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Simplified Approaches to Polynomial Design of Model Predictive Controllers

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***Abstract*— Two approaches are proposed for obtaining a regularly partitioned piecewise polynomial that approximates the optimal model predictive controller. They are advantageous to obtaining the optimal model predictive controller in that they do not require geometric computation, which is problematic when the predicted horizon is long, the plant is of high order, or the piecewise structure is nearly degenerate. In each of the approaches, the design is formulated as a robust optimization problem, which is solved with the sum-of-squares method. They are improvement of the approach previously proposed by the same author and can be performed with smaller computational cost. Between the proposed two approaches, the first one requires smaller computational cost while the second one gives evaluation of the quality of approximation. A numerical example is provided for illustration.**

***Keywords*—model predictive control, robust optimization problem, sum-of-squares method, computational cost**

I. Introduction

Model predictive control is one of the control methods most widely used in practice. In particular, the approach of Bemporad–Morari–Dua–Pistikopoulos [2] broadened its scope by the offline computation of the optimal control law as a piecewise affine function of the state of the plant. A problem here is that one has to compute the piecewise structure with geometric computation. Indeed, the piecewise structure may include many pieces, whose number is potentially exponential with respect to the length of the horizon. The piecewise structure is irregular in general and its computation becomes numerically difficult when it is nearly degenerate. Moreover, its computation requires special care and technique when the state space has a high dimension. In order to circumvent these difficulties, several authors proposed the use of a suboptimal control law described by a regularly partitioned piecewise affine function [1], [7], [8], [3] or a regularly partitioned piecewise polynomial [9].

In this paper, we present two approaches toward design of a suboptimal model predictive control law with a regularly partitioned piecewise polynomial. They are free from the mentioned problems on the irregular partition and are more flexible than the approaches with piecewise affine functions. An approach with a regularly partitioned piecewise polynomial has been presented by Kvasnica–Löfberg–Fikar [9]. An advantage of the present approaches is that they do not need to compute the optimal control law beforehand and thus do not require problematic geometric computation. The proposed two approaches are improvement of the approach of [13] given by the same author. All of these approaches are based on the same idea, that is, reducing a design of a model predictive controller to a robust optimization problem, which can be solved with the sum-of-squares method. The present two approaches are, however, computationally less demanding in that the sum-of-squares method produces smaller-sized matrix inequalities. Between these two approaches, the first one requires smaller computational cost than the second one. However, the second approach has an advantage that the distance to optimality can be estimated, which can be taken into account in the design process. A numerical example is presented for illustration.

This paper is organized as follows. Section II gives the problem to be solved. Sections III and IV present the proposed two approaches: the primal approach and the duality-gap approach, respectively. Section V is for comparison with the approach of [13]. Section VI provides a numerical example and Section VII concludes the paper.

The following notation is used. The symbols O and I stand for the zero matrix and the identity matrix, respectively, of appropriate size. The symbol $^{\mathrm{T}}$ expresses the transpose of a matrix or a vector. For a vector u, the inequalities $u > 0$ and $u \geq 0$ express the elementwise positivity and nonnegativity of u, respectively. The inequalities $u > v$ and $u \geq v$ are equivalent to $u - v > 0$ and $u - v \geq 0$, respectively. For a symmetric matrix Q, the inequalities $Q \succ O$ and $Q \succeq O$ stand for the positive definiteness and the positive semidefiniteness of Q, respectively. For a real number a, the symbol $\lceil a \rceil$ designates the smallest integer larger than or equal to a.

II. Problem

We consider control of a linear discrete-time plant

$$x(t+1) = Ax(t) + Bu(t) \quad (t = 0, 1, 2, \ldots),$$

which is subject to the constraints

$$u(t) \in U_0, \quad x(t) \in X_0$$

for $t = 0, 1, 2, \ldots$. Here U_0 and X_0 are given convex polytopes. The dimensions of the state $x(t)$ and the input $u(t)$ are denoted by p and p_u, respectively. In the model predictive control, the input $u(t)$ at the time t is computed by the following procedure. We first measure $x(t)$ and solve the following optimization problem parametrized by x with substituting $x := x(t)$:

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$$
\begin{aligned}
O_x: \quad & \text{minimize} \\
& \qquad \sum_{k=0}^{N-1}\Big(\frac{1}{2}x_k^{\mathrm{T}}Qx_k+\frac{1}{2}u_k^{\mathrm{T}}Ru_k\Big)+\frac{1}{2}x_N^{\mathrm{T}}Q_{\mathrm{f}}x_N \\
& \text{subject to} \\
& \quad x_0 = x, \\
& \quad x_{k+1} = Ax_k + Bu_k \quad (k=0,1,\dots,N-1), \\
& \quad u_k \in U_0 \quad (k=0,1,\dots,N-1), \\
& \quad x_k \in X_0 \quad (k=0,1,\dots,N-1), \\
& \quad x_N \in X_{\mathrm{f}}.
\end{aligned}
$$

In this problem, $Q \succeq O$, $R \succ O$, and $Q_{\mathrm{f}} \succeq O$ are given symmetric matrices and X_{f} is a given convex polytope. Then, u_0 in the optimal solution is applied as the input $u(t)$. In the following, we make the natural assumption that U_0, X_0, and X_{f} contain the origin in their interiors.

Repeated substitution of $x_{k+1} = Ax_k + Bu_k$ makes the problem O_x to have only $(u_0^{\mathrm{T}} \;\; u_1^{\mathrm{T}} \;\; \cdots \;\; u_{N-1}^{\mathrm{T}})^{\mathrm{T}}$ as the optimization variable. Moreover, some affine transformation of $(u_0^{\mathrm{T}} \;\; u_1^{\mathrm{T}} \;\; \cdots \;\; u_{N-1}^{\mathrm{T}})^{\mathrm{T}}$ to z gives the following standard quadratic programming problem:

$$
\begin{aligned}
P_x: \quad & \text{minimize} && \frac{1}{2}z^{\mathrm{T}}Hz \\
& \text{subject to} && Gz \le f + Fx.
\end{aligned}
$$

The inequality $Gz \le f + Fx$ is a concatenation of all the constraints on z. We write the dimension of z as $n(= Np_u)$ and that of f as $m(= N(m_u + m_x) + m_{\mathrm{f}})$, where m_u, m_x, and m_{f} are the numbers of inequalities necessary for the description of U_0, X_0, and X_{f}, respectively. Due to the assumptions on O_x, we have $H \succ O$ and $f > 0$. We also make the additional assumption that the problem P_x as well as the original problem O_x is considered only for $x \in X$ with X being a closed convex polytope in $\mathbb{R}^p$ having the origin in its interior.

Bemporad–Morari–Dua–Pistikopoulos [2] showed that the optimal solution of P_x is a piecewise affine function of x. Once this function is explicitly obtained for all $x \in X$, the input $u(t)$ is immediately computable from the current state $x(t)$. This approach requires however geometric computation, which has the mentioned drawbacks. Motivated by this observation, we will present two approaches that do not require geometric computation.

III. Primal Approach

A. Special case

We present in this section the first approach of ours, which is called a primal approach because it does not use the duality concept. In the approach, a (regular) partition of the domain X is supposed to be given and a piecewise polynomial defined for that partition is looked for so that it gives a suboptimal solution of P_x for each $x \in X$. To simplify the discussion, we first focus on the special case that the partition consists of only one subpolytope, that is, the domain X itself. The general case will be considered in Subsection III-B.

Suppose that $z(x)$ is a polynomial in x having some given degree $d \ge 1$. We choose its coefficients so as to solve the following optimization problem:

$$
\begin{aligned}
S_1: \quad & \text{minimize} && \int_X \frac{1}{2}z(x)^{\mathrm{T}}Hz(x)\,\mathrm{d}x \\
& \text{subject to} && Gz(x) \le f + Fx \quad (\forall x \in X).
\end{aligned}
$$

This is a robust optimization problem because the constraint has to be satisfied for all the possible values of $x \in X$. If a polynomial $z(x)$ is feasible in this problem, it gives a feasible solution of P_x for each $x \in X$. Moreover, if the degree d is high enough, the optimal solution of S_1 is expected to give a nearly optimal solution for each $x \in X$. This is because the integral is monotone with respect to the pointwise comparison of the integrant function. That is, if $(1/2)z_1(x)^{\mathrm{T}}Hz_1(x) \le (1/2)z_2(x)^{\mathrm{T}}Hz_2(x)$ $(\forall x \in X)$ for some polynomials $z_1(x)$ and $z_2(x)$, we have $\int_X (1/2)z_1(x)^{\mathrm{T}}Hz_1(x)\,\mathrm{d}x \le \int_X (1/2)z_2(x)^{\mathrm{T}}Hz_2(x)\,\mathrm{d}x$.

We reduce the problem S_1 to a solvable problem in two steps: transformation of the objective function and replacement of the constraint by its sufficient condition.

The transformation of the objective function is performed as follows. Recalling that $z(x)$ is a polynomial of degree d, we express it as $z(x) = Z\mu(x)$, where Z is a matrix consisting of the coefficients of $z(x)$ and $\mu(x)$ is the vector consisting of all the monomials of x whose degrees are less than or equal to d. The dimension of the vector $\mu(x)$ is $\binom{d+p}{p}$. The matrix Z has n rows and $\binom{d+p}{p}$ columns. Substitution of $z(x) = Z\mu(x)$ into the objective function gives

$$
\begin{aligned}
&\int_X \frac{1}{2}z(x)^{\mathrm{T}}Hz(x)\,\mathrm{d}x \\
&\quad = \int_X \frac{1}{2}\mu(x)^{\mathrm{T}}Z^{\mathrm{T}}HZ\mu(x)\,\mathrm{d}x \\
&\quad = \frac{1}{2}\operatorname{tr}\Big(H^{1/2}Z\int_X \mu(x)\mu(x)^{\mathrm{T}}\,\mathrm{d}x Z^{\mathrm{T}}H^{1/2}\Big).
\end{aligned}
$$

Here, $H^{1/2}$ is the matrix square root of H, which is positive definite. In the last expression, we write the integral $\int_X \mu(x)\mu(x)^{\mathrm{T}}\,\mathrm{d}x$ as Ξ, which is a positive semidefinite matrix. When the domain X is a hyperrectangle, as is often the case in practice, the elements of Ξ are easily computable. When X is a general polytope, its elements can be computed with the Monte-Carlo integration.

Now the minimization of $\int_X (1/2)z(x)^{\mathrm{T}}Hz(x)\,\mathrm{d}x = (1/2)\operatorname{tr}(H^{1/2}Z\Xi Z^{\mathrm{T}}H^{1/2})$ is equivalent to the minimization of $(1/2)\operatorname{tr}V$ subject to the constraint $H^{1/2}Z\Xi Z^{\mathrm{T}}H^{1/2} \preceq V$ or equivalently

$$
\begin{pmatrix} V & H^{1/2}Z\Xi \\ \Xi Z^{\mathrm{T}}H^{1/2} & \Xi \end{pmatrix} \succeq O.
$$

This constraint is tractable in the framework of semidefinite programming because it is affine in the matrix variables Z and V. The left-hand side matrix has $n + \binom{d+p}{p}$ rows and columns.

Next, the replacement of the constraint is considered. The constraint $Gz(x) \leq f + Fx$ $(\forall x \in X)$ is difficult to handle because it has to be satisfied for infinitely many values of x. We use the sum-of-squares method, which has made a remarkable progress in this decade [14], [10], [5], [15], [11], to obtain a sufficient condition of the constraint. Since this sufficient condition has no parameter x, its use in the problem S_1 in place of the original constraint makes the problem solvable with the standard method. Although the resulting problem is only an approximation of S_1, its solution is feasible also in S_1 and thus can be safely used for model predictive control. Moreover, the quality of approximation can be improved arbitrarily at the cost of increasing complexity of the sufficient condition.

Let us briefly see how the sufficient condition is obtained. Suppose that the domain X is expressed as $\{x \in \mathbb{R}^p \,|\, r_1(x) \geq 0,\ r_2(x) \geq 0,\ \ldots,\ r_\ell(x) \geq 0\}$ with scalar polynomials $r_1(x)$, $r_2(x)$, …, $r_\ell(x)$ whose degrees are d_1, d_2, …, d_ℓ, respectively. Then, the condition $f + Fx - Gz(x) \geq 0$ $(\forall x \in X)$ holds if the vector polynomial

$$f+Fx-Gz(x)-s_1(x)r_1(x)-s_2(x)r_2(x)-\cdots-s_\ell(x)r_\ell(x) \tag{1}$$

is nonnegative all over $\mathbb{R}^p$ for some vector polynomials $s_1(x)$, $s_2(x)$, …, $s_\ell(x)$ that are again nonnegative all over $\mathbb{R}^p$. We choose the degrees of $s_1(x)$, $s_2(x)$, …, $s_\ell(x)$ as $2(D - \lceil d_1/2\rceil)$, $2(D - \lceil d_2/2\rceil)$, …, $2(D - \lceil d_\ell/2\rceil)$, respectively, with some positive integer D larger than or equal to $d/2$, $d_1/2$, $d_2/2$, …, $d_\ell/2$, where d is the degree of $z(x)$. Then the polynomial (1) has the degree $2D$ at most. This polynomial is nonnegative all over $\mathbb{R}^p$ if each of its m elements is expressed as the sum of squares of polynomials and this sufficient condition is expressed as positive semidefiniteness of some matrix having $\binom{D+p}{p}$ rows and columns. Similarly, for each $i = 1, 2, \ldots, \ell$, nonnegativity of each element of $s_i(x)$ is expressed as positive semidefiniteness of a matrix having $\binom{D-\lceil d_i/2\rceil+p}{p}$ rows and columns. Thus, we obtain a sufficient condition for the constraint $Gz(x) \leq f + Fx$ $(\forall x \in X)$ in terms of positive semidefiniteness of appropriate matrices. These matrices are independent of x and are affine in the coefficients of $z(x)$. Hence, the use of this sufficient condition in S_1 gives a semidefinite programming problem solvable with the standard interior-point method. Moreover, the sufficient condition can be made arbitrarily tight by the increase of the positive integer D.

In Table I, we summarize the size of the matrix constraints to be handled. Here, the matrix constraint having n rows and columns is said to have the size n. In the leftmost column of the table, the third row indicates the constraint required in the transformation of the objective function. The first row is for the constraints composing the considered sufficient condition. Note that there are m sets of constraints because the original constraint $Gz(x) \leq f + Fx$ is m-dimensional. The table has the entries also for the duality-gap approach and the approach of [13], which will be discussed later. We can see that the constraints of the present approach are smallest both in number and size.

B. General case

In the general case that the domain X is partitioned into several convex polytopes, we apply the procedure discussed so far to each subpolytope. Since the procedure is independent for each subpolytope, the computational cost depends only linearly on the number of subpolytopes. This is a favorable property because we are often required to increase the number of subpolytopes for improvement of the solution to be obtained. One may notice that the resulting control law can have discontinuity on the boundary of the subpolytopes. However, when the quality of approximation is sufficiently good, this discontinuity should be small and have a limited effect in the discrete-time dynamics.

A drawback in the present approach is that the quality of approximation is not clear from the solution of the approximation problem. An easy way for evaluation of the quality is to solve P_x for several x sampled from X and to compare the solution with the obtained polynomial (See Section VI). It would be convenient, however, if we can evaluate the quality of approximation as a byproduct of a solution of an approximation problem. In the next section, such an approach will be presented.

IV. Duality-Gap Approach

We present in this section our second approach that gives evaluation of the quality of approximation. The idea is to consider not only the problem P_x but also its dual and to use the duality gap as the objective function. Since the duality gap is an upper bound on the distance to optimality, it is reasonable to minimize for good approximation. Also, the attained minimal value indicates the quality of approximation.

We again begin with the special case that the partition consists of the whole X.

We fix $x \in X$ for the time being and investigate the corresponding problem P_x. As is known as the Karush–Kuhn–Tucker condition, a necessary and sufficient condition for $z \in \mathbb{R}^n$ to become an optimal solution of P_x is the existence of $\lambda \in \mathbb{R}^m$ such that

$$Gz \leq f+Fx,\ \ \lambda \geq 0,\ \ Hz+G^{\mathrm{T}}\lambda = 0,\ \ \lambda^{\mathrm{T}}(f+Fx-Gz) = 0$$

[6, p. 340] [4, p. 366]. By solving the first equality as $z = -H^{-1}G^{\mathrm{T}}\lambda$ and substituting it to the remaining relations, we reach the problem:

$$\begin{aligned} &\text{minimize} && \lambda^{\mathrm{T}}(f + Fx + GH^{-1}G^{\mathrm{T}}\lambda) \\ &\text{subject to} && -GH^{-1}G^{\mathrm{T}}\lambda \leq f + Fx, \quad \lambda \geq 0. \end{aligned}$$

In fact, the same problem can be derived by considering the dual problem of P_x and minimizing the duality gap. Indeed, the vector λ is the dual variable in this derivation and $\lambda^{\mathrm{T}}(f+Fx+GH^{-1}G^{\mathrm{T}}\lambda)$ is the duality gap between the dual objective value attained by λ and the primal objective value attained by $z = -H^{-1}G^{\mathrm{T}}\lambda$. Since the duality gap is an upper bound on the distance to the optimal value, its minimization gives an optimal solution of P_x.

We now allow x to move in X. We suppose λ to be a polynomial in x of some given degree $d \geq 1$ and consider the following problem:

$$
\begin{aligned}
S_2: \quad & \text{minimize} && \int_X \lambda(x)^{\mathrm{T}}[f + Fx + GH^{-1}G^{\mathrm{T}}\lambda(x)]\,\mathrm{d}x \\
& \text{subject to} && -GH^{-1}G^{\mathrm{T}}\lambda(x) \leq f + Fx \quad (\forall x \in X), \\
& && \lambda(x) \geq 0 \quad (\forall x \in X).
\end{aligned}
$$

If a polynomial $\lambda(x)$ is feasible in this problem, the polynomial $z(x) = -H^{-1}G^{\mathrm{T}}\lambda(x)$ gives a feasible solution of P_x for each $x \in X$. Moreover, the attained objective value gives an upper bound on the distance to optimality integrated over X. It is hence reasonable to solve this problem S_2. Indeed, our approach presented in this section is to solve S_2 with some modification similar to that of S_1, that is, transformation of the objective function and replacement of the constraint.

The transformation of the objective function proceeds as follows. We write $\lambda(x) = \Lambda\mu(x)$ with the $\binom{d+p}{p}$-dimensional monomial vector $\mu(x)$, which has appeared in the primal approach, and a coefficient matrix Λ having m rows and $\binom{d+p}{p}$ columns. With this notation, we can transform the objective function as

$$
\begin{aligned}
&\int_X \lambda(x)^{\mathrm{T}}[f + Fx + GH^{-1}G^{\mathrm{T}}\lambda(x)]\,\mathrm{d}x \\
&\quad = \int_X \big[\mu(x)^{\mathrm{T}}\Lambda^{\mathrm{T}}f + \mu(x)^{\mathrm{T}}\Lambda^{\mathrm{T}}Fx + \\
&\qquad\qquad\qquad \mu(x)^{\mathrm{T}}\Lambda^{\mathrm{T}}GH^{-1}G^{\mathrm{T}}\Lambda\mu(x)\big]\,\mathrm{d}x \\
&\quad = \xi^{\mathrm{T}}\Lambda^{\mathrm{T}}f + \mathrm{tr}(\tilde{\Xi}\Lambda^{\mathrm{T}}F) + \mathrm{tr}(H^{-1/2}G^{\mathrm{T}}\Lambda\Xi\Lambda^{\mathrm{T}}GH^{-1/2}).
\end{aligned}
$$

Here, ξ^{T}, $\tilde{\Xi}$, and Ξ are defined by integration of $\mu(x)^{\mathrm{T}}$, $x\mu(x)^{\mathrm{T}}$, and $\mu(x)\mu(x)^{\mathrm{T}}$, respectively, over X. Minimization of the last expression can be performed by minimization of

$$\xi^{\mathrm{T}}\Lambda^{\mathrm{T}}f + \mathrm{tr}(\tilde{\Xi}\Lambda^{\mathrm{T}}F) + \mathrm{tr}\,V$$

subject to

$$\begin{pmatrix} V & H^{-1/2}G^{\mathrm{T}}\Lambda\Xi \\ \Xi\Lambda^{\mathrm{T}}GH^{-1/2} & \Xi \end{pmatrix} \succeq O.$$

The left-hand side matrix of the last expression has $n+\binom{d+p}{p}$ rows and columns.

We replace the two constraints of S_2 with their sufficient conditions using the sum-of-squares method. We omit the details because the procedure is similar to that of the primal approach. As in the primal approach, suppose that the domain X is described by the polynomials $r_1(x), r_2(x), \ldots, r_\ell(x)$ of degree $d_1, d_2, \ldots, d_\ell$, respectively. Then, the resulting sufficient conditions are expressed by positive semidefiniteness of $2m$ sets of $\ell+1$ matrices having the sizes $\binom{D+p}{p}$, $\binom{D-\lceil d_1/2\rceil+p}{p}$, …, $\binom{D-\lceil d_\ell/2\rceil+p}{p}$, respectively. The condition becomes tighter as the positive integer D increases.

With the modification above, the problem S_2 is converted to a solvable semidefinite programming problem. The sizes of the matrix constraints are summarized in Table I. Although the number of constraints is larger in the present approach than in the primal approach, the factor is only two. This disadvantage can be compensated by its advantage that the quality of approximation can be evaluated.

In the general case that the domain X is partitioned into multiple subpolytopes, a piecewise polynomial $\lambda(x)$ can be obtained by repetitive application of the procedure above. Just like the primal approach, the computational cost increases only linearly with the number of subpolytopes.

V. Comparison with the Previous Approach

In [13], the present author proposed a different approach, where the following optimization problem is considered in place of S_1 or S_2:

$$
\begin{aligned}
T: \quad & \text{minimize} \quad v \\
& \text{subject to} \\
& \quad -GH^{-1}G^{\mathrm{T}}\lambda(x) \leq f + Fx \quad (\forall x \in X), \\
& \quad \lambda(x) \geq 0 \quad (\forall x \in X), \\
& \quad \lambda(x)^{\mathrm{T}}[f + Fx + GH^{-1}G^{\mathrm{T}}\lambda(x)] \leq v \quad (\forall x \in X).
\end{aligned}
$$

The problem T is approximately solvable using the sum-of-squares method like S_1 and S_2. Although the problem T resembles the problem S_2, it is different in that the maximum of the duality gap is minimized in place of its integral.

The approach of [13] has its own advantage. Specifically, it can be extended so as to guarantee stability of the resulting closed-loop system. This extension utilizes the property of T that the maximum of the duality gap is minimized. The corresponding extension looks difficult with the present two approaches, where the integral is minimized.

On the other hand, the approach of [13] looks inferior to the present approaches on the computational cost. Comparison is found in Table I. In the first and the second rows, the matrix inequalities of T are comparable in number and size with those of the other two approaches. However, in the third row, the approach of [13] has $\ell+1$ matrix inequalities of size $(n+1)\times\binom{D+p}{p}$ and so on. Recall that n is equal to Np_u and $\binom{D+p}{p}$ is often a large positive integer. Hence, the above inequalities rapidly grow in size as the length of the horizon, N, increases. Note that the present approaches have only one inequality in the third row and its size is $n+\binom{d+p}{p}$, whose growth is much slower.

VI. Example

We applied the proposed two approaches together with the approach of [13] to a plant taken from Section 7.1 of [2], that is, $x(t+1) = Ax(t) + Bu(t)$ with

$$A = \begin{pmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{pmatrix}, \quad B = \begin{pmatrix} 0.0609 \\ 0.0064 \end{pmatrix},$$

which is subject to the constraint $-2 \leq u(t) \leq 2$ ($t = 0, 1, 2, \ldots$). The objective function in the original problem O_x was chosen with $Q = I$, $R = 0.01$, and Q_{f} being the solution of $Q_{\mathrm{f}} - A^{\mathrm{T}}Q_{\mathrm{f}}A = Q$. The domain X in the state space was set to $[-1,\ 1]^2$. All the semidefinite programming problems were solved with SeDuMi [16] through the modeling language YALMIP [12]. The used computer was

TABLE I

THE SIZE OF THE MATRIX INEQUALITIES REQUIRED BY THE THREE APPROACHES

	primal approach	duality-gap approach	approach of [13]
$Gz \le f + Fx$	m sets of $\ell+1$ ineqs. of size $\binom{D+p}{p}$, $\binom{D-\lceil d_1/2\rceil+p}{p}$, $\ldots$, $\binom{D-\lceil d_\ell/2\rceil+p}{p}$, resp.	m sets of $\ell+1$ ineqs. of size $\binom{D+p}{p}$, $\binom{D-\lceil d_1/2\rceil+p}{p}$, $\ldots$, $\binom{D-\lceil d_\ell/2\rceil+p}{p}$, resp.	m sets of $\ell+1$ ineqs. of size $\binom{D+p}{p}$, $\binom{D-\lceil d_1/2\rceil+p}{p}$, $\ldots$, $\binom{D-\lceil d_\ell/2\rceil+p}{p}$, resp.
$\lambda \ge 0$	—	m sets of $\ell+1$ ineqs. of size $\binom{D+p}{p}$, $\binom{D-\lceil d_1/2\rceil+p}{p}$, $\ldots$, $\binom{D-\lceil d_\ell/2\rceil+p}{p}$, resp.	m sets of $\ell+1$ ineqs. of size $\binom{D+p}{p}$, $\binom{D-\lceil d_1/2\rceil+p}{p}$, $\ldots$, $\binom{D-\lceil d_\ell/2\rceil+p}{p}$, resp.
others	one ineq. of size $n+\binom{d+p}{p}$	one ineq. of size $n+\binom{d+p}{p}$	$\ell+1$ ineqs. of size $(n+1)\times\binom{D+p}{p}$, $(n+1)\times\binom{D-\lceil d_1/2\rceil+p}{p}$, $\ldots$, $(n+1)\times\binom{D-\lceil d_\ell/2\rceil+p}{p}$, resp.

equipped with the Intel Core2 Duo P8800 CPU (2.66 GHz) and 4 GByte memory.

First, the length of the horizon was set as $N = 2$ and the three approaches were applied with various values of the degree d of the polynomials $z(x)$ and $\lambda(x)$ and various partitions of the domain X. The number D, which appears in the sum-of-squares method, was set to $d/2$. Table II shows the results of the duality-gap approach, where the degree d was set to 6, 8, and 10 and the domain X was not partitioned. It is seen that the optimal value became smaller as the degree d increased. This is understandable because the polynomials $z(x)$ and $\lambda(x)$ obtain more degree of freedom with a larger d. Table III presents the results of the same approach, where the degree d was fixed to 6 and the domain X was partitioned into 2 subpolytopes and 4 subpolytopes, respectively. In the first case, the partition consisted of $[-1,\ 0]\times[-1,\ 1]$ and $[0,\ 1]\times[-1,\ 1]$. In the second case, it consisted of $[-1,\ 0]^2$, $[-1,\ 0]\times[0,\ 1]$, $[0,\ 1]\times[-1,\ 0]$, and $[0,\ 1]^2$. We can see that the partition of X was more effective than the increase of d in the sense that it improved more the optimal value with shorter computational time. In the last case with 4 subpolytopes, we directly computed the distance to optimality integrated over X. Namely, we solved the pointwise problem P_x for 100 grid points in X, averaged the obtained optimal values, and multiplied the result by the area of X. The computed value was 0.021, which is reasonably close to the corresponding optimal value in the table, 0.038. It is natural that the optimal value in the table is larger because it includes the distance between the optimal value and the dual objective value.

The primal approach and the approach of [13] gave more or less similar results. Namely, the partition to the 4 subpolytopes gave the best result and the integrated distance to optimality computed from the 100 grid points was 0.015 and 0.035, respectively. Note that the primal approach by itself does not give the estimate of the distance to optimality and requires other means as above.

Next, we compared the computational time of the three approaches with choosing the length of the horizon as $N = 2, 3, \ldots, 6$. The degree d was fixed to 6 and the partition was the one consisting of the 4 subpolytopes.

TABLE II

THE RESULTS OF THE DUALITY-GAP APPROACH WHEN THE POLYNOMIAL DEGREE d WAS CHANGED AND THE PARTITION WAS FIXED.

poly. deg.	# of subpoly.	attained opt. val.	comp. time
6	1	21.057	10.5 s
8	1	0.765	56.0 s
10	1	0.221	472.1 s

TABLE III

THE RESULTS OF THE DUALITY-GAP APPROACH WHEN THE POLYNOMIAL DEGREE d WAS FIXED AND THE PARTITION WAS CHANGED.

poly. deg.	# of subpoly.	attained opt. val.	comp. time
6	1	21.057	10.5 s
6	2	0.213	17.2 s
6	4	0.038	51.1 s

The result is summarized in Figure 1. It is observed that the computational time was shorter in the proposed two approaches than in the approach of [13] especially when the length of the horizon was large. This is consistent with the analysis in Section V. Between the two approaches, the primal approach required shorter computational time. The distance to optimality computed from the 100 grid points was smaller than 0.05 in all the cases.

VII. CONCLUSION

Two approaches are presented for obtaining a regularly partitioned piecewise polynomial that approximates the optimal model predictive controller. In each of the approaches, the problem is reduced to a robust optimization problem, which is approximately solvable with the sum-of-squares method. In particular, the primal approach requires smaller computational cost while the duality-gap approach gives an estimate of the quality of approximation. They are compared with the approach of [13] and shown to be advantageous in the computational cost.

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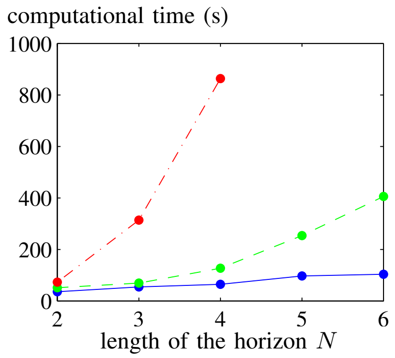

Fig. 1. Comparison of the three approaches in the computational time for various lengths of the horizon. The solid line represents the time of the primal approach, the dashed line that of the duality-gap approach, and the dash-dotted line that of the approach of [13].

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