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A new constructive logic: classical logic

JEAN-YVES GIRARD

Équipe de logique, UA 753 du CNRS Mathématiques, Université Paris VII, t.4555, 5° étage, 2, Place Jussieu 75251 Paris cedex 05, France

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0. Introduction

There are two ways to present this work; the most efficient is of course to start with the main syntactical definitions, and to end with the semantics: this is the presentation that we follow in the *body* of the text: section 1, syntax; section 2, semantics. Another possibility is to follow the order of discovery of the concepts, which (as expected) starts with the semantics and ends with the syntax; we adopt this second way for our introduction, hoping that this orthogonal look at the same object will help to apprehend the concepts.

0.1. *Classical proof-theory is inexistent*

This may seem paradoxical, since there are several books solely devoted to classical proof-theory, including one by the author of this provocative declaration. But it is true that, if we look carefully at classical proof-theory, which starts with the works of Herbrand and Gentzen in the early thirties, we get hardly more than a cut-elimination theorem (what Kreisel in his day proposed to call *Hauptsatz* in opposition to *normalization*) without a real procedure to do it. Proof-theory books can be rather big because this *Hauptsatz* has been extended to various systems of arithmetic, but these extensions, when restricted to pure propositional logic hardly say anything more than Gentzen's original result of 1934.

In sharp contrast intuitionistic proof-theory is organized along large avenues: let us mention natural deduction, Curry–Howard isomorphism, realizability, denotational semantics etc. More recently linear logic introduced another system with even large avenues—typically an involutive negation, like in classical logic.

This aspect of a well-organized proof-theory, common to intuitionistic and linear logics is the ultimate meaning of *constructivity*. Constructivity should not be confused with its ideological variant 'constructiv*ism*' which tries to build a kind of *countermathematics* by an *a priori* limitation of the methods of proofs; it should not either be characterized by a list of technical properties: e.g. disjunction and existence properties. Constructivity is the possibility of extracting the information *implicit* in proofs, i.e. constructivity is about *explicitation*. Typically we should accept as constructive any kind of system for which we can define a 'reasonable' *semantics* of proofs, compatible with cut-elimination. The example of the linear disjunction *par* ($\parr$) shows that the disjunction property is not a necessary condition for constructivity.

When we say that classical logic has no normalization procedure, we do not want to say that cut-elimination cannot be performed at the syntactical level, but that it is a very clumsy process. Typically, if we see cut-elimination as a form of rewriting, it is full of very bad *critical pairs* (the simplest one has been pointed out by Lafont, (see Girard *et al.*, 1989; pp 150–152); all these pairs come from the structural rules of weakening and contraction). The real problem here is not quite that the process is strongly non-deterministic (after all, why not?) but that we have not the slightest way to control—i.e. to understand—it. What is missing is a stock of *lemmas* analysing the input/output relation, ways of defining *modules* inside a proof etc, typically a *denotational semantics*.

0.2. *Denotational semantics*

Denotational semantics is the most efficient way of analysing cut-elimination. It is not to be confused with *categorical* semantics: surely a (good) categorical semantics is a denotational semantics, but the converse is by no means obvious; in fact our semantics is not a categorical semantics and if it was so difficult to put some order into classical proof-theory, this is precisely because there cannot be any serious categorical semantics for classical logic, as will be clear from the behaviour of classical conjunction.

The kind of semantics we are interested in is *concrete*, i.e. to each proof π we associate a *set* π^*. This map can be seen as a way to define an equivalence $\approx$ between proofs ($\pi \approx \pi'$ iff $\pi^* = \pi'^*$) of the same formulas (or sequents), which should enjoy the following:

(i) if π normalizes to π', then $\pi \approx \pi'$;
(ii) $\approx$ is non-degenerated, i.e. one can find a formula with at least two non-equivalent proofs;
(iii) $\approx$ is a congruence: this means that if π and π' have been obtained from λ and λ' by applying the same logical rule, and if $\lambda \approx \lambda'$, then $\pi \approx \pi'$;
(iv) certain canonical *isomorphisms* are satisfied; among those which are crucial let us mention
—involutivity of negation (hence De Morgan)
—associativity of disjunction (hence of conjunction).

Let us comment on these points:

(i) says that $\approx$ is about cut-elimination; however—as it happened in this work—the semantics is likely to be found first, and from it a *variant* of Gentzen's procedure enjoying (i) can be extracted.
(ii) Of course if all proofs of the same formula are declared to be equivalent, the contents of $\approx$ is empty; the condition has to be stated, to avoid certain jokes like 'symmetric Cartesian-closed categories', see appendix B1.
(iii) Is the analogue of a Church–Rosser property, and is the key to a modular approach to normalization.
(iv) Another key to modularity is commutation, which means that certain sequences of operations on proofs are equivalent w.r.t. $\approx$. It is clear that the more commutation we get the better, and that we cannot ask too much *a priori*. However, the two commutations mentioned are a strict minimum without which we would get a mess:

—involutivity of negation means that we have not to bother about double negations, i.e. that if we use the proof of $\neg\neg A \Leftrightarrow A$ to transform a proof λ of $\neg\neg A$ into a proof of A and again into a proof π of $\neg\neg A$, then $\lambda \approx \pi$.

—associativity of disjunction means that the bracketing of a ternary disjunction is inessential; furthermore, associativity renders possible the identification of $A \Rightarrow (B \Rightarrow C)$ with $(\mathrm{A} \wedge B) \Rightarrow C$.

0.3. *The solution: semantical aspects*

The obvious candidate for a classical semantics was of course *coherent spaces*, which had already given birth to *linear logic*; the main reason for choosing them was the presence of the involutive *linear* negation. However, the difficulty with classical logic is to accommodate structural rules (*weakening* and *contraction*); in linear logic, this is possible by considering coherent spaces of the form $?X$. But since classical logic allows contraction and weakening both on a formula and its negation, the solution seemed to require the linear negation of $?X$ to be of the form $?Y$, which is a nonsense (the negation of $?X$ is $!X^{\perp}$, which is by no means isomorphic to a space $?Y$). Attempts to find a self-dual variant $\S\, Y$ of $?Y$ (enjoying $(\S\, Y)^{\perp} = \S(Y^{\perp})$) systematically failed. The semantical study of classical logic stumbled on this problem of self-duality for years. Of course one way to remove the deadlock would have been to distinguish two kinds of coherent spaces, interchanged by negation. For instance the *negative* ones could be of the form $?Y$, and the positive ones of the form $!X$, etc, but how to stay within these two classes when forming binary connectives? It seems indeed impossible to form the conjunction of a positive and a negative space and to still get either a positive or a negative space. The real starting point of this work was to discover table 3 in section 2, which has the following properties:

(i) it starts with the definition of *correlation spaces*: a positive one is a comonad (typically any $\oplus_i !X_i$), a negative one is the dual of a comonad (typically any $\&_i ?Y_i$) i.e. a 'cocomonad' (unfortunately not a monad: since comonads are defined in terms of $\otimes$, their duals are defined in terms of the dual $⅋$ of $\otimes$, whereas monads are defined in terms of $\otimes$)... anyway this is a variation on the previous theme

(ii) between such spaces one can define classical operations, without going out of the class of correlation spaces

(iii) last but not least, in fact the only important point: disjunction is associative up to isomorphism; in fact it is even commutative and with a neutral element.

Such a tableau is definitely the proof of the constructive nature of classical logic; but the situation turns out to be much more surprising than expected...

0.4. *A positive surprise: the behaviour of disjunction*

The disjunction and existence properties of intuitionistic logic are essential to its *constructivity*—which means in concrete non-dogmatic terms, its computational aspect—. This is why typed λ-calculi like the system $\mathbb{F}$ of *second order λ-calculus* can be thought of as paradigms for computation. The most brutal ingredient of the relative success of $\mathbb{F}$ and its likes is the possibility of representing usual free data types by cut-free proofs of certain formulas. Typically if we want to represent booleans, we need two formulas $\mathbf{V}_1$ and $\mathbf{V}_2$, each of them with only one cut-free proof (these two formulas may be the same formula $\mathbf{V}$); then it is possible to represent booleans by means of the cut-free proofs of $bool := \mathbf{V}_1 \vee \mathbf{V}_2$, since by the disjunction property, a cut-free proof of *bool* must be a proof of one of the disjuncts.... This wonderful situation is likely to break down in the classical case, since the contraction rule allows a disjunction $A \vee B$ to be proved without deciding between them, as in $A \vee \neg A$. The first shock is therefore to discover, semantically and syntactically the fact—with the obvious choice for $\mathbf{V}$—that the property still holds classically! In other terms *classical booleans are standard* and the representation of current data types by means of proofs is still possible in a classical framework.

Investigating more carefully this surprising fact, it is possible to see a kind of *disjunction property* satisfied by classical logic: in a cut-free proof of $\vdash P \vee Q$ of a disjunction of positive formulas, one of the two disjuncts is 'preferred', and when P and Q are hereditarily positive, to prefer P means to prove P. This phenomenon of preference is not very conspicuous at the syntactical level, since it refers to the order of rules which may look a bit irrelevant, especially in **LK**, but semantically speaking, it comes from the extremely asocial behaviour of the linear connective '!'.

To sum up: not only classical logic is constructive, but it seems to be constructive in a way compatible with the usual intuitionistic handling of constructivity...we can even dream of a classical system $\mathbb{F}$....

0.5. *Bad news: classical conjunction is a mess*

Intuitionistic conjunction is a categorical product. This fact is extremely useful for constructive purposes. Unfortunately classical conjunction $A \wedge B$ when one of A and B is positive is by no means a product. Concretely, if we take a proof π of $A \wedge B$, then using the implications $A \wedge B \vdash A$, $A \wedge B \vdash B$, we can get two 'projections' π_1 (proof of A) and π_2 (proof of B) as in intuitionistic logic. But from π_1 and π_2 there is no way to recover π! In fact the equation $\pi = \langle \pi_1, \pi_2 \rangle$ is inconsistent...(this property—*surjective pairing*—is consistent with intuitionistic logic); even worse, when both A and B are positive, we get two pairing functions, i.e. two absolutely different ways of forming a proof of $A \wedge B$ from a proof of A and a proof of B, and these two ways cannot be identified without creating an inconsistency.

These complications about conjunction are just another way to state the *non-commutativity of cut*: given proofs of $\vdash \Gamma, P$ and $\vdash \Delta, Q$ and $\vdash \Lambda, \neg P, \neg Q$ we can form a proof of $\vdash \Gamma, \Delta, \Lambda$ by means of two cuts. But semantically speaking the result depends on the order of performance of the two cuts, when P and Q are positive. Since the general principle that our two cuts can be replaced by a single cut between $\vdash \Gamma, \Delta, P \wedge Q$ and $\vdash \Lambda, \neg P, \neg Q$ is still valid, this indicates that either the conjunction rule or the disjunction rule (in fact conjunction) should be refined into two cases; therefore if we want to fix **LK** it is not enough to distinguish polarities, but we must also introduce two conjunction rules when the conjuncts are positive: they are distinguished by the order in which the cuts on the conjuncts are performed during cut-elimination.

0.6. *Correlation domains and central cliques*

The denotational interpretation of classical proofs—once two rules for conjunction have been carefully distinguished—is not problematic: a proof π of A is interpreted by a *clique* in an associated correlation space. But it is slightly more delicate to get the semantics sharp enough so as to be able to prove from purely semantical grounds that for instance classical booleans are standard. The problem with usual denotational semantics is that it deals—for reasons of continuity—with partial objects for instance empty cliques; however the interpretation of proofs is most of the time not empty.... So it is possible to state a pure semantical property of the interpretation of proofs that would for instance exclude (in good cases) empty cliques?

Quite surprisingly, it can be done nicely: in a *correlation domain* we indicate which cliques are 'good', not in the correlation space itself, but in its disjunction with the constant $\mathbf{V}$. The correlation domain modelises therefore not the proofs of A itself (there are not enough of them) but of $\vdash A, \mathbf{V}$, i.e. of A in a context which is in turn provable. The proof of the fact that the interpretation π^* of a proof π belongs to the associated correlation domain could be quite easy if the problem of non-commutativity of cut were not still lingering: in order to associate a correlation domain to $\vdash A_1, \ldots, A_n$ we have to use duality and n cuts, and the order of these cuts should not matter for obvious reasons...

This is why *central cliques* come into the picture: roughly speaking a central clique corresponds to a proof of $\vdash \Gamma, P$ (P positive) where P has not been weakened or contracted. The main technicality about them is that the two ways of forming $\langle \pi_1, \pi_2 \rangle$ that we discovered above coincide when one of the π_i is central, or in other terms that if we perform cuts with central cliques, then the order of these cuts is irrelevant. At the conceptual level the interest of central cliques is that they are (traces of) linear maps preserving the structure of comonad. The notion of central clique is then taken as a central ingredient of the definition of *correlation domain*, and the theorem follows without too much trouble.

In fact the semantics of *correlation domains* is good, very good, so good indeed that it is difficult to imagine a clique which lies in a correlation domain without coming from a proof. This is why we propose a very challenging conjecture: prove that a clique in the correlation domain associated with A necessarily comes from a proof of A. The generalization of Gödel's completeness theorem can only hold when the completeness theorem (A true $\Rightarrow A$ provable) holds. This result looks technically difficult, but its interest is much more than technical: for the particular formulas for which the conjecture is stated the *subformula property* provides a kind of *absoluteness* of proofs, i.e. a cut-free proof of A in any 'reasonable' extension of **LK** is already a proof of **LK**: the conjecture would therefore be a quite satisfactory explanation of this absoluteness.

0.7. *A new sequent calculus: LC*

The main semantical *boulevards* being drawn, it remains to look at syntax, and the obvious proposal is to try to improve the cut-elimination procedure of **LK**, for instance by distinguishing between two conjunction rules...but this is not enough to fix **LK**.... For instance consider a ternary conjunction $P \wedge Q \wedge R$; we can decide to write it as

$P \wedge (Q \wedge R)$ or $(P \wedge Q) \wedge R$; in the former case the available conjunction rules yield four combinations corresponding to the orders PQR, PRQ, QRP, RQP, whereas in the latter the combinations correspond to PQR, QPR, RPQ, RQP. In fact the six combinations must be available in both cases (since conjunction is associative) in a cut-free way, and the only possibility is that in both cases the missing combinations are definable from the available ones. In fact what happens is that one can restrict the conjunction rules to the particular case where both sides have proofs interpreted by *central* cliques, in which case they coincide. This restriction amounts to making the structural rules commute with the conjunction rule in such a way that conjunction is performed between $\vdash \Gamma, P$ and $\vdash \Delta, Q$ only when both P and Q have not been weakened or contracted. This suggests a further refinement **LC** of **LK**: besides usual sequents (now denoted by $\vdash \Gamma;$) there will be sequents $\vdash \Gamma; P$ in which a positive formula is distinguished. Such sequents have no straightforward meaning in model-theoretic terms; their interest is purely proof-theoretic: a proof of $\vdash \Gamma; P$ is represented by a central clique.

Semantics easily yield a sequent calculus enjoying a nice cut-elimination theorem; observe the presence of two rules for cut, depending or not on the presence of the cut-formula in the *stoup*, i.e. the area delimited by ';'. The intricate relations between these two cut rules explain why classical cut-elimination is such a mess.

The calculus with *stoup* has an intuitionistic flavour, and this is why we eventually end with a translation of **LC** into intuitionistic logic. This translation—which seems to be obvious and that we took for pedagogic reasons as our starting point—has outstanding properties, expressed by our theorem 1. It is not completely incorrect to reduce **LC** to this translation—what we do for instance for technical properties—but also **LC** lives by itself. Moreover, even intuitionistic logic would benefit from a negative *stoup* (representing *headvariables*)....

Let us observe that, whereas the semantical aspects of classical logic are well-understood (but for the completeness conjecture), there are a lot of open syntactical problems, from the building of an efficient rewriting system to the extension to polymorphism, not to speak of the possibility of a new approach to Peano arithmetic for which a direct access to Π^0_2 sentences would be welcome.

At a general level, this work establishes deeper relation between classical, intuitionistic and linear logics. For instance section 1 gives an 'intuitionistic look' to classical syntax by means of the stoup, whereas section 2 explains the subtleties of classical connectives by means of the distinction additive/multiplicative. We cannot exclude the possibility of forming in the future a logical system in which classical, intuitionistic and linear features would live in harmony.... This idea is part of the background of *logical frameworks*; however, extant logical frameworks are unifying logic only at the '*metalevel*' and accept too many junk systems with not enough remarkable properties to deserve the name of *logic*: they are more a kind of *synchretic* approach than a true unification. The ideal logical framework should be a neutral logical system in which specific effects could be obtained by restriction to specific fragments.... Although we are far from being in position to build such a system, the new results of this paper are encouraging: they reinforce our strong belief in *the essential unity of logic*.

1. Classical sequent calculus

This section is devoted to the sequent calculus **LC**, an improvement of Gentzen's **LK**. Since **LC** has many more rules than **LK**, we decided to use a one-sided version, negation being thus a *defined* connective. The formulas of the predicate calculus we shall consider are built from atomic formulas and their negations by means of conjunction ($\wedge$), disjunction ($\vee$), universal ($\forall x$) and existential ($\exists x$) quantifiers. Among the atomic formulas there are two constants **V** and **F** for 'true' and 'false'; they are both atoms, so the four literals **V**, **F**, $\neg$**V**, $\neg$**F** are distinct.... Hence we did not identify **V** and $\neg$**F**: although these formulas are provably equivalent they have distinct proof-theoretic behaviours.

Negation will be defined through obvious De Morgan formulas:

$$\begin{array}{ll} \neg(a) := \neg a & \neg(\neg a) := a \text{ for } a \text{ atomic} \\ \neg(A \wedge B) := \neg A \vee \neg B & \neg(A \vee B) := \neg A \wedge \neg B \\ \neg\forall x A := \exists x \neg A & \neg\exists x A := \forall x \neg A \end{array}$$

in such a way that $\neg\neg A$ is identical to A; as usual $A \Rightarrow B$ will abbreviate $\neg A \vee B$.

Here it is time to open (and quickly close) a syntactical discussion. Our identification of $\neg\neg A$ with A forces negation to be involutive; but there are two obvious traps that one should avoid about this identification

Trap 1. The belief that it is enough to declare $\neg\neg A$ equal to A; cemeteries are full of unimaginative attempts starting with the adjunction of an equation to intuitionistic logic—typically variants on the theme of 'symmetrical Cartesian-closed categories' and other *Loch Ness* monsters—. Declaring $\neg\neg A$ isomorphic with A may simply produce inconsistencies, typically categories in which there is at most one arrow between objects etc, and which should rather be called... Boolean algebras.

Trap 2. The belief that this syntactical identification is essential to our approach; in reality this identification is just a facility to reduce the number of formulas, of rules etc. This is made possible by the presence of a denotational isomorphism between $\neg\neg A$ and A. The fact that this isomorphism can be seen as an identity simplifies presentation but plays no role. And by the way another crucial isomorphism is the associativity of disjunction which is by no way forced by an equality: $A \vee (B \vee C)$ and $(A \vee B) \vee C$ remain distinct formulas.

In spite of its great number of rules (six for each binary connective $\wedge$, $\vee$, $\Rightarrow$) the reader may be tempted to reconstitute a two-sided version of **LC** which may be more user-friendly. The essential novelty of **LC**, the *stoup*, will now occur twice in two-sided sequents: they are of the form $\Lambda; \Gamma \vdash \Delta; \Pi$ where the sequence Λ, Π consists of at most one formula, negative if in Λ, positive if in Π. This two-sided version is not only more user-friendly, it also yields via the negative stoup $\Lambda;\ldots$ a way to legalize a very important intuitionistic facility: what is usually called a *headvariable* in λ-calculus or a *main hypothesis* in natural deduction. This suggests the creation of an improvement **LI** of the familiar intuitionistic sequent calculus **LJ**.

1.1. *The polarity of a formula*

The notion of *polarity* of a formula is a way to get rid of the essential non-determinism of cut-elimination.

Definition 1. The *polarity* ($+$ or $-$) of a formula is defined as follows:

—atomic formulas (in particular **V** and **F**) are positive
—negation of atomic formulas (in particular $\neg\mathbf{V}$ and $\neg\mathbf{F}$) are negative
—for compound formulas polarities follow table 1.

Table 1. *Polarities*

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$\forall xA$	$\exists xA$	$\neg A$
$+$	$+$	$+$	$+$	$-$	$-$	$+$	$-$
$-$	$+$	$+$	$-$	$+$	$-$	$+$	$+$
$+$	$-$	$+$	$-$	$-$			
$-$	$-$	$-$	$-$	$-$			

Of course the polarities shown in the columns for $\Rightarrow$ and $\neg$ are not part of the definition, but should be read either as a lemma about the polarities associated with the defined connectives or as the definition to adopt in the two-sided version.

There are some remarks to make just about this meaningless table.

(1) **V** has not been identified with the negation of **F** because we want to have *positive* constants for the two truth values;

(2) atoms are declared positive; in reality in a pair $(A, \neg A)$ of literals one must be positive, and we decide to write this one as an atom...;

(3) polarities of the connectives $\wedge$, $\vee$, $\Rightarrow$, $\neg$ follow an obvious truth table ($+$ for false, $-$ for true). But this means very little: since we managed to keep many classical tautologies (De Morgan, associativity of disjunction etc) at the level of isomorphisms, a 'polarity table' is practically forced to be isomorphic to a truth table. The fact that **V** and **F** are both positive should refute the suspicion of a trivial relation to Boolean algebra.

(4) the case of quantifiers is quite unfortunate: one would for instance love to have $\forall xA$ positive when A is positive. This is not possible, for deep proof-theoretic reasons (related to the well-known fact that intuitionistically $\forall$ does not commute with $\sim\sim$).

(5) Observe that one can always replace a formula by an equivalent one of a given polarity. For instance $A \wedge \mathbf{V}$ (or $\exists xA$, x dummy) is equivalent to A and positive, $A \vee \neg\mathbf{V}$ (or $\forall xA$, x dummy) is a negative equivalent of A. So what does the limitation of (4) actually mean, since we could decide to introduce—say $(x)A$—positive with the meaning of $\forall xA \wedge \mathbf{V}$? This still means something: we can define $(x)A$ from $\forall xA$ but $\forall xA$ (i.e. its proof-theoretic behaviour) cannot be recovered from $(x)A$.

(6) The only serious limitation that we can observe is the following: if we want to respect proof-theoretic behaviour, it will not be possible to substitute a negative formula for a

positive atom; typically we can instanciate $p \Rightarrow p$ into $P \Rightarrow P$ but not into $N \Rightarrow N$; if we want something like $N \Rightarrow N$, we shall be forced to use $N^+ \Rightarrow N^+$ (see notations below). In particular, if we were working with second-order system, the quantification would range on predicates of a fixed polarity (positive for instance would be enough). In terms of *polymorphism* the reader may think that the fact that $\forall \alpha A$ refers only to half the available alphas is a strong limitation... but one has also to take into account that extant polymorphism (like the one of system $\mathbb{F}$) is based on the negative fragment of intuitionistic logic, which is negative in our sense (the confusion of terminology is on purpose).

Notation. In the sequel, P, Q, R,... will refer to positive formulas, L, M, N,... to negative ones. When we do not want to specify the polarity of a formula we use A, B, C,.... Atoms will be written p, q, r,... (they are positive) or $pt_1 \ldots t_n$ if we absolutely want to display their predicate structure.

A^+ will always refer to the positive formula $A \wedge \mathbf{V}$, A^- to the negative formula $A \vee \neg\mathbf{V}$.

1.2. *An interpretation inside intuitionistic logic*

In order to avoid confusion we shall adopt a different system of notations for intuitionistic connectives: $\cap$, $\cup$, $\supset$, (x), (Ex); our negation symbol will be $\sim$, but with the meaning of $A \supset \phi$, where ϕ is an unspecified formula; our constants will be *true* and *false*. With these conventions it will be easier to deal with classical, intuitionistic and linear features at the same time without generating confusion.

One of the main constructive tools related to classical logic has been by means of Gödel's $\sim\sim$-translation and its variants. The translation is well-known, and is based on the idea of adding 'enough' $\sim\sim$, typically on atoms, disjunctions and existences. But this translation has a very bad structure:

Defect 1. Not compatible with substitution; if A° denotes the translation, $A[B/\alpha]^\circ$ is not $A^\circ[B^\circ/\alpha]$: this is because the transformation is not identical on atoms.

Defect 2. Negation is not involutive; for instance $A[B/\alpha]^\circ$ and $A^\circ[B^\circ/\alpha]$ are related through the erasing of certain double negations (passing from $\sim\sim\sim A$ to $\sim A$); but this erasing is not harmless, and for instance if we try to restore the erased double negation, we arrive at a proof with a completely different behaviour from the original one. In terms of λ-calculus passing from $\sim\sim\sim A$ to $\sim\sim\sim A$ with a transit through $\sim A$ consists in replacing Z of type $\sim\sim\sim A$ by $Z' = \lambda w{:}\sim\sim A \cdot w(\lambda x{:}A \cdot Z(\lambda y{:}\sim A \cdot y(x))$. For instance Z may be of the form $\lambda w{:}\sim\sim A \cdot t$, with t of type ϕ, depending on certain variables, but not on w, and Z' is now $\lambda w{:}\sim\sim A \cdot w(\lambda x{:}A \cdot t)$. In particular, if a is of the form $\lambda z{:}\sim A \cdot u$, we get $Z(a) = t$ whereas $Z'(a) = u$. (The reader familiar with the Curry–Howard isomorphism will relate this to the example already mentioned of Yves Lafont; more precisely $Z(a)$ and $Z'(a)$ are two ways of interpreting the same classical cut between two weakenings; this example is also the Tarpeian Rock of Boolean categories, symmetric λ-calculi and other atrocities.)

Defect 3. Disjunction is not associative; we just saw how sensitive $\sim\sim$-translation is to removal/adjunction of double negations; in particular this poses a problem with ternary disjunction which is represented as $\sim\sim(\sim\sim(A \cup B) \cup C)$ or $\sim\sim(A \cup \sim\sim(B \cup C))$ depending on the bracketing; in particular extracting programs from proofs can be sensitive to nonsense like bracketing of a disjunction. In such an example, of course, the translation can be optimized into $\sim\sim(A \cup (B \cup C))$, and this kind of improvement may already be useful in practice... but precisely these kind of local, limited *cooking recipes* should be superseded by a general theory.

The basic idea will be to distinguish between formulas which are single negations and formulas which are double negations. This distinction cannot be justified on pure intuitionistic grounds and this is why we shall in fact work with pairs (A, ϵ) of a formula and a polarity ($+$ or $-$). In fact whereas A is arbitrary when $\epsilon = +1$ (it will be eventually double-negated, but the latter is the better), A must be a negation $\sim B$ when $\epsilon = -1$. Notationally speaking we shall content ourselves with a notational trick (which is awfully ambiguous, comrade Brezhnev!): P will denote a pair $(P, +)$ and $\sim P$ (deciding to display the negation) a pair $(\sim P, -)$. The ambiguity coming from the fact that nobody can prevent me from taking a negation $\sim Q$ for P disappears if one remarks that what we shall do is compatible with the *polarity table*; moreover, in practice, all positive formulas coming from our $\sim\sim$-translation will never be negations.

Definition 2. *Classical formulas are interpreted as follows in intuitionistic logic*:

(i) *atomic formulas* p, $\mathbf{V}$, $\mathbf{F}$ *by* p, *true*, *false*, *respectively*;
(ii) *their negations* $\neg p$, $\neg\mathbf{V}$, $\neg\mathbf{F}$ *by* $\sim p$, $\sim$*true*, $\sim$*false*;
(iii) *compound formulas by means of table* 2.

Table 2. $\sim\sim$*-translation*

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$\forall x A$	$\exists x A$	$\neg A$
P	Q	$P \cap Q$	$P \cup Q$	$\sim(P \cap \sim Q)$	$\sim Ex \sim P$	ExP	$\sim P$
$\sim P$	Q	$\sim P \cap Q$	$\sim(P \cap \sim Q)$	$P \cup Q$	$\sim ExP$	$Ex \sim P$	P
P	$\sim Q$	$P \cap \sim Q$	$\sim(\sim P \cap Q)$	$\sim(P \cap Q)$			
$\sim P$	$\sim Q$	$\sim(P \cup Q)$	$\sim(P \cap Q)$	$\sim(\sim P \cap Q)$			

(As with definition 1, the columns for $\neg$ and $\Rightarrow$ are not exactly part of the definition.)
The remarkable fact about this translation lies in the following result

Theorem 1. *The translation of definition* 2 *is compatible with substitutions respecting polarities*; *moreover it enjoys a certain number of remarkable isomorphisms in the category* **SET** *of sets*:

(i) *involutivity of negation*: $\neg\neg A \simeq A$.
(ii) *De Morgan isomorphisms*: $\neg(A \wedge B) \simeq \neg A \vee \neg B$; $\neg(A \vee B) \simeq \neg A \wedge \neg B$; $A \Rightarrow B \simeq \neg A \vee B$; $\neg\forall x A \simeq \exists x \neg A$; $\neg\exists x A \simeq \forall x \neg A$.

(iii) *Associativity isomorphisms*: $A \vee (B \vee C) \simeq (A \vee B) \vee C$; $A \wedge (B \wedge C) \simeq (A \wedge B) \wedge C$; *and therefore* $(A \wedge B) \Rightarrow C \simeq A \Rightarrow (B \Rightarrow C)$; $A \Rightarrow (B \vee C) \simeq (A \Rightarrow B) \vee C$.

(iv) *Neutrality isomorphisms*: $A \vee \mathbf{F} \simeq A$; $A \wedge \neg\mathbf{F} \simeq A$.

(v) *Commutativity isomorphisms*: $A \vee B \simeq B \vee A$; $A \wedge B \simeq B \wedge A$ (*hence* $A \Rightarrow B \simeq \neg\ B \Rightarrow \neg A$).

(v) *Distributivity isomorphisms with restriction on polarities*: $A \wedge (P \vee Q) \simeq A \wedge P) \vee (A \wedge Q)$; $A \vee (L \wedge M) \simeq (A \vee L) \wedge (A \vee M)$ (*therefore* $A \Rightarrow (L \wedge M) \simeq (A \Rightarrow L) \wedge (A \Rightarrow M)$; $(P \vee Q) \Rightarrow A \simeq (P \Rightarrow A) \wedge (Q \Rightarrow A)$).

(vii) *Idempotency isomorphisms*: $P^+ \simeq P$; $N^- \simeq N$; *and therefore* $A^{++} \simeq A^+$; $A^{--} \simeq A^-$.

(viii) *Quantifiers isomorphisms*: $A \wedge \exists xP \simeq \exists x(A \wedge P)$; $A \vee \forall xN \simeq \forall x(A \vee N)$; *and therefore* $A \Rightarrow \forall xN \simeq \forall x(A \Rightarrow N)$; $\exists xP \Rightarrow A \simeq \forall x(P \Rightarrow A)$.

Proof. The compatibility of the translation w.r.t. substitution is immediate. For the denotational isomorphisms, it is unfortunate that the denotational semantics of intuitionistic disjunction is quite bad, and only works in the category **SET** (which is a brutal denotational semantics); later on the semantics will be defined without any reference to intuitionistic logic, and theorem 2 will be restated in terms of more constructive semantics. In the category **SET**, *false* is the void set, *true* is a singleton, $X \supset Y$ is the function space, $X \cap Y$ is the Cartesian product, $X \cup Y$ is the disjoint union, quantifiers behave like infinitary versions of $\wedge$ and $\vee$.

(i) and (ii) have a clear meaning only if negation and implication are primitive; with our conventions they mean the correctness of the two unnecessary columns for $\neg$ and $\Rightarrow$.

(iii) by De Morgan we can content ourselves with disjunction. A few typical cases are enough: $P \cup (Q \cup R) \simeq (P \cup Q) \cup R$ (case $+, +, +$)

$\sim(P \cap \sim(Q \cup R)) \simeq \sim((P \cap \sim Q) \cap \sim R)$ (case $-, +, +$); essential use of $\sim(A \cup B) \approx \sim A \cap \sim B$

$\sim(P \cap (Q \cap \sim R)) \simeq \sim((P \cap Q) \cap \sim R)$ (case $-, -, +$)

$\sim(P \cap (Q \cap R)) \simeq \sim((P \cap Q) \cap R)$ (case $-, -, -$).

(iv) for instance for disjunction: $P \cup \varnothing \simeq P$ (case $+$), $\sim(P \cap \sim\varnothing) \simeq \sim P$ (case $-$); in the latter case the isomorphism $\sim\varnothing \simeq$ *true* (i.e. the fact that $\sim\varnothing$ is a singleton) is essential.

(v) is immediate from the symmetry of $\cap$ and $\cup$.

(vi) for instance distributivity of $\wedge$ over $\vee$ follows from: $A \cap (P \cup Q) \simeq (A \cap P) \cup (A \cap Q)$.

(vii) is obvious; it is worth remarking that P^- is $\sim(\sim P \cap \mathit{true})$ hence since *true* is a singleton, P^- is $\sim\sim P$, seen as $\sim(\sim P)$; in the same way $(\sim P)^+$ is $\sim P$, but in which the frontal negation is forgotten; since positive formulas are eventually used (see below) double-negated, the meaning of $(\sim P)^+$ is also to prefix a double negation.

(viii) is a quantifier analogue of (vi). □

Once and for all we can make the abuse of identifying a formula with its intuitionistic translation; however it makes a lot of difference whether we consider a formula as classical (inside **LK**) or intuitionistic (inside **LJ**): this is why we shall use two distinct symbols $\vdash^c$ and $\vdash^i$ to distinguish between classical and intuitionistic sequents. Later on, there will be no need to use $\vdash^c$ for **LC**, since the presence of the *stoup* indicates a that we are in **LC**.

Definition 3. A classical sequent $\vdash^c \Gamma$ of **LK** is interpreted as follows: let $\Gamma = P_1, \ldots, P_p, \sim Q_1, \ldots, \sim Q_n$ (we cheat with the exchange rule as usual); then $\vdash^c \Gamma$ is interpreted by the sequent $\neg\Gamma \vdash^i \phi$, ie. $\sim P_1, \ldots, \sim P_p, Q_1, \ldots, Q_n \vdash^i \phi$ (remember that ϕ has been used to define the 'negation').

It is time to make a few more essential remarks:

(1) for sequents consisting of one formula (which is what sequent calculus is eventually about), $\vdash^c \sim Q$ is interpreted as $Q \vdash^i \phi$, i.e. $\sim Q$ as expected, but $\vdash^c P$ is interpreted as $\sim P \vdash^i \phi$, i.e. eventually as $\sim \sim P$. In other terms double negations are not removed but only *delayed* as much as possible.

(2) The problem about the polarity of $\forall x P$ is just that since $\sim\sim$ does not commute with $\forall$, it is no longer possible to delay the formation of the double negation, and we are forced to produce $(x) \sim\sim P$, which is better written $\sim Ex \sim P$, i.e. negatively. This writing is the best, since in terms of sequent we mean $\sim Ex \sim P$ (i.e. $(x) \sim\sim P$) whereas the positive one would have the weaker meaning of $\sim\sim (x) \sim\sim P$. This weaker form should not be the only available expression of $\forall x P$.

To any proof in **LK** of a sequent $\vdash^c \Gamma$ one can associate a proof in **LJ** of $\neg\Gamma \vdash^i \phi$: since our translation is provably equivalent to usual $\sim\sim$-translation, the reader will find no difficulty in reducing this remark—as stated—to familiar results. □

However, what is the point of the remarkable isomorphisms of theorem 1, if we make no effort to exploit them? This is why we state the following.

Wrong theorem. *The translation of definition* 2 *and the denotational semantics of intuitionistic logic in* **SET** *yield a denotational semantics of* **LK** *in* **SET**.

Sloppy proof. By induction on a proof π of $\vdash^c \Gamma$ in **LK** we give a proof π' of the associated sequent $\neg\Gamma \vdash^i \phi$ in **LJ**, and we associate to π the denotational semantics of π'. The crucial case is that of a cut: we have to transform $\neg\Gamma, \sim P \vdash^i \phi$ and $\neg\Delta, P \vdash^i \phi$ into $\neg\Gamma, \neg\Delta \vdash^i \phi$; first prove $\neg\Delta \vdash^i \sim P$, and then by a cut, prove $\neg\Gamma, \neg\Delta \vdash^i \phi$. ¬□

This proof is extremely convincing for the following reason: the non-determinism of classical logic comes essentially from the cut-rule for which several cut-elimination protocols are possible; now the distinction positive/negative yields a unique way of interpreting cuts. *But wait a minute!* The rule which behaves very closely to a cut-rule is the introduction of conjunction: a cut between A and $\neg A$ is basically the same thing as introducing $A \wedge \neg A$. The step we went through during our sloppy proof corresponds to an introduction of a conjunction $P \wedge \sim P$ of two formulas of distinct polarities... if we look at the problem as a problem of conjunction rules, it is not unlikely that conjunction of formulas of the same polarity might be problematic.... In fact the sloppy proof stumbles on the rule of introduction of a conjunction of *positive* formulas (for negative ones not the slightest problem).

The problem, stated in intuitionistic terms, reduces to the following: we are given intuitionistic proofs of $\neg\Gamma, \sim P \vdash^i \phi$ and $\neg\Delta, \sim Q \vdash^i \phi$, and we want to produce a proof of $\neg\Gamma, \neg\Delta, \sim(P \cap Q) \vdash^i \phi$... suddenly we discover that we do not know how to do it! More precisely if we want to do it without introducing cuts, no obvious way! But at least we can do it with cuts; let us try: by $\vdash \supset$ we get $\neg\Gamma \vdash^i \sim\sim P$, $\neg\Delta \vdash i \sim\sim Q$, hence it suffices to remember that $\sim\sim P$, $\sim\sim Q$, $\sim(P \cap Q) \vdash^i \phi$ is provable:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash^i P$}
\AxiomC{}
\UnaryInfC{$Q \vdash^i Q$}
\BinaryInfC{$P, Q \vdash^i P \cap Q$}
\AxiomC{}
\UnaryInfC{$\phi \vdash^i \phi$}
\BinaryInfC{$P, Q, \sim(P \cap Q) \vdash^i \phi$}
\UnaryInfC{$P, \sim(P \cap Q) \vdash^i \sim Q$}
\AxiomC{}
\UnaryInfC{$\phi \vdash^i \phi$}
\BinaryInfC{$P, \sim\sim Q, \sim(P \cap Q) \vdash^i \phi$}
\UnaryInfC{$\sim\sim Q, \sim(P \cap Q) \vdash^i \sim P$}
\AxiomC{}
\UnaryInfC{$\phi \vdash^i \phi$}
\BinaryInfC{$\sim\sim P, \sim\sim Q, \sim(P \cap Q) \vdash^i \phi$}
\end{prooftree}
$$

and we are done ... NO!! Here we meet one of the big mysteries of classical logic, the reason why it cannot have any categorical semantics: $\sim\sim P$, $\sim\sim Q$, $\sim(P \cap Q) \vdash^i \phi$ is provable, *but in two different ways*: besides the proof θ_1 just written, there is another proof θ_2 of the same sequent in which $\sim\sim P$ is introduced before $\sim\sim Q$. As we shall see below there is no way to identify these two proofs: any attempt to do so will produce an immediate collapse of the system, like in those categories which are indeed Boolean algebras.

To the proof θ_1 we associate the λ-term (in the context X: $\sim\sim P, Y$: $\sim\sim Q$, Z: $\sim(P \cap Q)$):) $t_1[X, Y, z] := X(\lambda x{:}P.\, Y(\lambda y{:}Q.\, Z(\langle x, y\rangle)))$ where $\langle x, y\rangle$ is the pairing function, whereas θ_2 can be represented by $t_2[X, Y, Z] := Y(\lambda y{:}Q.\, X(\lambda x{:}P.\, Z(\langle x, y\rangle)))$. There is no possibility of equating these two terms. In fact if X and Y are respectively replaced by $A = \lambda z{:}{\sim}P.\, a$ and $B = \lambda w.\, {\sim}Q.\, b$, we get $t_1[A, B, Z] = a, t_2[A, B, Z] = b$.

This example is still related to the counterexample of Yves Lafont already twice quoted: typically if $\neg\Lambda$, $\neg A \vdash^i \phi$ and $\neg\Delta$, $\neg B \vdash^i \phi$ come both by weakenings from cut-free proofs of $\neg\Gamma \vdash^i \phi$ and $\neg\Delta \vdash^i \phi$ we must construct a cut-free proof of $\neg\Gamma$, $\neg\Delta$, $\neg(A \wedge B) \vdash^i \phi$ from what is available, i.e. our proofs of $\neg\Gamma \vdash^i \phi$ and $\neg\Delta \vdash^i \phi$. There is of course no good solution, unless we make some useful hypothesis, typically about polarities of A and B.... When A and B have been declared negative ($A = \sim P, B = \sim Q$) then $\neg(A \wedge B)$ is $P \vee Q$ and we can pass from $\neg\Gamma$, $P \vdash^i \phi$ and $\neg\Delta$, $Q \vdash^i \phi$ to $\neg\Gamma$, $\neg\Delta$, $P \vee Q \vdash^i \phi$ by usual left disjunction rule. When A $(= P)$ is positive and B $(= \sim Q)$ is negative, the use of the canonical proof of $\sim\sim P$, $\sim Q$, $\sim(P \cap \sim Q) \vdash^i \phi$ amounts to a weakening from $\neg\Delta \vdash^i \phi$ and symmetrically in the case negative/positive. But there is no reasonable way to decide the remaining case positive/positive.

This impossibility of getting a nice direct denotational semantics for **LK** is the reason for our introduction of a refined sequent calculus, **LC**.

1.3. *The sequent calculus* LC

A sequent of **LC** will be an expression $\vdash \Lambda; \Pi$ where Γ and Π are sequences of formulas, and Π is either empty or consists of exactly one positive formula. The space after ';' is called the *stoup*; the part Γ of the sequent is called the *body* and what is in the stoup is called the *head*. Usual sequents of **LK** correspond to sequents of **LC** with empty stoups.

Proof-theory has many reasons to distinguish one formula in a sequent:

(1) traditionally in intuitionistic sequent calculus, the right-hand side;
(2) in λ-calculus the notion of *headvariable* (or main hypothesis in natural deduction);

(3) in linear logic programming, Andreoli and Pareschi (1991) introduced something like a stoup to help proof-search;
(4) modalities like ! distinguish a unique formula in a sequent.

IDENTITY/NEGATION

$$\frac{}{\vdash \neg P; P}\ \textit{identity}$$

$$\frac{\vdash \Gamma; P \quad \vdash \neg P, \Delta; \Pi}{\vdash \Gamma, \Delta; \Pi}\ \textit{p-cut} \qquad \frac{\vdash \Gamma, N; \quad \vdash \neg N, \Delta; \Pi}{\vdash \Gamma, \Delta; \Pi}\ \textit{n-cut}$$

STRUCTURE

$$\frac{\vdash \Gamma; \Pi}{\vdash \sigma(\Gamma); \Pi}\ \textit{exchange} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, P;}\ \textit{dereliction}$$

$$\frac{\vdash \Gamma; \Pi}{\vdash \Gamma, A; \Pi}\ \textit{weakening} \qquad \frac{\vdash \Gamma, A, A; \Pi}{\vdash \Gamma, A; \Pi}\ \textit{contraction}$$

LOGIC

$$\frac{}{\vdash\ ; V} \qquad \frac{}{\vdash \Gamma, \neg F; \Pi}$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \wedge Q} \qquad \frac{\vdash \Gamma, A, B; \Pi}{\vdash \Gamma, A \vee B; \Pi}\ \text{when } A \vee B \text{ is negative}$$

$$\frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma, \Delta; P \wedge N} \qquad \frac{\vdash \Gamma, M; \quad \vdash \Gamma; Q}{\vdash \Gamma, \Delta; M \wedge Q}$$

$$\frac{\vdash \Gamma, M; \Pi \quad \vdash \Gamma, N; \Pi}{\vdash \Gamma, M \wedge N; \Pi} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma; P \vee Q} \qquad \frac{\vdash \Gamma; Q}{\vdash \Gamma; P \vee Q}$$

$$\frac{\vdash \Gamma, A; \Pi}{\vdash \Gamma, \forall x A; \Pi}\ \text{provided } x \text{ is not free in } \Gamma,\ \Pi \qquad \frac{\vdash \Gamma, N[t/x];}{\vdash \Gamma, \exists x N} \qquad \frac{\vdash \Gamma; P[t/x]}{\vdash \Gamma; \exists x P}$$

This calculus enables us to state our actual theorem.

Theorem 2. *The calculus* **LC** *has a denotational semantics in* **SET**.

Proof. There is a more or less transparent translation of **LC** sequents into intuitionistic ones: $\vdash \Gamma;$ becomes $\neg\Gamma \vdash^{i} \phi$ whereas $\vdash \Gamma;\ P$ is translated as $\neg\Gamma \vdash^{i} P$. All the rules of **LC** are easily interpreted in a non-ambiguous way using this translation. □

However, it is not clear that **LC** is a reasonable formalism for classical logic. In fact sequents $\vdash \Gamma; P$ are something new that cannot be tied to any classical intuition; but sequents $\vdash \Gamma;$ are the exact analogues of sequents $\vdash^{c} \Gamma$. In other terms, there is a translation from **LK** to **LC**. As far as provability is concerned the translation is faithful, i.e. $\vdash^{c} \Gamma$ is provable in **LK** iff $\vdash \Gamma;$ is provable in **LC**. Proof-theoretically speaking the situation is much more complex: this translation introduces cuts, i.e. certain proofs considered cut-free in **LK** will have cuts in **LC**. A step of this translation is conjunction introduction which splits into four cases, typically: from $\vdash \Gamma, P;$ and $\vdash \Delta, Q;$ it is possible to get $\vdash \Gamma, \Delta, P \wedge Q;$ by means of two n-cuts with the proof

$$\dfrac{\dfrac{\vdash \neg P; P \quad \vdash \neg Q; Q}{\vdash \neg P, \neg Q; P \wedge Q}}{\vdash \neg P, \neg Q, P \wedge Q;}$$

however the order of performance of these two *n-cuts does* matter, in other terms the passage from **LK** to **LC** is non-deterministic. But, once polarities have been fixed, non-determinism is wholly located inside introduction rules of formulas $P \wedge Q$.

Theorem 3. *The Hauptsatz holds for* **LC**. *Moreover the cut-elimination procedure leaves the denotational semantics invariant.*

Proof. It would be long and tedious to justify this fact that can be perfectly understood from the translation of **LC** into **LJ**. We just explain how the two cut-rules are handled:

(i) a p-cut (also called P-cut when P is specified)

$$\dfrac{\vdash \Gamma; P \quad \vdash \neg P, \Delta; \Pi}{\vdash \Gamma, \Delta; \Pi}$$

is eliminated by induction on the proof π' of the second premise $\vdash \neg P, \Delta; \Pi$; for induction loading one proves slightly more: π' is a proof of $\vdash \Lambda, \Delta; \Pi$ where Λ is a repetition of $\neg P$. The crucial step is when π' ends with a rule introducing an occurrence of $\neg P$ in Λ; of course a subordinated induction on the proof of $\vdash \Gamma; P$ is necessary in that case.

(ii) An n-cut (also called N-cut when N is specified)

$$\dfrac{\vdash \Gamma, N; \quad \vdash \neg N, \Delta, \Pi}{\vdash \Gamma, \Delta; \Pi}$$

is eliminated by induction on the proof π' of the second premise $\vdash \neg N, \Delta; \Pi$; for induction loading one proves slightly more: π' is a proof of $\vdash \Lambda, \Delta; \Pi$, where Λ is a repetition of $\neg N$. The crucial case is when π' ends with a dereliction rule introducing an occurrence of $\neg N$ in Λ in which case our n-cut is replaced by a p-cut. $\square$

As one can see the structure of cut-elimination is extremely complex with a lot of nested inductions... to illustrate this, let us come back to the problem of introduction of conjunction: if $\vdash \Gamma, A;$ and $\vdash \Delta, B;$ are cut-free provable, then by the provability of $\vdash \neg A, \neg B, A \wedge B;$ and two n-cuts, one gets a proof of $\vdash \Gamma, A \wedge B;$ that we can make cut-free by theorem 3.

Proposition 1. *If* $\vdash \Gamma, A;$ *and* $\vdash \Delta, B;$ *are cut-free provable, so is* $\vdash \Gamma, \Delta, A \wedge B;$.

Proof. We give a direct argument which is just an explicitation of the cut-elimination procedure; we consider the case where A and B are positive, i.e. $A = P, B = Q$.

Lemma 1. *Given cut-free proofs* π *of* $\vdash \Gamma, \Lambda; \Pi$, *where* Λ *is a repetition of* P, *and* λ *of* $\vdash \Delta; Q$ *one can build a cut-free proof* ρ *of* $\vdash \Gamma, \Delta, P \wedge Q; \Pi$.

Proof. Induction on π, the crucial case being that of a dereliction: from $\vdash \Gamma, \Lambda'; P$ conclude $\vdash \Gamma, \Lambda; \ldots$ the induction hypothesis yields ρ' of $\vdash \Gamma, \Delta, P \wedge Q;\ P$ and a conjunction between ρ' and λ yields $\vdash \Gamma, \Delta, P \wedge Q;\ P \wedge Q$ and one ends by dereliction + contraction. □

Lemma 2. *Given cut-free proofs π of $\vdash \Gamma, P$; and λ of $\vdash \Delta, \Lambda;\ \Pi$, where Λ is a repetition of Q one can construct a cut-free proof ρ of $\vdash \Gamma, \Delta,\ P \wedge Q;\ \Pi$.*

Proof. By induction on λ, the crucial case being that of a dereliction: from $\vdash \Delta, \Lambda';\ Q$ conclude $\vdash \Delta, \Lambda; \ldots$ then the induction hypothesis yields ρ' of $\vdash \Gamma, \Delta, P \wedge Q;\ Q$, and if we apply lemma 1 to π and ρ', we get $\vdash \Gamma, \Gamma, \Delta,\ P \wedge Q,\ P \wedge Q$; from which ρ is easily obtained by means of contractions. □

...the theorem is easily reduced to a particular case of lemma 2. □

The construction just given is violently asymmetrical in P and Q; in terms of cut-elimination, it depends on the order in which the two n-cuts have been performed.

1.4. *The disjunction property*

In this section we discuss the disjunction (and existence) properties of classical logic. These properties are well-known to fail classically and the existence of a sharper formalism like **LC** cannot change anything to that. However certain essential facts must be observed:

(i) Kreisel noticed the crucial fact that classical arithmetic enjoys existence property for Σ^0_1 formulas. This existence property is obtained by various techniques, but not by any immediate corollary of the *Hauptsatz*.
(ii) If we focus, not on provability, but on proofs, it is not unlikely that something can be said concerning the behaviour of disjunction (or existence).

This is why the following theorem is quite reasonable.

Theorem 4. (i) *If $\vdash ; P \vee Q$ is provable, then either $\vdash ; P$ or $\vdash ;\ Q$ is provable.*

(ii) *One can extract unambiguously from a cut-free proof of $\vdash P \vee Q$; either a proof of $\vdash P \vee Q;\ P$ or a proof of $\vdash P \vee Q;\ Q$.*

(iii) *When P and Q are purely positive* (i.e. *have no negative subformulas*) *we get more*: *when $\vdash P \vee Q$; is cut-free provable, we can extract unambiguously from the proof a proof of $\vdash ;\ P$ or $\vdash ; Q$*

Proof. (i) is immediate: no choice for the cut-free rules to be applied last.

(ii) is obtained as follows: we look for the last rules and they must be structural yielding proofs of $\vdash \Lambda; \Pi$ with Λ, Π a repetition of $P \vee Q \ldots$ the only way to break this is through a disjunction rule with premise $\vdash \Lambda; P$ or $\vdash \Lambda; Q$. The result follows from a structural rearrangement of Λ.

(iii) In order to prove the result, we work as in (ii) so as to arrive at—say—$\vdash \Lambda; Q$, then all sequents above $\vdash \Lambda; Q$ must be of the form $\vdash \Lambda'; R$ with R a subformula of Q and Λ' a repetition of $P \vee Q$: look at the rules. But the only axioms $\vdash \Lambda'; R$ are with Λ' empty, and we can simply erase all bodies of sequents in our proof...and we get a proof of $\vdash ; Q$. □

Part (iii) of our theorem is far from being very impressive since purely positive theorems are quite rare. However the result is already very interesting for a formula like $\mathbf{V} \vee \mathbf{V}$ whose cut-free proofs can therefore be used to program booleans: let us write it as $\mathbf{V}_1 \vee \mathbf{V}_2$ to distinguish the two occurrences of $\mathbf{V}$: we get the

Corollary. *A cut-free proof of* $\mathbf{V}_1 \vee \mathbf{V}_2$ *comes either from the axiom* $\vdash \mathbf{V}_1$ *or from the axiom* $\vdash \mathbf{V}_2$ *by structural manipulations.*

Proof. Corollary of (iii). □

What happens for existence? Roughly the same as for disjunction:

(i) if $\vdash ; \exists x A$ is probable then $\vdash A[t/x];$ is provable for some term t (moreover if A is positive we can claim that $\vdash ; A[t/x]$ is provable;
(ii) from a proof of $\vdash \exists x P;$ we can extract unambiguously a term t and a proof of $\vdash \exists x P;\ P[t/x];$
(iii) the case of a theorem $\vdash \exists x P;$ with P purely positive does not occur non-trivially in predicate calculus (however this case should become prominent in extensions to systems of arithmetic); in that case one extracts a proof of $\vdash ; P[t/x]$.

1.5. *Open problems in the syntax*

There are a lot of important questions left aside by this paper; let us mention:

(i) find a better syntax (which would be to **LC** what typed λ-calculus is to **LJ**) for normalization, typically with a Church–Rosser property. Using intuitionistic translation, natural deduction style (or λ-calculus style by Curry–Howard isomorphism), could be fine, but it stumbles on two major defects: the fact that disjunction and existence work badly, and the fact that the translation dualizes everything. A kind of proof-net could be the solution, and the fact that proof-nets are not available for full linear logic could be compensated by the fact that only certain linear configurations (see section 2) are used. While we are writing these lines, a new kind of syntax called *free deduction* is developed by Michel Parigot, and it is not unlikely that it could help to solve the problem.

(ii) Find the right formalization of Peano arithmetic with the property that Σ^0_1 formulas are purely positive. This involves a specific treatment of equality, of bounded quantifiers. The immediate output of this would be that a classical Π^0_2 theorem would be transformed into an algorithm by direct cut-elimination in a system of **LC**-sequents.

(iii) Find the right extension to polymorphism. The interest of a good polymorphic extension is that classical logic offers new facilities of programming, and that we now know that the basic meaning of specifications of the form 'data type D to data type D''—which is the core of the typed approach to programming and which is represented by proofs of $D \Rightarrow D'$—will be respected.

The classical quarrels about syntactical style—say between the Martin-Löf style and the polymorphic style of systems $\mathbb{F}$—will surely be influenced by the 'classical test':

(1) whereas polymorphism insists on defining data types by means of second-order universal quantification, the Martin-Löf style prefers inductive definitions. The classical

case will be better behaved with inductive definitions, since universal quantification yields *negative* formulas, whereas we would prefer to have data types positive (more: purely positive) which can be achieved in inductive definition style. However polymorphism retains all its usual interest.

(2) On the other hand Martin-Löf insisted on a unified treatment of implication and universal quantification by means of the primitive $\Pi x \in A\, B[x]$ and this seems to be quite unfortunate in classical terms since we are forced to declare this construction negative, whereas an implication can be positive... here classical logic seems to arbitrate against the proposal of Martin-Löf....

(iv) The retrospective study of the unbelievably complex '*cross-cut*' procedure of classical logic **LK**. For instance, if we take **LK** with polarities so as to distinguish between two conjunction rules, can we choose between the critical pairs of Gentzen's procedure so as to achieve a cut-elimination procedure which is denotationally invariant?

(v) Last but not least: put classical, intuitionistic and linear logic inside the same system (that one would call *LOGIC*) in such a way that these three systems appear as *fragments*. For instance the *stoup* seems to be a common feature (in linear logic something like that has been introduced by Andreoli and Pareschi for the purpose of *focalization* in linear logic programming). It is not possible to work directly with a translation in linear logic: the semantics of section 2 does not yield a *translation* classical $\mapsto$ linear since we assume something about the structure of atoms. In the long run, this is the most challenging question, whose answer would definitely increase the flexibility of logic.

2. The denotational semantics of classical logic

The denotational semantics that one can obtain through $\sim\sim$-interpretation is not quite satisfactory, essentially because it involves such a monster as the category **SET**. This is why we move to a more constructive semantics, in terms of the now familiar *coherent spaces* introduced in Girard (1986) under the name 'binary qualitative domains' and which are at the root of *linear logic*. The main reference is of course Girard (1987), but an annex with the main definitions and properties is included in this paper (the exponentials are slightly modified in order to get a semantic proof of the disjunction property, see theorem 9). Whereas section 1 emphasized the relation between intuitionistic and classical logic, we shall now mainly discover the deep links which tie linear and classical logics.

2.1. *Correlation spaces*

Definition 4. A *negative correlation space* (NCS) consists of a coherent space S together with:

(i) a clique $\perp(S) \sqsubset S$;
(ii) a clique $+_S \sqsubset (S \mathbin{⅋} S) \multimap S$; we use the notation $x + y \mapsto z$ to mean that $((x, y), z) \in +_S$.

These data must enjoy non-trivial properties:

(i) *neutrality of* $\perp(S)$: $\forall x\, \exists y \in \perp(S)\; x + y \mapsto x$;
(ii) *commutativity of* $+^S$: if $x + y \mapsto z$ then $y + x \mapsto z$;

(iii) *associativity of* $+_S$: if $y+z \mapsto t$ and $x+t \mapsto u$, then there is a v such that $x+y \mapsto v$ and $v+z \mapsto u$.

One can easily check that the y of (i) and the v of (iii) are unique. For instance in (iii) the solutions v of $x+y \mapsto v$ are pairwise coherent, whereas the solutions v of $v+z \mapsto u$ are pairwise incoherent.... As a consequence of (iii) one can write 'ternary sums' $x+y+z \mapsto u$, and more generally $\Sigma x_i \mapsto y$ (with two degenerated cases: $\Sigma x_i \mapsto y$ means $y = x_1$ when the sum consists of one element and means $y \in \bot(S)$ when the sum is empty).

Proposition 2. *The following are (naturally) equipped with a structure of NCS:*

(i) $\top, \bot$, *any space* $?X$;
(ii) $S \mathbin{\&} T$ *when* S *and* T *are already NCS;*
(iii) $S \parr T$ *when* S *and* T *are already NCS;*
(iv) *in particular coherent spaces which are 'with' of spaces* $?X_i$ *are NCS; this class is closed under (i), (ii) and (iii).*

Proof. (iv): the class of spaces $\&?X_i$ can be closed under $\&$ and $\parr$ only up to isomorphism; for instance the 'par' of two such spaces 'is' still of the same form because of the isomorphisms $?(X \oplus Y) \simeq ?X \parr ?Y$, and distributivity isomorphisms. For the other points, we give the precise definitions of the data $\bot(S)$ and $+_S$.

(i) $\bot(\top) = \varnothing$; $+_\top = \varnothing$,
$\bot(\bot) = \{0\}$; $+_\bot = \{((0,0),0)\}$,
$\bot(?X) = \{[\,]\}$; $+_{?X} = \{((s,s')), s+s'); s, s' \in |?X| \wedge s \asymp s'\}$;
(ii) $\bot(S \mathbin{\&} T) = \{(x,0);\ x \in \bot(S)\} \cup \{(y,1);\ y \in \bot(S)\}$ (i.e. $= \bot(S) \cup \bot(T)$) $(a,i)+(b,j) \mapsto (c,k)$ iff $i = j = k$ and $a+b \mapsto c$ (w.r.t. $+_S$ or $+_T$) depending on the common value 0 or 1 of i, j and k.
(iii) $\bot(S \parr T) = \bot(S) \times \bot(T)$

$$(x,',x'')+(y',y'') \mapsto (z',z'') \text{ iff } x'+y' \mapsto z' \text{ and } x''+y'' \mapsto z''.$$

The fact that these data enjoy the desired properties is immediate. $\square$

Definition 5. A *positive correlation space* (*PCS*) is the linear negation of a *NCS*. In other terms a PCS consists of a coherent space C equipped with:

(i) an anticlique $1(C) \sqsubset C^\perp$;
(ii) $+_C \sqsubset C \multimap (C \otimes C)$; one uses the notation $x \mapsto y+z$ for $(x,(y,z)) \in +_C$.

These data must enjoy *neutrality*, *commutativity* and *associativity* properties which are deduced from those of definition 4:

$$\forall x\, \exists y \in 1(C)\, x \mapsto x+y$$
$$x \mapsto y+z \text{ implies } x \mapsto z+y$$
$$u \mapsto x+t \text{ and } t \mapsto y+z \text{ implies the existence of } v \text{ such that } u \mapsto v+z \text{ and } v \mapsto x+y.$$

We also introduce the notation $x \mapsto \Sigma z_i$ symmetrically to the negative case; in particular $x \mapsto \Sigma s_i$ means $x \in \bot(C)$ or $x = x_1$ when there is zero or one summand.

Proposition 3. *The following are (naturally) equipped with a structure of PCS:*

(i) 0, 1, *any space* $!X$;
(ii) $C \oplus D$ *when* C *and* D *are already PCS;*
(iii) $C \otimes D$ *when* C *and* D *are already PCS;*
(iv) *in particular the coherent spaces which are 'plus' of spaces* $!X_i$ *are PCS; this class is closed under (i), (ii) and (iii).*

Proof. Immediately reduced to proposition 2. □

Proposition 4. *Let* C *be a PCS and assume that* $x \mapsto \Sigma x_i$; *then the atoms* $x, x_1, \ldots, x_n$ *are pairwise coherent.*

Proof. One constructs for each integer n a clique $+_{n_C}$ in $C \multimap \otimes_n C$:

— for $n = 0$, then $\otimes_0 C := 1$ and we define a clique in $C \multimap 1$ by $+_{0_C} := \{(x, 0);\ x \in 1(C)\}$;
— for $n = 1$, $+_{1_C}$ is the trace of the identity map of C;
— $+_{2_C} := +_C$;
— for greater values we can define $+_{n_C}$ from $+_C$ using the rules of linear logic; in fact there are for instance two candidates for $+_{3_C}$ but associativity equates them....

It is immediate that $(x,(x_1, \ldots, x_n)) \in +_{n_C}$ iff $x \mapsto \Sigma x_i$, and this shows that, given x, all sequences $(x_1, \ldots, x_n)$ such that $x \mapsto \Sigma x_i$ are pairwise coherent. If $x \mapsto \Sigma x_i$ then $x \mapsto \Sigma y_i$ for any permutation (y_i) of (x_i), hence $x_i \supset x_j$; also by neutrality, $x \mapsto \Sigma z_i$ where (z_i) is chosen so as to get $z_1 = x$, hence $x \supset x_1$ and by symmetry $x \supset x_1$. □

Definition 6. (i) Assume that the proof π of $\vdash \Gamma, A$ has been interpreted by a clique π^* and that Γ^* is a sequence of negative correlation spaces; then we can interpret the 'proof' ρ of $\vdash \Gamma, !A$ obtained from π by an *of course rule* by means of the set

$$\rho^* := \{\underline{x}[a_1, \ldots, a_k];\ \exists \underline{x}_1, \ldots, \underline{x}_k\ (\underline{x}_1 a_1, \ldots, \underline{x}_k a_k \in \pi^* \wedge \Sigma \underline{x}_i \mapsto \underline{x})\};$$

(ii) assume that the proof π of $\vdash \Gamma$ has been interpreted by a clique π^*; then if A is interpreted by a negative correlation space, we can interpret the 'proof' ρ of $\vdash \Gamma, A$ obtained from π by a *weakening rule* by the set $\rho^* := \{\underline{x}n; \underline{x} \in \pi^* \wedge n \in \perp(A)\}$.

(iii) Assume that the proof π of $\vdash \Gamma$, A, A has been interpreted by a clique π^* and that A has been interpreted by a negative correlation space; then the 'proof' ρ of $\vdash \Gamma$, A obtained from π by a *contraction rule* is interpreted by the set

$$\rho^* := \{\underline{x}c;\ \exists a, b\ (a + b \mapsto c \wedge \underline{x}ab \in \pi^*)\}.$$

The crucial point in (i) is that $\Sigma \underline{x}_i \mapsto \underline{x}$ forces the $\underline{x}_i$ to be pairwise incoherent (dual form of proposition 4), hence since the $\underline{x}_i a_i$ are pairwise coherent this forces $[a_1, \ldots, a_n] \in |\,!A|$.

Here we have to make an important categorical remark; as observed first by Yves Lafont (1988), the spaces $!X$ are commutative comonads (it is by the way unfortunate that the definitions of monad and comonad were both given in terms of tensor product: $?X$ is not a monad!). Now our last definition shows that in the interpretation of the rules for ?, only the rule of *dereliction* (which uses a relation between A and $?A$) is about the actual '?'; all the other ones work with arbitrary 'cocomonads'. In order to square the picture, it therefore suffices to observe that ? and ! are solution of an obvious universal problem.

Typically, $!X$ is the 'universal comonad (co)generated by X', and the rule 'of course' interprets precisely the solution to an obvious universal problem (the *cofree comonad* in Lafont, 1988).

Definition 7. (i) Let $A \sqsubset \Vdash \Xi, D$ where Ξ and D are respectively a sequence of *NCS* and a *PCS*; then A is said to be *central* when the following holds:

(1) if $\underline{x}_i y_i \in A$ for $i = 1, \dots, n$ and $\Sigma \underline{x}_i \mapsto \underline{x}$ then there is a y such that $\underline{x} y \in A$ and $y \mapsto \Sigma y_i$.

(2) if $\underline{x} y \in A$ and $y \mapsto \Sigma y_i$ then there are $\underline{x}_1, \dots, \underline{x}_n$ such that $\Sigma \underline{x}_i \mapsto \underline{x}$ and $\underline{x}_i y_i \in A$ for $i = 1, \dots, n$.

(ii) A clique $A \sqsubset C \multimap D$ (C and D *PCS*) is said to be *central* when the isomorphic clique $A' \sqsubset \Vdash C^{\perp}, D (A' = \{xy;\ (x, y) \in A\})$ is central.

(iii) A linear map f between *PCS* C and D is said to be *central* when $\mathrm{Tr}(f)$ is central as a clique in $C \multimap D$.

The notion of central clique is ... central in this paper. Definition (i) is the important one and (ii) is just a bureaucratic variant; (iii) indeed presents central cliques as the traces of certain maps; if one translates conditions (1) and (2) one will discover that central maps are just those preserving the 'sum'. More precisely f is central if for any n ($n = 0, 2$ suffice for obvious reasons) $+^{n}{}_{D} \circ f = \otimes_n f \circ +^{n}{}_{C}$ ($+^{n}{}_{C}$ has been introduced in the proof of proposition 4 as a clique, and here we refer to the associated linear map from C *to* $\otimes_n C$; $\otimes_n f$ is the linear map from $\otimes_n C$ to $\otimes_n D$ canonically induced by f). In terms of categories we get the following universal characterization of 'of course'.

Theorem 5. *Let X be a coherent space; then consider the PCS $!X$ and the linear map δ_X (δ for dereliction) from $!X$ to X defined by $Tr(\delta_X) = \{([x], x);\ x \in |X|\}$. Then given any PCS C and any linear map f from C to X, there is a unique map $!f$ from C to $!X$ enjoying*:

(i) $\delta_X \circ !f = f$;
(ii) *$!f$ is central.*

Proof. Up to trivial manipulations (write $\Vdash C^{\perp}, X$ or $\Vdash C^{\perp}, !X$ instead of $C \multimap X$ or $C \multimap !X$), $!f$ is obtained from f by an 'of course' rule. More precisely

$$\mathrm{Tr}(!f) = \{(c, [x_1, \dots, x_k]);\ \exists c_1, \dots, c_k\ ((c_1, x_1), \dots, (c_k, x_k) \in \mathrm{Tr}(f) \wedge c \mapsto \Sigma c_i)\}.$$

(i) is obtained as follows: $\mathrm{Tr}(\delta_X \circ !f) = \{(c, x);\ (c, [x]) \in \mathrm{Tr}(!f)\}$; but since $c_1 \mapsto c$ exactly when $c = c_1$, $(c, [x]) \in \mathrm{Tr}(1\phi)$ iff $(c, x) \in \mathrm{Tr}(f)$. Hence $\mathrm{Tr}(\delta_X \circ !f) = \mathrm{Tr}(f)$.

(ii) Splits into two parts:

(i) if $(c_i, s_i) \in \mathrm{Tr}(!f)$ for $i = 1, \dots, n$ and $c \mapsto \Sigma c_i$ we must find an s such that $(c, s) \in \mathrm{Tr}(!f)$ and $s \mapsto \Sigma s_i$. The c_i are pairwise coherent (proposition 4), hence since the (c_i, s_i) are pairwise coherent, the s_i must be pairwise coherent as well; then Σs_i is defined in $!X$ and we can take $s = \Sigma s_i$. Now writing $s_i = [x_{i1}, \dots, x_{ij}, \dots]$, we get $s = \Sigma_{ij} [x_{ij}]$; by the definition of $!f$ we can find c_{ij} such that $c_i \mapsto \Sigma_j c_{ij}$ and $(c_{ij}, x_{ij}) \in \mathrm{Tr}(f)$. If we observe that $c \mapsto \Sigma_{ij} c_{ij}$, we get $(c, s) \in \mathrm{Tr}(f)$.

(2) If $(c, s) \in \mathrm{Tr}(!f)$ and $s \mapsto \Sigma s_i$ then we must find $c_1, \dots, c_n$ such that $c \mapsto \Sigma c_i$ and $(c_i, s_i) \in \mathrm{Tr}(!f)$ for $i = 1, \dots, n$. We write $s_i = \Sigma_j [x_{ij}]$, and the definition of $!f$ yields c_{ij} such that $c \mapsto \Sigma_{ij} c_{ij}$ and $(c_{ij}, x_{ij}) \in \mathrm{Tr}(f)$. Now one easily checks the existence of (unique) c_i such

that $c \mapsto \Sigma_i c_i$ and $c_i \mapsto_j c_{ij}$: this is a consequence of associativity of $+_C$. Then the definition of $!f$ yields $(c_i, s_i) \in \mathrm{Tr}(!f)$.

The *unicity* of $!f$ is more or less obvious and therefore omitted. □

Proposition 5

(i) *The* $+^n{}_C$ *(seen as linear maps from* C *to* $\otimes_n C$*) are central;*
(ii) *if* f *is a central map from* C *to* D*, if* g *is a central map from* D *to* E *then* $g \circ f$ *is a central map from* C *to* E*;*
(iii) *if* f *and* f' *are central maps from* C *to* D *and* C' *to* D'*, then the (canonical) linear map* $f \otimes f'$ *from* $C \otimes C'$ *to* $D \otimes D'$ *is central;*
(iv) *the canonical maps from* C *to* $\oplus D$ *and* D *to* $C \oplus D$ *are central;*
(v) *if* f *is a central map from* C *to* D *and* g *is linear map from* D *to* X*, then* $!(g \circ f) = (!g) \circ f$.

Proof. All these properties are moreorless obvious; for instance (v) is a consequence of the unicity part of theorem 5. □

2.2. *The denotational semantics of LC*

Definition 8. To each formula A of classical logic we associate a correlation space A^*; when A is positive, A^* is a *PCS* and negative formulas are interpreted by means of *NCS*.

(i) We have to specify p^* when p is a proper atom of the language; we define $(\neg p)^* := p^{*\perp}$.
(ii) The logical atoms **V** and **F** are respectively interpreted by the trivial *PCS* 1 and 0.
(iii) Compound formulas by means of table 3.

Table 3. *Correlation semantics*

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$\forall x A$	$\exists x A$	$\neg A$
C	D	$C \otimes D$	$C \oplus D$	$C^\perp \parr ?D$	$\&?C_i$	$\oplus C_i$	$C^\perp$
S	D	$!S \otimes D$	$S \parr ?D$	$S^\perp \oplus D$	$\& S_i$	$\oplus !S_i$	$S^\perp$
C	T	$C \otimes !T$	$?C \parr T$	$C^\perp \parr T$			
S	T	$S \& T$	$S \parr T$	$?S^\perp \parr T$			

(As with the previous tables, the columns for $\neg$ and $\Rightarrow$ are not exactly part of the definition; C, D, E stand for arbitrary *PCS*, and S, T, U for arbitrary *NCS*. In the case of quantifiers, $A[x]$ is interpreted by a family (C_i) or (S_i) indexed by the domain of interpretation of terms and the interpretation involves generalized & or $\oplus$ over the domain.)

The actual starting point of this work is the next theorem.

Theorem 6. *The exact analogues of the isomorphism of theorem* 1 *hold for correlation spaces.*

Proof. Let us state these isomorphisms:

(i) involutivity of negation: $\neg\neg X \simeq X$;
(ii) De Morgan isomorphisms: $\neg(X \wedge Y) \simeq \neg X \vee \neg Y$; $\neg(X \vee Y) \simeq \neg X \wedge \neg Y$; $x \Rightarrow Y \simeq \neg X \vee Y$; $\neg\forall x X \simeq \exists x \neg X$; $\neg\exists x X \simeq \forall x \neg X$ etc;

(iii) associativity isomorphisms: $X \vee (Y \vee Z) \simeq (X \vee Y) \vee Z$; $X \wedge (Y \wedge Z) \simeq (X \wedge Y) \wedge Z$; and therefore $(X \wedge Y) \Rightarrow Z \simeq X \Rightarrow (Y \Rightarrow Z)$; $X \Rightarrow (Y \vee Z) \simeq (X \Rightarrow Y) \vee Z$;

(iv) neutrality isomorphisms: $X \vee 0 \simeq X$; $X \wedge \top \simeq X$;

(v) commutativity isomorphisms: $X \vee Y \simeq Y \vee X$; $X \wedge Y \simeq Y \wedge X$; hence $X \Rightarrow Y \simeq \neg Y \Rightarrow \neg X$;

(vi) distributivity isomorphisms with restriction on polarities: $X \wedge (C \vee D) \simeq (X \wedge C) \vee (X \wedge D)$; $X \vee (S \wedge T) \simeq (X \vee S) \wedge (X \vee T)$; and therefore $X \Rightarrow (S \wedge T) \simeq (X \Rightarrow S) \wedge (X \Rightarrow T)$; $(C \vee D) \Rightarrow X \simeq (C \Rightarrow S) \wedge (D \Rightarrow X)$;

(vii) idempotency isomorphisms: $C^+ \simeq C$; $S^- \simeq S$; and therefore $X^{++} \simeq X^+$; $X^{--} \simeq X^-$;

(viii) quantifiers isomorphisms: $X \wedge \exists x C \simeq \exists x(X \wedge C)$; $X \vee \forall x S \simeq \forall x(X \wedge S)$; and therefore $X \Rightarrow \forall x S \simeq \forall x(X \Rightarrow S)$; $\exists x C \Rightarrow X \simeq \forall x(C \Rightarrow X)$.

Proof. (i) and (ii) have a clear meaning only if negation and implication are primitive; with our conventions they mean the correctness of the two unnecessary columns for $\neg$ and $\Rightarrow$.

(iii) By De Morgan we can content ourselves with disjunction. A few typical cases are enough:

$$C \oplus (D \oplus E) \simeq (C \oplus D) \oplus E \text{ (case } +, +, +)$$

$$S \mathbin{⅋} ?(D \oplus E) \simeq (S \mathbin{⅋} ?D) \mathbin{⅋} ?E \text{ (case } -, +, +\text{); crucial use of } ?(X \oplus Y) \approx ?X \mathbin{⅋} ?Y$$

$$S \mathbin{⅋} (T \mathbin{⅋} ?E) \simeq (S \mathbin{⅋} T) \mathbin{⅋} ?E \text{ (case } -, -, +)$$

$$S \mathbin{⅋} (T \mathbin{⅋} U) \simeq (S \mathbin{⅋} T) \mathbin{⅋} U \text{ (case } -, -, -).$$

(iv) For instance for disjunction: $C \oplus 0 \simeq C$ (case $+$), $S \mathbin{⅋} ?0 \simeq S$ (case $-$); in the latter case the isomorphism $?0 \simeq \perp$ is essential.

(v) Is immediate from the symmetry of $\oplus$ and $⅋$.

(vi) For instance distributivity of $\wedge$ over $\vee$ follows from : $E \otimes (C \oplus D) \simeq (R \otimes C) \oplus (R \otimes D)$, $!S \otimes (C \oplus D) \simeq (!S \otimes C) \oplus (!S \otimes D)$.

(vii) Is obvious; it is worth remarking that C^- is $?C \mathbin{⅋} \perp$, hence since $\perp$ is neutral w.r.t. $⅋$, C^- is $?C$; S^- is $S \mathbin{⅋} \perp$ which is S. In the same way, C^+ is C whereas S^+ is $!S$.

(viii) is a quantifier analogue of (vi). □

Definition 9. If X is a correlation space, then X^- is defined as X if X is negative and as $?X$ if X is positive. If Ξ is a sequence of correlation spaces, then $\Vdash^c \Xi;$ denotes the coherent space $\Vdash \Xi^-$ (which is canonically equipped with a structure of *NCS*); if Ξ is a sequence of correlation spaces and C is a *PCS*, then $\Vdash^c \Xi; C$ denotes the coherent space $\Vdash \Xi^-, C$.

Theorem 7. *To any proof π of a sequent $\vdash \Gamma; \Pi$ of* **LC** *one can associate its denotation π^*; π^* is a clique in the coherent space $\Vdash^c \Gamma^*; \Pi^*$, and in case Π is non empty π^* is central.*

(1) *The denotation is non-trivial (there are two proofs of the same sequent with distinct denotations).*

(2) *The denotation is closed under logical rules (this means that when we apply a rule to a proof ρ to get a proof π, then π^* depends only on ρ^*).*

(3) *The denotation is invariant under cut-elimination.*

Proof. We now give the precise definition of π^* by induction on π:

(i) If π is an identity axiom $\vdash \neg P; P$, then $\pi^* := \{pp;\ p \in |P^*|\}$ is a central clique in

$\Vdash^c P^{*\perp}$; P^* $(= \Vdash P^{*\perp}, P^*)$ since it is (isomorphic to) the trace of the identity map from P^* to P^* which is central....

(ii) Case of a p-cut: we start with $\rho^* \sqsubset \Vdash \Gamma^{*-}, P^*$ and $\lambda^* \sqsubset \Vdash P^{*\perp}, \Delta^{*-}, \Pi^*$, then we can easily form a clique $\pi^* \sqsubset \Vdash \Gamma^{*-}, \Delta^{*-}, \Pi^*$ by applying the cut-rule of linear logic. However, in case Π^* is non-empty, we must check the *centrality* of π^*; this can be reduced to preservation of centrality by composition and tensorization (proposition 5).

(iii) Case of an n-cut: we start with $\rho^* \sqsubset \Vdash \Gamma^{*-}, N^*$ and $\lambda^* \sqsubset \Vdash ?N^{*\perp}, \Delta^{*-}, \Pi^*$, from which we first form a clique $!\rho^* \sqsubset \Vdash \Gamma^{*-}, !N^*$ (of course rule) and by cut with λ^* we get $\pi^* \sqsubset \Vdash \Gamma^{*-}, \Delta^{*-}, \Pi^*$. When Π is non-empty, the centrality of π^* is justified in the same sloppy way as in (ii) (the centrality of $!\rho^*$ is used).

(iv) From $\rho^* \sqsubset \Vdash \Gamma^*, \Pi^*$, we can define $\pi^* \sqsubset \Vdash \sigma(\Gamma^{*-}), \Pi^*$ by means of σ; if Π^* is non-empty, then the transformation preserves centrality.

(v) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, P^*$ we define $\pi^* \sqsubset \Vdash \Gamma^{*-}, ?P^*$ by $\pi^* = \{x[p];\ \underline{x}p \in \rho^*\}$.

(vi) From $\rho^* \sqsubset \Gamma^{*-}, \Pi^*$ we define $\pi^* \sqsubset \Vdash \Lambda^{*-}, A^{*-}, \Pi^*$ by means of a weakening rule; in case Π is non-empty, one must check that this operation preserves centrality: but weakening on A^{*-} behaves like a composition with the map $+_0$ from $(A^{*-})^\perp$ to 1 and which is central....

(vii) From $\rho^* \sqsubset \Gamma^{*-}, A^{*-}, A^{*-}, \Pi^*$ we define $\pi^* \sqsubset \Vdash \Gamma^{*-}, A^{*-}, \Pi^*$ by means of a contraction rule; in case Π is non-empty, one must check that this operation preserves centrality: but contraction behaves like a composition with the map $+_2$ from $(A^{*-})^\perp$ to $(A^{*-})^\perp \otimes (A^{*-})^\perp$ and which is central too....

(viii) The axiom $\vdash$; $\mathbf{V}$ is interpreted by the clique $\{0\} \sqsubset \Vdash 1$ which is central.

(ix) The axiom $\vdash \neg\mathbf{F}$; Π is interpreted by the clique $\varnothing \sqsubset \Vdash \top, \Pi^*$.

(x) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, P^*$ and $\lambda^* \sqsubset \Vdash \Delta^{*-}, Q^*$, one can form $\pi^* \sqsubset \Vdash \Gamma^{*-}, \Delta^{*-}, P^* \otimes Q^*$: $\pi^* {:=} \{\underline{xy}(p,q);\ \underline{x}p \in \rho^* \wedge \underline{x}q\, \lambda\, \Lambda^*\}$; if ρ^* and λ^* are central, so is π^* (proposition 5).

(xi) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, P^*$ and $\lambda^* \sqsubset \Vdash \Delta^{*-}, N^*$, one can form $\pi^* \sqsubset \Vdash \Gamma^{*-}, \Delta^{*-}, P^* \otimes !N^*$: in a first step form $!\lambda^* \sqsubset \Vdash \Delta^{*-}, !N^*$, then proceed as in x); centrality of π^* follows from the centrality of ρ^* and of any $!f$.

(xii) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, M^*$ and $\lambda^* \sqsubset \Vdash \Delta^{*-}, Q^*$, one can form $\pi^* \sqsubset \Vdash \Gamma^{*-}, !M^* \otimes Q^*$: this is symmetric to (xi).

(xiii) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, M^*, \Pi^*$ and $\lambda^* \sqsubset \Vdash \Gamma^{*-}, N^*, \Pi^*$ one can form $\pi^* \sqsubset \Vdash \Gamma^{*-}, M^* \& N^*, \Pi^*$ by applying a with rule; in case P is non-empty, centrality is preserved.

(xiv) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, A^{*-}, B^{*-}, \Pi^*$ we can define $\pi^* \sqsubset \Vdash \Gamma^{*-}, A^{*-} \parr B^{*-}, \Pi^*$. But if one of A and B is negative, then $(A \vee B)^{*-} = (A \vee B)^* = A^{*-} \parr B^{*-}$; this transformation preserves centrality in case Π is non-empty.

(xv) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, P^*$, we can define $\pi^* \sqsubset \Vdash \Gamma^{*-}, P^* \oplus Q^*$ by a left plus rule; this transformation preserves centrality (proposition 5).

(xvi) From $\rho^* \sqsubset \Vdash \Gamma^{*-}, Q^*$, we can define $\pi^* \sqsubset \Vdash \Gamma^{*-}, P^* \oplus Q^*$ by a right plus rule; this transformation preserves centrality.

(xvii)–(xix) The case of quantifiers is left to the reader ... no problem and limited interest.

(1) comes from the fact that the two proofs of $V \vee V$ mentioned in the corollary to

theorem 4 have distinct semantics; (2) is immediate; the precise verification of (3) is very long and without surprise. □

P-cuts commute with each other; *centrality* extends commutation to n-cuts:

Proposition 6. *Consider cliques* $a \sqsubset \Vdash^c \Xi, S, T;$ $b \sqsubset \Vdash^c \Xi; S^\perp$ $c \sqsubset \Vdash^c \Xi'', T^\perp; \Pi$ *with* b *central; then the cliques* $d, d' \sqsubset \Vdash^c \Xi, \Xi', \Xi''; \Pi$ *obtained by applying an n-cut and a p-cut (and which correspond to the two different orders of performance of these cuts) are equal.*

Proof. We give the proof only in a significant particular case, where $a \sqsubset \Vdash^c S, T;$ $b \sqsubset \Vdash^c E^\perp; S^\perp$ $c \sqsubset \Vdash^c T^\perp; D$. Then we can associate to a, b and c linear maps f from $S^\perp$ to T, g from E to $S^\perp$ and h from $!T$ to D, g and h being central. Now the problem is to compare $h \circ (!f) \circ g$ and $h \circ !(f \circ g)$; but proposition 5(v) yields $!(f \circ g) = (!f) \circ g$. □

Definition 10. *Let* Ξ be a sequence of NCS and let C be a PCS; a clique a $\sqsubset \Vdash \Xi, ?C$ is *central* (w.r.t. C) if one can write a as $\delta(b) := \{\underline{x}[c];\ \underline{x}c \in b\}$ for an appropriate central clique $b \sqsubset \Vdash \Xi, C$.

Proposition 7. *An n-cut between* $\rho^* \sqsubset \Vdash^c \Gamma^*, N^*;$ *and a central clique* $\delta(b) \sqsubset \Vdash^c \neg N^*, \Delta^*;$ *yields the same result as a p-cut between* $b \sqsubset \Vdash^c \Delta^*; \neg N^*$ *and* ρ^*.

Proof. Immediate computation using $\delta_X \circ !f = f$. □

Theorem 8. *Let* $a \sqsubset \Vdash^c \Xi, \Xi';$ *where* Ξ *is a sequence* $X_1, \ldots, X_k$ *of correlation spaces, and let* $b_i \sqsubset \Vdash^c \Xi_i, X_i^\perp;$ $(i = 1, \ldots, k)$ *be cliques such that for all negative* X_i *(except perhaps for one of them),* b_i *is central w.r.t.* $X_i^\perp$. *Then the clique in* $\Vdash^c \Xi', \Xi_1, \ldots, \mathrm{X}_k;$ *obtained by applying k n-cuts between a and the* b_i *does not depend on the order of performance of these cuts.*

Proof. Let $Y_i := X_i^-$, let $\Xi'' := Y_1, \ldots, Y_k$ (observe that $\Vdash^c \Xi, \Xi';$ is equal to $\Vdash^c \Xi'', \Xi';$) and let $c_i \sqsubset \Vdash^c \Xi_i;$ $Y^\perp{}_i$ be defined by $c_i := !b_i$ if X_i is positive and $b_i = \delta(c_i)$ if X_i is negative and b_i is central; if X_i is negative and b_i is not assumed to be central, then $c_i := b_i \sqsubset \Vdash^c \Xi_i, Y^\perp{}_i;$. Now our k n-cuts can be replaced by k cuts between a and the c_i, and all these cuts but perhaps one are p-cuts (by proposition 7 when X_i is negative and b_i is central and by the definition of the semantics of n-cut when X_i is positive). The result follows by an immediate generalization of proposition 6. □

It is time to give a last look at the problems of the conjunction in **LK**; the problem is (given cliques $a \sqsubset \Vdash^c A^*;$ and $b \sqsubset \Vdash^c B^*;$) to build a clique $c \sqsubset \Vdash^c \Gamma^*, \Delta^*, (A \wedge B)^*;$ and the obvious way to do it is by means of two n-cuts with the clique $\kappa^* \sqsubset \Vdash^c \neg A^*, \neg B^*, (A \wedge B)^*;$ which interprets the canonical proof κ of $\vdash \neg A, \neg B, A \wedge B;$. We denote the two possible cliques by $\langle a, b\rangle_1$ and $\langle a, b\rangle_2$. Now by theorem 8 these two 'pairs' are equal in the following cases:

(i) A (or B) is negative.
(ii) A and B are positive and a (or b) is central; of course when both a and b are central $a = \delta(a')$, $b = \delta(b')$, so is $c = \langle a, b\rangle_1 = \langle a, b\rangle_2$ and $c = \delta(c')$ where c' is obtained from a' and b' by means of the conjunction rule of **LC**.

Now traditional constructivism is based on an implicit *conjunction property*: a cut-free proof of $\vdash A \wedge B$ is the pair of a proof of $\vdash A$ and a proof of $\vdash B$. Now observe that:

(i) from a clique $c \sqsubset \Vdash^c A \wedge B$; we can recover cliques $pr_1(c) \sqsubset A$; and $pr_2(c) \sqsubset B$; (n-cut with $\vdash \neg(A \wedge B), A$; or $\vdash \neg(A \wedge B), B$;)

(ii) $pr_1(\langle a, b\rangle_i) = a$, $pr_2(\langle a, b\rangle_i) = b$, for $i = 1, 2$ in particular when both of A and B are positive, since we know that $\langle a, b\rangle_1$ may be distinct from $\langle a, b\rangle_2$ (see the refutation of our wrong theorem (section 1.2.): from $\theta_1^* = \theta_2^*$ we would get $a^* = b^*$); from the projections of a clique $c \sqsubset \Vdash^c (A \wedge B)^*$; there is no way to recover c.

(iii) In fact the conjunction property fails as soon one of A or B is positive; for instance take a clique $a \sqsubset \Vdash^c C, C$; and cliques $b, c \sqsubset \Vdash^c Y$; and form cliques $d' \sqsubset \Vdash^c C \wedge Y, C$; (use a and b), then $d'' \sqsubset \Vdash^c C \wedge Y, C \wedge Y$; (use d' and c) and finally $d \sqsubset \Vdash^c C \wedge Y$; (use contraction). $pr_2(d)$ is equal to b or to c (which one depends on which occurrence of C is *preferred* in the semantical form of the weak disjunction property, see next section). But if $pr_2(d) = b$ there will be no way of recovering c....

(iv) The conjunction property holds when both of A and B are negative; the reader might be curious to learn what happens to the counterexample of (iii) when C and Y are replaced by *NCS* S and T. We have first to observe that the formulation of conjunction is *additive* (S & T), i.e. that we have to modify contexts by weakenings: b is changed into $b' \sqsubset \Vdash^c T, S$; and c into $c' \sqsubset \Vdash^c S \,\&\, T, T$; : some asymmetry has been introduced by the order in which the conjunction rules have been performed. Now the clique $d'' \Vdash^c S \wedge T, S \wedge T$; is equal to

$$a \cup \{x_1 z;\ x_1 \in b,\ z \in \bot(S)\} \cup \{z x_2;\ z \in \bot(S) \cup \bot(T),\ x_2 \in c\}$$

(we assume $|S| \cap |T| = \varnothing$ to simplify notations).

Now if we apply contraction to d'' we obtain d which is a union $\sigma \cup \tau$ of a clique in $\Vdash^c S$; and a clique in $\Vdash^c T$;. The points $x_1 z$ do not contribute to τ because $x_1 \frown z$ (mod. S & T) when $z \in |S|$, and by proposition 4 (dualised) $x_1 + z \mapsto y$ is impossible; but one point $z x_2$ (with $z \in \bot(T)$) does contribute to τ, namely the unique $z \in \bot(T)$ such that $z + x_2 \mapsto x_2$ and yields x_2 in τ. We see that τ is equal to c, and the apparent contradiction is explained. This is another illustration of the sensitivity of the semantics—i.e. of the behaviour w.r.t. cut-elimination—to apparent innocuous permutation of rules.

The failure of the conjunction property (more precisely of any reasonable formulation of this informal idea) definitely establishes the impossibility of any direct categorical semantics of classical logic, since classical conjunction is not a product. On the other hand we insist on the fact that the categorical paradigm is still relevant to classical logic, typically through the use of central cliques, which are the morphisms of *PCS*. In some sense there is a categorical backbone (the category of *PCS*) of classical logic. More precisely we shall see that correlation domains are defined by dualization or bidualization of the notion of central clique.

2.3. *Correlation domains*

Our basic problem is the following: we can associate to each classical proof π of a sequent $\vdash \Gamma; \Pi$ a clique $\pi^* \sqsubset \Vdash^c \Gamma^*; \Pi^*$. However there are many cliques in $\Vdash^c \Gamma^*; \Pi^*$ that are not of the form π^*; for example, if A is not provable no clique in $\Vdash^c A^*$; is of the form π^*.

The idea would be to add to the definition of a correlation space X the notion of being a 'total clique', with the idea that total cliques are something like the interpretation of proofs of X. But this conflicts with consistency—if we want for instance to have total cliques in any space—. For that reason we shall not look at X but at $\Vdash^c X, 1;$ Formulas $\vdash A, \mathbf{V}:$ are always provable (first prove $\vdash \mathbf{V};$ then weaken on A), but there is also the possibility of proving $\vdash A;$ and then weakening on $\mathbf{V}$ which is the one which actually interests us, not to speak of intricate interleavings of both formulas.... This suggests the following definitions.

Definition 11. Let C be a positive correlation space; the *bilinear form* associates to any cliques $\alpha \sqsubset \Vdash^c C, 1;$ and $\beta \sqsubset \Vdash^c C^\perp, 1;$ a clique (α, β) in $\Vdash^c 1; :$ first perform an n-cut between α and β, then a contraction and get $(\alpha, \beta) \sqsubset \Vdash^c 1;$. Next observe that $\Vdash^c 1;$ is the coherent space $?1$ whose atoms are of the form $n.[0]$ for $n \in \mathbb{N}$ (pairwise incoherent): it follows that (α, β) is either empty or a singleton $\{n.[0]\}$. We define $\langle \alpha, \beta \rangle := n$ when $(\alpha, \beta) = \{n.[0]\}$ and $\langle \alpha, \beta \rangle := -\infty$ when $(\alpha, \beta) = \varnothing$.

The notation $\alpha \perp \beta$ is shorthand for $\langle \alpha, \beta \rangle = 1$. It means that (α, β) is the denotation of the canonical proof of $\vdash 1;$.

Proposition 8. *Let* $\alpha \sqsubset \Vdash^c C, 1;$ $\beta \sqsubset \Vdash^c C^\perp, 1; :$ *then* $\langle \alpha, \beta \rangle \neq -\infty$ *iff there are atoms* $z(m.[0]) \in \alpha$ *and* $z_1(p_1.[0]), \ldots, z_k(p_k[0]) \in \beta$ *such that* $z = [z_1, \ldots, z_k]$, *in which case* $\langle \alpha, \beta \rangle = m + p_1 + \ldots + p_k$.

Proof. Immediate denotational computation. □

Definition 12. A *negative correlation domain* (*NCD*) S is a negative correlation space S_s together with a set S_d (the *correlations* of S) of cliques in $\Vdash^c S_s, 1;$ which is the orthogonal of a set of *central* cliques (w.r.t. $S_s^\perp$) in $\Vdash^c S_s^\perp, 1;$

A *positive correlation domain* (*PCD*) C is a positive correlation space C_s together with a set C_d (the *correlations* of C) of cliques in $\Vdash^c C_s, 1;$ which is the biorthogonal of a set Z of central cliques (w.r.t. C_s) in $\Vdash^c C_s, 1;$ (Z can be taken as the set of central elements of C_d).

The nightmare of denotational semantics is empty domains. Here we can sleep well.

Proposition 9. *A correlation domain is never empty.*

Proof. By applying the criterion of proposition 8, we get $\langle \{[\,][0]\}, \varnothing \rangle = 1$, hence $\langle \{[\,][0]\}, \beta \rangle = 1$ for any $\beta \sqsubset \Vdash^c C_s^\perp, 1;$ and this shows that $\{[\,][0]\} \in C_d$ for any positive correlation domain. On the other hand, take the clique $\beta = \{z[0]; z \in \perp(S_s)\}$ in $\Vdash^c S_s\, 1;$ and cut it with a central clique α in $\Vdash^c S_s^\perp, 1; :$ α contains an atom $[z][\,]$ with $z \in \perp(S_s)$ and it is immediate that $\alpha \perp \beta$, hence negative correlation spaces are non-empty. □

Definition 13. (i) The negation of (C_s, C_d) is defined as $(C_s^\perp, C_d^\perp)$; the negation of (S_s, S_d) is defined as $(S_s^\perp, S_d^\perp)$ (in $C_d^\perp$, $S_d^\perp$ the symbol $(.)^\perp$ refers to the orthogonality defined in definition 11).

(ii) The conjunction of correlation domains is defined as follows:

(1) $(C_s, C_d) \wedge (D_s, D_d) = (C_s \otimes D_s, Z^{\perp\perp})$ where the set Z is obtained by taking all pairs $(\delta(\alpha), \delta(\beta))$ of central correlations in C and D, then applying *conjunction*, *contraction* and *dereliction* to (α, β) so as to get a central clique in $\Vdash^c C \otimes D, 1;$

(2) $(C_s, C_d) \wedge (T_s, T_d) = (C_s \otimes\, !T_s, Z^{\perp\perp})$ where Z is obtained by taking all pairs $(\delta(\alpha), \beta)$ of a central correlation in C and a correlation in T, then applying *of course* to β to get $!\beta$

then *conjunction*, *contraction* and *dereliction* to $(\alpha, !\beta)$ so as to get a central clique in $\Vdash^c C \otimes !T, 1$;

(3) $(S_s, S_d) \wedge (D_s, D_d)$ is defined in a symmetrical way;

(4) $(S_s, S_d) \wedge (T_s, T_d) = (S_s \,\&\, T_s, S_d \,\&\, T_d)$: $S_d \,\&\, T_d$ refers to all cliques that can be obtained by *conjunction* from a clique in S_d and a clique in T_d.

(iii) The disjunction of correlation domains is defined as follows:

(1) $(S_s, S_d) \vee (T_s, T_d) = (S_s ⅋ T_s, Z^\perp)$, where the set Z is obtained by taking all pairs $(\delta(\alpha), \delta(\beta))$ of central correlations in $S_s^\perp$ and $T_s^\perp$, then applying *conjunction*, *contraction* and *dereliction* to them so as to get a central clique in $\Vdash^c S_s^\perp \otimes T_s^\perp, 1$;

(2) $(S_s, S_d) \vee (D_s, D_d) = (S_s ⅋ ?D_s, Z^\perp)$ where Z is obtained by taking all pairs $(\delta(\alpha), \beta)$ of a central correlation in $S_s^\perp$ and a correlation in $D_s^\perp$, then applying *of course to* β to get $!\beta$ then *conjunction*, *contraction* and *dereliction* to $(\alpha, !\beta)$ so as to get a central clique in $\Vdash^c S_s^\perp \otimes !D_s, 1$;

(3) $(C_s, C_d) \vee (T_s, T_d)$ is defined in a symmetrical way;

(4) $(C_s, C_d) \vee (D_s, D_d) = (C_s \oplus D_s, Z^{\perp\perp})$ where Z refers to all cliques that can be obtained by *left disjunction* and *dereliction* from α such that $\delta(\alpha)$ is a central correlation in C_d or *right disjunction* and *dereliction* from β such that $\delta(\beta)$ is a central correlation in D_d.

(iv) the negative correlation domain $\neg\mathbf{F}$ is defined as $(\top, \neg\mathbf{F}_d)$, where $\neg\mathbf{F}_d$ consists in all cliques (i.e. only the empty one) in $\Vdash^c \top, 1$;. Its negation $(0, \mathbf{F}_d)$ is such that $\mathbf{F}_d$ contains only the clique $\{[][0]\}$ of $\Vdash^c 0, 1$; i.e. is the biorthogonal of the empty set.

(v) The positive correlation domain $\mathbf{V}$ is defined as $(1, \mathbf{V}_d)$; $\mathbf{V}_d$ is the biorthogonal of the set consisting only of the central clique $\{[0][]\}$. In fact $\mathbf{V}_d$ contains only another clique, namely $\{[][0]\}$. The negation $(\perp, \neg\mathbf{V}_d)$ of $\mathbf{V}$ is reduced to the only clique $\{0[0]\}$.

Proposition 10.

(i) the spaces defined are correlation domains;

(ii) all canonical isomorphisms of chapter 5 are still valid in terms of domains.

Proof. (i) is trivial except for case (4) of conjunction; if we use the notation $\&(\alpha, \beta)$ for the *conjunction* of $\alpha \in S_d$ with $\beta \in T_d$, and the notations $l \oplus (\alpha')$ and $r \oplus (\beta')$ for the left and right *disjunctions* applied to α', β' with $\delta(\alpha')$, $\delta(\beta')$ in $S_d\perp$ or $T_d^\perp$, then it is immediate that $\langle \delta(l \oplus (\alpha')), \&(\alpha, \beta)\rangle = \langle \delta(\alpha'), \alpha\rangle$ and $\langle \delta(r \oplus (\beta')), \&(\alpha, \beta)\rangle = \langle \delta(\beta'), \beta\rangle$. From this it is easy to show that $S_d \,\&\, T_d$ is the orthogonal of a set of central correlations.

(ii) this is trivial and very boring; moreover this can be obtained as a corollary to theorem 10. □

Definition 14. Let X be a correlation domain; a clique $a \sqsubset \Vdash^c X_s$; is said to be *total* when the clique $\theta(a) \sqsubset \Vdash^c X, 1$; obtained from a by weakening $(\theta(a) := \{x[];\ x \in a\})$ is a correlation of X. We shall implicitly use (especially in the proof of theorem 10) the remark that—when $\beta \sqsubset \Vdash^c \Xi, 1$—the clique $(\theta(\alpha), \beta)$ can be directly obtained through a cut between α and β (since weakening plus contraction simplify).

For the next results we adopt the following obvious generalization of our definitions:

(i) if $\Xi = A_1, \ldots, A_n$, then $\Vdash^c \Xi$; denotes the correlation domain isomorphic to $A_1^- ⅋ \ldots ⅋ A_n^-$; $(\Vdash^c \Xi;)_d$ is therefore a set of cliques in the coherent space $\Vdash^c \Xi, 1$;

(ii) we also transfer the concept of totality to sequents $\Vdash^c \Xi$; by isomorphism.

Proposition 11. *Let C be a positive correlation domain, and let $\alpha \sqsubset ?C$ be a total clique in C. Then there is a unique atom $z \in 1(C_s)$ such that $[z] \in \alpha$.*

Proof. Elements of $1(C_s)$ are pairwise incoherent, whereas those of α are pairwise coherent; therefore z must be unique if it exists. Now by proposition 9 the clique $\beta := \{z[0];\ z \in \perp(C_s^{\perp})\}$ is a correlation in $\neg C$, which implies that $\langle \theta(\alpha), \beta \rangle = 1$, and by proposition 8 there must be an atom $[z][]$ in $\theta(\alpha)$ with $z \in \perp(C_s^{\perp}) = 1(C_s)$. □

Theorem 9. *Let C and D be positive correlation domains, and let $\alpha \sqsubset (C \vee D)_s$ be a total clique in $C \vee D$. Then one and exactly one of the two domains is principal: this means that there is a unique atom $z \in 1(C \vee D)$ such that $[z] \in \alpha$, and the principal domain of α is the one from which z comes.*

Proof. This is just a restatement of proposition 11: for simplicity let us assume that $|C| \cap |D| = \varnothing$, so we can write $1(C \vee D) = 1(C \oplus D) = 1(C) \cup 1(D)$. Then $z \in 1(C)$ or $z \in 1(D)$. □

Theorem 9 is the semantic counterpart of the weak disjunction property (theorem 4(ii)). Technically speaking the change from sets to multisets arises in this theorem: if we had been working with sets, our bilinear form could only distinguish between $-\infty$, 0 and > 0 and it would be impossible to get a singleton $[z]$ in proposition 11. The semantic analogue of part (i) of the theorem (the fact that a central correlation in a sum $C \vee D$ is a correlation in one of the summands) is trivial, but a semantic interpretation of part (iii) is less obvious.

Definition 15. A positive correlation domain (C_s, C_a) is said to be *discrete* when the following holds:

(i) $1(C) = |C|$ (hence the atoms of C are pairwise incoherent and $x \mapsto x_1, \dots, x_n$ iff $x = x_1, \dots, x_n$);

(ii) C_a is defined as the biorthogonal of the set of singleton cliques $\{[z][]\}$: besides these cliques, C_a contains only another one, namely $\{[][0]\}$ (these cliques are all the total ones in $\Vdash^c C$, 1, see proposition 12).

The class of discrete *PCD* is closed under $\otimes$ and $\oplus$; moreover a clique in $?(C \oplus D)$ $(C, D$ discrete) is either a clique in C or a clique in D, and since the ambivalent clique $\{[]\}$ is not total, we get an analogue of theorem 4(iii). Discrete *PCD* are good candidates for a semantic interpretation of strict positivity. Are they really general enough? We think so, since usual data types (booleans, lists, trees) can be interpreted by discrete *PCD*.

Theorem 10. *Let π be a proof of $\vdash \Gamma;\ \Pi$ and let $\pi^* \sqsubset \Vdash^c \Gamma^*;\ \Pi^*$ be its interpretation according to theorem 7; then*

(i) if Π is empty, then π^* is total in $\Vdash^c \Gamma^*$;

(ii) if $\Pi = P$ then π^* is total in $\Vdash^c \Gamma^*, P^*;$.

Proof. The proof is by induction on a proof π of a sequent $\vdash \Gamma;\ \Pi$ and follows a rather uniform pattern, that we illustrate by an example: assume that π proves $\vdash \Gamma, \Delta;\ P \wedge Q$ from proofs λ of $\vdash \Gamma;\ P$ and ρ of $\vdash \Delta;\ Q$. The hypothesis is that $\delta(\lambda^*)$ and $\delta(\rho^*)$ are total and we need only to prove that $\delta(\pi^*)$ is total. Totality of $\delta(\lambda^*)$ means that if I take correlations $\alpha_1, \dots, \alpha_p$ in the spaces $A_1^{*\perp-}, \dots, A_p^{*\perp-}$ (we assume $\Gamma = A_1, \dots, A_p$) with α_i

central when A_i is negative, if I perform p n-cuts (irrelevant order, see theorem 8) between $\delta(\lambda^*)$ and $\alpha_1, \ldots, \alpha_k$ and then contractions, then the result—which is central by proposition 11 and theorem 7—is a correlation in $\Vdash^c P^*$, 1; which can be written $\delta((\lambda^*; \alpha_1, \ldots, \alpha_p))$. In the same way (if $\Delta = B_1, \ldots, B_q$) and $\beta_1, \ldots, \beta_q$ are correlations in the spaces $B_1^{*\perp-}, \ldots, B_q^{*\perp}$, then $\delta((\rho^*; \beta_1, \ldots, \beta_q))$ is a correlation in $\Vdash^c Q^*$, 1;. Finally observe that $\delta(\pi^*)$ is total iff for all $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ (chosen as above) then $\delta((\pi^*; \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, _q))$ is a correlation in $\Vdash^c (P \wedge Q)^*$, 1;. Now the important point to notice is all the operations involved (especially the cuts, see theorem 8) commute, and so

$$\delta((\pi^*; \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)) = \delta((\lambda^*; \alpha_1, \ldots, \alpha_p) \otimes (\rho^*; \beta_1, \ldots, \beta_q))$$

if we use $\otimes$ to denote the operation of definition 13(ii) (1) (without the final dereliction). The result is therefore reduced to the fact that if $\delta(a)$ and $\delta(b)$ are correlations in P^* and Q^*, then $\delta(a \otimes b)$ is a correlation in $(P \wedge Q)^*$, which is precisely what the definition was saying. $\square$

The robustness of the notion is expressed by the following result:

Proposition 12. *A clique α is total in $\Vdash^c X$, $\mathbf{V}$; iff α is a correlation in X.*

Proof. To say that $\alpha \sqsubset \Vdash X^-$, ?1 is total means that for any correlations β and γ in $\neg X$ and $\neg\mathbf{V}$ respectively (with β central in case X is negative) then the result of the two cuts and a contraction is the clique $\{[0]\}$. But γ is the clique $\{0[0]\}$ and cutting with γ has no effect. In other terms α is total iff $\langle \alpha, \beta \rangle = 1$ for all β in the appropriate set, i.e. if α is a correlation. $\square$

The proposition illustrates the fact that correlations are just total elements; but one must first 'square' the space by a disjunction with $\mathbf{V}$ to get 'enough' total elements.

We introduce our last definition:

Definition 16. We first introduce the notation cut$(!\alpha, \gamma)$: if $\alpha \sqsubset X^-$ is a total clique in the correlation domain X, then we can view α as a total clique in the *NCD* X^-; then we can form a central clique $!\alpha$ in the *PCD* X^{-+}; finally if γ is any total element in $\vdash^c \neg(X^{-+})$, $\mathbf{V}$, $\mathbf{V}$; we can form by cut a clique cut$(!\alpha, \gamma)$ in $\Vdash^c \mathbf{V}, \mathbf{V}$;. Now two total cliques α and β in X^- are said to be *observationally equivalent* (notation $\alpha \approx \beta$) when cut$(!\alpha, \gamma) =$ cut$(!\beta, \gamma)$ for any total clique γ in $\Vdash^c \neg S$, $\mathbf{V}$, $\mathbf{V}$;.

Proposition 13. *Assume that $\alpha \cup \beta$ is a clique; then $\alpha \approx \beta$.*

Proof. Simply observe that there are exactly two total cliques in $\Vdash^c \mathbf{V}, \mathbf{V}$; and that their union is not a clique. Hence if $\alpha \cup \beta$ is a clique, then cut$(!\alpha, \gamma) \cup$ cut$(!\beta, \gamma) \subset$ cut$(!(\alpha \cup \beta), \gamma)$ is a clique, and this forces cut$(!\alpha, \gamma) =$ cut$(!\beta, \gamma)$. $\square$

Theorem 11. *Observational equivalence is a congruence.*

Proof. In fact the result can be reduced to compatibility with cut, especially n-cut. In particular if $\alpha \approx \beta$ are total in $\Vdash^c X$; and γ is total in $\Vdash^c \neg X$, Y; we want to prove that cut$(!\alpha, \gamma) \approx$ cut$(!\beta, \gamma)$ and for this we must cut with a total γ' in $\Vdash^c \neg(Y^{-+})$, $\mathbf{V}, \mathbf{V}$;, i.e. show that cut$(!$cut$(!\alpha, \gamma), \gamma') =$ cut$(!$cut$(!$cut$(!\beta, \gamma), \gamma'))$. But this can be rewritten as cut$(!\alpha,$ cut$(!\gamma, \gamma')) =$ cut$(!\beta,$ cut$(!\gamma, \gamma'))$ (we strongly use the fact that $!\alpha$ and $!\beta$ are central), and the last equality is a consequence of $\alpha \approx \beta$. $\square$

The notion of observational equivalence is to pass 'boolean tests'; however just limiting them to something like $\mathrm{cut}(\alpha,\gamma) = \mathrm{cut}(\beta,\gamma)$ for any total γ in $\Vdash^c \neg\mathbf{Y}, \mathbf{V}, \mathbf{V}$; does not seem to be enough to ensure that $\approx$ is a congruence. Technically speaking $\mathrm{cut}(\alpha,\gamma)$ is linear in α when X is positive, and it seems impossible to restrict to linear tests.

2.4. *Open questions in the semantics*

(i) The immediate question is how to accommodate polymorphism. The problem is to explain how a construction may depend on a parameter α. The idea would be to follow the pattern of Girard (1986) and:

(1) to introduce a notion of embedding of *PCS*, namely a central map between X and Y which corresponds to isomorphisms between X and the restriction of Y to a subset. These embeddings must be central.
(2) To observe that the interpretation of proofs is functorial w.r.t. embeddings, something like $\pi^*(X) = f^{-1}(\pi^*(Y))$.
(3) Then using direct limit and pull-back preservation prove a normal form theorem.
(4) From the normal form theorem deduce a *NCS* (case of $\forall$) or a *PCS* (case of $\exists$). However it seems that here we meet some problems.
(5) It seems that *PCS* are not direct limits (w.r.t. the category of embeddings) of finite ones (problems with centrality); so one has to do something if one looks for a 'natural' semantics.
(6) It seems definitely impossible to have polymorphism in α without fixing the polarity; typically the semantic interpretation $\pi^*(X)$ of a cut between $\vdash \alpha, \mathbf{V}_1$; and $\vdash \neg\alpha, \mathbf{V}_2$; (where both cliques comes from $\vdash \mathbf{V}$; by weakening) is interpreted by a different clique in $\vdash \mathbf{V}_1, \mathbf{V}_2$; (there are only two total ones) depending on the polarity of α: therefore this cut can be polymorphic in X only if the category of embeddings (or anything playing the same role) is the disjoint union of a category of *PCS* and a category of *NCS*....

(ii) The most interesting problem is the completeness conjecture. Remember that Gödel's completeness theorem states that a classical first order formula which is true in all models is provable in classical predicate calculus. Now if we consider the universal closure of a first order formula w.r.t. its predicate and function constants, we get a second-order formula with no parameter at all. For these formulas, Gödel's theorem says 'if A is true, then A is provable'. Gödel's incompleteness theorem states that the result does not extend to wider classes, for instance to existential closures.

Gödel's theorem says that first-order predicate calculus contains all logically valid formulas and that it is impossible to add anything; but does it contain all logically valid proofs? Cut-elimination seems to answer 'yes': if we prove a 1st order formula A in a calculus enjoying cut-elimination, then we can use the subformula property to show that A is provable by first-order means. But would it be possible to give a pure semantical proof of this fact? This is why we state the following conjecture.

Completeness conjecture. Let A be a first-order formula depending on predicate and function symbols $p, q, \ldots$ and let $\alpha(C, D, \ldots) \sqsubset \Vdash^c A[C, D, \ldots/p, q, \ldots]^*$; be a family of total cliques depending functorially over the interpretations $C, D, \ldots$ of $p, q, \ldots$; then there is a proof π of $\vdash A$; in **LC** such that for all $C, D, \ldots$ the interpretation $\pi^*[C, D, \ldots]$ is observationally equivalent to $\alpha(C, D, \ldots)$.

If the interpretation of polymorphism is carried out properly, a slightly enhanced version can be stated; let A be a closed second order formula without negative (resp. positive) occurrences of second-order $\forall$ (resp. $\exists$), and let α be a total clique in $\Vdash^c A^*$;: then there exists a proof π of $\vdash A$; in **LC** such that $\alpha \approx \pi^*$.

This enhanced version is the strongest plausible: since the existence of a total clique in A implies the truth of the formula A, the conjecture of A implies Gödel completeness for A, which fails for more general formulas.

Appendix A. Coherent spaces

We present here the basic definitions of the *coherent semantics*, together with the denotational interpretation of linear proofs. We adopt a slightly different definition of '!' and '?' (the original one in Girard, 1987, was stated in terms of sets instead of multisets); this modification is an essential ingredient of the theory of *correlation domains*.

*A*1. *Coherent spaces*

Definition 17. A *coherent space* is a reflexive undirected graph. In other terms it consists of a set $|X|$ of *atoms* together with a compatibility or *coherence* relation between atoms, noted $x \frown\!\!\!\!\smile y$, or $x \frown\!\!\!\!\smile y$ [mod X] if there is any ambiguity as to X.

A *clique* a in X (notation $a \sqsubset X$) is a subset a of X made of pairwise coherent atoms: $a \sqsubset X$ iff $\forall x \forall y (x \in a \wedge y \in a \Rightarrow s \frown\!\!\!\!\smile y)$. In fact coherent space can be also presented as sets of cliques; when we want to emphasize the underlying graph $(|X|, \frown\!\!\!\!\smile)$ we call it the *web* of X.

Besides coherence we can also introduce

$$\begin{gathered}\textit{strict coherence}: x \frown y \text{ iff } x \frown\!\!\!\!\smile y \text{ and } x \neq y\\ \textit{incoherence}: x \asymp y \text{ iff } \neg(x \frown y)\\ \textit{strict incoherence}: x \smile y \text{ iff } \neg(x \frown\!\!\!\!\smile y).\end{gathered}$$

Any of these four relations can serve as a definition of coherent space. Observe the fact that $\asymp$ is the negation of $\frown$ and not of $\frown\!\!\!\!\smile$; this is due to the reflexivity of the web.

Definition 18. Given a coherent space X, its *linear negation* $X^\perp$ is defined by

$$\begin{gathered}|X^\perp| = |X|\\ x \frown\!\!\!\!\smile y \text{ [mod } X^\perp] \text{ iff } x \asymp y \text{ [mod } X]\end{gathered}$$

in other terms linear negation is nothing but the exchange of coherence and incoherence. It is obvious that linear negation is involutive: $X^{\perp\perp} = X$.

Definition 19. Given two coherent spaces X and Y, the *multiplicative connectives* $\otimes$, $⅋$, $\multimap$ define a new coherent space Z with $|Z| = |X| \times |Y|$; coherence is defined by:

$$\begin{gathered}(x, y) \frown\!\!\!\!\smile (x', y') \text{ [mod } X \otimes Y] \text{ iff } x \frown\!\!\!\!\smile x' \text{ [mod } X] \text{ and } y \frown\!\!\!\!\smile y' \text{ [mod } Y]\\ (x, y) \frown (x', y') \text{ [mod } X ⅋ Y] \text{ iff } x \frown x' \text{ [mod } X] \text{ or } y \frown y' \text{ [mod } Y]\\ (x, y) \frown (x', y') \text{ [mod } X \multimap Y] \text{ iff } x \frown\!\!\!\!\smile x' \text{ [mod } X] \text{ implies } y \frown y' \text{ [mod } Y]\end{gathered}$$

Observe that $\otimes$ is defined in terms of $\frown\!\!\smile$ but $\parr$ and $\multimap$ in terms of $\frown$. A lost of useful isomorphisms can be obtained:

(i) De Morgan equalities: $(X \otimes Y)^\perp = X^\perp \parr Y^\perp$; $(X \parr Y)^\perp = X^\perp \otimes Y^\perp$; $X \multimap Y = X^\perp \parr Y$;
(ii) commutativity isomorphisms: $X \otimes Y \simeq Y \otimes X$; $X \parr Y \simeq Y \parr X$; $X \multimap Y \simeq Y^\perp \multimap X^\perp$;
(iii) associativity isomorphisms: $X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$; $X \parr (Y \parr Z) \simeq (X \parr Y) \parr Z$ $X \multimap (y \multimap Z) \simeq (X \otimes Y) \multimap Z$; $X \multimap (Y \parr Z) \simeq (X \multimap Y) \parr Z$.

Definition 20. Up to isomorphism there is a unique coherent space whose web consists of one atom, 0; this space is self dual, i.e. equal to its linear negation. However the algebraic isomorphism between this space and its dual is logically meaningless, and we shall depending on the context use the notation 1 or the notation $\perp$ for this space, with the convention that $1^\perp = \perp$, $\perp^\perp = 1$. This space is neutral w.r.t. multiplicatives, namely

$$X \otimes 1 \simeq X,\ X \parr \perp \simeq X;\ 1 \multimap X \simeq X,\ X \multimap \perp \simeq X^\perp.$$

Once more this notational distinction is mere preciosity; we shall have to extend 1 and $\perp$ into correlation domains to see a genuine difference!

Definition 21. Given two coherent spaces X and Y, the *additive connectives* & and $\oplus$ define a new coherent space Z with $|Z| = |X| + |Y|$ $(= |X| \times \{0\} \cup |Y| \times \{1\})$

$$\begin{gathered}(x,0) \frown\!\!\smile (x',0)\ [\bmod Z] \text{ iff } x \frown\!\!\smile x'\ [\bmod X]\\ (y,1) \frown\!\!\smile (y',1)\ [\bmod Z] \text{ iff } y \frown\!\!\smile y'\ [\bmod Y]\\ (x,0) \frown (y,1)\ [\bmod X \& Y]\\ (x,0) \smile (y,1)\ [\bmod X \oplus Y].\end{gathered}$$

A lot of useful isomorphisms are immediately obtained:

(i) De Morgan equalities: $(X \& Y)^\perp = X^\perp \oplus Y^\perp$; $(X \oplus Y)^\perp = X^\perp \& Y^\perp$;
(ii) commutativity isomorphisms: $X \& Y \simeq Y \& X$; $X \oplus Y \simeq Y \oplus X$;
(iii) associativity isomorphisms: $X \& (Y \& Z) \simeq (X \& Y) \& Z$; $X \oplus (Y \oplus Z) \simeq (X \oplus Y) \oplus Z$;
(iv) distributivity isomorphisms: $X \otimes (Y \oplus Z) \simeq (X \otimes Y) \oplus (X \otimes Z)$; $X \parr (Y \& Z) \simeq (X \parr Y) \& (X \parr Z)$; $X \multimap (Y \& Z) \simeq (X \multimap Y) \& (X \multimap Z)$: $(X \oplus Y) \multimap Z \simeq (X \multimap Z) \& (Y \multimap Z)$.

The other distributivities fail; for instance $X \otimes (Y \& Z)$ is not isomorphic to $(X \otimes Y) \& (X \otimes Z)$.

Definition 22. There is a unique coherent space with an empty web. Although this space is also self dual, we shall use distinct notations for it and its negation, $\top$ and 0. These spaces are neutral w.r.t. additives

$$X \oplus 0 \simeq X;\ X \& \top \simeq X$$

and absorbing w.r.t. multiplicatives

$$X \otimes 0 \simeq 0;\ X \parr \top \simeq \top;\ 0 \multimap X \simeq \top;\ X \multimap \top \simeq \top.$$

A2. Linear sequent calculus

IDENTITY/NEGATION

$$\frac{}{\vdash A, A^{\perp}}\ (identity) \qquad \frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

STRUCTURE

$$\frac{\vdash \Gamma}{\vdash \Gamma'}$$

(*exchange*: Γ' *is a permutation of* Γ)

LOGIC

$$\frac{}{\vdash 1}\ (one) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (false)$$

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ (times) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \mathbin{⅋} B}\ (par)$$

$$\frac{}{\vdash \Gamma, \top}\ (true)$$

(*no rule for zero*)

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \mathbin{\&} B}\ (with) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (left\ plus) \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (right\ plus)$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (of\ course) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ (weakening) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (dereliction) \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ (contraction)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x A}\ (for\ all:\ x\ is\ not\ free\ in\ \Gamma) \qquad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x A}\ (\text{there is})$$

The syntactical conventions are as follows: formulas are built from literals $pt_1 \ldots t_n$ and their negations $pt_1 \ldots t_n{}^{\perp}$ by means of the connectives 1, $\perp$, $\top$, 0 (0-ary), !, ? (unary) and $\otimes$, ⅋, $\oplus$, & (binary) and the quantifiers $\forall x$ and $\exists x$. Negation is defined by immediate De Morgan formulas (exchanges $1/\perp$; $\top/0$; $!/?$; $\otimes/⅋$; $\oplus/\&$; $\forall x/\exists x$), and linear implication $A \multimap B$ as $A^{\perp} ⅋ B$. A sequent $\vdash \Gamma$ refers to a sequence $\vdash A_1, \ldots, A_n$ of formulas. The intended meaning is $A_1 ⅋ \ldots ⅋ A_n$, i.e. the comma is hypocrisy for '⅋'.

A3. Interpretation of rudimentary linear logic

Rudimentary linear logic is the fragment of linear logic without the exponentials '!' and '?'. In fact we shall content ourselves with the propositional part and omit quantifiers. If we wanted to treat quantifiers, the idea would be to essentially interpret $\forall x$ and $\exists x$ as respectively 'big' & and $\oplus$ indexed by the domain of interpretation of variables; the precise definition involves considerable bureaucracy for something completely straightforward. The treatment of second-order quantifiers is of course much more challenging and cannot be explained in a short appendix like here; it is better anyway to reserve the precise treatment of second-order connectives to further work.

Once we decided to ignore exponentials and quantifiers, everything is ready to interpret formulas of rudimentary propositional linear logic: more precisely, if we assume that the atomic propositions p, q, $r, \ldots$ of the language have been associated coherent spaces p^*, q^*, $r^*, \ldots$, then any formula A of the language is interpreted by a well-defined coherent space A^*; moreover this interpretation is consistent with the definitions of linear negation and implication (i.e. $A^{\perp *} = A^{*\perp}$, $(A \multimap B)^* = A^* \multimap B^*$). It remains to interpret sequents; the idea is to interpret $\vdash \Gamma$ $(= \vdash A_1, \ldots, A_n)$ as $A_1^* \mathbin{\wp} \ldots \mathbin{\wp} A_n^*$. More precisely

Definition 23. If $\Vdash \Xi$ $(= \Vdash X_1, \ldots, X_n)$ is a formal sequent made of coherent spaces, then the coherent space $\Vdash \Xi$ is defined by:

(i) $|\Vdash \Xi| = |X_1| \times \ldots \times |X_n|$; we use the notation $x_1 \ldots x_n$ for the atoms of $\Vdash \Xi$.
(ii) $x_1 \ldots x_n \frown y_1 \ldots y_n \Leftrightarrow \exists i\, x_i \frown y_i$.

If $\vdash \Gamma$ $(= \vdash A_1, \ldots, A_n)$ is a sequent of linear logic, then $\Vdash \Gamma^*$ will be the coherent space $\Vdash A_1^*, \ldots, A_n^*$.

The next step is to interpret proofs; the idea is that a proof π of $\vdash \Gamma$ will be interpreted by a *clique* $\pi^* \sqsubset \Vdash \Gamma^*$. In particular (since sequent calculus is eventually about proofs of singletons $\vdash A$) a proof of π of $\vdash A$ is interpreted by a clique in $\Vdash A^*$ i.e. a clique in A^*.

Definition 24. (i) *The identity axiom* $\vdash A, A^\perp$ of linear logic is interpreted by the set $\{xx; x \in |A^*|\}$.

(ii) Assume that the proofs π of $\vdash \Gamma$, A and λ of $\vdash A^\perp$, Δ have been interpreted by cliques π^* and γ^* in the associated coherent spaces; then the proof ρ of $\vdash \Gamma$, Δ obtained by means of a *cut rule* between π and λ is interpreted by the set $\rho^* = \{\underline{xx'};\ \exists z(\underline{x}z \in \pi^* \wedge z\underline{x}' \in \lambda^*)$.

(iii) Assume that the proof π of $\vdash \Gamma$ has been interpreted by a clique $\pi^* \sqsubset \Vdash \Gamma^*$, and that ρ is obtained from π by an *exchange rule* (permutation σ of Γ); then ρ^* is obtained from ρ by applying the same permutation $\rho^* = \{\sigma(\underline{x});\ \underline{x} \in \pi^*\}$.

All the sets constructed by our definition are cliques; let us remark that in the case of cut, the atom z of the formula is uniquely determined by $\underline{x}$ and $\underline{x}'$. □

Definition 25.

(i) The *axiom* $\vdash 1$ of linear logic is interpreted by the clique $\{0\}$ of 1 (if we call 0 the only atom of 1).
(ii) The *axioms* $\vdash \Gamma$, $\top$ of linear logic are interpreted by void cliques (since $\top$ has an empty web, the spaces $\Vdash \Gamma$, $\top^*$ have empty webs as well).
(iii) If the proof ρ of $\vdash \Gamma$, $\perp$ comes from a proof π of $\vdash \Gamma$ by a *falsum rule*, then we define $\rho^* = \{\underline{x}0\ ; \underline{x} \in \pi^*\}$.
(iv) If the proof ρ of $\vdash \Gamma$, $A \mathbin{\wp} B$ comes from a proof π of $\vdash \Gamma$, A, B by a *par rule*, then we define $\rho^* = \{\underline{x}(y, z)\ ; \underline{x}yz \in \pi^*\}$.

(v) If the proof ρ of $\vdash \Gamma$, $A \otimes B$, Δ comes from a proofs π of $\vdash \Gamma$, A and λ of $\vdash B$, Δ by a *times rule*, then we define $\rho^* = \{\underline{x}(y,z)\underline{x}' \ ; \underline{x}y \in \pi^* \wedge z\underline{x}' \in \lambda^*\}$.

(vi) If the proof ρ of $\vdash \Gamma$, $A \oplus B$ comes from a proof π of $\vdash \Gamma$, A by a *left plus rule*, then we define $\rho^* = \{\underline{x}(y,0) \ ; \underline{x}y \in \pi^*\}$; if the proof ρ of $\vdash \Gamma$, $A \oplus B$ comes from a proof π of $\vdash \Gamma$, B by a *right plus rule*, then we define $\rho^* = \{\underline{x}(y,1) \ ; \underline{x}y \in \pi^*\}$.

(vii) If the proof ρ of $\vdash \Gamma$, A & B comes from a proofs π of $\vdash \Gamma$, A and λ of $\vdash \Gamma$, B by a *with rule*, then we define $\rho^* = \{\underline{x}(y,0); \ \underline{x}y \in \pi^*\} \cup \{\underline{x}(y,1) \ ; \underline{x}y \in \lambda^*\}$.

Observe that (iv) is mainly a change of bracketing, i.e. does strictly nothing; if $|A| \cap |B| = \varnothing$ then one can define $A \& B$, $A \oplus B$ as unions, in which case (vi) is read $\rho^* = \pi^*$ in both cases, and (vii) is read $\rho^* = \pi^* \cup \lambda^*$.

It is of interest (since this is deeply hidden in definition 9) to stress the relation between *linear implication* and *linear maps*.

Definition 26. Let X and Y be coherent spaces; a *linear map* from X to Y consists in a function F such that:

(i) if $a \sqsubset X$ then $F(a) \sqsubset Y$;
(ii) if $\bigcup b_i = a \sqsubset X$ then $F(a) = \bigcup F(b_i)$;
(iii) if $a \cup b \sqsubset X$, then $F(a \cap b) = F(a) \cap F(b)$.

Proposition 14. *There is a* 1–1 *correspondence between linear maps from* X *to* Y *and cliques in* $X \multimap Y$; *more precisely*:

(i) *to any linear* F *from* X *to* Y, *associate* $\mathrm{Tr}(F) \sqsubset X \multimap Y$ (*the trace of* F) $\mathrm{Tr}(F) = \{(x,y); \ y \in F(\{x\})\}$.
(ii) *To any* $A \sqsubset X \multimap Y$ *associate a linear function* $A(.)$ *from* X *to* Y if $a \sqsubset X$, then $A(a) = \{y; \ \exists x \in a \ (x,y) \in A\}$.

Proof. The proofs that $\mathrm{Tr}(A)(.)) = A$ and $\mathrm{Tr}(F)(.) = F$ are left to the reader. In fact the structure of the space $X \multimap Y$ has been obtained so as to get this property and not the other way around. □

*A*4. *Exponentials*

This is the only section with some (limited) originality in this annex.

Definition 27. Let X be a coherent space; we define $\mu(X)$ to be the free commutative monoid generated by $|X|$. The elements of $\mu(X)$ are all the formal expressions $[x_1, \ldots, x_n]$ which are finite multisets of elements of $|X|$. This means that $[x_1, \ldots, x_n]$ is a sequence in $|X|$ defined up to the order. The difference with a subset of $|X|$ is that repetitions of elements matter. One easily defines the sum of two elements of $\mu(X)$: $[x_1, \ldots, x_n] + [y_1, \ldots, y_m] = [x_1, \ldots, x_n, y_1, \ldots, y_m]$, and the sum is generalized to any finite set. The neutral elements of $\mu(X)$ is written [].

If X is a coherent space, then $!X$ is defined as follows:

$$|!X| = \{[x_1, \ldots, x_n] \in \mu(X); \ x_i x_j \text{ for all } i \text{ and } j\}$$
$$\Sigma[x_i]\Sigma[y_j] \ [\mathrm{mod} \ !X] \text{ iff } x_i y_j \text{ for all indices } i \text{ and } j$$

If X is a coherent space, then $?X$ is defined as follows:

$$|?X| = \{[x_1, \ldots, x_n] \in \mu(X);\ x_i \asymp x_j \text{ for all } i \text{ and } j\}$$
$$\Sigma[x_i] \frown \Sigma[y_j] \ [\text{mod } ?X] \text{ iff } x_i \frown y_j \text{ for some pair of indices } i \text{ and } j.$$

Among remarkable isomorphisms let us mention:

(i) De Morgan equalities: $(!X)^\perp = ?(X^\perp)$; $(?X)^\perp = !(X^\perp)$;
(ii) exponentiation isomorphisms: $!(X \& Y) \simeq (!X) \otimes (!Y)$;$?(X \oplus Y) \simeq (?X) ⅋ (?Y)$, together with the 'particular cases' $!\top \simeq 1$; $?0 \simeq \perp$.

Definition 28. (i) Assume that the proof π of $\vdash ?\Gamma, A$ has been interpreted by a clique π^*; then the proof ρ of $\vdash ?\Gamma, !A$ obtained from π by an of *course rule* is interpreted by the set $\rho^* := \{\underline{x}_1 + \ldots + \underline{x}_k[a_1, \ldots, a_k]\ ;\ \underline{x}_1 a_1, \ldots, \underline{x}_k a_k \in \pi^*\}$. Some explanation about the notation: each if $?\Gamma$ is $?B^1, \ldots, ?B^n$, then $\underline{x}_i$ is $x_i{}^1, \ldots, x_i{}^n$ so $\underline{x}_1 + \ldots + \underline{x}_k$ is the sequence $x_1{}^1 + \ldots + x_k{}^1, \ldots, x_1{}^n + \ldots + x_k{}^n$; $[a_1, \ldots, a_k]$ refers to a multiset. What is implicit in the definition (but not obvious) is that we take only those expressions $\underline{x}_1 + \ldots + \underline{x}_k[a_1, \ldots, a_k]$ such that $\underline{x}_1 + \ldots + \underline{x}_k$ is defined (this forces $[a_1, \ldots, a_k] \in |\,!A|$).

(ii) Assume that the proof π of $\vdash \Gamma$ has been interpreted by a clique π^*; then the proof ρ of $\vdash \Gamma, ?A$ obtained from π by a *weakening rule* is interpreted by the set $\rho^* := \{\underline{x}[\,];\ \underline{x} \in \pi^*\}$.

(iii) Assume that the proof π of $\vdash \Gamma, ?A, ?A$ has been interpreted by a clique π^*; then the proof ρ of $\vdash \Gamma, ?A$ obtained from π by a *contraction rule* is interpreted by the set $\rho^* := \{\underline{x}(a+b);\ \underline{x}ab \in \pi^* \wedge a \asymp b\}$.

(iv) Assume that the proof π of $\vdash \Gamma, A$ has been interpreted by a clique π^*; then the proof ρ of $\vdash \Gamma, ?A$ obtained from π by a *dereliction rule* is interpreted by the set $\rho^* := \{\underline{x}[a];\ \underline{x}a \in \pi^*\}$.

Of course one can verify without difficulty that all sets constructed by definition 11 are cliques in the associated coherent spaces. However only part (iv) of definition 11 is actually used in our construction, since (i), (ii) and (iii) are generalized (see 2.1.) to the case of *NCS*. In fact correlation spaces (i.e. comonads and their dual) make exponential appear as the solutions to certain universal problems. The replacement of sets by multisets is the essential ingredient of our theorem 9.

Appendix B. Counterexamples and typical mistakes

Although the text is rather self-contained, it can be useful to examine some alternative possibilities and see directly what can be wrong with them.

B1. Symmetric closed-Cartesian categories

The basic idea of extending the notion of *Cartesian-closed category* (CCC)—which is a very good semantics for intuitionistic logic—to the classical case stumbles on a well-known argument of Joyal, that we shall sketch (the reader may also consult Lambert and Scott, 1986, especially p. 67).

Lemma 1. *If* 0 *is initial in a CCC* $\mathbf{K}$, *then any* $0 \times A$ *is initial too.*

Proof. C is initial in $\mathbf{K}$ if $\mathrm{Hom}(C, B)$ has exactly one element for any object B in $\mathbf{K}$; the result follows from the isomorphism $\mathrm{Hom}(0 \times A, B) \approx \mathrm{Hom}(0, A \Rightarrow B)$. $\square$

Lemma 2. *If* 0 *is initial in a CCC* $\mathbf{K}$ *and* $\mathrm{Hom}(A, 0) \neq \varnothing$, *then* A *is initial too.*

Proof. Let $f \in \mathrm{Hom}(A, 0)$; from f we deduce canonically a morphism $g \in \mathrm{Hom}(A, A \times 0)$. Now if $h \in \mathrm{Hom}(A, B)$, we can write $h = (h \circ \pi_1) \circ g$; but since $A \times 0$ is initial (lemma 1) $h \circ \pi_1$ is equal to the unique element k of $\mathrm{Hom}(A \times 0, B)$, hence $h = k \circ g$ and this proves that $\mathrm{Hom}(A, B)$ has exactly one element. $\square$

Before stating Joyal's theorem, let us remember that a category $\mathbf{K}$ is *degenerated* when $\mathrm{Hom}(A, B)$ has at most one element for any A and B. If $\mathbf{K}$ is degenerated we can identify $\mathbf{K}$ with a pre-ordered set, namely $Ob(\mathbf{K})$, with $A \leqslant B$ iff $\mathrm{Hom}(A, B) \neq \varnothing$. In other terms the morphisms convey no information. The meaning of Joyal's remark is that 'symmetric *CCC*', 'bi-*CCC*' etc. are just pedantic names for Boolean algebras. Logically speaking this means that the brutal generalization of *CCC* to the classical case is just a regression from a semantics of *proofs* to a semantics of *provability* (i.e. to a semantics in the familiar model-theoretic sense).

Theorem 12. *A CCC with an involution is degenerated.*

Proof. An involution is a functor $*$ from $\mathbf{K}$ to $\mathbf{K}^{\mathrm{op}}$ such that for all A and B in $Ob(\mathbf{K})$: $\mathrm{Hom}(A^*, B^*) \approx \mathrm{Hom}(B, A)$ (bijection defined by $f \mapsto f^*$) and $A^{**} = A$. If 1 is terminal in $\mathbf{K}$, then 1^* must be initial, and we can denote it by 0. Then observe that $\mathrm{Hom}(A, B) \approx \mathrm{Hom}(1 \times A, B) \approx \mathrm{Hom}(1, A \Rightarrow B) \approx \mathrm{Hom}(1, (A \Rightarrow B)^{**}) \approx \mathrm{Hom}((A \Rightarrow B)^*, 0)$; if $\mathrm{Hom}((A \Rightarrow B)^*, 0) \neq \varnothing$, then $(A \Rightarrow B)^*$ is initial and so $\mathrm{Hom}((A \Rightarrow B)^*, 0)$ has only one element. In all cases $\mathrm{Hom}(A, B)$ has at most one element and $\mathbf{K}$ is degenerated. $\square$

This shows that there is very little to expect from *CCC* (or λ-calculus, combinatory logic) in terms of classical logic, in fact there are only two possibilities:

—either one tries to weaken the axioms of a symmetric *CCC*: if we remove enough equations, then we can get a non-degenerated category, but what is the point if certain terms do not have normal forms? (If we start removing equations, then it will be possible to produce a morphism in $\mathrm{Hom}(a, a)$—a being an unspecified atom—which is not provably equal to the identity of a, the only cut-free morphism from a to a). So the only reasonable part of such *Loch Ness* categories is the part which does not use any of the new classical 'facilities'.

—Or one tries to cheat with the rule of β-conversion of λ-calculus (in a symmetrized λ-calculus, we cannot have Church–Rosser): for instance, if we adopt leftmost reduction, Church–Rosser is vacuously satisfied.... But this means that equality of normal forms is no longer a congruence, since we may have $u \mapsto u'$ without having $tu \mapsto tu'$. From this it is plain that the properties of such *non-monotonic* λ-calculi are severely limited; however—as above—if one carefully avoids the new 'classical' features, one gets a fairly good system....

Conditions (i) and (iii) in part 0.2. of our introduction were chosen to exclude such jokes.

B2. The impossibility of a symmetric cut-elimination

It is not difficult to adopt the notion of multiplicative proof-net (Girard, 1987, see also Girard *et al.* 1989, for a sloppy introduction) so as to get a notion of classical proof-net: in fact it suffices to introduce n-ary *contraction links*

$$\frac{A \ldots A}{A}$$

to get a reasonable representation of classical proofs.

Such links represent iterated structural rules; in order to get a nice notion, we can agree on the following:

(i) the number n of premises is never equal to 1 (in that case the rule is useless);
(ii) the conclusion of the link is not in turn the premise of another contraction link (all contractions are performed in one step);
(iii) the premises of the link are not ordered, like in cut and axiom links.

As far as we exclude 0-ary contraction links (i.e. weakening links) then there is a straightforward correctness condition for such proof-nets: it suffices to treat a n-ary contraction link as a n-ary 'par' link.... If 0-ary links are considered no non-trivial condition is known, but the notion still makes sense.

But if classical proof-nets make sense, classical cut-elimination does not make sense at the level of proof-nets: as soon as we want to normalize a cut whose premises are conclusions of structural links, then we run into endless problems.

Example 1. *Cut weakening/weakening*

$$\begin{array}{ccc} (\Pi) & & (\Lambda) \\ \vdots & \dfrac{\overline{a} \quad \overline{\neg a}}{\text{cut}} & \vdots \\ A & & B \end{array}.$$

This represents the result of weakening a proof of $\vdash A$ into $\vdash A, a$ then weakening a proof of $\vdash B$ into $\vdash B, \neg a$ and then making a cut on a. There are two ways to eliminate this cut:

$$\begin{array}{ccccc} (\Pi) & & & & (\Lambda) \\ \vdots & & & & \vdots \\ A & \overline{B} & \text{or} & \overline{A} & B \end{array}$$

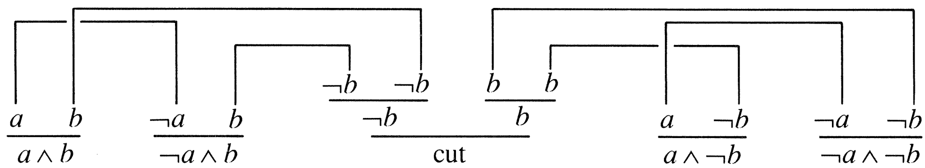
The original proof-net contains no information as to which solution should be chosen:

(i) the original proof-net is symmetrical w.r.t. a and $\neg a$ (if—say—a does not occur in Π or Λ);
(ii) even if we try to complicate the pattern and mention that a is 'linked' to A and $\neg a$ is 'linked' to B, then the choice between the two cut-free proofs will depend on a precedence between a and $\neg a$; we can for instance choose the left solution, but this choice is not compatible with a substitution $a \mapsto \neg a$... unless such a substitution is forbidden. But to forbid such a substitution is exactly the same as to introduce *polarities*.

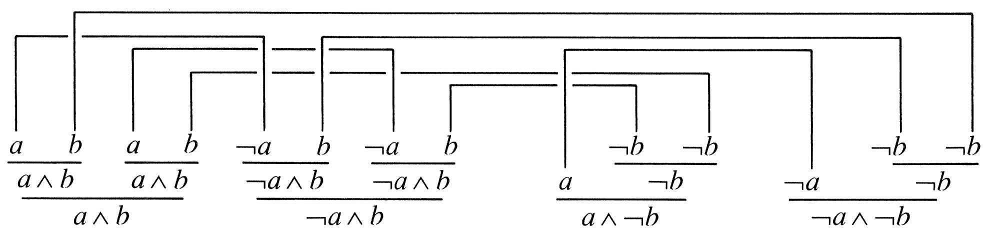
But should we only blame weakening? In fact if we refuse weakening (like in various relevance logics) we still get the same problem. In the next example, weakening (0-ary contraction) is not allowed, and we are still in trouble.

Example 2. *Cut contraction/contraction.*

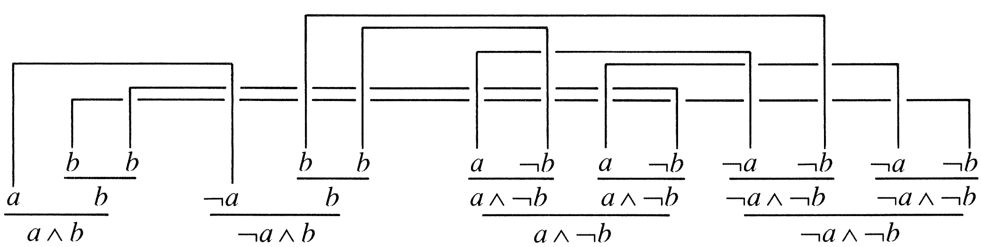
Let a and b be atoms; the following proof-net Π is associated with a proof of $\vdash a \wedge b,\ a \wedge \neg b,\ \neg a \wedge b, \neg a \wedge \neg b$



If I systematically exchange a and $\neg a$ in Π, then I get another proof-net with the same conclusions, which is exactly the original one; if I do the same with b and $\neg b$ the result is still the same. If there is any 'uniform' cut-elimination procedure, this procedure should commute with substitution, hence one should get a cut-free proof-net with the same conclusions and enjoying symmetries $a/\neg a, b/\neg b$. However one of the obvious choices, say Λ:



is invariant under the replacement $a \mapsto \neg a$, but not under $b \mapsto \neg b$ (we get



which is by no means the same as Λ). As we shall now see, there is no 'symmetrical proof-net' with these conclusions, hence no uniform cut-elimination procedure.

Consider the four element group **S** consisting of

$$\{\text{identity; exchange } a/\neg a;\ \text{exchange } b/\neg b;\ \text{exchange } a/\neg a \text{ and } b/\neg b\}$$

then **S** acts transitively on Π (seen as a graph); the fact that the premises of contraction and cut links and the conclusions of axom links are not ordered is essential. This group corresponds (in a one-sided version of classical logic) to the result of various substitutions. In particular, if there is any cut-elimination procedure compatible with substitution, then the same group **S** should also act transitively on the 'normal form' of our proof. Unfortunately

Theorem 13. *The only cut-free proof-net ending with (repetitions of) the formulas* $a \wedge b$, $a \wedge \neg b$, $\neg a \wedge b$, $\neg a \wedge \neg b$, a, $\neg a$, b, $\neg b$ *and on which* **S** *acts transitively are*:

$$\begin{array}{ccccc} (a) & & & (b) & \\ a & \neg a & \text{and} & b & \neg b \end{array}$$

Proof. For induction loading, we admit repetitions of conclusions and also subformulas of the original conclusions. Since weakening is not allowed, we have access to the correctness criterion of Girard (1987) and the proof is by induction on the number of links of the proof-net:

—one link: it must be an axiom link; since the conclusions A, $\neg A$ of this link must belong to the list given, this forces A to be a literal and therefore the proof-net is trivial;

—more than one link; two cases occur.

+ if there is a terminal contraction link then consider its *orbit* (which consists of 2 or 4 contraction links); then if we remove all these contraction links, we get another proof-net which is still symmetric w.r.t. **S**, and with less links. The induction hypothesis tells us that this proof-net is trivial, but there is no way to apply a contraction link to a trivial proof-net and therefore we reach a contradiction;

+ otherwise (since a proof-net is connected) there must be a terminal $\wedge$-link which *splits* (see Girard, 1987); this link has an orbit of four elements, hence we get four splitting $\wedge$-links. If we remove these four links, we get five connected components; if we consider these components as five points then **S** acts transitively on them. Therefore one of them must be invariant (5 must be the sum of the cardinalities of the orbits which can only be 1, 2 and 4, and there must be a 1 in this sum). Then this invariant component is symmetrical w.r.t. **S** and the induction hypothesis applies to it: it must be trivial, e.g. an axiom link between a and $\neg a$. But one of the two conclusions (say a) must be the premise of one of the severed $\wedge$-links (let us say with conclusion $a \wedge b$). The orbit of the conclusion has four elements, whereas the orbit of the premise a has only two elements, contradiction. □

The study of both weakening and contraction shows that formally symmetrical problems cannot receive symmetrical answers: cut-elimination must break the symmetry. The introduction of polarities is a way to indicate in which way the symmetry will be broken. If this conclusion is rather obvious, what was less obvious is that polarities can be chosen in a reasonably consistent way.

But as we said, we run into endless problems and the introduction of polarities is not enough to fix our 'classical proof-nets'. Polarities introduce hidden additive and exponential features in proof-nets and we cannot stay within a simple-minded multiplicative universe. The question of the right notion of classical proof-net is—as already stated in 1.5(i)—widely open.

Acknowledgements

From the beginning (winter 85–86) linear logic was seen as a way to analyse classical logic; however the translation *classical* $\mapsto$ *linear* given in Girard (1987) turned out to be a dead issue; for a long time the only evidence about classical logic were the negative examples of appendix B2 concerning the bad behaviour of cut-elimination in presence of structural rules. But the absence of progress was compensated by frequent—and sometimes passionate—discussions held with the group FORMEL of INRIA (especially Gérard Huet, Thierry Coquand, Benjamin Werner, Hugo Herbelin) about $\sim\sim$-translation, continuations, etc: the building up of a decent classical proof-theory was not only a logical puzzle, but also had many potential applications to computer science.

The decisive progress was made during a stay in Monash University at the invitation of John Crossley (December 90), where the crucial tables 1 and 3 were obtained. A first draft—mostly semantical and ending with **LC**—was ready by January 91. A second draft—starting from syntax and with table 2 as the main novelty—was ready by March 91. This second version benefitted from comments by Michel Parigot, Jacques van de Wiele, Vincent Danos, Jean H. Gallier and—last but not least—Yves Lafont: this printed version takes care of their various suggestions.

Finally many thanks are due to Peppe Longo for publishing this paper faster than light.

Added in print: the modifications between the second draft and this printed version are limited to correction of mistakes and clarification of obscurities. However the situation has evolved, since one of the main tasks proposed by the author—the building of a unified logical system in which (good) logical systems can interact freely, see section 1.5(*v*)*—has been carried out in Girard* (1991). *The striking property of the unified sequent calculus* **LU** *is that its classical, intuitionistic or linear fragments are better behaved than the calculi* **LK**, **LJ** *or* **LL**! *The price to pay for this is an increase of the number of rules, a price already noticed when replacing* **LK** *by* **LC**. *We did not make any effort to rewrite this paper in the light of the new paper* (*Girard* 1991), *the classical fragment of* **LU** *being a variant of* **LC**. *On the other hand Girard* (1991) *brings some new insight in some of our discussions*; *in particular the identification of intuitionism with negative polarity, turns out to be incorrect. Technically speaking a third polarity* (*neutral or* 0) *must be introduced. Classical logic uses polarities* -1 *and* $+1$, *intuitionistic logic uses* 0 *and* $+1$ (*only* 0 *for its negative fragment*) *and linear logic makes use of the three polarities* -1, 0 *and* $+1$.).

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