Worksheet 2

Foundations of Bayesian Methodology

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Exercise 3 (Conjugate Bayes: analytical derivation)

Assumptions:

$$
\begin{aligned}
y_1, \cdots, y_n \mid m &\overset{i.i.d}{\sim} \mathcal{N}(m, \kappa^{-1}) \\
m &\sim \mathcal{N}(\mu, \lambda^{-1})
\end{aligned}
$$

The Likelihood is equal

$$
\begin{aligned}
f(y_1, \cdots, y_n \mid m) &= \prod_{i=1}^{n} f(y_i \mid m) \\
&= \prod_{i=1}^{n} \left(\sqrt{\frac{\kappa}{2\pi}} \exp\left(-\frac{\kappa}{2}(y_i - m)^2 \right) \right) \\
&= \left(\frac{\kappa}{2\pi} \right)^{\frac{n}{2}} \exp\left(-\frac{\kappa}{2} \sum_{i=1}^{n} (y_i - m)^2 \right)
\end{aligned}
$$

Prior density distribution of m:

$$
f(m) = \sqrt{\frac{\lambda}{2\pi}} \exp\left(-\frac{\lambda}{2}(m - \mu)^2 \right)
$$

In a next step the likelihood and prior density function of m are multiplied:

$$
\begin{aligned}
& f(y_1,\cdots,y_n \mid m) f(m) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{\kappa}{2}\sum_{i=1}^{n}(y_i-m)^2\right)\cdot\sqrt{\frac{\lambda}{2\pi}}\exp\left(-\frac{\lambda}{2}(m-\mu)^2\right) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\kappa}{2}\sum_{i=1}^{n}(y_i-m)^2-\frac{\lambda}{2}(m-\mu)^2\right) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\kappa}{2}\sum_{i=1}^{n}(y_i^2-2y_im+m^2)-\frac{\lambda}{2}(m^2-2m\mu+\mu^2)\right) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\kappa}{2}\left(\sum_{i=1}^{n}y_i^2-2n\bar{y}m+nm^2\right)-\frac{\lambda}{2}(m^2-2m\mu+\mu^2)\right) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\left((n\kappa+\lambda)m^2-2m(\kappa n\bar{y}+\lambda\mu)+\kappa\sum_{i=1}^{n}y_i^2+\lambda\mu^2\right)\right) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{n\kappa+\lambda}{2}\left(m^2-2m\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}+\frac{\kappa\sum_{i=1}^{n}y_i^2+\lambda\mu^2}{n\kappa+\lambda}\right)\right) \\
&= \left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{n\kappa+\lambda}{2}\left(\left(m-\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2-\left(\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2+\frac{\kappa\sum_{i=1}^{n}y_i^2+\lambda\mu^2}{n\kappa+\lambda}\right)\right) \\
&= \underbrace{\left(\frac{\kappa}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(\frac{\kappa\sum_{i=1}^{n}y_i^2+\lambda\mu^2}{n\kappa+\lambda}-\left(\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2\right)}_{\text{constant}}\exp\left(-\frac{n\kappa+\lambda}{2}\left(m-\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2\right)
\end{aligned}
$$

According to the Bayes formula, which has been rewritten in terms of densities, we get:

$$
\begin{aligned}
f(m \mid y_1,\cdots,y_n) &= \frac{f(y_1,\cdots,y_n \mid m)f(m)}{f(y_1,\cdots,y_n)} \\
&= \frac{f(y_1,\cdots,y_n \mid m)f(m)}{\int_{-\infty}^{\infty} f(y_1,\cdots,y_n \mid m)f(m)\mathrm{d}m} \\
&= \frac{\exp\left(-\frac{n\kappa+\lambda}{2}\left(m-\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2\right)}{\int_{-\infty}^{\infty}\exp\left(-\frac{n\kappa+\lambda}{2}\left(m-\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2\right)} \\
&= \sqrt{\frac{n\kappa+\lambda}{2\pi}}\exp\left(-\frac{n\kappa+\lambda}{2}\left(m-\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\right)^2\right)
\end{aligned}
$$

$$
m \mid y_1,\cdots,y_n \sim \mathcal{N}\left(\frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda},\frac{1}{n\kappa+\lambda}\right)
$$

Note:

$$
\int_{-\infty}^{\infty} e^{-a(x+b)^2}\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}
$$

Source: Gaussian Integral

Exercise 4 (Conjugate Bayesian analysis in practice)

Assumptions:

$$
\begin{aligned}
y_1,\cdots,y_n \mid m &\overset{i.i.d}{\sim} \mathcal{N}(m,\kappa^{-1}) \\
m &\sim \mathcal{N}(\mu,\lambda^{-1})
\end{aligned}
$$

$$\kappa = \frac{1}{900}, \quad \mu = 161, \quad \lambda = \frac{1}{70}$$

```
height <- c(166, 168, 168, 177, 160, 170, 172, 159, 175, 164, 175, 167, 164)
table(height)
```

4(a)

```
## height
## 159 160 164 166 167 168 170 172 175 177
##   1   1   2   1   1   2   1   1   2   1
```

```
hist(height, breaks=length(height), freq=FALSE)
lines(density(height), col="red")
```

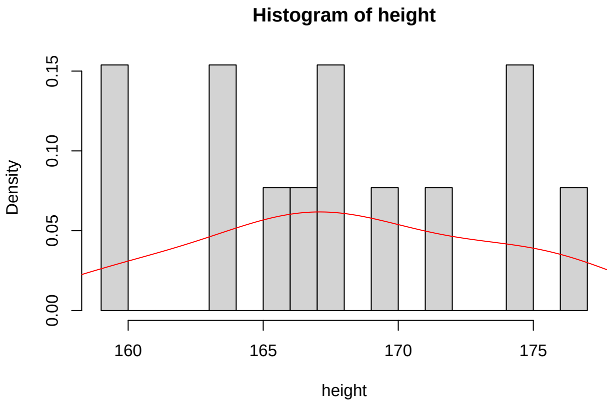


Figure 1: Histogram of the height measurements

```
library(car)
```

```
## Lade nötiges Paket: carData
```

```
qqPlot(height, main="Normal Q-Q Plot", ylab="Sample", xlab="Norm Quantiles")
```

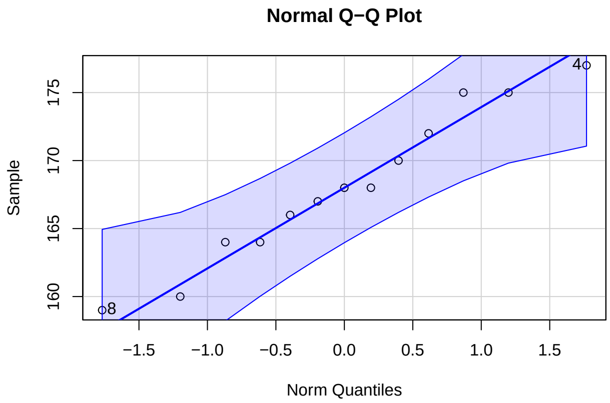


Figure 2: Q-Q Plot of the Measurements

```
## [1] 8 4
```

```
# or alternatively without library(car)
qqnorm(height)
qqline(height)
```

In the Q-Q Plot it can be seen that all sample values lie within the area where normal distribution of the

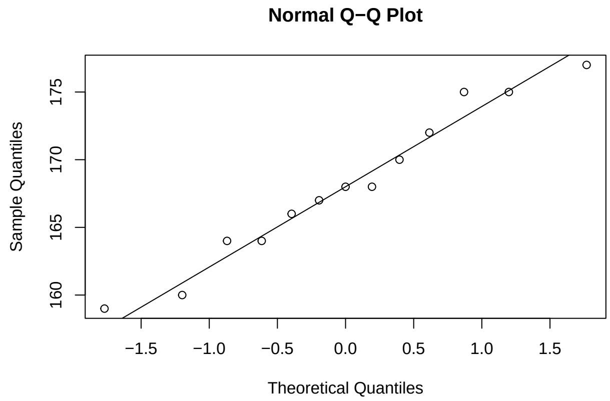


Figure 3: Q-Q Plot of the Measurements

data can be assumed. Due to the fact that we only have 13 observations the 95%-CI has been calculated assuming a t-distribution.

```
## Sample size
sample.n <- length(height); sample.n
```

```
## [1] 13
```

```
## Sample meam
sample.mean <- mean(height); sample.mean
```

```
## [1] 168.0769
```

```
## Sample standard deviation
sample.sd <- sd(height); sample.sd
```

```
## [1] 5.634145
```

```
## Sample standard error
sample.se <- sample.sd / sqrt(sample.n); sample.se
```

```
## [1] 1.562631
```

```
## Significance level
alpha <- 0.05

## Degrees of freedom
df <- sample.n - 1

## t-score
t.score <- qt(p=alpha/2, df=df, lower.tail=FALSE); t.score
```

```
## [1] 2.178813
```

```
## Contract the 95% confidence interval
lower.bound <- sample.mean - t.score * sample.se
upper.bound <- sample.mean + t.score * sample.se
print(c(lower.bound, upper.bound))
```

```
## [1] 164.6722 171.4816
```

Interpretation: Based on our data, the interval $(164.6722, 171.4816)$ gives us a range of parameters which are, based on this data, the best 95% parameters for the true unknown parameter of interest (mean). We do not know whether this Confidence Interval covers the parameter of interest but it is constructed in a way that if repeated sampling, estimation and construction of the confidence interval in 95% of all cases the confidence interval covers the true unknown parameter of interest.

```r
## Plot of the Prior Distribution:
m <- seq(101, 221)
plot(m, dnorm(m, mean=161, sd=sqrt(70)), type="l", ylab="density", col="red",
     ylim=c(0,0.07), main="Prior vs Posterior", lwd=2)
legend("topright", legend=c("prior: N(161,70)"),
       lty=1, col=c("red"), cex=.8)
```

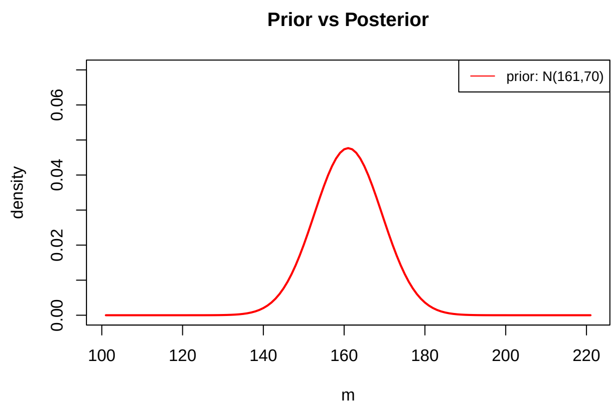


Figure 4: Prior Distribution

4(b)

```
$$
P[m>200]=1-P[m\leq200]
$$


```r
## 2.5%, 50%(i.e. median), 97.5% quantiles of N(161,70)
qnorm(c(0.025, 0.5, 0.975), mean=161, sd=sqrt(70))
```

```
## [1] 144.6018 161.0000 177.3982
```

```r
## P[m>200]
pnorm(200, mean=161, sd=sqrt(70), lower.tail=FALSE)
```

```
## [1] 1.570393e-06
```

- Plot: see Figure 5
- Summary statistics: see Table 1

4(c)

$$
\begin{aligned}
y_1, \cdots, y_n \mid m &\overset{i.i.d}{\sim} \mathcal{N}(m, \kappa^{-1}) \\
m &\sim \mathcal{N}(\mu, \lambda^{-1}) \\
m \mid y_1, \cdots, y_n &\sim \mathcal{N}\left(\frac{\kappa n\bar{y} + \lambda\mu}{n\kappa + \lambda}, (n\kappa + \lambda)^{-1}\right)
\end{aligned}
$$

```r
## Posterior mean
((1/900)*2185+(1/70)*161 ) / (13*(1/900)+(1/70) )
```

```
## [1] 164.558
```

```r
## Posterior variance
(13*(1/900)+(1/70))^(-1)
```

```
## [1] 34.80663
```
```

$$
\begin{aligned}
y_1, \cdots, y_n &\sim \mathcal{N}(m, 900) \\
m &\sim \mathcal{N}(161, 70) \\
m \mid y_1, \cdots, y_n &\sim \mathcal{N}(164.558, 34.80663)
\end{aligned}
$$

```
m <- seq(101, 221)
plot(m, dnorm(m, mean=161, sd=sqrt(70)), type="l", ylab="density", col="red",
     ylim=c(0,0.08), main="Prior vs Posterior")
lines(m, dnorm(m, mean=164.558, sd=sqrt(34.80663)), col="blue")
legend("topright", legend=c("prior: N(161,70)", "posterior: N(164.56, 34.81)"),
       lty=1, col=c("red", "blue"), cex=.8)
```

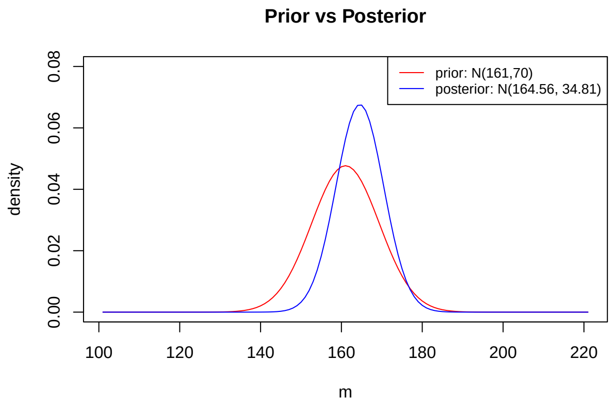


Figure 5: Prior vs. Posterior

```
qnorm(c(0.025, 0.5, 0.975), mean=164.558, sd=sqrt(34.80663))
```

```
## [1] 152.9948 164.5580 176.1212
```

Table 1: Summary statistics of the prior and posterior distributions

	Mean	SD	Median	Equi-tailed 95% CI/CrI
m	161.000	8.3666	161.000	$(144.6018, 177.3982)$
$m \mid y_i, \cdots, y_n$	164.558	5.8997	164.558	$(152.9948, 176.1212)$

Interpretation: there is a posterior probability of 95% that the true mean falls into the interval between 152.9948 and 176.1212, given a $\mathcal{N}(m, 900)$ prior is assumed.

4(d)

$$P[m > 200 | y_1, ..., y_n]$$

```
## P[m>200|y1,...,yn]
pnorm(200, mean=164.558, sd=sqrt(34.80663), lower.tail=FALSE)
```

```
## [1] 9.425552e-10
```

4(e)

$$
\begin{aligned}
\text{Prior} &\rightarrow \text{Posterior} \\
\mathcal{N}(161, 70) &\rightarrow \mathcal{N}(164.558, 34.80663) \\
P[m > 200] &\rightarrow P[m > 200 \mid y_1, \cdots, y_n] \\
1.570393 \times 10^{-6} &\rightarrow 9.425552 \times 10^{-10}
\end{aligned}
$$

It is noticeable that the variance in particular has decreased from prior to posterior distribution. Thus the dispersion is smaller. This leads to the distribution becoming narrower and the probabilities around the mean increase. Thus, the probability of observing a value greater than 200 also decreases.

Table 2: Comparision between prior and posterior

	Prior	Posterior
Mean	161	164.558
Variance	70	34.80663
$\mathcal{N}(\mu, \sigma^2)$	$\mathcal{N}(161, 70)$	$\mathcal{N}(164.558, 34.80663)$
$P[m > 200]$	1.570393×10^{-6}	9.425552×10^{-10}

Exercise 5 (Bayesian learning)

Prior distribution:

$$p \sim \text{Beta}(\alpha, \beta)$$

Posterior distribution:

$$p \mid y_1, \cdots, y_n \sim \text{Beta}(\alpha + n\bar{y}, \beta + n - n\bar{y})$$

5(a) $\alpha = \beta = 0.5, \quad \text{Beta}(0.5, 0.5)$

Table 3: Response rate for each stage under $B(0.5, 0.5)$ prior

Stage	n	Responders x (%)	Prior $\text{B}(\alpha, \beta)$	Posterior $\text{B}(\alpha + x, \beta + n - x)$
Interim	12	3 (25%)	$\text{B}(0.5, 0.5)$	$\text{B}(3.5, 9.5)$
Final	64	14 (21.875%)	$\text{B}(0.5, 0.5)$	$\text{B}(14.5, 50.5)$

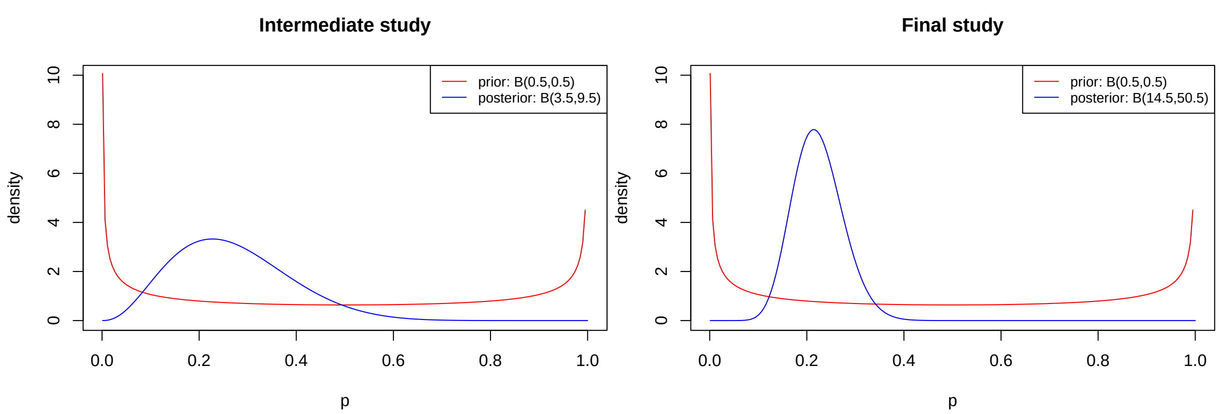
```r
p <- seq(1e-3,1, length=200)

plot(p, dbeta(p, 0.5, 0.5), type="l", ylab="density", col="red",
     ylim=c(0, 10), main="Intermediate study")
lines(p, dbeta(p, 3.5, 9.5), ylab="density", col="blue")

legend("topright", legend=c("prior: B(0.5,0.5)", "posterior: B(3.5,9.5)"),
       col=c("red", "blue"), lty=1, cex=.8)

plot(p, dbeta(p, 0.5, 0.5), type="l", ylab="density", col="red",
     ylim=c(0, 10), main="Final study")
lines(p, dbeta(p, 14.5, 50.5), ylab="density", col="blue")

legend("topright", legend=c("prior: B(0.5,0.5)", "posterior: B(14.5,50.5)"),
       col=c("red", "blue"), lty=1, cex=.8)
```



$$P[\text{response rate} > 0.4] = P[p > 0.4] = 1 - P[p \leq 0.4]$$

```r
# Intermediate study
pbeta(0.4, 3.5, 9.5, lower.tail=FALSE)
```

```
## [1] 0.1437649
```

```r
# Final study
pbeta(0.4, 14.5, 50.5, lower.tail=FALSE)
```

```
## [1] 0.001075757
```

```r
# Intermediate study
## 2.5%, 50%, 97.5% quantiles of Beta(3.5, 9.5)
qbeta(c(0.025, 0.5, 0.975), 3.5, 9.5)
```

```
## [1] 0.07594233 0.25711895 0.52919108
```

```r
# Final study
## 2.5%, 50%, 97.5% quantiles of Beta(14.5, 50.5)
qbeta(c(0.025, 0.5, 0.975), 14.5, 50.5)
```

```
## [1] 0.1312669 0.2202242 0.3310055
```

Table 4: Summary statistics of posterior distribution ($B(0.5, 0.5)$ prior)

Stage	$B(\alpha, \beta)$	Mean	Median	95% CrI
Interim	$B(3.5, 9.5)$	0.2692	0.2571	$(0.0759, 0.5292)$
Final	$B(14.5, 50.5)$	0.2292	0.2202	$(0.1313, 0.3310)$

5(b) $\alpha = 8, \beta = 24$, Beta$(8, 24)$

Table 5: Response rate for each stage under $B(8, 24)$ prior

Stage	n	Responders x (%)	Prior $\mathrm{B}(\alpha, \beta)$	Posterior $\mathrm{B}(\alpha + x, \beta + n - x)$
Interim	12	3 (25%)	$\mathrm{B}(8, 24)$	$\mathrm{B}(11, 33)$
Final	64	14 (21.875%)	$\mathrm{B}(8, 24)$	$\mathrm{B}(22, 74)$

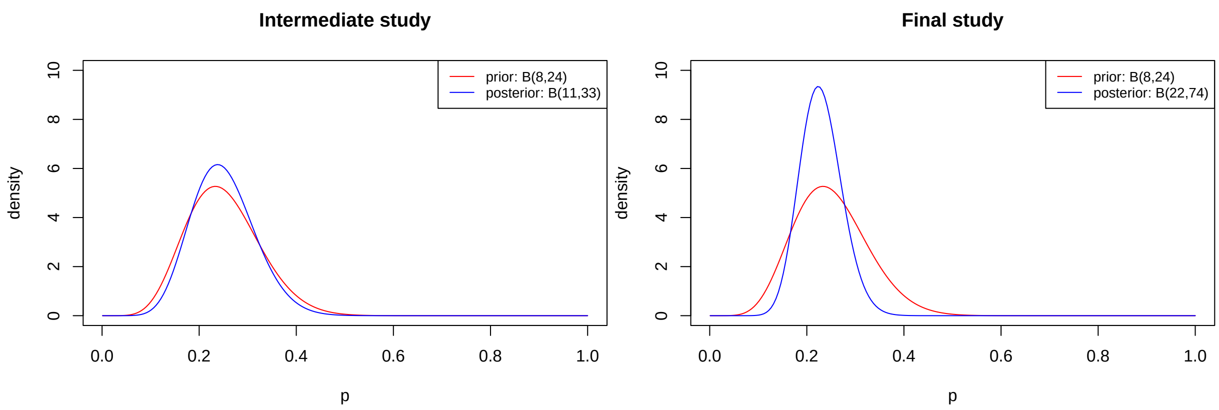
```r
p <- seq(1e-3,1, length=200)

plot(p, dbeta(p, 8, 24), type="l", ylab="density", col="red",
     ylim=c(0, 10), main="Intermediate study")
lines(p, dbeta(p, 11, 33), ylab="density", col="blue")

legend("topright", legend=c("prior: B(8,24)", "posterior: B(11,33)"),
       col=c("red", "blue"), lty=1, cex=.8)

plot(p, dbeta(p, 8, 24), type="l", ylab="density", col="red",
     ylim=c(0, 10), main="Final study")
lines(p, dbeta(p, 22, 74), ylab="density", col="blue")

legend("topright", legend=c("prior: B(8,24)", "posterior: B(22,74)"),
       col=c("red", "blue"), lty=1, cex=.8)
```



$$P[\text{response rate} > 0.4] = P[p > 0.4] = 1 - P[p \leq 0.4]$$

```r
# Intermediate study
pbeta(0.4, 11, 33, lower.tail=FALSE)
```

```
## [1] 0.01621346
```

```r
# Final study
pbeta(0.4, 22, 74, lower.tail=FALSE)
```

```
## [1] 0.0001727695
```

```r
# Intermediate study
## 2.5%, 50%, 97.5% quantiles of Beta(11, 33)
qbeta(c(0.025, 0.5, 0.975), 11, 33)
```

```
## [1] 0.1351860 0.2461854 0.3863082
```

```r
# Final study
## 2.5%, 50%, 97.5% quantiles of Beta(22, 74)
qbeta(c(0.025, 0.5, 0.975), 22, 74)
```

```
## [1] 0.1511774 0.2272801 0.3178360
```

Table 6: Summary statistics of posterior distribution ($B(8, 24)$ prior)

		Mean	Median	95% CrI
Stage	Posterior			
Interim	$B(11, 33)$	0.2500	0.2461	$(0.1352, 0.3863)$
Final	$B(22, 74)$	0.2292	0.2273	$(0.1512, 0.3178)$