by selecting a basis of m or less column vectors from the $m \times n$ matrix (a_{ij}). These vectors determine a feasible solution to the problem. If any feasible solution of less than m vectors can be found, the problem is said to be degenerate. We omit this important case from our discussion at this time.

In general, once a feasible solution has been determined the procedure is systematic in arriving at a minimum solution. Of course the problem may not have a finite minimum and this can also be detected by a simplex criterion.

Since there is a finite number of combinations (n,m), no more than this number of iterations (i.e., bringing in new vectors to substitute for old ones in a basis which does not lead to a minimum feasible solution) are necessary. Let some of the c_j's be linear functions of a parameter λ. When λ is introduced, the matrix (a_{ij}) from which the basis selection is made remains unchanged. Hence as λ varies over the real numbers only a finite number of solutions leading to minima for all values of λ will be necessary.

It has been possible to show how once a minimum solution is obtained for a fixed value of λ, then in a stepwise fashion all solutions yielding a minimum can be obtained. The values of λ, for which a basis leads to a minimum, form a closed interval. Sometimes it is only a point. All these intervals form a connected set which is a part or all of the real line. Thus once a solution is obtained for a certain value of λ, one can proceed to the right or to the left of that value obtaining a new solution. In the process of doing this, one also determines the interval of values of λ for which this solution is minimal. The new interval is contiguous to the previous one, i.e., both meeting at end points.

If a problem has a matrix with fixed elements (a_{ij}) while the cost coefficients c_j are unstable as linear functions of a parameter, it is now possible to solve the problem and obtain a table of solutions corresponding to all possible values that the c_j may have.

To determine the interval of values of λ corresponding to a minimum solution by the simplex algorithm, we note that in order for a feasible solution to be a minimum we must have $z_j - c_j \leq 0$ for all j. The z_j are defined as follows: if we denote the columns of the (a_{ij}) matrix by $P_1, P_2, \cdots P_n$, and the column vector of the b_i by P_0, and express all vectors P_j in terms of a basis consisting of m ordered vectors $P_1 \cdots P_m$, which form a feasible solution, i.e.,

$$P_j = y_{1j}P_1 + y_{2j}P_2 + \cdots + y_{mj}P_m, \qquad (j=1, 2, \cdots, n)$$

then
$$z_j = y_{1j}c_1 + y_{2j}c_2 + \cdots + y_{mj}c_m.$$

Note that in the last expression some of the z_j's will be linear functions of λ on account of some of the c_j's being so.

Now we can rewrite $z_j - c_j$ which is also a linear function of λ in the contracted form $\alpha_j + \lambda\beta_j$. Clearly, where no λ appears, $\beta_j = 0$.

Assume that we have a minimum feasible solution for $\lambda = \lambda_0$. Then as mentioned above

$$\alpha_j + \lambda_0\beta_j \leq 0. \qquad (j=1, 2, \cdots, n)$$

If $\beta_j = 0$, then we have $\alpha_j \leq 0$. For those j for which the corresponding vector is in the basis, we have $\alpha_j = \beta_j = 0$. From the above we have $\lambda_0 \leq -\alpha_j/\beta_j$ for all j for which $\beta_j > 0$; and $\lambda_0 \geq -\alpha_j/\beta_j$, for all j for which $\beta_j < 0$. The minimum solution for $\lambda = \lambda_0$ is then a minimum for all λ such that

$$\max_{\beta_j<0}(-\alpha_j/\beta_j) \leqq \lambda \leqq \min_{\beta_j>0}(-\alpha_j/\beta_j)$$

Note that if there are no $\beta_j > 0$ then the upper bound is $+\infty$; if there are no $\beta_j < 0$ then the lower bound is $-\infty$.

Let
$$\bar{\lambda} \equiv \min_{\beta_j>0}(-\alpha_j/\beta_j).$$

We shall indicate how to obtain a new minimum feasible solution for $\lambda \geq \bar{\lambda}$ and consequently determine the range of λ corresponding to the new solution. A similar argument applies for

$$\lambda \leq \underline{\lambda} \equiv \max_{\beta_j<0}(-\alpha_j/\beta_j).$$

In passing we may remark that if $\underline{\lambda} = \bar{\lambda}$ the iteration determines a minimum solution for this value of λ.

It can be readily shown that a new solution for $\lambda \geq \bar{\lambda}$ can be obtained by introducing any vector (using the simplex algorithm) for which $\alpha_j + \bar{\lambda}\beta_j = 0$. There will always be at least one. The new solution is a minimum for either $\lambda \geq \bar{\lambda}$ or $\lambda = \bar{\lambda}$. In the first case, the new upper bound can be determined as before. In the second case, we have no new upper bound but the process is again applied until a new solution for a range of λ is obtained. In either case the process is repeated. It is obvious, from the above discussion, that a finite number of minimum solutions for all possible values of λ can be obtained by the simplex procedure in a finite number of steps.

If a vector for which $\alpha_j+\bar{\lambda}\beta_j=0$ cannot be introduced into the basis because all its elements are less than or equal to zero,[3] there are no (finite) minimum solutions for $\bar{\lambda}<\lambda$. A similar argument applies if the vector for which $\alpha_j+\underline{\lambda}\beta_j=0$ cannot be introduced into the basis (i.e., there are no finite minimum solutions for $\lambda<\underline{\lambda}$).

The above procedure has been successfully employed on large electronic computers. A typical example consisted of 33 equations in 50 variables and a set of 19 solutions for all $\lambda \geq 0$ were generated.

Thus in an economy where the (a_{ij}) remain fixed but the costs fluctuate, it is now possible to tabulate optimal strategies (solutions) against ranges of the cost coefficients. Items that are difficult to obtain can be highly priced, and the corresponding solution obtained from the table. The problem can take on greater complexity by expressing each cost coefficient as a linear function of a different parameter. The two-parameter case has been studied. The method of solution and the correspondence between regions of parameter values and solutions do not yield to the direct analysis used above. The time element is prohibitive. A method for generalization is under study.

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