Answer

To prove $\vdash \Box(x = 1) \Rightarrow \Box(\Box(x = 1)))$ we must prove

$$[\![\Box(x = 1) \Rightarrow \Box(\Box(x = 1))]\!](\sigma)$$

for an arbitrary behavior σ. Here is the proof.

1. SUFFICES: $[\![\Box(x = 1)]\!](\sigma) \Rightarrow [\![\Box(\Box(x = 1))]\!](\sigma)$

   PROOF: Because $[\![F \Rightarrow G]\!](\sigma)$ is defined to equal $[\![F]\!](\sigma) \Rightarrow [\![G]\!](\sigma)$, for any F, G, and σ.

2. SUFFICES ASSUME: $[\![\Box(x = 1)]\!](\sigma)$ and τ a suffix of σ
   PROVE: $[\![\Box(x = 1)]\!](\tau)$

   PROOF: By step 1 and the definition of $[\![\Box F]\!](\sigma)$, with $F \leftarrow \Box(x = 1)$.

3. SUFFICES ASSUME: ρ a suffix of τ
   PROVE: $[\![x = 1]\!](\rho)$

   PROOF: By step 2 and the definition of $[\![F]\!](\sigma)$, with $F \leftarrow x = 1$.

4. Q.E.D.

   PROOF: ρ is a suffix of σ by steps 2 and 3 (since a suffix of a suffix of σ is a suffix of σ). Hence $[\![x = 1]\!](\rho)$ is true by the assumption $[\![\Box(x = 1)]\!](\sigma)$ of step 2 and the definition of $\Box$.

Observe that replacing $(x = 1)$ by F in the proof shows that $\Box F \Rightarrow \Box\Box F$ is a theorem, for any formula F.

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