PARAMETRIC OBJECTIVE FUNCTION (PART 2)—GENERALIZATION

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In Part 1,[1] the cost function in the general linear programming problem was parametrized with one parameter and the problem of generating solutions completely studied. In Part 2, a generalization to the n-parameter case is outlined, with special emphasis on the two-parameter case. An example is supplied for this case.

IT WAS shown in Part 1[1] that if one or several cost coefficients c_j in the objective function are replaced by a parameter, then the simplex method enables one to set up a correspondence between intervals of parameter values and bases yielding solutions to the linear programming problem. In fact, the procedure systematically exhausts the intervals, in a stepwise fashion, along with their corresponding solutions.

The Two-Parameter Case

The two-parameter problem is to find the values of x_j $(j=1, \cdots, n)$ that minimize the linear form

$$\sum_1^n (a_j+\lambda_1 b_j+\lambda_2 c_j)\, x_j \qquad (1)$$

and satisfy the conditions

$$x_j \geqq 0, \qquad (j=1, \cdots, n) \qquad (2)$$

$$\sum_{j=1}^n a_{ij}\, x_j = a_{i0}, \qquad (i=1, \cdots, m) \qquad (3)$$

where a_{ij}, a_{i0}, a_j, b_j, c_j are constants and λ_1 and λ_2 are parameters. The geometric interpretation of the problem is left for a separate study which would otherwise take us far afield.

Solution of the Problem

Assume that a basic feasible solution exists and is given. We wish to determine the values of λ_1 and λ_2 (if any) for which the solution minimizes the objective function. By the Simplex algorithm, a feasible solution is a minimum for those (λ_1, λ_2) satisfying the set of inequalities $z_j-(a_j+\lambda_1 b_j+\lambda_2 c_j)\leqq 0$. The z_j are defined as follows: if we denote the columns of the (a_{ij}) matrix by $P_1, P_2, \cdots, P_n$, and the column vector of the a_{i0} by P_0, and express all vectors P_j in terms of a basis consisting of m ordered vectors $P_1, \cdots, P_m$, which form a feasible solution, i.e.,

$$P_j=y_{1j}P_1+y_{2j}P_2+\cdots+y_{mj}P_m, \quad (j=1, 2, \cdots, n)$$

then

$$z_j=y_{1j}d_1+y_{2j}d_2+\cdots+y_{mj}d_m,$$

where the d's are the coefficients in (1). Note that in the last expression some of the z_j's will be linear functions of λ_1 and λ_2.

By substituting for z_j from this expression into the above inequality and grouping the coefficients of λ_1 and λ_2 and the constant terms, there results:

$$\alpha_j+\lambda_1\beta_j+\lambda_2\gamma_j\leqq 0. \qquad (j=1, \cdots, n) \quad (4)$$

For those j for which the corresponding vector is in the basis, $\alpha_j=\beta_j=\gamma_j=0$.

The solution of the two-parameter problem depends on the method employed in solving the set of inequalities (4). We have to determine the convex region in the (λ_1, λ_2) plane whose points satisfy (4). Two methods have been considered: The double descriptive method[2] and the two-dimensional graph of the inequalities. We will exhibit an example solved by the latter process.

The Iterative Process

Assume a minimum feasible solution is given and its corresponding convex region C_1 in the (λ_1, λ_2) plane. (These are termed a characteristic solution and a characteristic region respectively. The sides of the characteristic region are called characteristic boundaries.) We wish to generate by the Simplex process a set of characteristic solutions and their associated characteristic regions for all possible combinations of the parameters. The two-parameter problem is completed when we have examined all feasible solutions and determined the characteristic solutions, or when, to every point in the (λ_1, λ_2) plane, we have either determined a corresponding characteristic solution or shown for what points the problem is unbounded. The number of basic feasible solutions is finite, hence the number of characteristic regions is finite, each having a finite number of line segments for boundary.

The convex region C_1 is determined by the l inequalities (halfspaces)

$$Q_j=\alpha_j+\lambda_1\beta_j+\lambda_2\gamma_j\leqq 0, \quad (\lambda_1,\lambda_2)\in C_1, \quad (j=1, 2, \cdots, \nu, \cdots, l)$$

while the remaining inequalities

$$Q_k=\alpha_k+\lambda_1\beta_k+\lambda_2\gamma_k\leqq 0, \qquad (\lambda_1, \lambda_2)\in C_1, \qquad (k=l+1, \cdots, n)$$

are superfluous.

(a) If at least one element h_{iv} in the vth column (see example below) of the matrix expressing the remaining vectors as linear combinations of the given feasible basis, is greater than zero, we can introduce the vector A_v into the basis and obtain a new feasible solution which will be a minimum for at least those points of Q_v which bound C_1 (i.e., one of the characteristic boundaries of C_2 will be determined by Q_v and the vertices of C_1 lying on this boundary will also be vertices of C_2). Similarly, if the vectors associated with the other boundaries of C_1 can be introduced into the basis, a new characteristic solution will be determined for at least those points on the boundary (non-degeneracy assumed).

(b) If all $h_{iv} \leqq 0$, then there are no characteristic solutions for (λ_1, λ_2) in the halfspace $\alpha_v + \lambda_1\beta_v + \lambda_2\gamma_v > 0$.

It is clear that proceeding from one convex C_i to C_{i+1} in the manner suggested, we may reach an impasse without having considered all possible bases and their associated regions. Various methods have been suggested, but they have not proved computationally feasible. Definitions and theorems analogous to those in ref. 1 apply to the two-parameter problem and can be generalized to the n-parameter case.

If degeneracy occurs, we may generate a set of characteristic solutions whose characteristic regions have a region of the (λ_1, λ_2) plane in common. In the non-degenerate situation, two characteristic regions can have at most a boundary in common.

Example

$$\text{(a)} \quad \begin{cases} x_1 \quad - x_4 \qquad -2x_6 = 5 \\ x_2 + 2x_4 - 3x_5 + x_6 = 3 \\ x_3 + 2x_4 - 5x_5 + 6x_6 = 5 \end{cases}$$

$$x_1 + \lambda_1 x_2 + \lambda_2 x_3 = \text{minimum.}$$

I: EQUATIONS (a) IN SIMPLEX TABLEAU

Basis		$c_J=$	1	λ_1	λ_2	0	0	0
	A_0		A_1	A_2	A_3	A_4	A_5	A_6
A_1	5	(1)	1	0	0	-1	0	-2
A_2	3	(λ_1)	0	1	0	2	-3	1
A_3	5	(λ_2)	0	0	1	2	-5	6
h_{m+1}	5		0	0	0	-1	0	-2
h_{m+2}	3		0	0	0	2	-3	1
h_{m+3}	5		0	0	0	2	-5	6

II: VECTOR A_4 ELIMINATED VECTOR A_2

Basis		$c_J=$	1	λ_1	λ_2	0	0	0
	A_0		A_1	A_2	A_3	A_4	A_5	A_6
A_1	$13/2$	(1)	1	$1/2$	0	0	$-3/2$	$-3/2$
A_4	$3/2$	(0)	0	$1/2$	0	1	$-3/2$	$1/2$
A_3	2	(λ_2)	0	-1	1	0	-2	5
	$13/2$		0	$1/2$	0	0	$-3/2$	$-3/2$
	0		0	-1	0	0	0	0
	2		0	-1	0	0	-2	5

III: VECTOR A_6 ELIMINATED VECTOR A_3

A_1	$71/10$	(1)	1	$1/5$	$3/10$	0	$-21/10$	0
A_4	$13/10$	(0)	0	$3/5$[a]	$-1/10$	1	$-13/10$	0
A_6	$2/5$	(0)	0	$-1/5$	$1/5$	0	$-2/5$	1
	$71/10$		0	$1/5$	$3/10$	0	$-21/10$	0
	0		0	-1	0	0	0	0
	0		0	0	-1	0	0	0

[a] This fraction should be understood as circled.

IV: VECTOR A_2 ELIMINATED VECTOR A_4

A_1	$20/3$	(1)	1	0	$1/3$	$-1/3$	$-5/3$	0
A_2	$13/6$	(λ_1)	0	1	$-1/6$	$5/3$	$-13/6$	0
A_6	$5/6$	(0)	0	0	$1/6$	$1/3$	$5/6$	1
	$20/3$		0	0	$1/3$	$-1/3$	$-5/3$	0
	$13/6$		0	0	$-1/6$	$5/3$	$-13/6$	0
	0		0	0	-1	0	0	0

Step I is the system (a) in the usual Simplex tableau. No artificial vectors are needed. The first basis is (A_1, A_2, A_3) and the corresponding value of the objective function is $5+3\lambda_1+5\lambda_2$. The points in the (λ_1, λ_2) plane for which this solution is a minimum are determined by plotting the $[z_j-(a_j+\lambda_1 b_j+\lambda_2 c_j)]$'s of those vectors not in the basis. Here we plot the inequalities

$$-1+2\lambda_1+2\lambda_2\leqq 0, \qquad -3\lambda_1-5\lambda_2\leqq 0, \qquad -2+\lambda_1+6\lambda_2\leqq 0,$$

and determine the convex region that is defined by them. This is the triangle C_1 in Fig. 1. To obtain the next solution (Step II) we introduced vector A_4 into the basis. Vector A_6 would also have taken us into a new solution, but vector A_5 could not have been used as all $h_{i5}\leqq 0$. Therefore, the points which satisfy the inequality $3\lambda_1+5\lambda_2<0$ do not enter into the solution of the problem. The second solution has $13/2+2\lambda_2$ as the value of the objective function and it is a minimum for those points in the open region C_2. Here the minimum is also unbounded for all points in the region defined by $-3/2-2\lambda_2<0$. Introducing A_6 into the basis, gives $71/10$ as the value of the objective function (Step III) and the new solution is a minimum for the region C_3. Step IV brings us to the final characteristic solution and corresponding characteristic region C_4. The final value of the objective function is $20/3+13/6\lambda_1$. The minimum is also unbounded for those points satisfying the inequality $-5/3-13/6\lambda_1<0$.

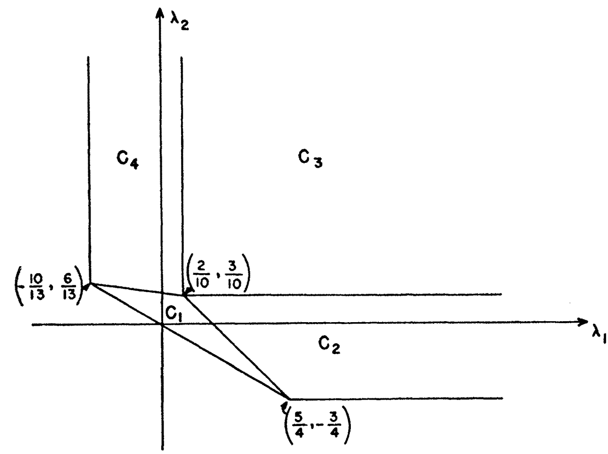

Figure 1

The Mesh Approach

There follows a remark on the use of the 1-dimensional case to construct 2-dimensional parameter regions. It is clear that when one parameter, say λ_2, is fixed at a constant value $\lambda_2=\lambda_2^0$ while λ_1 is allowed to vary, one is faced with solving a single parameter problem. Hence it is possible to generate the characteristic intervals corresponding to the characteristic solutions for the line $\lambda=\lambda_2^0$. These intervals have as boundary the intersection points of this line with the boundary of the mesh generated in the (λ_1, λ_2) plane by the convex parameter regions corresponding to minimum solutions.

It is natural to inquire whether it is possible to construct the (λ_1, λ_2) mesh using the one parameter procedure. We shall briefly point out how to generate a characteristic region by this method.

Consider the two lines $\lambda_2=\lambda_2^0$ and $\lambda_2=\lambda_2^0\pm\epsilon$ $(0<\epsilon\leqq 1)$. Generate two characteristic intervals for these two lines corresponding to the same characteristic solution. There is no loss in generality in assuming that the end points are all finite. Join neighboring end points with the lines A and B as in Fig. 2. It is clear that A and B lie on the boundary of the desired characteristic region. It is now possible to finish the region by varying the value of λ_2^0. When an interval appears whose end points do not lie on either A or B, one now considers a nearby value of λ_2 and, as above, draw the line joining corresponding end points. This process will terminate in a finite number of steps since the numbers of sides or vertices of any (λ_1, λ_2) region is finite.

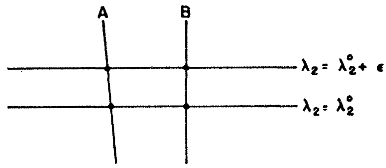

Figure 2

Generalization

The above discussion for the two-parameter case can be immediately extended to the multi-parameter case. However, one soon realizes the immensity of the problem, being required to solve by the double descriptive method a set of linear inequalities, in the parameters, at each iteration of the Simplex solution of the problem. Nevertheless, this is so far the only available technique for solving the problem. Most problems are sufficiently difficult to require the use of electronic computers.

It has been shown that the characteristic regions in n-parameter space are convex. Their totality forms a connected set, and fixing k of them is geometrically equivalent to piercing the n-dimensional mesh with an $(n-k)$-dimensional hyperplane.

The problem is now complete with one exception. Unlike the single-parameter case, the entire set of regions cannot be obtained in a systematic manner. When a region of parameter values corresponding to a solution has been generated, neighboring regions are generated by crossing one bounding hyperplane at a time.

For most practical purposes, when the problem is solved for a fixed value of the parameter, one is interested in the corresponding characteristic region and in those in its immediate neighborhood. If the entire mesh is desired, one must pay in time in exchange for the gain in generality.

Finally, it can be shown that the solution of a zero-sum two-person game, with each element of one row of its payoff matrix replaced by a parameter, can be obtained from the parametric linear programming study outlined above.[3]

REFERENCES

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2. *Symposium on Linear Inequalities and Programming,* Project Scoop, Publication No. 10 (1952).
3. S. I. Gass, "A Zero-Sum Two-Person Game and Its Equivalent Parametric Linear Programming Problem," an Air Force publication.