Categorial Symmetry

Categoriale Symmetrie

(met een samenvatting in het Nederlands)

Proefschrift

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I first met Vincent when taking his course on type theory during my Master's studies, having instilled me with a strong appreciation of constructive methods in mathematics. Vincent later became the second reader of my Master's thesis, and, now, one of my supervisors during my PhD candidacy. To me, what stands out the most

[1] Said article was later published and afterwards updated as [Bernardi and Moortgat, 2007, 2010].

is his deep enthusiasm for his research as well as his uncompromising dedication to furthering scientific discovery; qualities that he portrays to an extent that is rarely witnessed, even in academia. While I am convinced that academia is not always to be confused with science, as much as (written) history does not necessarily constitute a vehicle for truth, Vincent truly embodies the best of both worlds. I am as grateful for having come to know him as I am for his careful attention to my work and his detailed commenting thereupon, and I consider myself lucky for having him as my supervisor.

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Since the current work, as already mentioned, concludes not only three years of PhD studies, but another two and a half years going back to the writing of my Bachelor's thesis as well, I would like to use this opportunity to thank Peter Bougonje, Mathieu Klomp and Ben McKay. Without them as my co students, my three years of study for my Bachelor's degree would have been rather boring. And since the occasional distraction from work may often prove vital for the creative process, I would like to thank as well Ibtisam Daoudi, Arne Leer, Henk Meijer and Chun Shain Tan, all of whom, like Dirk, I have met at high school, and have since had the pleasure of maintaining regular contact with.

I wish to conclude by thanking my parents, Bert and Sylvia, as well as my sister, Esmeralda. Without their support, I would not have been where I am today, and this thesis would have remained unwritten, its contents never thought of.

(23-02-2013)

1

Introduction

The first chapter serves to acquaint the reader with the contents and contributions of this thesis. After a brief description of its subject matter in §1, we provide in §2 a more in-depth discussion of some of the materials dealt with. A brief outline of the subsequent chapters is provided in §3, while §4 provides a listing of the publications resulting from the efforts expended on the author's part in the course of this PhD project. §5 finally concludes with a listing of notational abbreviations.

1.1 Subject Matter

The present thesis constitutes an exploration into the field of *categorial type logics* (CTL, [Moortgat, 1997]), attempting the reduction of natural language grammar to proof theory. Thus, syntactic categories are replaced with propositions, or formulas, and a sentence's derivability is identified with the provability of its grammaticality. Among the main attractions of this approach we find:

1. **Lexicalism**. The lexicalist view proposes that all syntactic structure is projected from the lexicon, constituting the locus of linguistic variation. CTL realizes this position by identifying the rules of syntax with those of logical inference. Any language dependent information is therefore to be found in one's particular coupling of words and formulas, the latter, as the reader may recall, replacing the traditional notion of syntactic category.
2. **Transparent syntax-semantics interface**. When situated within the wider debate on the connections between logic and language, CTL may be considered a realization of Montague's [1970] Universal Grammar program, in the sense of offering a mathematical framework for studying both the syntax and semantics of natural and formal languages. In particular, CTL benefits of the close correspondence between the logician's formulas and the computer scientist's types, the latter used for categorizing algorithms according to the kinds of data they manipulate, as cemented in the Curry-Howard isomorphism [see Sørensen and Urzyczyn, 1998]. In other words, the meaning of some given expression may be explained using a program, a notion as mathematically rigid as that of formula or proof.
3. **High degree of formalization**. As already alluded to, CTL practices the maximum degree of mathematical rigour. As a consequence, implementation is relatively straightforward, particularly when targeting the higher-level logical and functional programming paradigms [Moortgat, 1988, Morrill, 1995].

The founding articles of the field may be traced back to several publications of Lambek in the late fifties and early sixties. In his first article on the topic, *The Mathematics of Sentence Structure* from 1958, Lambek [1958] proposed a reimagining according to proof-theoretic principles of the categorial grammar formalism, originating in the earlier works of Ajdukiewicz [1935] and Bar-Hillel [1953]. The result was a logic of strings, reflecting the importance of word-order in natural languages. In Lambek's [1961] follow-up article *On the Calculus of Syntactic Types* from 1961, this sensitivity to structure was further tightened to allow for reasoning with binary-branching trees, a concept prominently featured in syntactic literature. Historically, the two founding systems have come to be referred to by **L** and **NL**, abbreviating the *Lambek calculus* and the *non-associative Lambek calculus* respectively.

At this point, the reader might have gotten the impression that the field of CTL carries a distinctly plural flavour. And indeed, while the primary aim remains the aforementioned reduction of grammar to proof theory, many different systems have been put forward to this end, all building forth in one way or another on the traditional **L** or **NL**. The reasons for this divide originate in the long suspicion, originating in [Chomsky, 1963] and ultimately settled in the affirmative by Pentus [1999], that the original Lambek calculus recognizes only the context-free languages. And while the suitability of **(N)L** to the reasoning with strings or bracketings thereof has met with little critique, the question as to how their expressivity is to be improved upon in accordance with the empirical facts has generated considerably less consensus. Notable solutions include enriching the formula language with operations for extraction and infixation [Moortgat, 1992, Morrill et al., 2011], as well as introducing rules allowing for the controlled reorganization of syntactic structure [Morrill, 1990, Kurtonina and Moortgat, 1995, Moortgat, 1995].

Despite the many faces assumed by CTL within the literature, by far the most of its incarnations show an asymmetry that traces back to the original **(N)L**. Roughly, derivability is considered a relation between a possible multitude of hypotheses (the categories, or formulas, assigned to the individual words) and a unique conclusion (the formula categorizing the phrase made up from the hypotheses). One also speaks of an 'intuitionistic bias', in reference to a similar asymmetry found in intuitionistic logic. The late eighties and early nineties saw the first attempts at remedying this situation for the associative system **L**, treating hypotheses and conclusions alike. Key publications are Abrusci [1991] and Lambek [1993]. Again, one may speak of a 'classical turn', in reference to the symmetries found within classical logic, best expressed by its one-sided sequent system [see Schwichtenberg and Troelstra, 2000, §3.6]. With the dawn of the new millennium, **NL** finally followed suit with [De Groote and Lamarche, 2002]. At this point, studies into the viability of symmetry within CTL still mostly concerned proof-theoretic investigations, offering little empirical motivation for the change in perspective beyond what was already possible within the traditional systems.[1] As a result, despite their mathematical elegance, few of these systems have really caught on within computational linguistics.

In this thesis, we shall focus our attention on a recent contender within the CTL paradigm, derived from **NL** and referred to by the *Lambek-Grishin calculus* [Moortgat, 2009, **LG**], in reference to work of Grishin [1983], serving as the main source of inspiration besides Lambek's original papers. As with the various systems brought forth by the classical turn, it shows a bias towards symmetry. In contrast with its predecessors, however, **LG** successfully exploits said symmetry in pushing its expressivity beyond the context-free boundaries, essentially by allowing both sides of the derivability relation to communicate.

[1] As an exception, one may consider Abrusci and Ruet's [1999] non-commutative logic, allowing for both commutative and non-commutative logical operations to co-exist while also remaining conservative over **L**. A similar system was proposed, however, by de Groote [1996] within the traditional asymmetric setting.

The current document constitutes a linguistic and proof-theoretic investigation into **LG**, many whose formal properties have yet to be established due to its relative youth. True to the spirit of CTL, we strive for a mode of presentation where any advancements in our knowledge on proof theory are immediately investigated for their linguistic applicability. The next section provides a more in-depth discussion of **LG**, allowing for a finer description of our contributions in §3.

1.2 Mirror, mirror

The current section provides a slightly more in-depth introduction to the Lambek-Grishin calculus, describing the necessary background against which the particular research questions can be described that we shall tackle in the later chapters of this thesis. In our presentation, we shall opt for a two-step approach. Since **LG** is derived from **NL**, we first briefly introduce the latter. **LG** is subsequently presented as an exercise in restoring symmetry, identifying the derivability relation between formulas as the relevant notion of 'mirror'. By allowing both its sides to interact, we unearth the potential for increasing expressivity beyond the context-free boundaries.

1.2.1 Non-associative Lambek calculus

The non-associative Lambek calculus may arguably be considered the minimal conception of a categorial type logic, involving the least number of assumptions on its logical vocabulary and its notion of structure. As explained, CTL seeks the reduction of natural language grammar to proof theory, proposing in particular the identification between categories and (logical) formulas. Compared to traditional formalisms such as context-free rewrite grammars, the main benefit of this approach is found in the inductive definition of formulas, thus lessening the number of primitive categories one needs to assume (S, *NP*, N, *Det*, *VP*, *TV*, *IV*, *PP*, $\ldots$). In particular, we shall restrict the latter to sentences (s), noun phrases (*np*) and common nouns (n). From these, complex formulas are constructed, as may be explained by reference to the syntactic operations of (binary) merger and subcategorization. First, any phrases of some given categories A and B may be merged into a phrase of the category $(A \otimes B)$, pronounced A *tensor* B. The latter evaluates to C if A derives (C/B) (C *over* B) or B derives $(A\backslash C)$ (A *under* C). In other words, the complex formulas (C/B) and $(B\backslash C)$ categorize phrases requesting (i.e., subcategorizing for) merger with respect to the category B on their right, resp. left, projecting a phrase of category C. The following CFG for defining formulas A, B summarizes our previous discussion.

$$
\begin{array}{rcll}
A, B & ::= & s \mid np \mid n & \text{(Sentences, noun phrases, common nouns)} \\
 & \mid & (A \otimes B) & \text{(Binary merger)} \\
 & \mid & (A/B) & \text{(Right subcategorization)} \\
 & \mid & (B\backslash A) & \text{(Left subcategorization)}
\end{array}
$$

We next provide some examples of traditional categories and their translations to formulas, skipping those having atomic counterparts:

Category	Example	Formula
Det	the, a	np/n
Adj	big, scrawny	n/n
IV	walks, dreams	$np\backslash s$
TV	notices, overhears	$(np\backslash s)/np$
VP	walks, notices John	$np\backslash s$
VP	with, below	$(n\backslash n)/np$, $((np\backslash s)\backslash(np\backslash s))/n$

Note the use of two formulas for describing prepositions, allowing for modification of nouns and of verb phrases respectively. The well-known binoculars example, "John saw the man with the binoculars," illustrates this ambiguity: did John use the binoculars to see the man, or was the man seen by John carrying binoculars?

Besides the identification of categories and formulas, CTL dictates that syntactic derivations are proofs. The latter we shall take to establish *inequalities* $A \leq B$, expressing that any phrase of the category A is also of the category B. In particular, we shall want to prove the *cancellation schemas* $(C/B) \otimes B \leq A$ and $B \otimes (B\backslash C) \leq A$, in accordance with the readings associated to the various logical operations in the above discussion. Rules of inference include (Refl) and (Trans), abbreviating the *preorder laws* (i.e., reflexivity and transitivity) for $\leq$, together with (Res), relating $\otimes$ to $/$ and $\backslash$ by *residuation*:

$$
\begin{array}{rl}
A \leq A & \text{(Refl)} \\
\text{If } A \leq B \text{ and } B \leq C, \text{ then } A \leq C & \text{(Trans)} \\
B \leq A\backslash C \text{ iff } A \otimes B \leq C \text{ iff } A \leq C/B & \text{(Res)}
\end{array}
$$

While the justifiability of the preorder laws should be intuitively clear, the residuation laws are less immediate. The literature suggests the following possible interpretations: one arithmetical, and one logical.

1. **Arithmetical**. We read merger as multiplication and collapse the subcategorization operations into division, simply reading ≤ as 'less than or equal to':

$$x \leq \frac{y}{z} \text{ iff } x \times y \leq z$$

where the metavariables A, B, C have been replaced with the more usual x, y, z for denoting numbers. This was the interpretation pursued by Bar-Hillel [1953], predating Lambek, though, strictly speaking, targeting not the residuation laws as a whole, but rather restricting to the cancellation schemas $(C/B) \otimes B \leq A$ and $A \otimes (A\backslash C) \leq C$ discussed above.

2. **Logical**. This interpretation arises if we understand ⊗ as conjunction, while collapsing / and \ into implication and reading inequality as logical consequence (⊢):

$$\phi \vdash \psi \rightarrow \omega \text{ iff } \phi \wedge \psi \vdash \omega$$

this time replacing A, B, C with the more usual metavariables ϕ, ψ, ω for propositions. This is the interpretation pursued by Lambek, and underlies the enterprise of CTL.

While instructive, the above interpretations do not accurately reflect the sensitivity to syntactic structure inherent in **NL**. In particular, multiplication and conjunction both satisfy the following associativity and commutativity laws:

$$x \times (y \times z) = (x \times y) \times z \qquad \phi \wedge (\psi \wedge \omega) \dashv\vdash (\phi \wedge \psi) \wedge \omega \qquad \text{(Associativity)}$$
$$x \times y = y \times x \qquad \phi \wedge \psi \dashv\vdash \psi \wedge \phi \qquad \text{(Commutativity)}$$

In contrast, ⊗ remains sensitive to word order (rejecting commutativity) as well as to hierarchical structuring (rejecting associativity). With these explanations, we can finally check the desired $(C/B) \otimes B \leq A$ and $B \otimes (B\backslash C) \leq A$:

$$C/B \leq C/B \text{ by reflexivity, hence } (C/B) \otimes B \leq C \text{ by residuation}$$
$$B\backslash C \leq B\backslash C \text{ by reflexivity, hence } B \otimes (B\backslash C) \leq C \text{ by residutation}$$

E.g., we have $\mathit{np} \otimes (\mathit{np}\backslash s) \leq s$, motivating the use of $\mathit{np}\backslash s$ for the categorization of intransitive verbs or verb phrases, as well as $\mathit{np} \otimes (((\mathit{np}\backslash s)/\mathit{np}) \otimes \mathit{np}) \leq s$, likewise suggesting the use of $(\mathit{np}\backslash s)/\mathit{np}$ for transitive verbs.

1.2.2 Lambek-Grishin calculus

Grishin [1983], approaching Lambek's work from an algebraic perspective, first suggested adding new logical operations, dual to merger and subcategorization:

$$
\begin{array}{lcll}
A, B & ::= & s \mid np \mid n & \text{(Primitive categories)} \\
 & \mid & (A \otimes B) \mid (B \oplus A) & \text{(Combination vs. separation)} \\
 & \mid & (A/B) \mid (B \obslash A) & \text{(Right selection vs. left rejection)} \\
 & \mid & (B\backslash A) \mid (A \oslash B) & \text{(Left selection vs. right rejection)}
\end{array}
$$

'Duality' is here understood by inverting ≤ and replacing combination and selection by separation and rejection respectively, thus obtaining patterns of *co*residuation:

$$
\begin{array}{rcccll}
B \leq A\backslash C & \text{iff} & A \otimes B \leq C & \text{iff} & A \leq C/B & \text{(Res)} \\
C \oslash A \leq B & \text{iff} & C \leq B \oplus A & \text{iff} & B \obslash C \leq A & \text{(CoRes)}
\end{array}
$$

The extension of **NL** in accordance with Grishin's proposals (originally concerning **L**) is referred to by the Lambek-Grishin calculus [see Moortgat, 2009, **LG**]. As presented, it appears primarily as a formal exercise in abstraction, with little clear linguistic applicability. The main attraction of **LG**, however, is to be sought in the possibility of allowing communication between the residuated and coresiduated 'families' of logical operations. For example, while $\otimes$ and $\oplus$ individually reject associativity and commutativity, mixed variants of the latter could still be postulated without compromising overall sensitivity to syntactic structure. Grishin originally studied four groups of non-logical axioms, two of which, the first and fourth, involved the desired mix of logical operations. Thinking of these axioms as establishing communication, they are typically referred to by *interactions* of type I and IV.

	Type I	Type IV
Mixed Associativity	$(A \oplus B) \otimes C \leq A \oplus (B \otimes C)$ $A \otimes (B \oplus C) \leq (A \otimes B) \oplus C$	$(A\backslash B) \oslash C \leq A\backslash(B \oslash C)$ $A \obslash (B/C) \leq (A \obslash B)/C$
Mixed Commutativity	$A \otimes (B \oplus C) \leq B \oplus (A \otimes C)$ $(A \oplus B) \otimes C \leq (A \otimes C) \oplus B$	$A \obslash (B\backslash C) \leq B\backslash(A \obslash C)$ $(A/B) \oslash C \leq (A \oslash C)/B$

Though (the associative restrictions of) both the type I and IV interactions already reappeared in Lambek's [1993] previously cited work on bilinear logic, their actual linguistic applications were not considered until more recently, marking the birth of **LG**. Of note are Bernardi and Moortgat's [2010] analysis of scopal ambiguities, while, on a more formal note, Melissen [2011] showed that, when augmented with the full assortment of type IV interactions, **LG**'s expressivity already exceeds that of Lexicalized Tree Adjoining Grammars. Below, we provide a survey of some of the main results obtained thus far. As with Melissen's efforts, they mainly concern the addition with type IV interactions, the resulting system henceforth to be referred to by $\mathbf{LG}_{IV}$. Similarly, we can speak of $\mathbf{LG}_{\varnothing}$, referring to the 'base logic' without any of Grishin's axioms, $\mathbf{LG}_{I}$, denoting the augmentation by type I interactions, and finally $\mathbf{LG}_{I+IV}$, involving both the type I and IV interactions. Be warned, however, that the latter induces a collapse into associativity and commutativity of $\otimes$ and $\oplus$ independently (cf. chapter 3), so that its interest to linguistics is limited.

1. **Expressivity and complexity**. Moot [2007] first provided an embedding of LTAG inside **LG** together with type IV interactions. Melissen [2011], as already mentioned, subsequently showed that the lower bound on $\mathbf{LG}_{IV}$'s expressivity actually lies higher, though still leaving its upper bound a mystery. Complexity-wise, Bransen [2012, 2011] proved derivability in $\mathbf{LG}_{IV}$ is NP-complete, conjecturing the same result to hold of $\mathbf{LG}_{I}$. In contrast, Capelletti [2005] showed that $\mathbf{LG}_{\varnothing}$ can be decided in polynomial time.

2. **Model theory**. Kurtonina and Moortgat [2010] showed soundness and completeness w.r.t. a Kripke-style possible world interpretation, modeling n-ary logical operations by $(n + 1)$-ary relations. More recently, Chernilovskaya et al. [2012] proposed an alternative relational interpretation, better suited to the study of duality as found in lattice theory. On a different note, Moortgat and Pentus [2007] have conducted an investigation within $\mathbf{LG}_{IV}$ into the relation of *type similarity*, denoting the least equivalence relation on formulas containing derivability. In particular, while in general such patterns as commutativity and associativity for $\otimes$ and $\oplus$ individually are rejected by derivability, they do appear to hold at the level of type similarity. Soundness and completeness results were established w.r.t. an Abelian group interpretation.

3. **Formal semantics**. As mentioned in §1, the various incarnations of CTL typically enjoy a straightforward correspondence with a Montagovian style compositional semantics. In practice, however, this result is strongly linked to the intuitionistic bias of these systems. With **LG**, we require a more sophisticated translation back into an asymmetric setting, known in the computer science literature as a *continuation passing style translation* and in the proof theory literature as a *double negation translation*. Bernardi and Moortgat [2007, 2010] fill in the details, describing various linguistic applications.[2]

[2] Strictly speaking, [Bernardi and Moortgat, 2010] is a revised version of [Bernardi and Moortgat, 2007], and both papers indeed go by the same name. In practice, however, the differences are night and day, and we shall find more frequent occasion to refer to the original than to its revision.

1.3 Thesis outline

The first three chapters, excluding the current introduction, serve primarily as a condensed introduction to the field of CTL, concentrating on **NL** (chapter 2) and its more recent symmetric extensions, **LG** (chapter 3) and De Groote and Lamarche's [2002] *classical non-associative Lambek calculus* (chapter 4). The purposes of this exercise are twofold: first, to make the current work more accessible to the reader less familiar with the relevant background; and second, to identify some of the open research questions wherein the bulk of our contributions in the later chapters are to be found.

Chapter 5 defines labeled sequent derivations for both $\mathbf{LG}_{\varnothing}$ and **CNL**, applying them in proving context-freeness for the associated notions of grammars.

Chapter 6 revisits type similarity using a different notion of algebraic model, better reflecting the duality found within the source language compared to Moortgat and Pentus's [2007] use of Abelian groups. The relevant structures are dubbed *weakly distributive algebras*, in reference to work of Cockett and Seely [1991], and feature two associative and commutative binary operations, related to each other through laws similar to Grishin's interactions. This situation is to be contrasted with the more traditional, asymmetric incarnations of CTL, where derivability and similarity go hand in hand with respect to structural sensitivity. We describe several exploits of this discrepancy, showing how type similarity may be used in the analysis of extraction phenomena.

Chapter 7 concludes with investigations into formal semantics. As already noted, Bernardi and Moortgat [2007, 2010] previously defined translations of **LG** back into an intuitionistic language so as to benefit of the latter's (Curry-Howard) correspondence with the computer scientist's (functional) programming languages. In fact, several plausible witnesses for such a translation exist, and Bernardi and Moortgat describe two dual candidates, commonly known as the *call-by-name* (CBN) and *call-by-value* (CBV) interpretations. We generalize these results by replacing their target language with a substructural logic comparable in its resource sensitivity to the source, while furthermore investigating a third translation, adapting Girard's [1991] constructivization of classical logic. For all practical purposes, we may refer to this new target by a term language for **LG**, and we shall investigate its applicability to the study of quantifier scope ambiguities; arguably one of the great successes of the Montagovian approach to natural language semantics, besides intensionality. In particular, we shall find that even in omitting the Grishin interactions, the base logic $\mathbf{LG}_{\varnothing}$ already suffices for deriving all combinatorially possible scopal readings of a sentence.

1.4 Publications and presentations

The work put into this thesis has resulted in several publications and presentations, briefly described below.

1. Bastenhof [2010b] provides a context-freeness result for $\mathbf{LG}_{\varnothing}$ using labeled tableaux. Our chapter 5 is essentially an expanded version of this article, together with a shift in perspective from semantic tableaux to labeled sequent derivations.
2. The work on type similarity was only reported on in a talk, titled *an algebraic semantics for type similarity in symmetric categorial grammar*, held during the special session on substructural logics at the third world congress and school on Universal Logic in Lisbon, Portugal, 2010.
3. Bastenhof [2010a] briefly motivates the linguistic applicability of Girard's [1991] work on constructivizing classical logic, though presented secondary to a syntactic analysis of extraction.
4. Bastenhof [2012a], as a follow-up to the previous article, is entirely dedicated to the linguistic exploration of Girard's ideas, using **LG** for illustration.
5. Bastenhof [2011] provided further proof-theoretic justification for the previous two articles, demonstrating completeness using phase models.
6. Bastenhof [2012b], addresses the problem of quantifier scope ambiguities, arguing their satisfactory analysis need appeal to neither non-context-free mechanisms, nor to a relaxation of compositionality to a relation.

1.5 Notational preliminaries

In referring to a previously stated definition (lemma, theorem, corollary, figure, example) n, we shall often use the abbreviation D.n (L.n, T.n, C.n, F.n, E.n).

2

The Non-Associative Lambek Calculus

2.1 Introduction

The purpose of the current chapter is to provide a leisurely exposition on the type-logical approach to natural language syntax. We take the non-associative Lambek calculus of [Lambek, 1958, **NL**] as a case study; arguably the most conservative in both its choice of logical vocabulary and its assumptions on their structural properties. In doing so, we place special emphasis on Kurtonina and Moortgat's [2010] relational models, serving to describe the 'linguistic reality' with respect to which our systems of formal reasoning are to be justified. Said purpose is emphasized through the discussion of 'concrete' such models, parameterized over the more commonly known formalism of context-free rewrite grammars. While the latter require the grammar writer to devise rules specific to the natural language fragment under consideration, categorial type logics, by enriching the notion of a syntactic category with the inductive structure of a logical proposition, instead allow for the identification of the rule component with the universally available logical laws, leaving only the lexicon to be stipulated.

We shall proceed as follows. The first two subsequent sections focus on the description of several formalisms for conducting reasoning in **NL**, each with their own merits. The first that we shall discuss (§2) emphasizes the algebraic properties of the logical operations, and is presented in parallel with Kurtonina and Moortgat's relational models. Next, in §3, we treat natural deduction and sequent calculus, sharing their understanding of proofs as relating a possible multitude of hypotheses to a unique conclusion. The relation to linguistics is particularly clear here due to the structuring of hypotheses into binary-branching trees, reflecting the common linguists' assumptions on the syntactic structure of natural language. Both formalisms are furthermore founded upon certain symmetries, imparting them with a sense of naturality: while natural deduction compares the use of a formula with its inference, sequent calculus rather adopts the more abstract concept of introducing a formula as a hypothesis or as a conclusion. In practice, natural deduction best describes the kind of reasoning found in informal mathematical proofs, while sequent calculus has found applications in proof search. Continuing, we provide in §4 a brief introduction into the realization of formal semantics within **NL**. We conclude in §5 with an overview of some of the results on **NL**, including those highlighting its expressive shortcomings, paving the way for the introduction of **LG** in the next chapter. It should be noted that, save for the expository device of concrete frames encountered in §2, the current chapter essentially constitutes a selection of some of the main results found in the literature on **NL**, and thus offers little contribution of its own.

2.2 Ternary frames and non-associative Lambek calculus

The current section serves as a gradual introduction to **NL** through Kurtonina and Moortgat's [2010] relational models. Adhering to the classical tradition in logic, their purpose is to describe the 'external' linguistic reality with respect to which reasoning is shown to be both sound (every proven regularity is observed by reality) and complete (truth makes proof). We will emphasize a concrete instance of these models, based on the identification of linguistic reality with the (weak) generative capacity of a well-known simple descriptive grammar formalism, to wit context-free rewrite devices. While this of course does not provide the complete picture, given the non-context-freeness of natural language in general, it illustrates the transition from traditional methods in syntax to those based on proof-theory.

Most, if not all theories of natural language syntax classify linguistic expressions into *categories*, characterizing their distribution. Thus, by assigning any two utterances to the same category, it is predicted that they may always be interchanged without affecting grammaticality. As a next step, expressions are analyzed by projecting a hierarchical structure among the categories encountered within, as visualized by placing their words at the leaves of a rooted tree. E.g., if we agree upon the existence of noun phases *NP* (*Mary*, *the man*, ...), verb phrases *VP* (*strolls*, *likes to play chess*, ...) and finally sentences S (*The man strolls*, ...), we may posit the schema

$$\begin{array}{c} \text{S} \\ \mathit{NP} \quad \mathit{VP} \end{array}$$

meaning any combination of a noun phrase and a verb phrase yields a sentence. While inconclusive, with such mentioned words as *man* and *likes* having been left unanalyzed, the above discussion motivates the following definition, constituting a well known means of casting analysis trees into the mold of mathematical rigour.

Definition 1. A *context-free grammar* (CFG) G is a 4-tuple $\langle V, \Sigma, R, S\rangle$, where V is a set of *non-terminals* or *categories*, Σ is a set of *terminals* or *words* disjoint from V, $R \subseteq V \times (V \bigcup \Sigma)$, and $S \in \Sigma$ is the *start symbol.* The elements $\langle A, w\rangle$ of R are called *productions*, or *rewrite rules*, and are usually written $A \to w$.

Derivations are often depicted by trees, starting from a single node S (hence initially both root and leaf), and applying rules $A \to w$ as expansions of leafs A by daughters w. More formally,

Definition 2. Given $G = \langle V, \Sigma, R, S\rangle$ and strings u, v over $V \bigcup \Sigma$, we write $u \Rightarrow v$ and say u *yields* v if $u = u_1 A u_2$, $v = u_1 w u_2$ and $A \to w \in R$. A string w of words (i.e., over Σ) is said to be in the *language* $\mathscr{L}(G)$ iff $S \Rightarrow^* w$, being the reflexive-transitive closure of $\Rightarrow$.

Example 1. We define a CFG for analyzing the sentence *John saw the man with the binoculars.* As words Σ we then have the set {John, saw, the, man, with, binoculars}, while for the categories we adopt:

	Explanation	Examples
S	Sentences	John saw the man (with the binoculars)
Det	Determiners	the
N	Common nouns	man (with the binoculars), binoculars
NP	Noun phrases	John, the man, the binoculars
VP	Verb phrases	saw the man (with the binoculars)
TV	Transitive verbs	saw
Prep	Prepositions	with
PP	Prepositional phrases	with the binoculars

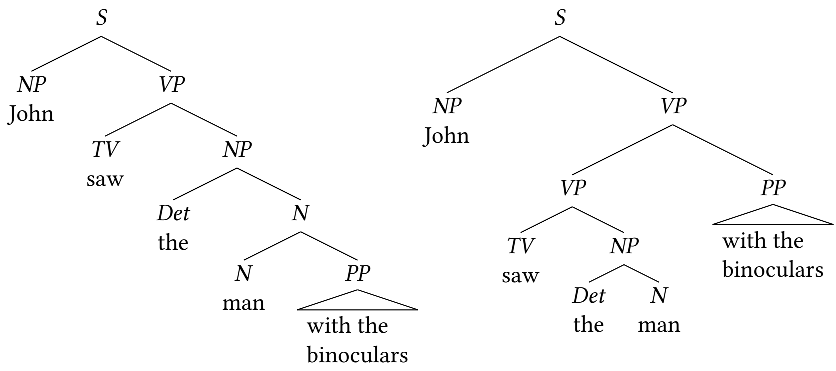
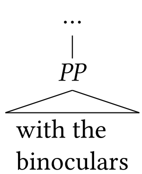


Figure 2.1: Analyses of *John saw the man with the binoculars.*

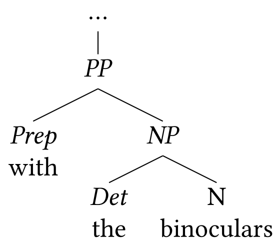
Finally, R consists of the following productions:

S → *NP VP*	*NP* → John
NP → *Det N*	*Det* → the
N → *N PP*	*N* → man
VP → *TV NP*	*N* → binoculars
VP → *VP PP*	*TV* → saw
PP → *Prep NP*	*Prep* → with

F.2.1 depicts two derivations for the target sentence in tree format, reflecting its different available readings: is the man being seen by John through the binoculars, or is the man with the binoculars being seen by John? Note we use the abbreviation



for



The above example features rules of two forms: binary productions $A \to B\ C$ for nonterminals $B, C \in V$, and unary productions $A \to w$ for a word $w \in \Sigma$, the latter referred to by rules for *lexical insertion.* As a result, derivations take on the form of binary-branching trees, save for lexical insertion at the leafs. In searching for *language universals*, it has been argued by Kayne [1994] that syntactic analyses of natural language need use only such binary-branching structures.[1] While certainly not uncontroversial, it has been widely adopted, especially in the literature on generative grammar. The developments below shall likewise adhere to the contents of Kayne's conjecture, although this is not to be considered a necessary restriction, as demonstrated in, e.g., [Buszkowski, 1989, Melissen, 2009].

The current thesis concerns the field of type-logical grammars, whose primary aim is to explain natural language syntax using *logic.* Thus, categories are formulas, or propositions, and the derivation of a sentence is a proof of its grammaticality. In the classical tradition of logic, reasoning always proceeds with respect to an external reality, being captured on print by the notion of *model.* Relational *frames* [cf. Dosen, 1992], as defined below, describe the structure of the particular models that we shall use for conducting linguistic reasoning.

Definition 3. A *frame* F is a pair $\langle W, \otimes \rangle$ consisting of a non-empty set W of *resources* or *worlds* $a, b, c \ldots$ and a ternary *accessibility* relation $\otimes \subseteq W \times W \times W$, called *merger.*

As a notational convention, we henceforth write $\otimes abc$ to express $\langle a, b, c \rangle \in \otimes$. The following concrete example of a frame is often cited.

Example 2. Given a (finite) alphabet Σ, the *flat* frame has its resources W defined by the set Σ^* of all strings over Σ, and for any $u, v, w \in \Sigma^*$, $\otimes uvw$ iff u is the concatenation of v with w.

In practice, one may consider Σ the set of words of some language fragment, Σ^* thus consisting of all possible phrases, grammatical and ungrammatical alike. We intend to pursue an understanding of frames as describing tree structures, with $\otimes uvw$ encoding a parent node u with daughters v and w. In the above example, however, we have $\otimes uvw$ for any decomposition of a string $u \in \Sigma^*$ into components v, w, hence resulting in a flattening of trees into strings. Below, we make another attempt at

[1]This claim is not to be confused with normalization into *Chomsky normal form*, whereby any CFG G may be rewritten so as to use only rules of the forms described above, and an additional $S \to \epsilon$ if the empty string ϵ is in the language. While $\mathscr{L}(G)$ is kept constant by this transformation, the analysis trees evidently are not preserved in general. As apparent from E.1 however, the latter serve as the input to semantic interpretation, and hence are not to be trivialized. Kayne's claim, besides not being restricted in scope to the particular formalism of CFG's, therefore concerns the actual analyses produced by a grammar (i.e., it's strong generative capacity), and not just the language it recognizes.

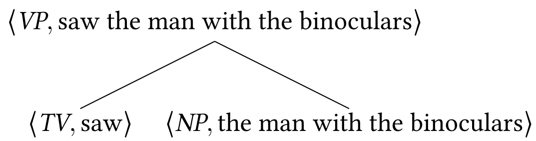
defining a natural enough concrete frame, this time assuming the various phrases of the language fragment under consideration to have already been analyzed into syntactic categories.

Definition 4. Let $G = \langle V, \Sigma, R, S\rangle$ be a CFG all whose productions are of the form $A \to B\ C$ for $B, C \in V$ or $A \to w$ for $w \in \Sigma$. Define the frame $F_G = \langle W_G, \otimes_G\rangle$ parameterized over G by taking the resources in W_G to be pairs $\langle A, w\rangle$ where $A \Rightarrow^* w$ for $w \in \Sigma^*$, while the productions serve as an extra check on $\otimes_G$:

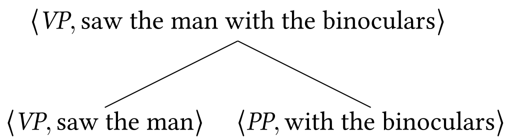
$$\begin{aligned} &W_G := \textstyle\bigcup_{A\in V}\{\langle A, w\rangle \mid w \in \Sigma^* \text{ and } A \Rightarrow^* w\} \\ &\otimes_G(\langle A, u\rangle, \langle B, v\rangle, \langle C, w\rangle) \text{ iff } u = vw \text{ and } A \to B\ C \in R \end{aligned}$$

We make the following observations on the above definition. First, a frame parameterized over a given CFG is always finitely generated, but not in general finite. Second, binary productions are modeled by ternary relations. More generally, we could state a rule $A \to B_1\ \dots\ B_n$ with $B_1, \dots, B_n \in V$ is modeled using $n+1$-ary relations. Lexical insertion, however, is ignored, seeing as this information is already compiled away into the resources.

Example 3. Consider the frame parameterized over the CFG found in E.1. In light of F.2.1, we see ⟨*VP*, saw the man with the binoculars⟩ stands in the relation $\otimes$ not only with ⟨*TV*, saw⟩ and ⟨*NP*, the man with the binoculars⟩, depicted graphically



but also with ⟨*VP*, saw the man⟩ and ⟨*PP*, with the binoculars⟩:



Our reason for discussing frames, ultimately, is so we can start reasoning about them, and particularly about their applications in natural language syntax. We thus require the means of making assertions about said structures, to which end we define an artificial unambiguous language whose members are otherwise referred to as *formulas* or *propositions*. A *model* essentially extends a frame by an interpretation of *atomic* formulas, constituting the base cases of an inductive definition.

Definition 5. Given a countable set *Atom* of atomic propositions (atoms, for short), a *model* $\mathscr{M}$ is a pair $\langle F, v\rangle$ of a frame F and a *valuation* $v : Atom \to \mathscr{P}(W)$. If $a \in v(p)$, we also say p is *true at* a, while, dually, p is *false at* a if $a \notin v(p)$.

Intuitively, an atom, and more generally a formula, is a property of resources, holding of those at which it is true. The current exposition, however, aims at a linguistic explanation of formulas, by which we may equivalently consider the latter to be syntactic categories. The following example motivates a particular choice of *Atom*.

Example 4. Of the categories featured in E.1, we consider only S, *NP* and N atomic. More specifically, to maintain a notational distinction between formulas and the categories of a CFG, we will write the former as s, *np*, n. The concrete frame of E.3 is now trivially extended to the following model:

$$\begin{array}{rcl} v(s) & := & \{\langle S, w\rangle \mid w \in \Sigma^* \text{ and } S \Rightarrow^* w\} \\ v(n) & := & \{\langle N, w\rangle \mid w \in \Sigma^* \text{ and } N \Rightarrow^* w\} \\ v(\mathit{np}) & := & \{\langle \mathit{NP}, w\rangle \mid w \in \Sigma^* \text{ and } \mathit{NP} \Rightarrow^* w\} \end{array}$$

Thus, s is 'true' exactly of the sentences, *np* of the noun phrases, etc.

Atomic formulas offer little opportunity for reasoning, their truth or falsity relative to any given resource having been stipulated. To allow for formal proofs comprising more than a single inference step, we require operations for deriving complex formulas, referred to by *connectives* or *logical constants*. The following example motivates three such operations through use of frames parameterized over CFG's.

Example 5. Given a CFG $G = \langle V, \Sigma, R, S\rangle$ of the usual restricted form, we define the following operations in the frame F_G (cf. D.4): for any $P, Q \subseteq W_G$,

$$\begin{array}{rcl} \langle C, u\rangle \in (P \otimes Q) & \text{iff} & \text{there exist } \langle A, v\rangle, \langle B, w\rangle \in W_G \text{ s.t.} \\ & & C \to A\ B \in R,\ u = vw,\ \langle A, v\rangle \in P \text{ and } \langle B, w\rangle \in Q \\ \langle A, v\rangle \in (P/Q) & \text{iff} & \text{for any } \langle B, w\rangle, \langle C, vw\rangle \in W_G \text{ s.t. } C \to A\ B \in R, \\ & & \text{if } \langle B, w\rangle \in Q, \text{ then also } \langle C, vw\rangle \in P \\ \langle B, w\rangle \in (Q\backslash P) & \text{iff} & \text{for any } \langle A, v\rangle, \langle C, vw\rangle \in W_G \text{ s.t. } C \to A\ B \in R, \\ & & \text{if } \langle A, v\rangle \in Q, \text{ then also } \langle C, vw\rangle \in P \end{array}$$

Note $(P \otimes Q)$ contains concatenations $\langle C, vw\rangle$ for any $\langle A, v\rangle \in P$ and $\langle B, w\rangle \in Q$ s.t. $C \to A\ B$. This definition essentially mirrors that of merger in F_G, hence the overloading of notation. In contrast, the operations (P/Q) and $(Q\backslash P)$ serve as requests for concatenation to the right and left respectively, provided the resulting expression obeys to the rules of the CFG. In case of the concrete model of E.4, we have the following results:

$$
\begin{aligned}
\{\langle \mathit{Det}, w\rangle \mid w \in \Sigma^* \text{ and } \mathit{Det} \Rightarrow^* w\} &= v(\mathit{np})/v(n) \\
\{\langle \mathit{VP}, w\rangle \mid w \in \Sigma^* \text{ and } \mathit{VP} \Rightarrow^* w\} &= v(\mathit{np})\backslash v(s) \\
\{\langle \mathit{TV}, w\rangle \mid w \in \Sigma^* \text{ and } \mathit{TV} \Rightarrow^* w\} &= (v(\mathit{np})\backslash v(s))/v(\mathit{np}) \\
\{\langle \mathit{PP}, w\rangle \mid w \in \Sigma^* \text{ and } \mathit{PP} \Rightarrow^* v\} &= v(n)\backslash v(n) \\
&= (v(\mathit{np})\backslash v(s))\backslash(v(\mathit{np})\backslash v(s)) \\
\{\langle \mathit{Prep}, w\rangle \mid w \in \Sigma^* \text{ and } \mathit{Prep} \Rightarrow^* w\} &= (v(n)\backslash v(n))/v(\mathit{np}) \\
&= ((v(\mathit{np})\backslash v(s))\backslash(v(\mathit{np})\backslash v(s)))/v(\mathit{np})
\end{aligned}
$$

To illustrate, we check the first claim. By definition, $\langle A, v\rangle \in v(\mathit{np})/v(n)$ iff for any $C \to A\ B \in R$ and $w \in \Sigma^*$, $\langle B, w\rangle \in v(n)$ implies $\langle C, vw\rangle \in v(\mathit{np})$. Recalling the definitions of $v(n)$ and $v(\mathit{np})$, C and B can be identified with *NP* and N respectively, so A must be *Det* since the only matching production is *NP* → *Det* N. Hence,

$$v(\mathit{np})/v(n) = \{\langle \mathit{Det}, v\rangle \mid \text{for any } w \in \Sigma^*,\ N \Rightarrow^* w \text{ implies } \mathit{NP} \Rightarrow^* vw \text{ and } vw \in \Sigma^*\}$$

the desired result being immediate from the fact that every determiner concatenates with a noun to form a noun phrase by the aforementioned production *NP* → *Det* N.

With the above motivation, we now provide the general definitions of formulas and of their interpretations inside arbitrary relational models.

Definition 6. The set of formulas $\mathscr{F}(\mathit{Atom})$ generated from the set *Atom* of atoms is obtained through closure under *multiplicative conjunction* (*tensor*) ⊗ and direction-sensitive *implications* (*divisions*) / and \:

$$
\begin{array}{lcll}
A, B & ::= & p & \text{(Atomic formulas in } \mathit{Atom}\text{)} \\
 & \mid & (A \otimes B) & \text{(Tensor)} \\
 & \mid & (A/B) & \text{(Right division)} \\
 & \mid & (B\backslash A) & \text{(Left division)}
\end{array}
$$

Definition 7. Given a model $\mathscr{M} = \langle F, v\rangle$, the valuation is inductively extended to a *forcing* relation ⊨ operating on arbitrary formulas, with target sets of resources:

$$
\begin{array}{rll}
a \vDash p & \text{iff} & a \in v(p) \\
a \vDash A \otimes B & \text{iff} & \text{there exist } b, c \text{ s.t. } \otimes abc,\ b \vDash A \text{ and } c \vDash B \\
b \vDash A/B & \text{iff} & \text{for all } a, c \text{ s.t. if } \otimes abc \text{ and } c \vDash B, \text{ then } a \vDash A \\
c \vDash B\backslash A & \text{iff} & \text{for all } a, b \text{ s.t. if } \otimes abc \text{ and } b \vDash B, \text{ then } a \vDash A
\end{array}
$$

Preorder laws: Reflexivity, Transitivity

$$\frac{}{A \leq A}\ Id \qquad \frac{A \leq B \quad B \leq C}{A \leq C}\ \circ$$

Residuation

$$\frac{A \otimes B \leq C}{\overline{A \leq C/B}}\ r \qquad \frac{A \otimes B \leq C}{\overline{B \leq A\backslash C}}\ r$$

Figure 2.2: The non-associative Lambek calculus.

Again, A is said to be *true at* $a \in W$ iff $a \vDash A$, and *false* otherwise.

Evidently, the interpretations of E.5 constitute special cases of the above definition. Having decided upon our logical vocabulary and the proper extension of valuations, we may state the rules of inference for reasoning with said vocabulary to derive truths concerning arbitrary models.

Definition 8. F.2.2 defines *non-associative Lambek calculus* (**NL**) using the judgement form $A \leq B$.

Intuitively, an inequality $A \leq B$ is intended as an assertion that for all models (w.r.t. $Atom$) and resources a therein, $a \vDash A$ implies $a \vDash B$.

Remark 1. While referred to as an axiomatization of judgements $A \leq B$, F.2.2 differs crucially from the standard Hilbert-style axiomatization for classical logic in that Modus Ponens (here, transitivity of $\leq$) is not the sole rule of inference. In fact, it has been demonstrated in a series of papers, starting with Zielonka [1988], that **NL** and its associative counterpart **L** are not finitely axiomatizable in this sense.

While a model-theoretic justification was claimed for the rules of F.2.2, its validation by the *soundness* result below [Dosen, 1992] is yet to be made.

Theorem 2.2.1. If $A \leq B$, then, for all models $\mathscr{M} = \langle F, v \rangle$ and resources a in (the set of resources W underlying) F, $a \vDash A$ implies $a \vDash B$.

Proof. By induction on the derivation witnessing $A \leq B$. The preorder laws obviously reflect reflexivity and transitivity of set inclusion. For the residuation laws, we check $A \otimes B \le C$ iff $A \le C/B$. Going from left to right first, suppose, for all models and worlds a, that $a \vDash A \otimes B$ implies $a \vDash C$, i.e., (a) $(\exists b, c)(\otimes abc \;\&\; b \vDash A \;\&\; c \vDash B)$ implies $a \vDash C$. To demonstrate, for any model and world b s.t. $b \vDash A$, also $b \vDash C/B$, it suffices by definition unfolding to show $a \vDash C$ on the additional assumptions $\otimes abc$ and $c \vDash B$. But this is easily deduced by backward chaining on (a). Conversely, suppose we know, for every model and world b, that $b \vDash A$ implies $b \vDash C/B$, i.e., (a') $b \vDash A$ implies $(\forall a, c)((\otimes abc \;\&\; c \vDash B) \Rightarrow a \vDash C)$. Now if for any model and world a, $a \vDash A \otimes B$, then in particular there exist b, c for which $\otimes abc$, $b \vDash A$ and $c \vDash B$. That also $a \vDash C$ is a matter of simply backward-chaining on (a'). □

The converse *completeness* result ensures that we do not miss out on any truths:

Theorem 2.2.2. If $a \vDash A$ implies $a \vDash B$ for all models and resources a, also $A \le B$.

Proof Idea. Using the standard proof method as found in [Dosen, 1992, Kurtonina, 1995], one first proves completeness for the *syntactic model*, defined by setting $W := \mathscr{F}(Atom)$, $v(p) := \{A \mid A \le p\}$ and $\otimes ABC$ iff $A \le B \otimes C$. One then proves the following *truth lemma* for arbitrary A, B, proceeding by induction on the former: $B \vDash A$ iff $B \le A$. Now, if, for every C s.t. $C \vDash A$, also $C \vDash B$, then in particular $A \vDash B$, since $A \vDash A$ iff $A \le A$, which holds by reflexivity. Hence, by the truth lemma, $A \le B$. The desired result follows from the observation that, if $a \vDash A$ implies $a \vDash B$ in every model, then in particular in the syntactic one. □

Further details, particularly the proof of the truth lemma, are found in the aforecited works. For now, we ask the following question: can we alternatively fulfill the role of the syntactic model in the above proof by a choice of a CFG G and valuation v on the frame F_G? We do not further pursue this topic here, restricting the use of frames F_G to that of an expository devise.

Example 6. F.2.3 lists some typical theorems and derived rules of inference. We demonstrate with monotonicity for /, and one of each of the Application, Co-Application and Lifting schemas:

$$\dfrac{\dfrac{\dfrac{\dfrac{C \le D \qquad \dfrac{\dfrac{\dfrac{}{A/D \le A/D}\,Id}{(A/D) \otimes D \le A}\,r}{D \le (A/D)\backslash A}\,r}{C \le (A/D)\backslash A}\,\circ}{(A/D) \otimes C \le A}\,r \qquad A \le B}{(A/D) \otimes C \le B}\,\circ}{A/D \le B/C}\,r$$

$$\dfrac{\dfrac{}{B\backslash A \le B\backslash A}\,Id}{B \otimes (B\backslash A) \le A}\,r$$

$$\dfrac{\dfrac{}{B \otimes A \le B \otimes A}\,Id}{A \le B\backslash(B \otimes A)}\,r$$

$$\dfrac{\dfrac{}{A \otimes (A\backslash B) \le B}\,App}{A \le B/(A\backslash B)}\,r$$

Application

$$A \otimes (A \backslash B) \leq B \qquad (A/B) \otimes B \leq A$$

Co-application

$$A \leq B \backslash (B \otimes A) \qquad A \leq (A \otimes B)/B$$

Lifting

$$A \leq (B/A) \backslash B \qquad A \leq B/(A \backslash B)$$

Monotonicity

$$\frac{A \leq B \quad C \leq D}{A \otimes C \leq B \otimes D} \otimes \qquad \frac{A \leq B \quad C \leq D}{D \backslash A \leq C \backslash B} \backslash \qquad \frac{A \leq B \quad C \leq D}{A/D \leq B/C} /$$

Figure 2.3: Some theorems and derived rules of inference for **NL**.

The application rules find immediate linguistic justification in E.5. E.g., one can show $\langle S, w \rangle \vDash s$ only if $\langle S, w \rangle \vDash \mathit{np} \otimes (\mathit{np} \backslash s)$, reflecting the fact that any concatenation of a noun phrase and a verb phrase yields a sentence. Similarly, $\langle \mathit{NP}, w \rangle \vDash \mathit{np}$ implies $\langle \mathit{NP}, w \rangle \vDash s/(\mathit{np} \backslash s)$ as well as $\langle \mathit{NP}, w \rangle \vDash (s/\mathit{np}) \backslash s$.

Remark 2. In practice, we often use the following consequences of the monotonicity laws, obtained by instantiating one of their premises by reflexivity.

$$\frac{A \leq B}{A \otimes C \leq B \otimes C} \otimes ? \qquad \frac{A \leq B}{C \otimes A \leq C \otimes B} ?\otimes$$

$$\frac{A \leq B}{A/C \leq B/C} /? \qquad \frac{A \leq B}{C/B \leq C/A} ?/ \qquad \frac{A \leq B}{B \backslash C \leq A \backslash C} \backslash ? \qquad \frac{A \leq B}{C \backslash A \leq C \backslash B} ?\backslash$$

Our motivation for discussing **NL** being linguistic, we now define the appropriate notion of grammar, illustrating with a final reference to the binoculars example.

Definition 9. An **NL** grammar G is a 4-tuple $\langle \Sigma, \mathit{Atom}, \mathit{Lex}, s \rangle$, consisting of a set of words Σ, a choice of atomic formulae *Atom* disjoint from Σ, a *lexicon* $\mathit{Lex} \subseteq \Sigma \times \mathscr{F}(\mathit{Atom})$ and a start symbol $s \in \mathit{Atom}$. The *language* $\mathscr{L}(G)$ is defined to be the set of strings $w_1 \dots w_n \in \Sigma^+$, s.t. for some $A_1, \dots, A_n$ with $\langle w_1, A_1 \rangle, \dots, \langle w_n, A_n \rangle \in$ *Lex* where $C \leq s$ for some bracketing C of $A_1 \otimes \dots \otimes A_n$.

Recalling F.1, CFG's rely primarily on non-logical axioms, i.e., productions, as necessitated by the lack of derived categories. In contrast, **NL** assumes a rich logical vocabulary, concentrating all language dependent information in the choice of atoms and the lexicon, while leaving derivability to the 'universal' rules of logic.

Example 7. As an example of an **NL** grammar, we consider yet again the binoculars example. Thus, identify Σ with the choice of vocabulary for the CFG of E.1, take *Atom* = $\{s, np, n\}$, let s be the start symbol, and define the lexicon by the following table (cf. E.5), including the original categories for reference purposes:

Word	Category	Formula
John	*NP*	np
the	*Det*	np/n
man, binoculars	*N*	n
saw	*TV*	$(np\backslash s)/np$
with	*Prep*	$(n\backslash n)/np$, $((np\backslash s)\backslash(np\backslash s))/np$

In particular, both $\langle \text{with}, (n\backslash n)/np \rangle, \langle \text{with}, ((np\backslash s)\backslash(np\backslash s))/np \rangle \in$ *Lex*, covering both $N \Rightarrow^* N$ *Prep NP* and *VP* $\Rightarrow^*$ *VP Prep NP* respectively. F.2.4 provides the derivations witnessing the grammaticality of *John saw the man with the binoculars*, using the formula $(n\backslash n)/np$ for *when* in case it is the man with the binoculars that is being seen by John, and $((np\backslash s)\backslash(np\backslash s))/n$ for when it is through the binoculars that the man is being seen by John.

While the current treatment of CFG's served only as an expository device to motivate the linguistic applicability of **NL**, chapter 5 goes more in-depth in proving context-freeness of grammars based on the *Lambek-Grishin* calculus (**LG**); a conservative extension of **NL** discussed in the next chapter. Analogous expressivity results for **NL** had already been the topic of several research papers, cf. [Buszkowski, 1988, Kandulski, 1988], among others.[2] So as not to give the reader the wrong idea, however, we note that the main attraction of **LG** is not to be found in it being yet another context-free formalism. Rather, its extended logical vocabulary suggests a natural class of structural rules that may be used to push its expressivity beyond the context-free boundaries.

[2]Research on **NL** aside, the real problem within the research community was long considered to be the conjectured context-freeness of the *associative* Lambek calculus, which Pentus [1999] ultimately settled in the affirmative.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$np \backslash s \leq np \backslash s$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$np \leq np$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$n \backslash n \leq n \backslash n$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$np/n \leq np/n$}
\RightLabel{r}
\UnaryInfC{$(np/n) \otimes n \leq np$}
\RightLabel{$/$}
\BinaryInfC{$(n \backslash n)/np \leq (n \backslash n)/((np/n) \otimes n)$}
\RightLabel{r}
\UnaryInfC{$((n \backslash n)/np) \otimes ((np/n) \otimes n) \leq n \backslash n$}
\RightLabel{r}
\UnaryInfC{$n \otimes (((n \backslash n)/np) \otimes ((np/n) \otimes n)) \leq n$}
\RightLabel{$/$}
\BinaryInfC{$np/n \leq np/(n \otimes (((n \backslash n)/np) \otimes ((np/n) \otimes n)))$}
\RightLabel{r}
\UnaryInfC{$(np/n) \otimes (n \otimes (((n \backslash n)/np) \otimes ((np/n) \otimes n))) \leq np$}
\RightLabel{$/$}
\BinaryInfC{$(np \backslash s)/np \leq (np \backslash s)/((np/n) \otimes (n \otimes (((n \backslash n)/np) \otimes ((np/n) \otimes n))))$}
\RightLabel{r}
\UnaryInfC{$((np \backslash s)/np) \otimes ((np/n) \otimes (n \otimes (((n \backslash n)/np) \otimes ((np/n) \otimes n)))) \leq np \backslash s$}
\RightLabel{r}
\UnaryInfC{$np \otimes (((np \backslash s)/np) \otimes ((np/n) \otimes (n \otimes (((n \backslash n)/np) \otimes ((np/n) \otimes n))))) \leq s$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$(np \backslash s) \backslash (np \backslash s) \leq (np \backslash s) \backslash (np \backslash s)$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$np/n \leq np/n$}
\RightLabel{r}
\UnaryInfC{$(np/n) \otimes n \leq np$}
\RightLabel{$/$}
\BinaryInfC{$((np \backslash s) \backslash (np \backslash s))/np \leq ((np \backslash s) \backslash (np \backslash s))/((np/n) \otimes n)$}
\RightLabel{r}
\UnaryInfC{$(((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n) \leq (np \backslash s) \backslash (np \backslash s)$}
\RightLabel{r}
\UnaryInfC{$(np \backslash s) \otimes ((((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n)) \leq np \backslash s$}
\RightLabel{r}
\UnaryInfC{$np \backslash s \leq (np \backslash s)/((((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n))$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$np/n \leq np/n$}
\RightLabel{r}
\UnaryInfC{$(np/n) \otimes n \leq np$}
\RightLabel{$/$}
\BinaryInfC{$(np \backslash s)/np \leq ((np \backslash s)/((((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n)))/((np/n) \otimes n)$}
\RightLabel{r}
\UnaryInfC{$((np \backslash s)/np) \otimes ((np/n) \otimes n) \leq (np \backslash s)/((((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n))$}
\RightLabel{r}
\UnaryInfC{$(((np \backslash s)/np) \otimes ((np/n) \otimes n)) \otimes ((((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n)) \leq np \backslash s$}
\RightLabel{r}
\UnaryInfC{$np \otimes ((((np \backslash s)/np) \otimes ((np/n) \otimes n)) \otimes ((((np \backslash s) \backslash (np \backslash s))/np) \otimes ((np/n) \otimes n))) \leq s$}
\end{prooftree}
$$

Figure 2.4: Two derivations for *John saw the man with the binoculars*, using the **NL** grammar of E.7 and the derived rules of F.2.3.

2.3 Natural deduction and sequent calculus

The previous section provided a means for reasoning about inequalities $A \leq B$, shown sound and complete w.r.t. relational models. We continue to discuss two alternative formalisms for describing proofs, closer to our intuitive understanding of logic. Rather than defining a preorder on formulas in order to describe such algebraic concepts as residuation, we shall describe provability as relating a possible multitude of hypotheses with a unique conclusion. The former are structured into binary-branching tree structures, as made explicit in the following definition.

Definition 10. *Structures* Γ, Δ compose formulas using binary *merger* $\bullet$:

$$\Gamma, \Delta \quad ::= \quad A \mid (\Gamma \bullet \Delta)$$

A structure Γ interprets as a formula $\Gamma^\bullet$ by replacing occurrences of $\bullet$ with $\otimes$. I.e., $A^\bullet = A$ and $(\Gamma \bullet \Delta)^\bullet = \Gamma^\bullet \otimes \Delta^\bullet$.

We shall occasionally want to refer to structures $\Gamma[\Delta]$ containing a distinguished occurrence of a substructure Δ. The following definition provides the technical means necessary for doing so.

Definition 11. A *context* $\Gamma[]$ is a structure featuring a unique occurrence of a *hole* $[]$, serving as the target location for insertion of any other structure.

$$\Gamma[], \Delta[] \quad ::= \quad [] \mid (\Gamma[] \bullet \Delta) \mid (\Gamma \bullet \Delta[])$$

Given $\Gamma[]$ and Δ, we write $\Gamma[\Delta]$ for the replacement of $[]$ with Δ in $\Gamma[]$, and we say Δ is a *substructure* of $\Gamma[\Delta]$.

The following definition defines two closely related judgement forms relating a structure Γ of hypotheses to a single conclusion A.

Definition 12. Figures 2.5 and 2.6 define the judgement forms $\Gamma \vdash A$ and $\Gamma \Rightarrow A$, both expressing the derivability of the *conclusion* A from the structural configuration of *hypotheses* Γ. The former's axiomatization is referred to by *natural deduction*, while the latter defines a *sequent calculus*.

Shared between natural deduction and sequent calculus are *initial sequents* (or *axioms*) $A \vdash A$ and $A \Rightarrow A$ (*Id*), expressing reflexivity of deducibility. On the other hand, transitivity ($\circ$), more commonly referred to in the current context as the *Cut rule*, is listed independently only in sequent calculus. Finally, both formalisms explain each connective by two inference rules, related through symmetry; *proof* and *use* in the case of natural deduction, and left- and right *introductions* in the case of sequent calculus. Further elaborated,

1. Natural deduction has *introduction-* and *elimination rules* (δI) and (δE) for each connective δ. The former 'defines' δ by specifying the conditions under which its occurrence as (the principal connective in) a conclusion may be *inferred*, i.e., when it may be *proven*. The elimination rule, on the other hand, has a formula $(A \,\delta\, B)$ appearing as a conclusion in one of the premises, showing how it may be *used* in an inference once derived.

$$\frac{}{A \vdash A}\ \mathit{Id}$$

$$\frac{\Gamma \vdash A/B \quad \Delta \vdash B}{\Gamma \bullet \Delta \vdash A}\ /E \qquad \frac{\Gamma \bullet B \vdash A}{\Gamma \vdash A/B}\ /I$$

$$\frac{\Delta \vdash B \quad \Gamma \vdash B\backslash A}{\Delta \bullet \Gamma \vdash A}\ \backslash E \qquad \frac{B \bullet \Gamma \vdash A}{\Gamma \vdash B\backslash A}\ \backslash I$$

$$\frac{\Delta \vdash A \otimes B \quad \Gamma[A \bullet B] \vdash C}{\Gamma[\Delta] \vdash C}\ \otimes E \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \bullet \Delta \vdash A \otimes B}\ \otimes I$$

Figure 2.5: Natural deduction presentation of **NL**.

$$\frac{}{A \Rightarrow A}\ \mathit{Id} \qquad \frac{\Delta \vdash B \quad \Gamma[B] \vdash A}{\Gamma[\Delta] \vdash A}\ \circ$$

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[A/B \bullet \Delta] \Rightarrow C}\ /L \qquad \frac{\Gamma \bullet B \Rightarrow A}{\Gamma \Rightarrow A/B}\ /R$$

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[\Delta \bullet B\backslash A] \Rightarrow C}\ \backslash L \qquad \frac{B \bullet \Gamma \Rightarrow A}{\Gamma \Rightarrow B\backslash A}\ \backslash R$$

$$\frac{\Gamma[A \bullet B] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C}\ \otimes L \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \bullet \Delta \Rightarrow A \otimes B}\ \otimes R$$

Figure 2.6: Sequent calculus presentation of **NL**.

2. Sequent calculus shares natural deduction's introduction rules (δI), but names them *right* introductions (δR) in light of the fact that they operate on the right-hand side of the derivability sign. In addition, *left* introductions (δL) are postulated, showing how to introduce a connective as a hypothesis.

We have informally motivated the pairs of rules associated with a connective by claiming them to be symmetric; be it 'proof vs. use' or 'left- vs. right introductions'. A precise characterization of when any given such pair adequately reflects said symmetries, however, is left wanted. We return to this issue below, first stating equivalence in provability for each of the three formalisms considered thus far.

$$\frac{\Gamma \vdash A/B \quad \Delta \vdash B}{\Gamma \bullet \Delta \vdash A}\,/E \quad \rightarrow \quad \frac{\dfrac{}{\Gamma \Rightarrow A/B}\,IH \quad \dfrac{\dfrac{}{\Delta \Rightarrow B}\,IH \quad \dfrac{}{A \Rightarrow A}\,Id}{A/B \bullet \Delta \Rightarrow A}\,/L}{\Gamma \bullet \Delta \Rightarrow A}\,\circ$$

$$\frac{\Delta \vdash B \quad \Gamma \vdash B\backslash A}{\Delta \bullet \Gamma \vdash A}\,\backslash E \quad \rightarrow \quad \frac{\dfrac{}{\Gamma \Rightarrow B\backslash A}\,IH \quad \dfrac{\dfrac{}{\Delta \Rightarrow B}\,IH \quad \dfrac{}{A \Rightarrow A}\,Id}{\Delta \bullet B\backslash A \vdash A}\,\backslash L}{\Delta \bullet \Gamma \Rightarrow A}\,\circ$$

$$\frac{\Gamma \vdash A \otimes B \quad \Delta[A \bullet B] \vdash C}{\Gamma[\Delta] \vdash C}\,\otimes E \quad \rightarrow \quad \frac{\dfrac{}{\Gamma \Rightarrow A \otimes B}\,IH \quad \dfrac{\dfrac{}{\Gamma[A \bullet B] \Rightarrow C}\,IH}{\Gamma[A \otimes B] \Rightarrow C}\,\otimes L}{\Gamma[\Delta] \Rightarrow C}\,\circ$$

Figure 2.7: Translating natural deduction derivations into sequent calculus.

Theorem 2.3.1. If $\Gamma \vdash A$, then $\Gamma \Rightarrow A$.

Proof. By induction on the derivation witnessing $\Gamma \vdash A$. Both the base case and the inductive cases corresponding to the introduction rules are immediate, so we need only check the elimination rules, as done in F.2.7. In general, the latter are derived by applying a Cut on the corresponding left introduction, possibly instantiating one of the latter's premises by an axiom. □

For the converse direction, we first prove natural deduction is closed under Cut.

Lemma 1. The following is an admissible rule in natural deduction

$$\frac{\Delta \vdash B \quad \Gamma[B] \vdash A}{\Gamma[\Delta] \vdash A}\,\circ$$

Proof. By showing Cut can be permuted upwards through the derivation of its second premise, finally disappearing upon the encounter of the axiom instance responsible for the occurrence of B singled out in $\Gamma[B]$. More formally, we proceed by induction on the derivation witnessing $\Gamma[B] \vdash A$. In the base case, $A = B$ and $\Gamma[] = []$, and we simply drop the resulting axiom, keeping only the derivation of $\Delta \vdash B$. For the inductive cases, we content ourselves by checking $(/E)$, the remaining rules receiving similar treatment. Thus, $\Gamma[B] = \Gamma_1 \bullet \Gamma_2$ with $\Gamma_1 \vdash A/C$ and $\Gamma_2 \vdash C$. Either B occurs in Γ_1 or in Γ_2. The two options don't make much of a difference for the continuation of our argument, so suppose that B is in Γ_1,

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[A/B \bullet \Delta] \Rightarrow C}\ /L \quad \rightarrow \quad \dfrac{\dfrac{\overline{A/B \vdash A/B}\ Id \quad \overline{\Delta \vdash B}\ IH}{A/B \bullet \Delta \vdash A}\ /E \quad \overline{\Gamma[A] \vdash C}\ IH}{\Gamma[A/B \bullet \Delta] \vdash C}\ \circ$$

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[\Delta \bullet B\backslash A] \Rightarrow C}\ \backslash L \quad \rightarrow \quad \dfrac{\dfrac{\overline{\Delta \vdash B}\ IH \quad \overline{B\backslash A \vdash B\backslash A}\ Id}{\Delta \bullet B\backslash A \vdash A}\ \backslash E \quad \overline{\Gamma[A] \vdash C}\ IH}{\Gamma[\Delta \bullet B\backslash A] \vdash C}\ \circ$$

$$\frac{\Gamma[A \bullet B] \Rightarrow C}{\Gamma[A \otimes B] \Rightarrow C}\ \otimes L \quad \rightarrow \quad \dfrac{\overline{A \otimes B \vdash A \otimes B}\ Id \quad \overline{\Gamma[A \bullet B] \vdash C}\ IH}{\Gamma[A \otimes B] \vdash C}\ \otimes E$$

Figure 2.8: Translating sequent calculus derivations into natural deduction.

i.e., $\Gamma_1 = \Gamma_1'[B]$. We then proceed as follows: on the left hand we find the original derivation, while the right hand shows the result of permuting the Cut upwards:

$$\dfrac{\Delta \vdash B \quad \dfrac{\Gamma_1'[B] \vdash A/C \quad \Gamma_2 \vdash C}{\Gamma_1'[B] \bullet \Gamma_2 \vdash A}\ /E}{\Gamma_1'[\Delta] \bullet \Gamma_2 \vdash A}\ \circ \qquad \dfrac{\dfrac{\Delta \vdash B \quad \Gamma_1'[B] \vdash A/C}{\Gamma_1'[\Delta] \vdash A/C}\ \circ \quad \Gamma_2 \vdash C}{\Gamma_1'[\Delta] \bullet \Gamma_2 \vdash A}\ /E$$

Note that, had B occurred in Γ_2 instead, then the Cut instance would have permuted up into the second premise of $(/E)$. □

Theorem 2.3.2. If $\Gamma \Rightarrow A$, then $\Gamma \vdash A$.

Proof. Proceeding by induction as usual, we note both the base case as well as the right introductions are immediate, while $(\circ)$ coincides with the previous lemma. Thus, we are left to check derivability of the left introductions. Again, we find that in each case we need merely apply Cut together with an elimination rule, instantiating one of the latter's premises by an axiom. In the case of $(\otimes L)$, the corresponding instance of Cut has an axiom in its right premise, so can be omitted altogether. The details are carried out in F.2.8. □

We continue to compare sequent calculus and algebraic derivability.

Theorem 2.3.3. If $\Gamma \Rightarrow A$, then $\Gamma^\bullet \leq A$.

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[A/B \bullet \Delta] \Rightarrow C} /L \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Delta^\bullet \leq B}\ IH}{A/B \leq A/\Delta^\bullet}\ ?/}{(A/B) \otimes \Delta^\bullet \leq A}\ r}{\Gamma[A/B \bullet \Delta]^\bullet \leq \Gamma[A]^\bullet}\ \Gamma[] \quad \overline{\Gamma[A]^\bullet \leq C}\ IH}{\Gamma[A/B \bullet \Delta]^\bullet \leq C} \circ$$

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[\Delta \bullet B\backslash A] \Rightarrow C} \backslash L \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Delta^\bullet \leq B}\ IH}{B\backslash A \leq \Delta^\bullet\backslash A}\ \backslash ?}{\Delta^\bullet \otimes (B\backslash A) \leq A}\ r}{\Gamma[\Delta \bullet B\backslash A]^\bullet \leq \Gamma[A]^\bullet}\ \Gamma[] \quad \overline{\Gamma[A]^\bullet \leq C}\ IH}{\Gamma[\Delta \bullet B\backslash A]^\bullet \leq C} \circ$$

Figure 2.9: Explaining sequent calculus in terms of algebraic derivability.

Proof. By induction on the derivation witnessing $\Gamma \Rightarrow A$. The base case is again immediate. As a lemma, we shall use the following derived rule

$$\frac{\Delta^\bullet \leq B}{\Gamma[\Delta]^\bullet \leq \Gamma[B]^\bullet} \Gamma[]$$

being easily demonstrable by induction on $\Gamma[\,]$, using monotonicity of $\otimes$ from F.2.3 in the inductive cases. The right introductions $(/R)$ and $(\backslash R)$ follow immediately from residuation, or, in the case of $(\otimes R)$, from monotonicity. For the left introductions, we check $(/L)$ and $(\backslash L)$ in F.2.9, with $(\otimes L)$ again being immediate. □

We conclude with the converse direction, the relation between natural deduction and algebraic derivability then being obtainable through composition.

Theorem 2.3.4. If $\Gamma^\bullet \leq A$, then $\Gamma \Rightarrow A$.

Proof. First, we show by induction that $A \leq B$ implies $A \Rightarrow B$. Hence, in particular $\Gamma^\bullet \Rightarrow A$ if $\Gamma^\bullet \leq A$. Successive applications of $(\otimes L)$ then suffice to derive the desired $\Gamma \Rightarrow A$. The cases (Id) and $(\circ)$ are immediate, so we are left with checking the residuation laws. F.2.10 checks two cases, the others being similar. □

Having ensured the equivalence (provability-wise) of the various presentations of **NL** surveyed thus far, we return to our discussion of the symmetries underlying both

$$\frac{A \leq C/B}{A \otimes B \leq C} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{}{A \Rightarrow C/B}\ IH \quad \dfrac{\dfrac{}{B \Rightarrow B}\ Id \quad \dfrac{}{C \Rightarrow C}\ Id}{C/B \bullet B \Rightarrow C}\ /L}{A \bullet B \Rightarrow C}\ \circ}{A \otimes B \Rightarrow C}\ \otimes L$$

$$\frac{A \otimes B \leq C}{A \leq C/B} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ Id \quad \dfrac{}{B \Rightarrow B}\ Id}{A \bullet B \Rightarrow A \otimes B}\ \otimes R \quad \dfrac{}{A \otimes B \Rightarrow C}\ IH}{A \bullet B \Rightarrow C}\ \circ}{A \Rightarrow C/B}\ /R$$

Figure 2.10: Translating algebraic derivability into sequent calculus.

the sequent calculus and natural deduction formalisms. In either case, we consider a rule pair for any connective *harmonious* if, within certain types of local configurations, its two members cancel each other out. Furthermore, the removal of each such 'bottleneck' within a given derivation is to terminate, with the result satisfying the following *analyticity* property: any provable judgement $\Gamma \vdash A$ ($\Gamma \Rightarrow A$) has a derivation containing only subformulas of A and of (the formulas found in) Γ.

On the metamathematical level, the above property has been explained by Girard [1999, annex B.2] as constituting a completeness property, not w.r.t. some notion of model, but rather in the sense that 'nothing is missing' from the rule set: in a proof of $\Gamma \vdash A$ or $\Gamma \Rightarrow A$, only the rule pairs for the subformulas found in Γ and A need be used. The consequences of this observation are not limited to the area of metamathematics, as it suggests in particular the viability of backward-chaining proof-search, i.e., starting from the desired conclusion and reasoning back towards the axioms. By this method, we can even show that provability in **NL** is decidable.

The concept of harmony is most easily understood within sequent calculus. Save for Cut, each rule satisfies a *local* subformula property, their premises containing only subformulas of the rule's conclusion. In other words, a 'bottleneck' consists of the two rules associated with any given connective being placed inside a Cut, e.g.,

$$\dfrac{\dfrac{\Delta \bullet B \Rightarrow A}{\Delta \Rightarrow A/B}\ /R \quad \dfrac{\Gamma' \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[A/B \bullet \Gamma'] \Rightarrow C}\ /L}{\Gamma[\Delta \bullet \Gamma'] \Rightarrow C}\ \circ$$

Cancelation proceeds via induction, the relevant measure of which we can reasonably assume to at least take into account the size of the sole active formula of the Cut, here A/B, henceforth referred to as the *Cut formula*. As such, we proceed by replacing with Cuts on the immediate subformulas. In the base case, we must ensure that Cuts at least one whose premises is an axiom can be removed. E.g., in the following case, we can simply keep only the derivation for the second premise:

$$\dfrac{\dfrac{}{A \Rightarrow A}\,\mathit{Id} \quad \Delta[A] \Rightarrow C}{\Gamma[A] \Rightarrow C}\,\circ$$

We make two observations w.r.t. the procedure just proposed.

1. Having laid claim on the length of the Cut formula as a necessary ingredient to our induction measure, we would expect the base case to concern atomic formulas, rather than dealing with arbitrary axioms (if only for aesthetic reasons). The solution is to restrict to the use of *atomic* axioms only. In other words, we can show that the claim $A \Rightarrow A$ for arbitrary A is admissible in the presence of axioms $p \Rightarrow p$ for atomic p only.

2. The survey of Cut instantiations found above is not exhaustive. To guarantee that the process of removing Cut terminates, we also have to consider cases where the Cut formula is not main in at least one of the premises, e.g.,

$$\dfrac{\dfrac{\Delta' \Rightarrow B \quad \Delta[A] \Rightarrow C}{\Delta[A/B \bullet \Delta'] \Rightarrow C}\,/L \quad \Gamma[C] \Rightarrow D}{\Gamma[\Delta[A/B \bullet \Delta']] \Rightarrow C}\,\circ$$

The solution here is to show that $(\circ)$ can be permuted over $(/L)$, headed either towards the rule deriving the Cut formula if non-atomic (thus appearing main again), or towards an axiom otherwise, where the base case applies. Note that, as a consequence, we cannot take the induction measure to consist of the length of the Cut formula only. In particular, even when restricting to atomic axioms, we can easily conceive of a situation where the Cut formula is atomic, yet neither premise is instantiated by an axiom, e.g.,

$$\dfrac{\dfrac{\dfrac{}{n \Rightarrow n}\,\mathit{Id} \quad \dfrac{}{np \Rightarrow np}\,\mathit{Id}}{np/n \bullet n \Rightarrow np}\,/L \quad \dfrac{\dfrac{}{np \Rightarrow np}\,\mathit{Id} \quad \dfrac{}{s \Rightarrow s}\,\mathit{Id}}{np \bullet np\backslash s \Rightarrow s}\,/L}{(np/n \bullet n) \bullet np\backslash s \Rightarrow s}\,\circ$$

The solution is to also take into account the lengths of the remaining, non-active formulas found inside an instantiation of Cut.

In contrast with the second observation, the first does not constitute a necessary ingredient to showing completeness of Cut-free derivations, although it does help clarify the underlying rationale. To this end, we first show

Lemma 2. $A \Rightarrow A$ (A arb.) is admissible in the presence of atomic axioms:

$$\frac{}{p \Rightarrow p}\ \mathit{Id}$$

Proof. By induction on A. The base case, where $A = p$, is immediate. The inductive cases are shown as follows:

$$\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ \mathit{IH} \quad \dfrac{}{A \Rightarrow A}\ \mathit{IH}}{A/B \bullet B \Rightarrow A}\ /L}{A/B \Rightarrow A/B}\ /R \qquad \dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ \mathit{IH} \quad \dfrac{}{A \Rightarrow A}\ \mathit{IH}}{B \bullet B\backslash A \Rightarrow A}\ \backslash L}{B\backslash A \Rightarrow B\backslash A}\ \backslash R \qquad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ \mathit{IH} \quad \dfrac{}{B \Rightarrow B}\ \mathit{IH}}{A \bullet B \Rightarrow A \otimes B}\ \otimes R}{A \otimes B \Rightarrow A \otimes B}\ \otimes L$$

□

Theorem 2.3.5. The Cut rule is eliminable: any derivation can be transformed so as to make no use of Cut, preserving the conclusion.

Proof. Note it suffices to show that Cut is *admissible*. I.e., given Cut-free derivations of $\Delta \Rightarrow A$ and $\Gamma[A] \Rightarrow B$, we have a Cut-free derivation of $\Gamma[\Delta] \Rightarrow B$. Given a derivation using an arbitrary number of Cuts, we transform it by removing Cuts bottom-up using the above result. Before proceeding, first assume the Cut's premises only contain atomic axioms, according to the previous lemma. We now proceed by induction on the lengths of the formula occurrences found in $\Gamma[\,]$, Δ, A and B combined (measured by the number of connectives). As already motivated above, we have three cases to consider.

1. In the base case, at least one of the premises is an axiom:

$$\dfrac{\dfrac{}{p \Rightarrow p}\ \mathit{Id} \quad \Gamma[p] \Rightarrow C}{\Gamma[p] \Rightarrow C}\ \circ \qquad \dfrac{\Delta \Rightarrow p \quad \dfrac{}{p \Rightarrow p}\ \mathit{Id}}{\Delta \Rightarrow p}\ \circ$$

   Cut is removed entirely by keeping only $\Gamma[p] \Rightarrow C$ and $\Delta \Rightarrow p$ respectively.

2. The Cut formula is main in both premises; we speak of *principal* Cuts. As an example, we return to the case ($/$) already discussed above:

$$\dfrac{\dfrac{\Delta \bullet B \Rightarrow A}{\Delta \Rightarrow A/B}\ /R \quad \dfrac{\Gamma' \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[A/B \bullet \Gamma'] \Rightarrow C}\ /L}{\Gamma[\Delta \bullet \Gamma'] \Rightarrow C}\ \circ$$

   We replace by Cuts on the immediate subformulas A and B:

$$\dfrac{\dfrac{\Gamma' \Rightarrow B \quad \Delta \bullet B \Rightarrow A}{\Delta \bullet \Gamma' \Rightarrow A}\circ \quad \Gamma[A] \Rightarrow C}{\Gamma[\Delta \bullet \Gamma'] \Rightarrow C}\circ$$

More accurately, by the induction hypothesis, we first construct a Cut-free derivation of $\Gamma[\Delta \bullet B] \Rightarrow C$, applying the induction hypothesis once more to conclude the desired $\Gamma[\Delta \bullet \Gamma'] \Rightarrow C$.

3. The Cut formula is not principal in at least one of the premises. We show that the Cut instance can be permuted upwards, speaking of a *commutative* Cut. In case the Cut formula is not main in the left premise, the latter's derivation must end in a left introduction. E.g.,

$$\dfrac{\dfrac{\Delta' \Rightarrow B \quad \Delta[A] \Rightarrow C}{\Delta[A/B \bullet \Delta'] \Rightarrow C}/L \quad \Gamma[C] \Rightarrow D}{\Gamma[\Delta[A/B \bullet \Delta']] \Rightarrow C}\circ$$

in which case we permute $(\circ)$ with $(/L)$:

$$\dfrac{\Delta' \Rightarrow B \quad \dfrac{\Delta[A] \Rightarrow C \quad \Gamma[C] \Rightarrow D}{\Gamma[\Delta[A]] \Rightarrow C}\circ}{\Gamma[\Delta[A/B \bullet \Delta']] \Rightarrow C}/L$$

In case the Cut formula is not principal in the right premise, the corresponding derivation may end in both a left introduction or a right one. As an example of the latter case, we have the following permutation

$$\dfrac{\Delta \Rightarrow C \quad \dfrac{\Gamma[C] \bullet B \Rightarrow A}{\Gamma[C] \Rightarrow A/B}/R}{\Gamma[\Delta] \Rightarrow A/B}\circ \quad \mapsto \quad \dfrac{\dfrac{\Delta \Rightarrow C \quad \Gamma[C] \bullet B \Rightarrow A}{\Gamma[\Delta] \bullet B \Rightarrow A}\circ}{\Gamma[\Delta] \Rightarrow A/B}/R$$

As a final example, consider a case where the right premise was derived using a left introduction, say $(/L)$:

$$\dfrac{\Delta \Rightarrow C \quad \dfrac{\Gamma'[C] \Rightarrow B \quad \Gamma[A] \Rightarrow D}{\Gamma[A/B \bullet \Gamma'[C]] \Rightarrow D}/L}{\Gamma[A/B \bullet \Gamma'[\Delta]] \Rightarrow D}\circ \quad \mapsto \quad \dfrac{\dfrac{\Delta \Rightarrow C \quad \Gamma'[C] \Rightarrow B}{\Gamma'[\Delta] \Rightarrow B}\circ \quad \Gamma[A] \Rightarrow D}{\Gamma[A/B \bullet \Gamma'[\Delta]] \Rightarrow D}/L$$

Note that the current schema is not exhaustive: C might as well have appeared elsewhere within $\Gamma[]$, in which case $(\circ)$ would have paired up with $(/L)$'s second premise instead. $\square$

Remark 3. While, by virtue of linearity, the procedure described above is easily seen to terminate, it is not deterministic. In particular, consider the case where the Cut formula is principal in neither the left nor the right premise:

$$\dfrac{\dfrac{\Delta[A \bullet B] \Rightarrow C}{\Delta[A \otimes B] \Rightarrow C}\ \otimes L \qquad \dfrac{\Gamma[C] \bullet E \Rightarrow D}{\Gamma[C] \Rightarrow D/E}\ /R}{\Gamma[\Delta[A \otimes B]] \Rightarrow D/E}\ \circ$$

Depending on whether we first permute $(\circ)$ over $(\otimes L)$ or over $(/R)$, we end up with different results:

$$\dfrac{\dfrac{\dfrac{\Delta[A \bullet B] \Rightarrow C \quad \Gamma[C] \bullet E \Rightarrow D}{\Gamma[\Delta[A \bullet B]] \bullet E \Rightarrow D}\ \circ}{\Gamma[\Delta[A \bullet B]] \Rightarrow D/E}\ /R}{\Gamma[\Delta[A \otimes B]] \Rightarrow D/E}\ \otimes L \qquad \dfrac{\dfrac{\dfrac{\Delta[A \bullet B] \Rightarrow C \quad \Gamma[C] \bullet E \Rightarrow D}{\Gamma[\Delta[A \bullet B]] \bullet E \Rightarrow D}\ \circ}{\Gamma[\Delta[A \otimes B]] \bullet E \Rightarrow D}\ \otimes L}{\Gamma[\Delta[A \otimes B]] \Rightarrow D/E}\ /R$$

Clearly, even after taking further steps towards removing the Cut there is no possibility of the two derivations converging. We will return to the problem of adapting sequent calculus so as to restore determinism of Cut elimination in chapter 7. For now, we refer the reader to Hepple [1990], Hendriks [1993] for solutions within associative Lambek calculus.

As already noted, sequent derivations without Cut satisfy a local subformula property: the formulas found within the premises of any rule application are subformulas of the rule's conclusion. In particular, we obtain a straightforward procedure for checking provability of any given claim $\Gamma \Rightarrow A$ through backward reasoning. By the completeness of Cut-free derivations, we obtain as a corollary the decidability of sequent provability in **NL**. By composing with Theorems 2.3.2 and 2.3.3, this result immediately extends to the other formalisms.

Theorem 2.3.6. Provability in **NL** is decidable.

We continue our discussion of harmony by returning to natural deduction. First, we note the translation found in F.2.8, used to prove T.2.3.2, is such that any derivation $\mathscr{D}$ in sequent calculus is taken to one in natural deduction using only formulas found in $\mathscr{D}$. In particular, when proceeding from a Cut-free source derivation, the subformula property is preserved in the target, in the usual global sense (i.e., not necessarily at the level of individual rule applications). Since analyticity served as the primary motivation for harmony, the latter's manifestation inside natural deduction can now be defined from its counterpart in sequent calculus. In carrying out this program, we keep our discussion informal, omitting any formal proofs of the subformula property and (strong) normalization. For further details, we refer the reader to [Prawitz, 2006] and [Girard et al., 1989, Ch.10].

Inside sequent calculus, a rule pair for a given connective was deemed harmonious if the corresponding principal Cut could be eliminated. For example,

$$\dfrac{\dfrac{\Delta \bullet B \Rightarrow A}{\Delta \Rightarrow A/B}\,/R \qquad \dfrac{\Gamma' \Rightarrow B \quad \Gamma[A] \Rightarrow C}{\Gamma[A/B \bullet \Gamma'] \Rightarrow C}\,/L}{\Gamma[\Delta \bullet \Gamma'] \Rightarrow C}\,\circ \quad \mapsto \quad \dfrac{\dfrac{\Gamma' \Rightarrow B \quad \Delta \bullet B \Rightarrow A}{\Delta \bullet \Gamma' \Rightarrow A}\,\circ \qquad \Gamma[A] \Rightarrow C}{\Gamma[\Delta \bullet \Gamma'] \Rightarrow C}\,\circ$$

Call the left-hand side the *redex* and the right-hand side the *contractum*, adapting terminology from term rewriting [see Baader and Nipkow, 1998]. Translated to natural deduction, the redex becomes

$$\dfrac{\dfrac{\Delta \bullet B \vdash A}{\Delta \vdash A/B}\,/I \qquad \dfrac{\dfrac{\dfrac{}{A/B \vdash A/B}\,Id \quad \Gamma' \vdash B}{A/B \bullet \Gamma' \vdash A}\,/E \qquad \Gamma[A] \vdash C}{\Gamma[A/B \bullet \Gamma'] \vdash C}\,\circ}{\Gamma[\Delta \bullet \Gamma'] \vdash C}\,\circ$$

Applying L.1, this reduces to the derivation on the lower left, while the contractum (lower right) remains unchanged.

$$\dfrac{\dfrac{\dfrac{\Delta \bullet B \vdash A}{\Delta \vdash A/B}\,/I \quad \Gamma' \vdash B}{\Delta \bullet \Gamma' \vdash A}\,/E \qquad \Gamma[A] \vdash C}{\Gamma[\Delta \bullet \Gamma'] \vdash C}\,\circ \qquad \dfrac{\dfrac{\Gamma' \vdash B \quad \Delta \bullet B \vdash A}{\Delta \bullet \Gamma' \vdash A}\,\circ \qquad \Gamma[A] \vdash C}{\Gamma[\Delta \bullet \Gamma'] \vdash C}\,\circ$$

Note the premise $\Gamma[A] \vdash C$ plays no role here, having been left dangling at the bottom of both derivations. Removing it, we identify in F.2.11 the relevant 'bottleneck' configurations inside natural deduction as the immediate use (elimination) of a proof (introduction) of δ.

We have yet to consider the natural deduction counterparts of permutative Cuts. Within the implicational fragment, the latter's elimination in sequent calculus, as outlined in the proof of T.2.3.5, follows exactly the Cut admissibility proof for natural deduction (Th.1), and hence can be ignored. Things get more complicated, however,

$$\dfrac{\dfrac{\Gamma \bullet B \vdash A}{\Gamma \vdash A/B}\,/I \quad \Delta \vdash B}{\Gamma \bullet \Delta \vdash A}\,/E \quad \mapsto \quad \dfrac{\Delta \vdash B \quad \Gamma \bullet B \vdash A}{\Gamma \bullet \Delta \vdash A}\,\circ$$

$$\dfrac{\Delta \vdash B \quad \dfrac{B \bullet \Gamma \vdash A}{\Gamma \vdash B\backslash A}\,\backslash I}{\Delta \bullet \Gamma \vdash A}\,\backslash E \quad \mapsto \quad \dfrac{\Delta \vdash B \quad B \bullet \Gamma \vdash A}{\Delta \bullet \Gamma \vdash A}\,\circ$$

$$\dfrac{\dfrac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta \bullet \Delta' \vdash A \otimes B}\,\otimes I \quad \Gamma[A \bullet B] \vdash C}{\Gamma[\Delta \bullet \Delta'] \vdash C}\,\otimes E \ \mapsto \ \dfrac{\Delta \vdash A \quad \dfrac{\Delta' \vdash B \quad \Gamma[A \bullet B] \vdash C}{\Gamma[A \bullet \Delta'] \vdash C}\,\circ}{\Gamma[\Delta \bullet \Delta'] \vdash C}\,\circ$$

Figure 2.11: Identifying the bottleneck configuration within natural deduction as the immediate elimination of an introduced connective.

when $(\otimes E)$ is taken into account. Roughly, its application may intervene between matching occurrences of an introduction and an elimination rule, preventing them from canceling each other out. To illustrate with $(/E)$:

$$\dfrac{\dfrac{\Delta \vdash C \otimes D \quad \dfrac{\Gamma[C \bullet D] \bullet B \vdash A}{\Gamma[C \bullet D] \vdash A/B}\,/I}{\Gamma[\Delta] \vdash A/B}\,\otimes E \quad \Theta \vdash B}{\Gamma[\Delta] \bullet \Theta \vdash A}\,/E$$

We can always push $(/E)$ over an application of $(\otimes E)$ in its main premise, thus closing the distance with $(/I)$. I.e.,

$$\dfrac{\dfrac{\Delta \vdash C \otimes D \quad \Gamma[C \bullet D] \vdash A/B}{\Gamma[\Delta] \vdash A/B}\,\otimes E \quad \Theta \vdash B}{\Gamma[\Delta] \bullet \Theta \vdash A}\,/E$$

is replaced by

$$\dfrac{\Delta \vdash C \otimes D \quad \dfrac{\Gamma[C \bullet D] \vdash A/B \quad \Theta \vdash B}{\Gamma[C \bullet D] \bullet \Theta \vdash A}\,/E}{\Gamma[\Delta] \bullet \Theta \vdash A}\,\otimes E$$

Similarly, suppose, in the following situation,

$$\dfrac{\dfrac{\Theta \vdash C \otimes D \quad \Delta[C \bullet D] \vdash A \otimes B}{\Delta[\Theta] \vdash A \otimes B} \otimes E \qquad \Gamma[A \bullet B] \vdash E}{\Gamma[\Delta[\Theta]] \vdash E} \otimes E$$

$\Delta[C \bullet D] \vdash A \otimes B$ was derived by $(\otimes I)$. The upper instance of $(\otimes E)$ then prevents cancellation with the lower occurrence. Again, we find that we can permute it:

$$\dfrac{\Theta \vdash C \otimes D \qquad \dfrac{\Delta[C \bullet D] \vdash A \otimes B \quad \Gamma[A \bullet B] \vdash E}{\Gamma[\Delta[C \bullet D]] \vdash E} \otimes E}{\Gamma[\Delta[\Theta]] \vdash E} \otimes E$$

With these remarks, we conclude our discussion of harmony inside natural deduction. In the next section, we point out the close correspondence with computation inside the simply-typed λ-calculus; a simple functional programming language figuring prominently in Montague's [1970] execution of his Universal Grammar program. As a consequence, we find that the natural deduction formalism lends itself particularly well to the formal analysis of natural language semantics.

2.4 Formal semantics

We have motivated natural deduction and sequent calculus as formalisms satisfying the subformula property: a kind of completeness result stating that, in proving a given judgement $\Gamma \vdash A$ or $\Gamma \Rightarrow A$, one need only take into account the subformulas found in Γ and A. While a significant metamathematical result implying the decidability of **NL**, we find that its actual proof is of equal, if not more, interest. Roughly, derivations may be considered amendable to computation: sticking with natural deduction for now, a procedure was presented for the stepwise transformation of derivations, resulting in the removal of any bottlenecks (or 'redexes', as we came to call them). The current section builds forth on this observation for the implicational fragment of **NL**, emphasizing the close correspondence with the field of formal semantics. As usual, we focus on communicating only the essential ideas, referring the reader to [Moortgat, 1997, §3] and references therein for further details.

We shall pursue an understanding of derivations as representing functions, here understood intensionally in the sense of an algorithm. In particular, it is not too difficult to find the corresponding notion of argument. Say we have proved the judgement $\Gamma \vdash A$. Taking Γ to be some bracketing over $A_1 \bullet \ldots \bullet A_n$, a trivial induction will show that the formulas $A_1, \ldots, A_n$ can be traced back to axioms:

$$\begin{array}{c} \dfrac{}{A_1 \vdash A_1}\,\mathit{Id} \quad \ldots \quad \dfrac{}{A_n \vdash A_n}\,\mathit{Id} \\ \vdots \\ \Gamma \vdash A \end{array}$$

While we have yet to settle upon the precise language for expressing algorithms, let us tentatively write the function encoding a derivation witnessing a given judgement on the latter's righthand side, labeling the conclusion. Axioms then correspond to free variables, serving as placeholders for arguments:

$$\begin{array}{c} \dfrac{}{A_1 \vdash x_1 : A_1}\,\mathit{Id} \quad \ldots \quad \dfrac{}{A_n \vdash x_n : A_n}\,\mathit{Id} \\ \vdots \\ \Gamma \vdash f(x_1, \ldots, x_n) : A \end{array}$$

Our notation can be further simplified if we agree to label the hypotheses occurring on the lefthand side of $\vdash$ by the variables naming the associated axioms. Disregarding bracketing, we can thus more succinctly write

$$x_1 : A_1 \bullet \ldots \bullet x_n : A_n \vdash f(x_1, \ldots, x_n) : A.$$

The linguistic relevance of the above discussion becomes clear when we recall axioms serve as placeholders for words that were assigned the corresponding formulas. Similarly, the variable x labeling an axiom may be considered a placeholder for a word's denotation. By extension, the function f in a derived judgement $x_1 : A_1 \bullet \ldots \bullet x_n : A_n \vdash f(x_1, \ldots, x_n) : A$ may be considered a recipe for combining the lexical meanings $x_1, \ldots, x_n$ into the denotation of the resulting phrase of 'category' A. Assuming its construction to proceed in accordance with the principle of compositionality adopted by Montague, each inference rule thus describes a procedure for combining the algorithms labeling its premises into one associated with its conclusion. Restated linguistically: the denotation of any derived expression is obtained as a function of the denotations associated with the immediate subexpressions, and of the particular syntactic operation (that is to say, inference rule) combining them.

The following definitions fix a language of *terms* to serve as notation for the algorithmic content underlying natural deduction. We start with the description of the *raw* such terms, existing independently of a derivability judgement.

$$\frac{}{x : A \vdash x : A}\ Id$$

$$\frac{\Gamma \vdash s : A/B \quad \Delta \vdash t : B}{\Gamma \bullet \Delta \vdash (s < t) : A}\ /E \qquad \frac{\Gamma \bullet x : B \vdash s : A}{\Gamma \vdash s/x^B : A/B}\ /I$$

$$\frac{\Delta \vdash t : B \quad \Gamma \vdash s : B\backslash A}{\Delta \bullet \Gamma \vdash (t > s) : A}\ \backslash E \qquad \frac{x : B \bullet \Gamma \vdash s : A}{\Gamma \vdash x^B\backslash s : B\backslash A}\ \backslash I$$

Figure 2.12: Labeling natural deduction by raw λ-terms.

Definition 13. *Raw terms* are defined by the following grammar.

$$s, t \quad ::= \quad x \mid (s < t) \mid (t > s) \mid s/x^A \mid x^A\backslash s$$

Terms of the shapes $(s < t)$ or $(t > s)$ and of s/x^A or $x^A\backslash s$ are henceforth referred to as *applications* and *abstractions* respectively. Intuitively, the latter serves as notation for functions in an argument x ranging over proofs of A, while the former applies such functions to t. Both operations are made sensitive to directionality, reflecting the existence of the two implications / and \.

The following definition cements the status of abstractions as binding operators.

Definition 14. Call an occurrence of a variable x in s *bound* if it occurs inside the *body* t of an abstraction $x^B\backslash t$ or t/x^B. Otherwise, said occurrence is said to be *free*.

The above term language for **NL** was first proposed by Buszkowski [1987], though using a different notation. Note that, by virtue of their definition independently of the derivability judgement for natural deduction, we can construct terms that do not correspond to any derivation. Hence the qualification 'raw'. The following definition singles out the meaningful terms by enriching the judgement $\Gamma \vdash A$ with the term coding its derivation, as already motivated above.

Definition 15. F.2.12 defines a labeling of natural deduction derivations by raw terms. Derivability judgements now take the shape $\Gamma \vdash s : A$ for Γ some bracketing over a sequence $x_1 : A_1 \bullet \ldots \bullet x_n : A_n$ of labeled formulas $x_i : A_i$, satisfying the linearity constraint that $x_i \neq x_j$ for each $1 \leq i, j \leq n$. As such, the variables found in Γ and Δ are required to be disjoint in instances of $(/E)$ and $(\backslash E)$.

$$\frac{\Delta \vdash t : B \quad \Gamma[x : B] \vdash s : A}{\Gamma[\Delta] \vdash s[t/x] : A} \circ$$

$$\left(\Delta \vdash t : B \ , \ \begin{array}{c} x : B \vdash x : B \\ \vdots \\ \Gamma[x : B] \vdash s : A \end{array} \right) \quad \mapsto \quad \begin{array}{c} \Delta \vdash t : B \\ \vdots \\ \Gamma[\Delta] \vdash s[t/x] : A \end{array}$$

Figure 2.13: Explaining Cut admissibility as substitution of λ-terms.

$$\frac{\dfrac{\Gamma \bullet x : B \vdash s : A}{\Gamma \vdash s/x^B : A/B}\,/I \qquad \Delta \vdash t : B}{\Gamma \bullet \Delta \vdash (s/x^B < t) : A}\,/E \quad \mapsto \quad \frac{\Delta \vdash t : B \quad \Gamma \bullet x : B \vdash s : A}{\Gamma \bullet \Delta \vdash s[t/x] : A} \circ$$

$$\frac{\Delta \vdash t : B \qquad \dfrac{x : B \bullet \Gamma \vdash s : A}{\Gamma \vdash x^B \backslash s : B \backslash A}\,/I}{\Delta \bullet \Gamma \vdash (t > x^B \backslash s) : A}\,\backslash E \quad \mapsto \quad \frac{\Delta \vdash t : B \quad x : B \bullet \Gamma \vdash s : A}{\Delta \bullet \Gamma \vdash s[t/x] : A} \circ$$

Figure 2.14: Identifying rules of computation by labeling proof transformations.

We proceed to discuss the rules for computing with terms, mirroring the procedure of the previous section for removing bottlenecks from derivations. Since the latter made extensive use of Cut admissibility for natural deduction, we first consider the corresponding operation inside the term language. Rather than doing the proof all over again in order to extract a precise definition, we settle in F.2.13 for an informal explanation. Rougly, we know any derivation for $\Gamma[x : A] \vdash s : B$ involves the use of an axiom $x : A \vdash x : A$. Thus, given another derivation for $\Delta \vdash s : B$, we can *substitute* the latter for said axiom, obtaining a derivation for $\Gamma[\Delta] \vdash s[t/x] : B$, the notation $s[t/x]$ referring to the substitution of t for the unique free occurrence of x in s (not to be confused with abstractions s/x^A). The reader is referred to chapter 7 on formal semantics for a more thorough treatment of substitution.

With substitution defined, we can extract the rules of computation for terms through a labeling of the proof transformations from the previous section, as is done in F.2.14. Played down to the level of raw terms, we obtain the following *rewrite* rules:

$$\begin{array}{lcl} (s/x^A < t) & \to & s[t/x] \\ (t > x^A \backslash s) & \to & s[t/x] \end{array}$$

Note that, when talking about raw terms, linearity conditions no longer apply, and so $s[t/x]$ is to be interpreted as the substitution of t for the free (unbound) occurrences of x in t. Traditionally, the undirected counterparts of these rules have been referred to as β-reductions [see Sørensen and Urzyczyn, 1998], and we shall henceforth adopt this practice. Note that we may reasonably demand of β-reduction that no free variables in t become bound after substitution. More generally, rules of computation are always restricted so as to preserve the free or bound status of variable occurrences. If any such restriction prevents the application of a rule, however, we currently do not yet have any means of proceeding with the computation. To remedy this situation, we take a closer look at the status of bound variable occurrences. Consider the following result, obtainable by a straightforward induction:

Lemma 3. If $\Gamma[x : B] \vdash s : A$, then also $\Gamma[y : B] \vdash s[y/x] : A$ for any fresh y.

In particular, if $\Gamma \vdash s/x^B : A/B$ or $\Gamma \vdash x^B\backslash s : B\backslash A$, then also $\Gamma \vdash s[y/x]/y^B : A/B$ and $\Gamma \vdash y^B\backslash s[y/x]$ respectively for any y fresh w.r.t. the variables found in Γ. We see no reason to distinguish between terms s/x^B ($x^B\backslash s$) and $s[y/x]/y^B$ ($y^B\backslash s[y/x]$), so that we can henceforth identify terms up to the renaming of bound variables. Unlike with β-reduction, however, the operation just described is obviously symmetric, so we choose a notation indicative of an equivalence relation:

$$\begin{array}{lcl} s/x^B & \equiv & s[y/x]/y^B \\ x^B\backslash s & \equiv & y^B\backslash s[y/x] \end{array}$$

where y is to be so chosen that it does not occur free in s. We also speak of α-*equivalence*, again following tradition. Using it, we can ensure that the names of bound variables are kept distinct from those of their free counterparts, thus ensuring β-reduction can always go through.

Besides β-reduction and α-equivalence, we obtain η-*reductions* by translating the admissibility proof of non-atomic axioms inside sequent calculus to natural deduction, as is done in F.2.15. Note we require x not to occur free in s:

$$\begin{array}{lcl} (s < x)/x^A & \rightarrow & s \\ x^A\backslash(x > s) & \rightarrow & s \end{array}$$

So far, we have reproduced several of the previous section's theorems and lemmas as operations inside the term language. Similarly, we can take the translation of natural deduction into sequent calculus (cf. T.2.7) to find a term labeling for the latter's derivations. Such is done in F.2.16, again using a labeling of the old derivability

$$\cfrac{\cfrac{\Gamma \vdash s : A/B \quad x : B \vdash x : B}{\Gamma \bullet x : B \vdash (s < x) : A/B}\ /E}{\Gamma \vdash (s < x)/x^B : A/B}\ /I \quad \mapsto \quad \Gamma \vdash s : A/B$$

$$\cfrac{\cfrac{x : B \vdash x : B \quad \Gamma \vdash s : B\backslash A}{x : B \bullet \Gamma \vdash (x > s) : B\backslash A}\ /E}{\Gamma \vdash x^B\backslash(x > s) : B\backslash A}\ \backslash I \quad \mapsto \quad \Gamma \vdash s : B\backslash A$$

Figure 2.15: Defining η-reductions as proof transformations.

$$\frac{}{x : p \Rightarrow x : p}\ I$$

$$\frac{\Delta \Rightarrow t : B \quad \Gamma[x : A] \Rightarrow s : C}{\Gamma[y : A/B \bullet \Delta] \Rightarrow s[(y < t)/x] : C}\ /L \qquad \frac{\Gamma \bullet x : B \Rightarrow s : A}{\Gamma \Rightarrow s/x^B : A/B}\ /R$$

$$\frac{\Delta \Rightarrow t : B \quad \Gamma[x : A] \Rightarrow s : C}{\Gamma[\Delta \bullet y : B\backslash A] \Rightarrow s[(t > y)/x] : C}\ \backslash L \qquad \frac{x : B \bullet \Gamma \Rightarrow s : A}{\Gamma \Rightarrow x^B\backslash s : B\backslash A}\ \backslash R$$

Figure 2.16: Deriving terms in long normal form using sequent derivations.

judgement. For expository purposes, we have left out Cut (interpreted by substitution) as well as non-atomic axioms, showing how sequent derivability can be used not only for deciding provability, but also for the mechanical construction of terms in what is called *long normal form*: no subterms amendable to β-reduction remain, and all possible *η-expansions* (inverting $\to$) are applied.

Having fixed a term language for denoting derivations, we revisit the claimed applicability to the field of natural language semantics with the following example.

Example 8. Consider sentence (1), illustrating *scopal ambiguities*:

(1) Everyone noticed someone.

Without further context, (1) may convey either one of the following *readings*:

(1a) For every person x, there exists a person y s.t. x noticed y

(1b) There exists a person y s.t. everyone noticed y

The difference in interpretation cannot be attributed to one in syntactic structure, as with the binoculars example from §2, nor do we have any reason to attribute it to a lexical ambiguity. Compositionality, however, leaves us with the additional possibility of a single syntactic structure receiving multiple interpretations whenever it has more than one candidate derivation; a so-called *derivational* ambiguity. As it stands, **NL** is incapable of a satisfactory analysis of the full range of scopal ambiguities encountered in linguistic reality. Nonetheless, at least for (1), we can still provide a lexicon manifesting the desired multitude of derivations. Roughly, we shall *lift* each occurrence of *np* found in the usual formulas for categorizing noun phrases and (transitive) verbs to $s/(np\backslash s)$, resulting in the following lexicon:

Word	Formula
everyone	$s/(np\backslash s)$
someone	$s/(np\backslash s)$
noticed	$((s/(np\backslash s))\backslash s)/(s/(np\backslash s))$

Note that, despite the new formulas for verbs, we can still use *np* for categorizing proper names by virtue of $np \vdash s/(np\backslash s)$. As F.2.17 shows, using the abbreviation *qnp* (*quantified noun phrase*) for $s/(np\backslash s)$, we now indeed obtain the desired multitude of derivations for witnessing the grammaticality of the same syntactic structure:

(2a) $(x < v^{np}\backslash((v > u)/u^{np\backslash s} > (y < z)))$

(2b) $(x > (y < z))$

Note that we have yet to substitute *lexical* denotations for the free variables. In particular, we shall later argue there to exist, for each word in the above lexicon, a single denotation such that (2a) and (2b) correspond to (1a) and (1b) respectively.

The foregoing example left us with the task of expanding the definition of a lexicon with a semantic component. Its realization takes us away from the description of proofs in **NL**, calling instead for a language to describe real world concepts. We thus need cling no further to **NL**'s resource-sensitivity, our means of referring to the world around us being not so restricted. To start with, we shall define a language of *(semantic) types*; just like **NL**'s formulas serve to syntactically categorize phrases, types delineate the possible kinds of denotations said phrases may have.

Definition 16. The following grammar defines (semantic) types:

$$\tau, \sigma \quad ::= \quad e \mid t \mid (\sigma \rightarrow \tau)$$

$$\dfrac{\dfrac{}{x : qnp \vdash x : qnp}\,Id \qquad \dfrac{\dfrac{}{y : (qnp\backslash s)/qnp \vdash y : (qnp\backslash s)/qnp}\,Id \quad \dfrac{}{z : qnp \vdash z : qnp}\,Id}{y : (qnp\backslash s)/qnp \bullet z : qnp \vdash (y < z) : qnp\backslash s}\,/E}{x : qnp \bullet (y : (qnp\backslash s)/qnp \bullet z : qnp) \vdash (x > (y < z)) : s}\,/E$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v : np \vdash v : np}\,Id \quad \dfrac{}{u : np\backslash s \vdash u : np\backslash s}\,Id}{v : np \bullet u : np\backslash s \vdash (v > u) : s}\,\backslash E}{v : np \vdash (v > u)/u^{np\backslash s} : s/(np\backslash s)}\,/I \qquad \dfrac{\dfrac{}{y : (qnp\backslash s)/qnp \vdash y : (qnp\backslash s)/qnp}\,Id \quad \dfrac{}{z : qnp \vdash z : qnp}\,Id}{y : (qnp\backslash s)/qnp \bullet z : qnp \vdash (y < z) : (s/(np\backslash s))\backslash s}\,/E}{\dfrac{v : np \bullet (y : (qnp\backslash s)/qnp \bullet z : qnp) \vdash ((v > u)/u^{np\backslash s} > (y < z)) : s}{y : (qnp\backslash s)/qnp \bullet z : qnp \vdash v^{np}\backslash((v > u)/u^{np\backslash s} > (y < z)) : np\backslash s}\,/I}\,\backslash E$$

$$\dfrac{\dfrac{}{x : qnp \vdash x : s/(np\backslash s)}\,Id \qquad \vdots}{x : qnp \bullet (y : (qnp\backslash s)/qnp \bullet z : qnp) \vdash (x < v^{np}\backslash((v > u)/u^{np\backslash s} > (y < z))) : s}\,/E$$

Figure 2.17: Analyzing scopal ambiguities in **NL**. The formula *qnp* is used as an abbreviation of $s/(np\backslash s)$.

Here, the *base* types e and t refer to *entities* and the boolean *truth values* respectively, while any given types σ and τ may combine into the *function type* $\sigma \to \tau$. We assume $\to$ to associate to the right, using $\sigma_1 \to \sigma_2 \to \tau$ to abbreviate $\sigma_1 \to (\sigma_2 \to \tau)$.

As noted, the purpose of semantic types is to 'categorize' denotations. We briefly consider a set-theoretic explanation of this idea. So far, we have had little to say about what kind of objects denotations actually are. The purpose of the current exercise is to mechanically associate words, and by extension the sentences containing them, with their real world references. In actuality, the best we can do is to provide a mathematical approximation of the reference of any given expression; what we have thus far named that expression's *denotation*. Let us tentatively assume denotations to be formalizable inside set-theory, certainly the most-well known among the proposed foundational languages for mathematics.

With the above background, we can understand any given type τ as a set of denotations, namely those considered to be 'of the type' τ. The construction of such sets proceeds by induction over τ, as taken up in the following definition.

Definition 17. We inductively define for each type τ its set D_τ of denotations, referred to by τ's *denotational domain.* For e, the choice of D_e is arbitrary, its inhabitants intuitively understood to represent the entities figuring in the particular discourse being interpreted. Next, $D_t := \{\mathbf{true}, \mathbf{false}\}$. Finally, given D_σ and D_τ, $D_{\sigma\to\tau}$ is defined by the set of all functions $f : D_\sigma \mapsto D_\tau$.

Example 9. We consider several examples of semantic types relevant for the study of natural language semantics. Since we have thus far considered $\to$ as the only type constructor, it helps, in light of its interpretation by function spaces, to first reformulate some standard set-theoretic concepts in terms of unary functions.

1. First, we shall make free use of the correspondence between sets and their characteristic functions. That is, given an overarching domain F, a subset $E \subseteq F$ can equivalently be presented by the function $f : F \mapsto D_t$ mapping $c \in F$ to **true** iff $c \in E$, picking **false** otherwise. Conversely, given a map $f : F \mapsto D_t$, we can define the set $E_f := \{c \in F \mid f(c) = \mathbf{true}\}$.
2. The above observation generalizes to n-ary relations R over F, corresponding to functions $f : F^n \mapsto D_t$ in n arguments. Seeing as we only have unary functions at our disposal, note f is isomorphic to the function f' taking only a single argument $c_1 \in F$, and mapping it to another function f'_c mapping its remaining $(n-1)$ arguments $\langle c_2, \ldots, c_n\rangle$ to $f(\langle c_1, c_2, \ldots, c_n\rangle)$. Through iteration, we can rewrite f as a function taking the same n arguments, but through n separate function applications.

Keeping these observations in mind, we present the following list of 'useful' types.

Type	Explanation
e	Entities
t	Truth values
$e \to t$	Properties (sets) of entities (1^{st}-order properties)
$e \to e \to t$	Binary relations over entities
$(e \to t) \to t$	Sets of properties of entities (2^{nd}-order properties)
$(e \to t) \to e$	Choice function for (sets of) entities

The inductive structure of semantic types reflects closely that of the formulas of **NL**, and indeed there exists a close correspondence between the formulas and types used to categorize phrases according to their syntactic and semantic properties.

Definition 18. We define a homomorphism $\sigma(\cdot)$ mapping formulas to types by collapsing / and \ into →, while associating s, np and n with t, e and $e \to t$:

$$\begin{array}{rclcrcl} \sigma(s) & := & t & & & & \\ \sigma(np) & := & e & & \sigma(A/B) & := & \sigma(B) \to \sigma(A) \\ \sigma(n) & := & e \to t & & \sigma(B\backslash A) & := & \sigma(B) \to \sigma(A) \end{array}$$

Consider the choices made for interpreting the base formulas: sentences denote truth values (a sentence is true or not), noun phrases denote entities, and finally nouns denote properties thereof. In the last case, consider as an example the noun 'king', denoting the set of those entities that happen to be a king.

Example 10. Recalling our lexicon from E.8, we can now apply the function $\sigma(\cdot)$ to find the types for the desired denotations.

Word	Formula	Type
everyone	$s/(np\backslash s)$	$(e \to t) \to t$
someone	$s/(np\backslash s)$	$(e \to t) \to t$
noticed	$((s/(np\backslash s))\backslash s)/(s/(np\backslash s))$	$((e \to t) \to t) \to ((e \to t) \to t) \to t$

First consider 'everyone', being interpreted by a 2^{nd}-order property. Intuitively, we shall want it to be true for all and only the 1^{st}-order properties that hold of all entities that are people. In contrast, 'someone', sharing its type with that of 'everyone', includes any 1^{st}-order property whose subset of people is non-empty.

While the above example already provides reasonable descriptions for the denotations of the quantified noun phrases, the situation for 'noticed' is less straightforward. In what is to follow, we first define a language of *typed λ-terms*, providing a convenient means, as made explicit through the appropriate notion of model, for referring to the various inhabitants of any given type's denotational domain.

Definition 19. We provide an inductive definition for the language of *simply-typed λ-terms*, describing in parallel the appropriate notion of *model* and interpretation therein. Our mode of presentation follows that of Church, in the sense that any given term always unambiguously belongs to a certain type τ. We shall assume our logical vocabulary to include at least:

1. For each type τ, a countably infinite set of *variables* $x_1^\tau, x_2^\tau, \ldots$ of that type.
2. For each type τ, a countable, possibly empty set of *constants* $\mathbf{c}_1^\tau, \mathbf{c}_2^\tau, \ldots$

A *model* is a pair $\mathscr{M} = \langle \{D_\sigma\}_\sigma, I \rangle$ of a family of denotational domains $\{D_\sigma\}_\sigma$ (fixed by the choice of D_e) and an *interpretation function* I mapping constants $\mathbf{c}_n^\tau$ to elements of D_τ, while a *valuation* is a map g interpreting variables by elements of the appropriate denotational domain. We inductively define the *typed λ-terms* s, t of any give type τ together with their *interpretations* $[\![s]\!]_I^g \in D_\tau$ in $\mathscr{M}$, as follows:

1. Any variable x_n^τ is a term of type τ, with $[\![x^\tau]\!]_I^g := g(x)$.
2. Any constant $\mathbf{c}^\tau$ is a term of type τ, with $[\![\mathbf{c}^\tau]\!]_I^g := I(\mathbf{c}^\tau)$.
3. If s and t are terms of types $\sigma \to \tau$ and σ respectively, then their *application* $(s\ t)$ is a term of type τ with $[\![(s\ t)]\!]_I^g := [\![s]\!]_I^g([\![t]\!]_I^g)$.
4. If s is a term of type τ, then for any variable x_n^σ, the (λ-)*abstraction* $\lambda x^\sigma s$ is a term of type $\sigma \to \tau$, with $[\![\lambda x^\sigma s]\!]_I^g$ the function mapping any $c \in D_\sigma$ to $[\![s]\!]_I^{g'}$ for $g'(x^\sigma) = c$ and $g'(y^\sigma) = g(y^\sigma)$ whenever $y^\sigma \neq x^\sigma$.

In practice, we assume applications to associate to the left. In other words, $(s\ t_1\ t_2)$ abbreviates $((s\ t_1)\ t_2)$. Furthermore, in writing variables, we often drop the numeral subscript in favor of using additional letters $y, z, \ldots$ Finally, we often omit the superscript of a variable or constant carrying its type whenever the latter can be inferred from context.

Thanks to the existence of the type t for boolean truth values, we can recover the usual connectives from classical propositional and predicate logic as special constants with fixed interpretations across models. Such is achieved with the following definition, and will prove particularly helpful when defining the semantical component of a lexicon.

Definition 20. In what is to follow, we shall assume to always have the following constants at our disposal.

Constant	Type	Explanation
$\wedge$	$t \to t \to t$	Conjunction
$\vee$	$t \to t \to t$	Disjunction
$\supset$	$t \to t \to t$	Implication
$\sim$	$t \to t$	Negation
$\exists_\tau$	$(\tau \to t) \to t$	Existential quantification
$\forall_\tau$	$(\tau \to t) \to t$	Universal quantification

We have used the notation '$\supset$' for implication to prevent confusion with the type constructor $\to$. Similarly, '$\sim$' is used for negation in order to keep '$\neg$' free for chapter 7. Finally, note the quantifiers are parameterized over the type of their domain, although usually we shall stick with e. The interpretations of these constants are assumed to be invariant across models, as specified by the following definitions.

- For any $c, d \in D_t$, $[\![\wedge]\!]_I^g(c)(d) = \textbf{true}$ iff $c = d = \textbf{true}$
- For any $c, d \in D_t$, $[\![\vee]\!]_I^g(c)(d) = \textbf{true}$ iff $c = \textbf{true}$ or $d = \textbf{true}$
- For any $c, d \in D_t$, $[\![\supset]\!]_I^g(c)(d) = \textbf{true}$ iff $c = \textbf{false}$ or $d = \textbf{true}$
- For any $c \in D_t$, $[\![\sim]\!]_I^g(c) = \textbf{true}$ iff $c = \textbf{false}$
- For any $c \in D_{\tau \to t}$, $[\![\exists_\tau]\!]_I^g(c) = \textbf{true}$ iff there exists $d \in D_\tau$ s.t. $c(d) = \textbf{true}$
- For any $c \in D_{\tau \to t}$, $[\![\forall_\tau]\!]_I^g(c) = \textbf{true}$ iff for all $d \in D_\tau$, $c(d) = \textbf{true}$

In practice, we often use the following abbreviations:

$$\begin{array}{rcl@{\qquad}rcl}
(s \wedge t) & := & (\wedge\ s\ t) & (s \vee t) & := & (\vee\ s\ t) \\
(s \supset t) & := & (\supset\ s\ t) & \sim s & := & (\sim\ s) \\
\forall x^\tau s & := & (\forall_\tau\ \lambda x^\tau s) & \exists x^\tau s & := & (\exists_\tau\ \lambda x^\tau s)
\end{array}$$

Example 11. We are finally in a position to provide the desired denotations for the lexical entries found in E.8. In doing so, we shall assume the existence of constants **person** and **noticed** of types $e \to t$ and $e \to e \to t$ respectively.

Word	Term
everyone	$\lambda P^{e \to t} \forall x^e((\textbf{person}\ x) \supset (P\ x))$
someone	$\lambda P^{e \to t} \exists y^e((\textbf{person}\ y) \wedge (P\ y))$
noticed	$\lambda Y^{(e \to t) \to t} \lambda X^{(e \to t) \to t}(Y\ \lambda y^e(X\ \lambda x^e(\textbf{noticed}\ y\ x)))$

While the denotations offered for the quantified noun phrases should provide little challenge, the one provided for 'noticed' is more involved. To see that it gets the job done, we require a means of inserting lexical interpretations for the corresponding free variables found in the terms derived in E.8, repeated for convenience:

(2a) $(x < v^{np}\backslash((v > u)/u^{np\backslash s} > (y < z)))$

(2b) $(x > (y < z))$

where x, y and z stand in for, respectively, the denotations of 'everyone', 'noticed' and 'someone'. The following definition shows that the terms used for referring to proofs in **NL** can be straightforwardly translated into typed λ-terms, reflecting the similar explanation of $\rightarrow$ as a collapse of / and \. A minor complication comes in the form of having to deal with labeled derivability judgements as opposed to raw terms, since the latter do not mention any typing information on variables.

Definition 21. Given a judgement $J = \Gamma \vdash s : A$ in **NL**, we can obtain a typed λ-term s' throught the following inductive procedure:

1. If J is an axiom $x : A \vdash x : A$, return $x^{\sigma(A)}$.
2. If J is of the form $\Gamma \bullet \Delta \vdash (s < t) : A$ or $\Delta \bullet \Gamma \vdash (s > t) : A$, being derived by $(/E)$ and $(\backslash E)$ respectively, return $(s'\ t')$.
3. If J is an abstraction $\Gamma \vdash s/x^B : A/B$ or $\Gamma \vdash x^B\backslash s : B\backslash A$, being derived by $(/I)$ and $(\backslash I)$ respectively, return $\lambda x^{\sigma(B)} s'$.

Note that in our treatment of variables we have implicitly assumed the existence of an injection for mapping variables x and formulas A of **NL** to typed variables $x^{\sigma(A)}$.

Definition 22. In light of the previous definition, the rules of β- and η-reduction straightforwardly extend to typed λ-terms, as does α-equivalence. Again, we require that no free variables become bound after application of these rules.

$$\begin{array}{rcll} \lambda x^\tau s & \equiv & \lambda y^\tau s[y/x] & (\alpha) \\ (\lambda x^\tau s\ t) & \rightarrow & s[t/x] & (\beta) \\ \lambda x^\tau (s\ x) & \rightarrow & s & (\eta) \end{array}$$

The reader is referred to [Carpenter, 1996] for a detailed proof of soundness w.r.t. set-theoretic models, in the sense that for any model $\langle\{D_\tau\}_\tau, I\rangle$ and valuation g,

$$\begin{array}{rcll} [\![\lambda x^\tau s]\!]_I^g & = & [\![\lambda y^\tau s[y/x]]\!]_I^g & (\alpha) \\ [\![(\lambda x^\tau s\ t)]\!]_I^g & = & [\![s[t/x]]\!]_I^g & (\beta) \\ [\![\lambda x^\tau (s\ x)]\!]_I^g & = & [\![s]\!]_I^g & (\eta) \end{array}$$

Example 12. Translated to λ-terms, the readings associated with E.8, still abstracting over the lexical denotations, are now as follows.

(2a') $(x\ \lambda v^e(y\ z\ \lambda u^e(u\ v)))$

(2b') $(y\ z\ x)$

Substituting the terms found in E.11, we finally obtain, after a series of β-reductions,

(3a) $\forall x^e((\mathbf{person}\ x) \supset \exists y^e((\mathbf{person}\ y) \wedge (\mathbf{noticed}\ y\ x)))$

(3b) $\exists y^e((\mathbf{person}\ y) \wedge \forall x^e((\mathbf{person}\ x) \supset (\mathbf{noticed}\ y\ x)))$

corresponding to the desired readings (1a) and (1b) respectively, repeated below:

(1a) For every person x, there exists a person y s.t. x noticed y

(1b) There exists a person y s.t. everyone noticed y

As already mentioned before, **NL**'s capabilities of dealing with scopal ambiguities are limited. Already, the analysis just proposed for the simple transitive clause 'Everyone noticed someone' exhibits several flaws:

1. While we have succeeded in deriving the two readings using a lexicon where each word associates with both a unique formula and interpretation, this success hinges upon the particular denotation proposed for the transitive verb. Had we used the following term instead

$$\lambda Y^{(e\to t)\to t}\lambda X^{(e\to t)\to t}(X\ \lambda x^e(Y\ \lambda y^e(\mathbf{noticed}\ y\ x)))$$

switching the order in which the variables X and Y are used inside the body of the term (i.e., inside the applications), both of the terms derived in E.8 would have collapsed into

$$\forall x^e((\mathbf{person}\ x) \supset \exists y^e((\mathbf{person}\ y) \wedge (\mathbf{noticed}\ y\ x)))$$

Ideally, we would like the readings associated with the **NL** terms to be fixed independently of the particular interpretation assigned in the lexicon.

2. Of the two-readings associated with 'Everyone noticed someone', the one where the subject takes wide scope (i.e., as in the (a) examples) is the more salient one. Nonetheless, it is also the reading for which the most efforts had to be expended, as witnessed by the difference in complexity found between (2a) and (2b). If our analyses are to transcend the descriptive, becoming instead explanations for the cognitive foundation of natural languages, this discrepancy is to be dealt with.

In chapter 7, we shall propose an alternative analysis of scopal ambiguities, covering a wider variety of empirical data. In doing so, we will particularly deal with the first complaint found above. The matter as to whether or not one's proposed lexicon provides an adequate reflection of the difference in salience between the various readings associated with any given sentence, on the other hand, is of a less concrete nature, and therefore more difficult to test. It shall thus not play a role in any of what is to follow. For an attempt at tackling this matter inside the associative Lambek calculus through use of proof nets, we refer the reader to [Morrill, 2000].

2.5 Concluding remarks

We have provided a brief survey of non-associative Lambek calculus, discussing its model theory as well as various formalisms for representing proofs. Throughout, we have attempted to highlight the applications to the study of natural languages: models were investigated for their encoding of linguistic reality, while natural deduction was found particularly suitable to conducting formal semantics. Nonetheless, **NL** hath its limitations. As already noted, being essentially a formalism for reasoning about trees, its expressivity does not exceed that of context-free grammars.[3] While already sufficient for dealing with many of the phenomena encountered in the field [see Pullum and Gazdar, 1982], they have proven incapable of explaining the cross-serial dependencies encountered in Swiss German and (though debatable) Dutch [Huybregts, 1976, Shieber, 1985]. While the various incarnations of typelogical grammar diverge with regard to their take on the proper extension of **NL** for dealing with such data, the next chapters will focus on the particular solution pursued by the *Lambek-Grishin* calculus.

[3] For a dispute of the claim that **NL** may be considered thé logic of trees, the reader is referred to the work of Venema [1996].

3

The Lambek-Grishin calculus

3.1 Introduction

We continue our exposition on categorial type logics with an overview of the Lambek-Grishin calculus, constituting the main focus for the remainder of this thesis. Our presentation roughly follows the structure of the previous chapter, proceeding through discussions of relational models, algebraic derivations and concluding with the sequent calculus and natural deduction formalisms.

LG is initially treated in §2 as a minimal enrichment of **NL**'s logical vocabulary to facilitate the existence of an order-reversing duality, witnessed by an involution $^{\infty}$ on formulas (meaning $A^{\infty\infty} = A$) s.t. $A \leq B$ iff $B^{\infty} \leq A^{\infty}$. Said exercise does not yet mark an improvement upon recognizing capacity, however, as argued in chapter 5. Instead, as discussed in §3, **LG**'s true promise shines in the availability of structural postulates for establishing communication between dual families of logical constants, a situation previously unthinkable within **NL**. Further explorations into the expressivity of said extensions are conducted in chapter 6 on type similarity. We continue in §4 by attempting to extend **NL**'s natural deduction and sequent calculus to **LG**; an attempt that ultimately fails, the reasons for which we shall attempt to clarify. Possible remedies, however, are also suggested. Finally, §5 concludes with a list of some of the open problems pertaining to **LG**, as well as including a brief comparison with some of its contenders within the field of CTL. As with chapter 2, besides §4.4 on nested sequent calculi, little contribution is to be found in the current chapter beyond the particular choice of presentation.

3.2 Relational models and algebraic derivability

Presented as a formal exercise, **LG** represents the search for a minimal (conservative) extension of **NL** admitting an order-reversing duality, i.e., the existence of an involution $\cdot^{\infty}$ on formulas (meaning $A^{\infty\infty} = A$ for any A) such that

$$A \leq B \text{ iff } B^{\infty} \leq A^{\infty}$$

The current section details the solution.[1] Beyond being a mere exercise in unearthing symmetry, however, **LG** represents foremost an exploration into the possibilities of establishing interaction between the old vocabulary and the new, always seeking to maximize linguistic coverage. This wider view is further pursued in §3.

Definition 23. **LG**'s logical vocabulary extends **NL**'s by a *multiplicative disjunction* (*par*) $\oplus$ and *co-implications* (*subtractions*) $\oslash, \oslash$. Thus, the set of formulas $\mathscr{F}(Atom)$ of **LG** over the (countable) set *Atom* is inductively defined by the following clauses:

$$\begin{array}{lcll}
A, B & ::= & p & \text{(Atomic formulas in } Atom\text{)} \\
 & | & (A \otimes B) \mid (A \oplus B) & \text{(Tensor vs. par)} \\
 & | & (A/B) \mid (B \obslash A) & \text{(Right division vs left subtraction)} \\
 & | & (B\backslash A) \mid (A \oslash B) & \text{(Left division vs. right subtraction)}
\end{array}$$

Definition 24. Made explicit, duality is realized by the map $\cdot^{\infty}$:

$$\begin{array}{rcl@{\qquad}rcl}
p^{\infty} & := & p & & & \\
(A \otimes B)^{\infty} & := & B^{\infty} \oplus A^{\infty} & (A \oplus B)^{\infty} & := & B^{\infty} \otimes A^{\infty} \\
(A/B)^{\infty} & := & B^{\infty} \obslash A^{\infty} & (B \obslash A)^{\infty} & := & A^{\infty}/B^{\infty} \\
(B\backslash A)^{\infty} & := & A^{\infty} \oslash B^{\infty} & (A \oslash B)^{\infty} & := & B^{\infty}\backslash A^{\infty}
\end{array}$$

Lemma 4. The map $\cdot^{\infty}$ is involutive: for all A, $A^{\infty\infty} = A$.

[1] A related proposal, discussed at length in chapters 4 and 5, is De Groote and Lamarche's [2002] *classical non-associative Lambek calculus* (**CNL**). Rather than proceeding from **NL** as is (like **LG** does), it rather opts for a different choice of logical vocabulary by which some of **NL**'s connectives arise as defined operations, while at the same time guaranteeing order-reversal.

We provide a brief digression into Grishin's original approach to duality. Rather than using $\cdot^{\infty}$ as primitive, he offered a decomposition into separate order-reversing and -preserving dualities $\cdot^{\circ}$ and $\cdot^{\sim}$ respectively, the latter written $\cdot^{\bowtie}$ by Moortgat [2009].

Definition 25. Define, for any A, the formula $A^{\sim}$ by induction:

$$
\begin{array}{rcl@{\qquad}rcl}
p^{\sim} & := & p & & & \\
(A \otimes B)^{\sim} & := & B^{\sim} \otimes A^{\sim} & (A \oplus B)^{\sim} & := & B^{\sim} \oplus A^{\sim} \\
(A/B)^{\sim} & := & B^{\sim} \backslash A^{\sim} & (B \obslash A)^{\sim} & := & A^{\sim} \oslash B^{\sim} \\
(B \backslash A)^{\sim} & := & A^{\sim}/B^{\sim} & (A \oslash B)^{\sim} & := & B^{\sim} \obslash A^{\sim}
\end{array}
$$

Similarly, for any A, let A° be defined

$$
\begin{array}{rcl@{\qquad}rcl}
p^{\circ} & := & p & & & \\
(A \otimes B)^{\circ} & := & A^{\circ} \oplus B^{\circ} & (A \oplus B)^{\circ} & := & A^{\circ} \otimes B^{\circ} \\
(A/B)^{\circ} & := & A^{\circ} \oslash B^{\circ} & (B \obslash A)^{\circ} & := & B^{\circ} \backslash A^{\circ} \\
(B \backslash A)^{\circ} & := & B^{\circ} \obslash A^{\circ} & (A \oslash B)^{\circ} & := & A^{\circ}/B^{\circ}
\end{array}
$$

The next result relates the various notions of duality surveyed thus far.

Lemma 5. For any A, $A^{\sim\sim} = A = A^{\circ\circ}$, $A^{\circ\sim} = A^{\infty} = A^{\sim\circ}$, $A^{\circ\sim} = A^{\infty} = A^{\sim\circ}$, and, finally, $A^{\infty\circ} = A^{\sim} = A^{\circ\infty}$.

Example 13. As a first approximation to interpreting the par and coimplications inside relational models, we shall attempt an understanding of $\cdot^{\infty}$ as set-difference w.r.t. W. We restrict to the concrete frame F_G parameterized over a CFG G in Chomsky-normal form, on which we defined the following operations in E.5:

$$
\begin{array}{rll}
\langle C, u\rangle \in (P \otimes Q) & \text{iff} & \text{there exist } \langle A, v\rangle, \langle B, w\rangle \in W_G \text{ s.t.} \\
 & & C \to A\ B \in R,\ u = vw,\ \langle A, v\rangle \in P \text{ and } \langle B, w\rangle \in Q \\
\langle A, v\rangle \in (P/Q) & \text{iff} & \text{for any } \langle B, w\rangle, \langle C, vw\rangle \in W_G \text{ s.t. } C \to A\ B \in R, \\
 & & \text{if } \langle B, w\rangle \in Q, \text{ then also } \langle C, vw\rangle \in P \\
\langle B, w\rangle \in (Q\backslash P) & \text{iff} & \text{for any } \langle A, v\rangle, \langle C, vw\rangle \in W_G \text{ s.t. } C \to A\ B \in R, \\
 & & \text{if } \langle A, v\rangle \in Q, \text{ then also } \langle C, vw\rangle \in P
\end{array}
$$

for $P, Q \subseteq W_G$. Writing $\neg P$ for $W_G/P := \{\langle A, w\rangle \in W_G \mid \langle A, w\rangle \notin P\}$, we define the following operations, assuming $\neg$ to bind more strongly than $\otimes$, $/$ and $\backslash$:

$$
\begin{array}{rcl}
P \oplus Q & := & \neg(\neg P \otimes \neg Q) \\
Q \obslash P & := & \neg(\neg Q \backslash \neg P) \\
P \oslash Q & := & \neg(\neg P / \neg Q)
\end{array}
$$

The correspondence with D.24 should be clear. Written out in full:

$$\begin{array}{rll}
\langle C,u\rangle \in (P \oplus Q) & \text{iff} & \text{for any } \langle A,v\rangle,\langle B,w\rangle \in W_G \text{ s.t. } u = vw \\
& & \text{and } C \to A\ B \in R,\ \langle A,v\rangle \in P \text{ or } \langle B,w\rangle \in Q \\
\langle B,w\rangle \in (Q \otimes P) & \text{iff} & \text{there exist } \langle A,v\rangle,\langle C,vw\rangle \in W_G \text{ s.t.} \\
& & C \to A\ B \in R,\ \langle A,v\rangle \notin Q \text{ and } \langle C,vw\rangle \in P \\
\langle A,v\rangle \in (P \oslash Q) & \text{iff} & \text{there exist } \langle B,w\rangle,\langle C,vw\rangle \in W_G \text{ s.t.} \\
& & C \to A\ B \in R,\ \langle w,B\rangle \notin Q \text{ and } \langle C,vw\rangle \in P
\end{array}$$

In words, given $C \to A\ B \in R$, any *separation* of $\langle C,u\rangle \in P \oplus Q$ into components $\langle A,v\rangle$ and $\langle B,w\rangle$ (i.e., with $u = vw$) is to be such that $\langle A,v\rangle \in P$ or $\langle B,w\rangle \in Q$. In case of $\langle B,w\rangle \in (Q \otimes P)$, we unearth a *rejection* of Q, in that we are to witness $\langle A,v\rangle \notin Q$ and $\langle C,vw\rangle \in P$ for which $C \to A\ B$. While the linguistic significance of such operations seems at present unclear, §3 provides empirical support by considering means of interaction with the tensor and implications.

While the identification between $\cdot^\infty$ and set-difference seems an attractive one conceptually, it falls down on the interpretation of atoms: while $p^\infty = p$, we cannot in general state that for any potential $P \subseteq W$ interpreting p, also $W/P = P$. We find that there are two ways out. First, we may alter the definition of formulae so as to accommodate atomic formulas p and $\bar{p}$ for each $p \in$ *Atom*, stating $p^\infty = \bar{p}$ and $\bar{p}^\infty = p$. In this case, $\cdot^\infty$ would be more suitably written $\cdot^\perp$. We shall further pursue this direction in the next chapter on classical **NL**. For now, we stick with the current definitions of formulas and the duality thereupon, rather opting for Kurtonina and Moortgat's [2010] revision of frames. Specifically, we consider a second relation $\oplus \subseteq W^3$ for interpreting the par and subtractions, thus breaking the set-theoretic duality between their interpretations and those of their images under $\cdot^\infty$.

Definition 26. Compared to D.3, a *frame* F for **LG** is now a triple $F = \langle W,\otimes,\oplus\rangle$, adding another ternary relation $\oplus \subseteq W \times W \times W$ besides $\otimes$. A *model* $\mathscr{M} = \langle F,v\rangle$ pairs F with a *valuation* v, mapping atoms to subsets of W. The latter extends to a *forcing* relation between resources and formulas, as follows:

$$\begin{array}{rll}
a \vDash p & \text{iff} & a \in v(p) \\
\\
a \vDash A \otimes B & \text{iff} & \text{there exist } b,c \text{ s.t. } \otimes abc,\ b \vDash A \text{ and } c \vDash B \\
b \vDash A/B & \text{iff} & \text{for all } a,c \text{ s.t. if } \otimes abc \text{ and } c \vDash B, \text{ then } a \vDash A \\
c \vDash B\backslash A & \text{iff} & \text{for all } a,b \text{ s.t. if } \otimes abc \text{ and } b \vDash B, \text{ then } a \vDash A \\
\\
a \vDash A \oplus B & \text{iff} & \text{for all } b,c \text{ s.t. if } \oplus abc, \text{ then } b \vDash A \text{ or } c \vDash B \\
b \vDash C \oslash B & \text{iff} & \text{there exist } a,c \text{ s.t. } \oplus abc,\ c \nvDash B \text{ and } a \vDash C \\
c \vDash A \otimes C & \text{iff} & \text{there exist } a,b \text{ s.t. } \oplus abc,\ b \nvDash A \text{ and } a \vDash C
\end{array}$$

Preorder laws

$$\frac{}{A \leq A}\, Id \qquad \frac{A \leq B \quad B \leq C}{A \leq C}\, \circ$$

(Co)residuation

$$\frac{A \otimes B \leq C}{A \leq C/B}\, r \qquad \frac{A \otimes B \leq C}{B \leq A\backslash C}\, r \qquad \frac{C \leq A \oplus B}{A \obslash C \leq B}\, cr \qquad \frac{C \leq A \oplus B}{C \oslash B \leq A}\, cr$$

Figure 3.1: The Lambek-Grishin calculus: Base logic.

We note E.13 still provides a concrete instance of the above definition, albeit one identifying $\otimes$ and $\oplus$. We turn our attention to derivability, updating our previous definition of said concept for **NL** to accommodate the extended logical vocabulary.

Definition 27. F.3.1 defines derivability judgements $A \leq B$ for **LG**, adding to **NL** the *coresiduated* family $\{\oplus, \obslash, \oslash\}$ of connectives.

The following theorem is now but a trivial inductive proof away:

Theorem 3.2.1. For any A, B, $A \leq B$ iff $B^{\infty} \leq A^{\infty}$, iff $A^{\sim} \leq B^{\sim}$, iff $B^{\circ} \leq A^{\circ}$.

Example 14. As before, monotonicity is derivable, and we illustrate with $\oplus$:

$$\frac{A \leq B \quad C \leq D}{A \oplus C \leq B \oplus D}\, \oplus \qquad \frac{A \leq B \quad C \leq D}{D \obslash A \leq C \obslash B}\, \obslash \qquad \frac{A \leq B \quad C \leq D}{A \oslash D \leq B \oslash C}\, \oslash$$

$$\frac{\dfrac{C \leq D \qquad \dfrac{\dfrac{\dfrac{A \leq B \qquad \dfrac{\dfrac{}{B \oplus D \leq B \oplus D}\, Id}{B \leq (B \oplus D) \oslash D}\, cr}{A \leq (B \oplus D) \oslash D}\, \circ}{A \oplus D \leq B \oplus D}\, cr}{D \leq A \obslash (B \oplus D)}\, cr}{C \leq A \obslash (B \oplus D)}\, \circ}{A \oplus C \leq B \oplus D}\, cr$$

Soundness w.r.t. arbitrary relational models is shown using a trivial extension of our previous argument for T.2.2.1.

Theorem 3.2.2. If $A \leq B$, then $a \vDash A$ implies $a \vDash B$ for arbitrary models and resources a therein.

The converse completeness theorem, however, is a little harder:

Theorem 3.2.3. If for all models and resources a, $a \models A$ only if $a \models B$, also $A \leq B$.

Wrong Proof. We employ an extension of the syntactic model used for proving T.2.2.2. Thus, the resources are instantiated by formulas, while we set $\otimes ABC$ iff $A \leq B \otimes C$, $\oplus ABC$ iff $A \leq B \oplus C$ and $A \in v(p)$ iff $A \leq p$. Now consider the truth lemma, $B \models A$ iff $B \leq A$ for all A, and specifically the case where $A = A_1 \oplus A_2$. Going from left to right, we show $B \leq A_1 \oplus A_2$ on the assumption that for all C, D, $C \oplus D \leq B$ implies $C \models A_1$ or $D \models A_2$. Now if we could prove the antecedent for $C = B \oslash A_2$ and $D = A_1 \obslash B$, then the desired result would follow: if $B \oslash A_2 \models A_1$, then by induction hypothesis $B \oslash A_2 \leq A_1$ and so $B \leq A_1 \oplus A_2$ by coresiduation, and similarly for when $A_1 \obslash B \models A_2$. One would be hard-pressed, however, to prove $B \leq (B \oslash A_2) \oplus (A_1 \obslash B)$ without any further assumptions. $\square$

From the above proof attempt, it should be clear one requires a richer notion of syntactic model. Such is the remedy suggested by Kurtonina and Moortgat, who take instead the resources to be sets of formulas, upward closed under derivability. The reader is referred to [Kurtonina and Moortgat, 2010] for further details.

Before continuing in the next section with a number of proposals for extending **LG**, let us first settle upon a notion of **LG** grammar. For now, we assume a straightforward generalization of the **NL** grammars of D.9. Chapter 5 will suggest a generalization, allowing other connectives besides $\otimes$ to combine words.

Definition 28. An **LG** grammar $G = \langle \Sigma, \mathit{Atom}, \mathit{Lex}, s \rangle$ consists of a set of words Σ, a choice of atomic formulae *Atom* disjoint from Σ, a *lexicon* $\mathit{Lex} \subseteq \Sigma \times \mathscr{F}(\mathit{Atom})$ and a start symbol $s \in \mathit{Atom}$. The *language* $\mathscr{L}(G)$ of G is defined by the set of strings $w_1 \ldots w_n \in \Sigma^+$ s.t. there exist $A_1 \in \mathit{Lex}(w_1), \ldots, A_n \in \mathit{Lex}(w_n)$ and a bracketing C of $A_1 \otimes \ldots \otimes A_n$ for which $C \leq s$, where $\mathit{Lex}(w_i) := \{A \mid \langle w_i, A \rangle\}$ for $1 \leq i \leq n$.

3.3 Grishin interactions

As defined, **LG** poses a harmless extension of **NL**, its recognizing capacity remaining within context-free boundaries (cf. Ch.5).[2] The current section discusses several possible extensions by groups of structural postulates, encoding variants of associativity and commutativity mixing connectives from the residuated and coresiduated families. Their discovery dates back to [Grishin, 1983], and were later partially

[2] In chapter 7, however, **LG** is shown particularly suitable to a type of Montagovian semantics that allows it to account for scopal ambiguities without any of the extensions that we shall presently discuss.

adopted by Lambek [1993] in his definition of bilinear logic. Though the latter works mainly represent exercises in abstract algebra and proof theory respectively, the linguistic interest of Grishin's efforts was later realized by Moortgat and associates.

Given the algebraic presentation of **LG** currently adopted, we can describe Grishin's postulates as applying two (co)residuation rules simultaneously. E.g., suppose $A \otimes B \leq C \oplus D$. We have four options of continuing using only (co)residuation:

$$\frac{A \otimes B \leq C \oplus D}{A \leq (C \oplus D)/B}\ r \quad \frac{A \otimes B \leq C \oplus D}{B \leq A\backslash(C \oplus D)}\ r \quad \frac{A \otimes B \leq C \oplus D}{(A \otimes B) \oslash D \leq C}\ cr \quad \frac{A \otimes B \leq C \oplus D}{C \obslash (A \otimes B) \leq D}\ cr$$

Now pick any two (r) and (cr) instances, and combine them into a new rule that applies them simultaneously. E.g., going from

$$\frac{A \otimes B \leq C \oplus D}{A \leq (C \oplus D)/B}\ r \quad \text{and} \quad \frac{A \otimes B \leq C \oplus D}{(A \otimes B) \oslash D \leq C}\ cr \quad \text{to} \quad \frac{A \otimes B \leq C \oplus D}{A \oslash D \leq C/B}$$

Considering each possible combination, we uncover three more rules

$$\frac{A \otimes B \leq C \oplus D}{C \obslash A \leq D/B} \quad \frac{A \otimes B \leq C \oplus D}{B \oslash D \leq A\backslash C} \quad \frac{A \otimes B \leq C \oplus D}{C \obslash B \leq A\backslash D}$$

The above example starts out from a premise allowing for both a residuation step on the left (i.e., going from premises to conclusion, moving material from the l.h.s. of $\leq$ to the r.h.s.) as well as a coresiduation step on the right. Three other combinatorially possible combinations of (co)residuation steps immediately present themselves, each again suggesting four structural rules. In other words, the 'recipe' outlined above foresees in four possible structural extensions in total, comprising four rules each.

Definition 29. F.3.2 lists four possible structural extensions of **LG**, originally appearing in Grishin's work, exhausting the combinatorially possible combinations of two (co)residuation steps into a single rule. The extensions under consideration are referred to by 'types' and are numbered using roman numerals, as follows:

I. Combines coresiduation on the left with residuation on the right.

II. Combines residuation steps on both the left and the right.

III. Combines coresiduation steps on both the left and the right.

IV. Combines residuation on the left with coresiduation on the right.

Type I

$$\frac{A \oslash B \leq C\backslash D}{C \otimes A \leq D \oplus B}\, A^1_I \qquad \frac{A \obslash B \leq C/D}{B \otimes D \leq A \oplus C}\, A^2_I$$

$$\frac{A \obslash B \leq C\backslash D}{C \otimes B \leq A \oplus D}\, C^1_I \qquad \frac{A \oslash B \leq C/D}{A \otimes D \leq C \oplus B}\, C^2_I$$

Type II

$$\frac{A \otimes B \leq C\backslash D}{C \otimes A \leq D/B}\, A^1_{II} \qquad \frac{A \otimes B \leq C/D}{B \otimes D \leq A\backslash C}\, A^2_{II}$$

$$\frac{A \otimes B \leq C\backslash D}{C \otimes B \leq A\backslash D}\, C^1_{II} \qquad \frac{A \otimes B \leq C/D}{A \otimes D \leq C/B}\, C^2_{II}$$

Type III

$$\frac{A \oslash B \leq C \oplus D}{C \obslash A \leq D \oplus B}\, A^1_{III} \qquad \frac{A \obslash B \leq C \oplus D}{B \oslash D \leq A \oplus C}\, A^2_{III}$$

$$\frac{A \obslash B \leq C \oplus D}{C \obslash B \leq A \oplus D}\, C^1_{III} \qquad \frac{A \oslash B \leq C \oplus D}{A \oslash D \leq C \oplus B}\, C^2_{III}$$

Type IV

$$\frac{A \otimes B \leq C \oplus D}{C \obslash A \leq D/B}\, A^1_{IV} \qquad \frac{A \otimes B \leq C \oplus D}{B \oslash D \leq A\backslash C}\, A^2_{IV}$$

$$\frac{A \otimes B \leq C \oplus D}{C \obslash B \leq A\backslash D}\, C^1_{IV} \qquad \frac{A \otimes B \leq C \oplus D}{A \oslash D \leq C/B}\, C^2_{IV}$$

Figure 3.2: Optional structural extensions: simultaneous (co)residuation

For any given extension found in F.3.2, the available rules are partitioned into those labeled A, denoting *associativity*, and those labeled C, denoting *commutavitiy*, with numbers further discriminating between the members of each given class. Our motivation for this choice of rule naming should be self-explanatory given the equivalent

Type I

$$
\begin{array}{llll}
A \otimes (B \oplus C) \leq (A \otimes B) \oplus C & (\alpha_I^1) & (A \oplus B) \otimes C \leq A \oplus (B \otimes C) & (\alpha_I^2) \\
A \otimes (B \oplus C) \leq B \oplus (A \otimes C) & (\gamma_I^1) & (A \oplus B) \otimes C \leq (A \otimes C) \oplus B & (\gamma_I^2)
\end{array}
$$

Type II

$$
\begin{array}{llll}
(A \otimes B) \otimes C \leq A \otimes (B \otimes C) & (\alpha_{II}^1) & A \otimes (B \otimes C) \leq (A \otimes B) \otimes C & (\alpha_{II}^2) \\
A \otimes (B \otimes C) \leq B \otimes (A \otimes C) & (\gamma_{II}^1) & (A \otimes B) \otimes C \leq (A \otimes C) \otimes B & (\gamma_{II}^2)
\end{array}
$$

Type III

$$
\begin{array}{llll}
(A \oplus B) \oplus C \leq A \oplus (B \oplus C) & (\alpha_{III}^1) & A \oplus (B \oplus C) \leq (A \oplus B) \oplus C & (\alpha_{III}^2) \\
A \oplus (B \oplus C) \leq B \oplus (A \oplus C) & (\gamma_{III}^1) & (A \oplus B) \oplus C \leq (A \oplus C) \oplus B & (\gamma_{III}^2)
\end{array}
$$

Type IV

$$
\begin{array}{llll}
(A \backslash B) \oslash C \leq A \backslash (B \oslash C) & (\alpha_{IV}^1) & A \obslash (B / C) \leq (A \obslash B) / C & (\alpha_{IV}^2) \\
A \obslash (B \backslash C) \leq B \backslash (A \obslash C) & (\gamma_{IV}^1) & (A / B) \oslash C \leq (A \oslash C) / B & (\gamma_{IV}^2)
\end{array}
$$

Figure 3.3: Equivalent axiomatic presentations of Grishin's type I-IV extensions.

axiomatic presentations covered in the following lemma, the (axiomatic renditions of the) A-rules being the only ones preserving order.

Lemma 6. F.3.3 presents equivalent axiomatic presentations for Grishin's rules.

Proof. The statement of the lemma follows from an extensive case analysis, showing that each rule (A_n^m) or (C_n^m) in F.3.2 for $n \in \{I, II, III, IV\}$ and $1 \leq m \leq 2$ is derivable using the corresponding (α_n^m) or (γ_n^m) in F.3.3 and vice versa. As typical cases, we compare (A_I^1) and (α_I^1). Going from right to left,

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{B \oslash C \leq B \oslash C}\,\mathit{Id}
}{B \leq (B \oslash C) \oplus C}\,cr
}{A \otimes B \leq A \otimes ((B \oslash C) \oplus C)}\,?\otimes
\qquad
\dfrac{}{A \otimes ((B \oslash C) \oplus C) \leq (A \otimes (B \oslash C)) \oplus C}\,\alpha_I^1
}{
\dfrac{A \otimes B \leq (A \otimes (B \oslash C)) \oplus C}{(A \otimes B) \oslash C \leq A \otimes (B \oslash C)}\,cr
}\,\circ
$$

whereas the other way around,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B\oplus C \leq B\oplus C}\,Id}{(B\oplus C)\oslash C \leq B}\,cr \qquad \dfrac{\dfrac{}{A\otimes B\leq A\otimes B}\,Id}{B\leq A\backslash(A\otimes B)}\,r}{(B\oplus C)\oslash C \leq A\backslash(A\otimes B)}\,\circ}{A\otimes(B\oplus C)\leq (A\otimes B)\oplus C}\,A_I^1 \qquad \Box$$

Remark 4. Putting things in historical perspective, the axiom-based presentation was used originally by Grishin, while its rule-based counterpart was later adopted by Moortgat [2009] and Moot [2007] in their proposals of proof nets and of display calculi, respectively. See also their combined effort [Moortgat and Moot, 2011].

For each of the four types of extensions considered, those referred to by II and III amount to regular same-sort associativity and (weak) commutativity of $\otimes$ and $\oplus$ respectively. Given the importance for the current enterprise of rejecting these laws (reflecting the sensitivity of natural language syntax to word order and hierarchical structuring), we shall find no use for them in the sequel, their inclusion in the current exposition having served only to show the rationale behind Grishin's proposals. The type I and IV extensions instead mix connectives from the residuated and coresiduated families, thus establishing communication. For this reason, we also tend to speak of *interactions* of type I and IV in the sequel.

The picture that arises is not that of a single logic **LG** containing both the remaining interactions of types I and IV. In fact, the latter are not mutually compatible, in the sense that their combination results in a structural collapse into same-sort associativity and -commutativity, as the following lemma shows.

Lemma 7. The following rules are derivable in **LG** augmented by both type I and IV interactions:

$$\begin{array}{c}(A\otimes B)\otimes C\leq D\oplus E\\ \hline\hline A\otimes(B\otimes C)\leq D\oplus E\end{array} \qquad \dfrac{A\otimes(B\otimes C)\leq D\oplus E}{B\otimes(A\otimes C)\leq D\oplus E} \qquad \dfrac{(A\otimes B)\otimes C\leq D\oplus E}{(A\otimes C)\otimes B\leq D\oplus E}$$

and, of course, dually,

$$\begin{array}{c}A\otimes B\leq (C\oplus D)\oplus E\\ \hline\hline A\otimes B\leq C\oplus(D\oplus E)\end{array} \qquad \dfrac{A\otimes B\leq C\oplus(D\oplus E)}{A\otimes B\leq D\oplus(C\oplus E)} \qquad \dfrac{A\otimes B\leq (C\oplus D)\oplus E}{A\otimes B\leq (C\oplus E)\oplus D}$$

Proof. Derivability is witnessed as follows:

$$
\begin{array}{cl}
(A \otimes B) \otimes C \leq D \oplus E & \\ \hline
D \obslash (A \otimes B) \leq E/C & A^1_{IV} \\ \hline
A \otimes B \leq D \oplus (E/C) & cr \\ \hline
B \oslash (E/C) \leq A\backslash D & A^2_{IV} \\ \hline
B \leq (A\backslash D) \oplus (E/C) & cr \\ \hline
(A\backslash D) \obslash B \leq E/C & cr \\ \hline
B \otimes C \leq (A\backslash D) \oplus E & A^2_{I} \\ \hline
(B \otimes C) \oslash E \leq A\backslash D & cr \\ \hline
A \otimes (B \otimes C) \leq D \oplus E & A^1_{I}
\end{array}
\qquad
\begin{array}{cl}
A \otimes (B \otimes C) \leq D \oplus E & \\ \hline
(B \otimes C) \oslash E \leq A\backslash D & A^1_{IV} \\ \hline
B \otimes C \leq (A\backslash D) \oplus E & cr \\ \hline
(A\backslash D) \obslash B \leq E/C & A^2_{IV} \\ \hline
B \leq (A\backslash D) \oplus (E/C) & cr \\ \hline
B \oslash (E/C) \leq A\backslash D & cr \\ \hline
A \otimes B \leq D \oplus (E/C) & A^2_{I} \\ \hline
D \obslash (A \otimes B) \leq E/C & cr \\ \hline
(A \otimes B) \otimes C \leq D \oplus E & A^1_{I}
\end{array}
$$

$$
\begin{array}{cl}
A \otimes (B \otimes C) \leq D \oplus E & \\ \hline
(B \otimes C) \oslash E \leq A\backslash D & A^2_{IV} \\ \hline
B \otimes C \leq (A\backslash D) \oplus E & cr \\ \hline
(A\backslash D) \obslash C \leq B\backslash E & C^1_{IV} \\ \hline
C \leq (A\backslash D) \oplus (B\backslash E) & cr \\ \hline
C \oslash (B\backslash E) \leq A\backslash D & cr \\ \hline
A \otimes C \leq D \oplus (B\backslash E) & A^1_{I} \\ \hline
D \obslash (A \otimes C) \leq B\backslash E & cr \\ \hline
B \otimes (A \otimes C) \leq D \oplus E & C^1_{I}
\end{array}
\qquad
\begin{array}{cl}
(A \otimes B) \otimes C \leq D \oplus E & \\ \hline
D \obslash (A \otimes B) \leq E/C & A^1_{IV} \\ \hline
A \otimes B \leq D \oplus (E/C) & cr \\ \hline
A \oslash (E/C) \leq D/B & C^2_{IV} \\ \hline
A \leq (D/B) \oplus (E/C) & cr \\ \hline
(D/B) \obslash A \leq E/C & cr \\ \hline
A \otimes C \leq (D/B) \oplus E & A^2_{I} \\ \hline
(A \otimes C) \oslash E \leq D/B & cr \\ \hline
(A \otimes C) \otimes B \leq D \oplus E & C^2_{I}
\end{array}
$$

□

We may conclude there is no one definitive way of extending **LG** by Grishin's interactions. Thus, rather than constituting a single logic, the Lambek-Grishin calculus serves as an umbrella term encompassing a hierarchy of substructural logics, each adding either one, both or none of the type I and IV interactions upon the *base logic* of §2, possibly further discriminating between the A- and C-type rules. Thus, we will speak of the following variants:

1. $\mathbf{LG}_{\varnothing}$ refers to the base logic, lacking both interactions of type I and IV.

2. $\mathbf{LG}_{I}$ refers to $\mathbf{LG}_{\varnothing}$ augmented by type I interactions.

3. $\mathbf{LG}_{IV}$ refers to $\mathbf{LG}_{\varnothing}$ augmented by type IV interactions.

4. $\mathbf{LG}_{I+IV}$ refers to $\mathbf{LG}_{\varnothing}$ augmented by both type I and type IV interactions.

$$\forall a,b,c,x,y((\otimes cbx \,\&\, \oplus cay) \Rightarrow \exists z(\oplus xzy \,\&\, \otimes abz))\ (A^1_I)$$
$$\forall a,b,c,x,y((\otimes bxc \,\&\, \oplus bya) \Rightarrow \exists z(\oplus xyz \,\&\, \otimes azc))\ (A^2_I)$$
$$\forall a,b,c,x,y((\otimes cbx \,\&\, \oplus cya) \Rightarrow \exists z(\oplus xyz \,\&\, \otimes abz))\ (C^1_I)$$
$$\forall a,b,c,x,y((\otimes bxc \,\&\, \oplus bay) \Rightarrow \exists z(\oplus xzy \,\&\, \otimes azc))\ (C^2_I)$$

$$\forall a,b,c,x,y((\otimes abc \,\&\, \oplus xcy) \Rightarrow \exists z(\oplus zay \,\&\, \otimes zbx))\ (A^1_{IV})$$
$$\forall a,b,c,x,y((\otimes abc \,\&\, \oplus xyb) \Rightarrow \exists z(\oplus zya \,\&\, \otimes zxc))\ (A^2_{IV})$$
$$\forall a,b,c,x,y((\otimes abc \,\&\, \oplus xyc) \Rightarrow \exists z(\oplus zya \,\&\, \otimes zbx))\ (C^1_{IV})$$
$$\forall a,b,c,x,y((\otimes abc \,\&\, \oplus xby) \Rightarrow \exists z(\oplus zay \,\&\, \otimes zxc))\ (C^2_{IV})$$

Figure 3.4: Frame constraints for the type I and IV interactions of F.3.2.

If we additionally wish to emphasize that we only adopt the A- or C-rules from the interaction types listed, we write $\mathbf{LG}^A_T$ and $\mathbf{LG}^C_T$ respectively for $T \in \{I, IV, I+IV\}$. Finally, we simply write **LG** to generalize over the above hierarchy as a whole.

So far, we have concentrated on the proof-theoretic aspects of the Grishin interactions. Model-theoretically, their introduction corresponds to the postulation of additional frame constraints, as specified in F.3.4. Showing soundness of the various incarnations of **LG** w.r.t. to the proper classes of frames remains a matter of applying a straightforward induction. For the revised completeness proof, we again refer the reader to [Kurtonina and Moortgat, 2010].

Lemma 8. Grishin interactions are sound w.r.t. the frame constraints of F.3.4.

Proof. We illustrate with (α^1_I). So suppose, for some model and resource a therein, that $a \models A \otimes (B \oplus C)$, i.e., there exist b, c s.t. $\otimes abc$, $b \models A$, and for any u, v s.t. $\oplus cuv$, also $u \models B$ or $v \models C$. We show $a \models (A \otimes B) \oplus C$. In other words, assume for some given m, n that $\oplus amn$ and $n \not\models C$. We then show $m \models A \otimes B$, i.e., there exist k, l s.t. $\otimes mkl$, $k \models A$ and $l \models B$. Applying the frame constraint of F.3.4 labeled (A^1_I) with $\otimes abc$ and $\oplus amn$, we know there exists t s.t. $\oplus ctn$ and $\otimes mbt$. We now witness the desired k, l by b, t respectively. Indeed, $b \models A$ by assumption, while $\oplus ctn$ and $n \not\models C$ imply $t \models B$ by the assumption that $c \models B \oplus C$. □

3.4 Sequent calculi and natural deduction

Given the almost effortless extension of algebraic derivability in **NL** with the coresiduation laws, one might expect the natural deduction and sequent calculus formalisms to follow suit. Unfortunately, we shall find that things don't run as smoothly as they did with **NL**: transitivity is admissible for neither sequent calculus nor natural deduction if still based on the same notion of structure as used in the previous chapter. In view of past literature, this outcome is not a surprising one, given that for Lambek's [1993] bilinear logic, closely related to **LG**, it was already observed by Abrusci [1991] that the usual two-sided sequent presentation lacks Cut admissibility. While the counterexamples discussed below differ from those found for bilinear logic, the underlying intuitions are similar.

Note the proviso in our previous claim: Cut admissibility may be recovered provided the 'old' concept of structure as a binary-branching tree of formulas is replaced with a richer notion. We discuss two possibilities: one in §4.3 by Moortgat [2009] using the display sequents of Goré [1998], while another one in §4.4 adapts the nested sequents of Goré et al. [2008]. First, however, we briefly discuss in §4.1 and §4.2 sequent and natural deduction calculi still based on the definition of structures used in chapter 2, explaining why Cut admissibility fails.

3.4.1 Sequent calculus

The definition of structures remains largely unchanged from that used for **NL**, the only difference being that formulas now range over those of **LG** instead. In what is to follow, we use additional metavariables Θ and Λ for referring to structures.

Definition 30. F.3.5 defines the judgement form $\Gamma \Rightarrow \Theta$ for sequent derivability in $\mathbf{LG}_{I+IV}$, where the occurrences of $\bullet$ in Γ and Θ are intuitively to be understood as the structural counterparts of the tensor and par respectively. In order to recover the more restrictive systems, dropping at least one of the interaction types, we place additional constraints on the use of contexts:

- $\mathbf{LG}_I$ requires $\Theta[] = []$ in $(/R)$ and $(\backslash R)$, and $\Gamma[] = []$ in $(\oslash L)$ and $(\oslash L)$.
- $\mathbf{LG}_{IV}$ requires $\Theta[] = []$ in $(/L)$ and $(\backslash L)$, $\Gamma[] = []$ in $(\oslash R)$ and $(\oslash R)$, $\Theta[] = \Lambda[] = []$ in $(\otimes R)$, and $\Gamma[] = \Delta[] = []$ in $(\oplus L)$. Finally, either $\Theta[] = []$ or $\Gamma[] = []$ in $(\circ)$, thus splitting into two rules

$$\frac{\Delta \Rightarrow B \quad \Gamma[B] \Rightarrow \Lambda}{\Gamma[\Delta] \Rightarrow \Lambda}\circ \quad \text{and} \quad \frac{\Delta \Rightarrow \Theta[B] \quad B \Rightarrow \Lambda}{\Delta \Rightarrow \Theta[\Lambda]}\circ$$

- $\mathbf{LG}_\varnothing$ combines the restrictions for *both* $\mathbf{LG}_I$ and $\mathbf{LG}_{IV}$.

$$\frac{}{A \Rightarrow A}\ Id \qquad \frac{\Delta \Rightarrow \Theta[B] \quad \Gamma[B] \Rightarrow \Lambda}{\Gamma[\Delta] \Rightarrow \Theta[\Lambda]}\ \circ$$

$$\frac{\Delta \Rightarrow \Theta[B] \quad \Gamma[A] \Rightarrow \Lambda}{\Gamma[A/B \bullet \Delta] \Rightarrow \Theta[\Lambda]}\ /L \qquad \frac{\Gamma[B] \Rightarrow \Lambda \quad \Delta \Rightarrow \Theta[A]}{\Gamma[\Delta] \Rightarrow \Theta[\Lambda \bullet B \varobslash A]}\ \varobslash R$$

$$\frac{\Delta \Rightarrow \Theta[B] \quad \Gamma[A] \Rightarrow \Lambda}{\Gamma[\Delta \bullet B\backslash A] \Rightarrow \Theta[\Lambda]}\ \backslash L \qquad \frac{\Gamma[B] \Rightarrow \Lambda \quad \Delta \Rightarrow \Theta[A]}{\Gamma[\Delta] \Rightarrow \Theta[A \oslash B \bullet \Lambda]}\ \oslash R$$

$$\frac{\Gamma \Rightarrow \Theta[A] \quad \Delta \Rightarrow \Lambda[B]}{\Gamma \bullet \Delta \Rightarrow \Theta[\Lambda[A \otimes B]]}\ \otimes R \qquad \frac{\Gamma[A] \Rightarrow \Theta \quad \Delta[B] \Rightarrow \Lambda}{\Gamma[\Delta[A \oplus B]] \Rightarrow \Theta \bullet \Lambda}\ \oplus L$$

$$\frac{\Gamma \bullet B \Rightarrow \Theta[A]}{\Gamma \Rightarrow \Theta[A/B]}\ /R \qquad \frac{\Gamma[A] \Rightarrow B \bullet \Theta}{\Gamma[B \varobslash A] \Rightarrow \Theta}\ \varobslash L$$

$$\frac{B \bullet \Gamma \Rightarrow \Theta[A]}{\Gamma \Rightarrow \Theta[B\backslash A]}\ \backslash R \qquad \frac{\Gamma[A] \Rightarrow \Theta \bullet B}{\Gamma[A \oslash B] \Rightarrow \Theta}\ \oslash L$$

$$\frac{\Gamma[A \bullet B] \Rightarrow \Theta}{\Gamma[A \otimes B] \Rightarrow \Theta}\ \otimes L \qquad \frac{\Gamma \Rightarrow \Theta[A \bullet B]}{\Gamma \Rightarrow \Theta[A \oplus B]}\ \oplus R$$

Figure 3.5: Sequent calculus for $\mathbf{LG}_{I+IV}$. Through appropriate restrictions on contexts, either one or both of the Grishin interactions may be dropped.

Example 15. We briefly illustrate the various restrictions listed in the above definition by showing how, in their absence, various characteristic theorems of both type I and type IV interactions become derivable. Starting with $\mathbf{LG}_I$, first note that, in the presence of $(\circ)$, we might as well drop the contexts $\Gamma[]$ and $\Delta[]$ from $(\varobslash R)$, $(\oslash R)$ and $(\otimes R)$, and similarly $\Theta[]$ and $\Lambda[]$ from $(/L)$, $(\backslash L)$ and $(\oplus L)$ without compromising derivability. For example, given $\Delta \Rightarrow \Theta[B]$ and $\Gamma[A] \Rightarrow \Lambda$, we derive $\Gamma[A/B \bullet \Delta] \Rightarrow \Theta[\Lambda]$ by using $(\circ)$ and the restricted form of $(/L)$:

$$\frac{\Delta \Rightarrow \Theta[B] \qquad \dfrac{\dfrac{}{B \Rightarrow B}\ Id \quad \Gamma[A] \Rightarrow \Lambda}{\Gamma[A/B \bullet B] \Rightarrow \Lambda}\ /L}{\Gamma[A/B \bullet \Delta] \Rightarrow \Theta[\Lambda]}\ \circ$$

In other words, the treatment of type I interactions comes down to the possibility of having an instance of $(\circ)$ with both $\Gamma \neq []$ and $\Theta[] \neq []$. Now, using $(\circ)$, it is easy to see how a typical instance of linear distributivity becomes derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,\mathit{Id} \quad \dfrac{}{C \Rightarrow C}\,\mathit{Id}}{B \oplus C \Rightarrow B \bullet C}\,\oplus L \qquad \dfrac{\dfrac{}{A \Rightarrow A}\,\mathit{Id} \quad \dfrac{}{B \Rightarrow B}\,\mathit{Id}}{A \bullet B \Rightarrow A \otimes B}\,\otimes R}{A \bullet B \oplus C \Rightarrow A \otimes B \bullet C}\,\circ}{A \bullet B \oplus C \Rightarrow (A \otimes B) \oplus C}\,\oplus R}{A \otimes (B \oplus C) \Rightarrow (A \otimes B) \oplus C}\,\otimes L$$

By allowing $\Theta[]$ ($\Gamma[]$) in $(/R)$ and $(\backslash R)$ ($(\obslash L)$ and $(\oslash L)$) to be non-empty, we can derive typical instances of the type IV interactions like (γ^1_{IV}):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,\mathit{Id} \quad \dfrac{\dfrac{}{A \Rightarrow A}\,\mathit{Id} \quad \dfrac{}{C \Rightarrow C}\,\mathit{Id}}{C \Rightarrow A \bullet A \obslash C}\,\obslash R}{B \bullet B\backslash C \Rightarrow A \bullet A \obslash C}\,\backslash L}{B \bullet A \obslash (B\backslash C) \Rightarrow A \obslash C}\,\obslash L}{A \obslash (B\backslash C) \Rightarrow B\backslash(A \obslash C)}\,\backslash R$$

Before formally stating completeness w.r.t. algebraic derivability, we prove

Lemma 9. For arbitrary $\Gamma[]$, $\Theta[]$, B and C, we have, in $\mathbf{LG}_I$:

(i) $\Gamma[A]^\bullet \oplus B \leq C$ and $A \oplus \Gamma[B]^\bullet \leq C$ imply $\Gamma[A \oplus B]^\bullet \leq C$.

(ii) $\Theta[\Gamma[B]^\bullet]^\circ \leq C$ implies $\Gamma[\Theta[B]^\circ]^\bullet \leq C$.

Proof. We first prove (i) by induction on $\Gamma[]$, using it as a stepping stone to demonstrate (ii) by induction on $\Theta[]$. The base case ($\Gamma[] = []$) is immediate. We further illustrate with $\Gamma[] = \Gamma' \bullet \Delta[]$, the case $\Gamma[] = \Gamma'[] \bullet \Delta$ being similar. We have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta[A]^\bullet \oplus B \leq \Delta[A]^\bullet \oplus B}\,\mathit{Id}}{(\Delta[A]^\bullet \oplus B) \oslash B \leq \Delta[A]^\bullet}\,cr}{\Gamma'^\bullet \otimes ((\Delta[A]^\bullet \oplus B) \oslash B) \leq \Gamma'^\bullet \otimes \Delta[A]^\bullet}\,?\otimes}{(\Delta[A]^\bullet \oplus B) \oslash B \leq \Gamma'^\bullet\backslash(\Gamma'^\bullet \otimes \Delta[A]^\bullet)}\,r}{\Gamma'^\bullet \otimes (\Delta[A]^\bullet \oplus B) \leq (\Gamma'^\bullet \otimes \Delta[A]^\bullet) \oplus B}\,A^1_I \quad (\Gamma'^\bullet \otimes \Delta[A]^\bullet) \oplus B \leq C}{\Gamma'^\bullet \otimes (\Delta[A]^\bullet \oplus B) \leq C}\,\circ}{\Delta[A]^\bullet \oplus B \leq \Gamma'^\bullet\backslash C}\,r}{\Delta[A \oplus B]^\bullet \leq \Gamma'^\bullet\backslash C}\,\mathit{IH}}{\Gamma'^\bullet \otimes \Delta[A \oplus B]^\bullet \leq C}\,r$$

Showing that $A \oplus (\Gamma'^{\bullet} \otimes \Delta[B]^{\bullet}) \leq C$ implies $\Gamma'^{\bullet} \otimes \Delta[A \oplus B]^{\bullet} \leq C$ works similarly. We prove (ii) by induction on $\Theta[]$. Again, the base case is immediate. We further check $\Theta = \Theta'[] \bullet \Lambda$, the remaining inductive case ($\Theta = \Theta' \bullet \Lambda[]$) being similar.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Theta'^{\circ} \oplus \Lambda[\Gamma[B]^{\bullet}]^{\circ} \leq C}{\Lambda[\Gamma[B]^{\bullet}]^{\circ} \leq \Theta'^{\circ} \obslash C}\; cr}{\Gamma[\Lambda[B]^{\circ}]^{\bullet} \leq \Theta'^{\circ} \obslash C}\; \mathit{IH}}{\Theta'^{\circ} \oplus \Gamma[\Lambda[B]^{\circ}]^{\bullet} \leq C}\; cr}{\Gamma[\Theta'^{\circ} \oplus \Lambda[B]^{\circ}]^{\bullet} \leq C}\; (i)$$

□

Theorem 3.4.1. Let $A^{\bullet} = A^{\circ} = A$, $(\Gamma \bullet \Delta)^{\bullet} = \Gamma^{\bullet} \otimes \Delta^{\bullet}$ and $(\Gamma \bullet \Delta)^{\circ} = \Gamma^{\circ} \oplus \Delta^{\circ}$. Then $\Gamma \Rightarrow \Theta$ iff $\Gamma^{\bullet} \leq \Theta^{\circ}$ for each of $\mathbf{LG}_{\varnothing}$, $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$.

Proof. The proof proceeds similar to those of T.2.3.3 and T.2.3.4 for **NL**, the main difference being that we now have to take into account the interaction postulates. We illustrate both directions of the theorem with some typical cases showing how said postulates come into play. Proceeding first from left to right, we check C_I^2 and C_{IV}^2. Starting with the former, suppose $A \oslash B \Rightarrow C/D$ by induction hypothesis:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\;\mathit{Id} \quad \dfrac{}{A \Rightarrow A}\;\mathit{Id}}{A \Rightarrow A \oslash B \bullet B}\;\oslash R \quad \dfrac{}{A \oslash B \Rightarrow C/D}\;\mathit{IH}}{A \Rightarrow C/D \bullet B}\;\circ \quad \dfrac{\dfrac{}{D \Rightarrow D}\;\mathit{Id} \quad \dfrac{}{C \Rightarrow C}\;\mathit{Id}}{C/D \bullet D \Rightarrow C}\;/L}{\dfrac{A \bullet D \Rightarrow C \bullet B}{A \bullet D \Rightarrow C \oplus B}\;\oplus R}\;\circ}{A \otimes D \Rightarrow C \oplus B}\;\otimes L$$

While for C_{IV}^2, we have as our induction hypothesis that $A \otimes D \Rightarrow C \oplus B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\;\mathit{Id} \quad \dfrac{}{D \Rightarrow D}\;\mathit{Id}}{A \bullet D \Rightarrow A \otimes D}\;\otimes R \quad \dfrac{\dfrac{}{A \otimes D \Rightarrow C \oplus B}\;\mathit{IH} \quad \dfrac{\dfrac{}{C \Rightarrow C}\;\mathit{Id} \quad \dfrac{}{B \Rightarrow B}\;\mathit{Id}}{C \oplus B \Rightarrow C \bullet B}\;\oplus L}{A \otimes D \Rightarrow C \bullet B}\;\circ}{A \bullet D \Rightarrow C \bullet B}\;\circ}{A \oslash B \bullet D \Rightarrow C}\;\oslash L}{A \oslash B \Rightarrow C/D}\;/R$$

Next, we turn to the other direction. To illustrate, we check algebraic derivability of unrestricted Cut (illustrating the use of type I interactions) and $(/R)$ (type IV). Starting with the former, suppose by induction hypothesis that $\Delta^{\bullet} \leq \Theta[B]^{\circ}$ and $\Gamma[B]^{\bullet} \leq \Lambda^{\circ}$. To show $\Gamma[\Delta]^{\bullet} \leq \Theta[\Lambda]^{\circ}$, we require, for arbitrary $\Gamma[]$, $\Theta[]$, B and C:

$$\frac{B \leq C}{\Gamma[B]^\bullet \leq \Gamma[C]^\bullet}\ \Gamma[] \qquad \frac{B \leq C}{\Theta[B]^\circ \leq \Theta[C]^\circ}\ \Theta[]$$

easily shown by induction on $\Gamma[]$ and $\Theta[]$, using monotonicity. We then apply L.9(ii) to derive the desired result, noting $\Gamma[\Delta]^\bullet = \Gamma[\Delta^\bullet]^\bullet$ and $\Theta[\Lambda]^\circ = \Theta[\Lambda^\circ]^\circ$:

$$\dfrac{\dfrac{\Delta^\bullet \leq \Theta[B]^\circ}{\Gamma[\Delta]^\bullet \leq \Gamma[\Theta[B]^\circ]^\bullet}\ \Gamma[] \qquad \dfrac{\dfrac{\Gamma[B]^\bullet \leq \Lambda^\circ}{\Theta[\Gamma[B]^\bullet]^\circ \leq \Theta[\Lambda]^\circ}\ \Theta[]}{\Gamma[\Theta[B]^\circ]^\bullet \leq \Theta[\Lambda]^\circ}\ L.9(ii)}{\Gamma[\Delta]^\bullet \leq \Theta[\Lambda]^\circ}\ \circ$$

To check $(/R)$, we show $\Gamma^\bullet \leq \Theta[A/B]^\circ$ on the assumption $\Gamma^\bullet \otimes B \leq \Theta[A]^\circ$. Proceeding by induction on $\Theta[]$, the base case is immediate, leaving us to check:

$$\dfrac{\dfrac{\dfrac{\Gamma^\bullet \otimes B \leq \Theta'[A]^\circ \oplus \Lambda^\circ}{\Gamma^\bullet \oslash \Lambda^\circ \leq \Theta'[A]^\circ/B}\ C^2_{IV}}{\Gamma^\bullet \oslash \Lambda^\circ \leq \Theta'[A/B]^\circ}\ IH}{\Gamma^\bullet \leq \Theta'[A/B]^\circ \oplus \Lambda^\circ}\ cr \qquad \dfrac{\dfrac{\dfrac{\Gamma^\bullet \otimes B \leq \Lambda^\circ \oplus \Theta'[A]^\circ}{\Lambda^\circ \obslash \Gamma^\bullet \leq \Theta'[A]^\circ/B}\ A^1_{IV}}{\Lambda^\circ \obslash \Gamma^\bullet \leq \Theta'[A/B]^\circ}\ IH}{\Gamma^\bullet \leq \Lambda^\circ \oplus \Theta'[A/B]^\circ}\ cr \qquad \square$$

We conclude our discussion of D.30's sequent calculus with an evaluation of its proof-theoretic viability. The good news is that *principal* Cuts are replaceable with Cuts on the immediate subformulas. Consider, for example,

$$\dfrac{\dfrac{\Delta_1 \Rightarrow \Lambda_1[A] \quad \Delta_2 \Rightarrow \Lambda_2[B]}{\Delta_1 \bullet \Delta_2 \Rightarrow \Lambda_1[\Lambda_2[A \otimes B]]}\ \otimes R \qquad \dfrac{\Gamma[A \bullet B] \Rightarrow \Theta}{\Gamma[A \otimes B] \Rightarrow \Theta}\ \otimes L}{\Gamma[\Delta_1 \bullet \Delta_2] \Rightarrow \Lambda_1[\Lambda_2[\Theta]]}\ \circ$$

being replaceable with Cuts on A and B:

$$\dfrac{\Delta_1 \Rightarrow \Lambda_1[A] \qquad \dfrac{\Delta_2 \Rightarrow \Lambda_2[B] \quad \Gamma[A \bullet B] \Rightarrow \Theta}{\Gamma[A \bullet \Delta_2] \Rightarrow \Lambda_2[\Theta]}\ \circ}{\Gamma[\Delta_1 \bullet \Delta_2] \Rightarrow \Lambda_1[\Lambda_2[\Theta]]}\ \circ$$

Unfortunately, Cut elimination in general still fails. We consider two counterexamples. The first applies to $\mathbf{LG}_{I+IV}$, while another one, due to Bernardi and Moortgat (though left unpublished), concerns $\mathbf{LG}_{\varnothing}$, $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$.[3]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\backslash B \Rightarrow A\backslash B}\,Id \quad \dfrac{}{C \Rightarrow C}\,Id}{C \Rightarrow A\backslash B \bullet (A\backslash B) \obslash C}\,\obslash R}{C \oslash ((A\backslash B) \obslash C) \Rightarrow A\backslash B}\,\oslash L \qquad \dfrac{\dfrac{}{A \Rightarrow A}\,Id \quad \dfrac{}{B \Rightarrow B}\,Id}{A \bullet A\backslash B \Rightarrow B}\,\backslash L}{A \bullet C \oslash ((A\backslash B) \obslash C) \Rightarrow B}\,\circ$$

Permuting Cut over $(\oslash L)$ violates the restrictions put on both rules in $\mathbf{LG}_{\varnothing}$, thus necessitating the use of interactions of both types I and IV:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\backslash B \Rightarrow A\backslash B}\,Id \quad \dfrac{}{C \Rightarrow C}\,Id}{C \Rightarrow A\backslash B \bullet (A\backslash B) \obslash C}\,\obslash R \qquad \dfrac{\dfrac{}{A \Rightarrow A}\,Id \quad \dfrac{}{B \Rightarrow B}\,Id}{A \bullet A\backslash B \Rightarrow B}\,\backslash L}{A \bullet C \Rightarrow B \bullet (A\backslash B) \obslash C}\,\circ}{A \bullet C \oslash ((A\backslash B) \obslash C) \Rightarrow B}\,\oslash L$$

We find that in $\mathbf{LG}_{I+IV}$ the resulting Cut can in fact be eliminated completely:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,Id \quad \dfrac{}{B \Rightarrow B}\,Id}{A \bullet A\backslash B \Rightarrow B}\,\backslash L \qquad \dfrac{}{C \Rightarrow C}\,Id}{A \bullet C \Rightarrow B \bullet (A\backslash B) \obslash C}\,\obslash R}{A \bullet C \oslash ((A\backslash B) \obslash C) \Rightarrow B}\,\oslash L$$

[3] Theirs is similar to a counterexample for Rauszer's [1974] sequent formalization of bi-intuitionistic logic, credited by Buisman and Goré [2007] to Uustalu via personal communication. Adapted to **LG**, it takes the form

$$\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,Id \quad \dfrac{}{p \Rightarrow p}\,Id}{p \Rightarrow q \bullet q \obslash p}\,\obslash R \qquad \dfrac{\dfrac{\dfrac{}{r \Rightarrow r}\,Id \quad \dfrac{}{q \obslash p \Rightarrow q \obslash p}\,Id}{r \bullet q \obslash p \Rightarrow r \otimes (q \obslash p)}\,\otimes R}{q \obslash p \Rightarrow r\backslash(r \otimes (q \obslash p))}\,\backslash R}{p \Rightarrow q \bullet r\backslash(r \otimes (q \obslash p))}\,\circ$$

The reasons for failure to eliminate Cut are similar to those discussed in the main text for Bernardi and Moortgat's sequent.

although Cut elimination in general still fails, as the following counterexample shows:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$B \Rightarrow B$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$C \Rightarrow C$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$D \Rightarrow D$}
\RightLabel{$\oplus L$}
\BinaryInfC{$C \oplus D \Rightarrow C \bullet D$}
\RightLabel{$/L$}
\BinaryInfC{$(C \oplus D)/B \bullet B \Rightarrow C \bullet D$}
\RightLabel{$/R$}
\UnaryInfC{$(C \oplus D)/B \Rightarrow C \bullet D/B$}
\RightLabel{$/L$}
\BinaryInfC{$((C \oplus D)/B)/A \bullet A \Rightarrow C \bullet D/B$}
\RightLabel{$/R$}
\UnaryInfC{$((C \oplus D)/B)/A \Rightarrow C/A \bullet D/B$}
\noLine
\UnaryInfC{$\vdots$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$B \Rightarrow B$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$D \Rightarrow D$}
\RightLabel{$/L$}
\BinaryInfC{$D/B \bullet B \Rightarrow D$}
\RightLabel{$\circ$}
\BinaryInfC{$((C \oplus D)/B)/A \bullet B \Rightarrow C/A \bullet D$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$C \Rightarrow C$}
\RightLabel{$/L$}
\BinaryInfC{$C/A \bullet A \Rightarrow C$}
\RightLabel{$\circ$}
\BinaryInfC{$(((C \oplus D)/B)/A \bullet B) \bullet A \Rightarrow C \bullet D$}
\end{prooftree}
$$

To see that this is indeed a counterexample, we single out that part of the above derivation relevant for reducing the upper instance of Cut:

$$
\begin{prooftree}
\AxiomC{$((C \oplus D)/B)/A \bullet A \Rightarrow C \bullet D/B$}
\RightLabel{$/R$}
\UnaryInfC{$((C \oplus D)/B)/A \Rightarrow C/A \bullet D/B$}
\AxiomC{$D/B \bullet B \Rightarrow D$}
\RightLabel{$\circ$}
\BinaryInfC{$((C \oplus D)/B)/A \bullet B \Rightarrow C/A \bullet D$}
\end{prooftree}
$$

Permuting $(\circ)$ over $(/R)$ necessitates Cutting $((C \oplus D)/B)/A \bullet A \Rightarrow C \bullet D/B$ with $D/B \bullet B \Rightarrow D$, resulting in $((C \oplus D)/B)/A \bullet B \Rightarrow C \bullet D$, and continuing with $(/R)$. The best we can do, however, is to derive $(((C \oplus D)/B)/A \bullet A) \bullet B \Rightarrow C \bullet D/B$, which is a different conclusion from what we started out with. More generally, the problem underlying this counterexample shows up in a typical Cut elimination proof when dealing with permutative Cuts. To see this, consider

$$
\begin{prooftree}
\AxiomC{$\Gamma \Rightarrow \Theta[C]$}
\AxiomC{$\Delta_1 \Rightarrow \Lambda_1[A]$}
\AxiomC{$\Delta_2[C] \Rightarrow \Lambda_2[B]$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\Delta_1 \bullet \Delta_2[C] \Rightarrow \Lambda_1[\Lambda_2[A \otimes B]]$}
\RightLabel{$\circ$}
\BinaryInfC{$\Delta_1 \bullet \Delta_2[\Gamma] \Rightarrow \Theta[\Lambda_1[\Lambda_2[A \otimes B]]]$}
\end{prooftree}
$$

Permuting $(\circ)$ over $(\otimes R)$ again provides us with a different conclusion:

$$
\begin{prooftree}
\AxiomC{$\Delta_1 \Rightarrow \Lambda_1[A]$}
\AxiomC{$\Gamma \Rightarrow \Theta[C]$}
\AxiomC{$\Delta_2[C] \Rightarrow \Lambda_2[B]$}
\RightLabel{$\circ$}
\BinaryInfC{$\Delta_2[\Gamma] \Rightarrow \Theta[\Lambda_2[B]]$}
\RightLabel{$\otimes R$}
\BinaryInfC{$\Delta_1 \bullet \Delta_2[\Gamma] \Rightarrow \Lambda_1[\Theta[\Lambda_2[A \otimes B]]]$}
\end{prooftree}
$$

3.4.2 Natural deduction

Unfortunately, things don't fair much better with natural deduction.

Definition 31. F.3.6 shows Natural deduction for $\mathbf{LG}_{I+IV}$ using judgement forms $\Gamma \vdash \Theta$. The new elimination- and introduction rules for the coimplications and coproducts are just the left- and right introductions of sequent calculus respectively, with an extra Cut added in the former cases. Again, we may posit extra restrictions to obtain the structurally more discriminative $\mathbf{LG}_{\varnothing}$, $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$:

- Dropping type *IV* interactions amounts to requiring $\Theta[] = []$ in $(/I)$ and $(\backslash I)$, and $\Gamma[] = []$ in $(\obslash E)$ and $(\oslash E)$.
- For $\mathbf{LG}_{IV}$, we require $\Theta[] = \Lambda[] = []$ in $(/E)$ and $(\backslash E)$, $\Gamma[] = []$ in $(\obslash I)$ and $(\oslash I)$, $\Theta[] = \Lambda[] = []$ in $(\otimes I)$, and $\Gamma[] = \Delta[] = []$ in $(\oplus E)$.
- $\mathbf{LG}_{\varnothing}$ combines the restrictions for $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$ listed above.

The current take on natural deduction for **LG** resembles Crolard's [2004] presentation of his 'subtractive logic'. Again, however, we find that the management of contexts clashes with normalization. Consider, for instance, the ND equivalent of the counterexample to Cut elimination in $\mathbf{LG}_{I+IV}$ used above:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$((C \oplus D)/B)/A \vdash ((C \oplus D)/B)/A$}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$A \vdash A$}
\RightLabel{$/E$}
\BinaryInfC{$((C \oplus D)/B)/A \bullet A \vdash (C \oplus D)/B$}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$B \vdash B$}
\RightLabel{$/E$}
\BinaryInfC{$(((C \oplus D)/B)/A \bullet A) \bullet B \vdash C \oplus D$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$C \vdash C$}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$D \vdash D$}
\RightLabel{$\oplus E$}
\TrinaryInfC{$(((C \oplus D)/B)/A \bullet A) \bullet B \vdash C \bullet D$}
\RightLabel{$/I$}
\UnaryInfC{$((C \oplus D)/B)/A \bullet A \vdash C \bullet D/B$}
\RightLabel{$/I$}
\UnaryInfC{$((C \oplus D)/B)/A \Rightarrow C/A \bullet D/B$}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$B \vdash B$}
\RightLabel{$/E$}
\BinaryInfC{$((C \oplus D)/B)/A \bullet B \vdash C/A \bullet D$}
\AxiomC{}
\RightLabel{\textit{Id}}
\UnaryInfC{$A \vdash A$}
\RightLabel{$/E$}
\BinaryInfC{$(((C \oplus D)/B)/A \bullet B) \bullet A \vdash C \bullet D$}
\end{prooftree}
$$

This derivation is not in normal form, containing a /-introduction as the main premise of a /-elimination. Any attempt at normalization, however, results in a derivation of $(((C \oplus D)/B)/A \bullet B) \bullet A \vdash C \oplus D$, which is a different conclusion. As another example, while $A \bullet C \oslash ((A \backslash B) \obslash C) \vdash B$ now has a normal form derivation in $\mathbf{LG}_{\varnothing}$,

$$\frac{}{A \vdash A}\ Id$$

$$\frac{\Gamma \vdash \Theta[A/B] \quad \Delta \vdash \Lambda[B]}{\Gamma \bullet \Delta \vdash \Theta[\Lambda[A]]}\ /E \qquad \frac{\Delta \vdash \Theta[B \oslash A] \quad \Gamma[A] \vdash B \bullet \Lambda}{\Gamma[\Delta] \vdash \Theta[\Lambda]}\ \oslash E$$

$$\frac{\Delta \vdash \Lambda[B] \quad \Gamma \vdash \Theta[B\backslash A]}{\Delta \bullet \Gamma \vdash \Theta[\Lambda[A]]}\ \backslash E \qquad \frac{\Delta \vdash \Theta[A \oslash B] \quad \Gamma[A] \vdash \Lambda \bullet B}{\Gamma[\Delta] \vdash \Theta[\Lambda]}\ \oslash E$$

$$\frac{\Delta \vdash \Theta[A \otimes B] \quad \Gamma[A \bullet B] \vdash \Lambda}{\Gamma[\Delta] \vdash \Theta[\Lambda]}\ \otimes E \qquad \frac{\Delta \vdash \Theta[A \oplus B] \ \ \Gamma_1[A] \vdash \Lambda_1 \ \ \Gamma_2[B] \vdash \Lambda_2}{\Gamma_1[\Gamma_2[\Delta]] \vdash \Theta[\Lambda_1 \bullet \Lambda_2]}\ \oplus E$$

$$\frac{\Gamma \bullet B \vdash \Theta[A]}{\Gamma \vdash \Theta[A/B]}\ /I \qquad \frac{\Gamma[B] \vdash \Lambda \quad \Delta \vdash \Theta[A]}{\Gamma[\Delta] \vdash \Theta[\Lambda \bullet B \oslash A]}\ \oslash I$$

$$\frac{B \bullet \Gamma \vdash \Theta[A]}{\Gamma \vdash \Theta[B\backslash A]}\ \backslash I \qquad \frac{\Gamma[B] \vdash \Lambda \quad \Delta \vdash \Theta[A]}{\Gamma[\Delta] \vdash \Theta[A \oslash B \bullet \Lambda]}\ \oslash I$$

$$\frac{\Gamma \vdash \Theta[A] \quad \Delta \vdash \Lambda[B]}{\Gamma \bullet \Delta \vdash \Theta[\Lambda[A \otimes B]]}\ \otimes I \qquad \frac{\Gamma \vdash \Theta[A \bullet B]}{\Gamma \vdash \Theta[A \oplus B]}\ \oplus R$$

Figure 3.6: Natural deduction for $\mathbf{LG}_{I+IV}$. Through appropriate restrictions on contexts, either one or both of the Grishin interactions may be dropped.

its dual, $B \vdash (C/(B \oslash A))\backslash C \bullet A$, still doesn't. In particular, we can only assume the desired derivation to end in an application of $(\backslash I)$ (being in normal form), which would violate the constraints on contexts posited for $\mathbf{LG}_{\varnothing}$.

In conclusion, while it is not too difficult to extend the natural deduction and sequent calculi for **NL** to the various incarnations of **LG**, we find that in neither case normalization is preserved. This effectively robs the resulting presentations of their proof-theoretic interest, thus making them of little use to us in the sequel. Chapter 5 proposes the use of relational labeling to recover Cut admissibility in sequent calculus, adapting a method previously employed by, among others, Kurtonina [1995] and Negri [2005]. For now, we present two alternative remedies. One novel, though inspired by previous work of Goré et al. [2008] on bi-intuitionistic logic, while the other, adapting Belnap's [1982] display calculus formalism, having previously been proposed by Moortgat [2009]. We start with the latter.

3.4.3 Display sequents

Derivability judgements in standard sequent calculi concern the provability of a disjunction of conclusions from a conjunction of hypotheses. In classical logic, both are represented by finite (multi)sets of formulae, with the turnstile sign $\Rightarrow$ serving disambiguation. Descending into the substructural hierarchy, the comma notation used for separating the elements of a set evolves into a binary bracketing thereof, serving as the structural counterpart for multiplicative conjunction and -disjunction.[4] As we have just seen, however, this monopoly of the upward monotone connectives over the structuring of hypotheses and conclusions proved an inhibition to normalization. Dealing with similar problems in intermediate and modal logics, Belnap [1982] first proposed extending the structural vocabulary by a representative for negation, naming the resulting formalism 'display logic', later renamed 'display calculus' by Goré [1998]. While the turnstile no longer can be said to mark the boundary between hypotheses and conclusions, a different property replaces it, serving as the formalism's namesake: through certain invertible structural rules, each hypothesis (conclusion) can be isolated ('displayed') to the left (right) of $\Rightarrow$.

We next briefly discuss how display calculus may be used to provide a Cut-free presentation of **LG**. We essentially recapitulate Moortgat's [2009] account, omitting only his 'focussing' rules and using slightly different notation, inspired in turn by Goré's [1998] revisioning of [Belnap, 1982] to cover a wider array of substructural logics. Typical of the latter is the use of structural counterparts for (co)implications, as opposed to their decomposition by Belnap into disjunction and negation.

Proceeding in the usual fashion, we first go through the definitions of sequents and derivations before dealing with our previous counterexamples to Cut elimination.

Definition 32. Structures are defined as follows:

$$\Gamma, \Delta \quad ::= \quad A \mid (\Gamma \bullet \Delta) \mid (\Gamma \bullet\!\!- \Delta) \mid (\Delta -\!\!\bullet \Gamma)$$

having in- and output interpretations $\cdot^\bullet$ and $\cdot^\circ$ inside **LG**'s formulas, depending on whether they occur as, respectively, the antecedent or succedent of a sequent:

$$\begin{array}{rcl@{\qquad}rcl}
A^\bullet & := & A & A^\circ & := & A \\
(\Gamma \bullet \Delta)^\bullet & := & \Gamma^\bullet \otimes \Delta^\bullet & (\Gamma \bullet \Delta)^\circ & := & \Gamma^\circ \oplus \Delta^\circ \\
(\Gamma \bullet\!\!- \Delta)^\bullet & := & \Gamma^\bullet \oslash \Delta^\circ & (\Gamma \bullet\!\!- \Delta)^\circ & := & \Gamma^\circ / \Delta^\bullet \\
(\Delta -\!\!\bullet \Gamma)^\bullet & := & \Delta^\circ \obslash \Gamma^\bullet & (\Delta -\!\!\bullet \Gamma)^\circ & := & \Delta^\bullet \backslash \Gamma^\circ
\end{array}$$

[4]Within the intuitionistic setting, Lambek [1958, 1961]) was the first to come to this insight, while Dosen [1988, 1989] provides further elaboration. The classical case was covered by Belnap [1982, 1990], among others.

Definition 33. F.3.7 defines a display calculus for **LG**, using judgements $\Gamma \Rightarrow \Delta$.

At the heart of F.3.7 are the display postulates; invertible structural rules relating the structural (co)implications to the conjunctions and disjunctions. Translated back into formulas, they are revealed as counterparts of the (co)residuation laws. Their use is in guaranteeing the aforementioned *display property*: for each hypothesis (conclusion) A occurring in it, a sequent may be brought into a form where A occurs as the sole formula on the left (right) of $\Rightarrow$. As a consequence, logical inferences may be formulated so as to have their main formula *displayed* in the above sense. The following definitions and lemmas are devoted to showing the display property.

Definition 34. We define, by mutual induction, the sets of positively and negatively monotone *contexts* over structures.

$$
\begin{array}{lcl}
\Gamma^{+}[\,], \Delta^{+}[\,] & ::= & [\,] \mid (\Gamma^{+}[\,] \bullet \Delta) \mid (\Gamma \bullet \Delta^{+}[\,]) \\
 & \mid & (\Gamma^{+}[\,] \mathbin{\bullet\!-} \Delta) \mid (\Gamma \mathbin{\bullet\!-} \Delta^{-}[\,]) \\
 & \mid & (\Delta \mathbin{-\!\bullet} \Gamma^{+}[\,]) \mid (\Delta^{-}[\,] \mathbin{-\!\bullet} \Gamma) \\
\Gamma^{-}[\,], \Delta^{-}[\,] & ::= & (\Gamma^{-}[\,] \bullet \Delta) \mid (\Gamma \bullet \Delta^{-}[\,]) \\
 & \mid & (\Gamma^{-}[\,] \mathbin{\bullet\!-} \Delta) \mid (\Gamma \mathbin{\bullet\!-} \Delta^{+}[\,]) \\
 & \mid & (\Delta \mathbin{-\!\bullet} \Gamma^{-}[\,]) \mid (\Delta^{+}[\,] \mathbin{-\!\bullet} \Gamma)
\end{array}
$$

Definition 35. For any $\Gamma^{+}[\,]$ and Θ ($\Gamma^{-}[\,]$ and Θ), define the structures $\Gamma^{+}[\,] \div \Theta$ and $\Gamma^{-}[\,] \div \Theta$ by mutual induction over $\Gamma^{+}[\,]$ and $\Gamma^{-}[\,]$, as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
[\,] \div \Delta & := & \Theta & & & \\
(\Gamma^{+}[\,] \bullet \Delta) \div \Theta & := & \Gamma^{+}[\,] \div (\Theta \mathbin{\bullet\!-} \Delta) & (\Gamma^{-}[\,] \bullet \Delta) \div \Theta & := & \Gamma^{-}[\,] \div (\Theta \mathbin{\bullet\!-} \Delta) \\
(\Gamma \bullet \Delta^{+}[\,]) \div \Theta & := & \Delta^{+}[\,] \div (\Gamma \mathbin{-\!\bullet} \Theta) & (\Gamma \bullet \Delta^{-}[\,]) \div \Theta & := & \Delta^{-}[\,] \div (\Gamma \mathbin{-\!\bullet} \Theta) \\
(\Gamma^{+}[\,] \mathbin{\bullet\!-} \Delta) \div \Theta & := & \Gamma^{+}[\,] \div (\Theta \bullet \Delta) & (\Gamma^{-}[\,] \mathbin{\bullet\!-} \Delta) \div \Theta & := & \Gamma^{-}[\,] \div (\Theta \bullet \Delta) \\
(\Gamma \mathbin{\bullet\!-} \Delta^{-}[\,]) \div \Theta & := & \Delta^{-}[\,] \div (\Theta \mathbin{-\!\bullet} \Gamma) & (\Gamma \mathbin{\bullet\!-} \Delta^{+}[\,]) \div \Theta & := & \Delta^{+}[\,] \div (\Theta \mathbin{-\!\bullet} \Gamma) \\
(\Delta \mathbin{-\!\bullet} \Gamma^{+}[\,]) \div \Theta & := & \Gamma^{+}[\,] \div (\Delta \bullet \Theta) & (\Delta \mathbin{-\!\bullet} \Gamma^{-}[\,]) \div \Theta & := & \Gamma^{-}[\,] \div (\Delta \bullet \Theta) \\
(\Delta^{-}[\,] \mathbin{-\!\bullet} \Gamma) \div \Theta & := & \Delta^{-}[\,] \div (\Gamma \mathbin{\bullet\!-} \Theta) & (\Delta^{+}[\,] \mathbin{-\!\bullet} \Gamma) \div \Theta & := & \Delta^{+}[\,] \div (\Gamma \mathbin{\bullet\!-} \Theta)
\end{array}
$$

Theorem 3.4.2. We have the following results:

$$
\begin{array}{ll@{\qquad}ll}
\text{(a)} & \Gamma^{+}[\Delta] \Rightarrow \Theta \text{ iff } \Delta \Rightarrow \Gamma^{+}[\,] \div \Theta & \text{(c)} & \Theta \Rightarrow \Gamma^{+}[\Delta] \text{ iff } \Gamma^{+}[\,] \div \Theta \Rightarrow \Delta \\
\text{(b)} & \Gamma^{-}[\Delta] \Rightarrow \Theta \text{ iff } \Gamma^{-}[\,] \div \Theta \Rightarrow \Delta & \text{(d)} & \Theta \Rightarrow \Gamma^{-}[\Delta] \text{ iff } \Delta \Rightarrow \Gamma^{-}[\,] \div \Theta
\end{array}
$$

Proof. By a tedious but trivial simultaneous induction, using only the display postulates. As illustration, consider the following instances of (a) and (c):

$$
\begin{array}{ll}
\text{(a)} & \Delta'^{-}[\Delta] \mathbin{-\!\bullet} \Gamma \Rightarrow \Theta \text{ iff } \Delta \Rightarrow \Delta'^{-}[\,] \div (\Gamma \mathbin{\bullet\!-} \Theta) \\
\text{(c)} & \Theta \Rightarrow \Gamma \mathbin{\bullet\!-} \Delta'^{-}[\Delta] \text{ iff } \Delta'^{-}[\,] \div (\Theta \mathbin{-\!\bullet} \Gamma) \Rightarrow \Delta
\end{array}
$$

Preorder laws

$$\frac{}{A \Rightarrow A}\ \mathit{Id} \qquad \frac{\Gamma \Rightarrow A \quad A \Rightarrow \Gamma}{\Gamma \Rightarrow \Delta}\ \circ$$

Display postulates ((Co)Residuation)

$$\frac{\Gamma \Rightarrow \Theta \mathbin{\bullet\!\!-} \Delta}{\Gamma \bullet \Delta \Rightarrow \Theta}\ dp \qquad \frac{\Delta \Rightarrow \Gamma \mathbin{-\!\!\bullet} \Delta}{\Gamma \bullet \Delta \Rightarrow \Delta}\ dp \qquad \frac{\Delta \mathbin{-\!\!\bullet} \Gamma \Rightarrow \Theta}{\Gamma \Rightarrow \Delta \bullet \Theta}\ dp \qquad \frac{\Gamma \mathbin{\bullet\!\!-} \Theta \Rightarrow \Delta}{\Gamma \Rightarrow \Delta \bullet \Theta}\ dp$$

Logical rules (Residuated family)

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \bullet \Delta \Rightarrow A \otimes B}\ \otimes R \qquad \frac{\Delta \Rightarrow B \quad A \Rightarrow \Gamma}{A/B \Rightarrow \Gamma \mathbin{\bullet\!\!-} \Delta}\ /L \qquad \frac{\Delta \Rightarrow B \quad A \Rightarrow \Gamma}{B\backslash A \Rightarrow \Delta \mathbin{-\!\!\bullet} \Gamma}\ \backslash L$$

$$\frac{A \bullet B \Rightarrow \Gamma}{A \otimes B \Rightarrow \Gamma}\ \otimes L \qquad \frac{\Gamma \Rightarrow A \mathbin{\bullet\!\!-} B}{\Gamma \Rightarrow A/B}\ /R \qquad \frac{\Gamma \Rightarrow B \mathbin{-\!\!\bullet} A}{\Gamma \Rightarrow B\backslash A}\ \backslash R$$

Logical rules (Coresiduated family)

$$\frac{A \Rightarrow \Gamma \quad B \Rightarrow \Delta}{A \oplus B \Rightarrow \Gamma \bullet \Delta}\ \oplus L \qquad \frac{B \Rightarrow \Delta \quad \Gamma \Rightarrow A}{\Delta \mathbin{-\!\!\bullet} \Gamma \Rightarrow B \obslash A}\ \obslash R \qquad \frac{B \Rightarrow \Delta \quad \Gamma \Rightarrow A}{\Gamma \mathbin{\bullet\!\!-} \Delta \Rightarrow A \oslash B}\ \oslash R$$

$$\frac{\Gamma \Rightarrow A \bullet B}{\Gamma \Rightarrow A \oplus B}\ \oplus R \qquad \frac{B \mathbin{-\!\!\bullet} A \Rightarrow \Gamma}{B \obslash A \Rightarrow \Gamma}\ \obslash L \qquad \frac{A \mathbin{\bullet\!\!-} B \Rightarrow \Gamma}{A \oslash B \Rightarrow \Gamma}\ \oslash L$$

Grishin interactions (Type I)

$$\frac{\Gamma_2 \mathbin{\bullet\!\!-} \Delta_2 \Rightarrow \Gamma_1 \mathbin{-\!\!\bullet} \Delta_1}{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}\ A_I^1 \qquad \frac{\Delta_1 \mathbin{-\!\!\bullet} \Gamma_1 \Rightarrow \Delta_2 \mathbin{\bullet\!\!-} \Gamma_2}{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}\ A_I^2$$

$$\frac{\Delta_1 \mathbin{-\!\!\bullet} \Gamma_2 \Rightarrow \Gamma_1 \mathbin{-\!\!\bullet} \Delta_2}{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}\ C_I^1 \qquad \frac{\Gamma_1 \mathbin{\bullet\!\!-} \Delta_2 \Rightarrow \Delta_1 \mathbin{\bullet\!\!-} \Gamma_2}{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}\ C_I^2$$

Grishin interactions (Type IV)

$$\frac{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}{\Gamma_2 \mathbin{\bullet\!\!-} \Delta_2 \Rightarrow \Gamma_1 \mathbin{-\!\!\bullet} \Delta_1}\ A_{IV}^1 \qquad \frac{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}{\Delta_1 \mathbin{-\!\!\bullet} \Gamma_1 \Rightarrow \Delta_2 \mathbin{\bullet\!\!-} \Gamma_2}\ A_{IV}^2$$

$$\frac{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}{\Delta_1 \mathbin{-\!\!\bullet} \Gamma_2 \Rightarrow \Gamma_1 \mathbin{-\!\!\bullet} \Delta_2}\ C_{IV}^1 \qquad \frac{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}{\Gamma_1 \mathbin{\bullet\!\!-} \Delta_2 \Rightarrow \Delta_1 \mathbin{\bullet\!\!-} \Gamma_2}\ C_{IV}^2$$

Figure 3.7: Display calculus for **LG**.

Their proofs are carried out as follows:

$$
\begin{prooftree}
\AxiomC{$\Delta \Rightarrow \Delta'^{-}[\,] \div (\Gamma \mathbin{\bullet\!-} \Theta)$}
\doubleLine
\RightLabel{$IH(d)$}
\UnaryInfC{$\Gamma \mathbin{\bullet\!-} \Theta \Rightarrow \Delta'^{-}[\Delta]$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$\Gamma \Rightarrow \Delta'^{-}[\Delta] \bullet \Theta$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$\Delta'^{-}[\Delta] \mathbin{-\!\bullet} \Gamma \Rightarrow \Theta$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\Delta'^{-}[\,] \div (\Theta \mathbin{-\!\bullet} \Gamma) \Rightarrow \Delta$}
\doubleLine
\RightLabel{$IH(b)$}
\UnaryInfC{$\Delta'^{-}[\Delta] \Rightarrow \Theta \mathbin{-\!\bullet} \Gamma$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$\Theta \bullet \Delta'^{-}[\Delta] \Rightarrow \Gamma$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$\Theta \Rightarrow \Gamma \mathbin{\bullet\!-} \Delta'^{-}[\Delta]$}
\end{prooftree}
\qquad \Box
$$

In the next example, we derive the sequent disproving Cut elimination in our previous naïve sequent calculus for $\mathbf{LG}_{\varnothing}$.

Example 16. The following is a derivation of $A \bullet C \oslash ((A \backslash B) \obslash C) \Rightarrow B$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$B \Rightarrow B$}
\RightLabel{$\backslash L$}
\BinaryInfC{$A \backslash B \Rightarrow A \mathbin{-\!\bullet} B$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$C \Rightarrow C$}
\RightLabel{$\obslash R$}
\BinaryInfC{$(A \mathbin{-\!\bullet} B) \mathbin{-\!\bullet} C \Rightarrow (A \backslash B) \obslash C$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$C \Rightarrow (A \mathbin{-\!\bullet} B) \bullet (A \backslash B) \obslash C$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$C \mathbin{\bullet\!-} (A \backslash B) \obslash C \Rightarrow A \mathbin{-\!\bullet} B$}
\RightLabel{$\oslash L$}
\UnaryInfC{$C \oslash ((A \backslash B) \obslash C) \Rightarrow A \mathbin{-\!\bullet} B$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$A \bullet C \oslash ((A \backslash B) \obslash C) \Rightarrow B$}
\end{prooftree}
$$

Display calculi benefit of a general Cut elimination theorem, first demonstrated by Belnap [1982] and later refined by Wansing [1995] and by Dawson and Goré [2002]. Roughly, for the latter result to apply to the particular incarnation under consideration, one need only check a number of structural properties, as well as show the reducibility of *principal* Cuts to Cuts on the immediate subformulas. This strategy was applied by Goré, while Moortgat instead showed normalization through use of an intermediate term language based on Curien and Herbelin's [2000] $\bar{\lambda}\mu\tilde{\mu}$-calculus.

3.4.4 Nested sequents

The previous section posited a solution to the failure of Cut elimination by enriching the notion of structure, culminating in the display property. Witnessing this result was a small collection of invertible rules that would deconstruct the context of a formula step by step, removing one structural connective at a time. As a result, derivations quickly become cluttered by information of a non-logical nature, pertaining only to the structural properties of the logic under consideration.

Goré et al. [2008] have proposed *nested sequents* as a refinement of their display calculus presentation for bi-intuitionistic logic, being a dualization of intuitionistic logic by coimplications. While their research questions rather concern weakening and contraction, we nonetheless find that their methods may be adapted to **LG**, providing a more succinct alternative to its display calculus presentation. The concept of a sequent is reconsidered yet again, replacing structural (co)implications by the possibility of having sequents contain others, or, more precisely, contexts thereof.

Definition 36. Structures and their contexts are defined as follows:

$$\begin{array}{rcll} \Gamma, \Delta & ::= & A \mid (\Gamma \bullet \Delta) \mid (\Gamma[] \multimap \Delta) & \text{(Structures)} \\ \Gamma[], \Delta[] & ::= & [] \mid (\Gamma[] \bullet \Delta) \mid (\Gamma \bullet \Delta[]) & \text{(Contexts)} \end{array}$$

We refer to $\Gamma[] \multimap \Delta$ by a *nested* sequent, and to the concept of a structure thus defined as a *nested structure*. As a notational convention, we shall write it as a sequent context $\Gamma[] \Rightarrow \Delta$ ($\Delta \Rightarrow \Gamma[]$) when occurring in antecedent (succedent) position.[5]

Definition 37. Nested sequent derivations are defined as in F.3.8.

Compared to §4.3, nested sequents allow to abbreviate multiple display postulate steps using $([]R)$ or $([]L)$, both parameterizing over arbitrary contexts. Despite this renewed 'display property', we have chosen the standard formulation of logical inferences found in §4.1, better accentuating the contribution of the (un)nesting rules thereupon. Finally, notice how both mixed associativity and -commutativity for each of the types I and IV interactions are captured by a single rule.

Example 17. Consider again the counterexample to Cut admissibility of $\mathbf{LG}_{\emptyset}$. Using the (un)nesting rules, we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow A}\ Id \quad \overline{B \Rightarrow B}\ Id}{A \bullet A \backslash B \Rightarrow B}\ \backslash L}{A \backslash B \Rightarrow (A \bullet [] \Rightarrow B)}\ []R \quad \overline{C \Rightarrow C}\ Id}{C \Rightarrow (A \bullet [] \Rightarrow B) \bullet (A \backslash B) \otimes C}\ \otimes R}{C \oslash ((A \backslash B) \otimes C) \Rightarrow (A \bullet [] \Rightarrow B)}\ \oslash L}{A \bullet C \oslash ((A \backslash B) \otimes C) \Rightarrow B}\ []R$$

[5] Another proof formalism using (unfilled) contexts was developed by de Groote [1999] for **NL**, subsequently having been adapted to its classical counterpart by De Groote and Lamarche [2002]. De Groote's work, however, involves contexts of formulas rather than of structures.

Preorder laws

$$\frac{}{A \Rightarrow A}\ Id \qquad \frac{\Delta \Rightarrow A \quad \Gamma[A] \Rightarrow \Theta}{\Gamma[\Delta] \Rightarrow \Theta}\ \circ \qquad \frac{\Gamma \Rightarrow \Delta[A] \quad A \Rightarrow \Theta}{\Gamma \Rightarrow \Delta[\Theta]}\ \circ$$

(Un)nesting

$$\begin{array}{c}\Gamma[\Delta] \Rightarrow \Theta \\ \hline\hline \Delta \Rightarrow (\Gamma[] \Rightarrow \Theta)\end{array}\ []R \qquad \begin{array}{c}\Theta \Rightarrow \Gamma[\Delta] \\ \hline\hline (\Theta \Rightarrow \Gamma[]) \Rightarrow \Delta\end{array}\ []L$$

Logical rules (Residuated family)

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \bullet \Delta \Rightarrow A \otimes B}\ \otimes R \qquad \frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow \Theta}{\Gamma[(A/B \bullet \Delta)] \Rightarrow \Theta}\ /L \qquad \frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow \Theta}{\Gamma[(\Delta \bullet B\backslash A)] \Rightarrow \Theta}\ \backslash L$$

$$\frac{\Gamma[(A \bullet B)] \Rightarrow \Delta}{\Gamma[A \otimes B] \Rightarrow \Delta}\ \otimes L \qquad \frac{\Gamma \bullet B \Rightarrow A}{\Gamma \Rightarrow A/B}\ /R \qquad \frac{B \bullet \Gamma \Rightarrow A}{\Gamma \Rightarrow B\backslash A}\ \backslash R$$

Logical rules (Coresiduated family)

$$\frac{A \Rightarrow \Gamma \quad B \Rightarrow \Delta}{A \oplus B \Rightarrow \Gamma \bullet \Delta}\ \oplus L \qquad \frac{B \Rightarrow \Delta \quad \Theta \Rightarrow \Gamma[A]}{\Theta \Rightarrow \Gamma[(\Delta \bullet B \obslash A)]}\ \obslash R \qquad \frac{B \Rightarrow \Delta \quad \Theta \Rightarrow \Gamma[A]}{\Theta \Rightarrow \Gamma[(A \oslash B \bullet \Delta)]}\ \oslash R$$

$$\frac{\Gamma \Rightarrow \Delta[(A \bullet B)]}{\Gamma \Rightarrow \Delta[A \oplus B]}\ \oplus R \qquad \frac{A \Rightarrow B \bullet \Gamma}{B \obslash A \Rightarrow \Gamma}\ \obslash L \qquad \frac{A \Rightarrow \Gamma \bullet B}{A \oslash B \Rightarrow \Gamma}\ \oslash L$$

Grishin interactions (Type I) Grishin interactions (Type IV)

$$\frac{(\Delta \Rightarrow \Theta[]) \Rightarrow (\Gamma[] \Rightarrow \Xi)}{\Gamma[\Delta] \Rightarrow \Theta[\Xi]}\ I \qquad \frac{\Gamma[\Delta] \Rightarrow \Theta[\Xi]}{(\Delta \Rightarrow \Theta[]) \Rightarrow (\Gamma[] \Rightarrow \Xi)}\ IV$$

Figure 3.8: Nested sequent presentation of **LG**.

To show Cut admissibility, we translate derivations in Cut-free display calculus, having already been shown complete, to their Cut-free nested sequent counterparts.

Definition 38. We map display structures Γ to their nested counterparts Γ^d_n:

$$\begin{array}{rlrl} A^d_n &:= A & (\Gamma \multimapdotinv \Delta)^d_n &:= ([] \bullet \Delta^d_n) \multimapdot \Gamma^d_n \\ (\Gamma \bullet \Delta)^d_n &:= \Gamma^d_n \bullet \Delta^d_n & (\Delta \multimapdot \Gamma)^d_n &:= (\Delta^d_n \bullet []) \multimapdot \Gamma^d_n \end{array}$$

$$\frac{\Gamma \Rightarrow \Theta \leftharpoondown \Delta}{\Gamma \bullet \Delta \Rightarrow \Theta}\, dp \quad \rightarrow \quad \frac{\Gamma^d_n \Rightarrow ([\,] \bullet \Delta^d_n \Rightarrow \Theta^d_n)}{\Gamma^d_n \bullet \Delta^d_n \Rightarrow \Theta^d_n}\, [\,]L$$

$$\frac{\Delta \Rightarrow B \quad A \Rightarrow \Gamma}{A/B \Rightarrow \Gamma \leftharpoondown \Delta}\, /L \quad \rightarrow \quad \frac{\dfrac{\dfrac{}{\Delta^d_n \Rightarrow B}\, IH \quad \dfrac{}{\Gamma^d_n \Rightarrow A}\, IH}{A/B \bullet \Delta^d_n \Rightarrow \Gamma^d_n}\, /L}{A/B \Rightarrow ([\,] \bullet \Delta^d_n \Rightarrow \Gamma^d_n)}\, [\,]L$$

$$\frac{\Gamma \Rightarrow A \leftharpoondown B}{\Gamma \Rightarrow A/B}\, /R \quad \rightarrow \quad \frac{\dfrac{\Gamma^d_n \Rightarrow ([\,] \bullet B \Rightarrow A)}{\Gamma^d_n \bullet B \Rightarrow A}\, [\,]L}{\Gamma^d_n \Rightarrow A/B}\, /R$$

$$\frac{\Gamma_2 \leftharpoondown \Delta_2 \Rightarrow \Gamma_1 \rightharpoondown \Delta_1}{\Gamma_1 \bullet \Gamma_2 \Rightarrow \Delta_1 \bullet \Delta_2}\, A^1_I \quad \rightarrow \quad \frac{({\Gamma_2}^d_n \Rightarrow [\,] \bullet {\Delta_1}^d_n) \Rightarrow ({\Gamma_1}^d_n \bullet [\,] \Rightarrow {\Delta_1}^d_n)}{{\Gamma_1}^d_n \bullet {\Gamma_2}^d_n \Rightarrow {\Delta_1}^d_n \bullet {\Delta_2}^d_n}\, I$$

Figure 3.9: From display sequents to nested counterparts.

Theorem 3.4.3. $\Gamma \Rightarrow \Delta$ in display calculus implies $\Gamma^d_n \Rightarrow \Delta^d_n$ in nested sequent calculus, implying the latter's completeness.

Proof. By induction. F.3.9 presents some typical cases. □

The opposite direction proceeds similarly, first comparing the two concepts of sequents. Roughly, nested sequents translate to structural coimplications via (co)residuation.

Definition 39. We associate with each nested structure Γ a display structure Γ^n_d, in so doing employing an auxiliary map $(\cdot,\cdot)^n_d$ taking a (nested) context and display structure as arguments and being defined by induction on the former.

$$\begin{array}{rlrl} A^n_d & := A & ([\,],\Theta)^n_d & := \Theta \\ (\Gamma \bullet \Delta)^n_d & := \Gamma^n_d \bullet \Delta^n_d & (\Gamma[\,] \bullet \Delta, \Theta)^n_d & := (\Gamma[\,], \Theta^n_d \leftharpoondown \Delta^n_d)^n_d \\ (\Gamma[\,] \rightharpoondown \Delta)^n_d & := (\Gamma[\,], \Delta^n_d)^n_d & (\Gamma \bullet \Delta[\,], \Theta)^n_d & := (\Delta[\,], \Gamma^n_d \rightharpoondown \Theta^n_d)^n_d \end{array}$$

We next tackle soundness of the (un)nesting rules and the Grishin interactions.

Lemma 10. For any nested structures Δ, Θ and context $\Gamma[\,]$, $\Gamma[\Delta]^n_d \Rightarrow \Theta^n_d$ iff $\Delta^n_d \Rightarrow (\Gamma[\,], \Theta^n_d)^n_d$, as well as $\Theta^n_d \Rightarrow \Gamma[\Delta]^n_d$ iff $(\Gamma[\,], \Theta^n_d)^n_d \Rightarrow \Delta^n_d$.

Proof. By induction on $\Gamma[]$. As a typical case, consider $\Gamma[] = \Lambda \bullet \Xi[]$, so that $(\Gamma[], \Theta^n_d)^n_d = (\Xi[], \Lambda^n_d \rightarrow \Theta^n_d)^n_d$. We have

$$\dfrac{\dfrac{\Delta^n_d \Rightarrow (\Xi[], \Lambda^n_d \rightarrow \Theta^n_d)^n_d}{\Xi[\Delta]^n_d \Rightarrow \Lambda^n_d \rightarrow \Theta^n_d}\,IH}{\Lambda^n_d \bullet \Xi[\Delta]^n_d \Rightarrow \Theta^n_d}\,dp \qquad \dfrac{\dfrac{(\Xi[], \Lambda^n_d \rightarrow \Theta^n_d)^n_d \Rightarrow \Delta^n_d}{\Lambda^n_d \rightarrow \Theta^n_d \Rightarrow \Xi[\Delta]^n_d}\,IH}{\Theta^n_d \Rightarrow \Lambda^n_d \bullet \Xi[\Delta]^n_d}\,dp$$ □

Prior to demonstrating the soundness of the Grishin interactions, we first prove

Lemma 11. For any nested (display) structures Γ, Δ (Θ) and context $\Lambda[]$, we have the following results in the presence of type I interactions, their converses being valid in the presence of those of type IV.

$$\dfrac{\Lambda[(\Delta \Rightarrow \Gamma \bullet [])]^n_d \Rightarrow \Theta}{\Gamma^n_d \rightarrow \Lambda[\Delta]^n_d \Rightarrow \Theta} \qquad \dfrac{\Lambda[(\Delta \Rightarrow [] \bullet \Gamma)]^n_d \Rightarrow \Theta}{\Lambda[\Delta]^n_d \leftarrow \Gamma^n_d \Rightarrow \Theta}$$

Proof. By induction on $\Lambda[]$. As a typical case, we check the left inference in the type I direction for $\Lambda[] = \Lambda_1 \bullet \Lambda_2[]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{{\Lambda_1}^n_d \bullet \Lambda_2[(\Delta \Rightarrow \Gamma \bullet [])]^n_d \Rightarrow \Theta}{\Lambda_2[(\Delta \Rightarrow \Gamma \bullet [])]^n_d \Rightarrow {\Lambda_1}^n_d \rightarrow \Theta}\,dp}{\Gamma^n_d \rightarrow_{\Rightarrow} \Lambda_2[\Delta]^n_d {\Lambda_1}^n_d \rightarrow \Theta}\,IH}{{\Lambda_1}^n_d \bullet \Lambda_2[\Delta]^n_d \Rightarrow \Gamma^n_d \bullet \Theta}\,C^1_I}{\Gamma^n_d \rightarrow ({\Lambda_1}^n_d \bullet \Lambda_2[\Delta]^n_d) \Rightarrow \Theta}\,dp$$ □

Lemma 12. For any nested structures Δ, Ξ and contexts $\Gamma[], \Theta[]$, $(\Theta[], \Delta^n_d)^n_d \Rightarrow (\Gamma[], \Xi^n_d)^n_d$ implies $\Gamma[\Delta]^n_d \Rightarrow \Theta[\Xi]^n_d$ in the presence of type I interactions, with the converse implication valid for those of type IV.

Proof. By induction on the pair $(\Gamma[], \Theta[])$. Note that the two sides of the implications coincide with identity (resp. (un)nesting rules) if both (resp., exactly one of) $\Gamma[], \Delta[]$ coincide with $[]$. Of the four remaining inductive cases, consider as a typical instance the situation $\Gamma[] = \Gamma_1[] \bullet \Gamma_2$ and $\Theta[] = \Theta_1 \bullet \Theta_2[]$. Then $(\Theta[], \Delta^n_d)^n_d = (\Theta_2[], {\Theta_1}^n_d \rightarrow \Delta^n_d)^n_d = (\Theta_2[], (\Delta \Rightarrow \Theta_1 \bullet [])^n_d)^n_d$, while similarly $(\Gamma[], \Xi^n_d)^n_d = (\Gamma_1[], ([] \bullet \Gamma_2 \Rightarrow \Xi)^n_d)^n_d$. We check the direction for type I, that of type IV following similarly.

$$\dfrac{\dfrac{\dfrac{(\Theta_2[], (\Delta \Rightarrow \Theta_1 \bullet [])^n_d)^n_d \Rightarrow (\Gamma_1[], ([] \bullet \Gamma_2 \Rightarrow \Xi)^n_d)^n_d}{\Gamma_1[(\Delta \Rightarrow \Theta_1 \bullet [])]^n_d \Rightarrow \Theta_2[([] \bullet \Gamma_2 \Rightarrow \Xi)]^n_d}\,IH}{{\Theta_1}^n_d \rightarrow \Gamma_1[\Delta]^n_d \Rightarrow \Theta_2[\Xi]^n_d \leftarrow {\Gamma_2}^n_d}\,L.11 \times 2}{\Gamma_1[\Delta]^n_d \bullet {\Gamma_2}^n_d \Rightarrow {\Theta_1}^n_d \bullet \Theta_2[\Xi]^n_d}\,A^2_I$$ □

Theorem 3.4.4. If $\Gamma \Rightarrow \Delta$ in nested sequent calculus, then also $\Gamma^n_d \Rightarrow \Delta^n_d$.

Proof. First, note that the following nested sequent rules

$$\frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow \Theta}{\Gamma[(A/B \bullet \Delta)] \Rightarrow \Theta} /L \qquad \frac{\Delta \Rightarrow B \quad \Gamma[A] \Rightarrow \Theta}{\Gamma[(\Delta \bullet B\backslash A)] \Rightarrow \Theta} \backslash L \qquad \frac{\Gamma[(A \bullet B)] \Rightarrow \Delta}{\Gamma[A \otimes B] \Rightarrow \Delta} \otimes L$$

$$\frac{B \Rightarrow \Delta \quad \Theta \Rightarrow \Gamma[A]}{\Theta \Rightarrow \Gamma[(\Delta \bullet B \obslash A)]} \obslash R \qquad \frac{B \Rightarrow \Delta \quad \Theta \Rightarrow \Gamma[A]}{\Theta \Rightarrow \Gamma[(A \oslash B \bullet \Delta)]} \oslash R \qquad \frac{\Gamma \Rightarrow \Delta[(A \bullet B)]}{\Gamma \Rightarrow \Delta[A \oplus B]} \oplus R$$

are equivalent, provability-wise, with the following in the presence of (un)nesting.

$$\frac{\Delta \Rightarrow B \quad A \Rightarrow \Gamma}{A/B \Rightarrow ([\,] \bullet \Delta \Rightarrow \Gamma)} /L' \qquad \frac{\Delta \Rightarrow B \quad A \Rightarrow \Gamma}{B\backslash A \Rightarrow (\Delta \bullet [\,] \Rightarrow \Gamma)} \backslash L' \qquad \frac{A \bullet B \Rightarrow \Gamma}{A \otimes B \Rightarrow \Gamma} \otimes L'$$

$$\frac{B \Rightarrow \Delta \quad \Gamma \Rightarrow A}{(\Gamma \Rightarrow \Delta \bullet [\,]) \Rightarrow B \obslash A} \obslash R' \qquad \frac{B \Rightarrow \Delta \quad \Gamma \Rightarrow \Gamma[A]}{(\Gamma \Rightarrow [\,] \bullet \Delta) \Rightarrow A \oslash B} \oslash R' \qquad \frac{\Gamma \Rightarrow A \bullet B}{\Gamma \Rightarrow A \oplus B} \oplus R'$$

For example, $(/L)$ and $(/L')$ are shown interderivable as follows:

$$\dfrac{\dfrac{\Delta \Rightarrow B \quad A \Rightarrow \Gamma}{A/B \bullet \Delta \Rightarrow \Gamma} /L}{A/B \Rightarrow ([\,] \bullet \Delta \Rightarrow \Gamma)} [\,]L \qquad \dfrac{\dfrac{\dfrac{\Delta \Rightarrow B \quad \dfrac{\Gamma[A] \Rightarrow \Theta}{A \Rightarrow (\Gamma[\,] \Rightarrow \Theta)} [\,]L}{A/B \Rightarrow ([\,] \bullet \Delta \Rightarrow (\Gamma[\,] \Rightarrow \Theta))} /L'}{A/B \bullet \Delta \Rightarrow (\Gamma[\,] \Rightarrow \Theta)} [\,]L}{\Gamma[(A/B \bullet \Delta)] \Rightarrow \Theta} [\,]L$$

Using this alternative presentation, the desired result follows by a straightforward induction, using previous results for dealing with structural rules. □

3.5 Open problems and competing proposals

We conclude with an overview of some of the open problems concerning **LG** (§5.1), as well as with a brief survey of some of the major competing proposals for extending (**N**)**L**'s expressivity. While all retain the intuitionistic bias underlying Lambek's original work, their strength derives through either one of the following methods:

1. **Multimodal Lambek calculus**. Distinguish between multiple copies of the residuated family $\{\otimes, /, \backslash\}$, and postulate associativity and commutativity principles mixing different copies of the tensor. (§5.2)
2. **Unary control**. Keep to a single residuated family, but add unary operators for controlling associativity and commutativity, similar to the use of exponents in linear logic for licensing weakening and contraction. (§5.3)
3. **Discontinuous Lambek calculus**. Again, keep to a single residuated family, but add logical constants for extraction and infixation. (§5.4)

3.5.1 Open problems

Below, we list several of the open problems concerning **LG**, as well as brief allusions to our contributions, as found in later chapters, where applicable.

1. **Expressivity**. When combined with type IV interactions, **LG** was shown by Moot [2007] to embed Lexicalized Tree Adjoining Grammars, although Melissen [2011] proved that its lower bound lies beyond. An upper bound, on the other hand, is unknown, while investigations into the expressivity of $\mathbf{LG}_\emptyset$ and $\mathbf{LG}_I$ are nonexistent. As a modest contribution, we will show in the next chapter that $\mathbf{LG}_\emptyset$ is context-free. Our method of proof proceeds via the interpolation method pioneered by Pentus [1999] for **L**, requiring a Cut-free sequent formalism. Said need is satisfied through the definition of a labeled sequent calculus, incorporating information from relational models.
2. **Complexity**. Derivability in $\mathbf{LG}_\emptyset$ has been shown to be decidable in polynomial time by Capelletti [2005], while Bransen [2012, 2011] proved $\mathbf{LG}_{IV}$ is NP-complete. Both results, however, concern the unit-free multiplicative fragment, the consequences for complexity of extending with units or additives currently unknown. Moreover, $\mathbf{LG}_I$ has been left uninvestigated, although its unit-free multiplicative fragment is again assumed to be NP-complete.
3. **Compositional semantics**. The previous chapter described a compositional Montagovian semantics for **NL**, emphasizing the close correspondence with natural deduction and normalization therein. On the other hand, attempts at extending natural deduction to **LG** (or sequent calculus, for that matter) have failed. Thus, it remains to conceive of a formalism for representing **LG** derivability that manifests the same close correspondence with a Montagovian semantics as found with natural deduction in **NL**. Thus far, Bernardi and Moortgat [2010] have come closest with their proposal for a mapping taking

the derivations of a display calculus presentation into intuitionistic multiplicative linear logic. However, theirs is such that a derivation without Cut may be mapped into a term that is not in normal form and as such does not yet express the desired correspondence.[6] We return to this issue in chapter 7, where we replace the target language with a logic expressing the same degree of resource sensitivity as found in the source.

3.5.2 Multimodal Lambek calculus

Ordinary (non-associative) Lambek calculus consists of a single residuated family $\{\otimes, /, \backslash\}$. Its multimodal extension, described at length by Moortgat [1995], rather allows for several copies $\{\otimes_i, /_i, \backslash_i\}$, obtained relative to a set I of indices i. Inference rules are adapted accordingly, with structures built using several instances $\bullet_i$ of merger for each $i \in I$. Restricting to natural deduction, we have:

$$\frac{}{A \vdash A}\ Id$$

$$\frac{\Gamma \vdash A/B \quad \Delta \vdash B}{\Gamma \bullet_i \Delta \vdash A}\ /_iE \qquad \frac{\Gamma \bullet_i B \vdash A}{\Gamma \vdash A/B}\ /_iI$$

$$\frac{\Delta \vdash B \quad \Gamma \vdash B\backslash A}{\Delta \bullet_i \Gamma \vdash A}\ \backslash_iE \qquad \frac{B \bullet_i \Gamma \vdash A}{\Gamma \vdash B\backslash A}\ \backslash_iI$$

$$\frac{\Delta \vdash A \otimes B \quad \Gamma[A \bullet_i B] \vdash C}{\Gamma[\Delta] \vdash C}\ \otimes_iE \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \bullet_i \Delta \vdash A \otimes B}\ \otimes_iI$$

At this point, no increase in expressivity is found. Rather, the strength of this approach lies in postulating variants of associativity and commutativity mixing different copies of the tensor, not too dissimilar from how **LG**'s interactions mix connectives from the residuated and coresiduated families. Thus, as an example, taken from the aforecited [Moortgat, 1995], we have the case where $I = \{i, j\}$ and the above inference rules are extended by

$$\frac{\Gamma[((\Delta_1 \bullet_i \Delta_2) \bullet_j \Delta_3)] \vdash A}{\Gamma[(\Delta_1 \bullet_j (\Delta_2 \bullet_i \Delta_3))] \vdash A}\ MA \qquad \frac{\Gamma[(\Delta_2 \bullet_j (\Delta_1 \bullet_i \Delta_3))] \vdash A}{\Gamma[(\Delta_1 \bullet_j (\Delta_2 \bullet_i \Delta_3))] \vdash A}\ MC$$

[6] To be fair, Bernardi and Moortgat [2010] have introduced a variation on Curien and Herbelin's [2000] $\bar{\lambda}\mu\widetilde{\mu}$-calculus as an intermediate step. The latter's purpose, however, was to highlight the causes for the nondeterminism of normalization in classical logic; a property that was inherited by Bernardi and Moortgat's adaptation.

Other types of structural rules naturally suggest themselves. For instance, we may have a fully associative and/or commutative tensor coexist with a non-associative and non-commutative one, or we may postulate inclusion principles, switching the index of $\bullet_i$ with some j.

As noted, the current approach bears resemblance to **LG**'s mixing of connectives from various families. Perhaps closest is the axiomatization of the type I interactions found in F.3.3. Repeated, for convenience:

$$
\begin{array}{ll}
(A \oplus B) \otimes C \leq A \oplus (B \otimes C) & A \otimes (B \oplus C) \leq (A \otimes B) \oplus C \\
A \otimes (B \oplus C) \leq B \oplus (A \otimes C) & (A \oplus B) \otimes C \leq (A \otimes C) \oplus B
\end{array}
$$

To better facilitate comparison, we list the axioms for (MA) and (MC):

$$
\begin{array}{ll}
(A \otimes_i B) \otimes_j C \leq A \otimes_j (B \otimes_i C) & A \otimes_j (B \otimes_i C) \leq (A \otimes_i B) \otimes_j C \\
A \otimes_j (B \otimes_i C) \leq B \otimes_j (A \otimes_i C) & B \otimes_j (A \otimes_i C) \leq A \otimes_j (B \otimes_i C)
\end{array}
$$

3.5.3 Unary control

A different approach to improving upon (**N**)**L**'s expressivity originates in the work of Morrill [1990], taking a hint from linear logic's treatment of contraction and weakening by adding a unary modality for controlling associativity and commutativity. We here briefly describe a more recent incarnation, introduced by Kurtonina and Moortgat [1995] and applied in a cross-linguistic investigation of wh-question formation by Vermaat [2005]. It proceeds from the observation that the residuation schema generalizes to operations of arbitrary arity n. In the particular case of $n = 1$, we may conceive of operators $\diamond$ (*diamond*) and $\Box$ (*box*), which, for expository purposes, we may think of as abbreviating, for some fixed B,

$$
\diamond A \;=\; B \otimes A \qquad \Box A \;=\; B\backslash A
$$

Thus, to obtain the corresponding inference rules, we can take those for the binary multiplicatives, and simply remove one of the arguments. Assuming a unary structural operator $\langle\cdot\rangle$, acting as the counterpart of $\diamond$, we obtain

$$
\frac{\Gamma \vdash \Box A}{\langle\Gamma\rangle \vdash A}\ \Box E \qquad\qquad \frac{\langle\Gamma\rangle \vdash A}{\Gamma \vdash \Box A}\ \Box I
$$

$$
\frac{\Delta \vdash \diamond A \quad \Gamma[\langle A\rangle] \vdash B}{\Gamma[\Delta] \vdash B}\ \diamond E \qquad\qquad \frac{\Gamma \vdash A}{\langle\Gamma\rangle \vdash \diamond A}\ \diamond I
$$

Associativity and commutativity may now be reinstated, though kept in check through the use of $\Diamond$. For example, the following rules are claimed by Vermaat [2005] to suffice for capturing the full range of cross-linguistic variation encountered in the wild, using a unary structural operator $\langle\cdot\rangle$, acting as the counterpart of $\Diamond$:

$$\frac{\Gamma[(\Delta_1 \bullet (\Delta_2 \bullet \langle\Delta_3\rangle))] \vdash C}{\Gamma[((\Delta_1 \bullet \Delta_2) \bullet \langle\Delta_3\rangle)] \vdash C}\, A_1 \qquad \frac{\Gamma[((\langle\Delta_1\rangle \bullet \Delta_2) \bullet \Delta_3)] \vdash C}{\Gamma[(\langle\Delta_1\rangle \bullet (\Delta_2 \bullet \Delta_3))] \vdash C}\, A_2$$

$$\frac{\Gamma[((\Delta_1 \bullet \langle\Delta_2\rangle) \bullet \Delta_3)] \vdash C}{\Gamma[((\Delta_1 \bullet \Delta_3) \bullet \langle\Delta_2\rangle)] \vdash C}\, C_1 \qquad \frac{\Gamma[(\Delta_1 \bullet (\langle\Delta_2\rangle \bullet \Delta_3))] \vdash C}{\Gamma[(\langle\Delta_2\rangle \bullet (\Delta_1 \bullet \Delta_3))] \vdash C}\, C_2$$

While centering on $\Diamond$, its counterpart $\Box$ is not without use. Particularly, we have $\langle\Box A\rangle \vdash A$. In other words, reasoning bottom-up, a formula A within some context $\Gamma[]$ may at any time be changed into $\langle\Box A\rangle$, thus readying it for extraction or infixation using the above structural rules.[7]

The comparison with **LG** is again best seen at the level of algebraic derivability. Expressed by two-formula sequents, the above rules are rendered as follows, where $\simeq$ abbreviates the conjunction of $\leq$ and $\geq$:

$$\begin{array}{ll} \Diamond A \otimes (B \otimes C) \simeq (\Diamond A \otimes B) \otimes C & (A \otimes B) \otimes \Diamond C \simeq A \otimes (B \otimes \Diamond C) \\ \Diamond A \otimes (B \otimes C) \simeq B \otimes (\Diamond A \otimes C) & (A \otimes B) \otimes \Diamond C \simeq (A \otimes \Diamond C) \otimes B \end{array}$$

To compare, the following are derivable using type I interactions, while reversing the inequality sign suffices for obtaining derivabilities through use of type IV:

$$\begin{array}{ll} A \otimes (B \otimes C) \leq (A \otimes B) \otimes C & (A \otimes B) \oslash C \leq A \otimes (B \oslash C) \\ A \otimes (B \otimes C) \leq B \otimes (A \otimes C) & (A \otimes B) \oslash C \leq (A \oslash C) \otimes B \end{array}$$

3.5.4 Discontinuous Lambek calculus

A more recent entry in the CTL family is the *discontinuous Lambek calculus* (DLC, for short; also referred to by the *displacement calculus*) of Morrill et al. [2011], although certain of the underlying ideas go back to Moortgat [1988]. Its definition rests upon

[7] Kurtonina and Moortgat [2010] employ an analogy involving $\Box$ serving as the key to opening the $\langle\cdot\rangle$-lock surrounding A.

a reimaging of the notion of structure based upon the addition of 'separators', understood roughly as marking the locations of gaps. A formal exposition would take us too far afield, so we suffice by an intuitive explanation.

The efforts of Morrill and associates concern the associative Lambek calculus, so connectives receive an explanation in terms of strings: $(A \otimes B)$ is interpreted by concatenation of strings in A and B, while (A/B) and $(B\backslash A)$ concatenate with B to their right and left respectively into a string of A. DLC adds a new residuated family of connectives, their explanation adapted from their traditional brethren by replacing the local operation of concatenation with wrapping. Thus, the discontinuous product $(A \odot B)$ wraps strings in A around those in B, while $(B \downarrow A)$ and $(A \uparrow B)$ act as infixes and circumfixes for strings inside B respectively. The formal definition employs the aforementioned separators, and distinguishes among several variants of DLC according to whether or not wrapping proceeds deterministically.

DLC has enjoyed application to a wide range of linguistic phenomena, among which may be found quantifier scope ambiguities, pied piping, ellipsis and discontinuous idioms. An implementation inside PROLOG has been proposed recently by Morrill [2011], adapting a parsing methodology for the associative Lambek calculus described independently in papers by Hepple [1990] and Hendriks [1993].

4

Towards classicality

4.1 Introduction

One motivation for introducing **LG** concerned the asymmetry inherent in traditional (non-associative) Lambek calculus. While derivations typically involved the conjunction of multiple hypotheses, only one conclusion, and one only, was allowed. By postulating a par connective $\oplus$ dual to the tensor $\otimes$, **LG** allowed for derivations to involve the (multiplicative) disjunction of multiple conclusions.

Predating **LG** in its aim at restoring symmetry to the Lambek calculus were the non-commutative versions of classical linear logic proposed by Lambek, Abrusci and Yetter [see Abrusci, 2002], the main difference found in their treatments of arrow-reversal. Within **LG**, $\cdot^{\infty}$ is not fixpoint-free, in that there exist formulas A for which $A^{\infty} = A$, to wit the identification $A = p$ for any atom p. In contrast, classicality pairs atoms p with their negations $\bar{p}$, thus making $\cdot^{\infty}$ into a classical negation $\cdot^{\perp}$, with $p^{\perp} = \bar{p}$ and $\bar{p}^{\perp} = p$. As a consequence, (co)implications need no longer be considered primitive, with, e.g., A/B defined as $A \oplus B^{\perp}$, and, dually $B \oslash A := B^{\perp} \otimes A$.

The first 'classical' Lambek calculi were proposed in the presence of same-sort associativity. Lambek's [1993] *bilinear logic*, discovered independently by Abrusci

[1991], built forth on the same paper by Grishin that inspired **LG**, though with different results. Compared to linear logic, it featured both a post- and retro-negation, later written $\cdot^r$ and $\cdot^l$ in [Lambek, 1997], with $A^{rl} = A = A^{lr}$, but not in general $A^{rr} = A$ or $A^{ll} = A$. Yetter's [1990] *cyclic linear logic* collapsed $\cdot^r$ and $\cdot^l$ into a single $\cdot^\perp$, necessitating the use of a structural rule for cycling through one's list of formulas.

With the new millennium, non-associativity finally followed suit through De Groote and Lamarche's [2002] *classical non-associative Lambek calculus*, or **CNL**, for short. Like **CyLL**, it uses but one negation, compensating with structural rules allowing to search through one's tree of formulas by turning it 'inside out'. In this chapter, we reconstruct **CNL** from the augmentation of $\mathbf{LG}_\varnothing$ by units and the definition of minimal (co)negations in terms thereof. As an intermediate step, we obtain non-associative bilinear logic, i.e., using both a post- and retro-negation, already alluded to in De Groote and Lamarche's [2002] introduction and having been described before using display calculus by Goré [1998, §5]. We discuss the respective sequent presentations, reconstructing De Groote and Lamarche's one-sided sequents from Belnap's [1990] display calculus for classical linear logic. Cut elimination is proven by model-theoretic means, inspired by Okada [2002].

We proceed as follows. §2 recapitulates, within a non-associative setting, earlier observations by Lambek and Grishin, defining minimal (co)negations in terms of units 1 and 0 for the tensor and par. using the Grishin interactions, the defining properties of a classical linear negation may be derived. Since both units as well as the presence of type I ánd IV interactions lead to overgeneration, we reintroduce in §3 our (co)negations as primitives, adapting Grishin's interactions accordingly. In contrast with similar efforts by Moortgat [2010], both type I and IV interactions are used in their associative restrictions, allowing for the (relevant) results of §2 to carry over straightforwardly.[1] Sequent presentations are discussed in §3, drawing from the aforecited works of De Groote and Lamarche, of Abrusci and of Belnap.

4.2 Classicality through units

The current section defines classical linear negation(s) by combining units 0 and 1 for the tensor and par together with the Grishin interactions. Similar investigations were previously conducted by both Lambek [1993] and Grishin [1983], though in both cases assuming associativity, while Goré [1998] subsequently reproved their

[1] Alternatively, much of the current exercise may be reproduced using only the commutative restrictions of the type I and IV interactions Hudelmaier and Schroeder-Heister [cf. 1995].

results in the absence thereof. We shall find, however, that both type I and IV interactions are needed in order to obtain classicality, a combination we have already observed lethal for our adherence to non-associativity and -commutativity. The current section therefore largely serves an expository purpose, with the next identifying the specific environments wherein the interactions are needed to operate and quarantining these inside new primitive connectives replacing the units, a presentation format previously employed by Moortgat [2010], though with a different outcome.

Skipping the obvious revision to the definition of formulas, we turn straight to the relevant inference rules, using algebraic derivations.

Definition 40. In the presence of units, the definition of algebraic derivations is extended with the following *push-* (read downward) and *pop* rules (upward):

$$\frac{A \leq B}{\overline{1 \otimes A \leq B}}\ 1 \qquad \frac{A \leq B}{\overline{A \otimes 1 \leq B}}\ 1 \qquad \frac{A \leq B}{\overline{A \leq B \oplus 0}}\ 0 \qquad \frac{A \leq B}{\overline{A \leq 0 \oplus B}}\ 0$$

Example 18. An oft-encountered argument against the linguistic relevance of units runs as follows. Traditionally, adjectives are categorized n/n, foreseeing in the derivability of, say, 'pink elephant' as a noun (i.e., $(n/n) \otimes n \leq n$). We can modify said adjective, say by 'mostly', by categorizing the latter $(n/n)/(n/n)$, deriving 'mostly pink elephant' as, again, a noun $((((n/n)/(n/n)) \otimes (n/n)) \otimes n \leq n)$. But now we must wrongfully conclude that 'mostly elephant' is a noun as well, given that $1 \leq n/n$ implies $((n/n)/(n/n)) \otimes n \leq n$. So either units were a mistake, or the categorization of adjectival modifiers by $(n/n)/(n/n)$ was.

Definition 41. In the presence of units, we may define four negations ${}^0A := 0/A$, $A^0 := A\backslash 0$, ${}^1A := 1 \oslash A$ and $A^1 := A \obslash 1$. We assume these to bind more strongly than the tensor and par, so that, e.g., ${}^0A \otimes B$ abbreviates $({}^0A) \otimes B$.

We first establish some negation-like properties within the minimal base logic.

Lemma 13. The following are derivable in $\mathbf{LG}_\varnothing$ plus units.

(i) $A \leq {}^0(A^0)$ and $A \leq ({}^0A)^0$ (Lifting)

(ii) $({}^1A)^1 \leq A$ and ${}^1(A^1) \leq A$ (Lowering)

(iii) $A \leq B$ implies ${}^0B \leq {}^0A$ and $B^0 \leq A^0$ (Monotonicity)

(iv) $A \leq B$ implies $B^1 \leq A^1$ and ${}^1B \leq {}^1A$ (Monotonicity)

(v) $A \leq {}^0B$ iff $B \leq A^0$ (Galois)

(vi) $A^1 \leq B$ iff ${}^1B \leq A$ (co-Galois)

Proof. Properties (i)-(iv) are immediate from earlier demonstrations of monotonicity, lifting and lowering, while (v) and (vi) abbreviate two (co)residuation steps. □

Combined with transitivity, (v) and (vi) suffice to derive (i)-(iv) when the negations are adopted as primitives, as shown by Areces et al. [2001], Moortgat [2010]. Note, however, that for no composition $\cdot^\sharp$ of the negations do we obtain a dualizing object, i.e., where both $A \leq A^\sharp$ and $A^\sharp \leq A$, nor can we derive any of the De Morgan laws. To do so requires both type I and IV interactions, as shown below. First, however, we consider the effects of the Grishin interactions on our newfound (co)negations.

Lemma 14. Using the mixed associative type I and IV interactions, we can derive

$$
\begin{array}{rcll@{\qquad}rcll}
{}^1A \leq B^0 & \Rightarrow & B \leq A & (A_I^{1\oslash,\backslash 0}) & B \leq A & \Rightarrow & {}^1A \leq B^0 & (A_{IV}^{1\oslash,\backslash 0}) \\
A^1 \leq {}^0B & \Rightarrow & B \leq A & (A_I^{\obslash 1,0/}) & B \leq A & \Rightarrow & A^1 \leq {}^0B & (A_{IV}^{\obslash 1,0/}) \\
{}^1A \leq B\backslash C & \Rightarrow & B \leq C \oplus A & (A_I^{1\oslash,\backslash}) & B \leq C \oplus A & \Rightarrow & {}^1A \leq B\backslash C & (A_{IV}^{1\oslash,\backslash}) \\
A^1 \leq B/C & \Rightarrow & C \leq A \oplus B & (A_I^{\obslash 1,/}) & C \leq A \oplus B & \Rightarrow & A^1 \leq B/C & (A_{IV}^{\obslash 1,/}) \\
A \oslash B \leq C^0 & \Rightarrow & C \otimes A \leq B & (A_I^{\oslash,\backslash 0}) & C \otimes A \leq B & \Rightarrow & A \oslash B \leq C^0 & (A_{IV}^{\oslash,\backslash 0}) \\
A \obslash B \leq {}^0C & \Rightarrow & B \otimes C \leq A & (A_I^{\obslash,0/}) & B \otimes C \leq A & \Rightarrow & A \obslash B \leq {}^0C & (A_{IV}^{\obslash,0/})
\end{array}
$$

Whereas for the commutative restrictions, we have,

$$
\begin{array}{rcll@{\qquad}rcll}
{}^1A \leq {}^0B & \Rightarrow & B \leq A & (C_I^{1\oslash,0/}) & B \leq A & \Rightarrow & {}^1A \leq {}^0B & (C_{IV}^{1\oslash,0/}) \\
A^1 \leq B^0 & \Rightarrow & B \leq A & (C_I^{\obslash 1,\backslash 0}) & B \leq A & \Rightarrow & A^1 \leq B^0 & (C_{IV}^{\obslash 1,\backslash 0}) \\
A^1 \leq B\backslash C & \Rightarrow & B \leq A \oplus C & (C_I^{\obslash 1,\backslash}) & B \leq A \oplus C & \Rightarrow & A^1 \leq B\backslash C & (C_{IV}^{\obslash 1,\backslash}) \\
{}^1A \leq B/C & \Rightarrow & C \leq B \oplus A & (C_I^{1\oslash,/}) & C \leq B \oplus A & \Rightarrow & {}^1A \leq B/C & (C_{IV}^{1\oslash,/}) \\
A \oslash B \leq {}^0C & \Rightarrow & A \otimes C \leq B & (C_I^{\oslash,0/}) & A \otimes C \leq B & \Rightarrow & A \oslash B \leq {}^0C & (C_{IV}^{\oslash,0/}) \\
A \obslash B \leq C^0 & \Rightarrow & C \otimes B \leq A & (C_I^{\obslash,\backslash 0}) & C \otimes B \leq A & \Rightarrow & A \obslash B \leq C^0 & (C_{IV}^{\obslash,\backslash 0})
\end{array}
$$

Proof. By combining the Grishin interactions with the push- and pop rules for the units. For example, $(A_I^{1\oslash,\backslash 0})$ and $(A_I^{1\oslash,\backslash})$ are derived thus:

$$
\dfrac{\dfrac{\dfrac{{}^1A \leq B^0}{B \otimes 1 \leq 0 \oplus A}\,A_I^{\oslash,\backslash}}{B \otimes 1 \leq A}\,0}{B \leq A}\,1
\qquad
\dfrac{\dfrac{{}^1A \leq B\backslash C}{B \otimes 1 \leq C \oplus A}\,A_I^{\oslash,\backslash}}{B \leq C \oplus A}\,1
$$
□

To start with, we find that the four negations collapse into two in both $\mathbf{LG}^A_{I+IV}$ and $\mathbf{LG}^C_{I+IV}$. As a corollary, only a single negation remains within the full $\mathbf{LG}_{I+IV}$.

Lemma 15. We have the following results:

(1)	$A^0 \leq {}^1A$	(2)	${}^0A \leq A^1$	$(\mathbf{LG}^A_I)$	(1')	${}^0A \leq {}^1A$	(2')	$A^0 \leq A^1$	$(\mathbf{LG}^C_I)$
(3)	${}^1A \leq A^0$	(4)	$A^1 \leq {}^0A$	$(\mathbf{LG}^A_{IV})$	(3')	${}^1A \leq {}^0A$	(4')	$A^1 \leq A^0$	$(\mathbf{LG}^C_{IV})$

Proof. As typical cases, we take (1) and (3), (g) and (cg) referring to L.13(v),(vi):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A^0 \leq A^0}\,Id}{A \leq {}^0(A^0)}\,g}{{}^1({}^0(A^0)) \leq {}^1A}\,?\oslash}{({}^1A)^1 \leq {}^0(A^0)}\,cg}{A^0 \leq {}^1A}\,A_I^{\otimes 1,0/} \qquad \dfrac{\dfrac{}{A \leq A}\,Id}{{}^1A \leq A^0}\,A_{IV}^{1\oslash,\backslash 0}$$

□

We next demonstrate the definability within $\mathbf{LG}^A_{I+IV}$ and $\mathbf{LG}^C_{I+IV}$ of the (co)implications in terms of the par, the tensor and the negations. Particularly, $A/B \simeq A \oplus {}^0B$, $B\backslash A \simeq B^0 \oplus A$, $B \oslash A \simeq B^1 \otimes A$ and $A \oslash B \simeq A \otimes {}^1B$ in $\mathbf{LG}^A_{I+IV}$, while $A/B \simeq {}^0B \oplus A$, $B\backslash A \simeq A \oplus B^0$, $B \oslash A \simeq A \otimes B^1$ and $A \oslash B \simeq {}^1B \otimes A$ in $\mathbf{LG}^C_{I+IV}$.

Lemma 16. We have the following results:

(1)	$A \oplus {}^0B \leq A/B$	(3)	$B \oslash A \leq B^1 \otimes A$	$(\mathbf{LG}^A_I)$
(2)	$B^0 \oplus A \leq B\backslash A$	(4)	$A \oslash B \leq A \otimes {}^1B$	
(1')	${}^0B \oplus A \leq A/B$	(3')	$B \oslash A \leq A \otimes B^1$	$(\mathbf{LG}^C_I)$
(2')	$A \oplus B^0 \leq B\backslash A$	(4')	$A \oslash B \leq {}^1B \otimes A$	
(5)	$A/B \leq A \oplus {}^0B$	(7)	$B^1 \otimes A \leq B \oslash A$	$(\mathbf{LG}^A_{IV})$
(6)	$B\backslash A \leq B^0 \oplus A$	(8)	$A \otimes {}^1B \leq A \oslash B$	
(5')	$A/B \leq {}^0B \oplus A$	(7')	$A \otimes B^1 \leq B \oslash A$	$(\mathbf{LG}^C_{IV})$
(6')	$B\backslash A \leq A \oplus B^0$	(8')	${}^1B \otimes A \leq A \oslash B$	

Proof. We illustrate with (1) and (5), the remaining cases being similar:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \oplus {}^0B \leq A \oplus {}^0B}\,Id}{A \oslash (A \oplus {}^0B) \leq {}^0B}\,cr}{(A \oplus {}^0B) \otimes B \leq A}\,A_I^{\oslash,0/}}{A \oplus {}^0B \leq A/B}\,r \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A/B \leq A/B}\,Id}{(A/B) \otimes B \leq A}\,r}{A \oslash (A/B) \leq {}^0B}\,A_{IV}^{\oslash,0/}}{A/B \leq A \oplus {}^0B}\,cr$$

□

We conclude with the derivation of the De Morgan laws.

$$
\begin{array}{cl}
\hline
A \otimes B \leq A \otimes B & Id \\
\hline
(A \otimes B) \obslash A \leq {}^0B & A_{IV}^{\obslash,0/} \\
\hline
A \leq (A \otimes B) \oplus {}^0B & cr \\
\hline
(A \otimes B)^1 \leq {}^0B/A & A_{IV}^{\obslash 1,/} \\
\hline
(A \otimes B)^1 \otimes A \leq {}^0B & r \\
\hline
{}^0B \obslash (A \otimes B)^1 \leq {}^0A & A_{IV}^{\obslash,0/} \\
\hline
(A \otimes B)^1 \leq {}^0B \oplus {}^0A & cr
\end{array}
\qquad
\begin{array}{cl}
\hline
{}^0B \oplus {}^0A \leq {}^0B \oplus {}^0A & Id \\
\hline
{}^0B \obslash ({}^0B \oplus {}^0A) \leq {}^0A & cr \\
\hline
({}^0B \oplus {}^0A) \otimes A \leq {}^0B & A_{I}^{\obslash,0/} \\
\hline
{}^0B \oplus {}^0A \leq {}^0B/A & r \\
\hline
{}^1({}^0B/A) \leq ({}^0B \oplus {}^0A)^0 & A_{IV}^{1\oslash,\backslash 0} \\
\hline
(({}^0B \oplus {}^0A)^0)^1 \leq {}^0B/A & cr \times 2 \\
\hline
A \leq ({}^0B \oplus {}^0A)^0 \oplus {}^0B & A_{I}^{\obslash 1,/} \\
\hline
({}^0B \oplus {}^0A)^0 \obslash A \leq {}^0B & cr \\
\hline
A \otimes B \leq ({}^0B \oplus {}^0A)^0 & A_{I}^{\obslash,0/} \\
\hline
(({}^0B \oplus {}^0A)^0)^1 \leq (A \otimes B)^1 & \obslash ? \\
\hline
{}^1((A \otimes B)^1) \leq ({}^0B \oplus {}^0A)^0 & cg \\
\hline
{}^0B \oplus {}^0A \leq (A \otimes B)^1 & A_{I}^{1\oslash,\backslash 0}
\end{array}
$$

Figure 4.1: Deriving the De Morgan laws inside $\mathbf{LG}^{A}_{I+IV}$ and $\mathbf{LG}^{C}_{I+IV}$.

Lemma 17. We have the following results:

(1)	$(A \otimes B)^1 \leq {}^0B \oplus {}^0A$	(3)	$B^1 \otimes A^1 \leq {}^0(A \oplus B)$	($\mathbf{LG}^{A}_{IV}$)
(2)	${}^1(A \otimes B) \leq B^0 \oplus A^0$	(4)	${}^1B \otimes {}^1A \leq (A \oplus B)^0$	
(1')	${}^1(A \otimes B) \leq {}^0A \oplus {}^0B$	(3')	${}^1A \otimes B^1 \leq {}^0(A \oplus B)$	($\mathbf{LG}^{C}_{IV}$)
(2')	$(A \otimes B)^1 \leq A^0 \oplus B^0$	(4')	$A^1 \otimes {}^1B \leq (A \oplus B)^0$	
(5)	${}^0B \oplus {}^0A \leq (A \otimes B)^1$	(7)	${}^0(A \oplus B) \leq B^1 \otimes A^1$	($\mathbf{LG}^{A}_{I+IV}$)
(6)	$B^0 \oplus A^0 \leq {}^1(A \otimes B)$	(8)	$(A \oplus B)^0 \leq {}^1B \otimes {}^1A$	
(5')	${}^0A \oplus {}^0B \leq {}^1(A \otimes B)$	(7')	${}^0(A \oplus B) \leq {}^1A \otimes B^1$	($\mathbf{LG}^{C}_{I+IV}$)
(6')	$A^0 \oplus B^0 \leq (A \otimes B)^1$	(8')	$(A \oplus B)^0 \leq A^1 \otimes {}^1B$	

Proof. As typical cases, we illustrate in F.4.1 with (1) and (5). Notice the use of interactions of both types I and IV with the latter. □

The above results pertaining to $\mathbf{LG}^{A}_{I+IV}$ have previously been employed by Lambek [1993] in the definition of bilinear logic, obtained through the further augmentation with same-sort associativity of $\otimes$ and $\oplus$. Note, however, that none of the results concerned relied on the existence of units beyond what is required for the derivability of the rules in Lemmas 41 and 14. By adopting as primitives both the (co)negations as well as the contents of said Lemmas pertaining to mixed associativity, we shall see in the next section how to define unit-free non-associative bilinear logic. By further adding those mixed commutative rules involving only the (co)negations, we obtain De Groote and Lamarche's [2002] classical non-associative Lambek calculus. Crucially, while the coexistence of both type I and IV interactions was previously shown to result in structural collapse, the same does not hold of their restricted instances found in L.14.

4.3 The Lambek-Grishin calculus classicalized

As alluded to, the results of §2 may be put to use if we quarantine the required instances of the Grishin interactions inside the (co)Galois connections, now considered primitive. We proceed to investigate two viable alternatives, using either two negations $\cdot^l$ and $\cdot^r$, or collapsing them into a single $\cdot^\perp$. In the first case, we speak of non-associative bilinear logic (**NBL**), in the second of classical non-associative Lambek calculus (**CNL**). In either case, we adopt the following formula language:

Definition 42. Formulas are redefined

$$\begin{array}{lcll} A, B & ::= & p & \text{(Atoms)} \\ & & (A \otimes B) \mid (A/B) \mid (B \backslash A) & \text{(Residuated family)} \\ & & (A \oplus B) \mid (A \oslash B) \mid (B \oslash A) & \text{(Coresiduated family)} \\ & & ({}^0A) \mid (A^0) & \text{(Galois connections)} \\ & & (A^1) \mid ({}^1A) & \text{(co-Galois connections)} \end{array}$$

We present both **NBL** and **CNL** by two axiomatizations each, one using explicit Grishin interactions for the above formula language, the other using a more restricted logical vocabulary, dispensing with the status of the (co)implications as primitives. Starting with the former, we have the following definition.

Definition 43. F.4.2 defines **NBL** and **CNL** using the formula language of D.42.

Note that the Grishin interactions used in F.4.2 hail straight from D.14. Since the latter were the sole instances used throughout the previous section, all of the latter's results regarding classicality carry over immediately. As a consequence, we find that we can make do with the following more concise formula languages.

Preorder laws

$$\frac{}{A \leq A}\ Id \qquad \frac{A \leq B \quad B \leq C}{A \leq C}\ \circ$$

(Co)residuation and (co)galois laws

$$\frac{A \leq C/B}{\overline{A \otimes B \leq C}}\ r \qquad \frac{B \leq A\backslash C}{\overline{A \otimes B \leq C}}\ r \qquad \frac{B \leq A^0}{\overline{A \leq {}^0B}}\ g$$

$$\frac{A \oslash C \leq B}{\overline{C \leq A \oplus B}}\ cr \qquad \frac{C \oslash B \leq A}{\overline{C \leq A \oplus B}}\ cr \qquad \frac{{}^1B \leq A}{\overline{A^1 \leq B}}\ cg$$

Grishin interactions (for both **NBL** and **CNL**)

$$\begin{array}{lcll} {}^1A \leq B^0 & \Leftrightarrow & B \leq A & (1\oslash, \backslash 0) \\ A^1 \leq {}^0B & \Leftrightarrow & B \leq A & (\oslash 1, 0/) \end{array} \quad \begin{array}{lcll} {}^1A \leq B\backslash C & \Leftrightarrow & B \leq C \oplus A & (1\oslash, \backslash) \\ A^1 \leq B/C & \Leftrightarrow & C \leq A \oplus B & (\oslash 1, /) \\ A \oslash B \leq C^0 & \Leftrightarrow & C \otimes A \leq B & (\oslash, \backslash 0) \\ A \oslash B \leq {}^0C & \Leftrightarrow & B \otimes C \leq A & (\oslash, 0/) \end{array}$$

Grishin interactions (for **CNL** only)

$$\begin{array}{lcll} {}^1A \leq {}^0B & \Leftrightarrow & B \leq A & (1\oslash, 0/) \\ A^1 \leq B^0 & \Leftrightarrow & B \leq A & (\oslash 1, \backslash 0) \end{array}$$

Figure 4.2: First axiomatizations of **NBL** and **CNL**.

Definition 44. Formulas for **NBL** are redefined:

$$A, B \quad ::= \quad p \mid (A \otimes B) \mid (A \oplus B) \mid A^l \mid A^r$$

A^l and A^r are referred to respectively by linear *post-* and *retro*-negation. For **CNL**, the latter collapse into a single $\cdot^{\perp}$:

$$A, B \quad ::= \quad p \mid (A \otimes B) \mid (A \oplus B) \mid A^{\perp}$$

Using the new vocabulary, equally concise axiomatizations may be devised.

Definition 45. F.4.3 defines **NBL** and **CNL** using the vocabulary of D.44.

Preorder laws

$$\frac{}{A \leq A}\ Id \qquad \frac{A \leq B \quad B \leq C}{A \leq C}\ \circ$$

	NBL		**CNL**
A1.	$C \leq A \oplus B$ iff $C \otimes B^r \leq A$	A1.	$C \leq A \oplus B$ iff $C \otimes B^\perp \leq A$
A2.	$A \otimes B \leq C$ iff $B \leq A^r \oplus C$	A2.	$A \otimes B \leq C$ iff $B \leq A^\perp \oplus C$
A3.	$A^{lr} \simeq A \simeq A^{rl}$	A3.	$A^{\perp\perp} \simeq A$
A4.	$A \leq B$ implies $B^l \leq A^l$ and $B^r \leq A^r$	A4.	$A \leq B$ implies $B^\perp \leq A^\perp$

Figure 4.3: Second axiomatizations of **NBL** and **CNL**.

Lemma 18. The following are derived rules of inference for **NBL**, adaptable to **CNL** by replacing $\cdot^r$ and $\cdot^l$ with $\cdot^\perp$.

A1'. $C \leq A \oplus B$ iff $A^l \otimes C \leq B$
A2'. $A \otimes B \leq C$ iff $A \leq C \oplus B^l$
A3'. $A \leq B$ iff $A^{lr} \leq B$ iff $A^{rl} \leq B$ iff $A \leq B^{lr}$ iff $A \leq B^{rl}$
A4'. $A \leq C$ and $B \leq D$ imply $A \otimes B \leq C \otimes D$ and $A \oplus B \leq C \oplus D$

Proof. As typical cases, we show one half for each of $(A1')$ and $(A4')$, noting $(A3')$ follows by $(A3)$ and $(\circ)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A^l \otimes C \leq B}{C \leq A^{lr} \oplus B}\ A2}{C \otimes B^r \leq A^{lr}}\ A1 \quad \dfrac{}{A^{lr} \leq A}\ A3}{C \otimes B^r \leq A}\ \circ}{C \leq A \oplus B}\ A1 \qquad \dfrac{\dfrac{A \leq C \quad \dfrac{\dfrac{\dfrac{B \leq D \quad \dfrac{\dfrac{}{C \otimes D \leq C \otimes D}\ Id}{D \leq C^r \oplus (C \otimes D)}\ A2}{B \leq C^r \oplus (C \otimes D)}\ \circ}{C \otimes B \leq C \otimes D}\ A2}{C \leq (C \otimes D) \oplus B^l}\ A2'}{A \leq (C \otimes D) \oplus B^l}\ \circ}{A \otimes B \leq C \otimes D}\ A2'$$

□

To demonstrate the two axiomatizations for each of **NBL** and **CNL** equivalent, we define translations taking the first formula language into the other two.

Definition 46. The translation $\cdot^\flat$ takes the formulas of D.42 to those of D.44 for **NBL**.

$$\begin{array}{lllllll} p^\flat := p & (A \otimes B)^\flat & := & A^\flat \otimes B^\flat & (A \oplus B)^\flat & := & A^\flat \oplus B^\flat \\ & (A/B)^\flat & := & A^\flat \oplus B^{\flat l} & (B \obslash A)^\flat & := & B^{\flat r} \otimes A^\flat \\ & (B \backslash A)^\flat & := & B^{\flat r} \oplus A^\flat & (A \oslash B)^\flat & := & A^\flat \otimes B^{\flat l} \\ & ({}^0 A)^\flat & := & A^{\flat l} & (A^1)^\flat & := & A^{\flat r} \\ & (A^0)^\flat & := & A^{\flat l} & ({}^1 A)^\flat & := & A^{\flat r} \end{array}$$

A second translation $\cdot^\sharp$ for **CNL** is constructed similarly, replacing all occurrences of $\cdot^l$ and $\cdot^r$ in the above definition by $\cdot^\perp$.

We proceed to compare the two axiomatizations of Figures 4.2 and 4.3.

Theorem 4.3.1. $A \leq B$ in **NBL** (**CNL**) according to F.4.2 iff $A^\flat \leq B^\flat$ ($A^\sharp \leq B^\sharp$).

Proof. We restrict to **NBL**, the desired result for **CNL** following similarly. Going from left to right, the preorder laws are immediate. The (co)residuation laws follow from (A1), (A2), (A1') and (A2'), while the (co-)Galois laws use (A3) and (A4). For example, to show that $A^\flat \leq B^{\flat l}$ implies $B^\flat \leq A^{\flat r}$, we proceed as follows:

$$\dfrac{\dfrac{}{B^\flat \leq B^{\flat l r}}\,A3 \quad \dfrac{A^\flat \leq B^{\flat l}}{B^{\flat l r} \leq A^{\flat r}}\,A4}{B^\flat \leq A^{\flat r}}\,\circ$$

We note $(1\oslash, \backslash 0)$ and $(\obslash 1, 0/)$ are immediate from the definition of $\cdot^\flat$. As a typical case among the remainder of the Grishin interactions, take $(\obslash 1, /)$:

$$\dfrac{\dfrac{C^\flat \leq A^\flat \oplus B^\flat}{A^{\flat l} \otimes C^\flat \leq B^\flat}\,A2'}{A^{\flat l} \leq B^\flat \oplus C^{\flat l}}\,A1$$

The converse direction uses the following equivalences, carrying over from §2:

$$\begin{array}{rclrcl} A^1 & \simeq & {}^0 A & {}^1 A & \simeq & A^0 \\ {}^0(A^0) & \simeq & A & ({}^0 A)^0 & \simeq & A \\ A \oplus {}^0 B & \simeq & A/B & B^1 \otimes A & \simeq & B \obslash A \\ B^0 \oplus A & \simeq & B \backslash A & A \otimes {}^1 B & \simeq & A \oslash B \\ (A \otimes B)^1 & \simeq & {}^0 B \oplus {}^0 A & {}^0(A \oplus B) & \simeq & B^1 \otimes A^1 \\ {}^1(A \otimes B) & \simeq & A^0 \oplus B^0 & (A \oplus B)^0 & \simeq & {}^1 B \otimes {}^1 A \end{array}$$

while for **CNL**, also $A^1 \simeq A^0$ and ${}^1 A \simeq {}^0 A$ using $(1\oslash, 0/)$ and $(\obslash 1, \backslash 0)$. A1 and A2 now follow easily, while A4 is just monotonicity. For A3, one makes use of $A \leq {}^0(A^0)$, $A \leq ({}^0 A)^0$, $({}^1 A)^1 \leq A$ and ${}^1(A^1) \leq A$, already within the base logic. □

4.4 Sequent calculi

We next describe sequent calculi for **NBL** and **CNL**. In the latter case, such a presentation was already provided by De Groote and Lamarche [2002]. Their work was predated, however, by Belnap's [1990] display calculus for classical linear logic, notable for making explicit both commutativity and associativity through structural rules, allowing these, in particular, to be dropped. When compared, Belnap's is a two-sided presentation using structural negations, whereas De Groote and Lamarche's sequents are right-sided. Given these previous results, we shall pay most of our attention to **NBL**. In doing so, we start out with a trivial adaptation of Belnap's efforts in order to accommodate two negations, transforming it stepwise into a presentation more akin to that of De Groote and Lamarche for **CNL**; an exercise hopefully serving an expository purpose in connecting the two works. Cut admissibility is proved by model-theoretic means, and a comparison is made with Abrusci's [1991] one-sided sequent calculus for (associative) bilinear logic.

4.4.1 Non-associative bilinear logic

Besides the differences in size of their logical vocabularies, **LG** and **NBL** differ most crucially in their treatment of atoms. Whereas **LG** considers them as fixpoints for its concept of duality, we cannot similarly claim that p^{rr} and p^{ll} are identifiable with p. Small as the difference may appear, the consequences are far-reaching. Particularly, one need no longer use both sides of the sequent turnstile. Disregarding the tree-like structure of sequents typical of non-associativity, instead of writing

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

for the derivabiliy of the disjunction of conclusions $B_1, \ldots, B_m$ from the conjunction of hypotheses $A_1, \ldots, A_n$, one may as well go *one-sided*, replacing either the hypotheses or conclusions by their negations at the other side of the turnstile:

$$\begin{array}{c} A_1, \ldots, A_n, B_m^r, \ldots, B_1^r \vdash, \text{ or } \vdash A_n^r, \ldots, A_1^r, B_1, \ldots, B_m \\ (B_m^l, \ldots, B_1^l, A_1, \ldots, A_n \vdash, \text{ or } \vdash B_1, \ldots, B_m, A_n^l, \ldots, A_1^l) \end{array}$$

noting that, informally, $(B_1 \oplus \ldots \oplus B_m)^r = B_m^r \otimes \ldots \otimes B_1^r$ and $(A_1 \otimes \ldots \otimes A_n)^r = A_n^r \oplus \ldots \oplus A_1^r$. In practice, the right-sided alternative has been the preferred choice in the literature. The exceptions to this unwritten rule are typically those concerning analytic tableaux, emphasizing refutability over provability. For now, we adhere to tradition in going right-sided, although in chapter 7 we shall also find reasons to use left-sided sequents instead. In contrast with work on tableaux, however, we shall understand the latter not in the model-theoretic sense, but rather in terms of the double negation translations used for constructivizing classical logics. Indeed, said translations will play a prominent role for natural language semantics, in that they can be used to assign **LG** a compositional semantics inside λ-calculus.

We shall proceed as follows. First, an adaptation of Belnap's (still two-sided) display calculus for classical linear logic is presented, using both post- and retronegation. As an intermediate step towards a presentation more reminiscent to that of **CNL**, we present a right-sided alternative still employing explicit structural counterparts for the negations. The latter's removal then brings us to the final judgement form, for which we also prove Cut admissibility using a model-theoretic argument.

NBL displayed

In contrast with chapter 3's display calculus for **LG**, the language of **NBL** requires structural counterparts for negations, as opposed to (co)implications.

Definition 47. *Structures* Γ, Δ are defined as follows:

$$\Gamma, \Delta, \Theta \quad ::= \quad A \mid (\Gamma \bullet \Delta) \mid (*\Gamma) \mid (\Gamma *)$$

Their translation to formulas uses mutually inductive maps $\cdot^{\bullet}$ and $\cdot^{\circ}$:

$$\begin{array}{rlrl}
A^{\bullet} & := A & A^{\circ} & := A \\
(\Gamma \bullet \Delta)^{\bullet} & := \Gamma^{\bullet} \otimes \Delta^{\bullet} & (\Gamma \bullet \Delta)^{\circ} & := \Gamma^{\circ} \oplus \Delta^{\circ} \\
(*\Gamma)^{\bullet} & := \Gamma^{\circ l} & (*\Gamma)^{\circ} & := \Gamma^{\bullet r} \\
(\Gamma *)^{\bullet} & := \Gamma^{\circ r} & (\Gamma *)^{\circ} & := \Gamma^{\bullet l}
\end{array}$$

We take $\cdot *$ and $* \cdot$ to bind more strongly than $\bullet$, dropping brackets accordingly.

Besides expressing (co)residuation, the structural rules are now required to reflect properties of classical negation as well.

Definition 48. F.4.4 defines the judgement form $\Gamma \vdash \Delta$, specifying derivability in the display calculus presentation of **NBL**.

Axioms

$$\frac{}{A \vdash A}\,Id \qquad \frac{\Gamma \vdash A \quad A \vdash \Delta}{\Gamma \vdash \Delta}\,\circ$$

Display postulates (dp)

(i)	$**\Gamma \vdash \Delta$ iff $\Gamma \vdash \Delta$ iff $\Gamma** \vdash \Delta$	(iii)	$\Gamma \vdash \Delta \bullet \Theta$ iff $\Delta* \bullet \Gamma \vdash \Theta$
(ii)	$*\Delta \vdash \Gamma*$ iff $\Gamma \vdash \Delta$ iff $\Delta* \vdash *\Gamma$	(iv)	$\Gamma \bullet \Delta \vdash \Theta$ iff $\Gamma \vdash \Theta \bullet *\Delta$

Logical rules

$$\frac{A* \vdash \Gamma}{A^l \vdash \Gamma}\,lL \qquad \frac{\Gamma \vdash *A}{\Gamma \vdash A^l}\,lR \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \bullet \Delta \vdash A \otimes B}\,\otimes R \qquad \frac{A \bullet B \vdash \Gamma}{A \otimes B \vdash \Gamma}\,\otimes L$$

$$\frac{*A \vdash \Gamma}{A^r \vdash \Gamma}\,rL \qquad \frac{\Gamma \vdash A*}{\Gamma \vdash A^r}\,rR \qquad \frac{A \vdash \Gamma \quad B \vdash \Delta}{A \oplus B \vdash \Gamma \bullet \Delta}\,\oplus L \qquad \frac{\Gamma \vdash A \bullet B}{\Gamma \vdash A \oplus B}\,\oplus R$$

Figure 4.4: Non-associative bilinear logic displayed.

Lemma 19. The following are some easily derived rules of inference:

(i')	$\Gamma \vdash \Delta$ iff $\Gamma \vdash **\Delta$ iff $\Gamma \vdash \Delta**$	(iii')	$\Gamma \vdash \Delta \bullet \Theta$ iff $\Gamma \bullet *\Theta \vdash \Delta$
(ii')	$*\Gamma \vdash \Delta$ iff $\Delta* \vdash \Gamma$, and $\Gamma \vdash \Delta*$ iff $\Delta \vdash *\Gamma$	(iv')	$\Gamma \bullet \Delta \vdash \Theta$ iff $\Delta \vdash \Gamma* \bullet \Theta$

We next show completeness of F.4.4 w.r.t. algebraic derivability.

Lemma 20. If $A \leq B$, also $A \vdash B$.

Proof. By induction. Cases (*Id*) and ($\circ$) are immediate, leaving us to check A1-A4.

A1. $C \leq A \oplus B$ iff $C \otimes B^r \leq A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash A \oplus B}\,IH \quad \dfrac{\dfrac{}{A \vdash A}\,Id \quad \dfrac{}{B \vdash B}\,Id}{A \oplus B \vdash A \bullet B}\,\oplus L}{C \vdash A \bullet B}\,\circ}{C \bullet *B \vdash A}\,(iii')}{*B \vdash C* \bullet A}\,(iv')}{B^r \vdash C* \bullet A}\,rL}{C \bullet B^r \vdash A}\,(iv')}{C \otimes B^r \vdash A}\,\otimes L \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}\,Id \quad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,Id}{*B \vdash B*}\,(ii)}{*B \vdash B^r}\,rR}{C \bullet *B \vdash C \otimes B^r}\,\otimes R \quad \dfrac{}{C \otimes B^r \vdash A}\,IH}{\dfrac{C \bullet *B \vdash A}{C \vdash A \bullet B}\,(iii')}\,\circ}{C \vdash A \oplus B}\,\oplus R$$

A2. $A \otimes B \leq C$ iff $B \leq A^r \oplus C$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{IH}
\UnaryInfC{$B \vdash A^r \oplus C$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash A$}
\RightLabel{(ii)}
\UnaryInfC{$*A \vdash A*$}
\RightLabel{rL}
\UnaryInfC{$A^r \vdash A*$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$C \vdash C$}
\RightLabel{$\oplus L$}
\BinaryInfC{$A^r \oplus C \vdash A* \bullet C$}
\RightLabel{$\circ$}
\BinaryInfC{$B \vdash A* \bullet C$}
\RightLabel{(iv')}
\UnaryInfC{$A \bullet B \vdash C$}
\RightLabel{$\otimes L$}
\UnaryInfC{$A \otimes B \vdash C$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$B \vdash B$}
\RightLabel{$\otimes R$}
\BinaryInfC{$A \bullet B \vdash A \otimes B$}
\AxiomC{}
\RightLabel{IH}
\UnaryInfC{$A \otimes B \vdash C$}
\RightLabel{$\circ$}
\BinaryInfC{$A \bullet B \vdash C$}
\RightLabel{(iv')}
\UnaryInfC{$B \vdash A* \bullet C$}
\RightLabel{(iii')}
\UnaryInfC{$B \bullet *C \vdash A*$}
\RightLabel{rR}
\UnaryInfC{$B \bullet *C \vdash A^r$}
\RightLabel{(iii')}
\UnaryInfC{$B \vdash A^r \bullet C$}
\RightLabel{$\oplus R$}
\UnaryInfC{$B \vdash A^r \oplus C$}
\end{prooftree}
$$

A3. $A^{lr} \simeq A \simeq A^{rl}$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash A$}
\RightLabel{(ii)}
\UnaryInfC{$A* \vdash *A$}
\RightLabel{lR}
\UnaryInfC{$A* \vdash A^l$}
\RightLabel{(ii')}
\UnaryInfC{$*A^l \vdash A$}
\RightLabel{rL}
\UnaryInfC{$A^{lr} \vdash A$}
\end{prooftree}
\quad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash A$}
\RightLabel{(ii)}
\UnaryInfC{$A* \vdash *A$}
\RightLabel{lL}
\UnaryInfC{$A^l \vdash *A$}
\RightLabel{(ii')}
\UnaryInfC{$A \vdash A^l*$}
\RightLabel{rR}
\UnaryInfC{$A \vdash A^{lr}$}
\end{prooftree}
\quad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash A$}
\RightLabel{(ii)}
\UnaryInfC{$*A \vdash A*$}
\RightLabel{rR}
\UnaryInfC{$*A \vdash A^r$}
\RightLabel{(ii')}
\UnaryInfC{$A^r* \vdash A$}
\RightLabel{lL}
\UnaryInfC{$A^{rl} \vdash A$}
\end{prooftree}
\quad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash A$}
\RightLabel{(ii)}
\UnaryInfC{$*A \vdash A*$}
\RightLabel{rL}
\UnaryInfC{$A^r \vdash A*$}
\RightLabel{(ii')}
\UnaryInfC{$A \vdash *A^r$}
\RightLabel{lR}
\UnaryInfC{$A \vdash A^{rl}$}
\end{prooftree}
$$

A4. $A \leq B$ implies $B^l \leq A^l$ and $B^r \leq A^r$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash B$}
\RightLabel{(ii)}
\UnaryInfC{$B* \vdash *A$}
\RightLabel{lR}
\UnaryInfC{$B* \vdash A^l$}
\RightLabel{lL}
\UnaryInfC{$B^l \vdash A^l$}
\end{prooftree}
\quad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \vdash B$}
\RightLabel{(ii)}
\UnaryInfC{$*B \vdash A*$}
\RightLabel{rR}
\UnaryInfC{$B^r \vdash A*$}
\RightLabel{rL}
\UnaryInfC{$B^r \vdash A^r$}
\end{prooftree}
$$

□

NBL redisplayed

As an intermediate step towards a sequent presentation lacking structural negations, we first present a one-sided retelling of the previous display calculus.

Definition 49. F.4.5 defines a right-sided display calculus for **NBL**.

Lemma 21. The following are derived rules of inference:

	(i') $\vdash \Gamma, \Delta$ iff $\vdash **\Gamma, \Delta$ iff $\vdash \Gamma**, \Delta$
(*Id'*) $\vdash A*, A$	(ii') $\vdash \Gamma, \Delta$ iff $\vdash \Delta, *(\Gamma*)$ iff $\vdash (*\Delta)*, \Gamma$
(∘') $\vdash \Gamma, \Delta$ if $\vdash \Gamma, *A$ and $\vdash \Delta, A$	(iii') $\vdash \Gamma, *(\Delta \bullet \Theta)$ iff $\vdash \Delta* \bullet \Gamma, *\Theta$

Preorder laws

$$\frac{}{\vdash A, *A}\ Id \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, *A}{\vdash \Gamma, \Delta}\ \circ$$

Display postulates (dp)

(i) $\vdash \Gamma, \Delta$ iff $\vdash \Gamma, **\Delta$ iff $\vdash \Gamma, \Delta**$
(ii) $\vdash \Gamma*, \Delta$ iff $\vdash \Delta, *\Gamma$
(iii) $\vdash \Gamma, *(\Delta \bullet \Theta)$ iff $\vdash \Gamma \bullet *\Theta, *\Delta$
(iv) $\vdash \Gamma, \Delta \bullet \Theta$ iff $\vdash \Gamma \bullet \Delta, \Theta$

Logical rules

$$\frac{\vdash \Gamma, A*}{\vdash \Gamma, A^r}\ Id \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, *A^r}\ *r \qquad \frac{\vdash \Gamma, *A \quad \vdash \Delta, *B}{\vdash \Gamma \bullet \Delta, *A \oplus B}\ *\oplus \qquad \frac{\vdash \Gamma, A \bullet B}{\vdash \Gamma, A \oplus B}\ \oplus$$

$$\frac{\vdash \Gamma, *A}{\vdash \Gamma, A^l}\ l \qquad \frac{\vdash A, \Gamma}{\vdash \Gamma, *A^l}\ *l \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Delta \bullet \Gamma, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, *B \bullet *A}{\vdash \Gamma, *A \otimes B}\ *\otimes$$

Figure 4.5: One-sided display calculus for **NBL**.

Proof. The only non-trivial case is (∘'), handled as follows:

$$\cfrac{\vdash \Delta, A \qquad \cfrac{\cfrac{\vdash \Gamma, A*}{\vdash A*, *(\Gamma*)}\ (ii')}{\vdash *(\Gamma*), *A}\ (ii)}{\cfrac{\vdash \Delta, *(\Gamma*)}{\vdash \Gamma, \Delta}\ (ii')}\ \circ$$

□

Lemma 22. If $\Gamma \vdash \Delta$, then $\vdash \Gamma*, \Delta$.

Proof. By induction. Note (*Id*) is immediate by (*Id'*) of the previous lemma, while (∘) is of nigh equal triviality. Furthermore, (rR) and (lR) translate directly to (*Id*) and (l) respectively. This leaves the following cases to check.

(i) $**\Gamma \vdash \Delta$ iff $\Gamma \vdash \Delta$ iff $\Gamma** \vdash \Delta$

$$\cfrac{\cfrac{\cfrac{\vdash \Gamma*, \Delta}{\vdash \Delta, *\Gamma}\ (ii)}{\vdash \Delta, ***\Gamma}\ (i)}{\vdash (**\Gamma)*, \Delta}\ (ii) \qquad \cfrac{\vdash \Gamma*, \Delta}{\vdash \Gamma***, \Delta}\ (i')$$

(ii) $*\Delta \vdash \Gamma*$ iff $\Gamma \vdash \Delta$ iff $\Delta* \vdash *\Gamma$

$$\begin{array}{cl} \vdash \Gamma*, \Delta & \\ \hline\hline \vdash \Gamma*, **\Delta & (i) \\ \hline\hline \vdash (*\Delta)*, \Gamma* & (ii) \end{array} \qquad \begin{array}{cl} \vdash \Gamma*, \Delta & \\ \hline\hline \vdash \Delta, *\Gamma & (ii) \\ \hline\hline \vdash \Delta**, *\Gamma & (i) \end{array}$$

(iii) $\Gamma \vdash \Delta \bullet \Theta$ iff $\Delta* \bullet \Gamma \vdash \Theta$ and (iv) $\Gamma \bullet \Delta \vdash \Theta$ iff $\Gamma \vdash \Theta \bullet *\Delta$

$$\begin{array}{cl} \vdash (\Delta* \bullet \Gamma)*, \Theta & \\ \hline\hline \vdash \Theta, *(\Delta* \bullet \Gamma) & (ii) \\ \hline\hline \vdash \Theta \bullet *\Gamma, *(\Delta*) & (iii) \\ \hline\hline \vdash \Delta, \Theta \bullet *\Gamma & (ii') \\ \hline\hline \vdash \Delta \bullet \Theta, *\Gamma & (iv) \\ \hline\hline \vdash \Gamma*, \Delta \bullet \Theta & (ii) \end{array} \qquad \begin{array}{cl} \vdash \Gamma*, \Theta \bullet *\Delta & \\ \hline\hline \vdash \Gamma* \bullet \Theta, *\Delta & (iv) \\ \hline\hline \vdash \Theta, *(\Gamma \bullet \Delta) & (iii') \\ \hline\hline \vdash (\Gamma \bullet \Delta)*, \Theta & (ii) \end{array}$$

$({}^lL)$ $A* \vdash \Gamma$ implies $A^l \vdash \Gamma$ and $({}^rL)$ $*A \vdash \Gamma$ implies $A^r \vdash \Gamma$

$$\begin{array}{cl} \vdash A**, \Gamma & \\ \hline \vdash A, \Gamma & (i') \\ \hline \vdash \Gamma, *A^l & l* \\ \hline \vdash A^l*, \Gamma & (ii) \end{array} \qquad \begin{array}{cl} \vdash (*A)*, \Gamma & \\ \hline \vdash \Gamma, A & (ii') \\ \hline \vdash \Gamma, *A^r & *r \\ \hline \vdash A^r*, \Gamma & (ii) \end{array}$$

$(\otimes R)$ $\Gamma \vdash A$ and $\Delta \vdash B$ imply $\Gamma \bullet \Delta \vdash A \otimes B$, and $(\otimes L)$ $A \bullet B \vdash \Gamma$ implies $A \otimes B \vdash \Gamma$

$$\begin{array}{cl} \vdash \Gamma*, A \quad \vdash \Delta*, B & \\ \hline \vdash \Delta* \bullet \Gamma*, A \otimes B & \otimes \\ \hline \vdash \Delta*, \Gamma* \bullet A \otimes B & (iv) \\ \hline \vdash \Gamma* \bullet A \otimes B, *\Delta & (ii) \\ \hline \vdash A \otimes B, *(\Gamma \bullet \Delta) & (iii') \\ \hline \vdash (\Gamma \bullet \Delta)*, A \otimes B & (ii) \end{array} \qquad \begin{array}{cl} \vdash (A \bullet B)*, \Gamma & \\ \hline \vdash \Gamma, *(A \bullet B) & (ii) \\ \hline \vdash \Gamma \bullet *A, *B & (iii) \\ \hline \vdash \Gamma, *A \bullet *B & (iv) \\ \hline \vdash \Gamma, *A \otimes B & *\otimes \\ \hline \vdash A \otimes B*, \Gamma & (ii) \end{array}$$

$(\oplus L)$ $A \vdash \Gamma$ and $B \vdash \Delta$ imply $A \oplus B \vdash \Gamma \bullet \Delta$ and $(\oplus R)$ $\Gamma \vdash A \bullet B$ implies $\Gamma \vdash A \oplus B$

$$\begin{array}{cl} \vdash \Gamma, *A \quad \vdash \Delta, *B & \\ \hline \vdash \Gamma \bullet \Delta, *A \oplus B & *\oplus \\ \hline \vdash A \oplus B*, \Gamma \bullet \Delta & (ii) \end{array} \qquad \begin{array}{cl} \vdash \Gamma*, A \bullet B & \\ \hline \vdash \Gamma*, A \oplus B & \oplus \end{array} \qquad \square$$

Preorder laws

$$\frac{}{\vdash A, A^l}\ Id \qquad \frac{\vdash \Pi, A \quad \vdash \Sigma, A^l}{\vdash \Pi, \Sigma}\ \circ$$

Negations

(i) $\vdash \Pi, A$ iff $\vdash \Pi, A^{lr}$ iff $\vdash \Pi, A^{rl}$
(ii) $\vdash \Pi, A^l$ iff $\vdash A^r, \Pi$
(iii) $\vdash \Pi \bullet \Sigma, \Upsilon$ iff $\vdash \Pi, \Sigma \bullet \Upsilon$

Logical rules

$$\frac{\vdash \Pi, A^l \quad \vdash \Sigma, B^l}{\vdash \Pi \bullet \Sigma, (A \oplus B)^l}\ \oplus^l \qquad \frac{\vdash \Pi, A \bullet B}{\vdash \Pi, A \oplus B}\ \oplus$$

$$\frac{\vdash \Pi, A \quad \vdash \Sigma, B}{\vdash \Sigma \bullet \Pi, A \otimes B}\ \otimes \qquad \frac{\vdash \Pi, B^l \bullet A^l}{\vdash \Pi, (A \otimes B)^l}\ \otimes^l$$

Figure 4.6: Sequent derivations for **NBL**.

Structural negations eliminated

We next define a sequent calculus omitting negations from the structural vocabulary.

Definition 50. Define *structures* Π, Σ by binary-branching trees of formulas:

$$\Pi, \Sigma \quad ::= \quad A \mid (\Pi \bullet \Sigma)$$

We have used metavariables different from the familiar Γ, Δ to prevent confusion.

Definition 51. F.4.6 defines sequent derivations for **NBL** using judgements $\vdash \Pi, \Sigma$. In what follows, we write $\vdash^{cf} \Pi, \Sigma$ if $\vdash \Pi, \Sigma$ without use of Cut.

We make the following observations regarding the above definition.

1. Evidently, the subformula property is violated. That said, negations here serve more of a structural than a logical role, as should be apparent from the previous discussion on display calculi. Note the latter are also notorious for their lack of a 'sub-structure' property [see Avron, 1996].

2. Contrary to usual one-sided presentations, ours couples each connective with two introductions instead of one. It is of interest to note that in classical tableaux as well, formulas tend to have introduction rules for both their affirmations and denials. That said, we will, below, consider an alternative formulation having but a single rule for each constant, resembling Abrusci's [1991] efforts for associative bilinear logic. Again, however, we will argue that, while less immediate, the latter still fails to satisfy the subformula property.

We shall want to compare the current sequent presentation with that of derivations in one-sided display calculus. To this end, we first require a means of interpreting structural negations in terms of the present notion of structure.

Definition 52. We define the operations $\cdot^L$ and $\cdot^R$ on structures, as follows:

$$
\begin{array}{rlrl}
A^L & := A^l & A^R & := A^r \\
(\Pi \bullet \Sigma)^L & := \Sigma^L \bullet \Pi^L & \quad (\Pi \bullet \Sigma)^R & := \Sigma^R \bullet \Pi^R
\end{array}
$$

Lemma 23. We claim, for any Π and Σ:

$$
\begin{array}{ll}
\text{(i')} & \vdash^{cf} \Pi, \Sigma \text{ iff } \vdash^{cf} \Pi, \Sigma^{LR} \text{ iff } \vdash^{cf} \Pi, \Sigma^{RL} \\
\text{(ii')} & \vdash^{cf} \Sigma^R, \Pi \text{ iff } \vdash^{cf} \Pi, \Sigma^L
\end{array}
$$

Proof. By a simultaneous induction on Σ. If $\Sigma = A$, then the desired results are immediate by (i) and (ii). So assume $\Sigma = \Sigma_1 \bullet \Sigma_2$:

$$
\begin{array}{cl}
\vdash \Pi, \Sigma_1 \bullet \Sigma_2 & \\ \hline\hline
\vdash \Pi \bullet \Sigma_1, \Sigma_2 & (ii) \\ \hline\hline
\vdash \Pi \bullet \Sigma_1, \Sigma_2^{RL} & IH(ii') \\ \hline\hline
\vdash \Sigma_2^{RR}, \Pi \bullet \Sigma_1 & IH(i') \\ \hline\hline
\vdash \Sigma_2^{RR} \bullet \Pi, \Sigma_1 & (ii) \\ \hline\hline
\vdash \Sigma_2^{RR} \bullet \Pi, \Sigma_1^{RL} & IH(ii') \\ \hline\hline
\vdash \Sigma_2^{RR}, \Pi \bullet \Sigma_1^{RL} & (ii) \\ \hline\hline
\vdash \Pi \bullet \Sigma_1^{RL}, \Sigma_2^{RL} & IH(i') \\ \hline\hline
\vdash \Pi, \Sigma_1^{RL} \bullet \Sigma_2^{RL} & (ii)
\end{array}
\qquad
\begin{array}{cl}
\vdash \Pi, \Sigma_2^L \bullet \Sigma_1^L & \\ \hline\hline
\vdash \Pi \bullet \Sigma_2^L, \Sigma_1^L & (ii) \\ \hline\hline
\vdash \Sigma_1^R, \Pi \bullet \Sigma_2^L & IH(i') \\ \hline\hline
\vdash \Sigma_1^R \bullet \Pi, \Sigma_2^L & (ii) \\ \hline\hline
\vdash \Sigma_2^R, \Sigma_1^R \bullet \Pi & IH(i') \\ \hline\hline
\vdash \Sigma_2^R \bullet \Sigma_1^R, \Pi & (ii)
\end{array}
$$

□

Lemma 24. If $\vdash \Gamma, \Delta$, then $\vdash \Pi, \Sigma$ where Π, Σ are obtained from Γ, Δ respectively by replacing occurrences of $*\cdot$ and $\cdot *$ with $\cdot^L$, resp. $\cdot^R$.

Proof. By induction. The rules for negations all reduce to identities, while the logical rules and preorder laws are similarly immediate. This leaves the display postulates, which can be trivially translated using the previous lemma. □

Theorem 4.4.1. If $A \leq B$, then $\vdash A^r, B$.

Proof. By composing Lemmas 20, 22 and 24. □

Next, we show how sequent derivations translate to their algebraic counterparts.

Definition 53. Structures Π interpret by dual formulas $\Pi^\bullet$ and Π°, as follows:

$$\begin{array}{rlrl} A^\bullet & := A^l & A^\circ & := A \\ (\Pi \bullet \Sigma)^\bullet & := \Sigma^\bullet \otimes \Pi^\bullet & (\Pi \bullet \Sigma)^\circ & := \Pi^\circ \oplus \Sigma^\circ \end{array}$$

An easy induction ensures

Lemma 25. For any Π, $\Pi^{\bullet r} \simeq \Pi^\circ$, and $\Pi^{\circ l} \simeq \Pi^\bullet$.

Lemma 26. If $\vdash \Pi, \Sigma$, then $\Pi^\bullet \leq \Sigma^\circ$.

Proof. By induction. The base case coincides with (Id), while for $(\circ)$, we have

$$\dfrac{\dfrac{\dfrac{}{\Pi^\bullet \leq A}\,IH \quad \dfrac{\dfrac{\dfrac{}{\Sigma^\bullet \leq A^l}\,IH}{A^{lr} \leq \Sigma^{\bullet r}}\,A4}{A \leq \Sigma^{\bullet r}}\,A3'}{\Pi^\bullet \leq \Sigma^{\bullet r}}\,\circ \qquad \dfrac{}{\Sigma^{\bullet r} \leq \Sigma^\circ}\,L.25}{\Pi^\bullet \leq \Sigma^\circ}\,\circ$$

Note (i) follows immediately from (A3'), while (ii) and (iii) are easy consequences of L.25, and, respectively, (A4) and (A2). Next, $(\otimes)$ and $(\oplus)$ are immediate from $(A4')$ and the induction hypothesis respectively, leaving us with $(\otimes^l)$ and $(\oplus^l)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Pi^\bullet \leq B^l \oplus A^l}\,IH}{\Pi^\bullet \otimes A \leq B^l}\,A2'}{A \leq \Pi^{\bullet r} \oplus B^l}\,A2}{A \otimes B \leq \Pi^{\bullet r}}\,A2'}{(A \otimes B)^{lr} \leq \Pi^{\bullet r}}\,A3'}{\Pi^\bullet \leq (A \otimes B)^l}\,A4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \leq \Pi^\circ}\,IH \; \dfrac{}{\Pi^\circ \leq \Pi^{\bullet r}}\,L.25}{A \leq \Pi^{\bullet r}}\,\circ \quad \dfrac{\dfrac{}{B \leq \Sigma^\circ}\,IH \; \dfrac{}{\Sigma^\circ \leq \Sigma^{\bullet r}}\,L.25}{B \leq \Sigma^{\bullet r}}\,\circ}{A \oplus B \leq \Pi^{\bullet r} \oplus \Sigma^{\bullet r}}\,A4'}{\Pi^\bullet \otimes (A \oplus B) \leq \Sigma^{\bullet r}}\,A2}{\Pi^\bullet \leq \Sigma^{\bullet r} \oplus (A \oplus B)^l}\,A2'}{\Sigma^\bullet \otimes \Pi^\bullet \leq (A \oplus B)^l}\,A2 \qquad \square$$

We briefly mention an alternative sequent presentation, due to Abrusci [1991]. While originally formulated for associative bilinear logic, we present its non-associative, unit-free version in F.4.7. Compared to F.4.6, each connective has only a single introduction rule, while the previous (i) and (ii) are replaced with (rr) and (ll). Crucially, the equivalences $A^{rl} = A = A^{lr}$ are used in *both* directions. Particularly, (ii) of F.4.6 ($\vdash \Pi, A^l$ iff $\vdash A^r, \Pi$) is simulated by (ll) and (rr) by *expanding* A^r (A^l) to A^{lrr} (resp. A^{rll}). Consequently, one may argue that F.4.7 no more satisfies the subformula property than F.4.6, although the latter is more explicit in this regard.

Preorder laws	Negations and structural rules	Logical rules
$\dfrac{}{\vdash A, A^l}\ Id$	$(rr) \vdash A^{rr}, \Pi$ implies $\vdash \Pi, A$	$\dfrac{\vdash \Pi, A \bullet B}{\vdash \Pi, A \oplus B}\ \oplus$
$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^l}{\vdash \Gamma, \Delta}\ \circ$	$(ll) \vdash \Pi, A^{ll}$ implies $\vdash A, \Pi$ (i) $\vdash \Pi \bullet \Sigma, \Upsilon$ iff $\vdash \Pi, \Sigma \bullet \Upsilon$	$\dfrac{\vdash \Pi, A \quad \vdash \Sigma, B}{\vdash \Sigma \bullet \Pi, A \otimes B}\ \otimes$

Figure 4.7: Abrusci's sequent presentation for bilinear logic adapted to **NBL**.

We show Cut admissibility for F.4.6's sequent presentation by model-theoretic means, following the method of Okada [2002]. Our models essentially constitute a concrete (and, for our purposes, the only relevant) instance of Abrusci's [1991] phase models for bilinear logic, here adapted to the unit-free, non-associative setting. Its interpretations $[\![A]\!]$ and $[\![\Pi]\!]$ of formulas A and structures Π consist of sets of structures. Writing $\vdash^{cf}$ for $\{\langle \Pi, \Sigma \rangle \mid \vdash^{cf} \Pi, \Sigma\}$, the desired result follows from composing

(i) $\Pi, \Sigma \vdash$ implies $[\![\Pi]\!] \times [\![\Sigma]\!] \subseteq \vdash^{cf}$; and
(ii) $\Pi \in [\![\Pi]\!]$ for arbitrary Π

As intermediate steps, we show $A \in [\![A]\!]$ and $[\![A]\!] \times \{A^l\} \subseteq \vdash^{cf}$ for all A. Writing $\mathscr{S}$ for the set of structures, we note the existence of a Galois connection on $\mathscr{P}(\mathscr{S})$.

Definition 54. Letting $X \subseteq \mathscr{S}$, define

$$X^l := \{\Sigma \mid \forall (\Pi \in X), \langle \Pi, \Sigma \rangle \in \vdash^{cf}\}$$
$$X^r := \{\Pi \mid \forall (\Sigma \in X), \langle \Pi, \Sigma \rangle \in \vdash^{cf}\}$$

Lemma 27. For any $X, Y \subseteq \mathscr{S}$, $X \times Y \subseteq \vdash^{cf}$ iff $Y \subseteq X^l$ iff $X \subseteq Y^r$.

Proof. By a trivial definition unfolding. □

Corollary 1. For any $X, Y \in \mathscr{P}(\mathscr{S})$, $X \subseteq Y^l$ iff $Y \subseteq X^r$.

Corollary 2. The compositions $\cdot^{lr}$ and $\cdot^{rl}$ define closure operations on $\mathscr{P}(\mathscr{S})$. I.e.,

1. $X \subseteq X^{lr}$ and $X \subseteq X^{rl}$
2. $X \subseteq Y$ implies $X^{lr} \subseteq Y^{lr}$ and $X^{rl} \subseteq Y^{rl}$
3. $X^{lrlr} \subseteq X^{lr}$ and $X^{rlrl} \subseteq X^{rl}$

Lemma 28. For any $X \subseteq \mathscr{S}$, $X^{lr} = X^{rl}$.

Proof. We treat ($\supseteq$), the converse ($\subseteq$) being similar. So let (a) $\Pi \in X^{rl}$, and (b) $\Sigma \in X^{l}$. We show $\vdash^{cf} \Pi, \Sigma$, iff $\vdash^{cf} \Pi, \Sigma^{RL}$ iff $\vdash^{cf} \Sigma^{RR}, \Pi$. By (a), it suffices to show $\Sigma^{RR} \in X^{r}$. So let $\Upsilon \in X$. By (b), $\vdash^{cf} \Upsilon, \Sigma$, hence $\vdash^{cf} \Upsilon, \Sigma^{RL}$ and $\vdash^{cf} \Sigma^{RR}, \Upsilon$. $\square$

Definition 55. The interpretations of structures and formulas are defined by the following clauses, writing $X \bullet Y$ for $\{\Pi \bullet \Sigma \mid \Pi \in X, \Sigma \in Y\}$ ($X, Y \subseteq \mathscr{S}$):

$$\begin{array}{rll} [\![p]\!] := \{\Pi \mid \vdash^{cf} \Pi, p^{l}\} & [\![A \otimes B]\!] := ([\![B]\!]^{r} \bullet [\![A]\!]^{r})^{l} & [\![A^{l}]\!] := [\![A]\!]^{l} \\ [\![\Pi \bullet \Sigma]\!] := [\![\Pi]\!] \bullet [\![\Sigma]\!] & [\![A \oplus B]\!] := ([\![A]\!] \bullet [\![B]\!])^{rl} & [\![A^{r}]\!] := [\![A]\!]^{r} \end{array}$$

Lemma 29. For all A, $[\![A]\!]^{lr} = [\![A]\!] = [\![A]\!]^{rl}$.

Proof. Immediate from C.2 and L.28. $\square$

Lemma 30. Derivability is sound. I.e., for all Π, Σ, $\vdash \Pi, \Sigma$ implies $[\![\Pi]\!] \times [\![\Sigma]\!] \in \vdash^{cf}$.

Proof. By induction on the derivation witnessing $\Pi, \Sigma \vdash$. We note the cases (i)-(iii) constitute trivial definition unfoldings.

- (R) We have $[\![A]\!] \times [\![A^{l}]\!] \subseteq \vdash^{cf}$ iff $[\![A]\!] \subseteq [\![A^{l}]\!]^{r}$. Noting $[\![A^{l}]\!] = [\![A]\!]^{l}$, the desired result is immediate from L.29.
- (T) By L.27, the induction hypotheses amount to $[\![A]\!] \subseteq [\![\Pi]\!]^{l}$ and $[\![\Sigma]\!] \subseteq [\![A^{l}]\!]^{r} = [\![A]\!]$. Hence, $[\![\Sigma]\!] \subseteq [\![\Pi]\!]^{l}$, as desired.
- ($\oplus$) By induction hypothesis, $[\![\Pi]\!] \subseteq ([\![A]\!] \bullet [\![B]\!])^{r}$. Since, for any $X \subseteq \mathscr{S}$, $X \subseteq X^{lr}$, also $[\![\Pi]\!] \subseteq ([\![A]\!] \bullet [\![B]\!])^{rlr} = [\![A \oplus B]\!]^{r}$, as desired.
- ($\otimes$) By induction hypothesis, $[\![\Pi]\!] \subseteq [\![A]\!]^{r}$ and $[\![\Sigma]\!] \subseteq [\![B]\!]^{r}$. Hence, $[\![\Sigma]\!] \bullet [\![\Pi]\!] (= [\![\Sigma \bullet \Pi]\!]) \subseteq [\![B]\!]^{r} \bullet [\![A]\!]^{r} \subseteq ([\![B]\!]^{r} \bullet [\![A]\!]^{r})^{lr} = [\![A \otimes B]\!]^{r}$.
- ($\oplus^{l}$) By induction hypothesis, $[\![\Pi]\!] \subseteq [\![A]\!]^{lr} = [\![A]\!]$ and $[\![\Sigma]\!] \subseteq [\![B]\!]^{lr} = [\![B]\!]$, hence $[\![\Pi]\!] \bullet [\![\Sigma]\!] (= [\![\Pi \bullet \Sigma]\!]) \subseteq [\![A]\!] \bullet [\![B]\!] \subseteq [\![A \oplus B]\!]$.
- ($\otimes^{l}$) By induction hypothesis, $[\![\Pi]\!] \subseteq ([\![B]\!]^{l} \bullet [\![A]\!]^{l})^{r}$. We show $[\![\Pi]\!] \subseteq [\![(A \oplus B)^{l}]\!]^{r} = [\![A \oplus B]\!]$. So let (a) $\Sigma \in ([\![B]\!]^{l} \bullet [\![A]\!]^{l})^{r}$. We show $\Sigma \in [\![A \oplus B]\!]$, implying the desired result. To this end, suppose (b) $\Upsilon_2 \in [\![B]\!]^{r}$, and (c) $\Upsilon_1 \in [\![A]\!]^{r}$. We show $\vdash^{cf} \Upsilon_2 \bullet \Upsilon_1, \Sigma$. By (b) and (c), $\Upsilon_1^{LL} \in [\![A]\!]^{l}$ and $\Upsilon_2^{LL} \in [\![B]\!]^{l}$. Hence, by (a), $\vdash^{cf} \Sigma, \Upsilon_2^{LL} \bullet \Upsilon_1^{LL}$. The desired result now follows from repeated applications of L.23. $\square$

Lemma 31. For any A, (a) $\Pi \in [\![A]\!]$ implies $\vdash^{cf} \Pi, A^l$; and (b) $A \in [\![A]\!]$.

Proof. By simultaneous induction on A, the base case being immediate.

Case $A \oplus B$.

(a) Let $\Pi \in [\![A \oplus B]\!]$, iff for every Σ s.t. $(\forall(\Upsilon_1 \in [\![A]\!])(\Upsilon_2 \in [\![B]\!]), \vdash^{cf} \Sigma, \Upsilon_1 \bullet \Upsilon_2)$, also $\vdash^{cf} \Sigma, \Pi$. We show the antecedent for $\Sigma := (A \oplus B)^r$, applying (ii). So let $\Upsilon_1 \in [\![A]\!], \Upsilon_2 \in [\![B]\!]$. By IH, $\vdash^{cf} \Upsilon_1, A^l$ and $\vdash^{cf} \Upsilon_2, B^l$, hence $\Upsilon_1 \bullet \Upsilon_2, (A \oplus B)^l$ by $(\oplus^l)$, and we apply (ii).

(b) Let Π be such that for any $\Sigma \in [\![A]\!]$ and $\Upsilon \in [\![B]\!]$, $\vdash^{cf} \Pi, \Sigma \bullet \Upsilon$. Since, by IH, $A \in [\![A]\!]$ and $B \in [\![B]\!]$, also $\vdash \Pi, A \bullet B$, and we apply $(\oplus)$.

Case $A \otimes B$.

(a) Let $\Pi \in [\![A \otimes B]\!]$, iff $\forall(\Sigma \in [\![A]\!]^r)(\Upsilon \in [\![B]\!]^r), \vdash^{cf} \Upsilon \bullet \Sigma, \Pi$. Since for any $\Xi \in [\![A]\!]$, $\Xi, A^l \vdash^{cf}$ by IH, $\vdash^{cf} A^r, \Xi$ by (ii) and hence $A^r \in [\![A]\!]^r$. Similarly, $B^r \in [\![B]\!]^r$. Consequently, $\vdash^{cf} B^r \bullet A^r, \Pi$, and we derive $\vdash^{cf} \Pi, B^l \bullet A^l$ through repeated application of $(ii), (iii)$, concluding with $(\otimes^l)$.

(b) Let $\Pi \in [\![A]\!]^r$, $\Sigma \in [\![B]\!]^r$. We show $\vdash^{cf} \Sigma \bullet \Pi, A \otimes B$. Since, by IH, $A \in [\![A]\!]$ and $B \in [\![B]\!]$, $\vdash^{cf} \Pi, A$ and $\vdash^{cf} \Sigma, B$, and we apply $(\otimes)$.

Case A^l.

(a) Let $\Pi \in [\![A]\!]^l$, iff $\forall(\Sigma \in [\![A]\!])$, $\vdash^{cf} \Sigma, \Pi$. Since $A \in [\![A]\!]$, also $\vdash^{cf} A, \Pi$, and $\vdash^{cf} A^{lr}, \Pi$ by (i), concluding with $\vdash^{cf} \Pi, A^{rr}$ by (ii).

(b) Let $\Pi \in [\![A]\!]$. By IH, $\vdash^{cf} \Pi, A^l$, as desired.

Case A^r.

(a) Let $\Pi \in [\![A^r]\!]$, iff $\forall(\Sigma \in [\![A]\!]), \vdash^{cf} \Pi, \Sigma$. Since $A \in [\![A]\!]$ by IH, $\Pi, A \vdash^{cf}$, and hence $\vdash^{cf} \Pi, A^{rl}$ by (i).

(b) Let $\Pi \in [\![A]\!]$. By IH, $\vdash^{cf} \Pi, A^l$, and $\vdash^{cf} A^r, \Pi$ by (ii). □

Lemma 32. For any Π, $\Pi \in [\![\Pi]\!]$.

Proof. By induction on Π. The base case is immediate from L.31. If $\Pi = \Sigma \bullet \Upsilon$, then by induction hypothesis, $\Sigma \in [\![\Sigma]\!]$ and $\Upsilon \in [\![\Upsilon]\!]$. Hence, $\Sigma \bullet \Upsilon \in [\![\Sigma]\!] \bullet [\![\Upsilon]\!] = [\![\Sigma \bullet \Upsilon]\!]$. □

Theorem 4.4.2. Cut is admissible.

Proof. Suppose $\vdash^{cf} \Pi, \Sigma$. By soundness, $\Pi', \Sigma' \vdash^{cf}$ for any $\Pi' \in [\![\Pi]\!]$ and $\Sigma' \in [\![\Sigma]\!]$. The previous lemma notes that in particular $\Pi \in [\![\Pi]\!]$ and $\Sigma \in [\![\Sigma]\!]$. □

Preorder laws	Structural rules	Logical rules
$\dfrac{}{\vdash A, A^{\perp}}\ Id$	(i) $\vdash \Gamma, \Delta$ iff $\vdash \Delta, \Gamma$	$\dfrac{\vdash \Gamma, A \bullet B}{\vdash \Gamma, A \oplus B}\ \oplus$
$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ \circ$	(ii) $\vdash \Gamma \bullet \Delta, \Theta$ iff $\vdash \Gamma, \Delta \bullet \Theta$	$\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Delta \bullet \Gamma, A \otimes B}\ \otimes$

Figure 4.8: Classical non-associative Lambek calculus

4.4.2 Classical non-associative Lambek calculus

Compared to **NBL**, **CNL**'s use of a single negation allows for a significantly simpler sequent presentation, due to De Groote and Lamarche. Bar minor notational changes, the main difference in our presentation is that, contrary to De Groote and Lamarche, we do not restrict to the Cut-free fragment. Like with **NBL**, normalization is shown by model-theoretic means.

Formulas are redefined to take advantage of the De Morgan laws and involutivity of negation, treating them as rewrite rules that have the effect of 'pushing' negations inward towards the atoms, thus leading us to distinguish, for each atom p, its affirmation (simply written p) and denial ($\bar{p}$).

Definition 56. Formulas for the sequent presentation of **CNL** are defined as follows:

$$A, B \quad ::= \quad p \mid \bar{p} \mid (A \otimes B) \mid (A \oplus B)$$

Linear negation becomes a defined operation:

$$\begin{array}{rclrcl} p^{\perp} & := & \bar{p} & \bar{p}^{\perp} & := & p \\ (A \otimes B)^{\perp} & := & B^{\perp} \oplus A^{\perp} & (A \oplus B)^{\perp} & := & B^{\perp} \otimes A^{\perp} \end{array}$$

Showing involutivity constitutes a trivial induction.

Structures (using metavariables Γ, Δ, Θ) are defined as with the final sequent presentation for **NBL**, being binary-branching trees of formulas.

Definition 57. F.4.8 defines sequent derivations for **CNL** using judgements $\vdash \Gamma, \Delta$.

The previous rules (i) and (ii) of **NBL**, relying heavily on negations, are replaced by a single structural rule exchanging the components of a sequent, while each connective now has a single introduction rule. Consequently, the subformula property is recovered, although an analogous substructure property still fails.

The exercise of translating between **CNL**'s algebraic and sequent presentations constitutes a simplification of what was already conducted for **NBL**. We thus omit its repetition, rather proceeding straight to Cut admissibility. Not unexpectedly, we again find that the collapse of $\cdot^{l}$ and $\cdot^{r}$ considerably simplifies matters. Again, write $\vdash^{cf}$ for the set of pairs $\langle\Gamma,\Delta\rangle$ s.t. $\vdash^{cf}\Gamma,\Delta$.

Definition 58. Letting $X\subseteq\mathscr{S}$, define $X^{\perp}$ by $\{\Gamma\mid\forall(\Delta\in X),\langle\Gamma,\Delta\rangle\in\vdash^{cf}\}$.

Lemma 33. $\cdot^{\perp}$ is a Galois connection on $\mathscr{P}(\mathscr{S})$, and $\cdot^{\perp\perp}$ a closure operation.

The next lemma again follows by definition unfolding, in addition to using (i).

Lemma 34. For any $X,Y\in\mathscr{S}$, $X\times Y\subseteq\vdash^{cf}$ iff $Y\subseteq X^{\perp}$ iff $X\subseteq Y^{\perp}$.

Definition 59. The evaluation function $[\![\cdot]\!]$ interprets formulas and structures by subsets of structures, as follows:

$$\begin{array}{ll}
[\![\Gamma\bullet\Delta]\!]:=[\![\Gamma]\!]\bullet[\![\Delta]\!] & \\
[\![A\otimes B]\!]:=([\![B]\!]^{\perp}\bullet[\![A]\!]^{\perp})^{\perp} & [\![p]\!]:=\{\Gamma\mid\ \vdash^{cf}\Gamma,\bar{p}\} \\
[\![A\oplus B]\!]:=([\![A]\!]\bullet[\![B]\!])^{\perp\perp} & [\![\bar{p}]\!]:=\{\Gamma\mid\ \vdash^{cf}\Gamma,p\}
\end{array}$$

One easily checks that for each A, $[\![A]\!]$ is closed w.r.t. $\cdot^{\perp\perp}$, i.e., $[\![A]\!]=[\![A]\!]^{\perp\perp}$.

Lemma 35. $\vdash\Gamma,\Delta$ implies $[\![\Gamma]\!]\times[\![\Delta]\!]\in\vdash^{cf}$, iff $[\![\Gamma]\!]\subseteq[\![\Delta]\!]^{\perp}$ iff $[\![\Delta]\!]\subseteq[\![\Gamma]\!]^{\perp}$.

Proof. By induction. Again, (i) and (ii) are trivial, as are (Id) and $(\circ)$, amounting, respectively, to reflexivity and transitivity of $\subseteq$. We are thus left to check

($\oplus$) By induction hypothesis, $[\![\Gamma]\!]\subseteq([\![A]\!]\bullet[\![B]\!])^{\perp}$, the latter equivalent with $([\![A]\!]\bullet[\![B]\!])^{\perp\perp\perp}=[\![A\oplus B]\!]^{\perp}$.

($\otimes$) By induction hypothesis, $[\![\Gamma]\!]\subseteq[\![A]\!]^{\perp}$ and $[\![\Delta]\!]\subseteq[\![B]\!]^{\perp}$. Hence, $[\![\Delta]\!]\bullet[\![\Gamma]\!](=[\![\Gamma\bullet\Delta]\!])\subseteq[\![B]\!]^{\perp}\bullet[\![A]\!]^{\perp}\subseteq([\![B]\!]^{\perp}\bullet[\![A]\!]^{\perp})^{\perp\perp}=[\![A\otimes B]\!]^{\perp}$. □

Lemma 36. For any A, $A\in[\![A]\!]$.

Proof. By induction on A, the base case being immediate.

($\oplus$) Let Γ be s.t. for any $\Delta\in[\![A]\!]$ and $\Theta\in[\![B]\!]$, $\vdash^{cf}\Gamma,\Delta\bullet\Theta$. Since, by IH, $A\in[\![A]\!]$ and $B\in[\![B]\!]$, also $\vdash^{cf}\Gamma,A\bullet B$, and we apply $(\oplus)$ and (i).

($\otimes$) Let $\Gamma\in[\![A]\!]^{\perp}$, $\Delta\in[\![B]\!]^{\perp}$. We show $\vdash^{cf}\Delta\bullet\Gamma,A\otimes B$, and apply (i). Since, by IH, $A\in[\![A]\!]$ and $B\in[\![B]\!]$, $\vdash^{cf}\Gamma,A$ and $\vdash^{cf}\Delta,B$, and we apply $(\otimes)$. □

Note we needed not show $\Gamma \in [\![A]\!]$ implies $\vdash \Gamma, A^{\perp}$. While an analogous statement was needed for **NBL** to prove $A^{l} \in [\![A^{l}]\!]$ and $A^{r} \in [\![A^{r}]\!]$, the defined nature of linear negation eliminates this need. This seems to constitute a simplification w.r.t. the methods of Okada [2002] as well. We next extend the above result to the level of structures, implying Cut admissibility by composition with soundness.

Lemma 37. For any Γ, $\Gamma \in [\![\Gamma]\!]$.

Proof. By a trivial induction, relying on the previous lemma for the base case. □

Theorem 4.4.3. Cut is admissible for **CNL**.

5

Labeled deduction and context-freeness

5.1 Introduction

As argued in chapter 3, the various understandings of the single denominator 'Lambek-Grishin calculus' together comprise a hierarchy of substructural logics, all constructed from a 'base' logic through modular extensions by structural rules licensing associativity and commutativity, whether same-sort or mixed. The current chapter explores the generative capacity of the base logic, proving that the languages recognized by the corresponding notion of grammar are context-free. This result remains in line with previous characterizations of expressivity for **LG**'s intuitionistic counterparts, while complementing Melissen [2011], where **LG** augmented by type IV interactions was shown to exceed LTAG in expressive power. Like in [Pentus, 1999, Buszkowski, 2002, Jäger, 2004], our proof method relies on the use of an interpolation lemma to construct a finite axiomatization for the restriction of **LG** obtained by putting an upper bound on the size of formulas. This is to be contrasted with [Buszkowski, 1988, Kandulski, 1988, Capelletti, 2005], who prove context-freeness by normalizing derivations.

Past proof-theoretic investigations into **LG** primarily concerned proof nets [Moot, 2007] and a display calculus [Moortgat, 2009]. The latter, already briefly described in chapter 3, generalizes ordinary sequent calculi by adding invertible structural rules for isolating the main formula of a logical rule. While facilitating Cut-free presentations of a great deal of modal and substructural logics [see Wansing, 1998, Goré, 1998], they have proved difficult for proving interpolation (but see [Brotherston and Goré, 2011] for recent progress). As an alternative, the current exposition explores *labeled deduction* [see Gabbay, 1996] for **LG**, being free of structural rules. Our approach stays close to the labeled calculi for modal logics and (**N**)**L** studied by Negri [2005] and Kurtonina [1995] respectively, who annotated derivations by information pertaining to relational models, to be contrasted with the algebraic labelings pursued by D'Agostino and Gabbay [1994]. That said, we put little emphasis on model theory, instead pursuing an intuition grounded in the theory of unrooted trees, already present in the aforecited literature on proof nets and described more abstractly by Andreoli [2004] and Lamarche [2003]. In addition, in line with Pinto and Uustalu [2010], we validate soundness constructively, constrasting with the usual classical demonstrations found in the literature on labeled deduction.

We proceed as follows. §2 explicates our labeling device, describing the structure imposed on formulas during inference. Labeled deduction is defined in §3, proven sound and complete w.r.t. algebraic derivability. §4 tackles interpolation and context-freeness, while §5 discusses the representation of Grishin's type I and IV interactions inside labeled deduction. In §6 we repeat the previous exercise for **CNL**. §7 concludes with a brief summary of our results and pointers to related literature.

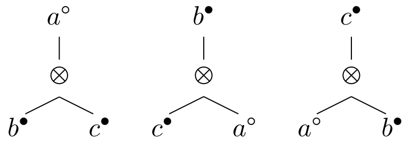
5.2 On varieties and presentations

LG being non-associative, reasoning is made contingent upon the structuring of the available formulas into trees. Furthermore, with the asymmetry between hypotheses and conclusions abolished, we no longer possess a canonical choice of root node, whereas for **NL** we could simply pick the unique output. The resulting *unrooted* trees already appeared prominently in the literature on non-associative proof nets, particularly as the *tensor trees* of Moot [2007] and as the *tree signatures* of Lamarche [2003], the latter building forth on De Groote and Lamarche's [2002] work on **CNL**. We aim to present a linearization through use of relation symbols, facilitating our interest in 'regular' sequent derivations. To motivate the general applicability of our approach, we work within the extension of **LG** with unary residuated modalities, adapting the concept of unrooted tree accordingly.

Prior to presenting our D.60 of the desired concept, we provide an informal exposition of its contents. Assume to have at our disposal a countably infinite set of *names* used for declaring nodes a, b, ... Those occurring as leaves act as locations for storing formulas (cf. §3), being categorized according to whether they can contain *inputs* (*hypotheses*) or *outputs* (*conclusions*). What constitutes a(n unrooted) tree $\boldsymbol{\gamma}$($\boldsymbol{\delta}$, $\boldsymbol{\epsilon}$) is defined using a small number of primitive such constructs, together with an operation for gluing these together. Starting with the former, we have four types to consider, one corresponding to a single node, the others to various hyperedges:[1]

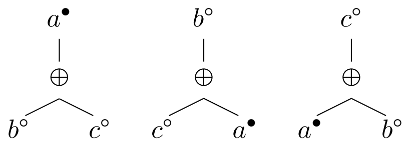
1. The simplest tree consists of a single node, assigned some arbitrary name a, being comparable to the (absent) structure of a two-formula sequent. As such, it is classified as both input and output, holding room for two formulas.

2. Hyperedges $\otimes abc$ connect nodes $a \neq b \neq c$, with b and c acting as input, while a constitutes the unique output. Graphically, an unrooted tree may be presented by as many rooted counterparts as there are nodes to pick as root. We currently have three such options, using counter-clockwise ordering:

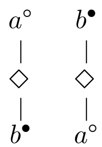


   Note we have identified in- and outputs by superscripting with $^\bullet$ and $^\circ$ respectively. Exclusive usage of this type of hyperedge is typical of the **NL** fragment.

3. The situation dual to that of the previous case features hyperedges $\oplus abc$, with a appearing as the unique input opposite outputs b and c. Depicted graphically, again assuming a counter-clockwise ordering of the leaves:

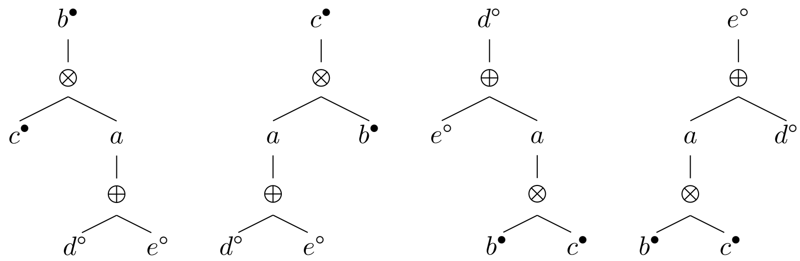


4. Finally, hyperedges $\Diamond\, ab$ foresee in structure for reasoning with modalities:

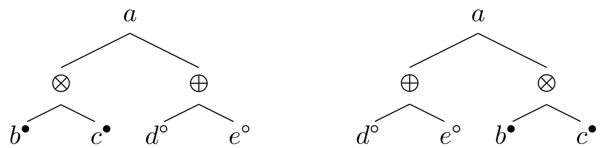


[1] While frequently borrowing terminology from the theory of hypergraphs [see Voloshin, 2009], our set-theoretic execution in D.60 will differ from standard convention.

Having described our inventory of building blocks, we next discuss how to combine them. Suppose γ and $\boldsymbol{\delta}$ are trees intersecting on a single leaf a, occurring as output in one (say γ) while as input in the other ($\boldsymbol{\delta}$). Then $(\gamma \lhd_a \boldsymbol{\delta})$ denotes the result of joining γ and $\boldsymbol{\delta}$ at a. We illustrate with $(\otimes abc \lhd_a \oplus ade)$, having four presentations where one of the leaves acts as root



If internal nodes are also considered to be chosen as root, two more presentations arise for the choice of c, being invariant under cyclic permutations of the leaves.



With these introductory remarks, we next turn to our formal definition. For more thorough discussions on the two-dimensional representation of unrooted trees, We refer the reader to the aforementioned works of Moot [2007], Lamarche [2003] and De Groote and Lamarche [2002] on proof nets.

Definition 60. A *relational frame* is a quadruple $\langle N, \otimes, \oplus, \Diamond \rangle$ with N a non-empty set of *nodes*, $\otimes, \oplus \subseteq N^3$ and $\Diamond \subseteq N^2$. A *variety* $\gamma = \langle N, \otimes, \oplus, \Diamond, I, O \rangle$ is any frame augmented by sets $I, O \subseteq N$ of *in-* and *output nodes* respectively, and which may be built using the constructions of the following table.

Trees γ	a	$\otimes abc$	$\oplus abc$	$\Diamond ab$	$(\boldsymbol{\delta} \lhd_a \boldsymbol{\epsilon})$
Conditions	-	$a \neq b \neq c$	$a \neq b \neq c$	$a \neq b$	$N_{\boldsymbol{\delta}} \cap N_{\boldsymbol{\epsilon}} = \{a\}$ $a \in O_{\boldsymbol{\delta}}$, $a \in I_{\boldsymbol{\epsilon}}$
Nodes N_γ	$\{a\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b\}$	$N_{\boldsymbol{\delta}} \cup N_{\boldsymbol{\epsilon}}$
$\otimes_\gamma$	$\varnothing$	$\{\langle a,b,c\rangle\}$	$\varnothing$	$\varnothing$	$\otimes_{\boldsymbol{\delta}} \cup \otimes_{\boldsymbol{\epsilon}}$
$\oplus_\gamma$	$\varnothing$	$\varnothing$	$\{\langle a,b,c\rangle\}$	$\varnothing$	$\oplus_{\boldsymbol{\delta}} \cup \oplus_{\boldsymbol{\epsilon}}$
$\Diamond_\gamma$	$\varnothing$	$\varnothing$	$\varnothing$	$\{\langle a,b\rangle\}$	$\Diamond_{\boldsymbol{\delta}} \cup \Diamond_{\boldsymbol{\epsilon}}$
Inputs I_γ	$\{a\}$	$\{b,c\}$	$\{a\}$	$\{b\}$	$(I_{\boldsymbol{\delta}}/\{a\}) \cup I_{\boldsymbol{\epsilon}}$
Outputs O_γ	$\{a\}$	$\{a\}$	$\{b,c\}$	$\{a\}$	$O_{\boldsymbol{\delta}} \cup (O_{\boldsymbol{\epsilon}}/\{a\})$

As noted, relations $\otimes, \oplus, \Diamond$ describe hyperedges between nodes, with the constructions found in the table ensuring each node connects to at most two edges. Note that in a tree described by a single node a, the latter occurs both as hypothesis and conclusion. In practice, we refer to the union $E_{\gamma} = I_{\gamma} \bigcup O_{\gamma}$ of any γ by the *external nodes* of γ, while N_{γ}/E_{γ} is referred to by γ's *internal nodes*.

Remark 5. We will speak interchangeably of varieties and *(unrooted) trees*, our terminology constituting a loose adaptation of that of Andreoli [2004] in his treatment of substructural sequents. There, inspired by Abrusci and Ruet's [1999] noncommutative logic, he introduced the opposition between varieties and presentations, the latter understood as 'presenting' the former from the point of view of one of its locations (here, nodes). While intended as a general framework, we are here particularly interested in the reading, mentioned as an example in [Andreoli, 2004], where varieties are taken to be unrooted trees of formulas, while their presentations are the rooted (binary-branching) trees obtained by designating an arbitrary node as root. The further examples and definitions below will continue to build forth on Andreoli's terminological distinction, though restricting to **LG** and using definitions that, while similar in spirit, are strictly speaking incompatible. Another closely related work is [Lamarche, 2003].

The identification of some given γ with a set of rooted trees, their yields equivalent under cyclic permutations, can be made further precise. In preparation, we first construct a number of derived operations on trees.

Definition 61. We define the following operations on trees using $\otimes$:

$$\begin{array}{rcll}
(\gamma \otimes_{a}^{b,c} \boldsymbol{\delta}) & := & (\gamma \lhd_b (\boldsymbol{\delta} \lhd_c \otimes abc)) & (a \notin N_{\gamma} \bigcup N_{\boldsymbol{\delta}}, b \in O_{\gamma}, c \in O_{\boldsymbol{\delta}}) \\
(\boldsymbol{\delta} \backslash_{c}^{b,a} \gamma) & := & (\boldsymbol{\delta} \lhd_b (\otimes abc \lhd_a \gamma)) & (c \notin N_{\gamma} \bigcup N_{\boldsymbol{\delta}}, a \in I_{\gamma}, b \in O_{\boldsymbol{\delta}}) \\
(\gamma /_{b}^{a,c} \boldsymbol{\delta}) & := & (\boldsymbol{\delta} \lhd_c (\otimes abc \lhd_a \gamma)) & (b \notin N_{\gamma} \bigcup N_{\boldsymbol{\delta}}, a \in I_{\gamma}, c \in O_{\boldsymbol{\delta}})
\end{array}$$

Dually, using $\oplus$:

$$\begin{array}{rcll}
(\gamma \oplus_{a}^{b,c} \boldsymbol{\delta}) & := & ((\oplus abc \lhd_b \gamma) \lhd_c \boldsymbol{\delta}) & (a \notin N_{\gamma} \bigcup N_{\boldsymbol{\delta}}, b \in I_{\gamma}, c \in I_{\boldsymbol{\delta}}) \\
(\gamma \oslash_{c}^{a,b} \boldsymbol{\delta}) & := & ((\gamma \lhd_a \oplus abc) \lhd_c \boldsymbol{\delta}) & (b \notin N_{\gamma} \bigcup N_{\boldsymbol{\delta}}, a \in O_{\gamma}, c \in I_{\boldsymbol{\delta}}) \\
(\boldsymbol{\delta} \obslash_{c}^{b,a} \gamma) & := & ((\gamma \lhd_a \oplus abc) \lhd_b \boldsymbol{\delta}) & (c \notin N_{\gamma} \bigcup N_{\boldsymbol{\delta}}, a \in O_{\gamma}, b \in I_{\boldsymbol{\delta}})
\end{array}$$

And finally, using $\Diamond$:

$$\begin{array}{rcll}
(\Diamond_{a}^{b} \gamma) & := & (\gamma \lhd_b \Diamond ab) & (a \notin N_{\gamma}, b \in O_{\gamma}) \\
(\Box_{b}^{a} \gamma) & := & (\Diamond ab \lhd_a \gamma) & (b \notin N_{\gamma}, a \in I_{\gamma})
\end{array}$$

If tree composition is read $\leq$, then the above operations give rise to the familiar (co)residuation laws, replacing bi-implication by equality:

Lemma 38. We have the following identifications:

$$
\begin{aligned}
(\boldsymbol{\delta} \lhd_c (\boldsymbol{\gamma} \backslash^{b,a}_{c} \boldsymbol{\epsilon})) = ((\boldsymbol{\gamma} \otimes^{b,c}_{a} \boldsymbol{\delta}) \lhd_a \boldsymbol{\epsilon}) &= (\boldsymbol{\gamma} \lhd_b (\boldsymbol{\epsilon} /^{a,c}_{b} \boldsymbol{\delta}))\\
((\boldsymbol{\gamma} \oslash^{a,c}_{b} \boldsymbol{\epsilon}) \lhd_b \boldsymbol{\delta}) = (\boldsymbol{\gamma} \lhd_a (\boldsymbol{\delta} \oplus^{b,c}_{a} \boldsymbol{\epsilon})) &= ((\boldsymbol{\delta} \obslash^{b,a}_{c} \boldsymbol{\gamma}) \lhd_c \boldsymbol{\epsilon})\\
((\Diamond^{b}_{a} \boldsymbol{\gamma}) \lhd_a \boldsymbol{\delta}) &= (\boldsymbol{\gamma} \lhd_b (\Box^{a}_{b} \boldsymbol{\delta}))
\end{aligned}
$$

together with the following special cases:

$$
\begin{array}{lll}
(\boldsymbol{\gamma} \otimes^{a,b}_{c} b) = (\boldsymbol{\gamma} \backslash^{a,c}_{b} c) & (\boldsymbol{\gamma} \oplus^{a,b}_{c} b) = (\boldsymbol{\gamma} \obslash^{a,c}_{b} c) & (\Diamond^{b}_{a} b) = (\Box^{a}_{b} a)\\
(a \otimes^{a,b}_{c} \boldsymbol{\delta}) = (c /^{c,b}_{a} \boldsymbol{\gamma}) & (a \oplus^{a,b}_{c} \boldsymbol{\delta}) = (c \oslash^{c,b}_{a} \boldsymbol{\gamma}) &
\end{array}
$$

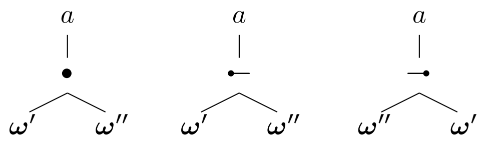
Proof. By a routine definition unfolding. □

D.78 below describes the possible rooted presentations for a variety (metavariables $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$, $\boldsymbol{\omega}''$, $\ldots$), again using the same set of names $a, b, c, \ldots$ for denoting nodes in doing so. Continuing to adapt terminology of Andreoli [2004], we shall speak interchangeably of (rooted) trees and *presentations*. As with our previous discussion on varieties, we first provide an informal explanation.

Each presentation $\boldsymbol{\omega}$ is assigned a *root* $H_{\boldsymbol{\omega}}$ from among its set of nodes $N_{\boldsymbol{\omega}}$, while its leaves are parameterized into those occurring in *positive* and *negative* positions ($+_{\boldsymbol{\omega}}$, resp. $-_{\boldsymbol{\omega}}$). Being defined inductively, we proceed (informally) as follows:

1. The base case describes a tree $\boldsymbol{\omega}$ consisting of a single node a, thus constituting the only choice for the root node ($H_{\boldsymbol{\omega}} = a$), while furthermore considered to occur in positive position ($+_{\boldsymbol{\omega}} = \{a\}$, $-_{\boldsymbol{\omega}} = \varnothing$).

2. Given presentations $\boldsymbol{\omega}, \boldsymbol{\omega}'$ with disjoint sets of nodes, as well as a fresh a, we can build presentations $\boldsymbol{\omega}_1 = (\boldsymbol{\omega}' \bullet_a \boldsymbol{\omega}'')$, $\boldsymbol{\omega}_2 = (\boldsymbol{\omega}' \bullet\!\!-_a \boldsymbol{\omega}'')$ and $\boldsymbol{\omega}_3 = (\boldsymbol{\omega}'' -\!\!\bullet_a \boldsymbol{\omega}')$, in each case featuring a as the root node. Visualized:



$(\boldsymbol{\omega}' \bullet_a \boldsymbol{\omega}'')$ $\quad$ $(\boldsymbol{\omega}' \bullet\!\!-_a \boldsymbol{\omega}'')$ $\quad$ $(\boldsymbol{\omega}'' -\!\!\bullet_a \boldsymbol{\omega}')$

The positive and negative leaves are identified upon the stipulations that $\bullet$ is upward monotone in both arguments, while $\mathbin{\bullet\!\!-}$ and $\mathbin{-\!\!\bullet}$ are upward monotone in the first, resp. second argument, and negative in the remaining position:

$$\begin{array}{ll} +_{\omega_1} := +_{\omega'} \bigcup +_{\omega''} & -_{\omega_1} := -_{\omega'} \bigcup -_{\omega''} \\ +_{\omega_2} := +_{\omega'} \bigcup -_{\omega''} & -_{\omega_2} := -_{\omega'} \bigcup +_{\omega''} \\ +_{\omega_3} := +_{\omega'} \bigcup -_{\omega''} & -_{\omega_3} := -_{\omega'} \bigcup +_{\omega''} \end{array}$$

3. Given $\boldsymbol{\omega}'$ and a fresh leaf a, we can build the tree $\boldsymbol{\omega} = \langle \boldsymbol{\omega}' \rangle_a$ with root a:

$$\begin{array}{c} a \\ | \\ \langle\rangle \\ | \\ \boldsymbol{\omega}' \end{array}$$

$\langle\rangle$ is considered upward monotone, so that $+_{\boldsymbol{\omega}} = +_{\boldsymbol{\omega}'}$ and $-_{\boldsymbol{\omega}} = -_{\boldsymbol{\omega}'}$.

Later, we shall define, for each presentation $\boldsymbol{\omega}$, two varieties $[\![\boldsymbol{\omega}]\!]^\bullet$ and $[\![\boldsymbol{\omega}]\!]^\circ$, depending on whether we interpret $H_{\boldsymbol{\omega}}$ as an output, respectively input. In the former case, the positive leaves become inputs while the negative leaves become outputs, with the converse situation arising if $H_{\boldsymbol{\omega}}$ is chosen as an input. Had we chosen to define presentations unambiguously, internalizing the input/output polarity of their roots, the number of operations defined upon them would have doubled. Note the current situation is comparable to the overloading of structural connectives in two-sided sequent calculi. With these introductory remarks, we present the desired definition.

Definition 62. We define in the following table, by induction, the set of *presentations* $\boldsymbol{\omega}(, \boldsymbol{\omega}', \ldots)$, with *root* $H_{\boldsymbol{\omega}}$, *occurrence set* $N_{\boldsymbol{\omega}}$ and disjoint sets of *positive-* and *negative leaves* $+_{\boldsymbol{\omega}}, -_{\boldsymbol{\omega}} \in \mathscr{P}(N_{\boldsymbol{\omega}})$ respectively.

Presentations $\boldsymbol{\omega}$	a	$(\boldsymbol{\omega}' \bullet_a \boldsymbol{\omega}'')$	$(\boldsymbol{\omega}' \mathbin{\bullet\!\!-}_a \boldsymbol{\omega}'')$ $(\boldsymbol{\omega}'' \mathbin{-\!\!\bullet}_a \boldsymbol{\omega}')$	$\langle \boldsymbol{\omega}' \rangle_a$
Conditions	-	$a \notin N_{\boldsymbol{\omega}} \bigcup N_{\boldsymbol{\omega}'}$, $N_{\boldsymbol{\omega}} \bigcap N_{\boldsymbol{\omega}'} = \varnothing$	$a \notin N_{\boldsymbol{\omega}} \bigcup N_{\boldsymbol{\omega}'}$, $N_{\boldsymbol{\omega}} \bigcap N_{\boldsymbol{\omega}'} = \varnothing$	$a \notin N_{\boldsymbol{\omega}}$
Nodes $N_{\boldsymbol{\omega}}$	$\{a\}$	$N_{\boldsymbol{\omega}'} \bigcup N_{\boldsymbol{\omega}''} \bigcup \{a\}$	$N_{\boldsymbol{\omega}'} \bigcup N_{\boldsymbol{\omega}''} \bigcup \{a\}$	$N_{\boldsymbol{\omega}'} \bigcup \{a\}$
Root $H_{\boldsymbol{\omega}}$	a	a	a	a
Positive leaves $+_{\boldsymbol{\omega}}$	$\{a\}$	$+_{\boldsymbol{\omega}'} \bigcup +_{\boldsymbol{\omega}''}$	$+_{\boldsymbol{\omega}'} \bigcup -_{\boldsymbol{\omega}''}$	$+_{\boldsymbol{\omega}'}$
Negative leaves $-_{\boldsymbol{\omega}}$	$\varnothing$	$-_{\boldsymbol{\omega}'} \bigcup -_{\boldsymbol{\omega}''}$	$-_{\boldsymbol{\omega}'} \bigcup +_{\boldsymbol{\omega}''}$	$-_{\boldsymbol{\omega}'}$

We now make precise the correspondence between varieties and presentations. Given a variety γ and $a \in E_\gamma$, we seek to define the presentation $(\gamma \downarrow a^\bullet)$ (if $a \in I_\gamma$) or $(\gamma \downarrow a^\circ)$ (if $a \in O_\gamma$) of γ from the point of view of a. Conversely, given any presentation $\boldsymbol{\omega}$, we can define two varieties $[\![\boldsymbol{\omega}]\!]^\bullet$ and $[\![\boldsymbol{\omega}]\!]^\circ$, depending, as the reader may recall, on whether we interpret $H_{\boldsymbol{\omega}}$ as an output, respectively input.

Definition 63. For any $\gamma, \boldsymbol{\delta}$ and $a \in I_{\gamma}$, $b \in O_{\boldsymbol{\delta}}$, define, by mutual induction, the elements $(\gamma \downarrow a^{\bullet})$ and $(\boldsymbol{\delta} \downarrow b^{\circ})$, as follows:

$$
\begin{array}{rlrl}
(a \downarrow a^{\bullet}) & := a & ((\boldsymbol{\delta} \otimes_a^{b,c} \boldsymbol{\epsilon}) \downarrow a^{\circ}) & := ((\boldsymbol{\delta} \downarrow b^{\circ}) \bullet_a (\boldsymbol{\epsilon} \downarrow c^{\circ})) \\
(a \downarrow a^{\circ}) & := a & ((\boldsymbol{\delta} \oplus_a^{b,c} \boldsymbol{\epsilon}) \downarrow a^{\bullet}) & := ((\boldsymbol{\delta} \downarrow b^{\bullet}) \bullet_a (\boldsymbol{\epsilon} \downarrow c^{\bullet})) \\
((\Diamond_a^b \boldsymbol{\delta}) \downarrow a^{\circ}) & := \langle (\boldsymbol{\delta} \downarrow b^{\circ}) \rangle_a & ((\boldsymbol{\delta} \oslash_a^{b,c} \boldsymbol{\epsilon}) \downarrow a^{\circ}) & := ((\boldsymbol{\delta} \downarrow b^{\circ}) \leftarrow_a (\boldsymbol{\epsilon} \downarrow c^{\bullet})) \\
((\Box_a^b \boldsymbol{\delta}) \downarrow a^{\bullet}) & := \langle (\boldsymbol{\delta} \downarrow b^{\bullet}) \rangle_a & ((\boldsymbol{\delta} /_a^{b,c} \boldsymbol{\epsilon}) \downarrow a^{\bullet}) & := ((\boldsymbol{\delta} \downarrow b^{\bullet}) \leftarrow_a (\boldsymbol{\epsilon} \downarrow c^{\circ})) \\
 & & ((\boldsymbol{\epsilon} \obslash_a^{c,b} \boldsymbol{\delta}) \downarrow a^{\circ}) & := ((\boldsymbol{\epsilon} \downarrow c^{\bullet}) \rightarrow_a (\boldsymbol{\delta} \downarrow b^{\circ})) \\
 & & ((\boldsymbol{\epsilon} \backslash_a^{c,b} \boldsymbol{\delta}) \downarrow a^{\bullet}) & := ((\boldsymbol{\epsilon} \downarrow c^{\circ}) \rightarrow_a (\boldsymbol{\delta} \downarrow b^{\bullet}))
\end{array}
$$

Definition 64. Given $\boldsymbol{\omega}$, define, by induction, the varieties $[\![\boldsymbol{\omega}]\!]^{\bullet}$ and $[\![\boldsymbol{\omega}]\!]^{\circ}$:

$$
\begin{array}{rlrl}
[\![a]\!]^{\bullet} & := a & [\![(\boldsymbol{\omega} \bullet_a \boldsymbol{\omega}')]\!]^{\bullet} & := ([\![\boldsymbol{\omega}]\!]^{\bullet} \otimes_a^{H_{\boldsymbol{\omega}}, H_{\boldsymbol{\omega}'}} [\![\boldsymbol{\omega}']\!]^{\bullet}) \\
[\![a]\!]^{\circ} & := a & [\![(\boldsymbol{\omega} \bullet_a \boldsymbol{\omega}')]\!]^{\circ} & := ([\![\boldsymbol{\omega}]\!]^{\circ} \oplus_a^{H_{\boldsymbol{\omega}}, H_{\boldsymbol{\omega}'}} [\![\boldsymbol{\omega}']\!]^{\circ}) \\
[\![\langle \boldsymbol{\omega} \rangle_a]\!]^{\bullet} & := (\Diamond_a^{H_{\boldsymbol{\omega}}} [\![\boldsymbol{\omega}]\!]^{\bullet}) & [\![(\boldsymbol{\omega} \leftarrow_a \boldsymbol{\omega}')]\!]^{\bullet} & := ([\![\boldsymbol{\omega}]\!]^{\bullet} \oslash_a^{H_{\boldsymbol{\omega}}, H_{\boldsymbol{\omega}'}} [\![\boldsymbol{\omega}']\!]^{\circ}) \\
[\![\langle \boldsymbol{\omega} \rangle_a]\!]^{\circ} & := (\Box_a^{H_{\boldsymbol{\omega}}} [\![\boldsymbol{\omega}]\!]^{\circ}) & [\![(\boldsymbol{\omega} \leftarrow_a \boldsymbol{\omega}')]\!]^{\circ} & := ([\![\boldsymbol{\omega}]\!]^{\circ} /_a^{H_{\boldsymbol{\omega}}, H_{\boldsymbol{\omega}'}} [\![\boldsymbol{\omega}']\!]^{\bullet}) \\
 & & [\![(\boldsymbol{\omega}' \rightarrow_a \boldsymbol{\omega})]\!]^{\bullet} & := ([\![\boldsymbol{\omega}']\!]^{\circ} \obslash_a^{H_{\boldsymbol{\omega}'}, H_{\boldsymbol{\omega}}} [\![\boldsymbol{\omega}]\!]^{\bullet}) \\
 & & [\![(\boldsymbol{\omega}' \rightarrow_a \boldsymbol{\omega})]\!]^{\circ} & := ([\![\boldsymbol{\omega}']\!]^{\bullet} \backslash_a^{H_{\boldsymbol{\omega}'}, H_{\boldsymbol{\omega}}} [\![\boldsymbol{\omega}]\!]^{\circ})
\end{array}
$$

The following is an easy induction.

Theorem 5.2.1. For any $\gamma, \boldsymbol{\delta}$ and $a \in I_{\gamma}$, $b \in O_{\boldsymbol{\delta}}$, we have $[\![(\gamma \downarrow a^{\bullet})]\!]^{\circ} = \gamma$ and $[\![(\boldsymbol{\delta} \downarrow b^{\circ})]\!]^{\bullet} = \boldsymbol{\delta}$. Conversely, for any $\boldsymbol{\omega}$, $([\![\boldsymbol{\omega}]\!]^{\bullet} \downarrow H_{\boldsymbol{\omega}}^{\circ}) = ([\![\boldsymbol{\omega}]\!]^{\circ} \downarrow H_{\boldsymbol{\omega}}^{\bullet}) = \boldsymbol{\omega}$.

We next seek an equivalence relation $\equiv$ on presentations s.t. $\boldsymbol{\omega} \equiv \boldsymbol{\omega}'$ iff $[\![\boldsymbol{\omega}]\!]^{\bullet} = [\![\boldsymbol{\omega}']\!]^{\bullet}$, iff $[\![\boldsymbol{\omega}]\!]^{\circ} = [\![\boldsymbol{\omega}']\!]^{\circ}$. Suppose this holds, and refer by γ to $[\![\boldsymbol{\omega}]\!]^{\bullet} = [\![\boldsymbol{\omega}']\!]^{\bullet}$. Then $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ can only differ in case $\boldsymbol{\omega} = (\gamma \downarrow a^{*1})$ and $\boldsymbol{\omega}' = (\gamma \downarrow b^{*2})$ $(*1, *2 \in \{\bullet, \circ\})$ for different choices of a^{*1}, b^{*2}. As such, we define an operation $\cdot^{-a}$ on presentations $\boldsymbol{\omega}$ for any choice of leaf $a \in +_{\boldsymbol{\omega}} \cup -_{\boldsymbol{\omega}}$, propagating a to the status of root while ensuring $[\![\boldsymbol{\omega}]\!]^{\bullet} = [\![\boldsymbol{\omega}^{-a}]\!]^{\bullet}$. Given then any $\boldsymbol{\omega}, \boldsymbol{\omega}'$, we set $\boldsymbol{\omega} \equiv \boldsymbol{\omega}'$ iff $\boldsymbol{\omega} = \boldsymbol{\omega}'^{-H_{\boldsymbol{\omega}}}$ $(\boldsymbol{\omega}' = \boldsymbol{\omega}^{-H_{\boldsymbol{\omega}'}})$. In preparation, we first define the notion of a context on presentations for measuring the distance between the current root and any choice of leaf.

Definition 65. Presentation *contexts* $\boldsymbol{\omega}[], \boldsymbol{\omega}'[], \ldots$ are defined as follows:

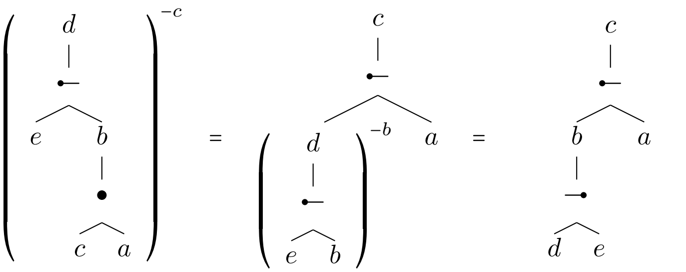
$$
\begin{array}{rcl}
\boldsymbol{\omega}[], \boldsymbol{\omega}'[] & ::= & [] \mid \langle \boldsymbol{\omega}[] \rangle_a \\
 & \mid & (\boldsymbol{\omega}[] \bullet_a \boldsymbol{\omega}') \mid (\boldsymbol{\omega} \bullet_a \boldsymbol{\omega}'[]) \\
 & \mid & (\boldsymbol{\omega}[] \leftarrow_a \boldsymbol{\omega}') \mid (\boldsymbol{\omega} \leftarrow_a \boldsymbol{\omega}'[]) \\
 & \mid & (\boldsymbol{\omega}' \rightarrow_a \boldsymbol{\omega}[]) \mid (\boldsymbol{\omega}'[] \rightarrow_a \boldsymbol{\omega})
\end{array}
$$

Denote by $\boldsymbol{\omega}[\boldsymbol{\omega}']$ the partial operation of substituting $\boldsymbol{\omega}'$ for $[]$ in $\boldsymbol{\omega}$, having a value if the result complies with D.78.

Definition 66. Given some $\boldsymbol{\omega}[a]$, define the presentation $\boldsymbol{\omega}[a]^{-a}$ by induction on the depth of $[]$:

$$\begin{array}{rcl rcl}
a^{-a} & := & a & \boldsymbol{\omega}[(a \bullet\!\!-_b \boldsymbol{\omega}')]^{-a} & := & (\boldsymbol{\omega}[b]^{-b} \bullet_a \boldsymbol{\omega}') \\
\boldsymbol{\omega}[\langle a\rangle_b]^{-a} & := & \langle \boldsymbol{\omega}[b]^{-b}\rangle_a & \boldsymbol{\omega}[(\boldsymbol{\omega}' \bullet\!\!-_b a)]^{-a} & := & (\boldsymbol{\omega}[b]^{-b} -\!\!\bullet_a \boldsymbol{\omega}') \\
\boldsymbol{\omega}[(a \bullet_b \boldsymbol{\omega}')]^{-a} & := & (\boldsymbol{\omega}[b]^{-b} -\!\!\bullet_a \boldsymbol{\omega}') & \boldsymbol{\omega}[(\boldsymbol{\omega}' -\!\!\bullet_b a)]^{-a} & := & (\boldsymbol{\omega}' \bullet_a \boldsymbol{\omega}[b]^{-b}) \\
\boldsymbol{\omega}[(\boldsymbol{\omega}' \bullet_b a)]^{-a} & := & (\boldsymbol{\omega}' \bullet\!\!-_a \boldsymbol{\omega}[b]^{-b}) & \boldsymbol{\omega}[(a -\!\!\bullet_b \boldsymbol{\omega}')]^{-a} & := & (\boldsymbol{\omega}' \bullet\!\!-_a \boldsymbol{\omega}[b]^{-b})
\end{array}$$

Example 19. Intuitively, the above definition turns the surrounding context $\boldsymbol{\omega}[]$ inside out, proceeding bottom-up. To illustrate, we calculate $(e \bullet\!\!-_d (c \bullet_b a))^{-c}$.



$(e \bullet\!\!-_d (c \bullet_b a))^{-c} \qquad ((e \bullet\!\!-_d b)^{-b} \bullet\!\!-_c a) \qquad ((d -\!\!\bullet_b e) \bullet\!\!-_c a)$

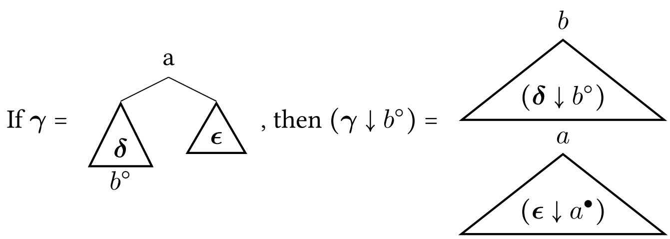
Definition 67. For any $\boldsymbol{\omega}, \boldsymbol{\omega}'$, define $\boldsymbol{\omega} \equiv \boldsymbol{\omega}'$ iff $\boldsymbol{\omega}^{-H_{\boldsymbol{\omega}'}} = \boldsymbol{\omega}'$ and $\boldsymbol{\omega}'^{-H_{\boldsymbol{\omega}}} = \boldsymbol{\omega}$.

One easily shows $\equiv$ is an equivalence relation. Before tackling the desired result, we first establish a few lemmas.

Lemma 39. Let $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ be such that $N_{\boldsymbol{\delta}} \cap N_{\boldsymbol{\epsilon}} = \{a\}$, $a \in O_{\boldsymbol{\delta}}$ and $a \in I_{\boldsymbol{\epsilon}}$. Then, for $\gamma = (\boldsymbol{\delta} \triangleleft_a \boldsymbol{\epsilon})$,

1. For any $b \in O_{\boldsymbol{\delta}}$, $(\gamma \downarrow b^\circ) = (\boldsymbol{\delta} \downarrow b^\circ)[(\boldsymbol{\epsilon} \downarrow a^\bullet)/a]$;
2. For any $b \in I_{\boldsymbol{\delta}}$, $(\gamma \downarrow b^\bullet) = (\boldsymbol{\delta} \downarrow b^\bullet)[(\boldsymbol{\epsilon} \downarrow a^\bullet)/a]$;
3. For any $b \in O_{\boldsymbol{\epsilon}}$, $(\gamma \downarrow b^\circ) = (\boldsymbol{\epsilon} \downarrow b^\circ)[(\boldsymbol{\delta} \downarrow a^\circ)/a]$;
4. For any $b \in I_{\boldsymbol{\epsilon}}$, $(\gamma \downarrow b^\bullet) = (\boldsymbol{\epsilon} \downarrow b^\bullet)[(\boldsymbol{\delta} \downarrow a^\circ)/a]$

To illustrate the statement of the lemma, we can visualize (1) as follows:



Proof. By simultaneous induction on the unique path from a to b in $\boldsymbol{\delta}$ or $\boldsymbol{\epsilon}$. The base cases, where $\boldsymbol{\delta} = a = b$ ((1) and (2)) or $\boldsymbol{\epsilon} = a = b$ ((3) and (4)), are immediate. Each of 1-4 gives rise to seven inductive cases, summarized in the following tables:

(1) and (2)	$b \in O_{\boldsymbol{\delta}_1}$	$b \in O_{\boldsymbol{\delta}_2}$	$b \in I_{\boldsymbol{\delta}_1}$	$b \in I_{\boldsymbol{\delta}_2}$
$\boldsymbol{\delta} = (\boldsymbol{\delta}_1 \otimes_a^{c,d} \boldsymbol{\delta}_2)$	1a	1b	2a	2b
$\boldsymbol{\delta} = (\boldsymbol{\delta}_1 \oslash_a^{c,d} \boldsymbol{\delta}_2)$	1c	1d	2c	2d
$\boldsymbol{\delta} = (\boldsymbol{\delta}_2 \obslash_a^{d,c} \boldsymbol{\delta}_1)$	1e	1f	2e	2f
$\boldsymbol{\delta} = (\Diamond_a^c \boldsymbol{\delta}_1)$	1g	n.a.	2g	n.a.

(3) and (4)	$b \in O_{\boldsymbol{\epsilon}_1}$	$b \in O_{\boldsymbol{\epsilon}_2}$	$b \in I_{\boldsymbol{\epsilon}_1}$	$b \in I_{\boldsymbol{\epsilon}_2}$
$\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1 \oplus_a^{c,d} \boldsymbol{\epsilon}_2)$	3a	3b	4a	4b
$\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_2 \backslash_a^{d,c} \boldsymbol{\epsilon}_1)$	3c	3d	4c	4d
$\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1 /_a^{c,d} \boldsymbol{\epsilon}_2)$	3e	3f	4e	4f
$\boldsymbol{\epsilon} = (\Box_a^c \boldsymbol{\epsilon}_1)$	3g	n.a.	4g	n.a.

$\square$

For practical purposes, these may be reduced to two typical cases, the others being handled similarly. Thus, we check (1a) and (1g).

1a. Note $\boldsymbol{\gamma} = (\boldsymbol{\delta}_1 \triangleleft_c (\boldsymbol{\epsilon} /_c^{a,d} \boldsymbol{\delta}_2))$, so by induction hypothesis $(\boldsymbol{\gamma} \downarrow b^\circ) = (\boldsymbol{\delta}_1 \downarrow b^\circ)[((\boldsymbol{\epsilon} /_c^{a,d} \boldsymbol{\delta}_2) \downarrow c^\bullet)/c] = (\boldsymbol{\delta}_1 \downarrow b^\circ)[((\boldsymbol{\epsilon} \downarrow a^\bullet) \multimapinv_c (\boldsymbol{\delta}_2 \downarrow d^\circ))/c] = (\boldsymbol{\delta}_1 \downarrow b^\circ)[(a \multimapinv_c (\boldsymbol{\delta}_2 \downarrow d^\circ))/c][(\boldsymbol{\epsilon} \downarrow a^\bullet)/a] = (\boldsymbol{\delta}_1 \downarrow b^\circ)[((a /_c^{a,d} \boldsymbol{\delta}_2) \downarrow c^\bullet)/c][(\boldsymbol{\epsilon} \downarrow a^\bullet)/a]$. Again, by induction hypothesis, this equals $((\boldsymbol{\delta}_1 \triangleleft_c (a /_c^{a,d} \boldsymbol{\delta}_2)) \downarrow b^\circ)[(\boldsymbol{\epsilon} \downarrow a^\bullet)/a] = ((\boldsymbol{\delta}_1 \otimes_a^{c,d} \boldsymbol{\delta}_2) \downarrow b^\circ)[(\boldsymbol{\epsilon} \downarrow a^\bullet)/a]$, as desired.

1g. Note $\boldsymbol{\gamma} = (\boldsymbol{\delta}_1 \triangleleft_c (\Box_c^a \boldsymbol{\epsilon}))$. By induction hypothesis $(\boldsymbol{\gamma} \downarrow b^\circ) = (\boldsymbol{\delta}_1 \downarrow b^\circ)[((\Box_c^a \boldsymbol{\epsilon}) \downarrow c^\bullet)/c] = (\boldsymbol{\delta}_1 \downarrow b^\circ)[\langle(\boldsymbol{\epsilon} \downarrow a^\bullet)\rangle_c/c] = (\boldsymbol{\delta}_1 \downarrow b^\circ)[\langle a\rangle_c/c][(\boldsymbol{\epsilon} \downarrow a^\bullet)/a] = (\boldsymbol{\delta}_1 \downarrow b^\circ)[((\Box_c^a a) \downarrow c^\bullet)/c][(\boldsymbol{\epsilon} \downarrow a^\bullet)/a]$. By induction hypothesis, this equals $(\boldsymbol{\delta}_1 \triangleleft_c (\Box_c^a a))[(\boldsymbol{\epsilon} \downarrow a^\bullet)/a] = (\Diamond_a^c \boldsymbol{\delta}_1)[(\boldsymbol{\epsilon} \downarrow a^\bullet)/a]$.

Lemma 40. For any $\boldsymbol{\gamma}$ and $a, b \in E_{\boldsymbol{\gamma}}$, we have the following results, depending on whether a and b are in- or outputs:

1. $(\boldsymbol{\gamma} \downarrow a^\circ)^{-b} = (\boldsymbol{\gamma} \downarrow b^\circ)$ $(a, b \in O_{\boldsymbol{\gamma}})$
2. $(\boldsymbol{\gamma} \downarrow a^\bullet)^{-b} = (\boldsymbol{\gamma} \downarrow b^\circ)$ $(a \in I_{\boldsymbol{\gamma}}, b \in O_{\boldsymbol{\gamma}})$
3. $(\boldsymbol{\gamma} \downarrow a^\bullet)^{-b} = (\boldsymbol{\gamma} \downarrow b^\bullet)$ $(a, b \in I_{\boldsymbol{\gamma}})$
4. $(\boldsymbol{\gamma} \downarrow a^\circ)^{-b} = (\boldsymbol{\gamma} \downarrow b^\bullet)$ $(a \in O_{\boldsymbol{\gamma}}, b \in I_{\boldsymbol{\gamma}})$

Proof. By simultaneous induction on the length of the unique path from b to a. If already $a = b$, we are done. The remaining inductive cases are as follows:

1 and 2	$a \in O_{\boldsymbol{\delta}}$	$a \in O_{\boldsymbol{\epsilon}}$	$a \in I_{\boldsymbol{\delta}}$	$a \in I_{\boldsymbol{\epsilon}}$
$\gamma = (\boldsymbol{\delta} \otimes_b^{c,d} \boldsymbol{\epsilon})$	1a	1b	2a	2b
$\gamma = (\boldsymbol{\delta} \oslash_b^{c,d} \boldsymbol{\epsilon})$	1c	1d	2c	2d
$\gamma = (\boldsymbol{\epsilon} \obslash_b^{d,c} \boldsymbol{\delta})$	1e	1f	2e	2f
$\gamma = (\Diamond_b^c \boldsymbol{\delta})$	1g	n.a.	2g	n.a.

3 and 4	$a \in O_{\boldsymbol{\delta}}$	$a \in O_{\boldsymbol{\epsilon}}$	$a \in I_{\boldsymbol{\delta}}$	$a \in I_{\boldsymbol{\epsilon}}$
$\gamma = (\boldsymbol{\delta} \oplus_b^{c,d} \boldsymbol{\epsilon})$	1a	1b	2a	2b
$\gamma = (\boldsymbol{\epsilon} \backslash_b^{d,c} \boldsymbol{\delta})$	1c	1d	2c	2d
$\gamma = (\boldsymbol{\delta} /_b^{c,d} \boldsymbol{\epsilon})$	1e	1f	2e	2f
$\gamma = (\Box_b^c \boldsymbol{\delta})$	1g	n.a.	2g	n.a.

We check 1a and 1g, the others being treated similarly.

1a. Since $\gamma = (\boldsymbol{\delta} \triangleleft_c (b /_c^{b,d} \boldsymbol{\epsilon}))$, $(\gamma \downarrow a^\circ) = (\boldsymbol{\delta} \downarrow a^\circ)[(b \leftarrow\!\!\bullet_c (\boldsymbol{\epsilon} \downarrow d^\circ))/c]$ by the previous lemma, and $(\gamma \downarrow a^\circ)^{-b} = ((\boldsymbol{\delta} \downarrow a^\circ)^{-c} \bullet_b (\boldsymbol{\epsilon} \downarrow d^\circ))$. By induction hypothesis, $(\boldsymbol{\delta} \downarrow a^\circ)^{-c} = (\boldsymbol{\delta} \downarrow c^\circ)$, from which the desired result follows after noting $(\gamma \downarrow b^\circ) = ((\boldsymbol{\delta} \downarrow c^\circ) \bullet_b (\boldsymbol{\epsilon} \downarrow d^\circ))$.

1g. Note $\gamma = (\boldsymbol{\delta} \triangleleft_c (\Box_c^b b))$. Hence, by the previous lemma, $(\gamma \downarrow a^\circ) = (\boldsymbol{\delta} \downarrow a^\circ)[\langle b \rangle_c / c]$, so that $(\gamma \downarrow a^\circ)^{-b} = \langle (\boldsymbol{\delta} \downarrow a^\circ)^{-c} \rangle_b$. By induction hypothesis, $(\boldsymbol{\delta} \downarrow a^\circ)^{-c} = (\boldsymbol{\delta} \downarrow c^\circ)$, so that the desired result follows after realizing $(\gamma \downarrow b^\circ) = \langle (\boldsymbol{\delta} \downarrow c^\circ) \rangle_b$. □

Theorem 5.2.2. $\boldsymbol{\omega} \equiv \boldsymbol{\omega}'$ iff $\boldsymbol{\omega}^\bullet = \boldsymbol{\omega}'^\bullet$, or, equivalently, $\boldsymbol{\omega}^\circ = \boldsymbol{\omega}'^\circ$.

Proof. Below, we argue $\boldsymbol{\omega}^{-H_{\boldsymbol{\omega}'}} = \boldsymbol{\omega}'$ on the assumption $[\![\boldsymbol{\omega}]\!]^\bullet = [\![\boldsymbol{\omega}']\!]^\bullet$ (with $\boldsymbol{\omega}'^{-H_{\boldsymbol{\omega}}} = \boldsymbol{\omega}$ shown similarly), as well as the converse.

$$
\begin{array}{lllll}
\boldsymbol{\omega}^{-H_{\boldsymbol{\omega}'}} & & [\![\boldsymbol{\omega}]\!]^\bullet & \\
= ([\![\boldsymbol{\omega}]\!]^\bullet \downarrow H^\circ_{\boldsymbol{\omega}})^{-H_{\boldsymbol{\omega}'}} & \text{(L.5.2.1)} & = [\![([\![\boldsymbol{\omega}]\!]^\bullet \downarrow H^\circ_{\boldsymbol{\omega}'})]\!]^\bullet & \text{(L.5.2.1)} \\
= ([\![\boldsymbol{\omega}]\!]^\bullet \downarrow H^\circ_{\boldsymbol{\omega}'}) & \text{(L.40)} & = [\![([\![\boldsymbol{\omega}]\!]^\bullet \downarrow H^\circ_{\boldsymbol{\omega}})^{-H_{\boldsymbol{\omega}'}}]\!]^\bullet & \text{(L.40)} \\
= ([\![\boldsymbol{\omega}']\!]^\bullet \downarrow H^\circ_{\boldsymbol{\omega}'}) & \text{(Assumption)} & = [\![\boldsymbol{\omega}^{-H_{\boldsymbol{\omega}'}}]\!]^\bullet & \text{(L.5.2.1)} \\
= \boldsymbol{\omega}' & \text{(L.5.2.1)} & = [\![\boldsymbol{\omega}']\!]^\bullet & \text{(Assumption)}
\end{array}
$$
□

5.3 Labeled deduction

We proceed to consider two derivability judgements, one operating directly on varieties while the other targeting its presentations. We concentrate our efforts on $\mathbf{LG}_{\varnothing}$, leaving the treatment of the Grishin interactions for §5. We first provide further specification of the basic units of derivability.

Definition 68. By a *signed formula* (metavariables $\phi, \psi, \ldots$) we understand a formula suffixed by $\cdot^{\bullet}$ or $\cdot^{\circ}$. We also speak of *inputs*, or *hypotheses* $A^{\bullet}$, and of *outputs*, or *conclusions* A°. A *labeled signed formula* (*lsf*), relative to a variety or presentation, pairs a signed formula with an external node, and is written $a : A^{\bullet}$ or $a : A^{\circ}$. We write $\phi^{\perp}$ for $A^{\bullet}$ if $\phi = A^{\circ}$, while else, if $\phi = A^{\bullet}$, $\phi^{\perp}$ denotes A°.

To facilitate a more compact definition of derivability, we adapt Smullyan's [1995] *unified notation* in providing a classification of signed formulas into four categories, organizing them according to similarity in proof-theoretic behavior.

Definition 69. Signed formulas are organized into four categories, as follows. Those of binary arity are divided between types α and β,

α / β	α_1/β_1	α_2/β_2	$\alpha(\boldsymbol{\omega}, b, a)$	$\beta(\boldsymbol{\omega}, \boldsymbol{\omega}', a)$	$\alpha_{a,b,c}(\boldsymbol{\gamma}, \boldsymbol{\delta}, c)$ $= \beta_{a,b,c}(\boldsymbol{\gamma}, \boldsymbol{\delta}, c)$
$A \otimes B^{\bullet}$ / $A \otimes B^{\circ}$	$A^{\bullet}/A^{\circ}$	$B^{\bullet}/B^{\circ}$	$(\boldsymbol{\omega} \mathbin{\bullet\!\!-}_a b)$	$(\boldsymbol{\omega} \bullet_a \boldsymbol{\omega}')$	$((\boldsymbol{\gamma} \otimes_c^{a,b} \boldsymbol{\delta}) \lhd_c \boldsymbol{\epsilon})$
A/B° / $A/B^{\bullet}$	$A^{\circ}/A^{\bullet}$	$B^{\bullet}/B^{\circ}$	$(\boldsymbol{\omega} \bullet_a b)$	$(\boldsymbol{\omega} \mathbin{\bullet\!\!-}_a \boldsymbol{\omega}')$	$(\boldsymbol{\epsilon} \lhd_c (\boldsymbol{\gamma} /_c^{a,b} \boldsymbol{\delta}))$
$B\backslash A^{\circ}$ / $B\backslash A^{\bullet}$	$A^{\circ}/A^{\bullet}$	$B^{\bullet}/B^{\circ}$	$(b \bullet_a \boldsymbol{\omega})$	$(\boldsymbol{\omega}' \mathbin{-\!\!\bullet}_a \boldsymbol{\omega})$	$(\boldsymbol{\epsilon} \lhd_c (\boldsymbol{\delta} \backslash_c^{b,a} \boldsymbol{\gamma}))$
$A \oplus B^{\circ}/A \oplus B^{\bullet}$	$A^{\circ}/A^{\bullet}$	$B^{\circ}/B^{\bullet}$	$(\boldsymbol{\omega} \mathbin{\bullet\!\!-}_a b)$	$(\boldsymbol{\omega} \bullet_a \boldsymbol{\omega}')$	$(\boldsymbol{\epsilon} \lhd_c (\boldsymbol{\gamma} \oplus_c^{a,b} \boldsymbol{\delta}))$
$B \oslash A^{\bullet}$ / $B \oslash A^{\circ}$	$A^{\bullet}/A^{\circ}$	$B^{\circ}/B^{\bullet}$	$(b \bullet_a \boldsymbol{\omega})$	$(\boldsymbol{\omega}' \mathbin{-\!\!\bullet}_a \boldsymbol{\omega})$	$((\boldsymbol{\delta} \oslash_c^{b,a} \boldsymbol{\gamma}) \lhd_c \boldsymbol{\epsilon})$
$A \oslash B^{\bullet}$ / $A \oslash B^{\circ}$	$A^{\bullet}/A^{\circ}$	$B^{\circ}/B^{\bullet}$	$(\boldsymbol{\omega} \bullet_a b)$	$(\boldsymbol{\omega} \mathbin{\bullet\!\!-}_a \boldsymbol{\omega}')$	$((\boldsymbol{\gamma} \oslash_c^{a,b} \boldsymbol{\delta}) \lhd_c \boldsymbol{\epsilon})$

Formulas of types γ and δ adapt the α/β distinction to unary connectives,

γ / δ	γ_1/δ_1	$\gamma(\boldsymbol{\omega}, a)$ $= \delta(\boldsymbol{\omega}, a)$	$\gamma_{a,b}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ $= \delta_{a,b}(\boldsymbol{\gamma}, \boldsymbol{\delta})$
$\Diamond A^{\bullet}$ / $\Diamond A^{\circ}$	$A^{\bullet}/A^{\circ}$	$\langle \boldsymbol{\omega} \rangle_a$	$((\Diamond_b^a \boldsymbol{\gamma}) \lhd_b \boldsymbol{\omega}')$
$\Box A^{\circ}$ / $\Box A^{\bullet}$	$A^{\circ}/A^{\bullet}$	$\langle \boldsymbol{\omega} \rangle_a$	$(\boldsymbol{\delta} \lhd_b (\Box_b^a \boldsymbol{\gamma}))$

Definition 70. Figures 5.1 and 5.2 define two types of labeled deduction, based on the use of varieties and of presentations respectively. We use three judgement forms, with Γ, in each case, representing a multiset of lsf's:

Preorder laws

$$\frac{}{a \vdash a : \phi, a : \phi^{\perp}}\ Id \qquad \frac{\boldsymbol{\gamma} \vdash \Gamma, a : A^{\circ} \quad \boldsymbol{\delta} \vdash \Delta, a : A^{\bullet}}{(\boldsymbol{\gamma} \triangleleft_a \boldsymbol{\delta}) \vdash \Gamma, \Delta}\ \circ$$

Logical rules

$$\frac{\alpha_{a,b,c}(a, b, \boldsymbol{\gamma}) \vdash \Gamma, a : \alpha_1, b : \alpha_2}{\boldsymbol{\gamma} \vdash \Gamma, c : \alpha}\ \alpha \qquad \frac{\boldsymbol{\gamma} \vdash \Gamma, a : \beta_1 \quad \boldsymbol{\delta} \vdash \Delta, b : \beta_2}{\beta_{a,b,c}(\boldsymbol{\gamma}, \boldsymbol{\delta}, c) \vdash \Gamma, \Delta, c : \beta}\ \beta$$

$$\frac{\gamma_{a,b}(a, \boldsymbol{\gamma}) \vdash \Gamma, a : \gamma_1}{\boldsymbol{\gamma} \vdash \Gamma, b : \gamma}\ \gamma \qquad \frac{\boldsymbol{\gamma} \vdash \Gamma, a : \delta_1}{\delta_{a,b}(\boldsymbol{\gamma}, b) \vdash \Gamma, b : \delta}\ \delta$$

Figure 5.1: Variety-based deduction.

1. Variety-based derivability judgements $\boldsymbol{\gamma} \vdash \Gamma$, s.t. $\boldsymbol{\gamma}$'s inputs (outputs) label exactly Γ's hypotheses (conclusions).
2. Presentation-based derivability judgements $\boldsymbol{\omega} \vdash A^{\bullet}; \Gamma$, s.t. the elements of $-_{\boldsymbol{\omega}}$ $(+_{\boldsymbol{\omega}})$ label exactly Γ's hypotheses (conclusions), while $A^{\bullet}$ is implicitly understood to be labeled by $H_{\boldsymbol{\omega}}$.
3. Presentation-based derivability judgements $\boldsymbol{\omega} \vdash A^{\circ}; \Gamma$, s.t. the elements of $+_{\boldsymbol{\omega}}$ $(-_{\boldsymbol{\omega}})$ label exactly Γ's hypotheses (conclusions), while A° is implictly understood to be labeled by $H_{\boldsymbol{\omega}}$.

The comparison between the two judgement forms is an easy one in light of the results established in the previous section.

Theorem 5.3.1. $\boldsymbol{\gamma} \vdash \Gamma, a : A^x$ iff $(\boldsymbol{\gamma} \downarrow a^x) \vdash A^x; \Gamma$ for any $a \in I_{\boldsymbol{\gamma}}$, $x \in \{\bullet, \circ\}$, preserving Cut-free proofs in both directions.

Proof. Going from left to right first, it suffices, by L.40 and (a), to show the desired result only w.r.t. the external node labeling the main formula. The result is a trivial induction. Likewise, going from right to left, the only difficult subcase is (a), which is an immediate consequence of T.5.6.1. □

We next check proof-theoretic well-behavedness: Cut is admissible, as are arbitrary axiom instances in the presence of their atomic restrictions. We concentrate on variety-based deduction, preferred for its absence of structural rules, deriving similar results for its presentation-based counterpart by composing with T.5.3.1.

Preorder laws and change-of-perspective

$$\frac{}{a \vdash \phi^{\perp}; a : \phi}\ Id \qquad \frac{\boldsymbol{\omega} \vdash \phi; \Gamma, a : \psi}{\boldsymbol{\omega}^{-a} \vdash \psi; \Gamma, H_{\boldsymbol{\omega}} : \phi}\ a$$

$$\frac{\boldsymbol{\omega} \vdash A^{\circ}; \Gamma \quad \boldsymbol{\omega}' \vdash \phi; \Gamma, H_{\boldsymbol{\omega}} : A^{\bullet}}{\boldsymbol{\omega}'[\boldsymbol{\omega}/H_{\boldsymbol{\omega}}] \vdash \phi; \Gamma, \Delta}\ \circ$$

Logical rules

$$\frac{\alpha(\boldsymbol{\omega}, b, a) \vdash \alpha_1; \Gamma, b : \alpha_2}{\boldsymbol{\omega} \vdash \alpha; \Gamma}\ \alpha \qquad \frac{\boldsymbol{\omega} \vdash \beta_1; \Gamma \quad \boldsymbol{\omega}' \vdash \beta_2; \Delta}{\beta(\boldsymbol{\omega}, \boldsymbol{\omega}', a) \vdash \beta; \Gamma, \Delta}\ \beta$$

$$\frac{\gamma(\boldsymbol{\omega}, a) \vdash \gamma_1; \Gamma}{\boldsymbol{\omega} \vdash \gamma; \Gamma}\ \gamma \qquad \frac{\boldsymbol{\omega} \vdash \delta_1; \Gamma}{\delta(\boldsymbol{\omega}, a) \vdash \delta; \Gamma}\ \delta$$

Figure 5.2: Presentation-based deduction.

Lemma 41. Non-atomic axioms are admissible for variety-based deduction.

Proof. We show, for any formula A and node a, that $a \vdash a : A^{\bullet}, a : A^{\circ}$ using atomic axioms only, using induction on A. The desired result is immediate for $A = p$. The remaining inductive cases can be summarized using unified notation, as follows.

$$\frac{\dfrac{\dfrac{}{b \vdash b : \alpha_1, b : \alpha_1^{\perp}}\ IH \quad \dfrac{}{c \vdash c : \alpha_2, c : \alpha_2^{\perp}}\ IH}{\alpha_{b,c,a}(b, c, a) \vdash b : \alpha_1, c : \alpha_2, a : \alpha^{\perp}}\ \beta}{a \vdash a : \alpha, a : \alpha^{\perp}}\ \alpha \qquad \frac{\dfrac{\dfrac{}{b \vdash b : \gamma_1, b : \gamma_1^{\perp}}\ IH}{\gamma_{b,a}(b, a) \vdash b : \gamma_1, a : \gamma^{\perp}}\ \delta}{a \vdash a : \gamma, a : \gamma^{\perp}}\ \gamma$$

□

Theorem 5.3.2. Cut is admissible for variety-based deduction.

Proof. By induction on the number of connectives found in the Cut instance. We assume only atomic instances of axioms are used, distinguish the following cases: (1) one of the Cut's premises is an axiom; the Cut formula is not main in (2) the left premise, or (3) the right premise; and (4) the Cut formula is main in both premises.

1. One of the premises is an axiom. E.g.,

$$\frac{\dfrac{}{a \vdash a : A^{\bullet}, a : A^{\circ}}\ Id \quad \boldsymbol{\gamma} \vdash \Gamma, a : A^{\bullet}}{(a \lhd_a \boldsymbol{\gamma}) \vdash \Gamma, a : A^{\bullet}}\ \circ$$

Since $(a \lhd_a \boldsymbol{\gamma}) = \boldsymbol{\gamma}$, we keep only the derivation for the right premise.

2. The Cut formula is not main in the left premise. We have four subcases to consider, depending on the Cut formula's type. If an α,

$$\dfrac{\dfrac{\alpha_{a,b,c}(a,b,\boldsymbol{\gamma}) \vdash \Gamma, a:\alpha_1, b:\alpha_2, d:C^\circ}{\boldsymbol{\gamma} \vdash \Gamma, c:\alpha, d:C^\circ}\,\alpha \qquad \boldsymbol{\delta} \vdash \Delta, d:C^\bullet}{(\boldsymbol{\gamma} \lhd_d \boldsymbol{\epsilon}) \vdash \Gamma, \Delta, c:\alpha}\,\circ$$

If a β (assuming $d \in O_{\boldsymbol{\gamma}}$, the case where $d \in O_{\boldsymbol{\delta}}$ being treated similarly),

$$\dfrac{\dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\alpha_1, d:C^\circ \qquad \boldsymbol{\delta} \vdash \Delta, b:\alpha_2}{\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},c) \vdash \Gamma, \Delta, c:\beta, d:C^\circ}\,\beta \qquad \boldsymbol{\epsilon} \vdash \Theta, d:C^\bullet}{(\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},c) \lhd_d \boldsymbol{\epsilon}) \vdash \Gamma, \Delta, \Theta, c:\beta}\,\circ$$

If the Cut formula is a γ,

$$\dfrac{\dfrac{\gamma_{a,b}(a,\boldsymbol{\gamma}) \vdash \Gamma, a:\gamma_1, c:C^\circ}{\boldsymbol{\gamma} \vdash \Gamma, b:\gamma, c:C^\circ}\,\gamma \qquad \boldsymbol{\delta} \vdash \Delta, c:C^\bullet}{(\boldsymbol{\gamma} \lhd_c \boldsymbol{\delta}) \vdash \Gamma, \Delta, b:\gamma}\,\circ$$

While, finally, if the Cut formula is a δ,

$$\dfrac{\dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\delta_1, c:C^\circ}{\delta_{a,b}(\boldsymbol{\gamma},b) \vdash \Gamma, b:\delta, c:C^\circ}\,\delta \qquad \boldsymbol{\delta} \vdash \Delta, c:C^\bullet}{(\delta_{a,b}(\boldsymbol{\gamma},b) \lhd_c \boldsymbol{\delta}) \vdash \Gamma, \Delta, b:\delta}\,\circ$$

In each case, we proceed by permuting Cut with (α). To this end, we require the following equivalences, all following from definitional unfolding:

$$\begin{aligned}
(\alpha_{a,b,c}(a,b,\boldsymbol{\gamma}) \lhd_d \boldsymbol{\delta}) &= \alpha_{a,b,c}(a,b,(\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta}))\\
(\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},c) \lhd_d \boldsymbol{\epsilon}) &= \beta_{a,b,c}((\boldsymbol{\gamma} \lhd_d \boldsymbol{\epsilon}),\boldsymbol{\delta},c) \text{ (if } d \in O_{\boldsymbol{\gamma}})\\
(\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},c) \lhd_d \boldsymbol{\epsilon}) &= \beta_{a,b,c}(\boldsymbol{\gamma},(\boldsymbol{\delta} \lhd_c \boldsymbol{\epsilon}),c) \text{ (if } d \in O_{\boldsymbol{\delta}})\\
(\gamma_{a,b}(a,\boldsymbol{\gamma}) \lhd_c \boldsymbol{\delta}) &= \gamma_{a,b}(a,(\boldsymbol{\gamma} \lhd_c \boldsymbol{\delta}))\\
(\delta_{a,b}(\boldsymbol{\gamma},b) \lhd_c \boldsymbol{\delta}) &= \delta_{a,b}((\boldsymbol{\gamma} \lhd_c \boldsymbol{\delta}),b)
\end{aligned}$$

As examples, consider the permutations of $(\circ)$ with (α) and (β):

$$\dfrac{\dfrac{\alpha_{a,b,c}(a,b,\boldsymbol{\gamma}) \vdash \Gamma, a:\alpha_1, b:\alpha_2, d:C^\circ \qquad \boldsymbol{\delta} \vdash \Delta, d:C^\bullet}{(\alpha_{a,b,c}(a,b,\boldsymbol{\gamma}) \lhd_d \boldsymbol{\delta}) \vdash \Gamma, \Delta, a:\alpha_1, b:\alpha_2}\,\circ}{(\boldsymbol{\gamma} \lhd_d \boldsymbol{\epsilon}) \vdash \Gamma, \Delta, c:\alpha}\,\alpha$$

$$\dfrac{\dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\beta_1, d:C^\circ \qquad \boldsymbol{\epsilon} \vdash \Theta, d:C^\bullet}{(\boldsymbol{\gamma} \lhd_d \boldsymbol{\epsilon}) \vdash \Gamma, \Theta, a:\beta_1}\,\circ \qquad \boldsymbol{\delta} \vdash \Delta, b:\beta_2}{\beta_{a,b,c}((\boldsymbol{\gamma} \lhd_d \boldsymbol{\epsilon}),\boldsymbol{\delta},c) \vdash \Gamma, \Delta, \Theta, c:\beta}\,\beta$$

3. The Cut formula is not main in the right premise. Treated similarly.
4. The Cut formula is main in both premises. There are two subcases, corresponding to Cuts of a β against an α, and of a δ against a γ:

$$\dfrac{\dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\beta_1 \quad \boldsymbol{\delta} \vdash \Delta, b:\beta_2}{\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},c) \vdash \Gamma,\Delta, c:\beta}\,\beta \qquad \dfrac{\beta_{a,b,c}(a,b,\boldsymbol{\epsilon}) \vdash \Theta, a:\beta_1^\perp, b:\beta_2^\perp}{\boldsymbol{\epsilon} \vdash \Theta, c:\beta^\perp}\,\alpha}{\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\epsilon}) \vdash \Gamma,\Delta,\Theta}\,\circ$$

$$\dfrac{\dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\delta_1}{\delta_{a,b}(\boldsymbol{\gamma},b) \vdash \Gamma, b:\delta}\,\delta \qquad \dfrac{\delta_{a,b}(a,\boldsymbol{\delta}) \vdash \Delta, a:\delta_1^\perp}{\boldsymbol{\delta} \vdash \Delta, b:\delta^\perp}\,\gamma}{\delta_{a,b}(\boldsymbol{\gamma},\boldsymbol{\delta}) \vdash \Gamma,\Delta}\,\circ$$

We proceed by replacing with Cuts on the immediate subformulas.

$$\dfrac{\boldsymbol{\delta} \vdash \Delta, b:\beta_2 \qquad \dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\beta_1 \quad \beta_{a,b,c}(a,b,\boldsymbol{\epsilon}) \vdash \Theta, a:\beta_1^\perp, b:\beta_2^\perp}{\beta_{a,b,c}(\boldsymbol{\gamma},b,\boldsymbol{\epsilon}) \vdash \Gamma,\Theta, b:\beta_2^\perp}\,\circ}{\beta_{a,b,c}(\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\epsilon}) \vdash \Gamma,\Delta,\Theta}\,\circ$$

$$\dfrac{\boldsymbol{\gamma} \vdash \Gamma, a:\delta_1 \quad \delta_{a,b}(a,\boldsymbol{\delta}) \vdash \Delta, a:\delta_1^\perp}{\delta_{a,b}(\boldsymbol{\gamma},\boldsymbol{\delta}) \vdash \Gamma,\Delta}\,\circ$$

□

Composition of previous results gives

Corollary 3. Cut is admissible for presentation-based deduction.

We proceed to make the comparison with algebraic derivability.

Theorem 5.3.3. If $A \leq B$, then $a \vdash a : A^\bullet, b : B^\circ$.

Proof. By induction. Reflexivity and transitivity follow from axioms and Cut. We illustrate the remaining cases with the residuation law $A \otimes B \leq C$ only if $A \leq C/B$. We have, by induction hypothesis, that $a \vdash a : A^\bullet, a : C/B^\circ$. Hence,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{c \vdash c:C^\bullet, c:C^\circ}\,\mathit{Id} \quad \dfrac{}{b \vdash b:B^\bullet, b:B^\circ}\,\mathit{Id}}{\otimes cab \vdash c:C^\circ, b:B^\bullet, a:C/B^\circ}\,\beta \qquad \dfrac{}{a \vdash a:A^\bullet, a:C/B^\circ}\,\mathit{IH}}{\otimes cab \vdash a:A^\bullet, b:B^\bullet, c:C^\circ}\,\circ}{c \vdash c: A\otimes B^\bullet, c:C^\circ}\,\alpha$$

□

For soundness of labeled deduction, note that the presentation-based format is easily translated inside chapter 3's display calculus.

5.4 Context-freeness

We (re)define an **LG** grammar and prove that the language recognized by any such grammar is context free.

Definition 71. Given γ and $a \in E_\gamma$, the *yield* of γ w.r.t. a is defined by the yield of $(\gamma \downarrow a^\bullet)$ or $(\gamma \downarrow a^\circ)$ (depending on whether $a \in I_\gamma$ or $a \in O_\gamma$), i.e., the string read off from the leaves (excluding the root).

Definition 72. An **LG** grammar G is a tuple $\langle \Sigma, Atom, L, \phi \rangle$ consisting of: a set of *words* Σ; a finite set of *atomic formulae Atom* generating $\mathscr{F}$; a *lexicon* L mapping words to finite sets of lsf's and an atomic signed *goal* formula ϕ. The *language* $\mathscr{L}(G)$ recognized by $\mathscr{L}$ is the set of lists $w_1, \ldots, w_n \in \Sigma^+$ of words $w_i \in \Sigma$ $(1 \leq i \leq n)$ s.t., for some $\phi_1 \in L(w_1), \ldots, \phi_n \in L(w_n)$ and variety γ with yield $a_1, \ldots, a_n$ relative to a, $\gamma \vdash a_1 : \phi_1, \ldots, a_n : \phi_n, a : \phi$.

Remark 6. Restricted to the **NL** fragment, the goal formula is always an output, whereas the formulas in the codomain of L are inputs. The language $\mathscr{L}(G)$ recognized by an **NL** grammar G is defined by the set of lists $w_1, \ldots, w_n$ of words w_i $(1 \leq i \leq n)$ such that, for some $A_1 \in L(w_1), \ldots, A_n \in L(w_n)$ and tree γ built without use of $\oplus abc$ and with yield $a_1, \ldots, a_n$ relative to a, $\gamma \vdash a_1 : A_1^\bullet, \ldots, a_n : A_n^\bullet, a : g^\circ$. Since γ in this case describes a rooted tree (there always being a unique output), we have a canonical choice for a bracketing C of $A_1 \otimes \ldots \otimes A_n$ s.t.

$$\frac{\gamma \vdash a_1 : A_1^\bullet, \ldots, a_n : A_n^\bullet, a : g^\circ}{a \vdash a : C^\bullet, a : g^\circ}$$

via applications of (α). By letting go of the restriction to a unique output, we obtain our previous definition of an **LG** grammar from chapter 3. Our current notion of the latter concept may thus be considered a generalization, obtained by further dropping the restriction on γ to have been built without use of $\oplus abc$.

We proceed to show context-freeness of **LG** grammars. Our strategy for doing so follows closely that of Jäger [2004], inspired in turn by Pentus [1999]. We first prove an interpolation property, its statement modeled after Jağer's for $\mathbf{NL}\Diamond$.

Theorem 5.4.1. Suppose $(\gamma \lhd_d \delta) \vdash \Gamma$. Then there exists a partitioning $\Gamma = \Gamma_1, \Gamma_2$ and C s.t., first, $\gamma \vdash \Gamma_1, d : C^\circ$ and $\delta \vdash \Gamma_2, d : C^\bullet$, and second, for some B in Γ, C contains no more connectives than found in B.

Proof. By induction on the witness for $(\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta}) \vdash \Gamma$. In the base case, $(\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta}) = d$ (i.e., $\boldsymbol{\gamma} = \boldsymbol{\delta} = d$) and $\Gamma = d : A^\bullet, d : A^\circ$, and we choose $\Gamma_1 = d : A^\bullet$, $\Gamma_2 = d : A^\circ$ and $C = A$. If a logical rule was applied, the leaf a labeling the main formula occurs in either $E_{\boldsymbol{\gamma}}$ or $E_{\boldsymbol{\delta}}$, or in both if $a = d$. In the latter case, we simply pick $C = A$, so henceforth assume otherwise. We check for $a \in E_{\boldsymbol{\gamma}}$, and thus $a \notin E_{\boldsymbol{\delta}}$, the situation where $a \in E_{\boldsymbol{\delta}}$ being treated similarly. There are four subcases, detailed below. In what is to follow, we sometimes lazily equate a formula A with a *signed* formula $B^\bullet$ or B°, in which case we mean to put $A = B$.

1. The introduced formula is an α. Then $\Gamma = \Gamma', c : \alpha$:

$$\frac{\alpha_{a,b,c}(a, b, (\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta})) \vdash \Gamma', a : \alpha_1, b : \alpha_2}{(\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta}) \vdash \Gamma', c : \alpha}\ \alpha$$

   Since $\alpha_{a,b,c}(a, b, (\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta})) = (\alpha_{a,b,c}(a, b, \boldsymbol{\gamma}) \lhd_d \boldsymbol{\delta})$, we can apply the induction hypothesis to obtain Γ_1, Γ_2 and a C' with degree bounded above by $\Gamma' = \Gamma'_1, \Gamma'_2$, s.t. (a) $(\alpha_{a,b,c}(a, b, \boldsymbol{\gamma}) \lhd_d \boldsymbol{\delta}) \vdash \Gamma'_1, a : \alpha_1, b : \alpha_2, d : C'^\circ$, and (b) $\boldsymbol{\delta} \vdash \Gamma'_2, d : C'^\bullet$. We now put $\Gamma_1 = \Gamma'_1, c : \alpha$, $\Gamma_2 = \Gamma'_2$ and $C = C'$, applying (α) on (a) to obtain $\boldsymbol{\gamma} \vdash \Gamma'_1, c : \alpha, d : C'^\circ$.

2. The introduced formula is a β, in which case $(\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta}) = \beta_{a,b,c}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, c)$, and $\boldsymbol{\gamma}$ can be decomposed into $\beta_{a,b,c}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, c)$, where either: (a) $d \in O_{\boldsymbol{\gamma}_1}$, in which case $\boldsymbol{\epsilon}_1 = (\boldsymbol{\gamma}_1 \lhd_d \boldsymbol{\delta})$ and $\boldsymbol{\epsilon}_2 = \boldsymbol{\gamma}_2$, or (b) $d \in O_{\boldsymbol{\gamma}_2}$, in which case $\boldsymbol{\epsilon}_1 = \boldsymbol{\gamma}_1$ and $\boldsymbol{\epsilon}_2 = (\boldsymbol{\gamma}_2 \lhd_d \boldsymbol{\delta})$. We check (a), with (b) being similar. Then $\Gamma = \Gamma'_1, \Gamma'_2, c : \beta$:

$$\frac{(\boldsymbol{\gamma}_1 \lhd_d \boldsymbol{\delta}) \vdash \Gamma'_1, a : \beta_1 \qquad \boldsymbol{\gamma}_2 \vdash \Gamma'_2, b : \beta_2}{\beta_{a,b,c}((\boldsymbol{\gamma}_1 \lhd_d \boldsymbol{\delta}), \boldsymbol{\gamma}_2, c) \vdash \Gamma'_1, \Gamma'_2, c : \beta}\ \beta$$

   If $d = a$ (in which case $\boldsymbol{\gamma}_1 = a$ as $a = d$ is exterior in $\boldsymbol{\gamma} = \beta_{a,b,c}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, c)$), we pick $C = \beta_1$, $\Gamma_1 = \Gamma'_1$ and $\Gamma_2 = \Gamma'_2, c : \beta$, deriving $\boldsymbol{\gamma} \vdash \Gamma_2, a : \beta_1^\perp$ by applying (β), as follows:

$$\frac{\dfrac{}{a \vdash a : \beta_1, a : \beta_1^\perp}\ \mathit{Id} \qquad \boldsymbol{\gamma}_2 \vdash \Gamma'_2, b : \beta_2}{\beta_{a,b,c}(a, \boldsymbol{\gamma}_2, c) \vdash \Gamma'_2, c : \beta, a : \beta_1^\perp}\ \beta$$

   If instead $d \neq a$, we apply the induction hypothesis on the left premise to find a decomposition $\Gamma'_1 = \Gamma'_{11}, \Gamma'_{12}$ and a suitable C' for which (a) $\boldsymbol{\gamma}_1 \vdash \Gamma'_{11}, a : beta_1, d : C^\circ$ and (b) $\boldsymbol{\delta} \vdash \Gamma'_{12}, d : C^\circ$. We pick $\Gamma_1 = \Gamma'_{11}, \Gamma'_2, c : \beta$, $\Gamma_2 = \Gamma'_{12}$ and $C = C'$, deriving $\beta_{a,b,c}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, c) \vdash \Gamma_1, d : C'^\circ$ by applying (β), as follows:

$$\frac{\boldsymbol{\gamma}_1 \vdash \Gamma'_{11}, a : \beta_1, d : C'^\circ \qquad \boldsymbol{\gamma}_2 \vdash \Gamma'_2, b : \beta_2}{\beta_{a,b,c}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, c) \vdash \Gamma_1, d : C'^\circ}\ \beta$$

3. The introduced formula is a γ. Then $\Gamma = \Gamma', b : \gamma$:
$$\frac{\gamma_{a,b}(a, (\boldsymbol{\gamma} \lhd_c \boldsymbol{\delta})) \vdash \Gamma', a : \gamma_1}{(\boldsymbol{\gamma} \lhd_c \boldsymbol{\delta}) \vdash \Gamma', b : \gamma}\ \gamma$$
Since $\gamma_{a,b}(a, (\boldsymbol{\gamma} \lhd_c \boldsymbol{\delta})) = (\gamma_{a,b}(a, \boldsymbol{\gamma}) \lhd_c \boldsymbol{\delta})$, the induction hypothesis provides a decomposition $\Gamma' = \Gamma'_1, \Gamma'_2$ and a suitable C' for which (a) $(\gamma_{a,b}(a, \boldsymbol{\gamma}) \lhd_c \boldsymbol{\delta}) \vdash \Gamma'_1, a : \gamma_1, d : C'^{\circ}$ and (b) $\boldsymbol{\delta} \vdash \Gamma'_2, d : C'^{\circ}$. We put $\Gamma_1 = \Gamma'_1, b : \gamma$, $\Gamma_2 = \Gamma'_2$ and $C = C'$, deriving $\boldsymbol{\gamma} \vdash \Gamma_1, d : C^{\circ}$ by applying (γ) on (a).

4. The introduced formula is a δ, in which case $\boldsymbol{\gamma} = \delta_{a,b}(\boldsymbol{\gamma}', b)$ and $(\boldsymbol{\gamma} \lhd_d \boldsymbol{\delta}) = \delta_{a,b}((\boldsymbol{\gamma}' \lhd_d \boldsymbol{\delta}), b)$. Then $\Gamma = \Gamma', b : \delta$:
$$\frac{(\boldsymbol{\gamma}' \lhd_d \boldsymbol{\delta}) \vdash \Gamma', a : \gamma_1}{\delta_{a,b}((\boldsymbol{\gamma}' \lhd_d \boldsymbol{\delta}), b) \vdash \Gamma', b : \delta}\ \delta$$
If $d = a$, $\boldsymbol{\gamma}' = f$ (since a is exterior in $\boldsymbol{\gamma} = \delta_{a,b}(\boldsymbol{\gamma}', b)$), and we set $\Gamma_1 = \Gamma'$, $\Gamma_2 = b : \delta$ and $C' = \delta_1$, deriving $\delta_{a,b}(a, b) \vdash \Gamma_2, a : \delta_1^{\perp}$ using (δ):
$$\frac{\dfrac{}{a \vdash a : \delta_1, a : \delta_1^{\perp}}\ \mathit{Id}}{\delta_{a,b}(a, b) \vdash b : \delta, a : \delta_1^{\perp}}\ \delta$$
If instead $d \neq a$, we apply the induction hypothesis to find a decomposition $\Gamma' = \Gamma'_1, \Gamma'_2$ and a suitable C' for which (a) $\boldsymbol{\gamma}' \vdash \Gamma'_1, a : \delta_1, d : C'^{\circ}$ and (b) $\boldsymbol{\delta} \vdash \Gamma'_2 d : C^{\bullet}$. We put $\Gamma_1 = \Gamma'_1, b : \delta$, $\Gamma_2 = \Gamma'_2$ and $C = C'$. We then derive $\delta_{a,b}(a, \boldsymbol{\gamma}') \vdash$ from $\Gamma_1, d : C'^{\circ}$ by applying (δ) on (a). □

We next define Cut-rule axiomatizations for the restrictions of **LG** to formulas of at most some degree n, the set of which is henceforth denoted $\mathscr{F}^n$.

Definition 73. For any given positive natural n, we define the hypothetical judgement form $\boldsymbol{\gamma} \vdash^n \Gamma$, as follows. We construct the axiom set Ax^n of pairings of varieties with multisets of lsf's by specifying $\langle \boldsymbol{\gamma}, \Gamma \rangle \in Ax^n$ iff $\boldsymbol{\gamma} \vdash \Gamma$ for

1. $\boldsymbol{\gamma} = a$ and $\Gamma = a : A^{\bullet}, b : B^{\circ}$, $A, B \in \mathscr{F}^n$; or
2. $\boldsymbol{\gamma} = \otimes abc$ and $\Gamma = a : A^{\circ}, b : B^{\bullet}, c : C^{\bullet}$, $A, B, C \in \mathscr{F}^n$; or
3. $\boldsymbol{\gamma} = \oplus abc$ and $\Gamma = a : A^{\bullet}, b : B^{\circ}, c : C^{\circ}$, $A, B, C \in \mathscr{F}^n$; or
4. $\boldsymbol{\gamma} = \Diamond\, ab$ and $\Gamma = a : A^{\circ}, b : B^{\bullet}$, $A, B \in \mathscr{F}^n$.

Observe that Ax^n is both finite (assuming the set of atomic formulas is), and, in light of Cut admissibility, decidable for any choice of n. Next, the rules of inference:

$$\frac{\langle \boldsymbol{\gamma}, \Gamma \rangle \in Ax^n}{\boldsymbol{\gamma} \vdash^n \Gamma}\ Ax \qquad \frac{\boldsymbol{\gamma} \vdash^n a : A^{\circ} \quad \boldsymbol{\delta} \vdash^n a : A^{\bullet}}{(\boldsymbol{\gamma} \lhd_a \boldsymbol{\delta}) \vdash^n \Gamma, \Delta}\ MP$$

Lemma 42. If $\boldsymbol{\gamma} \vdash \Gamma$ with all formulas of Γ in $\mathscr{F}^n$, also $\boldsymbol{\gamma} \vdash^n \Gamma$.

Proof. By induction on the size of $\boldsymbol{\gamma}$. In the base case, $\boldsymbol{\gamma}$ equals a, $\otimes abc$, $\oplus abc$ or $\diamond ab$, and $\langle \gamma, \Gamma \rangle \in Ax^n$ by definition. So suppose $\boldsymbol{\gamma} = (\boldsymbol{\delta} \triangleleft_a \boldsymbol{\epsilon})$, with $\boldsymbol{\delta} \neq a$ and $\boldsymbol{\epsilon} \neq a$. By interpolation, there exists a decomposition $\Gamma = \Gamma_1, \Gamma_2$ and a C of size at most n s.t. $\boldsymbol{\gamma} \vdash \Gamma_1, a : C^\circ$ and $\boldsymbol{\delta} \vdash \Gamma_2, a : C^\bullet$. We now need only apply the induction hypothesis twice, followed by (*MP*). □

Definition 74. Given an **LG**-grammar $G_1 = \langle \Sigma, Atom, L, \phi \rangle$, with $L \subseteq \Sigma \times \mathscr{F}^n$, define the following CFG G_2: its set of terminals coincides with Σ, its set of nonterminals with $\mathscr{F}^n$, its startsymbol is ϕ, and its productions R are given by

1. If $\psi \in L(w)$, then $\psi^\perp \to w$.
2. If $a \vdash^n a : A^\bullet, a : B^\circ$, then $B^\bullet \to A^\bullet \in R$ and $A^\circ \to B^\circ \in R$.
3. If $\otimes abc \vdash^n a : A^\circ, b : B^\bullet, c : C^\bullet$, then $A^\circ \to B^\circ\ C^\circ \in R$, $B^\bullet \to C^\circ\ A^\bullet \in R$ and $C^\bullet \to A^\bullet\ B^\circ \in R$.
4. If $\oplus abc \vdash^n a : A^\bullet, b : B^\circ, c : C^\circ$, then $A^\bullet \to B^\bullet\ C^\bullet \in R$, $B^\circ \to C^\bullet\ A^\circ \in R$ and $C^\circ \to A^\circ\ B^\bullet \in R$.
5. If $\diamond ab \vdash^n a : A^\circ, b : B^\bullet$, then $A^\circ \to B^\circ \in R$ and $B^\bullet \to A^\bullet \in R$.

Theorem 5.4.2. For any **LG**-grammar G_1, $\mathscr{L}(G_1) = \mathscr{L}(G_2)$, where G_2 is the CFG constructed from G_1 as in the preceding definition.

Proof. We first prove $\mathscr{L}(G_1) \subseteq \mathscr{L}(G_2)$, followed by the inverse direction.

1. Assume G_1 recognizes $w_1, \ldots, w_n$. Then for some $\phi_1 \in L(w_1), \ldots, \phi_n \in L(w_n)$ and $\boldsymbol{\gamma}$ with yield $a_1, \ldots, a_n$ w.r.t. some a, $\boldsymbol{\gamma} \vdash a_1 : \phi_1, \ldots, a_n : \phi_n, a : \phi$ (ϕ being the goal lsf). By L.42, also $\boldsymbol{\gamma} \vdash^n a_1 : \phi_1, \ldots, a_n : \phi_n, a : \phi$, where n is the maximum size of the formulas found in G_1. We claim $\phi \to^* \phi_1^\perp\ \ldots\ \phi_n^\perp$, and hence $\phi \to w_1\ \ldots\ w_n$. The desired result follows from an inductive argument, showing, for arbitrary $\boldsymbol{\delta}, \psi_1, \ldots \psi_m, \psi$ with $b_1, \ldots, b_n$ the yield of $\boldsymbol{\delta}$ w.r.t. b, that if $\boldsymbol{\delta} \vdash^n b_1 : \psi_1, \ldots, b_m : \psi_m, b : \psi$ with all lsf's involved drawing from $\mathscr{F}^n$, then $\psi \to \psi_1^\perp\ \ldots\ \psi_m^\perp$. The base cases follow from the construction of G_2, while the sole inductive case depends on the transitivity of $\to^*$.
2. Conversely, suppose $\phi \to^* w_1\ \ldots\ w_n$. Then, by the construction of G_2, $g \to^* \phi_1\ \ldots\ \phi_n$ for some $\phi_1 \in L(w_1), \ldots, \phi_n \in L(w_n)$. Hence, since all production rules involved draw from elements of Ax^n, there exists $\boldsymbol{\gamma}$ with yield $a_1, \ldots, a_n$ w.r.t. some a s.t. $\boldsymbol{\gamma} \vdash^n a_1 : \phi_1^\perp, \ldots, a_n : \phi^\perp, a : \phi$. By L.42, we are then done. □

Type I

$$\frac{((\gamma_2 \oslash_c^{b,e} \boldsymbol{\delta}_2) \lhd_c (\gamma_1 \backslash_c^{a,d} \boldsymbol{\delta}_1)) \vdash \Gamma}{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma} A_I^1 \qquad \frac{((\boldsymbol{\delta}_1 \obslash_c^{d,a} \gamma_1) \lhd_c (\boldsymbol{\delta}_2 /_c^{e,b} \gamma_2)) \vdash \Gamma}{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma} A_I^2$$

$$\frac{((\boldsymbol{\delta}_1 \obslash_c^{d,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 /_c^{d,b} \gamma_2)) \vdash \Gamma}{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma} C_I^1 \qquad \frac{((\boldsymbol{\delta}_1 \obslash_c^{d,b} \gamma_2) \lhd_c (\gamma_1 \backslash_c^{a,e} \boldsymbol{\delta}_2)) \vdash \Gamma}{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma} C_I^2$$

Type IV

$$\frac{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma}{((\gamma_2 \oslash_c^{b,e} \boldsymbol{\delta}_2) \lhd_c (\gamma_1 \backslash_c^{a,d} \boldsymbol{\delta}_1)) \vdash \Gamma} A_{IV}^1 \qquad \frac{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma}{((\boldsymbol{\delta}_1 \obslash_c^{d,a} \gamma_1) \lhd_c (\boldsymbol{\delta}_2 /_c^{e,b} \gamma_2)) \vdash \Gamma} A_{IV}^2$$

$$\frac{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma}{((\boldsymbol{\delta}_1 \obslash_c^{d,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 /_c^{d,b} \gamma_2)) \vdash \Gamma} C_{IV}^1 \qquad \frac{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma}{((\boldsymbol{\delta}_1 \obslash_c^{d,b} \gamma_2) \lhd_c (\gamma_1 \backslash_c^{a,e} \boldsymbol{\delta}_2)) \vdash \Gamma} C_{IV}^2$$

Figure 5.3: Type I and IV Grishin interactions for variety-based deduction.

5.5 Grishin interactions

While the present concept of labeled deduction served as a means of showing context-freeness for the base logic $\mathbf{LG}_\varnothing$, it allows as well for modular extensions by the various structural postulates discussed in the previous chapter, although the aforementioned expressivity results then of course no longer go through. We here concentrate on variety-based deduction, ensuring preservation of Cut admissibility.

Definition 75. F.5.1 extends variety-based deduction with the Grishin interactions.

We ensure Cut admissibility is preserved.

Theorem 5.5.1. Variety-based deduction for the Lambek-Grishin hierarchy enjoys Cut admissibility.

Proof. In addition to the proof already constructed for $\mathbf{LG}_\varnothing$, we need only check that Cut permutes with each of the Grishin interactions. To illustrate, consider the situation where (A_I^1) derives the left premise:

$$\frac{\dfrac{((\gamma_2 \oslash_c^{b,e} \boldsymbol{\delta}_2) \lhd_c (\gamma_1 \backslash_c^{a,d} \boldsymbol{\delta}_1)) \vdash \Gamma, f : A^\circ}{((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \vdash \Gamma, f : A^\circ} A_I^1 \qquad \boldsymbol{\epsilon} \vdash \Delta, f : A^\bullet}{(((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\boldsymbol{\delta}_1 \oplus_c^{d,e} \boldsymbol{\delta}_2)) \lhd_f \boldsymbol{\epsilon}) \vdash \Gamma, \Delta} \circ$$

Clearly, $(((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\delta_1 \oplus_c^{d,e} \delta_2)) \lhd_f \epsilon)$ equals either one of

$$\begin{array}{l}
(((\gamma_1 \lhd_f \epsilon) \otimes_c^{a,b} \gamma_2) \lhd_c (\delta_1 \oplus_c^{d,e} \delta_2)) \text{ if } f \in O_{\gamma_1} \\
((\gamma_1 \otimes_c^{a,b} (\gamma_2 \lhd_f \epsilon)) \lhd_c (\delta_1 \oplus_c^{d,e} \delta_2)) \text{ if } f \in O_{\gamma_2} \\
((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c ((\delta_1 \lhd_f \epsilon) \oplus_c^{d,e} \delta_2)) \text{ if } f \in O_{\delta_1} \\
((\gamma_1 \otimes_c^{a,b} \gamma_2) \lhd_c (\delta_1 \oplus_c^{d,e} (\delta_2 \lhd_f \epsilon))) \text{ if } f \in O_{\delta_2}
\end{array}$$

validating permutability. For example, if $f \in O_{\gamma_1}$, we can proceed thus,

$$\dfrac{\dfrac{((\gamma_2 \oslash_c^{b,e} \delta_2) \lhd_c (\gamma_1 \backslash_c^{a,d} \delta_1)) \vdash \Gamma, f : A^{\circ} \quad \epsilon \vdash \Delta, f : A^{\bullet}}{((\gamma_2 \oslash_c^{b,e} \delta_2) \lhd_c ((\gamma_1 \lhd_f \epsilon) \backslash_c^{a,d} \delta_1)) \vdash \Gamma, \Delta} \circ}{(((\gamma_1 \lhd_f \epsilon) \otimes_c^{a,b} \gamma_2) \lhd_c (\delta_1 \oplus_c^{d,e} \delta_2)) \vdash \Gamma, \Delta} A_I^1 \qquad \square$$

5.6 Classical non-associative Lambek calculus

The previous developments can be adapted to **CNL** by changing the definitions of varieties and presentations. We here sketch the outline of such an exercise. We present the relevant definitions and the statements of the main theorems and lemmas, though foregoing the latter's proofs in light of their similarity to (in fact, simplification of) the demonstrations already provided for their counterparts in **LG**.

Definition 76. A *relational frame* is a triple $\langle N, T, B\rangle$ with N a non-empty set of *nodes*, $T \subseteq N^3$ and $B \subseteq N^2$ ternary and binary relations on N respectively. A *variety* $\gamma = \langle N, T, B, E\rangle$ is any frame augmented by a *multi*set E of *exterior nodes*, containing (possibly multiple) copies of elements from N, that may be built using the constructions of the following table, where ab abbreviates $\langle a, b\rangle$.

Trees γ	a	$Tabc$	Bab	$(\delta \circ_x \epsilon)$
Conditions	-	$a \neq b \neq c$	$a \neq b$	$N_\delta \bigcap N_\epsilon = \{a\}$ $a \in E_\delta$, $a \in E_\epsilon$
Nodes N_γ	$\{a\}$	$\{a, b, c\}$	$\{a, b\}$	$N_\delta \bigcup N_\epsilon$
Ternary T_γ	$\varnothing$	$\{\{ab, bc, ca\}\}$	$\varnothing$	$T_\delta \bigcup T_\epsilon$
Binary B_γ	$\varnothing$	$\varnothing$	$\{\{ab, ba\}\}$	$B_\delta \bigcup B_\epsilon$
Exterior E_γ	$\{a, a\}$	$\{a, b, c\}$	$\{a, b\}$	$(E_\gamma/\{x\}) \bigcup (E_\delta/\{a\})$

Remark 7. Note that for any a, b, c, $Tabc = Tcab = Tbca$ and $Bab = Bba$.

Definition 77. For any trees γ and δ, we define:

$$
\begin{array}{rcll}
(\gamma \oplus_a^{b,c} \delta) & := & ((Tabc \circ_b \gamma) \circ_c \delta) & (N_\gamma \cap N_\delta \cap \{a\} = \varnothing, b \in E_\gamma, c \in E_\delta) \\
(\Box_a^b \gamma) & := & (Bab \circ_b \gamma) & (N_\gamma \cap \{a\} = \varnothing, b \in E_\gamma)
\end{array}
$$

Lemma 43. We have the following three equivalences shown on the left, together with the two special cases on the right:

$$
\begin{array}{rclcrcl}
(\gamma \circ_a \delta) & = & (\delta \circ_a \gamma) & & (\gamma \oplus_a^{b,c} c) & = & (a \oplus_c^{a,b} \gamma) \\
((\gamma \oplus_a^{b,c} \delta) \circ_a \epsilon) & = & (\gamma \circ_b (\delta \oplus_b^{c,a} \epsilon)) & & (\Box_a^b b) & = & (\Box_b^a a) \\
((\Box_a^b \gamma) \circ_a \delta) & = & (\gamma \circ_b (\Box_b^a \delta)) & & & &
\end{array}
$$

Definition 78. We define below, by induction, the set of *presentations* $\omega(, \omega', \ldots)$, with *root* H_ω, *occurrence set* N_ω and disjoint sets of *positive* and *negative leaves* $+_\omega, -_\omega \in \mathscr{P}(N_\omega)$ respectively.

Presentations ω	a	$(\omega' \bullet_a \omega'')$	$\langle \omega' \rangle_a$
Conditions	-	$a \notin N_\omega \cup N_{\omega'}$, $N_\omega \cap N_{\omega'} = \varnothing$	$N_\omega \cap \{a\} = \varnothing$
Nodes N_ω	$\{a\}$	$N_{\omega'} \cup N_{\omega''} \cup \{a\}$	$N_{\omega'} \cup \{a\}$
Root H_ω	a	a	a
Exterior E_ω	$\{a\}$	$E_{\omega'} \cup E_{\omega''}$	$E_{\omega'}$

Definition 79. For any γ and $a \in E_\gamma$, define, by induction on γ, the element $(\gamma \downarrow a)$ in $F_{E_\gamma/\{a\}}$, as follows: $(a \downarrow a) = a$, $((\delta \oplus_a^{b,c} \epsilon) \downarrow a) = ((\delta \downarrow b) \bullet_a (\epsilon \downarrow c))$ and $((\Box_a^b \delta) \downarrow a) = \langle (\delta \downarrow b) \rangle_a$.

Definition 80. Given ω, define the variety $\overline{\omega}$ inductively by $\overline{a} = a$, $\overline{(\omega \bullet_a \omega')} = (\overline{\omega} \oplus_a^{H_\omega, H_{\omega'}} \overline{\omega'})$ and $\overline{\langle \omega \rangle_a} = (\Box_a^{H_\omega} \overline{\omega})$.

Lemma 44. For any ω, $\omega = (\overline{\omega} \downarrow H_\omega)$, while for any γ and $a \in E_\gamma$, $\gamma = \overline{(\gamma \downarrow a)}$.

Definition 81. Presentation *contexts* $\omega[], \omega'[], \ldots$ are defined as follows:

$$
\omega[], \omega'[] \quad ::= \quad [] \mid (\omega[] \bullet_a \omega') \mid (\omega \bullet_a \omega'[]) \mid \langle \omega[] \rangle_a
$$

Definition 82. Given a presentation $\omega[a]$, we define the presentation $\omega[a]^{-a}$ by induction on the depth of $[]$:

$$
\begin{array}{rclrcl}
a^{-a} & := & a & \omega[(a \bullet_b \omega')]^{-a} & := & (\omega' \bullet_a \omega[b]^{-b}) \\
\omega[\langle a \rangle_b]^{-a} & := & \langle \omega[b]^{-b} \rangle_a & \omega[(\omega' \bullet_b a)]^{-a} & := & (\omega[b]^{-b} \bullet_a \omega')
\end{array}
$$

$$\frac{}{a \vdash a : A, a : A^\perp}\ \textit{Id} \qquad \frac{\gamma \vdash \Gamma, a : A \quad \boldsymbol{\delta} \vdash \Delta, a : A^\perp}{(\gamma \circ_a \boldsymbol{\delta}) \vdash \Gamma, \Delta}\ \circ$$

$$\frac{\gamma \vdash \Gamma, a : A \quad \boldsymbol{\delta} \vdash \Delta, b : B}{(\boldsymbol{\delta} \oplus_c^{b,a} \gamma) \vdash \Gamma, \Delta, c : A \otimes B}\ \otimes \qquad \frac{(Tcab \lhd_c \gamma) \vdash \Gamma, a : A, b : B}{\gamma \vdash \Gamma, c : A \oplus B}\ \oplus$$

$$\frac{\gamma \vdash \Gamma, a : A}{(\Box_b^a \gamma) \vdash \Gamma, b : \Diamond A}\ \Diamond \qquad \frac{(Bab \lhd_b \gamma) \vdash \Gamma, a : A}{\gamma \vdash \Gamma, b : \Box A}\ \Box$$

Figure 5.4: Variety-based deduction for **CNL**.

$$\frac{}{a \vdash A; a : A^\perp}\ \textit{Id}$$

$$\frac{\boldsymbol{\omega} \vdash A; \Gamma \quad \boldsymbol{\omega}' \vdash B; \Delta, a : A^\perp}{\boldsymbol{\omega}'[\boldsymbol{\omega}/H_{\boldsymbol{\omega}}] \vdash B; \Gamma, \Delta}\ \circ \qquad \frac{\boldsymbol{\omega}^{-a} \vdash A; \Gamma, H_{\boldsymbol{\omega}} : B}{\boldsymbol{\omega} \vdash B; \Gamma, a : A}\ a$$

$$\frac{\boldsymbol{\omega} \vdash A; \Gamma \quad \boldsymbol{\omega}' \vdash B; \Delta}{(\boldsymbol{\omega}' \bullet_a \boldsymbol{\omega}) \vdash A \otimes B; \Gamma, \Delta}\ \otimes \qquad \frac{(b \bullet_a \boldsymbol{\omega}) \vdash A; \Gamma, b : B}{\boldsymbol{\omega} \vdash A \oplus B; \Gamma}\ \oplus$$

$$\frac{\boldsymbol{\omega} \vdash A; \Gamma}{\langle \boldsymbol{\omega} \rangle_a \vdash \Diamond A; \Gamma}\ \Diamond \qquad \frac{\langle \boldsymbol{\omega} \rangle_a \vdash A; \Gamma}{\boldsymbol{\omega} \vdash \Box A; \Gamma}\ \Box$$

Figure 5.5: Presentation-based deduction for **CNL**.

Definition 83. Define an equivalence relation $\equiv$ on presentations by $\boldsymbol{\omega} \equiv \boldsymbol{\omega}'$ iff $\boldsymbol{\omega}^{-H_{\boldsymbol{\omega}'}} = \boldsymbol{\omega}'$ and $\boldsymbol{\omega}'^{-H_{\boldsymbol{\omega}}} = \boldsymbol{\omega}$.

Theorem 5.6.1. $\boldsymbol{\omega} \equiv \boldsymbol{\omega}'$ iff $\overline{\boldsymbol{\omega}} = \overline{\boldsymbol{\omega}'}$.

Definition 84. Figures 5.4 and 5.5 describe derivability judgements $\gamma \vdash \Gamma$ and $\boldsymbol{\omega} \vdash A; \Gamma$ for variety- and presentation-based deduction respectively. In either case, Γ is a multiset consisting of pairs $a : A$ for each $a \in E_\gamma$ ($a \in E_{\boldsymbol{\omega}}$). The distinguished formula occurrence A in $\boldsymbol{\omega} \vdash \Gamma A$ is implicitly assumed to be labeled by $H_{\boldsymbol{\omega}}$.

Theorem 5.6.2. $\gamma \vdash \Gamma, a : A$ iff $(\gamma \downarrow a) \vdash A; \Gamma$ for any $a \in E_\gamma$.

Theorem 5.6.3. Cut is admissible in both Figures 5.4 and 5.5.

Theorem 5.6.4. Variety-based deduction satisfies interpolation. I.e., suppose we have $(\boldsymbol{\gamma} \circ_a \boldsymbol{\delta}) \vdash \Gamma$. Then there exists a decomposition $\Gamma = \Gamma_1, \Gamma_2$ and a A s.t. $\boldsymbol{\gamma} \vdash \Gamma, a : A$, $\boldsymbol{\delta} \vdash \Delta, a : A^{\perp}$ and for some B in Γ, A is of size at most B.

Definition 85. For any given positive natural n, we define, as follows, the hypothetical judgement form $\boldsymbol{\gamma} \vdash^n \Gamma$, where Γ contains only formulas of size at most n (the set of which is denoted $\mathscr{F}^n$). First, the set of axioms Ax^n, consisting of pairings $\langle \gamma, \Gamma \rangle$ for which $\boldsymbol{\gamma} \vdash \Gamma$ s.t. either one of the following holds:

1. $\gamma = a$ and $\Gamma = a : A, b : B$, $A, B \in \mathscr{F}^n$; or
2. $\gamma = Tabc$ and $\Gamma = a : A, b : B, c : C$, $A, B, C \in \mathscr{F}^n$; or
3. $\gamma = Bab$ and $\Gamma = a : A, b : B$, $A, B \in \mathscr{F}^n$.

Rules of inference are as follows:

$$\frac{\langle \gamma, \Gamma \rangle \in Ax^n}{\boldsymbol{\gamma} \vdash^n \Gamma}\ Ax \qquad \frac{\boldsymbol{\gamma} \vdash^n \Gamma, a : A \quad \boldsymbol{\delta} \vdash^n \Delta, a : A^{\perp}}{(\boldsymbol{\gamma} \lhd_a \boldsymbol{\delta}) \vdash^n \Gamma, \Delta}\ \circ$$

Lemma 45. If $\boldsymbol{\gamma} \vdash \Gamma$ with Γ containing only formulas in $\mathscr{F}^n$, also $\boldsymbol{\gamma} \vdash^n \Gamma$.

Definition 86. By a **CNL** grammar G we understand a tuple $\langle \Sigma, \mathit{Atom}, L, g \rangle$ consisting of: a set of *words* Σ; a finite set of atomic formula *Atom* generating $\mathscr{F}$; a *lexicon* L mapping elements of Σ to finite subsets of $\mathscr{F}$; and an atomic *goal* formula $g \in \mathit{Atom}$. The *language* $\mathscr{L}(G)$ recognized by G we define by the set of strings $w_1 \dots w_n \in \Sigma^+$ s.t. for some $A_1 \in L(w_1), \dots, A_n \in L(w_n)$ and variety $\boldsymbol{\gamma}$ with $E_{\boldsymbol{\gamma}} = \{a_1, \dots, a_n, a\}$ and yield $a_n, \dots, a_1$ w.r.t. a (identified with the yield of $(\boldsymbol{\gamma} \downarrow a)$), $\boldsymbol{\gamma} \vdash a_1 : A_1, \dots, a_n : A_n$.

Definition 87. Given a **CNL**-grammar $G_1 = \langle \Sigma, \mathit{Atom}, L, g \rangle$ with the range of L a subset of $\mathscr{F}^n$, define the following CFG G_2: its set of terminals coincides with Σ, its set of nonterminals with $\mathscr{F}^n$, its startsymbol is g, and its productions are

$$\begin{array}{ll} & \{A^{\perp} \to w \mid A \in L(w)\} \\ \cup & \{A \to B^{\perp} \mid a \vdash^n a : A, a : B\} \\ \cup & \{A \to B^{\perp} \mid Bab \vdash^n a : A, b : B\} \\ \cup & \{A \to C^{\perp}\ B^{\perp} \mid Tabc \vdash^n a : A, b : B, c : C\} \end{array}$$

Theorem 5.6.5. For any **CNL**-grammar G_1, $\mathscr{L}(G_1) = \mathscr{L}(G_2)$, where G_2 is the CFG constructed from G_1 as in the preceding definition.

5.7 Discussion

We briefly mention related literature on the topics of labeling and of the distinction between varieties and presentations.

5.7.1 Labeled deduction

Labeling-wise, the closest proposals to ours are those of Negri [2005], working in modal logic, and of Kurtonina [1995], targeting the intuitionistic Lambek hierarchy. Both incorporate relation symbols in their derivations, in direct reference to the underlying models. We followed Pinto and Uustalu [2010], however, in constructing our labeling device as a mathematical object more rich in structure than a mere set of relation symbols, as well as in the use of a constructive argument for witnessing soundness relative to a form of deduction using explicit structural rules.

5.7.2 Structads and coloured linear logics

Inspired, respectively, by the proof nets of De Groote and Lamarche [2002] and Moot and Puite [2002] and by Abrusci and Ruet's [1999] order varieties of non-commutative logic, both Lamarche [2003] and Andreoli [2004] sought to devise a general theory of structure in logic, subsuming previous accounts of substructural logics, among which the type-logical hierarchy of the Lambek calculus. While the approaches differed, the former employing category theory while the latter stuck closer to Girard's [2001] then recent invention of Ludics, their efforts to a large extent were found compatible, at least in spirit. The main idea underlying both works was the generalization of the distinction between an unrooted tree structure and the various presentations obtained from it by picking an arbitrary node as root. While our interest in this chapter was in the latter concrete case, as exemplified in §2, we have benefited of the various terminological distinctions and notations used in the works mentioned, particularly Andreoli's.

6
Type similarity

Chapter 5 proved the context-freeness of languages recognized by grammars founded upon the base logic $\mathbf{LG}_{\emptyset}$. For $\mathbf{LG}_{IV}$, Melissen already pointed out a significant increase in recognizing capacity, exceeding that of lexicalized Tree Adjoining Grammars. The current chapter continues the study into the expressivity of the Grishin interactions, though this time we shall measure by a different standard. Besides model-theoretic investigations into derivability, people have sought to similarly characterize its symmetric-transitive closure. Thus, we consider A and B *type similar*, written $\vdash A \sim B$, iff there exists a sequence of formulae $C_1 \ldots C_n$ s.t. $C_1 = A$, $C_n = B$, and either $C_i \leq C_{i+1}$ or $C_{i+1} \leq C_i$ for each $1 \leq i < n$.[1] For the traditional intuitionistic calculi, one finds their level of resource sensitivity reflected in the algebraic models for the corresponding notions of $\sim$, as summarized in the following table:

Calculus	Models	Reference
NL	quasigroup	Foret [2003]
L	group	Pentus [1993]
LP	Abelian group	Pentus [1993]

[1]Our terminology is adapted from Moortgat and Pentus [2007], revising that of Pentus [1993] and van Benthem van Benthem [1995], who previously spoke of *type conjoinability*.

With $\mathbf{LG}_{IV}$, however, Moortgat and Pentus [2007, henceforth M&P] found that, while same-sort associativity and -commutativity remain underivable, the latter principles do hold at the level of type similarity. More specifically, we find there exist formulas serving as common ancestors or descendants (in terms of derivability) for $A \otimes B$ and $B \otimes A$, or $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$:

$$\begin{tikzcd}[column sep=small]
 & D \arrow[dl, dashed] \arrow[dr, dashed] & \\
A \otimes B \arrow[dr, dashed] & & B \otimes A \arrow[dl, dashed] \\
 & C &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & D \arrow[dl, dashed] \arrow[dr, dashed] & \\
(A \otimes B) \otimes C \arrow[dr, dashed] & & A \otimes (B \otimes C) \arrow[dl, dashed] \\
 & C &
\end{tikzcd}$$

In general, we may prove type similarity coincides with the existence of such *meets* D or *joins* C, as referred to by M&P (though not to be confused with terminology from lattice theory). With respect to linguistic applications, these findings suggest the possibility of tapping in on the flexibility of **L** and **LP** without compromising overall resource-sensitivity, simply by assigning the relevant joins or meets inside one's lexicon.

The current chapter defines a class of models w.r.t. which we prove soundness and completeness of type similarity in $\mathbf{LG}_{\emptyset}$ extended by type I or IV interactions, both with and without units. While M&P already provided a non-standard interpretation of $\mathbf{LG}_{IV}$ inside Abelian groups, we here consider a notion of model better reflecting the presence of dual (co)residuated families of connectives. Such results still leave open, however, the matter of deciding the existence of joins or meets. We first solve this problem for the specific case of $\mathbf{LG}_{I}$ together with units 0 and 1, taking a hint from M&P's Abelian group interpretation for $\mathbf{LG}_{IV}$. Decidability for type similarity in the remaining incarnations of **LG** is derived as a corollary.

We proceed as follows. We first briefly discuss the extension of the formula language with (non-distributive) additive conjunction and -disjunction, paying particular attention to their use in reducing lexical ambiguity. A number of concerns are raised, however, regarding their effects upon expressivity and complexity, leading us to consider type similarity as a viable alternative. We provide various equivalent definitions in §3, illustrating the notion with some typical instances in $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$. Models for ~ in the presence of type I or IV interactions are defined in §4, along with proofs of soundness and completeness. An algorithm for generating joins in $\mathbf{LG}_{I}$ is detailed in §5, with §6 suggesting linguistic applications, focussing on extraction.

6.1 Additives

We discuss the extension of the formula language with the lattice operations of *meet* ($\wedge$) and *join* ($\vee$), also referred to by *additive* conjunction and -disjunction respectively. While in intuitionistic and classical logic they are often understood to satisfy the distributivity laws below, we do not make any such assumptions here.

$$
\begin{array}{ll}
A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C) & \quad (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C) \\
A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) & \quad (A \wedge B) \vee C = (A \vee C) \wedge (B \vee C)
\end{array}
$$

Definition 88. Additives are defined by the following inference rules:

$$
\frac{C \leq A \quad C \leq B}{C \leq A \wedge B} \wedge \qquad \frac{A \leq C}{A \wedge B \leq C} \wedge^l \qquad \frac{C \leq A}{C \leq A \vee B} \vee^l
$$

$$
\frac{A \leq C \quad B \leq C}{A \vee B \leq C} \vee \qquad \frac{B \leq C}{A \wedge B \leq C} \wedge^r \qquad \frac{C \leq B}{C \leq A \vee B} \vee^r
$$

Remark 8. Using double inference lines to indicate derivability may be read in both directions, the above rules are more succinctly rendered thus:

$$
\genfrac{}{}{0pt}{}{C \leq A \quad C \leq B}{\overline{\overline{}}} \!\!\!\! \;
\frac{\;C \leq A \quad C \leq B\;}{C \leq A \wedge B}\!\!\!\!\!\!\;\; \wedge \qquad \frac{A \leq C \quad B \leq C}{A \vee B \leq C} \vee
$$

Example 20. Writing $A \simeq B$ to abbreviate the conjunction $A \leq B$ and $B \leq A$, we can derive the following distributivity results:

$$
\begin{array}{cc}
A \otimes (B \vee C) \simeq (A \otimes B) \vee (A \otimes C) & (A \vee B) \otimes C \simeq (A \otimes C) \vee (B \otimes C) \\
(A \wedge B) \oplus C \simeq (A \oplus C) \wedge (B \oplus C) & A \oplus (B \wedge C) \simeq (A \oplus B) \wedge (A \oplus C) \\
A/(B \vee C) \simeq (A/B) \wedge (A/C) & (A \vee B)\backslash C \simeq (A\backslash C) \wedge (B\backslash C) \\
(A \wedge B)/C \simeq (A/C) \wedge (B/C) & C\backslash(A \wedge B) \simeq (C\backslash A) \wedge (C\backslash B) \\
(A \wedge B) \otimes C \simeq (A \otimes C) \vee (B \otimes C) & A \oslash (B \wedge C) \simeq (A \oslash B) \vee (A \oslash C) \\
C \otimes (A \vee B) \simeq (C \otimes A) \vee (C \otimes B) & (A \vee B) \oslash C \simeq (A \oslash C) \vee (B \oslash C)
\end{array}
$$

Note, however, that we cannot in general derive distributivity laws relating $\wedge$ and $\vee$. We illustrate with derivations for $A/(B \vee C) \simeq (A/B) \wedge (A/C)$:

$$
\frac{\dfrac{\dfrac{\dfrac{}{B \leq B}\,Id}{B \leq B \vee C}\vee}{A/(B \vee C) \leq A/B}\,?/}{A/(B \vee C) \leq (A/B) \wedge (A/C)} \wedge
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A/B \leq A/B}\ Id}{(A/B) \wedge (A/C) \leq A/B}\ \wedge}{B \leq ((A/B) \wedge (A/C))\backslash A}\ r \times 2 \quad \dfrac{\dfrac{\dfrac{}{A/C \leq A/C}\ Id}{(A/B) \wedge (A/C) \leq A/C}\ \wedge}{C \leq ((A/B) \wedge (A/C))\backslash A}\ r \times 2}{\dfrac{B \vee C \leq ((A/B) \wedge (A/C))\backslash A}{(A/B) \wedge (A/C) \leq A/(B \vee C)}\ r \times 2}\ \vee$$

Example 21. One obvious benefit of the additive conjunction lies in reducting lexical ambiguity, replacing multiple assignments of categories $A_1, \ldots, A_n$ to a single word w by their conjunction $A_1 \wedge \cdots \wedge A_n$.

At the level of models, we might at first try to interpret additive conjunction and -disjunction through (classical) set intersection and -union:

$x \vDash A \wedge B$ iff $x \in \{x \in W \mid x \vDash A\} \bigcap \{x \in W \mid x \vDash B\}$ (i.e., iff $x \vDash A$ and $x \vDash B$)
$x \vDash A \vee B$ iff $x \in \{x \in W \mid x \vDash A\} \bigcup \{x \in W \mid x \vDash B\}$ (i.e., iff $x \vDash A$ or $x \vDash B$)

Obviously, if this line of thought is to be further pursued, we would have to adopt additional axioms expressing distributivity of $\wedge$ over $\vee$ and vice versa in order to guarantee completeness w.r.t. D.88. Sticking with non-distributivity, we may resort to either one of the following alternative conceptions of models.

1. As with the multiplicatives, we could interpret $\wedge$ and $\vee$ using ternary relations, assuming a number of additional frame constraints in order to match the duplication of formula occurrences inherent in the inference rules above. The work of MacCaull [1998] is an example of this approach.

2. Another solution is to interpret formulas by sets with *structure*. Typically, one demands the addition upon frames of a *closure operation* $C : W \to W$, satisfying, for any $X, Y \in \mathscr{P}(W)$, $X \subseteq C(X)$, $C(X) \subseteq C(Y)$ if $X \subseteq Y$, and finally $C(C(X)) \subseteq C(X)$.[2] When extended to a model, formulas no longer interpret by arbitrary sets $X \subseteq W$ of resources, but rather by the *closed* such sets, i.e., for which $X = C(X)$. In particular, to interpret the additives, we ensure the space of closed subsets of W constitutes a non-distributive lattice, with meet and join $X \wedge Y$ and $X \vee Y$ for $X, Y \subseteq W$ defined by $X \bigcap Y$ and $C(X \bigcup Y)$ respectively, restoring completeness. Restall [2000] provides more in-depth discussion within the broader field of substructural logics, while Kurtonina [1995] considers specifically the Lambek calculus, discussing a hierarchy of structured sets of worlds for relational models to cover various additional connectives, among which the additives.

[2] Strictly speaking, another condition is required that demands commutation of C with the interpretation(s) of tensor (and par) [see Buszkowski and Farulewski, 2009].

Within the base logic, Buszkowski and Farulewski [2009] showed that the presence of additives (together with distributivity) does not increase expressivity beyond the context-free. Already with associativity in place, however, Kanazawa [1992] showed that any finite intersection of (associative) Lambek grammars staying within the multiplicative fragment can, through use of additives, be replaced by an equivalent one without any lexical ambiguity. We conjecture that additives may interact similarly with the Grishin interactions. Finally, while to the author's knowledge no complexity results have been obtained as of yet for Lambek calculi augmented with additives, multiplicative-additive linear logic was already shown PSPACE-complete by Lincoln et al. [1992]. At least where the reduction of lexical ambiguity is concerned, however, some of the benefits of additives may be reproduced within the multiplicative fragment through the relation of type similarity.

6.2 Diamond property and examples

We write $\mathbf{T} \vdash A \leq B$ for $\mathbf{T} \in \{\mathbf{LG}_I, \mathbf{LG}_{IV}, \mathbf{LG}_I^{0,1}, \mathbf{LG}_{IV}^{0,1}\}$ to claim derivability in the calculi under consideration, the superscripts 0 and 1 indicating augmentation with units, as described in chapter 4. In the case of statements valid for arbitrary choice of $\mathbf{T}$, we simply write $A \leq B$.

Definition 89. Given $\mathbf{T} \in \{\mathbf{LG}_I, \mathbf{LG}_{IV}, \mathbf{LG}_I^{0,1}, \mathbf{LG}_{IV}^{0,1}\}$, we say A, B are *type similar* in $\mathbf{T}$, written $\mathbf{T} \vdash A \sim B$, iff $\exists C, \mathbf{T} \vdash A \leq C$ and $\mathbf{T} \vdash B \leq C$.

Following convention established by Pentus [1993], Moortgat and Pentus [2007], we say that the C witnessing $\mathbf{T} \vdash A \sim B$ is a *join* for A, B, not to be confused with terminology from lattice theory. Keeping with tradition, we write $\vdash A \sim B$ in case a statement is independent of the particular choice of $\mathbf{T}$. We have the following equivalent definition.

Lemma 46. Formulas A, B are type similar iff there exists D s.t. $D \leq A$ and $D \leq B$.

Proof. The following table provides for each choice of $\mathbf{T}$ the solution for D in case the join C is known, and conversely. Note q refers to an arbitrary atom.

$\mathbf{T}$	Solution for C	Solution for D
$\mathbf{LG}_I$	$((B \otimes B) \oplus (B \oslash A))/D$	$C \oslash ((A/B) \otimes (B \oplus B))$
$\mathbf{LG}_{IV}$	$((D/B)\backslash q) \oplus ((D/A)\backslash(q \oslash D))$	$((C/q) \oslash (A \oslash C)) \otimes (q \oslash (B \oslash C))$
$\mathbf{LG}_I^{0,1}$	$(1 \oslash D) \oplus (A \otimes B)$	$(B \oplus A) \otimes (C\backslash 0)$
$\mathbf{LG}_{IV}^{0,1}$	$(A \otimes B)/(1 \oslash (D\backslash 0))$	$((1 \oslash C)\backslash 0) \oslash (B \oplus A)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$B \obslash A \leq B \obslash A$}
\RightLabel{cr}
\UnaryInfC{$A \leq B \oplus (B \obslash A)$}
\RightLabel{cr}
\UnaryInfC{$A \oslash (B \obslash A) \leq B$}
\AxiomC{$D \leq B$}
\RightLabel{$\otimes$}
\BinaryInfC{$(A \oslash (B \obslash A)) \otimes D \leq B \otimes B$}
\RightLabel{r}
\UnaryInfC{$A \oslash (B \obslash A) \leq (B \otimes B)/D$}
\RightLabel{C_I^2}
\UnaryInfC{$A \otimes D \leq (B \otimes B) \oplus (B \obslash A)$}
\RightLabel{r}
\UnaryInfC{$A \leq ((B \otimes B) \oplus (B \obslash A))/D$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$D \leq A$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$B \leq B$}
\RightLabel{$\obslash$}
\BinaryInfC{$B \obslash D \leq B \obslash A$}
\RightLabel{cr}
\UnaryInfC{$D \leq B \oplus (B \obslash A)$}
\RightLabel{cr}
\UnaryInfC{$D \oslash (B \obslash A) \leq B$}
\RightLabel{$?\otimes$}
\UnaryInfC{$B \otimes (D \oslash (B \obslash A)) \leq B \otimes B$}
\RightLabel{r}
\UnaryInfC{$D \oslash (B \obslash A) \leq B \backslash (B \otimes B)$}
\RightLabel{A_I^1}
\UnaryInfC{$B \otimes D \leq (B \otimes B) \oplus (B \obslash A)$}
\RightLabel{r}
\UnaryInfC{$B \leq ((B \otimes B) \oplus (B \obslash A))/D$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$q \obslash D \leq q \obslash D$}
\RightLabel{cr}
\UnaryInfC{$D \leq q \oplus (q \obslash D)$}
\AxiomC{$D \leq B$}
\RightLabel{$/$}
\BinaryInfC{$D/B \leq (q \oplus (q \obslash D))/D$}
\RightLabel{r}
\UnaryInfC{$(D/B) \otimes D \leq q \oplus (q \obslash D)$}
\RightLabel{A_{IV}^2}
\UnaryInfC{$D \oslash (q \obslash D) \leq (D/B) \backslash q$}
\RightLabel{cr}
\UnaryInfC{$D \leq ((D/B) \backslash q) \oplus (q \obslash D)$}
\RightLabel{$/?$}
\UnaryInfC{$D/A \leq (((D/B) \backslash q) \oplus (q \obslash D))/A$}
\RightLabel{r}
\UnaryInfC{$(D/A) \otimes A \leq ((D/B) \backslash q) \oplus (q \obslash D)$}
\RightLabel{C_{IV}^1}
\UnaryInfC{$((D/B) \backslash q) \obslash A \leq (D/A) \backslash (q \obslash D)$}
\RightLabel{cr}
\UnaryInfC{$A \leq ((D/B) \backslash q) \oplus ((D/A) \backslash (q \obslash D))$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$q \obslash D \leq q \obslash D$}
\RightLabel{cr}
\UnaryInfC{$D \leq q \oplus (q \obslash D)$}
\AxiomC{$D \leq A$}
\RightLabel{$/$}
\BinaryInfC{$D/A \leq (q \oplus (q \obslash D))/D$}
\RightLabel{r}
\UnaryInfC{$(D/A) \otimes D \leq q \oplus (q \obslash D)$}
\RightLabel{A_{IV}^2}
\UnaryInfC{$q \obslash D \leq (D/A) \backslash (q \obslash D)$}
\RightLabel{cr}
\UnaryInfC{$D \leq q \oplus ((D/A) \backslash (q \obslash D))$}
\RightLabel{$/?$}
\UnaryInfC{$D/B \leq (q \oplus ((D/A) \backslash (q \obslash D)))/B$}
\RightLabel{r}
\UnaryInfC{$(D/B) \otimes B \leq q \oplus ((D/A) \backslash (q \obslash D))$}
\RightLabel{A_{IV}^2}
\UnaryInfC{$B \oslash ((D/A) \backslash (q \obslash D)) \leq (D/B) \backslash q$}
\RightLabel{cr}
\UnaryInfC{$B \leq ((D/B) \backslash q) \oplus ((D/A) \backslash (q \obslash D))$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$1 \leq 1$}
\AxiomC{$D \leq B$}
\RightLabel{$\oslash$}
\BinaryInfC{$1 \oslash B \leq 1 \oslash D$}
\RightLabel{cr}
\UnaryInfC{$1 \leq (1 \oslash D) \oplus B$}
\RightLabel{cr}
\UnaryInfC{$(1 \oslash D) \obslash 1 \leq B$}
\RightLabel{$?\otimes$}
\UnaryInfC{$A \otimes ((1 \oslash D) \obslash 1) \leq A \otimes B$}
\RightLabel{r}
\UnaryInfC{$(1 \oslash D) \obslash 1 \leq A \backslash (A \otimes B)$}
\RightLabel{C_I^1}
\UnaryInfC{$A \otimes 1 \leq (1 \oslash D) \oplus (A \otimes B)$}
\RightLabel{1}
\UnaryInfC{$A \leq (1 \oslash D) \oplus (A \otimes B)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$1 \leq 1$}
\AxiomC{$D \leq A$}
\RightLabel{$\oslash$}
\BinaryInfC{$1 \oslash A \leq 1 \oslash D$}
\RightLabel{cr}
\UnaryInfC{$1 \leq (1 \oslash D) \oplus A$}
\RightLabel{cr}
\UnaryInfC{$(1 \oslash D) \obslash 1 \leq A$}
\RightLabel{$\otimes ?$}
\UnaryInfC{$((1 \oslash D) \obslash 1) \otimes B \leq A \otimes B$}
\RightLabel{r}
\UnaryInfC{$(1 \oslash D) \obslash 1 \leq (A \otimes B)/B$}
\RightLabel{A_I^2}
\UnaryInfC{$1 \otimes B \leq (1 \oslash D) \oplus (A \otimes B)$}
\RightLabel{1}
\UnaryInfC{$B \leq (1 \oslash D) \oplus (A \otimes B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$D \leq B$}
\RightLabel{0}
\UnaryInfC{$D \leq B \oplus 0$}
\RightLabel{1}
\UnaryInfC{$D \otimes 1 \leq B \oplus 0$}
\RightLabel{C_{IV}^1}
\UnaryInfC{$B \obslash 1 \leq D \backslash 0$}
\RightLabel{cr}
\UnaryInfC{$1 \leq B \oplus (D \backslash 0)$}
\RightLabel{cr}
\UnaryInfC{$1 \oslash (D \backslash 0) \leq B$}
\RightLabel{$?\otimes$}
\UnaryInfC{$A \otimes (1 \oslash (D \backslash 0)) \leq A \otimes B$}
\RightLabel{r}
\UnaryInfC{$A \leq (A \otimes B)/(1 \oslash (D \backslash 0))$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$D \leq A$}
\RightLabel{$\otimes ?$}
\UnaryInfC{$D \otimes B \leq A \otimes B$}
\RightLabel{0}
\UnaryInfC{$D \otimes B \leq (A \otimes B) \oplus 0$}
\RightLabel{C_{IV}^1}
\UnaryInfC{$(A \otimes B) \obslash B \leq D \backslash 0$}
\RightLabel{cr}
\UnaryInfC{$B \leq (A \otimes B) \oplus (D \backslash 0)$}
\RightLabel{1}
\UnaryInfC{$B \otimes 1 \leq (A \otimes B) \oplus (D \backslash 0)$}
\RightLabel{A_{IV}^2}
\UnaryInfC{$1 \oslash (D \backslash 0) \leq B \backslash (A \otimes B)$}
\RightLabel{r}
\UnaryInfC{$B \otimes (1 \oslash (D \backslash 0)) \leq A \otimes B$}
\RightLabel{r}
\UnaryInfC{$B \leq (A \otimes B)/(1 \oslash (D \backslash 0))$}
\end{prooftree}
$$

Figure 6.1: Derivations for the joins constructed in the proof of T.46.

Note the solutions for $\mathbf{LG}_{I}$ and $\mathbf{LG}_{IV}$ apply as well to $\mathbf{LG}_{I}^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$ respectively, although the latter calculi also allow for simpler choices of C and D. We provide in F.6.1 the derivations for the joins, those concerning D being essentially dual. $\square$

L.46 is commonly referred to by the *diamond* property, in reference to the following equivalent diagrammatic representation:

$$\begin{tikzcd}[column sep=small]
 & \exists D \arrow[dl, dashed] \arrow[dr, dashed] & \\
A \arrow[dr] & & B \arrow[dl] \\
 & C &
\end{tikzcd}
\quad \text{iff} \quad
\begin{tikzcd}[column sep=small]
 & D \arrow[dl] \arrow[dr] & \\
A \arrow[dr, dashed] & & B \arrow[dl, dashed] \\
 & \exists C &
\end{tikzcd}$$

The formula D is also referred to as a *meet* for A, B. In case C is known, we write $A \sqcap_C B$ for the meet constructed in L.46, while conversely we write $A \sqcup_D B$ for the join if the meet D is known. Clearly, we have that if $A \sqcup_D B \leq E$ ($E \leq A \sqcap_C B$), then also $A \leq E$ ($E \leq A$), $B \leq E$ ($E \leq B$) and $D \leq E$ ($E \leq C$).

Remark 9. M&P provided another solution for $\mathbf{LG}_{IV}$: given a join C for A and B, let $A \sqcap_C B = (A/C) \otimes (C \oslash (B \obslash C))$, while conversely $A \sqcup_D B = ((D/B)\backslash D) \otimes (D\backslash A)$. Though slightly smaller when compared to the solution proposed here (-2, when measured in terms of atoms), the latter allows for easier generalization. For example, in the following event, suppose we wish to find a meet for A_1 and A_4:

$$\begin{tikzcd}[column sep=small]
A_1 \arrow[dr] & & A_2 \arrow[dl] \arrow[dr] & & A_3 \arrow[dl] \arrow[dr] & & A_4 \arrow[dl] \\
 & C_1 & & C_2 & & C_3 &
\end{tikzcd}$$

Normally, we would suffice by repeated applications of the diamond property:

$$\begin{tikzcd}[column sep=small]
 & & & D_6 \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
 & & D_4 \arrow[dl, dashed] \arrow[dr, dashed] & & D_5 \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & D_1 \arrow[dl, dashed] \arrow[dr, dashed] & & D_2 \arrow[dl, dashed] \arrow[dr, dashed] & & D_3 \arrow[dl, dashed] \arrow[dr, dashed] & \\
A_1 \arrow[dr] & & A_2 \arrow[dl] \arrow[dr] & & A_3 \arrow[dl] \arrow[dr] & & A_4 \arrow[dl] \\
 & C_1 & & C_2 & & C_3 &
\end{tikzcd}$$

In doing so, however, we get more than we bargained for: D_6 derives each of A_1, A_2, A_3, A_4. Restricting to A_1 and A_4, we have a much shorter solution

$$((C_2/q)\oslash(A_1\obslash C_3))\otimes(((q/q)\oslash(A_4\obslash C_1))\otimes(((q/q)\oslash(A_2\obslash C_2))\otimes(q\oslash(A_3\obslash C_2))))$$

which obviously generalizes $A \sqcap_C B = ((C/q) \oslash (A \obslash C)) \otimes (q \oslash (B \obslash C))$.

Lemma 47. Already in the base logic $\mathbf{LG}_{\emptyset}$, type similarity satisfies

1. *Reflexivity.* $\vdash A \sim A$
2. *Transitivity.* $\vdash A \sim B$ and $\vdash B \sim C$ imply $\vdash A \sim C$
3. *Reflexivity.* $\vdash A \sim B$ implies $\vdash B \sim A$
4. *Congruence.* $\vdash A_1 \sim A_2$ and $\vdash B_1 \sim B_2$ imply $\vdash A_1\ \delta\ B_1 \sim A_2\ \delta\ B_2$ for any $\delta \in \{\otimes, /, \backslash, \oplus, \obslash, \oslash\}$.

Proof. Reflexivity and symmetry are immediate, while congruence is a consequence of monotonicity. Finally, transitivity follows from repeated applications of the diamond property. □

The introduction gave a slightly different definition of type similarity. We now show that the two notions coincide.

Definition 90. Say formulas A, B are *type-alike*, written $\vdash A \sim' B$, iff there exists a sequence of formulae $C_1 \ldots C_n$ s.t. $C_1 = A$, $C_n = B$, and either $C_i \leq C_{i+1}$ or $C_{i+1} \leq C_i$ for each $1 \leq i < n$.

Theorem 6.2.1. For any A, B, $\vdash A \sim B$ iff $\vdash A \sim' B$.

Proof. The direction from left to right is immediate. Going from right to left, one proceeds by induction on the length of the chain $C_1 \ldots C_n$ used in witnessing $\vdash A \sim' B$, repeatedly applying the diamond property to obtain a join or meet for C_1 and C_i at each step i. □

We consider some examples illustrating the expressivity of ~ in the presence of type I or IV interactions. Some of these were already considered by M&P for $\mathbf{LG}_{IV}$, although each of the solutions for said calculus provided here is either shorter or provides a join (meet) while M&P gave a meet (join).

Lemma 48. *Neutrals.* $\vdash C\backslash C \sim D/D$

Proof. We have a join $(((C\backslash C)\oslash D)\oplus((D\oslash C)\oplus C))/(C\backslash C)$ for $\mathbf{LG}_I$ and a meet $(C \oslash (C \obslash (C \otimes D))) \otimes (D \oslash (D \obslash (C \otimes D)))$ for $\mathbf{LG}_{IV}$, as shown in F.6.2. □

The next few lemmas detail associativity and commutativity properties; underivable, but still valid at the level of type similarity.

Lemma 49. *Symmetry.* $\vdash A\backslash B \sim B/A$

Proof. For $\mathbf{LG}_I$ we have a join $(((A\backslash B) \oslash A) \oplus B)/(A\backslash B)$ (F.6.3), while for $\mathbf{LG}_{IV}$, we have a meet $A \oslash (B \obslash (A \otimes A))$ (F.6.4). □

Lemma 50. *Rotations.* $\vdash A\backslash(C/B) \sim (A\backslash C)/B$ and $\vdash A\backslash(B\backslash C) \sim B\backslash(A\backslash C)$

Proof. In $\mathbf{LG}_I$, we have $A\backslash(B\backslash((B \oslash A) \oplus ((A \oslash B) \oplus C)))$ as a join for $A\backslash(B\backslash C)$ and $B\backslash(A\backslash C)$ (F.6.3). To derive $\vdash A\backslash(C/B) \sim (A\backslash C)/B$, we proceed as follows:

1.	$\mathbf{LG}_I \vdash A\backslash(C/B) \sim (C/B)/A$	(L.49)
2.	$\mathbf{LG}_I \vdash (C/B)/A \sim (C/A)/B$	(shown above)
3.	$\mathbf{LG}_I \vdash (C/A)/B \sim (A\backslash C)/B$	(L.49 and L.47(4))
4.	$\mathbf{LG}_I \vdash A\backslash(C/B) \sim (A\backslash C)/B$	(L.47(2), 1,2,3)

For $\mathbf{LG}_{IV}$, we have a meet $((C \oslash (C/B)) \obslash q) \otimes ((C \oslash (A\backslash C)) \obslash (q\backslash C))$ witnessing $\mathbf{LG}_{IV} \vdash A\backslash(C/B) \sim (A\backslash C)/B$, and similarly $\mathbf{LG}_{IV} \vdash A\backslash(B\backslash C) \sim B\backslash(A\backslash C)$ with meet $((C \oslash (A\backslash C)) \obslash q) \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C))$ (F.6.4). □

Lemma 51. *Distributivity.* $\vdash A \otimes (B/C) \sim (A \otimes B)/C$

Proof. For $\mathbf{LG}_I$, note $\vdash A \otimes (B/C) \sim A \otimes (C\backslash B)$ and $\vdash (A \otimes B)/C \sim C\backslash(A \otimes B)$ by L.49 and L.47(4). We then obtain a join $C\backslash((A \oslash C) \oplus (C \otimes B))$ for witnessing $\mathbf{LG}_I \vdash A \otimes (C\backslash B) \sim C\backslash(A \otimes B)$ (F.6.5). For $\mathbf{LG}_{IV}$, we have a direct solution with the meet $A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B))))$ (F.6.5). □

Lemma 52. *Commutativity.* $\vdash A \otimes B \sim B \otimes A$

Proof. We have a join $(A \oslash B) \oplus (B \otimes B)$ for $\mathbf{LG}_I$ (F.6.6) and a meet $(B \otimes B) \obslash (B \otimes ((B \oplus A) \otimes B))$. For $\mathbf{LG}_{IV}$, we have a meet $(((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A$ (F.6.6). □

Lemma 53. *Associativity.* $\vdash (A \otimes B) \otimes C \sim A \otimes (B \otimes C)$

Proof. For $\mathbf{LG}_I$, we have a join $(B \oslash ((A\backslash q) \oplus (Q/C))) \oplus (q \oplus q)$ and a meet $(A \otimes A) \obslash (A \otimes ((A \otimes (A \oplus B)) \otimes C))$ (F.6.7). For $\mathbf{LG}_{IV}$, we apply the diamond property, as follows, using L.51 to find D_1:

$$\begin{tikzcd}
 & & D_2 \arrow[dl, dashed] \arrow[dr, dashed] & \\
A \otimes (B \otimes C) & D_1 \arrow[dl, dashed] \arrow[dr, dashed] & & (A \otimes B) \otimes C \arrow[dl] \\
((A \otimes (B \otimes C))/C) \otimes C \arrow[u] & & (A \otimes ((B \otimes C)/C)) \otimes C &
\end{tikzcd}$$

□

Remark 10. While the above lemmas immediately extend to $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$, the presence of units often allows for simpler joins and meets. For example, we have the following joins (J) and meets (M) in $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$:

Lemma	$\mathbf{LG}_I^{0,1}$	$\mathbf{LG}_{IV}^{0,1}$
Neutrals	J. $C\backslash((1 \oslash D) \oplus (C \otimes D))$	J. $((1 \oslash C)\backslash 0)\backslash((C \otimes D)/D)$
Symmetry	J. $(1 \oslash A) \oplus B$	M. $1 \oslash (B \obslash A)$
Commutativity	J. $(1 \oslash (1/A)) \oplus B$	M. $((1 \oslash A)\backslash 0) \otimes B$
Associativity	J. $(1 \oslash (1/A)) \oplus (B \otimes C)$	M. $A \otimes (((1 \oslash B)\backslash 0) \otimes C)$

6.3 Completeness results

We shall consider models built upon algebraic structures featuring two binary operations $\times$ and $+$, related by linear distributivity. Their definition is derived from Cockett and Seely's [1991] weakly (or, as they were later called, linear) distributive categories by turning their arrows into equivalences.

Definition 91. A *linearly distributive algebra* $\mathscr{A} = \langle A, \top, \bot, \cdot^{\bot}, \times, + \rangle$ pairs the carrier set A with constants $\top, \bot$, a unary operation $\cdot^{\bot}$ and operators $\times, +$ satisfying

1. *Associativity.* $(A \times B) \times C = A \times (B \times C)$; $(A + B) + C = A + (B + C)$
2. *Commutativity.* $A \times B = B \times A$; $A + B = B + A$
3. *Units.* $A \times \top = A$; $A + \bot = A$
4. *Inverses.* $A^{\bot} \times A = \bot$; $A^{\bot} + A = \top$
5. *Linear distributivity.* $A \times (B + C) = (A \times B) + C$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(C\backslash C)\oslash D\leq(C\backslash C)\oslash D}\,\mathit{Id}}{((C\backslash C)\oslash D)\obslash(C\backslash C)\leq D}\,cr\times 2}{(((C\backslash C)\oslash D)\obslash(C\backslash C))\oslash C\leq D\oslash C}\,\oslash ?}{(D\oslash C)\obslash(((C\backslash C)\oslash D)\obslash(C\backslash C))\leq C}\,cr\times 2 \qquad \dfrac{\dfrac{}{C\backslash C\leq C\backslash C}\,\mathit{Id}}{C\leq C/(C\backslash C)}\,r\times 2}{\dfrac{(D\oslash C)\obslash(((C\backslash C)\oslash D)\obslash(C\backslash C))\leq C/(C\backslash C)}{\dfrac{(((C\backslash C)\oslash D)\obslash(C\backslash C))\otimes(C\backslash C)\leq(D\oslash C)\oplus C}{\dfrac{((C\backslash C)\oslash D)\obslash(C\backslash C)\leq((D\oslash C)\oplus C)/(C\backslash C)}{\dfrac{(C\backslash C)\otimes(C\backslash C)\leq((C\backslash C)\oslash D)\oplus((D\oslash C)\oplus C)}{C\backslash C\leq(((C\backslash C)\oslash D)\oplus((D\oslash C)\oplus C))/(C\backslash C)}\,r}\,A_I^2}\,r}\,A_I^2}\,\circ$$

$$\dfrac{\dfrac{\dfrac{}{D\oslash C\leq D\oslash C}\,\mathit{Id}}{D\leq(D\oslash C)\oplus C}\,cr \qquad \dfrac{\dfrac{}{(C\backslash C)\oslash D\leq(C\backslash C)\oslash D}\,\mathit{Id}}{((C\backslash C)\oslash D)\obslash(C\backslash C)\leq D}\,cr\times 2}{\dfrac{D/D\leq((D\oslash C)\oplus C)/(((C\backslash C)\oslash D)\obslash(C\backslash C))}{\dfrac{((C\backslash C)\oslash D)\obslash(C\backslash C)\leq(D/D)\backslash((D\oslash C)\oplus C)}{\dfrac{(D/D)\otimes(C\backslash C)\leq((C\backslash C)\oslash D)\oplus((D\oslash C)\oplus C)}{D/D\leq(((C\backslash C)\oslash D)\oplus((D\oslash C)\oplus C))/(C\backslash C)}\,r}\,C_I^1}\,r\times 2}\,/$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\obslash(C\otimes D)\leq D\obslash(C\otimes D)}\,\mathit{Id}}{C\otimes D\leq D\oplus(D\obslash(C\otimes D))}\,cr}{D\oslash(D\obslash(C\otimes D))\leq C\backslash D}\,A_{IV}^2}{C\otimes(D\oslash(D\obslash(C\otimes D)))\leq D}\,r}{C\otimes(C\otimes(D\oslash(D\obslash(C\otimes D))))\leq C\otimes D}\,?\otimes}{C\obslash(C\otimes(C\otimes(D\oslash(D\obslash(C\otimes D)))))\leq C\obslash(C\otimes D)}\,?\obslash}{C\otimes(C\otimes(D\oslash(D\obslash(C\otimes D))))\leq C\oplus(C\obslash(C\otimes D))}\,cr}{(C\otimes(D\oslash(D\obslash(C\otimes D))))\oslash(C\obslash(C\otimes D))\leq C\backslash C}\,A_{IV}^2}{C\otimes(D\oslash(D\obslash(C\otimes D)))\leq(C\backslash C)\oplus(C\obslash(C\otimes D))}\,cr}{C\oslash(C\obslash(C\otimes D))\leq(C\backslash C)/(D\oslash(D\obslash(C\otimes D)))}\,C_{IV}^2}{(C\oslash(C\obslash(C\otimes D)))\otimes(D\oslash(D\obslash(C\otimes D)))\leq C\backslash C}\,r$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C\obslash(C\otimes D)\leq C\obslash(C\otimes D)}\,\mathit{Id}}{C\otimes D\leq C\oplus(C\obslash(C\otimes D))}\,cr}{C\oslash(C\obslash(C\otimes D))\leq C/D}\,C_{IV}^2}{(C\oslash(C\obslash(C\otimes D)))\otimes D\leq C}\,r}{((C\oslash(C\obslash(C\otimes D)))\otimes D)\otimes D\leq C\otimes D}\,\otimes ?}{D\obslash(((C\oslash(C\obslash(C\otimes D)))\otimes D)\otimes D)\leq D\obslash(C\otimes D)}\,?\obslash}{((C\oslash(C\obslash(C\otimes D)))\otimes D)\otimes D\leq D\oplus(D\obslash(C\otimes D))}\,cr}{((C\oslash(C\obslash(C\otimes D)))\otimes D)\oslash(D\obslash(C\otimes D))\leq D/D}\,A_{IV}^2}{(C\oslash(C\obslash(C\otimes D)))\otimes D\leq(D/D)\oplus(D\obslash(C\otimes D))}\,cr}{D\oslash(D\obslash(C\otimes D))\leq(C\oslash(C\obslash(C\otimes D)))\backslash(D/D)}\,A_{IV}^2}{(C\oslash(C\obslash(C\otimes D)))\otimes(D\oslash(D\obslash(C\otimes D)))\leq D/D}\,r$$

Figure 6.2: Deriving a join (meet) in $\mathbf{LG}_I$ ($\mathbf{LG}_{IV}$) witnessing $\vdash C\backslash C \sim D/D$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A\backslash B) \oslash A \leq (A\backslash B) \oslash A}\;\mathit{Id}}{((A\backslash B) \oslash A) \obslash (A\backslash B) \leq A}\;cr \times 2}{A\backslash B \leq (((A\backslash B) \oslash A) \obslash (A\backslash B))\backslash B}\;\backslash ?}{((A\backslash B) \oslash A) \obslash (A\backslash B) \leq B/(A\backslash B)}\;r \times 2}{(A\backslash B) \otimes (A\backslash B) \leq ((A\backslash B) \oslash A) \oplus B}\;A_I^2}{A\backslash B \leq (((A\backslash B) \oslash A) \oplus B)/(A\backslash B)}\;r
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A\backslash B) \oslash A \leq (A\backslash B) \oslash A}\;\mathit{Id}}{((A\backslash B) \oslash A) \obslash (A\backslash B) \leq A}\;cr \times 2}{B/A \leq B/(((A\backslash B) \oslash A) \obslash (A\backslash B))}\;?/}{((A\backslash B) \oslash A) \obslash (A\backslash B) \leq (B/A)\backslash B}\;r \times 2}{(B/A) \otimes (A\backslash B) \leq ((A\backslash B) \oslash A) \oplus B}\;C_I^1}{B/A \leq (((A\backslash B) \oslash A) \oplus B)/(A\backslash B)}\;r
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \obslash A \leq B \obslash A}\;\mathit{Id}}{(B \oslash A) \obslash B \leq A}\;cr \times 2}{((B \oslash A) \obslash B) \oslash B \leq A \oslash B}\;\oslash ?}{(A \oslash B) \obslash ((B \oslash A) \obslash B) \leq B}\;cr \times 2}{B\backslash C \leq ((A \oslash B) \obslash ((B \oslash A) \obslash B))\backslash C}\;\backslash ?}{A\backslash(B\backslash C) \leq A\backslash(((A \oslash B) \obslash ((B \oslash A) \obslash B))\backslash C)}\;?\backslash}{(A \oslash B) \obslash ((B \oslash A) \obslash B) \leq C/(A \otimes (A\backslash(B\backslash C))}\;r \times 3}{((B \oslash A) \obslash B) \otimes (A \otimes (A\backslash(B\backslash C))) \leq (A \oslash B) \oplus C}\;A_I^2}{(B \oslash A) \obslash B \leq ((A \oslash B) \oplus C)/(A \otimes (A\backslash(B\backslash C)))}\;r}{B \otimes (A \otimes (A\backslash(B\backslash C))) \leq (B \oslash A) \oplus ((A \oslash B) \oplus C)}\;A_I^2}{A\backslash(B\backslash C) \leq A\backslash(B\backslash((B \oslash A) \oplus ((A \oslash B) \oplus C)))}\;r \times 2
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \oslash A \leq B \oslash A}\;\mathit{Id}}{(B \oslash A) \obslash B \leq A}\;cr \times 2}{A\backslash C \leq ((B \oslash A) \obslash B)\backslash C}\;\backslash ? \qquad \dfrac{\dfrac{}{A \oslash B \leq A \oslash B}\;\mathit{Id}}{(A \oslash B) \obslash A \leq B}\;cr \times 2}{B\backslash(A\backslash C) \leq ((A \oslash B) \obslash A)\backslash(((B \oslash A) \obslash B)\backslash C)}\;\backslash}{(A \oslash B) \obslash A \leq (((B \oslash A) \obslash B)\backslash C)/(B\backslash(A\backslash C))}\;r \times 2}{A \otimes (B\backslash(A\backslash C)) \leq (A \oslash B) \oplus (((B \oslash A) \obslash B)\backslash C)}\;A_I^2}{(A \oslash B) \obslash (A \otimes (B\backslash(A\backslash C))) \leq ((B \oslash A) \obslash B)\backslash C}\;cr}{((B \oslash A) \obslash B) \otimes (A \otimes (B\backslash(A\backslash C))) \leq (A \oslash B) \oplus C}\;C_I^1}{(B \oslash A) \obslash B \leq ((A \oslash B) \oplus C)/(A \otimes (B\backslash(A\backslash C)))}\;r}{B \otimes (A \otimes (B\backslash(A\backslash C))) \leq (B \oslash A) \oplus ((A \oslash B) \oplus C)}\;A_I^2}{B\backslash(A\backslash C) \leq A\backslash(B\backslash((B \oslash A) \oplus ((A \oslash B) \oplus C)))}\;r \times 2
$$

Figure 6.3: Joins witnessing $\mathbf{LG}_I \vdash A\backslash B \sim B/A$ and $\mathbf{LG}_I \vdash A\backslash(B\backslash C) \sim B\backslash(A\backslash C)$.

$$\begin{array}{cl}
\hline
B \obslash (A \otimes A) \leq B \obslash (A \otimes A) & Id \\ \hline
A \otimes A \leq B \oplus (B \obslash (A \otimes A)) & cr \\ \hline
A \oslash (B \obslash (A \otimes A)) \leq A\backslash B & A^2_{IV}
\end{array}
\qquad
\begin{array}{cl}
\hline
B \obslash (A \otimes A) \leq B \obslash (A \otimes A) & Id \\ \hline
A \otimes A \leq B \oplus (B \obslash (A \otimes A)) & cr \\ \hline
A \oslash (B \obslash (A \otimes A)) \leq B/A & C^2_{IV}
\end{array}$$

$$\begin{array}{cl}
\hline
C \oslash (A\backslash C) \leq C \oslash (A\backslash C) & Id \\ \hline
A \otimes ((C \oslash (A\backslash C)) \obslash C) \leq C & cr \times 2, r \\ \hline
(A \otimes ((C \oslash (A\backslash C)) \obslash C)) \oslash (B\backslash C) \leq C \oslash (B\backslash C) & \oslash ? \\ \hline
C \leq (C \oslash (A\backslash C)) \oplus (A\backslash((C \oslash (B\backslash C)) \oplus (B\backslash C))) & r, cr, r \\ \hline
q\backslash C \leq q\backslash((C \oslash (A\backslash C)) \oplus (A\backslash((C \oslash (B\backslash C)) \oplus (B\backslash C)))) & ?\backslash \\ \hline
q \otimes (q\backslash C) \leq (C \oslash (A\backslash C)) \oplus (A\backslash((C \oslash (B\backslash C)) \oplus (B\backslash C))) & r \\ \hline
(C \oslash (A\backslash C)) \obslash q \leq (A\backslash((C \oslash (B\backslash C)) \oplus (B\backslash C)))/(q\backslash C) & A^1_{IV} \\ \hline
A \otimes (((C \oslash (A\backslash C)) \obslash q) \otimes (q\backslash C)) \leq (C \oslash (B\backslash C)) \oplus (B\backslash C) & r \times 2 \\ \hline
(C \oslash (B\backslash C)) \obslash (((C \oslash (A\backslash C)) \obslash q) \otimes (q\backslash C)) \leq A\backslash(B\backslash C) & C^1_{IV} \\ \hline
((C \oslash (A\backslash C)) \obslash q) \otimes (q\backslash C) \leq (C \oslash (B\backslash C)) \oplus (A\backslash(B\backslash C)) & cr \\ \hline
((C \oslash (B\backslash C)) \obslash (q\backslash C)) \leq ((C \oslash (A\backslash C)) \obslash q)\backslash(A\backslash(B\backslash C)) & C^1_{IV} \\ \hline
((C \oslash (A\backslash C)) \obslash q) \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C)) \leq A\backslash(B\backslash C) & r
\end{array}$$

$$\begin{array}{cl}
\hline
C \oslash (B\backslash C) \leq C \oslash (B\backslash C) & Id \\ \hline
B \otimes ((C \oslash (B\backslash C)) \obslash C) \leq C & cr \times 2, r \\ \hline
(B \otimes ((C \oslash (B\backslash C)) \obslash C)) \oslash (A\backslash C) \leq C \oslash (A\backslash C) & \oslash ? \\ \hline
C \leq (C \oslash (B\backslash C)) \oplus (B\backslash((C \oslash (A\backslash C)) \oplus (A\backslash C))) & r, cr, r \\ \hline
q\backslash C \leq q\backslash((C \oslash (B\backslash C)) \oplus (B\backslash((C \oslash (A\backslash C)) \oplus (A\backslash C)))) & ?\backslash \\ \hline
q \otimes (q\backslash C) \leq (C \oslash (B\backslash C)) \oplus (B\backslash((C \oslash (A\backslash C)) \oplus (A\backslash C))) & r \\ \hline
(C \oslash (B\backslash C)) \obslash (q\backslash C) \leq q\backslash(B\backslash((C \oslash (A\backslash C)) \oplus (A\backslash C))) & A^1_{IV} \\ \hline
B \otimes (q \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C))) \leq (C \oslash (A\backslash C)) \oplus (A\backslash C) & r \times 2 \\ \hline
(C \oslash (A\backslash C)) \obslash (q \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C))) \leq B\backslash(A\backslash C) & C^1_{IV} \\ \hline
q \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C)) \leq (C \oslash (A\backslash C)) \oplus (B\backslash(A\backslash C)) & cr \\ \hline
((C \oslash (A\backslash C)) \obslash q) \leq (A\backslash(B\backslash C))/((C \oslash (B\backslash C)) \obslash (q\backslash C)) & A^1_{IV} \\ \hline
((C \oslash (A\backslash C)) \obslash q) \otimes ((C \oslash (B\backslash C)) \obslash (q\backslash C)) \leq B\backslash(A\backslash C) & r
\end{array}$$

Figure 6.4: Meets for $\mathbf{LG}_{IV} \vdash A\backslash B \sim B/A$ and $\mathbf{LG}_{IV} \vdash A\backslash(B\backslash C) \sim B\backslash(A\backslash C)$.

$$
\begin{array}{cl}
\hline A \oslash C \leq A \oslash C & \mathit{Id} \\
\hline (A \oslash C) \obslash A \leq C & \mathit{cr} \\
\hline C \backslash B \leq ((A \oslash C) \obslash A) \backslash B & \backslash ? \\
\hline (A \oslash C) \obslash A \leq B / (C \backslash B) & r \times 2 \\
\hline A \otimes (C \backslash B) \leq (A \oslash C) \oplus B & A^2_I \\
\hline (A \oslash C) \obslash (A \otimes (C \backslash B)) \leq B & \mathit{cr} \\
\hline C \otimes ((A \oslash C) \obslash (A \otimes (C \backslash B))) \leq C \otimes B & ?\otimes \\
\hline (A \oslash C) \obslash (A \otimes (C \backslash B)) \leq C \backslash (C \otimes B) & r \\
\hline C \otimes (A \otimes (C \backslash B)) \leq (A \oslash C) \oplus (C \otimes B) & C^1_I \\
\hline A \otimes (C \backslash B) \leq C \backslash ((A \oslash C) \oplus (C \otimes B)) & r
\end{array}
$$

$$
\begin{array}{cl}
\hline A \oslash C \leq A \oslash C & \mathit{Id} \\
\hline (A \oslash C) \obslash A \leq C & \mathit{cr} \\
\hline ((A \oslash C) \obslash A) \otimes B \leq C \otimes B & \otimes ? \\
\hline (A \oslash C) \obslash A \leq (C \otimes B) / B & r \\
\hline A \otimes B \leq (A \oslash C) \oplus (C \otimes B) & A^2_I \\
\hline C \backslash (A \otimes B) \leq C \backslash ((A \oslash C) \oplus (C \otimes B)) & ?\backslash
\end{array}
$$

$$
\begin{array}{cl}
\hline ((A \otimes B)/C) \obslash (A \otimes B) \leq ((A \otimes B)/C) \obslash (A \otimes B) & \mathit{Id} \\
\hline A \otimes B \leq ((A \otimes B)/C) \oplus (((A \otimes B)/C) \obslash (A \otimes B)) & \mathit{cr} \\
\hline B \oslash (((A \otimes B)/C) \obslash (A \otimes B)) \leq A \backslash ((A \otimes B)/C) & A^1_{IV} \\
\hline (A \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B)))) \otimes C \leq A \otimes B & r \times 2 \\
\hline B \obslash ((A \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B)))) \otimes C) \leq B \obslash (A \otimes B) & ?\obslash \\
\hline (A \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B)))) \otimes C \leq B \oplus (B \obslash (A \otimes B)) & \mathit{cr} \\
\hline (A \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B)))) \oslash (B \obslash (A \otimes B)) \leq B/C & C^2_{IV} \\
\hline A \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B))) \leq (B/C) \oplus (B \obslash (A \otimes B)) & \mathit{cr} \\
\hline A \oslash (B \obslash (A \otimes B)) \leq (B/C)/(B \oslash (((A \otimes B)/C) \obslash (A \otimes B))) & C^2_{IV} \\
\hline (A \oslash (B \obslash (A \otimes B))) \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B))) \leq B/C & r \\
\hline A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B)))) \leq A \otimes (B/C) & ?\otimes
\end{array}
$$

$$
\begin{array}{cl}
\hline B \obslash (A \otimes B) \leq B \obslash (A \otimes B) & \mathit{Id} \\
\hline A \otimes B \leq B \oplus (B \obslash (A \otimes B)) & \mathit{cr} \\
\hline A \oslash (B \obslash (A \otimes B)) \leq B/B & C^2_{IV} \\
\hline (A \oslash (B \obslash (A \otimes B))) \otimes B \leq B & r \\
\hline A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes B) \leq A \otimes B & ?\otimes \\
\hline ((A \otimes B)/C) \obslash (A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes B)) \leq ((A \otimes B)/C) \obslash (A \otimes B) & ?\obslash \\
\hline A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes B) \leq ((A \otimes B)/C) \oplus (((A \otimes B)/C) \obslash (A \otimes B)) & \mathit{cr} \\
\hline ((A \oslash (B \obslash (A \otimes B))) \otimes B) \oslash (((A \otimes B)/C) \obslash (A \otimes B)) \leq A \backslash ((A \otimes B)/C) & A^2_{IV} \\
\hline (A \oslash (B \obslash (A \otimes B))) \otimes B \leq (A \backslash ((A \otimes B)/C)) \oplus (((A \otimes B)/C) \obslash (A \otimes B)) & \mathit{cr} \\
\hline B \oslash (((A \otimes B)/C) \obslash (A \otimes B)) \leq (A \oslash (B \obslash (A \otimes B))) \backslash (A \backslash ((A \otimes B)/C)) & A^2_{IV} \\
\hline A \otimes ((A \oslash (B \obslash (A \otimes B))) \otimes (B \oslash (((A \otimes B)/C) \obslash (A \otimes B)))) \leq (A \otimes B)/C & r \times 2
\end{array}
$$

Figure 6.5: Derivations accompanying L.51.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \oslash B \leq A \oslash B}\;\mathit{Id}}{(A \oslash B) \obslash A \leq B}\;cr \times 2}{((A \oslash B) \obslash A) \otimes B \leq B \otimes B}\;\otimes ?}{(A \oslash B) \obslash A \leq (B \otimes B)/B}\;r}{A \otimes B \leq (A \oslash B) \oplus (B \otimes B)}\;A^2_I$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \oslash B \leq A \oslash B}\;\mathit{Id}}{(A \oslash B) \obslash A \leq B}\;cr \times 2}{B \otimes ((A \oslash B) \obslash A) \leq B \otimes B}\;? \otimes}{(A \oslash B) \obslash A \leq B\backslash(B \otimes B)}\;r}{B \otimes A \leq (A \oslash B) \oplus (B \otimes B)}\;C^1_I$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \otimes B \leq A \otimes B}\;\mathit{Id}}{B \leq A\backslash(A \otimes B)}\;r \qquad \dfrac{\dfrac{}{A/B \leq A/B}\;\mathit{Id}}{B \leq (A/B)\backslash A}\;r \times 2}{((A/B)\backslash A) \obslash B \leq B \obslash (A\backslash(A \otimes B))}\;\backslash}{(A/B) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B)))) \leq A}\;cr \times 2, r}{((A/B) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A \leq A \otimes A}\;\otimes ?}{(A \otimes B) \obslash (((A/B) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A) \leq (A \otimes B) \obslash (A \otimes A)}\;? \obslash}{((A/B) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A \leq (A \otimes B) \oplus ((A \otimes B) \obslash (A \otimes A))}\;cr}{((A/B) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \oslash ((A \otimes B) \obslash (A \otimes A)) \leq (A \otimes B)/A}\;C^2_{IV}}{(A/B) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B)))) \leq ((A \otimes B)/A) \oplus ((A \otimes B) \obslash (A \otimes A))}\;cr}{(A/B) \oslash ((A \otimes B) \obslash (A \otimes A)) \leq ((A \otimes B)/A)/(B \oslash (B \obslash (A\backslash(A \otimes B))))}\;C^2_{IV}}{(((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A \leq A \otimes B}\;r \times 2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A/B \leq A/B}\;\mathit{Id}}{(A/B) \otimes B \leq A}\;r}{A \otimes ((A/B) \otimes B) \leq A \otimes A}\;? \otimes}{(A \otimes B) \obslash (A \otimes ((A/B) \otimes B)) \leq (A \otimes B) \obslash (A \otimes A)}\;? \obslash}{A \otimes ((A/B) \otimes B) \leq (A \otimes B) \oplus ((A \otimes B) \obslash (A \otimes A))}\;cr}{((A/B) \otimes B) \oslash ((A \otimes B) \obslash (A \otimes A)) \leq A\backslash(A \otimes B)}\;A^2_{IV}}{(A/B) \otimes B \leq (A\backslash(A \otimes B)) \oplus ((A \otimes B) \obslash (A \otimes A))}\;cr}{(A/B) \oslash ((A \otimes B) \obslash (A \otimes A)) \leq (A\backslash(A \otimes B))/B}\;C^2_{IV}}{((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes B \leq A\backslash(A \otimes B)}\;r}{B \obslash (((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes B) \leq B \obslash (A\backslash(A \otimes B))}\;? \obslash}{((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes B \leq B \oplus (B \obslash (A\backslash(A \otimes B)))}\;cr}{B \oslash (B \obslash (A\backslash(A \otimes B))) \leq ((A/B) \oslash ((A \otimes B) \obslash (A \otimes A)))\backslash B}\;A^2_{IV}}{((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B)))) \leq B}\;r}{(((A/B) \oslash ((A \otimes B) \obslash (A \otimes A))) \otimes (B \oslash (B \obslash (A\backslash(A \otimes B))))) \otimes A \leq B \otimes A}\;\otimes ?$$

Figure 6.6: Witnessing $\vdash A \otimes B \sim B \otimes A$ for $\mathbf{LG}_I$ and $\mathbf{LG}_{IV}$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \otimes A \leq A \otimes A$}
\RightLabel{r}
\UnaryInfC{$A \leq A\backslash(A \otimes A)$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$(A \otimes B) \otimes C \leq (A \otimes B) \otimes C$}
\RightLabel{$r \times 2$}
\UnaryInfC{$B \leq A\backslash(((A \otimes B) \otimes C)/C)$}
\RightLabel{$\oplus$}
\BinaryInfC{$A \oplus B \leq (A\backslash(A \otimes A)) \oplus (A\backslash(((A \otimes B) \otimes C)/C))$}
\RightLabel{cr}
\UnaryInfC{$(A\backslash(A \otimes A)) \obslash (A \oplus B) \leq A\backslash(((A \otimes B) \otimes C)/C)$}
\RightLabel{C_I^1}
\UnaryInfC{$A \otimes (A \oplus B) \leq (A\backslash(A \otimes A)) \oplus (((A \otimes B) \otimes C)/C)$}
\RightLabel{cr}
\UnaryInfC{$(A\backslash(A \otimes A)) \obslash (A \otimes (A \oplus B)) \leq ((A \otimes B) \otimes C)/C$}
\RightLabel{A_I^2}
\UnaryInfC{$(A \otimes (A \oplus B)) \otimes C \leq (A\backslash(A \otimes A)) \oplus ((A \otimes B) \otimes C)$}
\RightLabel{cr}
\UnaryInfC{$((A \otimes (A \oplus B)) \otimes C) \oslash ((A \otimes B) \otimes C) \leq A\backslash(A \otimes A)$}
\RightLabel{A_I^1}
\UnaryInfC{$A \otimes ((A \otimes (A \oplus B)) \otimes C) \leq (A \otimes A) \oplus ((A \otimes B) \otimes C)$}
\RightLabel{cr}
\UnaryInfC{$(A \otimes A) \obslash (A \otimes ((A \otimes (A \oplus B)) \otimes C)) \leq (A \otimes B) \otimes C$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \otimes A \leq A \otimes A$}
\RightLabel{r}
\UnaryInfC{$A \leq A\backslash(A \otimes A)$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \otimes (B \otimes C) \leq A \otimes (B \otimes C)$}
\RightLabel{$r \times 2$}
\UnaryInfC{$B \leq ((A\backslash(A \otimes (B \otimes C)))/C)$}
\RightLabel{$\oplus$}
\BinaryInfC{$A \oplus B \leq (A\backslash(A \otimes A)) \oplus ((A\backslash(A \otimes (B \otimes C)))/C)$}
\RightLabel{cr}
\UnaryInfC{$(A \oplus B) \oslash ((A\backslash(A \otimes (B \otimes C)))/C) \leq A\backslash(A \otimes A)$}
\RightLabel{A_I^1}
\UnaryInfC{$A \otimes (A \oplus B) \leq (A \otimes A) \oplus ((A\backslash(A \otimes (B \otimes C)))/C)$}
\RightLabel{cr}
\UnaryInfC{$(A \otimes A) \obslash (A \otimes (A \oplus B)) \leq (A\backslash(A \otimes (B \otimes C)))/C$}
\RightLabel{A_I^2}
\UnaryInfC{$(A \otimes (A \oplus B)) \otimes C \leq (A \otimes A) \oplus (A\backslash(A \otimes (B \otimes C)))$}
\RightLabel{cr}
\UnaryInfC{$(A \otimes A) \obslash ((A \otimes (A \oplus B)) \otimes C) \leq A\backslash(A \otimes (B \otimes C))$}
\RightLabel{C_I^1}
\UnaryInfC{$A \otimes ((A \otimes (A \oplus B)) \otimes C) \leq (A \otimes A) \oplus (A \otimes (B \otimes C))$}
\RightLabel{cr}
\UnaryInfC{$(A \otimes A) \obslash (A \otimes ((A \otimes (A \oplus B)) \otimes C)) \leq A \otimes (B \otimes C)$}
\end{prooftree}
$$

Figure 6.7: Witnessing $\mathbf{LG}_I \vdash (A \otimes B) \otimes C \sim A \otimes (B \otimes C)$

Definition 92. A *model* $\mathscr{M}$ for ~ is a pair $\langle \mathscr{A}, v \rangle$ of a linearly distributive algebra $\mathscr{A}$ and a *valuation* v mapping atoms into $\mathscr{A}$'s, being inductively extended to an *interpretation* $[\![\cdot]\!]$ operating on arbitrary formulas:

$$
\begin{array}{rcl@{\qquad}rcl}
[\![p]\!] & := & v(p) & & & \\
[\![1]\!] & := & \top & [\![0]\!] & := & \bot \\
[\![A \otimes B]\!] & := & [\![A]\!] \times [\![B]\!] & [\![A \oplus B]\!] & := & [\![A]\!] + [\![B]\!] \\
[\![A/B]\!] & := & [\![A]\!] + [\![B]\!]^{\bot} & [\![B \obslash A]\!] & := & [\![B]\!]^{\bot} \times [\![A]\!] \\
[\![B\backslash A]\!] & := & [\![B]\!]^{\bot} + [\![A]\!] & [\![A \oslash B]\!] & := & [\![A]\!] \times [\![B]\!]^{\bot}
\end{array}
$$

Remark 11. Note that, for arbitrary A, $[\![{}^1A]\!] = [\![A^1]\!] = [\![{}^0A]\!] = [\![A^0]\!] = [\![A]\!]^{\bot}$. E.g., $[\![A^1]\!] = [\![A \obslash 1]\!] = [\![A]\!]^{\bot} \times \top = [\![A]\!]^{\bot}$, and $[\![{}^0A]\!] = [\![0/A]\!] = \bot + [\![A]\!]^{\bot} = [\![A]\!]^{\bot}$.

Model-theoretic investigations into type similarity within the Lambek-Grishin 'hierarchy' were already conducted by M&P for the concrete case of $\mathbf{LG}_{IV}$. Their interpretation takes as target the *free* Abelian group generated by the atomic formulae and an additional element $\oplus$,

$$
\begin{array}{rcl@{\qquad}rcl}
[\![p]\!]' & := & p & & & \\
[\![A \otimes B]\!]' & := & [\![A]\!]' \cdot [\![B]\!]' & [\![A \oplus B]\!]' & := & [\![A]\!]' \cdot \oplus^{-1} \cdot [\![B]\!]' \\
[\![A/B]\!]' & := & [\![A]\!]' \cdot [\![B]\!]'^{-1} & [\![B \oslash A]\!]' & := & [\![B]\!]'^{-1} \cdot \oplus \cdot [\![A]\!]' \\
[\![B\backslash A]\!]' & := & [\![B]\!]'^{-1} \cdot [\![A]\!]' & [\![A \oslash B]\!]' & := & [\![A]\!]' \cdot \oplus \cdot [\![B]\!]'^{-1}
\end{array}
$$

writing 1 for unit and $^{-1}$ for inverse. While not reconcilable with Def.92 in that it cannot be presented as a concrete instance of a linearly distributive algebra, the decidability of the word problem in free Abelian groups implies the decidability of type similarity as a corollary of completeness. The current investigation rather aims at a more 'natural' concept of model, better reflecting the coexistence of residuated and coresiduated families of connectives in the source language. While we can still prove type similarity complete w.r.t. the freely generated such model, as shown in Lem.58, the inference of its decidability requires some additional steps. Specifically, we use M&P's models as inspiration to define, for each formula, a 'normal form', possibly involving units, w.r.t. which it is found type similar. We then decide type similarity at the level of such normal forms by providing an algorithm for generating joins, settling the word problem in the freely generated linear distributive algebra as a corollary, ensuring, in turn, the desired result.

Lemma 54. If $A \leq B$, then $[\![A]\!] = [\![B]\!]$ in every model.

Proof. By induction on the derivation witnessing $A \leq B$. □

Theorem 6.3.1. If $\vdash A \sim B$, then $[\![A]\!] = [\![B]\!]$ in every model.

Proof. Since, by L.54, $A \leq A \sqcup_C B$ and $B \leq A \sqcup_C B$ imply $[\![A]\!] = [\![C]\!]$ and $[\![B]\!] = [\![C]\!]$, we get $[\![A]\!] = [\![B]\!]$ by transitivity. □

To prove completeness, we define a syntactic model wherein the interpretations of formulae are (constructively) shown to coincide with their equivalence classes under $\sim$. In defining said model, we shall use the following lemmas.

Lemma 55. We have $\vdash (A\backslash A) \oslash A \sim (A \oslash A)/A$.

Proof. By L.49, we have meets D_1, D_2 for $\vdash (A\backslash A) \oslash A \sim A \oslash (A\backslash A)$ resp. $\vdash (A \oslash A)/A \sim A\backslash(A \oslash A)$. Hence, we have a join C for $\vdash A \oslash (A\backslash A) \sim A\backslash(A \oslash A)$, with another use of the diamond property sufficing for finding the desired meet D_3:

$$\begin{tikzcd}[column sep=small]
 & & & D \arrow[dll, dashed] \arrow[drr, dashed] & & & \\
 & D_1 \arrow[dl, dashed] \arrow[dr, dashed] & & & & D_2 \arrow[dl, dashed] \arrow[dr, dashed] & \\
(A\backslash A) \oslash A & & A \obslash (A\backslash A) \arrow[dr, dashed] & & A\backslash(A \obslash A) \arrow[dl, dashed] & & A\backslash(A \oslash A) \\
 & & & C & & &
\end{tikzcd}$$ ☐

Lemma 56. $\vdash 1 \oslash A \sim A \obslash 1 \sim 0/A \sim A\backslash 0$ in $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$.

Proof. That $\vdash 1 \oslash A \sim A \obslash 1$ and $\vdash 0/A \sim A\backslash 0$ are immediate consequences of L.49. Furthermore, $\mathbf{LG}_{IV}^{0,1} \vdash 1 \oslash A \to A\backslash 0$, as shown on the left, while for $\mathbf{LG}_I$ we apply the diamond property, as shown on the right,

$$\dfrac{\dfrac{\dfrac{\overline{A \le A}\;Id}{A \le 0 \oplus A}\;0}{A \otimes 1 \le 0 \oplus A}\;1}{1 \oslash A \le A\backslash 0}\;A^2_{IV}
\qquad
\begin{tikzcd}[column sep=small]
 & & D \arrow[dll, dashed] \arrow[dr, dashed] & \\
A\backslash((A \otimes 1) \oslash A) \arrow[d] \arrow[dr] & & & 1 \oslash A \arrow[dll] \\
A\backslash 0 & A\backslash(A \otimes (1 \oslash A)) & &
\end{tikzcd}$$ ☐

Definition 93. We construct a *syntactic model* by building a linearly distributive algebra upon the set of equivalence classes $[A]_\sim := \{B \mid\ \vdash A \sim B\}$ of formulae w.r.t. $\sim$. The various operations of the algebra are defined as in the following table, providing separate (simpler) solutions for $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$:

$$\begin{array}{rcl}
\top & := & [A\backslash A]_\sim \\
\bot & := & [A \obslash A]_\sim \\
[A]_\sim^\bot & := & [(A\backslash A) \oslash A]_\sim \\
 & = & [(A \obslash A)/A]_\sim \\
[A]_\sim \times [B]_\sim & := & [A \otimes B]_\sim \\
[A]_\sim + [B]_\sim & := & [A \oplus B]_\sim
\end{array}$$

For $\mathbf{LG}_{I/IV}^{0,1}$, the following simpler definitions suffice for the first three operations:

$$\begin{array}{rcl}
\top & := & [1]_\sim \\
\bot & := & [0]_\sim \\
[A]_\sim^\bot & := & [1 \oslash A]_\sim = [A \obslash 1]_\sim = [0/A]_\sim = [A\backslash 0]_\sim
\end{array}$$

Finally, we define the valuation by $v(p) := [p]_\sim$ for arbitrary atom p.

Lemma 57. The syntactic model is well-defined.

Proof. We check the equations of Def.91. Note that, by definition unfolding, (1) reduces to showing $\vdash (A \otimes B) \otimes C \sim A \otimes (B \otimes C)$ and $\vdash (A \oplus B) \oplus C \sim A \oplus (B \oplus C)$. The former was shown in Lem.53, while for the latter we note that we obtain a join (meet) by taking the dual of a meet (join) for $\vdash C^{\infty} \otimes (B^{\infty} \otimes A^{\infty}) \sim (C^{\infty} \otimes B^{\infty}) \otimes A^{\infty}$ under $^{\infty}$. Similarly, (2) and (4) are immediate consequences of Lem.52 and Lem.55 respectively (Lem.56 in the presence of units), while case (3) is equally straightforward. This leaves us to check (5), requiring us to show $\vdash A \otimes (B \oplus C) \sim (A \otimes B) \oplus C$. We actually have $\mathbf{LG}_I \vdash A \otimes (B \oplus C) \leq (A \otimes B) \oplus C$, while for $\mathbf{LG}_{IV}$ we use the diamond property:

$$\begin{tikzcd}
& D \arrow[dl, dashed] \arrow[dr, dashed] & & \\
A \otimes (B \oplus C) \arrow[dr] & & A \otimes (A \backslash ((A \otimes B) \oplus C)) \arrow[dl] \arrow[dr] & \\
& A \otimes ((A \backslash (A \otimes B)) \oplus C) & & (A \otimes B) \oplus C
\end{tikzcd}$$ □

While we could proceed to prove $[\![A]\!] = [A]_{\sim}$ in the syntactic model for all A, we prove a slightly more involved statement, the increase in complexity paying off when proving decidability of type similarity in T.6.4.2. Write $\mathscr{A}(Atom)$ for the linear distributive algebra freely generated by the atoms. We then have

Lemma 58. If $[\![A]\!] = [\![B]\!]$ in $\mathscr{A}(Atom)$, then also $\vdash A \sim B$.

Proof. We proceed according to the strategy pioneered by Pentus [1993]. Consider the homomorphic extension h of the map taking atoms p into the equivalence classes $[p]_{\sim}$ of the syntactic model (cf. D.93). We prove, for arbitrary A, that the image under h of the interpretation $[\![A]\!]$ of A in $\mathscr{A}(Atom)$ coincides with $[A]_{\sim}$. As a consequence, if $[\![A]\!] = [\![B]\!]$ in $\mathscr{A}(Atom)$, then also $h([\![A]\!]) = h([\![B]\!])$, hence $[A]_{\sim} = [B]_{\sim}$, and so $\vdash A \sim B$. Proceeding by induction, the cases $A = p$ and (in the presence of units) $A = 1$, $A = 0$ follow by definition, while a straightforward definitional unfolding suffices if $A = A_1 \otimes A_2$ or $A = A_1 \oplus A_2$:

$$\begin{aligned}
h([\![A \otimes B]\!]) &= h([\![A]\!]) \times h([\![B]\!]) = [A]_{\sim} \times {\sim}[B]_{\sim} = [A \otimes B]_{\sim} \\
h([\![A \oplus B]\!]) &= h([\![A]\!]) + h([\![B]\!]) = [A]_{\sim} + {\sim}[B]_{\sim} = [A \oplus B]_{\sim}
\end{aligned}$$

The cases $A = A_1/A_2$, $A = A_2 \backslash A_1$, $A = A_1 \oslash A_2$ and $A = A_2 \obslash A_1$ are all alike, differing primarily in the number of applications of L.49. We shall demonstrate with $A = A_1/A_2$. In $\mathbf{LG}_I$ and $\mathbf{LG}_{IV}$, (i.e., when lacking units), we have

$$h([\![A_1/A_2]\!]) = h([\![A_1]\!]) + h([\![A_2]\!])^{\perp} = [A_1]_{\sim} + [A_2]_{\sim}^{\perp} = [A_1 \oplus ((A_2 \obslash A_2)/A_2)]_{\sim}$$

Thus, we have to show $\vdash A_1 \oplus ((A_2 \obslash A_2)/A_2) \sim A_1/A_2$. Note that in the presence of units, we may suffice by proving the simpler $\vdash A_1 \oplus (0/A_2) \sim A_1/A_2$.

1.	$\vdash A_2 \obslash A_2 \sim A_1 \oslash A_1$	(L.48)	
2.	$\vdash A_1 \oslash A_1 \sim A_1 \obslash A_1$	(L.49)	
3.	$\vdash A_2 \obslash A_2 \sim A_1 \obslash A_1$	(Transitivity, 1, 2)	
4.	$\vdash A_1 \oplus ((A_2 \obslash A_2)/A_2) \sim A_1 \oplus ((A_1 \obslash A_1)/A_2)$	(Congruence, 3)	$\square$
5.	$\vdash A_1 \oplus ((A_1 \obslash A_1)/A_2) \sim (A_1 \oplus (A_1 \obslash A_1))/A_2$		
6.	$\vdash (A_1 \oplus (A_1 \obslash A_1))/A_2 \geq A_1/A_2$		
7.	$\vdash A_1 \oplus ((A_2 \obslash A_2)/A_2) \sim A_1/A_2$	(Transitivity, 4, 5, 6)	

Theorem 6.3.2. If $[\![A]\!] = [\![B]\!]$ in every model, then $\vdash A \sim B$.

Proof. In particular, $[\![A]\!] = [\![B]\!]$ in $\mathscr{A}(Atom)$, and hence $\vdash A \sim B$ by L.58. $\square$

6.4 Generating joins

We present an algorithm for generating joins for $\mathbf{LG}_I^{0,1}$, deriving decidability for the remaining incarnations of **LG** as a corollary. First, we shall define for each formula A a 'normal form' $\|A\|^\circ$ w.r.t. which it is shown type similar by some join C_A. Whether or not some given A and B are type similar is then decided at the level of $\|A\|^\circ$ and $\|B\|^\circ$, an affirmative answer, witnessed by some meet D, implying the existence of a join C for A and B by the diamond property. Summarized:

$$\begin{tikzcd}
& & & D \arrow[dl] \arrow[dr] & & & \\
A \arrow[dr] & & \|A\|^\circ \arrow[dl] & & \|B\|^\circ \arrow[dr] & & B \arrow[dl] \\
& C_A \arrow[drr, dashed] & & & & C_B \arrow[dll, dashed] & \\
& & & C & & &
\end{tikzcd}$$

Definition 94. We define the maps $\|\cdot\|^\circ$ and $\|\cdot\|^\bullet$ by mutual induction:

$$\begin{array}{rcl@{\qquad}rcl}
\|p\|^\circ & := & p & \|p\|^\bullet & := & 1/p \\
\|1\|^\circ & := & 1 & \|1\|^\bullet & := & 1 \\
\|0\|^\circ & := & 0 & \|0\|^\bullet & := & 1/0 \\
\|A \otimes B\|^\circ & := & \|A\|^\circ \otimes \|B\|^\circ & \|A \otimes B\|^\bullet & := & \|A\|^\bullet \otimes \|B\|^\bullet \\
\|A/B\|^\circ & := & \|A\|^\circ \otimes \|B\|^\bullet & \|A/B\|^\bullet & := & \|A\|^\bullet \otimes \|B\|^\circ \\
\|B\backslash A\|^\circ & := & \|A/B\|^\circ & \|B\backslash A\|^\bullet & := & \|A/B\|^\bullet \\
\|A \oplus B\|^\circ & := & \|A\|^\circ \otimes ((1/0) \otimes \|B\|^\circ) & \|A \oplus B\|^\bullet & := & \|A\|^\bullet \otimes (\|B\|^\bullet \otimes 0) \\
\|A \oslash B\|^\circ & := & \|A\|^\circ \otimes (0 \otimes \|B\|^\bullet) & \|A \oslash B\|^\bullet & := & \|A\|^\bullet \otimes ((1/0) \otimes \|B\|^\circ) \\
\|B \obslash A\|^\circ & := & \|A \oslash B\|^\circ & \|B \obslash A\|^\bullet & := & \|A \oslash B\|^\bullet
\end{array}$$

The above definition should be compared to the Abelian group interpretation of M&P: multiplications $A \cdot B$ and inverses A^{-1} are rendered as $A \otimes B$ and $1/A$, while 0 replaces the special atom $\oplus$. Rather than having to stipulate the defining equations for Abelian groups, we now rely on their justification at the level of type similarity. In particular, we need only solve the problem of generating joins for the formulas in the images of $\|\cdot\|^\circ$ and $\|\cdot\|^\bullet$, relying on the result, proved presently, that $\vdash A \sim \|A\|^\circ$ and $\vdash 1/A \sim \|A\|^\bullet$.

Lemma 59. There exist maps $f(\cdot)$ and $g(\cdot)$ mapping any given A to joins witnessing $\vdash A \sim \|A\|^\circ$ and $\vdash 1/A \sim \|A\|^\bullet$ respectively.

Proof. The (mutual) inductive definitions of f and g are presented together with the proofs of their correctness. In the base cases, $A = p$, $A = 1$ or $A = 0$ and we set

$$\begin{array}{lllllllll} f(p) & := & p & f(1) & := & 1 & f(0) & := & 0 \\ g(p) & := & 1/p & g(1) & := & 1 & g(0) & := & 1/0 \end{array}$$

Correctness is nigh immediate, noting $1/1 \leq 1$ for $g(1)$. Most of the inductive cases resort to the diamond property. To illustrate, consider the cases $A = A_1 \oslash A_2$ and $A = A_2 \obslash A_1$, handled similarly. Starting with the map $f(\cdot)$, we have, by induction hypothesis, formulas $f(A_1)$ and $g(A_2)$ acting as joins witnessing $\vdash A_1 \sim \|A_1\|^\circ$ and $\vdash (1/A_2) \sim \|A_2\|^\bullet$ respectively. Hence, by L.47(4), we have a join $f(A_1) \otimes (0 \otimes g(A_2))$ for $\vdash (A_1 \otimes (0 \otimes (1/A_2)) \sim (\|A_1\|^\circ \otimes (0 \otimes \|A_2\|^\bullet))$. In addition, we have joins

$$\begin{tikzcd}[column sep=tiny] A_1 \oslash A_2 \arrow[dr] & & A_1 \otimes (0 \otimes (1/A_2)) \arrow[dl] \\ & ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) & \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=tiny] A_2 \obslash A_1 \arrow[dr] & & A_1 \otimes (0 \otimes (1/A_2)) \arrow[dl] \\ & (A_2 \oplus (A_2 \obslash A_1)) \oplus ((A_2 \oplus 1) \obslash 1) & \end{tikzcd}$$

We show $A_1 \otimes (0 \otimes (1/A_2)) \leq ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)$ in F.6.8, which can be a bit tricky. With these findings, we may now define $f(A_1 \oslash A_2)$ through the diamond property:

$$\begin{tikzcd}[column sep=tiny] A_1 \oslash A_2 \arrow[dr] & & A_1 \otimes (0 \otimes (1/A_2)) \arrow[dl] \arrow[dr] & & \|A_1\|^\circ \otimes (0 \otimes \|A_2\|^\bullet) \arrow[dl] \\ & ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1) \arrow[dr, dashed] & & f(A_1) \otimes (0 \otimes g(A_2)) \arrow[dl, dashed] & \\ & & f(A_1 \oslash A_2) & & \end{tikzcd}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A_1 \oslash A_2 \leq A_1 \oslash A_2$}
\RightLabel{cr}
\UnaryInfC{$A_1 \leq (A_1 \oslash A_2) \oplus A_2)$}
\RightLabel{1}
\UnaryInfC{$A_1 \otimes 1 \leq (A_1 \oslash A_2) \oplus A_2)$}
\RightLabel{r}
\UnaryInfC{$1 \leq (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2))$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$0 \leq 0$}
\RightLabel{1}
\UnaryInfC{$0 \otimes 1 \leq 0$}
\RightLabel{r}
\UnaryInfC{$1 \leq 0 \backslash 0$}
\RightLabel{$/A_2$}
\UnaryInfC{$1/A_2 \leq (0 \backslash 0)/A_2$}
\RightLabel{$r \times 2$}
\UnaryInfC{$A_2 \leq (1/A_2) \backslash (0 \backslash 0)$}
\RightLabel{$\oplus$}
\BinaryInfC{$1 \oplus A_2 \leq (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)) \oplus ((1/A_2) \backslash (0 \backslash 0))$}
\RightLabel{cr}
\UnaryInfC{$(A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)) \obslash (1 \oplus A_2) \leq (1/A_2) \backslash (0 \backslash 0)$}
\RightLabel{C_I^1}
\UnaryInfC{$(1/A_2) \otimes (1 \oplus A_2) \leq (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)) \oplus (0 \backslash 0)$}
\RightLabel{cr}
\UnaryInfC{$(A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)) \obslash ((1/A_2) \otimes (1 \oplus A_2)) \leq 0 \backslash 0$}
\RightLabel{C_I^1}
\UnaryInfC{$0 \otimes ((1/A_2) \otimes (1 \oplus A_2)) \leq (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)) \oplus 0$}
\RightLabel{0}
\UnaryInfC{$0 \otimes ((1/A_2) \otimes (1 \oplus A_2)) \leq A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)$}
\RightLabel{$r \times 2$}
\UnaryInfC{$1 \oplus A_2 \leq (1/A_2) \backslash (0 \backslash (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)))$}
\RightLabel{$\obslash ?$}
\UnaryInfC{$((1/A_2) \backslash (0 \backslash (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)))) \obslash 1 \leq (1 \oplus A_2) \obslash 1$}
\RightLabel{$cr \times 2$}
\UnaryInfC{$1 \oslash ((1 \oplus A_2) \obslash 1) \leq (1/A_2) \backslash (0 \backslash (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)))$}
\RightLabel{A_I^1}
\UnaryInfC{$(1/A_2) \otimes 1 \leq (0 \backslash (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2))) \oplus ((1 \oplus A_2) \obslash 1)$}
\RightLabel{1}
\UnaryInfC{$1/A_2 \leq (0 \backslash (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2))) \oplus ((1 \oplus A_2) \obslash 1)$}
\RightLabel{cr}
\UnaryInfC{$(1/A_2) \oslash ((1 \oplus A_2) \obslash 1) \leq 0 \backslash (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2))$}
\RightLabel{A_I^1}
\UnaryInfC{$0 \otimes (1/A_2) \leq (A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)) \oplus ((1 \oplus A_2) \obslash 1)$}
\RightLabel{cr}
\UnaryInfC{$(0 \otimes (1/A_2)) \oslash ((1 \oplus A_2) \obslash 1) \leq A_1 \backslash ((A_1 \oslash A_2) \oplus A_2)$}
\RightLabel{A_I^1}
\UnaryInfC{$A_1 \otimes (0 \otimes (1/A_2)) \leq ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)$}
\end{prooftree}
$$

Figure 6.8: Showing $A_1 \otimes (0 \otimes (1/A_2)) \leq ((A_1 \oslash A_2) \oplus A_2) \oplus ((1 \oplus A_2) \obslash 1)$.

while $f(A_2 \obslash A_1)$ is similarly defined

$$(A_2 \oplus (A_2 \obslash A_1)) \oplus ((A_2 \oplus 1) \obslash 1) \sqcup_{A \otimes (0 \otimes (1/A_2))} (f(A_1) \otimes (0 \otimes g(A_2)))$$

The same strategy is used to define $g(A_1 \oslash A_2)$ and $g(A_2 \obslash A_1)$, employing joins $(1 \oslash A_1) \oplus (1 \oplus (0 \obslash A_2))$ and $(1 \oslash A_1) \oplus ((A_2 \oslash 0) \oplus 1)$ witnessing $\vdash 1/(A_1 \oslash A_2) \sim (1/A_1) \otimes ((1/0) \otimes A_2)$ and $\vdash 1/(A_2 \obslash A_1) \sim (1/A_1) \otimes ((1/0) \otimes A_2)$ respectively. In the same vein, we can handle a significant portion of the remaining cases:

$$
\begin{aligned}
g(A_1 \otimes A_2) &:= ((1 \oplus (A_2 \obslash 1)) \oplus (A_1 \obslash 1)) \sqcup_{(1/A_1) \otimes (1/A_2)} (g(A_1) \otimes g(A_2)) \\
g(A_1/A_2) &:= (1 \oplus ((A_1/A_2) \obslash 1)) \sqcup_{(1/A_1) \otimes A_2} (g(A_1) \otimes f(A_2)) \\
g(A_2 \backslash A_1) &:= (1 \oplus ((A_2 \backslash A_1) \obslash 1)) \sqcup_{(1/A_2) \otimes A_2} (g(A_1) \otimes f(A_2)) \\
f(A_1 \oplus A_2) &:= (A_1 \oplus (0 \obslash A_2)) \sqcup_{A_1 \otimes ((1/0) \otimes A_2)} f(A_1) \otimes ((1/0) \otimes f(A_2)) \\
g(A_1 \oplus A_2) &:= (1 \oplus ((A_1 \oplus A_2) \obslash 1)) \sqcup_{(1/A_1) \otimes ((1/A_2) \otimes 0)} (g(A_1) \otimes (g(A_2) \otimes 0))
\end{aligned}
$$

To show $(1/A_1) \otimes ((1/A_2) \otimes 0) \leq 1 \oplus ((A_1 \oplus A_2) \varobslash 1)$ with the definition of $g(A_1 \oplus A_2)$ can be a bit tricky, so we give the derivation in F.6.9. We are left with the following cases, handled without use of the diamond property:

$$\begin{array}{rcl} f(A_1 \otimes A_2) & := & f(A_1) \otimes f(A_2) \\ f(A_1/A_2) & := & ((1 \oplus (1 \varobslash f(A_1))) \oplus (A_2 \varobslash g(A_2))) \oplus ((1/A_2) \varobslash 1) \\ f(A_2 \backslash A_1) & := & ((1 \oplus (1 \varobslash f(A_1))) \oplus (A_2 \varobslash g(A_2))) \oplus ((1/A_2) \varobslash 1) \end{array}$$

We demonstrate well-definedness of $f(A_1/A_2)$ in F.6.10, with $f(A_2 \backslash A_1)$ handled similarly. □

We shall decide type similarity by reference to the following invariants.

Definition 95. For arbitrary p, we define by mutual inductions the functions $|\cdot|_p^+$ and $|\cdot|_p^-$ counting, respectively, the numbers of positive and negative occurrences of p inside their arguments. First, the positive count:

$$\begin{array}{rcl@{\qquad}rcl} |p|_p^+ & := & 1 & |q|_p^+ & := & 0 \text{ if } q \neq p \\ |1|_p^+ & := & 0 & |0|_p^+ & := & 0 \\ |A \otimes B|_p^+ & := & |A|_p^+ + |B|_p^+ & |A \oplus B|_p^+ & := & |A|_p^+ + |B|_p^+ \\ |A/B|_p^+ & := & |A|_p^+ + |B|_p^- & |B \varobslash A|_p^+ & := & |A|_p^+ + |B|_p^- \\ |B\backslash A|_p^+ & := & |A|_p^+ + |B|_p^- & |A \oslash B|_p^+ & := & |A|_p^+ + |B|_p^- \end{array}$$

and similarly, the negative count:

$$\begin{array}{rcl@{\qquad}rcl} |p|_p^- & := & 0 & |q|_p^- & := & 0 \text{ if } q \neq p \\ |1|_p^- & := & 0 & |0|_p^- & := & 0 \\ |A \otimes B|_p^- & := & |A|_p^- + |B|_p^- & |A \oplus B|_p^- & := & |A|_p^- + |B|_p^- \\ |A/B|_p^- & := & |A|_p^- + |B|_p^+ & |B \varobslash A|_p^- & := & |A|_p^- + |B|_p^+ \\ |B\backslash A|_p^- & := & |A|_p^- + |B|_p^+ & |A \oslash B|_p^- & := & |A|_p^- + |B|_p^+ \end{array}$$

The *atomic count* $|A|_p$ for p is defined $|A|_p^+ - |A|_p^-$. In a similar fashion, we define positive and negative counts $|A|_0^+$ and $|A|_0^-$ for occurrences of the unit 0 inside A, and set $|A|_0 := |A|_0^+ - |A|_0^-$.

In practice, however, the previously defined counts shall prove only of interest with arguments of the form $\|A\|^\circ$. In the case of arbitrary formulas, we therefore define (by mutual induction) the positive and negative *operator counts* $|A|_\oplus^+$ and $|A|_\oplus^-$ (resembling, though slightly differing from, a concept of Moortgat and Pentus [2007] bearing the same name), recording the values of $|\|A\|^\circ|_0^+$ and $|\|A\|^\circ|_0^-$ respectively.

$$
\begin{array}{cl}
\hline
0 \leq 0 & Id \\
\hline
1 \otimes 0 \leq 0 & 1 \\
\hline
1 \leq 0/0 & r \\
\hline
1/A_2 \leq (0/0)/A_2 & /A_2 \\
\hline
A_2 \leq (1/A_2)\backslash(0/0) & r \times 2 \\
\hline
A_1 \oplus A_2 \leq A_1 \oplus ((1/A_2)\backslash(0/0)) & ?\oplus \\
\hline
A_1 \obslash (A_1 \oplus A_2) \leq (1/A_2)\backslash(0/0) & cr \\
\hline
(1/A_2) \otimes (A_1 \oplus A_2) \leq A_1 \oplus (0/0) & C_I^1 \\
\hline
A_1 \obslash ((1/A_2) \otimes (A_1 \oplus A_2)) \leq 0/0 & cr \\
\hline
((1/A_2) \otimes (A_1 \oplus A_2)) \otimes 0 \leq A_1 \oplus 0 & A_I^2 \\
\hline
((1/A_2) \otimes (A_1 \oplus A_2)) \otimes 0 \leq A_1 & 0 \\
\hline
A_1 \oplus A_2 \leq (1/A_2)\backslash(A_1/0) & r \times 2 \\
\hline
((1/A_2)\backslash(A_1/0)) \obslash 1 \leq (A_1 \oplus A_2) \obslash 1 & \obslash? \\
\hline
1 \oslash ((A_1 \oplus A_2) \obslash 1) \leq (1/A_2)\backslash(A_1/0) & cr \times 2 \\
\hline
(1/A_2) \otimes 1 \leq (A_1/0) \oplus ((A_1 \oplus A_2) \obslash 1) & A_I^1 \\
\hline
((1/A_2) \otimes 1) \oslash ((A_1 \oplus A_2) \obslash 1) \leq A_1/0 & cr \\
\hline
((1/A_2) \otimes 1) \otimes 0 \leq A_1 \oplus ((A_1 \oplus A_2) \obslash 1) & C_I^2 \\
\hline
(1/A_2) \otimes 1 \leq (A_1 \oplus ((A_1 \oplus A_2) \obslash 1))/0 & r \\
\hline
1/A_2 \leq (A_1 \oplus ((A_1 \oplus A_2) \obslash 1))/0 & 1 \\
\hline
(1/A_2) \otimes 0 \leq A_1 \oplus ((A_1 \oplus A_2) \obslash 1) & r \\
\hline
((1/A_2) \otimes 0) \oslash ((A_1 \oplus A_2) \obslash 1) \leq A_1 & cr \\
\hline
1/A_1 \leq 1/(((1/A_2) \otimes 0) \oslash ((A_1 \oplus A_2) \obslash 1)) & 1/ \\
\hline
((1/A_2) \otimes 0) \oslash ((A_1 \oplus A_2) \obslash 1) \leq (1/A_1)\backslash 1 & r \times 2 \\
\hline
(1/A_1) \otimes ((1/A_2 \otimes 0) \leq 1 \oplus ((A_1 \oplus A_2) \obslash 1) & A_I^1
\end{array}
$$

Figure 6.9: Showing $(1/A_1) \otimes ((1/A_2 \otimes 0) \leq 1 \oplus ((A_1 \oplus A_2) \obslash 1)$ to establish the well-definedness of $g(A_1 \oplus A_2)$.

Definition 96. For arbitrary A, the positive and negative operator counts $|A|^+_\oplus$ and $|A|^-_\oplus$ are defined by induction over A, as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
|p|^+_\oplus & := & 0 & & & \\
|1|^+_\oplus & := & 0 & |0|^+_\oplus & := & 1 \\
|A \otimes B|^+_\oplus & := & |A|^+_\oplus + |B|^+_\oplus & |A \oplus B|^+_\oplus & := & |A|^+_\oplus + |B|^+_\oplus \\
|A/B|^+_\oplus & := & |A|^+_\oplus + |B|^-_\oplus & |B \obslash A|^+_\oplus & := & |A|^+_\oplus + |B|^-_\oplus + 1 \\
|B\backslash A|^+_\oplus & := & |A|^+_\oplus + |B|^-_\oplus & |A \oslash B|^+_\oplus & := & |A|^+_\oplus + |B|^-_\oplus + 1
\end{array}
$$

$$
\begin{prooftree}
\AxiomC{$A_1 \leq f(A_1)$}
\RightLabel{$?\obslash$}
\UnaryInfC{$1 \obslash A_1 \leq 1 \obslash f(A_1)$}
\RightLabel{cr}
\UnaryInfC{$A_1 \leq 1 \oplus (1 \obslash f(A_1))$}
\AxiomC{$1/A_2 \leq g(A_2)$}
\RightLabel{$\obslash ?$}
\UnaryInfC{$g(A_2) \obslash 1 \leq (1/A_2) \obslash 1$}
\RightLabel{$cr \times 2$}
\UnaryInfC{$1 \oslash ((1/A_2) \obslash 1) \leq g(A_2)$}
\RightLabel{$?\obslash$}
\UnaryInfC{$A_2 \obslash (1 \oslash ((1/A_2) \obslash 1)) \leq A_2 \obslash g(A_2)$}
\RightLabel{$cr \times 2$}
\UnaryInfC{$(1 \oslash ((1/A_2) \obslash 1)) \oslash (A_2 \obslash g(A_2)) \leq A_2$}
\RightLabel{$/$}
\BinaryInfC{$A_1/A_2 \leq (1 \oplus (1 \obslash f(A_1)))/((1 \oslash ((1/A_2) \obslash 1)) \oslash (A_2 \obslash g(A_2)))$}
\RightLabel{$r \times 2$}
\UnaryInfC{$(1 \oslash ((1/A_2) \obslash 1)) \oslash (A_2 \obslash g(A_2)) \leq (A_1/A_2) \backslash (1 \oplus (1 \obslash f(A_1)))$}
\RightLabel{A_I^1}
\UnaryInfC{$(A_1/A_2) \otimes (1 \oslash ((1/A_2) \obslash 1)) \leq (1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))$}
\RightLabel{r}
\UnaryInfC{$1 \oslash ((1/A_2) \obslash 1) \leq (A_1/A_2) \backslash ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2)))$}
\RightLabel{A_I^1}
\UnaryInfC{$(A_1/A_2) \otimes 1 \leq ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)$}
\RightLabel{1}
\UnaryInfC{$A_1/A_2 \leq ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\|A_1\|^\circ \leq f(A_1)$}
\RightLabel{$?\obslash$}
\UnaryInfC{$1 \obslash \|A_1\|^\circ \leq 1 \obslash f(A_1)$}
\RightLabel{cr}
\UnaryInfC{$\|A_1\|^\circ \leq 1 \oplus (1 \obslash f(A_1))$}
\RightLabel{1}
\UnaryInfC{$\|A_1\|^\circ \otimes 1 \leq 1 \oplus (1 \obslash f(A_1))$}
\RightLabel{r}
\UnaryInfC{$1 \leq \|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1)))$}
\AxiomC{$\|A_2\|^\bullet \leq g(A_2)$}
\RightLabel{$?\obslash$}
\UnaryInfC{$A_2 \obslash \|A_2\|^\bullet \leq A_2 \obslash g(A_2)$}
\RightLabel{$cr \times 2$}
\UnaryInfC{$\|A_2\|^\bullet \oslash (A_2 \obslash g(A_2)) \leq A_2$}
\RightLabel{$/$}
\BinaryInfC{$1/A_2 \leq (\|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1))))/(\|A_2\|^\bullet \oslash (A_2 \obslash g(A_2)))$}
\RightLabel{$\obslash ?$}
\UnaryInfC{$((\|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1))))/(\|A_2\|^\bullet \oslash (A_2 \obslash g(A_2)))) \obslash 1 \leq (1/A_2) \obslash 1$}
\RightLabel{$cr \times 2$}
\UnaryInfC{$1 \oslash ((1/A_2) \obslash 1) \leq (\|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1))))/(\|A_2\|^\bullet \oslash (A_2 \obslash g(A_2)))$}
\RightLabel{$r \times 2$}
\UnaryInfC{$\|A_2\|^\bullet \oslash (A_2 \obslash g(A_2)) \leq (1 \oslash ((1/A_2) \obslash 1)) \backslash (\|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1))))$}
\RightLabel{A_I^1}
\UnaryInfC{$(1 \oslash ((1/A_2) \obslash 1)) \otimes \|A_2\|^\bullet \leq (\|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1)))) \oplus (A_2 \obslash g(A_2))$}
\RightLabel{cr}
\UnaryInfC{$((1 \oslash ((1/A_2) \obslash 1)) \otimes \|A_2\|^\bullet) \oslash (A_2 \obslash g(A_2)) \leq \|A_1\|^\circ \backslash (1 \oplus (1 \obslash f(A_1)))$}
\RightLabel{A_I^1}
\UnaryInfC{$\|A_1\|^\circ \otimes ((1 \oslash ((1/A_2) \obslash 1)) \otimes \|A_2\|^\bullet) \leq (1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))$}
\RightLabel{r}
\UnaryInfC{$1 \oslash ((1/A_2) \obslash 1) \leq (\|A_1\|^\circ \backslash ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))))/\|A_2\|^\bullet$}
\RightLabel{C_I^2}
\UnaryInfC{$1 \otimes \|A_2\|^\bullet \leq (\|A_1\|^\circ \backslash ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2)))) \oplus ((1/A_2) \obslash 1)$}
\RightLabel{cr}
\UnaryInfC{$(1 \otimes \|A_2\|^\bullet) \oslash ((1/A_2) \obslash 1) \leq \|A_1\|^\circ \backslash ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2)))$}
\RightLabel{A_I^1}
\UnaryInfC{$\|A_1\|^\circ \otimes (1 \otimes \|A_2\|^\bullet) \leq ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)$}
\RightLabel{r}
\UnaryInfC{$1 \otimes \|A_2\|^\bullet \leq \|A_1\|^\circ \backslash (((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1))$}
\RightLabel{1}
\UnaryInfC{$\|A_2\|^\bullet \leq \|A_1\|^\circ \backslash (((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1))$}
\RightLabel{r}
\UnaryInfC{$\|A_1\|^\circ \otimes \|A_2\|^\bullet \leq ((1 \oplus (1 \obslash f(A_1))) \oplus (A_2 \obslash g(A_2))) \oplus ((1/A_2) \obslash 1)$}
\end{prooftree}
$$

Figure 6.10: Proving well-definedness of $f(A_1/A_2)$.

and

$$\begin{array}{rcl@{\qquad}rcl}
|p|^-_\oplus & := & 0 & & & \\
|1|^-_\oplus & := & 0 & |0|^-_\oplus & := & 0 \\
|A \otimes B|^-_\oplus & := & |A|^-_\oplus + |B|^-_\oplus & |A \oplus B|^-_\oplus & := & |A|^-_\oplus + |B|^-_\oplus + 1 \\
|A/B|^-_\oplus & := & |A|^-_\oplus + |B|^+_\oplus & |B \oslash A|^-_\oplus & := & |A|^-_\oplus + |B|^+_\oplus \\
|B\backslash A|^-_\oplus & := & |A|^-_\oplus + |B|^+_\oplus & |A \oslash B|^-_\oplus & := & |A|^-_\oplus + |B|^+_\oplus
\end{array}$$

Finally, $|A|_\oplus := |A|^+_\oplus - |A|^-_\oplus$.

Lemma 60. For arbitrary A, we have

$$\begin{array}{c}
|A|^+_\oplus = |\|A\|^\circ|^+_0 = |\|A\|^\bullet|^-_0 \\
|A|^-_\oplus = |\|A\|^\bullet|^+_0 = |\|A\|^\circ|^-_0
\end{array}$$

Proof. By a straightforward induction. □

Lemma 61. If $A \leq B$, then $|A|_\oplus = |B|_\oplus$, and $|A|_p = |B|_p$ for all p_i.

Proof. Again, by a straightforward induction. □

Corollary 4. If $\vdash A \sim B$, then $|A|_\oplus = |B|_\oplus$, and $|A|_p = |B|_p$ for all p_i.

The remainder of this section is dedicated to proving the inverse of the above corollary. Our aim is to define a meet for $\|A\|^\circ$ and $\|B\|^\circ$, entering into the construction of a join for A and B through application of the diamond property together with $f(A)$ and $f(B)$. To this end, we first require a few more definitions and lemmas. The following is an easy observation.

Lemma 62. Formulas $\|C\|^\circ, \|C\|^\bullet$ for any C are included in the proper subset of $\mathscr{F}(Atom)$ generated by the following grammar:

$$\begin{array}{rcl}
\phi & ::= & 0 \mid p_i \mid (1/0) \mid (1/p_i) \\
A^{nf}, B^{nf} & ::= & 1 \mid \phi \mid (A^{nf} \otimes B^{nf})
\end{array}$$

Thus, positive and negative occurrences of units 0 (atoms p_i) take the forms 0 (p_i) and $1/0$ ($1/p_i$) respectively, being glued together through $\otimes$ only. The following definition details the corresponding notion of *context*. Through universal quantification over said concept in stating derivability of certain rules pertaining to the Grishin interactions (cf. L.64), we obtain the non-determinacy required for the construction of the desired meet.

Definition 97. A *context* $A^{\otimes}[]$ is a bracketing of a series of formulae connected through $\otimes$, containing a unique occurrence of a *hole* $[]$:

$$A^{\otimes}[], B^{\otimes}[] \quad ::= \quad [] \mid (A^{\otimes}[] \otimes B) \mid (A \otimes B^{\otimes}[])$$

Given $A^{\otimes}[], B$, let $A^{\otimes}[B]$ denote the substitution of B for $[]$ in $A^{\otimes}[]$.

Intuitively, a context singles out a subformula that may be 'displayed' as the whole of the antecedent of the inequality sign. Thus, if $A^{\otimes}[B] \leq C$, then there exists D s.t. $B \leq D$ (and vice versa). The following definition constructs the desired D.

Definition 98. Given $A^{\otimes}[], C$, define the formula $A^{\otimes}[] \div C$ by induction on $A^{\otimes}[]$:

$$\begin{array}{rcl} [] \div C & := & C \\ (A^{\otimes}[] \otimes B) \div C & := & A^{\otimes}[] \div C/B \\ (A \otimes B^{\otimes}[]) \div C & := & B^{\otimes}[] \div A\backslash C \end{array}$$

Lemma 63. For any $A^{\otimes}[], B, C$, $A^{\otimes}[B] \leq C$ iff $B \leq A^{\otimes}[] \div C$.

Proof. By induction on $A^{\otimes}[]$. The base case is immediate. The inductive cases are shown as follows, where $A^{\otimes}[] = A_1^{\otimes}[] \otimes A_2$ or $A^{\otimes}[] = A_1 \otimes A_2^{\otimes}[]$.

$$\dfrac{\dfrac{B \leq A_1^{\otimes}[] \div C/A_2}{A_1^{\otimes}[B] \leq C/A_2}\,IH}{A_1^{\otimes}[B] \otimes A_2 \leq C}\,r \qquad \dfrac{\dfrac{B \leq A_2^{\otimes}[] \div A_1\backslash C}{A_2^{\otimes}[B] \leq A_1\backslash C}\,IH}{A_1 \otimes A_2^{\otimes}[B] \leq C}\,r \qquad \square$$

The following lemma characterizes the type I Grishin interaction using contexts.

Lemma 64. The following rules are derivable:

$$\text{(i)}\ \dfrac{A^{\otimes}[B \obslash C] \leq D}{B \obslash A^{\otimes}[C] \leq D} \qquad \text{(ii)}\ \dfrac{A^{\otimes}[C \oslash B] \leq D}{A^{\otimes}[C] \oslash B \leq D}$$

Proof. By (separate) induction(s) on $A^{\otimes}[]$. We illustrate with (i), (ii) being similar. The desired result is immediate in the base case, while the inductive cases are handled as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{A_1^{\otimes}[B \obslash C] \otimes A_2 \leq D}{A_1^{\otimes}[B \obslash C] \leq D/A_2}\,r}{B \obslash A_1^{\otimes}[C] \leq D/A_2}\,IH}{A_1^{\otimes}[C] \otimes A_2 \leq B \oplus D}\,A_I^2}{B \obslash (A_1^{\otimes}[C] \otimes A_2) \leq D}\,cr \qquad \dfrac{\dfrac{\dfrac{\dfrac{A_1 \otimes A_2^{\otimes}[B \obslash C] \leq D}{A_2^{\otimes}[B \obslash C] \leq A_1\backslash D}\,r}{B \obslash A_2^{\otimes}[C] \leq A_1\backslash D}\,IH}{A_1 \otimes A_2^{\otimes}[C] \leq B \oplus D}\,A_I^2}{B \obslash (A_1 \otimes A_2^{\otimes}[C]) \leq D}\,cr \qquad \square$$

The nondeterminacy required for the construction of our desired meet is obtained through the liberty of choosing one's context in instantiating the above rules. In practice, however, we only require the restricted form of (i) below.

Lemma 65. The following rules are derivable:

$$\text{(i)}\quad \frac{A^{\otimes}[B] \leq C}{(1 \oslash B) \obslash A^{\otimes}[1] \leq C} \qquad \text{(ii)}\quad \frac{A \leq C}{((1/B) \otimes B) \otimes A \leq C}$$

Proof. As follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A^{\otimes}[B] \leq C}{B \leq A^{\otimes}[\,] \div C}\;L.63}{1 \oslash (A^{\otimes}[\,] \div C) \leq 1 \oslash B}\;?\oslash}{(1 \oslash B) \obslash 1 \leq A^{\otimes}[\,] \div C}\;cr \times 2}{A^{\otimes}[(1 \oslash B) \obslash 1] \leq C}\;L.63}{(1 \oslash B) \obslash A^{\otimes}[1] \leq C}\;L.64(i) \qquad \cfrac{\cfrac{\cfrac{\cfrac{A \leq C}{1 \otimes A \leq C}\;1}{1 \leq C/A}\;r}{1/B \leq (C/A)/B}\;/?}{((1/B) \otimes B) \otimes A \leq C}\;r \times 2$$

□

Finally, we prove the desired

Theorem 6.4.1. $\vdash A \sim B$ if $|A|_{\oplus} = |B|_{\oplus}$ and $|\|A\|^{\circ}|_p = |\|B\|^{\circ}|_p$ for all p.

Proof. First, we require some notation. We shall write a (possible empty) *list* of formulas $[A_1, \ldots, A_n]$ to denote the right-associative bracketing of $A_1 \obslash \ldots A_n \obslash 1$. Further, given $n \geq 0$, let A^n denote the list of n repetitions of A. Finally, we write ++ for list concatenation. Now let there be given an enumeration $p_1, p_2, \ldots p_n$ of all the atoms occurring in A and B. Define

$$\begin{array}{rcl}
k & := & \mathit{max}(|\|A\|^{\circ}|_0^+, |\|B\|^{\circ}|_0^+) = \mathit{max}(|A|_{\oplus}^+, |B|_{\oplus}^+) \\
l & := & \mathit{max}(|\|A\|^{\circ}|_0^-, |\|B\|^{\circ}|_0^-) = \mathit{max}(|A|_{\oplus}^-, |B|_{\oplus}^-) \\
k(i) & := & \mathit{max}(|\|A\|^{\circ}|_{p_i}^+, |\|B\|^{\circ}|_{p_i}^+)\ (1 \leq i \leq n) \\
l(i) & := & \mathit{max}(|\|A\|^{\circ}|_{p_i}^-, |\|B\|^{\circ}|_{p_i}^-)\ (1 \leq i \leq n)
\end{array}$$

We now witness $\vdash \|A\|^{\circ} \sim \|B\|^{\circ}$ by a meet

$$\begin{array}{rclcl}
D & := & (1 \oslash p_1)^{k(1)} & ++ & (1 \oslash (1/p_1))^{l(1)} \\
 & ++ & \ldots & & \\
 & ++ & (1 \oslash p_n)^{k(n)} & ++ & (1 \oslash (1/p_n))^{l(n)} \\
 & ++ & (1 \oslash 0)^{k} & ++ & (1 \oslash (1/0))^{l}
\end{array}$$

Since we know from L.59 that $\vdash A \sim \|A\|^\circ$ and $\vdash B \sim \|B\|^\circ$ with joins $f(A)$ and $f(B)$, we can construct a join $f(A) \sqcup_D f(B)$ witnessing $\vdash A \sim B$. Suffice it to show that D, as defined above, is indeed a meet for $\|A\|^\circ$ and $\|B\|^\circ$. W.l.o.g., we show $D \leq \|A\|^\circ$. We proceed from $\|A\|^\circ \leq \|A\|^\circ$, instantiating reflexivity.

1. First, we bring the antecedent of $\leq$ into a form where it contains $k(i)$ (k) and $l(i)$ (l) occurrences of p_i (0) and $1/p_i$ (1/0) respectively. Thus, for $i = 1$ to n, apply L.65(ii) $k(i) - |\|A\|^\circ|^+_{p_i}$ ($= l(i) - |\|A\|^\circ|^-_{p_i}$, by $|A|_{p_i} = |B|_{p_i}$) times, instantiating B with p_i. Apply L.65(ii) an additional $k - |\|A\|^\circ|^+_\oplus$ ($= l - |\|A\|^\circ|^-_\oplus$, since $|A|_\oplus = |B|_\oplus$) times, this time instantiating B by 0.
2. For $i = 1$ to n, apply the following procedure. Apply L.65(i) $k(i)$ times, instantiating B with p_i, followed by $l(i)$ applications of L.65(i), instantiating B with $1/p_i$, all the while using coresiduation steps in between as needed. Finally, we repeat the above procedure one last time with the positive and negative occurrences of 0.
3. We conclude by removing the superfluous occurrences of 1 left behind by L.65(ii), applying (co)residuation as needed. □

As a corollary of the above theorem, we can prove the decidability of the word problem in $\mathscr{A}(Atom)$. L.58 in turn implies decidability of type similarity in each of the variants of the Lambek-Grishin calculus discussed in this chapter.

Lemma 66. For any expression ϕ in $\mathscr{A}(Atom)$, there exists A in $\mathbf{LG}_I^{0,1}$ s.t. $[\![A]\!] = \phi$.

Proof. We define the map $[\![\cdot]\!]^{-1}$ taking ϕ to a formula, as follows:

$$\begin{array}{rcl@{\qquad}rcl}
[\![p]\!]^{-1} & := & p & [\![\phi^\perp]\!]^{-1} & := & 0/[\![\phi]\!]^{-1} \\
[\![\top]\!]^{-1} & := & 1 & [\![\perp]\!]^{-1} & := & 0 \\
[\![\phi \times \psi]\!]^{-1} & := & [\![\phi]\!]^{-1} \otimes [\![\psi]\!]^{-1} & [\![\phi + \psi]\!]^{-1} & := & [\![\phi]\!]^{-1} \oplus [\![\psi]\!]^{-1}
\end{array}$$

An easy induction ensures $[\![[\![\phi]\!]^{-1}]\!] = \phi$. To illustrate, consider the case $\phi^\perp$: $[\![[\![\phi^\perp]\!]^{-1}]\!] = [\![0/[\![\phi]\!]^{-1}]\!] = \perp + [\![[\![\phi]\!]^{-1}]\!]^\perp = \perp + \phi^\perp = \phi^\perp$. □

Lemma 67. For any $\phi, \psi \in \mathscr{A}(Atom)$, we can decide whether or not $\phi = \psi$.

Proof. By T.6.4.1, we can decide $\mathbf{LG}_I^{0,1} \vdash [\![\phi]\!]^{-1} \sim [\![\psi]\!]^{-1}$ through comparison of atomic- and operator counts. If affirmative, then also $[\![[\![\phi]\!]^{-1}]\!] = [\![[\![\psi]\!]^{-1}]\!]$ by T.6.3.1, i.e., $\phi = \psi$ by L.66. If instead $\mathbf{LG}_I^{0,1} \nvdash [\![\phi]\!]^{-1} \sim [\![\psi]\!]^{-1}$, then also $[\![[\![\phi]\!]^{-1}]\!] \neq [\![[\![\psi]\!]^{-1}]\!]$, i.e., $\phi \neq \psi$ by T.6.3.2. □

Theorem 6.4.2. Type similarity is decidable for each of $\mathbf{LG}_I$, $\mathbf{LG}_{IV}$, $\mathbf{LG}_I^{0,1}$ and $\mathbf{LG}_{IV}^{0,1}$.

Proof. Use L.67 to decide whether or not $[\![A]\!] = [\![B]\!]$ in $\mathscr{A}(Atom)$. If so, then $\vdash A \sim B$ by L.58. Otherwise, $\nvdash A \sim B$ by T.6.3.1. □

6.5 Linguistic applications

We will illustrate the linguistic applications of type similarity with a simple case analysis involving extraction. Consider first the result of transforming (1) below into a relative clause through extraction of 'John'.

(1) John offered the lady a drink.

(2) man who offered the lady a drink

Both sentences may be shown grammatical w.r.t. the goal formulas s and n respectively, using the lexicon below. For expository purposes, we have added subscripts to facilitate matching with grammatical roles: *su*, *do* and *io* abbreviate *subject*, *direct object* and *indirect object* respectively.

Word	Formula
John	np_{su}
who	$(n\backslash n)/(np_{su}\backslash s)$
offered	$((np_{su}\backslash s)/np_{do})/np_{io}$
the	np_{io}/n
a	np_{do}/n
man, lady, drink	n

We offer in F.6.11 a derivation of (1) and (2) inside $\mathbf{LG}_\varnothing$. The latter's systems capacities are limited, however, when it comes to non-peripheral extraction (or with extraction from object position, given the current categorization of transitive verbs). Consider, for example, the extraction of an indirect object:

(3) lady whom John offered a drink

The best we can do within $\mathbf{LG}_\varnothing$ is to categorize 'whom' by

$$(n\backslash n)/((((np_{su}\backslash s)/np_{do})/np_{io}) \otimes ((np_{do}/n) \otimes n)),$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{np\backslash s \le np\backslash s}\,\mathit{Id}
\quad
\dfrac{\dfrac{}{np/n \le np/n}\,\mathit{Id}}{(np/n)\otimes n \le np}\,r
}{(np\backslash s)/np \le (np\backslash n)/((np/n)\otimes n)}\,/
\quad
\dfrac{\dfrac{}{np/n \le np/n}\,\mathit{Id}}{(np/n)\otimes n \le np}\,r
}{((np\backslash s)/np)/np \le ((np\backslash n)/((np/n)\otimes n))/((np/n)\otimes n)}\,/
}{(((np\backslash s)/np)/np)\otimes((np/n)\otimes n) \le (np\backslash n)/((np/n)\otimes n)}\,r
}{((((np\backslash s)/np)/np)\otimes((np/n)\otimes n))\otimes((np/n)\otimes n) \le np\backslash n}\,r
}{\underset{\text{John}}{np} \otimes ((\underset{\text{offered}}{(((np\backslash s)/np)/np)} \otimes (\underset{\text{the}}{(np/n)} \otimes \underset{\text{lady}}{n})) \otimes (\underset{\text{a}}{(np/n)} \otimes \underset{\text{drink}}{n})) \le n}\,r
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{n\backslash n \le n\backslash n}\,\mathit{Id}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{np\backslash s \le np\backslash s}\,\mathit{Id}
\quad
\dfrac{\dfrac{}{np/n \le np/n}\,\mathit{Id}}{(np/n)\otimes n \le np}\,r
}{(np\backslash s)/np \le (np\backslash n)/((np/n)\otimes n)}\,/
\quad
\dfrac{\dfrac{}{np/n \le np/n}\,\mathit{Id}}{(np/n)\otimes n \le np}\,r
}{((np\backslash s)/np)/np \le ((np\backslash n)/((np/n)\otimes n))/((np/n)\otimes n)}\,/
}{(((np\backslash s)/np)/np)\otimes((np/n)\otimes n) \le (np\backslash n)/((np/n)\otimes n)}\,r
}{((((np\backslash s)/np)/np)\otimes((np/n)\otimes n))\otimes((np/n)\otimes n) \le np\backslash n}\,r
}{(n\backslash n)/(np\backslash s) \le (n\backslash n)/(((((np\backslash s)/np)/np)\otimes((np/n)\otimes n))\otimes((np/n)\otimes n))}\,/
}{((n\backslash n)/(np\backslash s))\otimes(((((np\backslash s)/np)/np)\otimes((np/n)\otimes n))\otimes((np/n)\otimes n)) \le n\backslash n}\,r
}{\underset{\text{man}}{n} \otimes (\underset{\text{who}}{((n\backslash n)/(np\backslash s))} \otimes ((\underset{\text{offered}}{(((np\backslash s)/np)/np)} \otimes (\underset{\text{the}}{(np/n)} \otimes \underset{\text{lady}}{n})) \otimes (\underset{\text{a}}{(np/n)} \otimes \underset{\text{drink}}{n}))) \le n}\,r
$$

Figure 6.11: Deriving sample sentences (1) and (2) inside $\mathbf{LG}_{\varnothing}$.

obviously lacking the desired generality. We shall provide a solution making crucial use of type similarity in the presence of type IV interactions. Our analysis complies with chapter 3's description of Lambek-Grishin grammars, granting the tensor a monopoly on projecting syntactic structure from the lexicon. Our previous definition of a context $A^{\otimes}[]$ (cf. D.97) will therefore come in handy again:

$$A^{\otimes}[], B^{\otimes}[] \quad ::= \quad [] \mid (A^{\otimes}[] \otimes B) \mid (A \otimes B^{\otimes}[])$$

We will initially motivate our formula assignment using Moortgat's [1992] infixation operator, proposed as a primitive connective allowing the categorization of phrases involving discontinuity. While initially motivated w.r.t. the analysis of scopal ambiguities, we here pursue its application to extraction. Furthermore, as shown by Bernardi and Moortgat [2007], there exists within $\mathbf{LG}_{IV}$ a decomposition into a nesting of coimplications, thus taking away the necessity of postulating the logical constant under consideration as primitive. Our analysis initially results in a limited lexical ambiguity: a predicate is assigned two formulas, depending on whether or not an argument has been extracted. However, said formulas are shown to be type similar, as witnessed by a meet preventing overgeneration.

Infixation as a logical constant

Roughly, Moortgat's infixation formulas $A\left[\begin{smallmatrix} C \\ B \end{smallmatrix}\right]$ allow for abstraction over the syntactic context of an expression. Given the prevalence assigned to the tensor in composing lexical material, the latter concept is best formalized through the previous definition of a context $A^{\otimes}[]$ (cf. D.97). The inference rule then reads

$$\frac{Y \leq A\left[\begin{smallmatrix} C \\ B \end{smallmatrix}\right] \quad X^{\otimes}[A] \leq B}{X^{\otimes}[Y] \leq C} \, []$$

Explained in words: the embedded constituent Y with syntactic distribution characterized by A may seize scope over its embedding context $X^{\otimes}[]$ of category B, with outcome an expression of category C.[3] Moortgat proposes assigning formulas $np\left[\begin{smallmatrix} s \\ s \end{smallmatrix}\right]$ to quantified noun phrases, abstracting over the sentential domain defining their scope. Here, we consider instead a more surface oriented manifestation of discontinuity: extraction. A predicate an argument of which is extracted we lexically

[3] Moortgat writes instead $q(A, B, C)$, giving inference rules for a sequent calculus. Our notation is borrowed from Shan [2002].

assign the formula $A\left[{B \div ((C/q) \otimes q) \atop B}\right]$, parameterizing over the types: A of its syntactic distribution; C of its extracted argument (instantiating $B \div ((C/q) \otimes q)$ by $B/((C/q) \otimes q)$ if it occurs in a *right* branch, and $((C/q) \otimes q)\backslash B$ otherwise); and B of the extraction domain. Here, we have used $((C/q) \otimes q)$, understanding by q some fixed atomic formula, as opposed to plain C, having in mind the prevention of overgeneration. In particular, we have $(C/q) \otimes q \leq C$ as well as $C \nleq (C/q) \otimes q$. Consequently, constituents assigned the formula C do not directly combine with gapped clauses when categorized $B \div ((C/q) \otimes q)$, seeing as they do not derive $(C/q) \otimes q$. In practice, we will abbreviate the latter formula by C'. Now reconsider our example (3), repeated for convenience:

(3) lady whom John offered a drink

We find that the following lexicon now suffices, as witnessed in F.6.12:

Word	Formula
John	np_{su}
whom	$(n\backslash n)/(s/np'_{do})$
offered	$((np_{su}\backslash s)/np_{io})\left[{s/np'_{do} \atop s}\right]$
a	np_{do}/n
lady, drink	n

We make the following observations:

1. Our schema for extraction places no constraints on the location of the gap, seeing as (T') operates at an arbitrarily deep level within a context.

2. We rely on limited lexical ambiguity: formulas $A\left[{B \div C' \atop B}\right]$ are lexically assigned next to the usual formulas for when no extraction takes place. Although the ambiguity is well under control (it being of finite quantity), we establish it reducible in §4.2.3 through type similarity.

3. Our analysis is, of course, not limited to the case where the extraction domain is a sentence. For example, with wh-extraction in English the gap shows up at the level of a yes-no question [Vermaat, 2005, chapter 3]. Combining a sample sentence with the corresponding lexicon, we have:

Whom	did	John	offer	something?
$wh/(q/np')$	q/s_{inf}	np	$((np\backslash s_{inf})/np)\left[{q/np' \atop q}\right]$	np

Here q, *wh* categorize yes-no and wh-questions respectively.

$$
\dfrac{
  \dfrac{
    \dfrac{}{n\backslash n \le n\backslash n}\,Id
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{np\backslash s \le np\backslash s}\,Id
            \quad
            \dfrac{\dfrac{}{np/n \le np/n}\,Id}{((np/n)\otimes n) \le np}\,r
          }{(np\backslash s)/np \le (np\backslash s)/((np/n)\otimes n)}\,/
        }{((np\backslash s)/np)\otimes((np/n)\otimes n) \le np\backslash s}\,r
      }{np\otimes(((np\backslash s)/np)\otimes((np/n)\otimes n)) \le s}\,r
    }{np\otimes(((np\backslash s)/np)\left[\begin{smallmatrix} s/np' \\ s \end{smallmatrix}\right]\otimes((np/n)\otimes n)) \le s/np'}\,[\,]
  }{
    \dfrac{(n\backslash n)/(s/np) \le (n\backslash n)/(np\otimes(((np\backslash s)/np)\left[\begin{smallmatrix} s/np' \\ s \end{smallmatrix}\right]\otimes((np/n)\otimes n)))}{((n\backslash n)/(s/np'))\otimes(np\otimes(((np\backslash s)/np)\left[\begin{smallmatrix} s/np' \\ s \end{smallmatrix}\right]\otimes((np/n)\otimes n))) \le n\backslash n}\,r
  }\,/
}{n\otimes(((n\backslash n)/(s/np'))\otimes(np\otimes(((np\backslash s)/np)\left[\begin{smallmatrix} s/np' \\ s \end{smallmatrix}\right]\otimes((np/n)\otimes n)))) \le n}\,r
$$

Figure 6.12: Deriving (3) using Moortgat's infixation connective. For reasons of space, the first premise of (T') is ommitted, being an axiom.

Infixation decomposed

In the presence of type IV interactions, formulas $A\left[\begin{smallmatrix} C \\ B \end{smallmatrix}\right]$ decompose by $(B \oslash C) \owedge A$, as observed by Bernardi and Moortgat [2007]. The following two lemmas ensure derivability of ([]).

Lemma 68. For every $X^{\otimes}[], A, B, C$, we have

$$
\dfrac{B \owedge X^{\otimes}[A] \le C}{X^{\otimes}[B \owedge A] \le C}\ (i)
$$

Proof. We proceed by induction on $X^{\otimes}[]$. If identifiable with [], the desired result is immediate. We illustrate the inductive cases with $X^{\otimes}[] = X \otimes Y^{\otimes}[]$:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{B \owedge (X \otimes Y^{\otimes}[A]) \le C}{X \otimes Y^{\otimes}[A] \le B \oplus C}\,cr
    }{B \owedge Y^{\otimes}[A] \le X\backslash C}\,C^1_{IV}
  }{Y^{\otimes}[B \owedge A] \le X\backslash C}\,IH
}{X \otimes Y^{\otimes}[B \owedge A] \le C}\,r
$$

□

Lemma 69. The following is an admissible rule of inference:

$$\dfrac{Y \leq (B \oslash C) \obslash A \quad X^{\otimes}[A] \leq B}{X^{\otimes}[Y] \leq C}$$

Proof. As follows:

$$\dfrac{Y \leq (B \oslash C) \obslash A \quad \dfrac{\dfrac{\dfrac{\dfrac{X^{\otimes}[A] \leq B}{A \oslash C \leq B \oslash C}\,\oslash ?}{X^{\otimes}[A] \leq (B \oslash C) \oplus C}\,cr}{(B \oslash C) \obslash X^{\otimes}[A] \leq C}\,cr}{X^{\otimes}[(B \oslash C) \obslash A] \leq C}\,L.68}{X^{\otimes}[Y] \leq C}\,T'$$

□

We often use the following corollary, instantiating the left premise by (*Id*).

Corollary 5. The following is an admissible rule of inference:

$$\dfrac{X^{\otimes}[A] \leq B}{X^{\otimes}[(B \oslash C) \obslash A] \leq C}$$

In light of the above results, we may revise our lexicon of §4.2.1 as follows:

Word	Formula
John	np_{su}
whom	$(n\backslash n)/(s/np'_{do})$
offered	$(s \oslash (s/np'_{do})) \obslash ((np_{su}\backslash s)/np_{io})$
a	np_{do}/n
lady, drink	n

Lexical ambiguity resolved

Seeing as we wish our grammar to retain coverage of (1) and (2) as well as (3), we are forced to adopt a limited lexical ambiguity for 'offered', ending up with

Word	Formula
John	np_{su}
who	$(n\backslash n)/(np_{su}\backslash s)$
whom	$(n\backslash n)/(s/np'_{do})$
offered	$((np_{su}\backslash s)/np_{do})/np_{io}$
	$(s \oslash (s/np'_{do})) \obslash ((np_{su}\backslash s)/np_{io})$
a	np_{do}/n
man, lady, drink	n

The first formula assigned to 'offered' figures in the non-extraction environments described in (1) and (2), while extraction of the direct object in (3) necessitates the assignment of the second formula. A type and operator count will, however, reveal that the two formulas assigned to 'offered' are type similar. We may thus question the existence of a sufficiently small meet D, having in mind its substitution for its descendants within the lexicon. A first approximation is the following, taking inspiration from L.50:

$$D = \left(\begin{array}{ll} & \left(\begin{array}{ll} & ((s \oslash (np_{su} \backslash s)) \obslash q) \\ \otimes & ((s \oslash (s / np_{do})) \obslash q) \end{array} \right) \\ \otimes & ((s \oslash (s / np_{io})) \obslash q) \end{array} \right) \otimes (((q \otimes q) \otimes q) \backslash s)$$

F.6.13 ensures D is indeed a meet. Nonetheless, the current solution is suboptimal. Not only are the formulas already lexically assigned to 'offered' derivable from D, but also the following:

$$\begin{array}{l} D \leq (np_{su} \backslash (s / np_{io})) / np_{do} \\ D \leq ((np_{su} \backslash s) / np_{do}) / np_{io} \end{array}$$

Strictly speaking, the suffixes su, do and io have been assigned no formal status, there thus being no visual difference between $((np_{su} \backslash s) / np_{do}) / np_{io}$ and $((np_{su} \backslash s) / np_{io}) / np_{do}$ upon their removal. The fact, however, remains that the grammatical roles io and do have been switched, having severe repercussions for semantic interpretation. We thus seek a refinement of D that lacks the shortcomings of our previous attempt, preventing overgeneration. Luckily, we have just the thing. Abbreviating $((np_{su} \backslash s) / np_{do}) / np_{io}$ as *dtv*:

$$D = \left(\ ((s \oslash dtv) \obslash (s/s)) \ \otimes \ ((s \oslash (s / np'_{io})) \obslash s) \ \right) \otimes np'_{io}$$

We demonstrate its suitability as a meet in F.6.14. That $D \nleq (np_{su} \backslash (s / np_{io})) / np_{do}$ and $D \nleq ((np_{su} \backslash s) / np_{do}) / np_{io}$ follows from **LG**'s decidability [see Moortgat, 2007].

We conclude with the following observation. The currently proposed meet serves only to categorize the various occurrences of 'offered' found in (1)-(3). More generally, we can find a D for formulas A/B and $(C \oslash (C/B')) \obslash A$, where C is the first non-implicational subformula encountered by traversing the positive monotone positions of $/, \backslash$ (e.g., for $A = (r \backslash (p \otimes q)) / (t \backslash s)$, $C = (p \otimes q)$).

$$D = \left(\ ((C \oslash (A/B)) \obslash (C/C)) \ \otimes \ ((C \oslash (C/B')) \obslash C) \ \right) \otimes B'$$

$$
\begin{array}{cl}
\hline
((q \otimes q) \otimes q)\backslash s \leq ((q \otimes q) \otimes q)\backslash s & Id \\
\hline
((q \otimes q) \otimes q) \otimes (((q \otimes q) \otimes q)\backslash s) \leq s & r \\
\hline
((q \otimes q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s) \leq s/np & L.5 \\
\hline
(((q \otimes q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np \leq s & r \\
\hline
(((q \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np \leq s/np & L.5 \\
\hline
((((q \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np) \otimes np \leq s & r \\
\hline
((((((s \oslash (np\backslash s)) \obslash q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np) \otimes np \leq np\backslash s & L.5 \\
\hline
(((((s \oslash (np\backslash s)) \obslash q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np \leq (np\backslash s)/np & r \\
\hline
((((s \oslash (np\backslash s)) \obslash q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s) \leq ((np\backslash s)/np)/np & r
\end{array}
$$

$$
\begin{array}{cl}
\begin{array}{cl}
\hline
((q \otimes q) \otimes q)\backslash s \leq ((q \otimes q) \otimes q)\backslash s & Id \\
\hline
((q \otimes q) \otimes q) \otimes (((q \otimes q) \otimes q)\backslash s) \leq s & r \\
\hline
((((s \oslash (np\backslash s)) \obslash q) \otimes q) \otimes q) \otimes (((q \otimes q) \otimes q)\backslash s) \leq s/np & L.5 \\
\hline
(((((s \oslash (np\backslash s)) \obslash q) \otimes q) \otimes q) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np \leq s & r \\
\hline
(((((s \oslash (np\backslash s)) \obslash q) \otimes q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \otimes np \leq np\backslash s & L.5 \\
\hline
((((s \oslash (np\backslash s)) \obslash q) \otimes q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s) \leq (np\backslash s)/np & r
\end{array}
\quad
\begin{array}{cl}
\hline
np/q \leq np/q & Id \\
\hline
(np/q) \otimes q \leq np & r \\
\hline
s/np \leq s/np' & ?/ \\
\hline
s \oslash (s/np') \leq s \oslash (s/np) & ?\oslash
\end{array}
& \\
\hline
(s \oslash (s/np)) \obslash (((((s \oslash (np\backslash s)) \obslash q) \otimes q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)) \leq (s \oslash (s/np')) \obslash ((np\backslash s)/np) & \obslash \\
\hline
((((s \oslash (np\backslash s)) \obslash q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s) \leq (s \oslash (s/np')) \obslash ((np\backslash s)/np) & L.68
\end{array}
$$

Figure 6.13: $D = ((((s \oslash (np\backslash s)) \obslash q) \otimes ((s \oslash (s/np)) \obslash q)) \otimes ((s \oslash (s/np)) \obslash q)) \otimes (((q \otimes q) \otimes q)\backslash s)$ as a meet for $((np\backslash s)/np)/np$ and $(s \oslash (s/np')) \obslash ((np\backslash s)/np)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{s/s \leq s/s}\;Id}{(s/s) \otimes s \leq s}\;r}{(s/s) \otimes ((s \oslash (s/np')) \obslash s) \leq s/np'}\;L.5}{((s/s) \otimes ((s \oslash (s/np')) \obslash s)) \otimes np' \leq s}\;r}{(((s \oslash dtv) \obslash (s/s)) \otimes ((s \oslash (s/np')) \obslash s)) \otimes np' \leq dtv}\;L.5
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{s/s \leq s/s}\;Id}{(s/s) \otimes s \leq s}\;r \qquad \dfrac{\dfrac{\dfrac{}{np/q \leq np/q}\;Id}{(np/q) \otimes q \leq np}\;r}{((np\backslash s)/np)/np \leq ((np\backslash s)/np)/np'}\;?/}{((s/s) \otimes s) \oslash (((np\backslash s)/np)/np') \leq s \oslash dtv}\;\oslash}{(s/s) \otimes s \leq (s \oslash dtv) \oplus (((np\backslash s)/np)/np')}\;cr}{(s \oslash dtv) \obslash ((s/s) \otimes s) \leq ((np\backslash s)/np)/np'}\;cr}{((s \oslash dtv) \obslash (s/s)) \otimes s \leq ((np\backslash s)/np)/np'}\;L.68}{(((s \oslash dtv) \obslash (s/s)) \otimes s) \otimes np' \leq (np\backslash s)/np}\;r}{(s \oslash (s/np')) \obslash ((((s \oslash dtv) \obslash (s/s)) \otimes s) \otimes np') \leq (s \oslash (s/np')) \obslash ((np\backslash s)/np)}\;?\obslash}{(((s \oslash dtv) \obslash (s/s)) \otimes ((s \oslash (s/np')) \obslash s)) \otimes np' \leq (s \oslash (s/np')) \obslash ((np\backslash s)/np)}\;L.68
$$

Figure 6.14: $D = (((s \oslash dtv) \obslash (s/s)) \otimes ((s \oslash (s/np')) \obslash s)) \otimes np'$ as a meet for $((np\backslash s)/np)/np$ and $(s \oslash (s/np')) \obslash ((np\backslash s)/np)$.

7

Natural deduction and Montagovian semantics

7.1 Introduction

The final chapter concerns the definition of a compositional semantics for **LG**. As noted in chapter 3's concluding remarks, our main obstacle is the apparent incompatibility with the inherent asymmetry of the intuitionistic target language. Similar difficulties have plagued attempts at unearthing the constructive contents of classical proofs, having led to the definitions of various double negation translations, essentially replacing the multitude of conclusions with the hypothesis of their conjoined negations. At the other end of the Curry-Howard isomorphism, such embeddings are known as *continuation-passing style* translations (CPS, for short). While mainly used within computer science for compiling programming languages with side effects, their importance for the study of natural language semantics has in recent years been stressed by de Groote [2001], Barker and Shan [2006a,b] and, specifically within **LG**, by Bernardi and Moortgat [2007, 2010], among others. The solution pursued here builds forth upon these efforts by writing the source derivations for **LG** directly in continuation-passing style. Several benefits obtain:

1. Whereas CPS translations such as employed by Bernardi and Moortgat [2007] typically abound in administrative redexes, ours preserves normal forms.
2. While provability-wise nothing changes, we obtain a finer distinction between proofs themselves. Specifically, we transfer the semantic expressivity found in Bernardi and Moortgat's [2007] target language to the source, leaving its degree of resource-sensitivity intact. As such, we can account for scopal ambiguities without recourse to any machinery beyond the context-free base logic.

The contents of this chapter are organized as follows:

1. §2 first defines syntactic types. While one can go back and forth, the current concept is not a mere notational variation on that of formulas. While each type is related uniquely to two dual formulas, the same formula can have many types that correspond to it. The explanation for this discrepancy is postponed until §6, where we compare the direction from formulas to types with the multitude of plausible double negation translations for classical logic, as well as with various strategies for backward-chaining proof search.
2. In §3, we define a language of raw proof terms together with the associated notion of α-equivalence and (capture avoiding) substitution.
3. §4 defines our typing rules for **LG**, identifying the well-typed instances among the raw proof terms. Substitution is shown to respect the inference rules, while translations back and forth are defined w.r.t. algebraic derivability.
4. §5 tackles the issue of normalization. As opposed to the standard operational approach, we define normal forms inductively, while substituting an equivalence relation for the usual rewriting rules. Initially, normalization is defined as a translation into a Cut-free sequent calculus. As a characterization of normal forms, however, the latter shows fallacies, particularly in the existence of a number of trivial rule permutations. In response, we define canonical representatives for derivations differing only by such permutations, inspired by Andreoli's [1992] focussing strategy for backward-chaining proof search inside full linear logic. We next introduce phase models, and (re)define normalization by composing the constructive contents extracted from soundness and completeness proofs. The appeal to an intuitionistic metalanguage for relegating normalization is typical of *normalization by evaluation* [see Berger and Schwichtenberg, 1991, Coquand, 1993], and our result may be considered an instance thereof.
5. In §6, we relate our previous efforts to three known continuation-passing style translations, two of which coincide with the call-by-name and call-by-value

interpretations of Bernardi and Moortgat [2007], while another, as of yet unstudied for its linguistic applications, adapts Girard's [1991] constructive interpretation of classical logic, notable for its parsimonious use of double negations. In contrast with Bernardi and Moortgat's efforts, we adapt each of the translations mentioned so as to lack recourse to administrative redexes, to which end we shall require keeping a tighter grip on the sequential ordering of the inference rules in the source calculus. As a result, we can flesh out a correspondence with various strategies for proof-search in Cut-free sequent calculi, most notably Andreoli's aforementioned focussing method.

6. While our proof terms may be used as the target for CPS translations, they share their degree of resource sensitivity with traditional Lambek calculi and can hence also be used to formulate syntactic analyses in directly. Such is done in §7 for data exemplifying scopal ambiguities, having traditionally been a main source of motivation for the Montagovian school of semantics. We shall see that $\mathbf{LG}_{\varnothing}$ suffices to account for all combinatorially possible readings using a finite lexicon, thus leaving the Grishin interactions out of the equation. This contrasts heavily with traditional categorial accounts, typically either employing additional structural postulates which increase generative capacity beyond the context-free, or, as proposed by Hendriks [1993], relaxing compositionality to a relation. In other words, we show that context-free expressivity still suffices to account for scopal ambiguities (at least at the sentential level) under a strict interpretation of compositionality.

7.2 Syntactic types

Our intention being to define a term language for **LG**, we shall refer to its formulas as (syntactic) *types* accordingly.[1] We present two alternative definitions, the first sticking closer to the language of formulas as used in previous chapters, while the second is more economic by taking advantage of duality, though at the cost of losing clarity in its correspondence to previous developments.

Syntactic types, in their first conception, come in two flavours, referred to by positive(ly) and negative(ly polar) respectively. Roughly, the positive types are those whose occurrences as inputs were classified as α in chapter 5, and whose output occurrences were classified as β, with the converse situation applying to negative types. In other words, the distinction is one of duality in proof-theoretic behaviour,

[1] The difference is not one of mere terminology, however, in that we shall later find several plausible translations to exist taking our previous notion of formulas to the present concept of types.

classifying formulas according to the invertibility of their inference rules. It can be further extended to the level of atoms by allowing each atom p to occur with either positive or negative *bias* (written p and $\bar{p}$ respectively). In addition, we shall adopt two new unary connectives $\downarrow$ and $\uparrow$, originating in [Girard, 2000] and referred to by (polarity) *shifts*, their purpose being to transfer from negative to positive types and vice versa. Interestingly, in light of the positive/negative distinction having been applied to atoms as well, arrow-reversal $\cdot^{\infty}$ can now be formulated fixpoint-free by setting $p^{\infty} = \bar{p}$ and $\bar{p}^{\infty} = p$. Consequently, a more natural notation to use would be that of linear negation $\cdot^{\perp}$. In addition, we shall in this chapter again make use of the diamond- and box operators (cf. §6.3 of Ch.3, as well as Ch.5), suggesting their usage in the analysis of scopal ambiguities in §7.

Definition 99. (Syntactic) types for **LG** are defined by the following grammar:

$$\begin{array}{rcl} P, Q & ::= & p \mid (P \otimes Q) \mid (P \oslash N) \mid (N \obslash P) \mid \Diamond P \mid \downarrow M \\ M, N & ::= & \bar{p} \mid (M \oplus N) \mid (Q \backslash M) \mid (M / Q) \mid \Box M \mid \uparrow P \end{array}$$

Duality, now written using linear negation in light of being fixpoint-free, maps positives into negatives and vice-versa:

$$\begin{array}{rclrcl} p^{\perp} & := & \bar{p} & \bar{p}^{\perp} & := & p \\ (P \otimes Q)^{\perp} & := & Q^{\perp} \oplus P^{\perp} & (M \oplus N)^{\perp} & := & N^{\perp} \otimes M^{\perp} \\ (P \oslash N)^{\perp} & := & N^{\perp} \backslash P^{\perp} & (Q \backslash M)^{\perp} & := & M^{\perp} \oslash Q^{\perp} \\ (N \obslash P)^{\perp} & := & P^{\perp} / N^{\perp} & (M / Q)^{\perp} & := & Q^{\perp} \obslash M^{\perp} \\ (\Diamond P)^{\perp} & := & \Box P^{\perp} & (\Box M)^{\perp} & := & \Diamond M^{\perp} \\ (\downarrow M)^{\perp} & := & \uparrow M^{\perp} & (\uparrow P)^{\perp} & := & \downarrow P^{\perp} \end{array}$$

Now that duality has come to manifest as classical linear negation, a more succinct notation seems in reach. Rather than having positive and negative types coexist in perfect symmetry, we decide to use only one, writing the others using their De Morgan duals. Thus, for example, implications could be equivalently expressed using positive types only by $(Q^{\perp} \obslash P)^{\perp}$ and $(P \oslash Q^{\perp})^{\perp}$. Continuing to favor positives, types A, B are then defined as consisting of either atoms p, or of being of one of the shapes $(A \otimes B)$, $(A \oslash B^{\perp})$, $(B^{\perp} \obslash A)$, $\Diamond A$ or $\uparrow A^{\perp}$. Note $\cdot^{\perp}$ is used in argument positions that otherwise require instantiation by a negative type. For the sake of aesthetics, we pursue a slightly different choice of syntax.

Definition 100. Granting positive formulas prevalence over their negative counterparts, the definition of types is revised accordingly.

$$A, B \quad ::= \quad p \mid (A \otimes_i B) \mid (\Diamond A) \mid A^{\perp} \quad \text{(where } i \in \{1, 2, 3\}\text{)}$$

Instead of using hybrid connectives $(\cdot \oslash \cdot^{\perp})$ and $(\cdot^{\perp} \obslash \cdot)$, we instead hide negation inside the translation back into the formula language, to be made precise shortly. Consequently, we end up with three tensor-like connectives, so that it makes sense to use a single notation while expressing the distinction using indices. Finally, $\downarrow A^{\perp}$ is simplified to $A^{\perp}$. The result is a formula language previously investigated by Venema [1996], Roorda [1991], Kurtonina [1995], though in each case from the perspective of model theory. Here, instead, we pursue a proof-theoretic analysis, having in mind the definition of a Montagovian semantics for **LG**. The next definition shows how to go back from types to formulas.

Definition 101. Types are mapped into formulas using dual operations $\cdot^{+}$ and $\cdot^{-}$:

$$\begin{array}{rcl@{\qquad}rcl}
p^{+} & := & p & p^{-} & := & p \\
(A \otimes_1 B)^{+} & := & A^{+} \otimes B^{+} & (A \otimes_1 B)^{-} & := & B^{-} \oplus A^{-} \\
(A \otimes_2 B)^{+} & := & A^{+} \oslash B^{-} & (A \otimes_2 B)^{-} & := & B^{+} \backslash A^{-} \\
(A \otimes_3 B)^{+} & := & A^{-} \obslash B^{+} & (A \otimes_3 B)^{-} & := & B^{-} / A^{+} \\
(\Diamond A)^{+} & := & \Diamond A^{+} & (\Diamond A)^{-} & := & \Box A^{-} \\
A^{\perp +} & := & A^{-} & A^{\perp -} & := & A^{+}
\end{array}$$

The following is an easy induction.

Lemma 70. For any type A, $A^{+} = A^{-\infty}$, or, equivalently, $A^{-} = A^{+\infty}$.

For the converse direction, we can use negations to account for the downward monotonicity of the (co)implications. Note, however, that negations in the type language can be iterated indefinitely, so that the number of possible translations in this direction, at least to the extent that composing with $\cdot^{+}$ or $\cdot^{-}$ returns the original, is unbounded. We return to this issue in §6, presenting three alternatives, corresponding, as we shall find later, to different continuation-passing style translations or proof search strategies for the original formula-based presentation of **LG**.

7.3 Raw terms

Types may be understood as describing the range of possible kinds of denotations. The following definitions describe the language used for referring to a type's inhabitants, that is to say, encodings for the concrete derivations associated to it. First, we shall define the *raw* such expressions. While appearing without type assignment, they are often more amendable to use in various inductively defined operations,

$$
\begin{array}{rcl}
FV(x) & := & \{x\} \\
FV(\lambda x s) & := & FV(s)/\{x\} \\
FV((s\,t)) & := & FV(s) \cup FV(t) \\
FV(\Diamond\, s) & := & FV(s) \\
FV(\langle t \mid \Diamond\, x.s\rangle) & := & FV(t) \cup (FV(s)/\{x\}) \\
FV((s \otimes_i t)) & := & FV(s) \cup FV(t) \\
FV(\langle t \mid (x \otimes_i y).s\rangle) & := & FV(t) \cup (FV(s)/\{x,y\})
\end{array}
$$

Figure 7.1: Defining the set of free variables $FV(s)$ of s.

leaving the *typeable* expressions to be singled out later through a natural deduction calculus. In practice, we shall often first define operations on raw terms, subsequently proving a lemma to the extent that said operations respect the typing rules.

Definition 102. Assume to have at our disposal a countably infinite set of variables $x, y, \ldots$. The language of *raw* λ*-terms* s, t is defined through the grammar below.

$$
\begin{array}{rcl}
s, t & ::= & x \\
& \mid & \lambda x^A s \mid (s\,t) \\
& \mid & (\Diamond\, s) \mid \langle t \mid \Diamond\, y^A.s\rangle \\
& \mid & (s \otimes_i t) \mid \langle t \mid (x^A \otimes_i y^B).s\rangle \quad (i \in \{1,2,3\})
\end{array}
$$

Here, in $\langle t \mid (x^A \otimes_i y^B).s\rangle$, we require $x \neq y$.

In practice, terms of the form $\lambda x^A s$ are referred to as *(*λ*-)abstractions*, $(s\,t)$ as *applications*, $(s \otimes_i t)$ as *pairings*, while the remaining terms (besides variables) describe forms of *case analyses*, their notation suggested by the Cuts of Curien and Herbelin [2000]. As a notational convention, we often (informally) drop the types found in the above definition. Abstractions and case analyses have the capacity to bind variables. The following definition provides the formal analogue.

Definition 103. Given a (raw) term s, the set $FV(s)$ of *free* variables in s is defined inductively as in F.7.1.

Earlier, we described our term language as an encoding for derivations inside **LG**. The definitions of raw terms and of the sets of free variables thereof, however, certainly do not reflect its resource sensitivity: a single operator can bind multiple occurrences of the same variable, and the same variable may have multiple free occurrences as well. While we could enforece these restrictions already at the level of raw terms, we choose instead to postpone these until the definition of the typing rules. First, we consider another useful operation: capture-avoiding substitution.

$$
\begin{aligned}
x[t/x] &:= t \\
y[t/x] &:= y \text{ if } y \neq x \\
(\lambda x s)[t/x] &:= \lambda x s \\
(\lambda y s)[t/x] &:= \lambda y s[t/x] \\
&\text{(provided } x \notin FV(s) \text{ or } y \notin FV(t)) \\
(\Diamond\, s)[t/x] &:= \Diamond\, s[t/x] \\
(s' \otimes_i s)[t/x] &:= (s'[t/x] \otimes_i s[t/x]) \\
(s\ s')[t/x] &:= (s[t/x]\ s'[t/x]) \\
(\langle s \mid \Diamond\, x.s' \rangle)[t/x] &:= \langle s[t/x] \mid \Diamond\, x.s' \rangle \\
(\langle s \mid \Diamond\, y.s' \rangle)[t/x] &:= \langle s[t/x] \mid \Diamond\, y.s'[t/x] \rangle \\
&\text{(provided } x \notin FV(s') \text{ or } y \notin FV(t)) \\
(\langle s \mid (x \otimes_i v).s' \rangle)[t/x] &:= \langle s[t/x] \mid (x \otimes_i v).s' \rangle \\
(\langle s \mid (u \otimes_i x).s' \rangle)[t/x] &:= \langle s[t/x] \mid (u \otimes_i x).s' \rangle \\
(\langle s \mid (u \otimes_i v).s' \rangle)[t/x] &:= \langle s[t/x] \mid (u \otimes_i v).s'[t/x] \rangle \\
&\text{(provided } x \notin FV(s') \text{ or } u, v \notin FV(t))
\end{aligned}
$$

Figure 7.2: Defining capture-avoiding substitution for raw terms.

Definition 104. For s, t raw terms and x a variable, the *capture-avoiding substitution* $s[t/x]$, defined in F.7.2, denotes the substitution of t for the free occurrences of x in s on the proviso that no free variables in t become bound in the process.

Remark 12. Note many of the clauses in D.104, being defined on raw terms, duplicate information. For example, $(s\ s')[t/x] := (s[t/x]\ s'[t/x])$. Once we have defined the typable terms of the form $(s\ s')$, we can be sure that x occurs free in only s or s' (by virtue of the side conditions on context merger), and hence $(s[t/x]\ s'[t/x])$ evaluates to $(s[t/x]\ s')$ or $(s\ s'[t/x])$.

Note that a substitution $s[t/x]$ is not defined in case any free variables in t would become bound in the process. In what is to follow, we do not distinguish between terms that differ only in the names of bound variables. As such, we can always introduce fresh names for any bound variables prior to applying a substitution, guaranteeing that the latter operation exits with a result.

Definition 105. Raw terms are considered *α-equivalent* when invariant under renaming of bound variables. In other words, $\equiv_\alpha$ refers to the least compatible equivalence relation satisfying

$$
\begin{array}{llll}
(\alpha_{\neg}) & \lambda x s \equiv_\alpha \lambda y s[y/x] & & (y \notin FV(s)) \\
(\alpha_{\Diamond}) & \langle t \mid \Diamond\, x.s \rangle \equiv_\alpha \langle t \mid \Diamond\, y.s[y/x] \rangle & & (y \notin FV(s)) \\
(\alpha_{\otimes_i}) & \langle t \mid (x \otimes_i y).s \rangle \equiv_\alpha \langle t \mid (u \otimes_i v).s[u/x][v/y] \rangle & & (u, v \notin FV(s))
\end{array}
$$

Note that the side conditions are easily met by simply choosing the new name(s) for the bound variable(s) fresh.

In stating the theorems and lemmas to follow, we often do not explicitly mention the required applications of α-equivalences, assuming that in the background something like Barendregt's *variable convention* is in effect: names for bound variables are always chosen fresh, and in particular kept disjoint from the names of free variables. To truly solve the problem, however, we have several solutions at our disposal. Most notably, we may choose to use the 'nameless dummy' notation of de Bruijn [1972], identifying an occurrence of a bound variable by the number of steps to the corresponding binding operator, or we may choose to consider the above definitions as pertaining to 'pre-terms', defining the actual terms as equivalence classes of pre-terms under $\equiv_\alpha$ [see Sørensen and Urzyczyn, 1998].

7.4 Typing rules

We proceed to define the *typable* terms, encoding derivations inside a *natural deduction calculus.* Roughly, we prohibit multiple free occurrences of the same variable, as well as demand that each binding operator binds exactly one variable occurrence.

Definition 106. *Structures* of variable/formula pairs $x : A$ capture the structural configurations relative to which deduction is to take place:

$$\Gamma, \Delta \quad ::= \quad x : A \mid (\Gamma \bullet_i \Delta) \mid \langle \Gamma \rangle \quad (i \in \{1, 2, 3\})$$

$FV(\Gamma)$ denotes the set of variables found in Γ. A *context* $\Gamma[]$ is defined as follows:

$$\Gamma[], \Delta[] \quad ::= \quad [] \mid (\Gamma[] \bullet_i \Delta) \mid (\Gamma \bullet_i \Delta[]) \mid \langle \Gamma[] \rangle \quad (i \in \{1, 2, 3\})$$

Definition 107. F.7.3 defines the judgement form $\Pi \vdash s : \Sigma$, where Σ is a set containing at most one formula, s.t. if empty (written $\perp$) Π is a pair (Γ, Δ) (the outer brackets usually omitted in practice), while if a singleton, Π denotes a single structure Γ. The notation for contexts $\Pi[]$ is used accordingly. I.e., if $\Sigma = \varnothing$ in a judgement $\Pi[\Delta] \vdash s : \Sigma$, $\Pi[]$ is of the shape $(\Gamma'[], \Gamma)$ or $(\Gamma, \Gamma'[])$, while otherwise Π denotes some $\Gamma[]$. We require that in instances of $({}^{\perp}E)$, $(\Diamond E)$, $(\otimes_i E)$ and $(\otimes_i I)$, the premises employ disjoint sets of variables.

Axioms

$$\frac{}{x : A \vdash x : A}\ V$$

Structural rules

$$\frac{\Delta, \Gamma \vdash s : \bot}{\Gamma, \Delta \vdash s : \bot}\ dp1 \qquad \frac{\langle \Gamma \rangle, \Delta \vdash s : \bot}{\Gamma, \langle \Delta \rangle \vdash s : \bot}\ dp2$$

$$\frac{\Gamma \bullet_2 \Delta, \Theta \vdash s : \bot}{\Gamma, \Delta \bullet_1 \Theta \vdash s : \bot}\ dp3 \qquad \frac{\Gamma \bullet_1 \Delta, \Theta \vdash s : \bot}{\Gamma, \Delta \bullet_3 \Theta \vdash s : \bot}\ dp4$$

Logical Rules

$$\frac{\Delta \vdash s : A^{\bot} \quad \Gamma \vdash t : A}{\Gamma, \Delta \vdash (s\ t) : \bot}\ {}^{\bot}E \qquad \frac{\Gamma, x : A \vdash s : \bot}{\Gamma \vdash \lambda x^A s : A^{\bot}}\ {}^{\bot}I$$

$$\frac{\Delta \vdash t : \Diamond A \quad \Pi[\langle y : A \rangle] \vdash s : \Sigma}{\Pi[\Delta] \vdash \langle t \mid \Diamond y^A . s \rangle : \Sigma}\ \Diamond E \qquad \frac{\Gamma \vdash s : A}{\langle \Gamma \rangle \vdash \Diamond s : \Diamond A}\ \Diamond I$$

$$\frac{\Delta \vdash t : A \otimes_i B \quad \Pi[(y : A \bullet_i z : B)] \vdash s : \Sigma}{\Pi[\Delta] \vdash \langle t \mid (y^A \otimes_i z^B).s \rangle : \Sigma}\ \otimes_i E \qquad \frac{\Gamma \vdash s : A \quad \Delta \vdash t : B}{\Gamma \bullet_i \Delta \vdash (s \otimes_i t) : A \otimes_i B}\ \otimes_i I$$

Type I Grishin interactions

$$\frac{\Gamma_2 \bullet_2 \Delta_2, \Delta_1 \bullet_2 \Gamma_1 \vdash s : \bot}{\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash s : \bot}\ A^1_I$$

$$\frac{\Delta_1 \bullet_3 \Gamma_1, \Gamma_2 \bullet_3 \Delta_2 \vdash s : \bot}{\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash s : \bot}\ A^2_I$$

$$\frac{\Delta_1 \bullet_3 \Gamma_2, \Delta_2 \bullet_2 \Gamma_1 \vdash s : \bot}{\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash s : \bot}\ C_I$$

Type IV Grishin interactions

$$\frac{\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash s : \bot}{\Gamma_2 \bullet_2 \Delta_2, \Delta_1 \bullet_2 \Gamma_1 \vdash s : \bot}\ A^1_{IV}$$

$$\frac{\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash s : \bot}{\Delta_1 \bullet_3 \Gamma_1, \Gamma_2 \bullet_3 \Delta_2 \vdash s : \bot}\ A^2_{IV}$$

$$\frac{\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash s : \bot}{\Delta_1 \bullet_3 \Gamma_2, \Delta_2 \bullet_2 \Gamma_1 \vdash s : \bot}\ C_{IV}$$

Figure 7.3: Typing rules.

We make the following observations regarding the above definition.

1. We interchangeably refer to the contents of F.7.3 as typing rules and as rules of natural deduction, tying in to the Curry-Howard isomorphism for intuitionistic logic. Seen in this light, while (V) is meant to abbreviate the introduction of a *variable*, we also often refer to it as an axiom.

2. The case where $\Sigma = \varnothing$ intuitively corresponds to inhabitation of absurdity, motivating our choice of writing $\perp$. We have refrained from granting it the status of an actual type constant, nor is the empty conclusion to be understood intuitionistically, in that anything can be derived from absurdity. Rather, $\perp$ may be understood as in minimal logic, its importance tying in to our later use of the current typing rules to define double-negation translations for **LG**.

3. The influence of De Groote and Lamarche's [2002] sequent calculus for **CNL** shows in the presence of $(dp1)$, $(dp3)$ and $(dp4)$, referred to, as before, by display postulates given their similarity both in formulation and in purpose to Belnap's [1982] rules of the same name. In practice, we abbreviate zero or more successive applications by a single (dp).

4. Compared to [De Groote and Lamarche, 2002], we use left-sided sequents; a design decision grounded in our pursuit of intuitionistic interpretations. More important, however, is our use of both sides of the turnstile, though treated asymetrically to restore the tight connection with λ-calculus. The newly added shift connectives (conveniently written as negation in our D.100) serve solely to control traffic across $\vdash$. Provability-wise, these are harmless extensions over $\mathbf{LG}_\varnothing$, as we shall demonstrate in T.7.4.1 below. At the level of the actual proofs, however, shifts, together with the structural rules, allow abstraction over any of the variables found in a sequent without compromising structure-sensitivity. In contrast, **NL** severely restricts the abstraction mechanism, accounting for its limited semantic expressivity.

The next definition and subsequent lemma prove the familiar display property.

Definition 108. Given $\Gamma[\,], \Theta$, define inductively on $\Gamma[\,]$ the structure $\Gamma[\,] \div \Theta$:

$$\begin{array}{rr}
[\,] \div \Theta = \Theta & \langle \Gamma[\,] \rangle \div \Theta = \Gamma[\,] \div \langle \Theta \rangle \\
(\Gamma[\,] \bullet_1 \Delta) \div \Theta = \Gamma[\,] \div (\Delta \bullet_3 \Theta) & (\Gamma \bullet_1 \Delta[\,]) \div \Theta = \Delta[\,] \div (\Theta \bullet_2 \Gamma) \\
(\Gamma[\,] \bullet_2 \Delta) \div \Theta = \Gamma[\,] \div (\Delta \bullet_1 \Theta) & (\Gamma \bullet_2 \Delta[\,]) \div \Theta = \Delta[\,] \div (\Theta \bullet_3 \Gamma) \\
(\Gamma[\,] \bullet_3 \Delta) \div \Theta = \Gamma[\,] \div (\Delta \bullet_2 \Theta) & (\Gamma \bullet_3 \Delta[\,]) \div \Theta = \Delta[\,] \div (\Theta \bullet_1 \Gamma)
\end{array}$$

Lemma 71. For any $\Gamma[]$, Δ, Θ and s, $\Gamma[\Delta], \Theta \vdash s : \bot$ iff $\Gamma[] \div \Theta, \Delta \vdash s : \bot$.

Proof. By a straightforward induction, using $(dp1)$, $(dp3)$ and $(dp4)$. □

We proceed to show closure of derivability under α-equivalence and substitution.

Lemma 72. Derivability is closed under α-equivalence. I.e., $\Pi \vdash s : \Sigma$ implies $\Pi \vdash t : \Sigma$ for any $t \equiv_\alpha s$. In addition, $\Pi[x : A] \vdash s : \Sigma$ implies $\Pi[y : A] \vdash s[y/x] : \Sigma$ for any y free for x in s, and not already used in $\Pi[]$.

Proof. By a straightforward induction. □

Lemma 73. Substitution respects the typing rules, (S) being an admissible rule:

$$\frac{\Delta \vdash t : C \quad \Pi[x : C] \vdash s : \Sigma}{\Pi[\Delta] \vdash s[t/x] : \Sigma}\ S$$

Proof. By induction on the derivation witnessing $\Pi[x : C] \vdash s : \Sigma$. The proof is not difficult, but we nevertheless consider several cases in detail. Obviously, the base case, where the right premise of (S) instantiates an axiom, is immediate, while the structural rules, including the display postulates and Grishin interactions, are equally trivial. This leaves us to check the logical rules. In proceeding, we consider the rules for $\cdot^\bot$ and $\otimes_i$, those for $\diamond$ being treated similarly.

1. The last applied rule was $({}^\bot E)$. Then $s = (s'\ s'')$, while $\Pi[x : C] = (\Gamma_1, \Gamma_2)$, with either $x \in FV(\Gamma_1)$ or $x \in FV(\Gamma_2)$. We check the latter situation, the former being dealt with similarly. Thus, $\Gamma_2 = \Gamma_2'[x : C]$, and we have

$$\frac{\Delta \vdash t : C \quad \dfrac{\Gamma_2'[x : C] \vdash s' : A^\bot \quad \Gamma_1 \vdash s'' : A}{\Gamma_1, \Gamma_2'[x : C] \vdash (s'\ s'') : \bot}\ {}^\bot E}{\Gamma_1, \Gamma_2'[\Delta] \vdash (s'\ s'')[t/x] : \bot}\ S$$

   Since $x \in FV(s'')$ and not $x \in FV(s')$, $(s'\ s'')[t/x] = (s'\ s''[t/x])$, and we proceed as follows:

$$\frac{\dfrac{\Delta \vdash t : C \quad \Gamma_2'[x : C] \vdash s' : A^\bot}{\Gamma_2'[\Delta] \vdash s'[t/x] : A^\bot}\ S \quad \Gamma_1 \vdash s'' : A}{\Gamma_1, \Gamma_2'[\Delta] \vdash (s'[t/x]\ s'') : \bot}\ {}^\bot E$$

2. The last applied rule was $({}^\bot I)$. Then $s = \lambda y s'$,

$$\frac{\Delta \vdash t : C \quad \dfrac{\Gamma[x : C], y : A \vdash s' : \bot}{\Gamma[x : C] \vdash \lambda y s' : A^\bot}\ {}^\bot I}{\Gamma[\Delta] \vdash (\lambda y s')[t/x] : A^\bot}\ S$$

Note $y \neq x$ in light of linearity. Consequently, $(\lambda y s')[t/x] = \lambda y s'[t/x]$, and we continue as follows:

$$\dfrac{\dfrac{\Delta \vdash t : C \quad \Gamma[x : C], y : A \vdash s' : \bot}{\Gamma[\Delta], y : A \vdash s'[t/x] : \bot}\, S}{\Gamma[\Delta] \vdash \lambda y s'[t/x] : A^{\bot}}\, {}^{\bot}I$$

3. The last applied rule was $(\otimes_i E)$. Then $s = \langle s' \mid (y \otimes_i z).s'' \rangle$,

$$\dfrac{\Delta \vdash t : C \quad \dfrac{\Gamma[x : C] \vdash s' : A \otimes_i B \quad \Pi'[(y : A \bullet_i z : B)] \vdash s'' : \Sigma}{\Pi'[\Gamma[x : C]] \vdash \langle s' \mid (y \otimes_i z).s'' \rangle : \Sigma}\, \otimes_i E}{\Pi'[\Gamma[\Delta]] \vdash (\langle s' \mid (y \otimes_i z).s'' \rangle)[t/x] : \Sigma}\, S$$

Alternatively, $x : C$ occurs in $\Pi'[]$. While treated similarly, said situation admittedly requires some more acrobatics with notation. Thus, $\Pi[] = \Pi'[\Gamma[]]$, and $(\langle s' \mid (y \otimes_i z).s'' \rangle)[t/x] = \langle s'[t/x] \mid (y \otimes_i z).s'' \rangle$.

$$\dfrac{\dfrac{\Delta \vdash t : C \quad \Gamma[x : C] \vdash s' : A \otimes_i B}{\Gamma[\Delta] \vdash s'[t/x] : A \otimes_i B}\, S \quad \Pi'[(y : A \bullet_i z : B)] \vdash s'' : \Sigma}{\Pi'[\Gamma[\Delta]] \vdash \langle s'[t/x] \mid (y \otimes_i z).s'' \rangle : \Sigma}\, \otimes_i E$$

4. The last applied rule was $(\otimes_i I)$. Then $s = (s' \otimes_i s'')$, $\Pi[x : C] = \Gamma_1 \bullet_i \Gamma_2$ with either $x \in FV(\Gamma_1)$ or $x \in FV(\Gamma_2)$. Assume the former, the latter situation being treated similarly. Then $\Gamma_1 = \Gamma_1'[x : C]$,

$$\dfrac{\Delta \vdash t : C \quad \dfrac{\Gamma_1'[x : C] \vdash s' : A \quad \Gamma_2 \vdash s'' : B}{\Gamma_1'[x : C] \bullet_i \Gamma_2 \vdash (s' \otimes_i s'') : A \otimes_i B}\, \otimes_i I}{\Gamma_1'[\Delta] \bullet_i \Gamma_2 \vdash (s' \otimes_i s'')[t/x] : A \otimes_i B}\, S$$

Now $(s' \otimes_i s'')[t/x] = (s'[t/x] \otimes_i s'')$, and we proceed as follows:

$$\dfrac{\dfrac{\Delta \vdash t : C \quad \Gamma_1'[x : C] \vdash s' : A}{\Gamma_1'[\Delta] \vdash s'[t/x] : A}\, S \quad \Gamma_2 \vdash s'' : B}{\Gamma_1'[\Delta] \bullet_i \Delta_2 \vdash (s'[t/x] \otimes_i s'') : A \otimes_i B}\, \otimes_i I \qquad \square$$

We proceed to show soundness w.r.t. algebraic derivability. To this end, we must first specify the interpretation of structures by formulas.

Definition 109. We lift the maps $\cdot^{+}$ and $\cdot^{-}$ from D.101 to the level of structures:

$$\begin{array}{rlrl}
(\Gamma \bullet_1 \Delta)^{+} & := \Gamma^{+} \otimes \Delta^{+} & (\Gamma \bullet_1 \Delta)^{-} & := \Delta^{-} \oplus \Gamma^{-} \\
(\Gamma \bullet_2 \Delta)^{+} & := \Gamma^{+} \oslash \Delta^{-} & (\Gamma \bullet_2 \Delta)^{-} & := \Delta^{+} \backslash \Gamma^{-} \\
(\Gamma \bullet_3 \Delta)^{+} & := \Gamma^{-} \obslash \Delta^{+} & (\Gamma \bullet_3 \Delta)^{-} & := \Delta^{-} / \Gamma^{+} \\
\langle \Gamma \rangle^{+} & := \Diamond \Gamma^{+} & \langle \Gamma \rangle^{-} & := \Box \Gamma^{-}
\end{array}$$

Theorem 7.4.1. We have the following implications:

1. $\Gamma, \Delta \vdash s : \bot$ implies $\Gamma^+ \leq \Delta^-$ and $\Delta^+ \leq \Gamma^-$
2. $\Gamma \vdash s : A$ implies $\Gamma^+ \leq A^+$ and $A^- \leq \Gamma^-$

Proof. By mutual induction. The base case (Id) is trivial. We skip $(\Diamond E)$ and $(\Diamond I)$, their treatment being similar to that of $(\otimes_i E)$ and $(\otimes_i I)$.

1. (dp). We illustrate with $(dp3)$. Depending on which direction we check, we have, by induction hypothesis, $\Gamma^+ \oslash \Delta^- \leq \Theta^-$ and $\Theta^+ \leq \Delta^+ \backslash \Gamma^-$, or $\Gamma^+ \leq \Theta^- \oplus \Delta^-$ and $\Delta^+ \otimes \Theta^+ \leq \Gamma^-$. The desired result follows from (co)residation:
$$\frac{\Theta^+ \leq \Delta^+ \backslash \Gamma^-}{\Delta^+ \otimes \Theta^+ \leq \Gamma^-}\, r \qquad \frac{\Gamma^+ \oslash \Delta^- \leq \Theta^-}{\Gamma^+ \leq \Theta^- \oplus \Delta^-}\, cr$$
2. $({}^{\perp}E)$. By induction hypothesis, $\Delta^+ \leq A^-$ $(A^+ \leq \Delta^-)$ and $\Gamma^+ \leq A^+$ $(A^- \leq \Gamma^-)$. The desired results then follow from the use of transitivity.
$$\frac{\dfrac{}{\Gamma^+ \leq A^+}\, IH \quad \dfrac{}{A^+ \leq \Delta^-}\, IH}{\Gamma^+ \leq \Delta^-}\, \circ \qquad \frac{\dfrac{}{\Delta^+ \leq A^-}\, IH \quad \dfrac{}{A^- \leq \Gamma^-}\, IH}{\Delta^+ \leq \Gamma^-}\, \circ$$
3. $({}^{\perp}I)$. Immediate by the induction hypotheses.
4. $(\otimes E_i)$. We check the instantiation for $i = 3$. First, suppose Θ is empty. In light of L.71, we can assume, w.l.o.g., that $\Pi[] = (\Gamma, [])$ for some Γ. Thus, by induction hypothesis, $\Delta^+ \leq A^- \oslash B^+$ $(B^-/A^+ \leq \Delta^-)$ and $\Gamma^+ \leq B^-/A^+$ $(A^- \oslash B^+ \leq \Gamma^-)$. The desired results follow from transitivity.
$$\frac{\dfrac{}{\Gamma^+ \leq B^-/A^+}\, IH \quad \dfrac{}{B^-/A^+ \leq \Delta^-}\, IH}{\Gamma^+ \leq \Delta^-}\, \circ \qquad \frac{\dfrac{}{\Delta^+ \leq A^- \oslash B^+}\, IH \quad \dfrac{}{A^- \oslash B^+ \leq \Gamma^-}\, IH}{\Delta^+ \leq \Gamma^-}\, \circ$$
Now suppose Θ is non-empty (say, $\Theta = C$) and $\Pi[] = \Gamma[]$ for some $\Gamma[]$. By induction hypothesis, $\Delta^+ \leq A^- \oslash B^+$ $(B^-/A^+ \leq \Delta^-)$, as before, and $\Gamma[A \otimes_3 B]^+ \leq C^+$ $(C^- \leq \Gamma[A \otimes_3 B]^-)$. We proceed as follows:
$$\frac{\dfrac{}{C^- \leq \Gamma[A \otimes_3 B]^-}\, IH \quad \begin{array}{c} \dfrac{}{B^-/A^+ \leq \Delta^-}\, IH \\ \vdots\ (mon) \\ \Gamma[A \otimes_3 B]^- \leq \Gamma[\Delta]^- \end{array}}{C^- \leq \Gamma[\Delta]^-}\, \circ$$
While dually,
$$\frac{\begin{array}{c} \dfrac{}{\Delta^+ \leq A^- \oslash B^+}\, IH \\ \vdots\ (mon) \\ \Gamma[\Delta]^+ \leq \Gamma[A \otimes_3 B]^+ \end{array} \quad \dfrac{}{\Gamma[A \otimes_3 B]^+ \leq C^+}\, IH}{\Gamma[\Delta]^+ \leq C^+}\, \circ$$

5. $(\otimes_i I)$. We again check for $i = 3$. By induction hypothesis, $\Gamma^+ \leq A^+$ $(A^- \leq \Gamma^-)$ and $\Delta^+ \leq B^+$ $(B^- \leq \Delta^-)$. Hence, using monotonicity,

$$\frac{\Delta^+ \leq B^+ \quad A^- \leq \Gamma^-}{\Gamma^- \oslash \Delta^+ \leq A^- \oslash B^+}\, m \qquad \frac{B^- \leq \Delta^- \quad \Gamma^+ \leq A^+}{B^-/A^+ \leq \Delta^-/\Gamma^+}\, m$$

6. Of the Grishin interactions, we check (C_I). By induction hypothesis, $\Delta_1^- \oslash \Gamma_2^+ \leq \Gamma_1^+ \backslash \Delta_2^-$ and $\Delta_2^+ \oslash \Gamma_1^- \leq \Gamma_2^-/\Delta_1^+$. The desired results now follow by applying (C_I^1) and (C_I^2).

$$\frac{\Gamma_1^+ \otimes \Gamma_2^+ \leq \Delta_1^- \oplus \Delta_2^-}{\Delta_1^- \oslash \Gamma_2^+ \leq \Gamma_1^+ \backslash \Delta_2^-}\, C_I^1 \qquad \frac{\Delta_2^+ \otimes \Delta_1^+ \leq \Gamma_2^- \oplus \Gamma_1^-}{\Delta_2^+ \oslash \Gamma_1^- \leq \Gamma_2^-/\Delta_1^+}\, C_I^2 \qquad \square$$

Again, we postpone discussing the converse direction until §6.

7.5 Normalization

While differing from ordinary simply-typed λ-calculus in both the choice of type-forming operators as well as the adherence to linearity in variable binding, our term language for **LG** may still be considered a notation for algorithms, i.e., being amendable to computation. Restated proof-theoretically, derivations can contain detours, and said detours can always be removed to obtain a derivation in *normal form*. The remainder of this section is dedicated to an in-depth discussion of this procedure.

The algorithmic content of λ-calculi is typically understood in terms of a rewriting relation $\rightarrow$ on terms. The process of transforming a term by successive rewritings is dubbed *reduction*, while in certain cases going against the orientation of $\rightarrow$ is referred to by *expansion*. By defining a term to be in *normal form* iff it cannot be further reduced, one proceeds to show *strong normalization*, meaning all reduction sequences end in a normal form, as well as the *Church-Russer property*, implying normal forms are unique (up to α-equivalence).

An alternative approach to the one sketched above dispenses with the rewriting mechanism, using an equivalence relation instead. Rather than being understood operationally as irreducible terms, normal forms are now defined inductively, while normalization becomes a function that maps an arbitrary term to an equivalent normal form. One reason for favoring the rewrite-free approach is when one's term language is host to matching constructs, as exemplified in the following example.

Example 22. Suppose $\Delta_1 \vdash t_1 : \Diamond A$, $\Delta_2 \vdash t_2 : \Diamond B$ and $\langle x : A\rangle, \langle y : B\rangle \vdash s : \bot$. To find an s' for which $\langle \Delta_1\rangle, \langle \Delta_2\rangle \vdash s' : \bot$, we can proceed as follows:

$$\dfrac{\Delta_1 \vdash t_1 : \Diamond A \qquad \dfrac{\Delta_2 \vdash t_2 : \Diamond B \quad \langle x : A\rangle, \langle y : B\rangle \vdash s : \bot}{\langle x : A\rangle, \Delta_2 \vdash \langle t_2 \mid \Diamond\, y.s\rangle : \bot}\,\Diamond E}{\Delta_1, \Delta_2 \vdash \langle t_1 \mid \Diamond\, x.\langle t_2 \mid \Diamond\, y.s\rangle\rangle : \bot}\,\Diamond E$$

Alternatively,

$$\dfrac{\Delta_2 \vdash t_2 : \Diamond B \qquad \dfrac{\Delta_1 \vdash t_1 : \Diamond A \quad \langle x : A\rangle, \langle y : B\rangle \vdash s : \bot}{\Delta_1, \langle y : B\rangle \vdash \langle t_1 \mid \Diamond\, x.s\rangle : \bot}\,\Diamond E}{\Delta_1, \Delta_2 \vdash \langle t_2 \mid \Diamond\, y.\langle t_1 \mid \Diamond\, x.s\rangle\rangle : \bot}\,\Diamond E$$

While there seems no reason to distinguish between these derivations, their similarity cannot be captured by a terminating rewriting relation, seeing as

$$\langle t_1 \mid \Diamond\, x.\langle t_2 \mid \Diamond\, y.s\rangle\rangle \to \langle t_2 \mid \Diamond\, y.\langle t_1 \mid \Diamond\, x.s\rangle\rangle$$

obviously introduces cycles.

Similar problems such as described above arise within the simply-typed λ-calculus with strong sums, where Lindley [2007] pursues a solution involving rewriting modulo an equivalence relation identifying terms differing by the order of matching constructs. An adaptation of these techniques to the current case study goes beyond the scope of this work. Instead, we provide two rewrite-free proofs of normalization, one syntactic, the other model-theoretic:

1. We define normal forms by the Cut-free proofs of a sequent calculus, while identifying normalization with the algorithm underlying Cut admissibility, shown to respect a suitable equivalence relation on terms. At this stage, the problem exposed in the foregoing example is to a large extent treated, though not yet in full.
2. The definition of normal forms is further refined by providing canonical representatives for derivations differing only by the ordering of matching constructs. Their relation to natural deduction is mediated by phase models, with normalization obtained by extracting the constructive content from the composition of soundness (relative to natural deduction) and completeness (producing normal forms).

$$\frac{}{x : p \vdash^{nf} x : p}\ V$$

$$\frac{\Gamma \vdash^{nf} t : A}{\Gamma, x : A^{\perp} \vdash^{nf} (x\ t) : \perp}\ {}^{\perp}L \qquad \frac{\Gamma, x : A \vdash^{nf} s : \perp}{\Gamma \vdash^{nf} \lambda x^{A} s : A^{\perp}}\ {}^{\perp}R$$

$$\frac{\Pi[\langle y : A \rangle] \vdash^{nf} s : \Sigma}{\Pi[x : \Diamond A] \vdash^{nf} \langle x \mid \Diamond y^{A}.s \rangle : \Sigma}\ \Diamond L \qquad \frac{\Gamma \vdash^{nf} s : A}{\langle \Gamma \rangle \vdash^{nf} \Diamond s : \Diamond A}\ \Diamond R$$

$$\frac{\Pi[(y : A \bullet_i z : B)] \vdash^{nf} s : \Sigma}{\Pi[x : A \otimes_i B] \vdash^{nf} \langle x \mid (y^{A} \otimes_i z^{B}).s \rangle : \Sigma}\ \otimes_i L \quad \frac{\Gamma \vdash^{nf} s : A \quad \Delta \vdash^{nf} t : B}{\Gamma \bullet_i \Delta \vdash^{nf} (s \otimes_i t) : A \otimes_i B}\ \otimes_i R$$

Figure 7.4: Long normal forms defined; display postulates and Grishin interactions carry over from F.7.3 by replacing $\vdash$ with $\vdash^{nf}$.

Both approaches are discussed in turn in the next two subsections. The second in particular closely resembles the literature on "normalization by evaluation" [Berger and Schwichtenberg, 1991], as clearly explained by Coquand [1993] w.r.t. Kripke frames for intuitionistic logic.

7.5.1 Cut elimination

In the current subsection we shall be concerned with the definition of long normal forms. From a rewriting perspective, these would comprise the fully η-expanded terms devoid of β-redexes. As is well-known for intuitionistic logic, they may as well be defined inductively by Cut-free sequent derivations using only atomic instances of axioms. With this in mind, we present

Definition 110. F.7.4 defines the judgement form $\Pi \vdash^{nf} s : \Sigma$, where Π and Σ are defined as before. The terms thus derived we refer to as being in (long) *normal form*.

From the above definition, it is easy to see that a natural deduction derivation produces a normal form iff all major premises of elimination rules are axioms. In other words, if the corresponding type-forming operators are introduced as hypotheses. Couple this with the fact that the introduction rules similarly derive a compound type in the conclusion, and we have obtained the derivations of a sequent calculus, motivating our choice of rule names. From these observations, the next lemma can be proved using a particularly simple induction.

Lemma 74. If $\Gamma, \Delta \vdash^{nf} s : \bot$ ($\Gamma \vdash^{nf} s : A$), also $\Gamma, \Delta \vdash s : \bot$ (resp. $\Gamma \vdash s : A$).

Remark 13. Recall the shape of long normal forms in simply-typed λ-calculus:

$$\lambda x_1^{\sigma_1} \ldots \lambda x_n^{\sigma_n} (((x\, t_1)\ \ldots\ )\, t_m)$$

where the body of the term is of atomic type, and $t_1, \ldots, t_m$ are in long normal form. Comparison to the terms constructed by the Cut-free derivations of F.7.4 is facilitated through uncurrying. Specifically, in the presence of product types, the long normal forms in the →-fragment may be isomorphically rendered by an application of a variable x to a pairing of terms $t_1, \ldots, t_n$, followed by an abstraction over a variable of product type:

$$\lambda \langle x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \rangle (x\ \langle t_1, \ldots, t_m \rangle)$$

where paired abstractions $\lambda\langle x^{\sigma_1}, y^{\sigma_2}\rangle s$ abbreviate $\lambda z^{\sigma_1 \times \sigma_2} s[\pi_1(z)/x, \pi_2(z)/y]$. Within the $\{\otimes, \cdot^{\bot}\}$-fragment of the type-language for **LG**, essentially the same construct arises from a Cut-free derivation if paired abstraction $\lambda(x^A \otimes_i y^B)s$ is interpreted instead by $\lambda z^{A \otimes B}\langle z \mid (x^A \otimes_i y^B).s\rangle$.

We proceed to define a procedure for transforming a given term of **LG** into a normal form, taking the following steps in doing so:

1. We define an equivalence relation ≡ on (typed) terms, covering the usual β-equivalences.
2. We prove Cut admissibility for normal form derivations, in the sense that if $\Delta \vdash^{nf} t : A$ and $\Pi[x : A] \vdash^{nf} s : \Sigma$, then there exists $s' \equiv s[t/x]$ for which $\Pi[\Delta] \vdash^{nf} s' : \Sigma$.
3. Using the previous result, we show that if $\Pi \vdash s : \Sigma$, then there exists $t \equiv s$ s.t. $\Pi \vdash^{nf} s : \Sigma$.

Definition 111. We define in F.7.5 the judgement form $\Pi \vdash s \equiv t : \Sigma$, assuming $\Pi \vdash s : \Sigma$ and $\Pi \vdash t : \Sigma$. We require the following side conditions:

1. In $(\beta_\neg)$ and $(\beta_\Diamond)$, t is to be free for y in s.
2. In $(\beta_\otimes)$, t_1 and t_2 are be free for y and z respectively in s.
3. In $(\eta_\bot)$, y is not to occur free in s.

We abbreviate $\Pi \vdash s \equiv t : \Sigma$ by $s \equiv t$ when Π and Σ are inferable from context.

Reflexivity, transitivity, symmetry

$$\frac{\Pi \vdash s : \Sigma}{\Pi \vdash s \equiv s : \Sigma}\ \textit{Refl} \quad \frac{\Pi \vdash s \equiv s' : \Sigma \quad \Pi \vdash s' \equiv s'' : \Sigma}{\Pi \vdash s \equiv s'' : \Sigma}\ \textit{Trans} \quad \frac{\Pi \vdash t \equiv s : \Sigma}{\Pi \vdash s \equiv t : \Sigma}\ \textit{Symm}$$

Congruence

$$\frac{\Delta \vdash s \equiv s' : A^{\perp} \quad \Gamma \vdash t \equiv t' : A}{\Gamma, \Delta \vdash (s\ t) \equiv (s'\ t') : \perp}\ C_E^{\perp} \qquad \frac{\Gamma, x : A \vdash s \equiv s' : \perp}{\Gamma \vdash \lambda x s \equiv \lambda x s' : A^{\perp}}\ C_I^{\perp}$$

$$\frac{\Delta \vdash t \equiv t' : \Diamond A \quad \Pi[\langle y : A \rangle] \vdash s \equiv s' : \Sigma}{\Pi[\Delta] \vdash \langle t \mid \Diamond y.s \rangle \equiv \langle t' \mid \Diamond y.s' \rangle : \Sigma}\ C_E^{\Diamond} \quad \frac{\Gamma \vdash s \equiv s' : A}{\Gamma \vdash \Diamond s \equiv \Diamond s' : \Diamond A}\ C_I^{\Diamond}$$

$$\frac{\Delta \vdash t \equiv t' : A \otimes B \quad \Pi[(y : A \bullet_i z : B)] \vdash s \equiv s' : \Sigma}{\Pi[\Delta] \vdash \langle t \mid (y \otimes_i z).s \rangle \equiv \langle t' \mid (y \otimes_i z).s' \rangle : \Sigma}\ C_E^{\otimes_i}$$

$$\frac{\Gamma \vdash s \equiv s' : A \quad \Delta \vdash t \equiv t' : B}{\Gamma \bullet_i \Delta \vdash (s \otimes_i t) \equiv (s' \otimes_i t') : A \otimes B}\ C_I^{\otimes_i}$$

β-equivalences

$$\frac{\Delta, y : A \vdash s : \perp \quad \Gamma \vdash t : A}{\Gamma, \Delta \vdash (\lambda y s\ t) \equiv s[t/x] : \perp}\ \beta_{\perp} \quad \frac{\Delta \vdash t : A \quad \Pi[\langle y : A \rangle] \vdash s : \Sigma}{\Pi[\langle \Delta \rangle] \vdash \langle \Diamond t \mid \Diamond y.s \rangle \equiv s[t/y] : \Sigma}\ \beta_{\Diamond}$$

$$\frac{\Delta_1 \vdash t_1 : A \quad \Delta_2 \vdash t_2 : B \quad \Pi[(y : A \bullet_i z : B)] \vdash s : \Sigma}{\Pi[(\Delta_1 \bullet_i \Delta_2)] \vdash \langle (t_1 \otimes_i t_2) \mid (y \otimes_i z).s \rangle \equiv s[t_1/y][t_2/z] : \Sigma}\ \beta_{\otimes_i}$$

η-equivalences

$$\frac{\Gamma \vdash s : A^{\perp}}{\Gamma \vdash \lambda y (s\ y) \equiv s : A^{\perp}}\ \eta_{\perp} \quad \frac{\Gamma \vdash s : \Diamond A}{\Gamma \vdash \langle s \mid \Diamond y.y \rangle \equiv s : \Diamond A}\ \eta_{\Diamond}$$

$$\frac{\Gamma \vdash s : A \otimes_i B}{\Gamma \vdash \langle s \mid (y \otimes_i z).(y \otimes_i z) \rangle \equiv s : A \otimes_i B}\ \eta_{\otimes_i}$$

γ-equivalences

$$\frac{\Delta \vdash t : \Diamond A \quad \Gamma[\langle y : A \rangle] \vdash s : B \quad \Pi[x : B] \vdash s' : \Sigma}{\Pi[\Gamma[\Delta]] \vdash s'[\langle t \mid \Diamond y.s \rangle / x] \equiv \langle t \mid \Diamond y.s'[s/x] \rangle : \Sigma}\ \gamma_{\Diamond}$$

$$\frac{\Delta \vdash t : A \otimes_i B \quad \Gamma[(y : A \bullet_i z : B)] \vdash s : B \quad \Pi[x : B] \vdash s' : \Sigma}{\Pi[\Gamma[\Delta]] \vdash s'[\langle t \mid (y \otimes_i z).s \rangle / x] \equiv \langle t \mid (y \otimes_i z).s'[s/x] \rangle : \Sigma}\ \gamma_{\otimes_i}$$

Figure 7.5: Defining an equivalence relation on terms.

The previous definition introduces additional γ-equivalences to handle permutations of matching constructs. Their successfulness at doing so, however, is only partial, with coverage of E.22 still left wanting. We return to this issue in the next section, for now focussing on proving Cut admissibility. Given $\Pi \vdash^{nf} s : \Sigma$, we shall often speak of some t for which $t \equiv s$, meaning $\Pi \vdash s \equiv t : \Sigma$, as justified by L.74.

Lemma 75. If $\Theta \vdash^{nf} t : C$ and $\Pi[u : C] \vdash^{nf} s : \Sigma$, also $\Pi[\Theta] \vdash^{nf} t' : \Sigma$ for some $t' \equiv s[t/u]$.

Proof. By induction on the number of connectives found in the conclusion and on the lengths of the derivations for the premises. We consider five cases: one of the premises corresponds to an axiom, one of the premises instantiates a structural rule (i.e., a display postulate or Grishin interaction), the Cut formula does not occur principal in one of the premises (constituting two separate cases), and the Cut formula occurs principal in both premises. For the base case, note that the term found in the conclusion is identical to that found in the other premise. The cases corresponding to structural rules are equally trivial, being handled by simple permutations. This leaves us to check

1. The Cut formula does not occur principal in the right premise. Depending on the latter's inference, we have six logical rules to consider, each handled by a simple permutation. First, consider $({}^{\perp}L)$:

$$\frac{\Gamma[u : C] \vdash^{nf} s : A}{\Gamma[u : C], x : A^{\perp} \vdash^{nf} (x\ s) : \perp}\ {}^{\perp}L$$

By induction hypothesis, there exists $s'' \equiv s[t/u]$ s.t. $\Gamma[\Theta] \vdash^{nf} s'' : A$. We then put $s' = (x\ s'')$, where $s' \equiv (x\ s)[t/x]$ by congruence:

$$\frac{\dfrac{\Theta \vdash^{nf} t : C \quad \Gamma[u : C] \vdash^{nf} s : A}{\Gamma[\Theta] \vdash^{nf} s'' : A}\ IH}{\Gamma[\Theta], x : A \vdash^{nf} (x\ s'') : \perp}\ {}^{\perp}L$$

Next, $({}^{\perp}R)$.

$$\frac{\Gamma[u : C], x : A \vdash^{nf} s : \perp}{\Gamma[u : C] \vdash^{nf} \lambda x s : A^{\perp}}\ {}^{\perp}R$$

By induction hypothesis, there exists $s'' \equiv s[t/u]$ s.t. $\Gamma[\Theta], x : A \vdash^{nf} s'' : \perp$, and we put $s' = \lambda x s''$.

$$\frac{\dfrac{\Theta \vdash^{nf} t : C \quad \Gamma[u : C], x : A \vdash^{nf} s : \perp}{\Gamma[\Theta], x : A \vdash^{nf} s'' : \perp}\ IH}{\Gamma[\Theta] \vdash^{nf} \lambda x s'' : A^{\perp}}\ {}^{\perp}R$$

Next, $(\otimes_i L)$. W.l.o.g., we can assume the following instantiation:

$$\frac{\Pi[(u : C \bullet_j (y : A \bullet_i z : B))] \vdash^{nf} s : \Sigma}{\Pi[(u : C \bullet_j x : A \otimes_i B)] \vdash^{nf} \langle x \mid (y \otimes_i z).s\rangle : \Sigma} \otimes_i L$$

By induction hypothesis, there exists $s'' \equiv s[t/u]$ s.t. $\Pi[(\Theta \bullet_j (y : A \bullet_i z : B))] \vdash^{nf} s'' : \Sigma$, and we put $s' = \langle x \mid (y \otimes_i z).s''\rangle$.

$$\frac{\dfrac{\Theta \vdash^{nf} t : C \quad \Pi[(u : C \bullet_j (y : A \bullet_i z : B))] \vdash^{nf} s : \Sigma}{\Pi[(\Theta \bullet_j (y : A \bullet_i z : B))] \vdash^{nf} s'' : \Sigma} \mathit{IH}}{\Pi[(\Theta \bullet_j x : A \otimes_i B)] \vdash^{nf} \langle x \mid (y \otimes_i z).s''\rangle : \Sigma} \otimes L$$

Finally, $(\otimes_i R)$.

$$\frac{\Gamma \vdash^{nf} s_1 : A \quad \Delta \vdash^{nf} s_2 : B}{\Gamma \bullet_i \Delta \vdash^{nf} (s_1 \otimes_i s_2) : A \otimes_i B} \otimes_i R$$

Either $u : C$ occurs in Γ or in Δ. Assume the former situation, the other being handled similarly. Then $\Gamma = \Gamma'[u : C]$, and we have, by induction hypothesis, some $s'' \equiv s_1[t/u]$ s.t. $\Gamma'[\Theta] \vdash^{nf} s'' : A$, and we put $s' = (s'' \otimes_i s_2)$.

$$\frac{\dfrac{\Theta \vdash^{nf} t : C \quad \Gamma'[u : C] \vdash^{nf} s_1 : A}{\Gamma'[\Theta] \vdash^{nf} s'' : A} \mathit{IH} \quad \Delta \vdash^{nf} s_2 : B}{\Gamma'[\Theta] \bullet_i \Delta \vdash^{nf} (s'' \otimes_i s_2) : A \otimes B} \otimes_i R$$

The cases for $\Diamond$ are treated similarly to those for $\otimes_i$.

2. The Cut formula does not occur principal in the left premise. We have only two candidates, corresponding to $(\Diamond L)$ and $(\otimes_i L)$. We check only the latter, the former being treated similarly. Thus, we have the following two premises:

$$\frac{\Gamma[(y : A \bullet_i z : B)] \vdash^{nf} t : C}{\Gamma[x : A \otimes_i Bi] \vdash^{nf} \langle x \mid (y \otimes_i z).t\rangle : C} \otimes_i L \qquad \Pi[u : C] \vdash^{nf} s : \Sigma$$

By induction hypothesis, there exists $s'' \equiv s[t/u]$ s.t. $\Pi[\Gamma[(y : A \bullet_i z : B)]] \vdash^{nf} s'' : \Sigma$, and we put $s' = \langle x \mid (y \otimes_i z).s''\rangle$. Note $s' \equiv s[\langle x \mid (y \otimes_i z).t\rangle/u]$ by congruence and γ-equivalence.

$$\frac{\dfrac{\Gamma[(y : A \bullet_i z : B)] \vdash^{nf} t : C \quad \Pi[u : C] \vdash^{nf} s : \Sigma}{\Pi[\Gamma[(y : A \bullet_i z : B)]] \vdash^{nf} s'' : \Sigma} \mathit{IH}}{\Pi[\Gamma[x : A \otimes_i B]] \vdash^{nf} \langle x \mid (y \otimes_i z).s''\rangle : \Sigma} \otimes_i L$$

3. The Cut formula is principal in both premises. We replace with Cuts on the immediate subformulas using β-equivalences. We have four cases to check, one for each type-forming operation. In the case of a Cut formula $A^\perp$, we have

$$\frac{\Delta, x : A \vdash^{nf} s : \bot}{\Delta \vdash^{nf} \lambda xs : A^\perp}\ {}^\perp R \qquad \frac{\Gamma \vdash^{nf} t : A}{\Gamma, x : A^\perp \vdash^{nf} (x\ t) : \bot}\ {}^\perp L$$

By induction hypothesis, there exists $s'' \equiv s[t/x]$ s.t. $\Delta, \Gamma \vdash^{nf} s'' : \bot$, and we put $s' = s''$. Clearly, $s' \equiv (\lambda xs\ t)$ by β-equivalence.

$$\frac{\dfrac{\Gamma \vdash^{nf} t : A \quad \Delta, x : A \vdash^{nf} s : \bot}{\Delta, \Gamma \vdash^{nf} s'' : \bot}\ \mathit{IH}}{\Gamma, \Delta \vdash^{nf} s'' : \bot}\ \mathit{dp}$$

Next, $A \otimes B$, with $\Diamond A$ receiving the same treatment as usual.

$$\frac{\Delta_1 \vdash^{nf} t_1 : A \quad \Delta_2 \vdash^{nf} t_2 : B}{\Delta_1 \bullet_i \Delta_2 \vdash^{nf} (t_1 \otimes_i t_2) : A \otimes_i B}\ \otimes_i R$$

$$\frac{\Pi[(y : A \bullet_i z : B)] \vdash^{nf} s : \Sigma}{\Pi[x : A \otimes_i B] \vdash^{nf} \langle x \mid (y \otimes_i z).s\rangle : \Sigma}\ \otimes_i L$$

By induction hypothesis, there exist $s_1'' \equiv s[t_1/y]$ and $s_2'' \equiv s_1''[t_2/z]$ s.t. $\Pi[(\Delta_1 \bullet_i z : B)] \vdash^{nf} s_1'' : \Sigma$, resp. $\Pi[(\Delta_1 \bullet_i \Delta_2)] \vdash^{nf} s_1'' : \Sigma$, and we put $s' = s_2''$. Clearly, $s' \equiv \langle (t_1 \otimes_i t_2) \mid (y \otimes_i z).s\rangle$ by β-equivalence.

$$\frac{\Delta_2 \vdash^{nf} t_2 : B \qquad \dfrac{\Delta_1 \vdash^{nf} t_1 : A \quad \Pi[(y : A \bullet_i z : B)] \vdash^{nf} s : \Sigma}{\Pi[(\Delta_1 \bullet_i z : B)] \vdash^{nf} s_1'' : \Sigma}\ \mathit{IH}}{\Pi[(\Delta_1 \bullet_i \Delta_2)] \vdash^{nf} s_2'' : \Sigma}\ \mathit{IH}$$

□

Lemma 76. For any A and x, there exists $s \equiv x$ s.t. $x : A \vdash^{nf} s : A$.

Proof. By induction on A. The base case is immediate. Next, $A = B^\perp$. By induction hypthesis, $y : B \vdash^{nf} s' : B$ for some $s' \equiv y$. We put $s = \lambda y(x\ s')$, where $s \equiv x$ by η-equivalence and congruence.

$$\frac{\dfrac{\dfrac{\overline{y : B \vdash^{nf} s' : B}\ \mathit{IH}}{y : B, x : B^\perp \vdash^{nf} (x\ s') : \bot}\ {}^\perp L}{x : B^\perp, y : B \vdash^{nf} (x\ s') : \bot}\ \mathit{dp}}{x : B^\perp \vdash^{nf} \lambda y(x\ s') : B^\perp}\ {}^\perp R$$

We next check $A = B \otimes_i C$, with $A = \Diamond B$ being similar. We have, by induction hypothesis, that $y : B \vdash^{nf} s_1 : B$ and $z : C \vdash^{nf} s_2 : C$ for some $s_1 \equiv y$ and $s_2 \equiv z$. We put $s = \langle x \mid (y \otimes_i z).(s_1 \otimes_i s_2)\rangle$, where $s \equiv x$ by η-equivalences and congruence.

$$\dfrac{\dfrac{\dfrac{}{y : B \vdash^{nf} s_1 : B}\ \mathit{IH} \qquad \dfrac{}{z : C \vdash^{nf} s_2 : C}\ \mathit{IH}}{y : B \bullet_i z : C \vdash^{nf} B \otimes_i C : (s_1 \otimes_i s_2)}\ \otimes_i R}{x : B \otimes_i C \vdash^{nf} \langle x \mid (y \otimes_i z).(s_1 \otimes_i s_2)\rangle : B \otimes_i C}\ \otimes_i L$$ □

We conclude with the statement and proof of the desired normalization result.

Theorem 7.5.1. If $\Pi \vdash s : \Sigma$, then there exists $t \equiv s$ s.t. $\Pi \vdash^{nf} t : \Sigma$.

Proof. By induction. Axioms are handled by the previous lemma, while structural rules and introductions are all immediate. This leaves us with the elimination rules, treated by retrieving the lost premise using the previous theorem. For example, consider the case $(\otimes_i E)$.

$$\dfrac{\Delta \vdash t : A \otimes_i B \quad \Pi[(y : A \bullet_i z : B)] \vdash s : \Sigma}{\Pi[\Delta] \vdash \langle t \mid (y \otimes_i z).s\rangle : \Sigma}\ \otimes_i E$$

By induction hypothesis, there exist $t' \equiv t$ and $s' \equiv s$ s.t. $\Delta \vdash^{nf} t' : A \otimes_i B$ and $\Pi[(y : A \bullet_i z : B)] \vdash^{nf} s' : \Sigma$. Hence, $\Pi[x : A \otimes_i B] \vdash^{nf} \langle x \mid (y \otimes_i z).s'\rangle : \Sigma$ by $(\otimes_i L)$, and we can apply L.75 to obtain $s'' \equiv \langle t' \mid (y \otimes_i z).s'\rangle$ for which $\Pi[\Delta] \vdash^{nf} s'' : \Sigma$. Clearly, $s'' \equiv \langle t \mid (y \otimes_i z).s\rangle$ by congruence. □

7.5.2 Normalization by evaluation

While the normalization procedure just described incorporates all the usual β- and η-equivalences, to ensure Cut admissibility respects $\equiv$ we had to postulate additional γ-equivalences, identifying, to a certain extent, rule permutations involving $(\otimes_i E)$ and $(\Diamond E)$. Note, however, that still not all the desired identifications were made, a notable exception being found in E.22. We settle this issue describing precisely the types of rule permutations for which we see no reason to distinguish between the associated proof terms, following up by providing canonical representatives for derivations differing solely by said permutations. We proceed from judgement forms $\Pi \vdash^{nf} s : \Sigma$, these being the ones we claim to possess shortcomings in describing the notion of normal form. Disregarding $(\cdot^{\perp} L)$ and $(\cdot^{\perp} R)$ for the moment, we have the following cases to consider:

1. Permutations between left- and right introductions. For example, given

$$\dfrac{\dfrac{\Gamma[(y : A \bullet_i z : B)] \vdash s : C}{\Gamma[x : A \otimes_i B] \vdash \langle x \mid (y \otimes_i z).s\rangle : C}\otimes_i L \quad \Delta \vdash t : D}{\Gamma[x : A \otimes_i B] \bullet_j \Delta \vdash (\langle x \mid (y \otimes_i z).s\rangle \otimes_j t) : C \otimes_j D}\otimes_j R$$

   there seems no reason to make the distinction with

$$\dfrac{\dfrac{\Gamma[(y : A \bullet_i z : B)] \vdash s : C \quad \Delta \vdash t : D}{\Gamma[(y : A \bullet_i z : B)] \bullet_j \Delta \vdash (s \otimes_j t) : C \otimes_j D}\otimes_j R}{\Gamma[x : A \otimes_i B] \bullet_j \Delta \vdash \langle x \mid (y \otimes_i z).(s \otimes_j t)\rangle : C \otimes_j D}\otimes_i L$$

2. Permutations between two left introductions, e.g., E.22. Recast into judgement forms $\Pi \vdash^{nf} s : \Sigma$, we see no reason to distinguish between

$$\dfrac{\dfrac{\langle x : A\rangle, \langle y : B\rangle \vdash s : \bot}{\langle x : A\rangle, v : \Diamond B \vdash \langle v \mid \Diamond y.s\rangle : \bot}\Diamond L}{u : \Diamond A, v : \Diamond B \vdash \langle u \mid \Diamond x.\langle v \mid \Diamond y.s\rangle\rangle : \bot}\Diamond L$$

   and

$$\dfrac{\dfrac{\langle x : A\rangle, \langle y : B\rangle \vdash s : \bot}{u : \Diamond A, \langle y : B\rangle \vdash \langle u \mid \Diamond x.s\rangle : \bot}\Diamond L}{u : \Diamond A, v : \Diamond B \vdash \langle v \mid \Diamond y.\langle u \mid \Diamond x.s\rangle\rangle : \bot}\Diamond L$$

Note that permutations between two right introductions do not arise by virtue of Σ having been restricted to containing at most one formula. Furthermore, the rules $(\cdot^{\bot}L)$ and $(\cdot^{\bot}R)$ take a special position, in that they provide the sole means of crossing the turnstile. Both, however, still freely permute with $(\otimes_i L)$ and $(\Diamond L)$.

The issue of inessential rule permutability received extensive coverage in literature on backward-chaining Cut-free proof search, i.e., when attempting to reason from the desired conclusion back to the axioms. Of particular interest we consider Andreoli's [1992] proposals, attempting the further 'normalization' of Cut-free sequent derivations in full linear logic by imposing canonical rule orderings to avoid unwanted permutability of inferences. Adapted to the problems we are currently facing, the first issue addressed above is settled by always applying $(\otimes_i L)$ and $(\Diamond L)$ as soon as possible, taking a backward chaining perspective. Again, justification derives from proof search: since the rules in question are invertible (as demonstrated immediately below for $\otimes_i$), no information is lost by their application.

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}V \quad \dfrac{}{z : B \vdash z : B}V}{y : A \bullet_i z : B \vdash^{nf} (y \otimes_i z) : A \otimes_i B}\otimes_i R \quad \Pi[x : A \otimes_i B] \vdash^{nf} s : \Sigma}{\exists t \equiv s[(y \otimes_i z)/x], \Pi[(y : A \bullet_i z : B)] \vdash^{nf} t : \Sigma}\text{L.75}$$

The second issue is settled by grouping together successive applications of $(\otimes_i L)$ and $(\diamond L)$ into a single inference step, applying them all at once. The resulting derivations are referred to as *strongly focalized*, adopting terminology from Laurent [2004]. The classification *weakly* focalized, from the same source, may in retrospect be applied to judgement forms $\Pi \vdash^{nf} s : \Sigma$ by virtue of lacking rule permutations between inferences that are not in general invertible.

We proceed as follows. First, §5.2.1 defines a judgement form adhering to the above specification of strong focalization, together with a translation back into 'weakly' focalized derivations. §5.2.2 proves the desired normalization result by appealing to an intuitionistic metalanguage to extract the constructive content of soundness and completeness results w.r.t. phase models, the former shown for weakly focalized derivations, the latter for its strongly focalized counterparts. As noted in §1, this strategy is more generally referred to by 'normalization by evaluation'. Its origins can be traced back to Martin-Löf [1975], and has been further developed by Berger and Schwichtenberg [1991], Coquand and Dybjer [1997], among others. Coquand [1993] particularly clarifies the connection with soundness and completeness results for intuitionistic logic w.r.t. Kripke frames.

Strong focalization

The key difference between weak and strong focalization is that, from a backward chaining perspective, invertible rules are applied as soon as possible in one full swoop. The next definition provides the necessary ingredients for decomposing a left occurrence of a formula into its structural equivalent, compiling away the effects of successive individual applications of $(\otimes_i L)$ and $(\diamond L)$.

Definition 112. For each A, $\|A\|$ decomposes A into its structural counterpart.

$$\begin{array}{rcrl} \|p\| := p & \quad & \|A^{\perp}\| := A^{\perp} \\ \|\diamond A\| := \langle \|A\| \rangle & \quad & \|A \otimes_i B\| := (\|A\| \bullet_i \|B\|) \end{array}$$

In defining strong focalization we shall omit term annotations, leaving the proof terms to be recovered back into judgement forms $\Pi \vdash^{nf} s : \Sigma$.

Definition 113. F.7.6 defines, by mutual induction, judgement forms $\Gamma, \Delta \vdash^{Nf}$ and $\Delta \vdash^{Nf} \Gamma$, where it is understood that Γ and Δ contain only formulas of the shapes p or $A^{\perp}$. More specifically, $\Gamma, \Delta ::= p \mid A^{\perp} \mid (\Gamma \bullet_i \Delta) \mid \langle \Gamma \rangle$.

We make the following observations regarding the above definition.

$$\frac{}{p \vdash^{Nf} p}\ Id$$

$$\frac{\Gamma \vdash^{Nf} \|A\|}{\Gamma, A^{\perp} \vdash^{Nf}}\ {}^{\perp}L \qquad \frac{\Gamma, \|A\| \vdash^{Nf}}{\Gamma \vdash^{Nf} A^{\perp}}\ {}^{\perp}R$$

$$\frac{\Delta \vdash^{Nf} \Gamma}{\langle\Delta\rangle \vdash^{Nf} \langle\Gamma\rangle}\ \langle\rangle \qquad \frac{\Gamma' \vdash^{Nf} \Gamma \quad \Delta' \vdash^{Nf} \Delta}{\Gamma' \bullet_i \Delta' \vdash^{Nf} \Gamma \bullet_i \Delta}\ \bullet_i$$

Figure 7.6: Defining strong focalization. Display postulates and Grishin interactions remain unchanged from F.7.3.

1. Left introductions of $\otimes_i$ and $\Diamond$ are compiled away into $({}^{\perp}R)$, using the map $\|\cdot\|$ from D.112. Furthermore, by restricting formulas on the left hand side to either atoms or negations, we can guarantee that invertible inferences are always applied as soon as they become available.
2. The map $\|\cdot\|$ plays a double role in also subsuming the right introductions of $\otimes_i$ and $\Diamond$, replacing them by inferences $(\langle\rangle)$ and $(\bullet_i)$.

We next translate strongly focalized derivations back into their weak counterparts, in so doing again omitting the term labeling in the target, being easily reconstructed.

Lemma 77. For any Γ, A, $\Gamma, \|A\| \vdash^{nf}$ implies $\Gamma, A \vdash^{nf}$.

Proof. By induction on A. If $A = p$ or $A = \neg B$, $\|A\| = A$ and the desired result is immediate. If $A = \Diamond B$ or $A = B \otimes_i C$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \langle\|B\|\rangle \vdash^{nf}}{\langle\Gamma\rangle, \|B\| \vdash^{nf}}\ dp}{\langle\Gamma\rangle, B \vdash^{nf}}\ IH}{\Gamma, \langle B\rangle \vdash^{nf}}\ dp}{\Gamma, \Diamond B \vdash^{nf}}\ \Diamond L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \|B\| \bullet_i \|C\| \vdash^{nf}}{(\|B\| \bullet_i [\,]) \div \Gamma, \|C\| \vdash^{nf}}\ dp}{(\|B\| \bullet_i [\,]) \div \Gamma, C \vdash^{nf}}\ IH}{([\,] \bullet_i C) \div \Gamma, \|B\| \vdash^{nf}}\ dp}{([\,] \bullet_i C) \div \Gamma, B \vdash^{nf}}\ IH}{\Gamma, B \bullet_i C \vdash^{nf}}\ dp}{\Gamma, B \otimes_i C \vdash^{nf}}\ \otimes_i L$$

□

Definition 114. Define, by induction, the formula Γ^f corresponding to a structure Γ, as follows: $(\Gamma \bullet_i \Delta)^f = \Gamma^f \otimes_i \Delta^f$ and $\langle\Gamma\rangle^f = \Diamond \Gamma^f$.

Theorem 7.5.2. $\Gamma, \Delta \vdash^{Nf}$ and $\Delta \vdash^{Nf} \Gamma$ imply, respectively, $\Gamma, \Delta \vdash^{nf}$ and $\Gamma \vdash^{nf} \Delta^{f}$.

Proof. By induction. The only non-trivial case is $({}^{\perp}R)$, being handled by composing $({}^{\perp}R)$ in the target with the contents of L.77. □

Phase models and soundness

We next proceed to define phase models, and to prove their soundness and completeness w.r.t. weak- and strongly focalized derivations respectively. By composing the two results, we obtain the desired normalization result. By furthermore ensuring no appeal is made to classical principles, we guarantee that an actual algorithm can be extracted from our proof. We begin by the stepwise definition of a model.

Definition 115. A *phase space* is a 6-tuple $\langle P, \bullet_1, \bullet_2, \bullet_3, \langle\rangle, \perp\rangle$ where:

1. P is a non-empty set of *phases* with operations $\bullet_1, \bullet_2, \bullet_3 : P^2 \to P$ and $\langle\rangle : P \to P$. We use metavariables x, y, z for denoting elements of P and A, B, C for denoting subsets of P.

2. $\perp \subseteq P \times P$ s.t.

$$\begin{aligned} \langle x, y\rangle \in \perp &\Rightarrow \langle y, x\rangle \in \perp \\ \langle \langle x\rangle, y\rangle \in \perp &\Leftrightarrow \langle x, \langle y\rangle\rangle \in \perp \\ \langle x \bullet_2 y, z\rangle \in \perp &\Leftrightarrow \langle x, y \bullet_1 z\rangle \in \perp \\ \langle x \bullet_1 y, z\rangle \in \perp &\Leftrightarrow \langle x, y \bullet_3 z\rangle \in \perp \end{aligned}$$

3. A phase space may be required to satisfy further conditions depending on which of the Grishin interactions are added to the base logic. For type I, we require

$$\begin{aligned} \langle x_2 \bullet_2 y_2, y_1 \bullet_2 x_1\rangle \in \perp &\Rightarrow \langle x_1 \bullet_1 x_2, y_2 \bullet_1 y_1\rangle \in \perp \quad (A_I^1) \\ \langle y_1 \bullet_3 x_1, x_2 \bullet_3 y_2\rangle \in \perp &\Rightarrow \langle x_1 \bullet_1 x_2, y_2 \bullet_1 y_1\rangle \in \perp \quad (A_I^2) \\ \langle y_1 \bullet_3 x_2, y_2 \bullet_2 x_1\rangle \in \perp &\Rightarrow \langle x_1 \bullet_1 x_2, y_2 \bullet_1 y_1\rangle \in \perp \quad (C_I) \end{aligned}$$

   while for type IV,

$$\begin{aligned} \langle x_1 \bullet_1 x_2, y_2 \bullet_1 y_1\rangle \in \perp &\Rightarrow \langle x_2 \bullet_2 y_2, y_1 \bullet_2 x_1\rangle \in \perp \quad (A_{IV}^1) \\ \langle x_1 \bullet_1 x_2, y_2 \bullet_1 y_1\rangle \in \perp &\Rightarrow \langle y_1 \bullet_3 x_1, x_2 \bullet_3 y_2\rangle \in \perp \quad (A_{IV}^2) \\ \langle x_1 \bullet_1 x_2, y_2 \bullet_1 y_1\rangle \in \perp &\Rightarrow \langle y_1 \bullet_3 x_2, y_2 \bullet_2 x_1\rangle \in \perp \quad (C_{IV}) \end{aligned}$$

As usual, we often identify a phase space by its carrier set P. Given a phase space P, we define $\cdot^{\perp} : \mathscr{P}(P) \to \mathscr{P}(P)$ by mapping $A \subseteq P$ to $\{x \mid (\forall y \in A)(\langle x, y\rangle \in \perp)\}$.

The following are some easy observations on the operation $\cdot^{\perp}$ on phase spaces.

Lemma 78. Given P, we have $A \subseteq B^\perp$ iff $B \subseteq A^\perp$ ($A, B \in \mathscr{P}(P)$). Equivalently, $A \subseteq A^{\perp\perp}$, $A \subseteq B$ implies $B^\perp \subseteq A^\perp$ and $A^{\perp\perp\perp} \subseteq A^\perp$. In other words, $\cdot^\perp$ is a *Galois connection*, and hence $\cdot^{\perp\perp}$ a closure operator, meaning (at the cost of some redundancy), $A \subseteq A^{\perp\perp}$, $A \subseteq B$ implies $A^{\perp\perp} \subseteq B^{\perp\perp}$, $(A^{\perp\perp})^{\perp\perp} \subseteq A^{\perp\perp}$.

Formulas will be interpreted by *facts*: subsets $A \subseteq P$ s.t. $A = A^{\perp\perp}$.

Definition 116. A *model* consists of a phase space P and a valuation v taking positive atoms p into facts. v extends inductively to a map $[\![\cdot]\!]$ accepting arbitrary formulas, as follows:

$$\begin{array}{rr} [\![p]\!] := v(p) & [\![A^\perp]\!] := [\![A]\!]^\perp \\ [\![\Diamond A]\!] := \Diamond[\![A]\!] & [\![A \otimes_i B]\!] := [\![A]\!] \otimes_i [\![B]\!] \end{array}$$

We have employed the following operations, evidently facts by L.78:

$$\begin{array}{rl} \otimes_i : \mathscr{P}(P)^2 \to \mathscr{P}(P), & \langle A, B\rangle \mapsto \{(x \bullet_i y) \mid x \in A^\perp,\ y \in B^\perp\}^\perp \\ \Diamond : \mathscr{P}(P) \to \mathscr{P}(P), & A \mapsto \{\langle x\rangle \mid x \in A^\perp\}^\perp \end{array}$$

At the level of structures, $[\![\cdot]\!]$ is extended as follows:

$$[\![\Gamma \bullet_i \Delta]\!] := [\![\Gamma]\!] \otimes_i [\![\Delta]\!] \qquad [\![\langle\Gamma\rangle]\!] := \Diamond\, [\![\Gamma]\!]$$

Remark 14. It may be instructive to consider a reformulation of the above definition of a model for formulas with explicit shifts:

$$\begin{array}{rcll} P, Q & ::= & p \mid (P \otimes_i Q) \mid \Diamond P \mid \downarrow M & (i \in \{1,2,3\}) \\ M, N & ::= & \bar{p} \mid (M \oplus_i N) \mid \Box M \mid \uparrow P & (i \in \{1,2,3\}) \end{array}$$

We now have a pair of interpretation functions $[\![\cdot]\!]_+$ and $[\![\cdot]\!]_-$, acting on positive and negative formulas respectively, and defined by mutual induction.

$$\begin{array}{rclrcl} [\![p]\!]_+ & := & v(p) & [\![\bar{p}]\!]_- & := & v(p) \\ [\![\Diamond P]\!]_+ & := & \Diamond[\![P]\!]_+ & [\![\Box M]\!]_- & := & \Diamond[\![M]\!]_- \\ [\![P \otimes_i Q]\!]_+ & := & [\![P]\!]_+ \otimes_i [\![Q]\!]_+ & [\![M \oplus_i N]\!]_- & := & [\![N]\!]_- \otimes_i [\![M]\!]_- \\ [\![\downarrow N]\!]_+ & := & [\![N]\!]_-^\perp & [\![\uparrow P]\!]_- & := & [\![P]\!]_+^\perp \end{array}$$

Note that negative formulas are interpreted by reference to their De Morgan duals. Furthermore, one easily shows $[\![P]\!]_+ = [\![P^\perp]\!]_-$ and (dually) $[\![N]\!]_- = [\![N^\perp]\!]_+$.

Lemma 79. For any facts A, B, $A^\perp \subseteq B$ iff $B^\perp \subseteq A$.

Proof. Going from left to right, $A^\perp \subseteq B$ implies $B^\perp \subseteq A^{\perp\perp}$ by L.78, while $A^{\perp\perp} = A$ by virtue of being a fact. The other direction is shown similarly. □

We state and prove soundness.

Theorem 7.5.3. If $\Gamma, \Delta \vdash^{nf}$ ($\Gamma \vdash^{nf} A$), then $[\![\Gamma]\!]^\perp \subseteq [\![\Delta]\!]$ ($[\![A]\!] \subseteq [\![\Gamma]\!]$).

Proof. We proceed by induction, making free use of L.79. The base case is clear. In addition, $(\otimes_i L)$ and $(\Diamond L)$ are immediate from their induction hypotheses, while $({}^\perp L)$ and $({}^\perp R)$ follow directly from L.79. This leaves us to check the following cases.

Case (*dp*). As a typical instance, we check $\Gamma, \Delta \bullet_3 \Theta \vdash^{Nf}$ only if $\Gamma \bullet_1 \Delta, \Theta \vdash^{Nf}$. The following hypotheses will be used:

(*IH*)	$[\![\Gamma]\!]^\perp \subseteq [\![\Delta]\!] \otimes_3 [\![\Theta]\!]$	(b)	$y \in [\![\Gamma]\!]^\perp$
(a)	$x \in [\![\Theta]\!]^\perp$	(c)	$z \in [\![\Delta]\!]^\perp$

(*IH*) being the induction hypothesis. We desire $[\![\Theta]\!]^\perp \subseteq [\![\Gamma]\!] \otimes_1 [\![\Delta]\!] = [\![\Delta]\!]^{\perp\perp}$. So assume (a)-(c). We show $\langle x, y \bullet_1 z\rangle \in \perp$, iff $\langle z \bullet_3 x, y\rangle \in \perp$. By (b) and (*IH*), $y \in \{z \bullet_3 x \mid z \in [\![\Delta]\!]^\perp, x \in [\![\Theta]\!]^\perp\}^\perp$, so we apply (a) and (c).

Case $A^{1/2}_{I/IV}, C^{1/2}_{I/IV}$. As a typical instance, we check (A^1_I), i.e., $\Gamma_1 \bullet_1 \Gamma_2, \Delta_2 \bullet_1 \Delta_1 \vdash^{nf}$ if $\Gamma_2 \bullet_2 \Delta_2, \Delta_1 \bullet_2 \Gamma_1 \vdash^{nf}$. We use the following hypotheses:

(*IH*)	$([\![\Gamma_2]\!] \otimes_2 [\![\Delta_2]\!])^\perp \subseteq [\![\Delta_1]\!] \otimes [\![\Gamma_1]\!]$	(c)	$z \in [\![\Delta_1]\!]^\perp$
(a)	$x \in ([\![\Gamma_1]\!] \otimes_1 [\![\Gamma_2]\!])^\perp$	(d)	$u \in [\![\Gamma_1]\!]^\perp$
(b)	$y \in [\![\Delta_2]\!]^\perp$	(e)	$v \in [\![\Gamma_2]\!]^\perp$

□

we must show $([\![\Gamma_1]\!] \otimes_1 [\![\Gamma_2]\!])^\perp \subseteq [\![\Delta_2]\!] \otimes_1 [\![\Delta_1]\!]$. Thus, we establish $\langle x, y \bullet_1 z\rangle \in \perp$ on the assumptions (a)-(c). By (a), it suffices to show $\langle y \bullet_1 z, u \bullet_1 v\rangle \in \perp$ given (d), (e). By (A^1_I), this further reduces to $\langle z \bullet_2 u, v \bullet_2 y\rangle \in \perp$, iff $\langle v \bullet_2 y, z \bullet_2 u\rangle \in \perp$. By (*IH*), (c) and (d), the desired result follows from $v \bullet_2 y \in ([\![\Gamma_2]\!] \otimes [\![\Delta_2]\!])^\perp$. But this is a consequence of (b), (e) and the fact that $\{v \bullet_2 y \mid v \in [\![\Gamma_2]\!]^\perp, y \in [\![\Delta_2]\!]^\perp\} \subseteq ([\![\Gamma_2]\!] \otimes [\![\Delta_2]\!])^\perp$ by L.78.

Case $(\Diamond R)$. By induction hypothesis, $[\![A]\!] \subseteq [\![\Gamma]\!]$. Hence, $[\![\Gamma]\!]^\perp \subseteq [\![A]\!]^\perp$, and $\{\langle x\rangle \mid x \in [\![A]\!]^\perp\}^\perp \subseteq \{\langle y\rangle \mid y \in [\![\Gamma]\!]^\perp\}^\perp$, as desired.

Case $(\otimes_i R)$. By induction hypothesis, $[\![A]\!] \subseteq [\![\Gamma]\!]$ and $[\![B]\!] \subseteq [\![\Delta]\!]$. Hence, $[\![\Gamma]\!]^\perp \subseteq [\![A]\!]^\perp$ and $[\![\Delta]\!]^\perp \subseteq [\![A]\!]^\perp$, and $\{x \bullet_i y \mid x \in [\![A]\!]^\perp, y \in [\![B]\!]^\perp\}^\perp \subseteq \{u \bullet_i v \mid u \in [\![\Gamma]\!]^\perp, v \in [\![\Delta]\!]^\perp\}$, as desired.

Completeness is shown w.r.t the following concrete model.

Definition 117. Define the *syntactic* model by instantiating P with the set of structures, setting $\langle \Gamma, \Delta \rangle \in \bot$ iff $\Gamma, \Delta \vdash^{Nf}$ and defining $v(p) := \{\Gamma \mid \Gamma, p \vdash^{Nf}\}$.

The following is our central lemma, resembling results of Okada [2002] and Herbelin and Lee [2009] for linear and intuitionistic logic respectively.

Lemma 80. For arbitrary A, Γ, Δ, the syntactic model satisfies:

1. $\Gamma \in [\![A]\!]$ implies $\Gamma, \|A\| \vdash^{Nf}$.
2. If $\Theta, \Delta \vdash^{Nf}$ for every Θ s.t. $\Theta \vdash^{Nf} \|A\|$, then $\Delta \in [\![A]\!]$.

Proof. First, note that if $\Gamma \vdash^{Nf} \|A\|$, also $\Gamma, A^{\bot} \vdash^{Nf}$ by $({}^{\bot}L)$. Hence, (2) implies $A^{\bot} \in [\![A]\!]$. When invoking the induction hypothesis for (2), we often immediately instantiate by the latter consequence. We proceed by simultaneous induction on A.

Case p. Since $\|p\| = p$, both (1) and (2) follow by definition of $v(p)$, noting for (2) that $\Theta \vdash^{Nf} p$ implies $\Theta = p$.

Case $A^{\bot}$. Since $\|A^{\bot}\| = A^{\bot}$, it suffices to show $\Gamma \in [\![A^{\bot}]\!]$ implies $\Gamma, A^{\bot} \vdash^{Nf}$ for (1), and if $\Gamma \vdash^{Nf} A^{\bot}$ implies $\Gamma, \Delta \vdash^{Nf}$, then $\Delta \in [\![A^{\bot}]\!]$.

1. Suppose $\Gamma \in [\![A^{\bot}]\!] = [\![A]\!]^{\bot}$. By IH(2), $A^{\bot} \in [\![A]\!]$, so that $\Gamma, A^{\bot} \vdash^{Nf}$.
2. We show $\Delta \in [\![A^{\bot}]\!] = [\![A]\!]^{\bot}$, assuming (a) $\Gamma \vdash^{Nf} A^{\bot}$ implies $\Gamma, \Delta \vdash^{Nf}$, Γ arb. Letting (b) $\Theta \in [\![A]\!]$, it suffices to ensure $\Delta, \Theta \vdash^{Nf}$. IH(1) and (b) imply $\Theta, \|A\| \vdash^{Nf}$, so $\Theta \vdash^{Nf} A^{\bot}$ by $({}^{\bot}R)$. Thus, $\Theta, \Delta \vdash^{Nf}$ by (a), and we apply $(dp1)$.

Case $\Diamond A$.

1. Let (a) $\Gamma \in \Diamond[\![A]\!]$. We show $\Gamma, \langle \|A\| \rangle \vdash^{Nf}$. By (a), it suffices to ensure $\|A\| \in [\![A]\!]^{\bot}$, i.e., $\Delta, \|A\| \vdash^{Nf}$ for every $\Delta \in [\![A]\!]$. But this holds by IH(1).
2. The following hypotheses are used:

   (a) $\Gamma \vdash^{Nf} \langle \|A\| \rangle$ implies $\Gamma, \Delta \vdash^{Nf}$, for all Γ
   (b) $\Theta \in [\![A]\!]^{\bot}$
   (c) If, for every Γ, $\Gamma \vdash^{Nf} \|A\|$ implies $\Gamma, \langle \Delta \rangle \vdash^{Nf}$, then $\langle \Delta \rangle \in [\![A]\!]$
   (d) $\Gamma' \vdash^{Nf} \|A\|$

   Assuming (a), we show $\Delta \in \Diamond[\![A]\!]$. So let (b). Since $\Delta, \langle \Theta \rangle \vdash^{Nf}$ iff $\Theta, \langle \Delta \rangle \vdash^{Nf}$ by (dp), it suffices by (b) to prove $\langle \Delta \rangle \in [\![A]\!]$. By (c), i.e., IH(2), we need only prove $\Gamma', \langle \Delta \rangle \vdash^{Nf}$ assuming (d), iff $\langle \Gamma' \rangle, \Delta \vdash^{Nf}$. By (a), this follows from $\langle \Gamma' \rangle \vdash^{Nf} \langle \|A\| \rangle$ provided $\langle \Gamma' \rangle \vdash^{Nf} \langle \|A\| \rangle$, shown by applying $(\langle\rangle)$ on (d).

Case $A \otimes_i B$

1. Let (a) $\Gamma \in [\![A]\!] \otimes_i [\![B]\!]$. We show $\Gamma, \|A\| \bullet_i \|B\| \vdash^{Nf}$. By (a), it suffices to ensure $\|A\| \in [\![A]\!]^\perp$ and $\|B\| \in [\![B]\!]^\perp$, which are easy consequences of IH(1).

2. The following hypotheses are used:

   (a) $\Gamma \vdash^{Nf} \|A\| \bullet_i \|B\|$ implies $\Gamma, \Delta \vdash^{Nf}$, for all Γ
   (b) $\Theta_1 \in [\![A]\!]^\perp$
   (c) $\Theta_2 \in [\![B]\!]^\perp$
   (d) If, for every Γ_1, $\Gamma_1 \vdash^{Nf} \|A\|$ implies $\Gamma_1, ([\,] \bullet_i \Theta_2) \div \Delta \vdash^{Nf}$, then $([\,] \bullet_i \Theta_2) \div \Delta \in [\![A]\!]$
   (e) $\Gamma_1 \vdash^{Nf} \|A\|$
   (f) If, for every Γ_2, $\Gamma_2 \vdash^{Nf} \|B\|$ implies $\Gamma_2, (\Gamma_1 \bullet_i [\,]) \div \Delta \vdash^{Nf}$, then $(\Gamma_1 \bullet_i [\,]) \div \Delta \in [\![B]\!]$
   (g) $\Gamma_2 \vdash^{Nf} \|B\|$

   Assuming (a), we show $\Delta \in [\![A]\!] \otimes_i [\![B]\!]$. So let (b) and (c). Since $\Delta, \Theta_1 \bullet_i \Theta_2 \vdash^{Nf}$ iff $\Theta_1, ([\,] \bullet_i \Theta_2) \div \Delta \vdash^{Nf}$ by (dp), it suffices by (b) to prove $([\,] \bullet_i \Theta_2) \div \Delta \in [\![A]\!]$. By (d), i.e., IH(2), we need only prove, on the assumption (e), that $\Gamma_1, ([\,] \bullet_i \Theta_2) \div \Delta \vdash^{Nf}$, iff $\Theta_2, (\Gamma_1 \bullet_i [\,]) \div \Delta \vdash^{Nf}$ by (dp). By (d), it suffices to show $(\Gamma_1 \bullet_i [\,]) \div \Delta \in [\![B]\!]$, which, by (f), i.e., IH(2), follows if we can prove, on the assumption (g), that $\Gamma_2, (\Gamma_1 \bullet_i [\,]) \div \Delta \vdash^{Nf}$, iff $\Gamma_1 \bullet_i \Gamma_2, \Delta \vdash^{Nf}$ by (dp). The latter is a consequence of (a) by applying $(\bullet_i)$ on (e) and (g). □

Lemma 81. For any A, $\|A\| \in [\![A]\!]^\perp$.

Proof. Since $[\![A]\!]^\perp = [\![A^\perp]\!]$, we apply L.80(2) to derive the desired result by showing $\Gamma, \|A\| \vdash^{Nf}$, assuming $\Gamma \vdash^{Nf} A^\perp$ (noting $\|A^\perp\| = A^\perp$). The latter can only have been derived from the former using $({}^\perp R)$, and we are done. □

We state completeness w.r.t. the syntactic model, implying in particular completeness w.r.t. all phase models.

Theorem 7.5.4. For arbitrary A, B, if $[\![A]\!]^\perp \subseteq [\![B]\!]$, then $\|A\|, \|B\| \vdash^{Nf}$.

Proof. By L.81, $\|A\| \in [\![A]\!]^\perp$. Hence, $\|A\| \in [\![B]\!]$, and $\|A\|, \|B\| \vdash^{Nf}$ by L.80. □

Corollary 6. $\Gamma, \Delta \vdash^{nf}$ iff $\|\Gamma^f\|, \|\Delta^f\| \vdash^{Nf}$.

Proof. From left to right, note $\Gamma, \Delta \vdash^{nf}$ implies $\Gamma^f, \Delta^f \vdash^{nf}$ by applications of $(\diamond L)$ and $(\otimes_i L)$, so that the desired result follows by composing soundness with completeness. Going from right to left, we have $\|\Gamma^f\|, \|\Delta^f\| \vdash^{nf}$ by T.7.5.2, the desired result again following by applications of $(\diamond L)$ and $(\otimes_i L)$. □

7.6 From categories to types

§3 saw the definition of a term language, its soundness relative to **LG** having been shown in T.7.4.1. To extend this result to a two-way correspondence, we show the converse direction by using negations to account for the downward monotonicity of the (co)implications. Note, however, that negations in the type language can be iterated indefinitely, so that the number of possible translations in this direction, at least to the extent that composing with $\cdot^{+}$ or $\cdot^{-}$ returns the original, is unbounded. The current section presents three alternatives:

1. The first we cover is an adaptation of the Andreoli/Girard translation (§6.1), having independent origins in literature on proof-search [see Andreoli, 1992] and on constructive interpretations of classical logic [see Girard, 1991]. Its main characteristic is the lack of subtypes $A^{\perp\perp}$ in its image.
2. In addition, we discuss in §6.2 the call-by-name and call-by-value translations previously investigated by Bernardi and Moortgat [2007], adapting those of Curien and Herbelin [2000].

Each translation can be approached through either one of two perspectives. If used to assign a derivational semantics to standard display calculus, they manifest as continuation-passing style translations. This was the point of view adopted, for instance, by Bernardi and Moortgat [2007]. If instead one attempts the modification of sequent derivations with the aim of deriving only terms in their normal forms, the result is a strategy for dealing with irrelevant rule permutations in Cut-free backward-chaining proof search, while at the same time having been used in classical logic to 'tame' the nondeterminism of Cut elimination. These observations, made by Curien and Munch-Maccagnoni [2010], among others, are especially relevant for the dichotomy between the logical and functional programming paradigms, emphasizing, respectively, computation through proof search and through proof normalization. Reconciliation has in particular been suggested recently by Girard [2001] through the concept of Ludics, incorporating many of the themes discussed in the remainder of this section.

7.6.1 Focussing proofs

On permutations and polarities

In §5.2, we already briefly covered focussing and inessential rule permutations in the context of characterizing normal form λ-terms featuring matching constructs. The same phenomenon can be investigated from the perspective of normalizing goal-driven proof search inside the display calculus from chapter 3, pertaining, in particular, to formulas. To understand the problem, we first require a better understanding of the naïve such strategy. As already pointed out in chapter 2, Cut admissibility establishes a local analyticity property: every remaining inference rule has only subformulas of its conclusion appear in its premises. Consequently, the provability of a sequent can be investigated by reasoning backwards towards the axioms. In more detail, we start by nondeterministically selecting a formula from inside the initial goal sequent, applying the inference rule corresponding to its outermost connective and adding any premises as new goals. We succeed if the set of goal sequents is reduced to the empty set, and fail if the latter condition is not met despite all applicable inference rules having been exhausted. In the latter case, we may still attempt backtracking, choosing formulas in a different order or possibly instantiate the applied inference rules differently if context-splitting is non-deterministic. Since the latter process, at least for the types of Lambek calculi studied in the present thesis, cannot continue indefinitely, we indeed fail or succeed in finite time.

While getting the job done, the above procedure still suffers from inessential non-determinism, originating in the free choice of a major formula. In particular, neighboring logical inference steps involving disjoint active and main formulas freely permute, making the choice of their relative ordering meaningless for settling provability. The solutions proposed in the literature roughly fall into two camps.

1. **Proof nets**. Girard [1996] considered the problem as being inherent to the sequentialization of rule applications. Thus, he developed a parallel representation of proofs, called *proof nets*, wherein the trivially permuting inference steps could be untangled and presented side by side.

2. **Focalization**. Andreoli [1992] rather pursued a solution internal to sequent calculus, seeking a canonical rule ordering for deciding between trivial permutations. He referred to his solution as *focalization*: always apply invertible inference rules first whenever available (preserving provability of the conclusion in the premises), while the active formulas appearing in the premises of non-invertible inferences always are to be principal. In other words, once chosen as main, a formula is 'focussed upon' in the sense of fixing the choice for subsequent rule applications to those targeting its subformulas.

Proof nets have already seen widespread application within CTL, starting with [Roorda, 1991]. Lamarche and Retoré [1996] provide an overview, though predating Moot and Puite [2002]; another notable publication in the area. For **LG**, a theory of proof nets was provided by Moot [2007], and was subsequently refined by Moortgat and Moot [2011] to take formal semantics into account. In contrast, focalization has enjoyed less success within CTL, whatever initial productivity having quickly become overshadowed by the growing interest in proof nets. The main results are found in [Hendriks, 1993, Ch.4] and in [Hepple, 1990], neither of which explicitly mention focalization, rather having been discovered independently. Related works are [Moortgat, 1990], deriving a proof search strategy similar to focalization from the partial execution of a PROLOG implementation, and [Morrill, 1995], concerning the formalization of Hendriks' and Hepple's results inside a higher-order linear logic programming language. A more recent contribution is that of Morrill [2011], targeting a fragment of his displacement calculus.

Around the same time as [Andreoli, 1992], Girard [1991] independently published on a similar sequent calculus for classical logic, with the much different aim of restoring the Church-Rosser property for Cut elimination, bypassing the critical pairs of Girard et al. [1989, Appendix B.1].[2] In particular, Girard's results inspired a novel translation into intuitionistic logic, achieving parsimony by making the introduction of negations contingent upon the proof-theoretic behavior of a formula's main connective.

Below, we first provide a more in-depth look into the types of rule permutations addressed by focussing, using the two-sided display calculus from chapter 3. The main focus for the remainder of this subsection will then be on adapting Girard's results to **LG**.

On rules and permutations

We will subsequently parameterize over the following partitioning of formulas, speaking of a positive/negative polarity:[3]

$$\begin{array}{rll} \text{Positive(ly polar):} & p,\, A \otimes B,\, A \oslash B,\, B \obslash A,\, \Diamond A & \text{(Metavariables } P, Q) \\ \text{Negative(ly polar):} & A \oplus B,\, A/B,\, B\backslash A,\, \Box A & \text{(Metavariables } M, N) \end{array}$$

[2] Lafont's critical pair arises in classical logic when both of the Cut formulas have been introduced through weakening. Using a one-sided presentation:

$$\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, A}\, W \quad \dfrac{\begin{array}{c}\mathscr{D}_2\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \neg A}\, W}{\vdash \Gamma}\, \circ$$

We may proceed by keeping either $\mathscr{D}_1$ or $\mathscr{D}_2$, thus disproving confluency. For critical pairs arising in the reduction of principal Cuts, the reader is referred to Danos et al. [1997].

[3] Terminology derives from Girard [1991]. Andreoli [1992] instead spoke of a synchronous/ asynchronous distinction.

While the same terminology was applied before on types, cf. §2, several differences obtain. First, no separate shift connectives are used for recording polarity switches. Second, we have arbitrarily chosen to consider all atoms as being of positive polarity, as opposed to splitting each atom into a positive and negative counterpart.

Expressed in terms of the α/β classification, positive inputs and negative outputs together comprise formulas of type α, while those of type β consist of negative inputs and positive outputs. Proof-theoretically, we observe that the logical inferences associated with the former are always invertible, meaning premises and conclusion may be interchanged, while the same cannot in general be said for formulas of β-type. For example, invertibility of $(\otimes L)$ is witnessed as follows:

$$\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \bullet B \Rightarrow A \otimes B}\,\otimes R \qquad A \otimes B \Rightarrow \Gamma}{A \bullet B \Rightarrow \Gamma}\,\circ$$

Using this classification, we can encounter permutations between rules both introducing an α (1), both introducing a β (2) or mixed (3). Translated to the terminological practice of positive/negative (+/−) inputs/outputs ($^\bullet/^\circ$):

	$(+,^\bullet)$	$(+,^\circ)$	$(-,^\bullet)$	$(-,^\circ)$
$(+,^\bullet)$	1	3	3	1
$(+,^\circ)$	3	2	2	3
$(-,^\bullet)$	3	2	2	3
$(-,^\circ)$	1	3	3	1

Below, each type of permutation is briefly illustrated.

1. Type 1 (α/α). E.g.,

$$\dfrac{\dfrac{A \bullet B \Rightarrow C \leftharpoonup D}{A \bullet B \Rightarrow C/D}\,/R}{A \otimes B \Rightarrow C/D}\,\otimes L \qquad \dfrac{\dfrac{A \bullet B \Rightarrow C \leftharpoonup D}{A \otimes B \Rightarrow C \leftharpoonup D}\,\otimes L}{A \otimes B \Rightarrow C/D}\,/R$$

2. Type 2 (β/β). E.g.,

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta \Rightarrow B \quad A \Rightarrow C}{A/B \Rightarrow C \leftharpoonup \Delta}\,/L}{A/B \bullet \Delta \Rightarrow C}\,dp \qquad D \Rightarrow \Gamma}{C\backslash D \Rightarrow (A/B \bullet \Delta) \rightharpoonup \Gamma}\,\backslash L}{(A/B \bullet \Delta) \bullet C\backslash D \Rightarrow \Gamma}\,dp \qquad \dfrac{\dfrac{\Delta \Rightarrow B \qquad \dfrac{\dfrac{A \Rightarrow C \quad D \Rightarrow \Gamma}{C\backslash D \Rightarrow A \rightharpoonup \Gamma}\,\backslash L}{A \Rightarrow \Gamma \leftharpoonup C\backslash D}\,dp}{A/B \Rightarrow (\Gamma \leftharpoonup C\backslash D) \leftharpoonup \Delta}\,/L}{(A/B \bullet \Delta) \bullet C\backslash D \Rightarrow \Gamma}\,dp$$

3. Type 3 (α/β). E.g.,

$$\dfrac{\dfrac{\dfrac{C \bullet D \Rightarrow B}{C \otimes D \Rightarrow B}\ \otimes L \qquad A \Rightarrow \Gamma}{A/B \Rightarrow \Gamma \leftharpoondown C \otimes D}\ /L}{\overline{\overline{A/B \bullet C \otimes D \Rightarrow \Gamma}}}\ dp \qquad \dfrac{\dfrac{\dfrac{C \bullet D \Rightarrow B \quad A \Rightarrow \Gamma}{A/B \Rightarrow \Gamma \leftharpoondown C \bullet D}\ /L}{\overline{\overline{C \bullet D \Rightarrow A/B \rightharpoondown \Gamma}}}\ dp}{\dfrac{C \otimes D \Rightarrow A/B \rightharpoondown \Gamma}{\overline{\overline{A/B \bullet C \otimes D \Rightarrow \Gamma}}}\ dp}\ \otimes L$$

The sequent calculus we shall define primarily targets the type 2 permutations by requiring the active formulas involved to be principal: once a main formula is chosen, at least stick with it. Said strategy was referred to by Laurent [2004] as *weak* focalization, a classification similarly applicable to Girard's [1991] efforts concerning classical logic. *Full* focalization, as explored by Andreoli [1992], further eliminates the type 1 and 3 permutations by greedily applying the available invertible inferences all at once. Our current interest, however, is in formal semantics, for which weak focalization suffices. Where proof-search is concerned, more efficient strategies than focussing have been proposed, adopting formalisms other than that of sequent calculus. For example, Trautwein and Aarts [1995] proved the polynomiality of **NL**'s implicational fragment through successive transformations of its sequent presentation to better reflect context-free rewrite grammars. Their results were improved upon through the context calculi of de Groote [1999], De Groote and Lamarche [2002], used to deal with the full fragment of (**C**)**NL**.

Translation

We next proceed by defining, in parallel, a weakly focussed display calculus for **LG**, together with its translation back into the term language of §4. To start with, we adapt Girard's translation of formulas, notable for its parsimony in the use of double negations through the parameterization over the formulas' polarities.

Definition 118. The *Andreoli/Girard translation* of formulas A by types $[\![A]\!]$ is defined by parameterizing over A's polarity $\epsilon(A)$. I.e., $\epsilon(A \otimes B) = \epsilon(A \oslash B) = \epsilon(B \varobslash A) = \epsilon(\Diamond A) = +$, while $\epsilon(A \oplus B) = \epsilon(A/B) = \epsilon(B\backslash A) = \epsilon(\Box A) = -$. Finally, $\epsilon(p)$ is arbitrarily fixed by $+$. The desired translation is now as follows. For atoms, we set $[\![p]\!] = p$. For the residuated family of connectives, proceed as follows:

$\epsilon(A)$	$\epsilon(B)$	$[\![A \otimes B]\!]$	$[\![B\backslash A]\!]$	$[\![A/B]\!]$
$+$	$+$	$[\![A]\!] \otimes_1 [\![B]\!]$	$[\![A]\!]^\perp \otimes_2 [\![B]\!]$	$[\![B]\!] \otimes_3 [\![A]\!]^\perp$
$+$	$-$	$[\![A]\!] \otimes_1 [\![B]\!]^\perp$	$[\![A]\!]^\perp \otimes_2 [\![B]\!]^\perp$	$[\![B]\!]^\perp \otimes_3 [\![A]\!]^\perp$
$-$	$+$	$[\![A]\!]^\perp \otimes_1 [\![B]\!]$	$[\![A]\!] \otimes_2 [\![B]\!]$	$[\![B]\!] \otimes_3 [\![A]\!]$
$-$	$-$	$[\![A]\!]^\perp \otimes_1 [\![B]\!]^\perp$	$[\![A]\!] \otimes_2 [\![B]\!]^\perp$	$[\![B]\!]^\perp \otimes_3 [\![A]\!]$

For the coresiduated family, we define

$\epsilon(A)$	$\epsilon(B)$	$[\![A \oplus B]\!]$	$[\![A \oslash B]\!]$	$[\![B \obslash A]\!]$
$+$	$+$	$[\![B]\!]^\perp \otimes_1 [\![A]\!]^\perp$	$[\![A]\!] \otimes_2 [\![B]\!]^\perp$	$[\![B]\!]^\perp \otimes_3 [\![A]\!]$
$+$	$-$	$[\![B]\!]^\perp \otimes_1 [\![A]\!]$	$[\![A]\!] \otimes_2 [\![B]\!]$	$[\![B]\!] \otimes_3 [\![A]\!]$
$-$	$+$	$[\![B]\!] \otimes_1 [\![A]\!]^\perp$	$[\![A]\!]^\perp \otimes_2 [\![B]\!]^\perp$	$[\![B]\!]^\perp \otimes_3 [\![A]\!]^\perp$
$-$	$-$	$[\![B]\!] \otimes_1 [\![A]\!]$	$[\![A]\!]^\perp \otimes_2 [\![B]\!]$	$[\![B]\!] \otimes_3 [\![A]\!]^\perp$

While finally, for the modalities,

$\epsilon(A)$	$[\![\Diamond A]\!]$	$[\![\Box A]\!]$
$+$	$\Diamond[\![A]\!]$	$\Diamond[\![A]\!]^\perp$
$-$	$\Diamond[\![A]\!]^\perp$	$\Diamond[\![A]\!]$

Roughly, a connective expects its polarity to be preserved by an argument when upward monotone, while reversed when downward monotone. Deviations are recorded by marking the offending argument by $^\perp$.

Lemma 82. For any formula A, $[\![A]\!] = [\![A^\infty]\!]$.

Proof. By a long but straightforward induction. The base case is clear. As an example of an inductive case, consider $A = B \otimes C$, with both B, C positive. Then $[\![B \otimes C]\!] = [\![B]\!] \otimes_1 [\![C]\!]$, and since B^∞ and C^∞ are either negative or atomic, $[\![C^\infty \oplus B^\infty]\!] = [\![B^\infty]\!] \otimes_1 [\![C^\infty]\!]$, and we apply the induction hypothesis twice. □

Lemma 83. For any formula A, $A = [\![A]\!]^-$ if $\epsilon(A) = -$, and $A = [\![A]\!]^+$ if $\epsilon(A) = +$. Conversely, for any type A, $A = [\![A^+]\!] = [\![A^-]\!]$.

Proof. By a straightforward but tedious case analysis. □

Structures are as before, their leaves now carrying labeled formulas $x : A$.

Definition 119. The association of types $[\![A]\!]$ with formulas A extends to structures using two maps $[\![\cdot]\!]^\bullet$ and $[\![\cdot]\!]^\circ$, depending on whether a structure occurs as input, resp. output. In the base case, i.e., $x : A$, we parameterize over $\epsilon(A)$, as follows:

$$[\![x : A]\!]^\bullet = \begin{cases} x : [\![A]\!] \text{ if } \epsilon(A) = + \\ x : [\![A]\!]^\perp \text{ if } \epsilon(A) = - \end{cases} \qquad [\![x : A]\!]^\circ = \begin{cases} x : [\![A]\!]^\perp \text{ if } \epsilon(A) = + \\ x : [\![A]\!] \text{ if } \epsilon(A) = - \end{cases}$$

For the inductive cases, we proceed as follows.

$$\begin{array}{ll} [\![(\Gamma \bullet \Delta)]\!]^\bullet := ([\![\Gamma]\!]^\bullet \bullet_1 [\![\Delta]\!]^\bullet) & [\![(\Gamma \bullet \Delta)]\!]^\circ := ([\![\Delta]\!]^\circ \bullet_1 [\![\Gamma]\!]^\circ) \\ [\![(\Gamma \mathbin{\bullet\!\!-} \Delta)]\!]^\bullet := ([\![\Gamma]\!]^\bullet \bullet_2 [\![\Delta]\!]^\circ) & [\![(\Gamma \mathbin{\bullet\!\!-} \Delta)]\!]^\circ := ([\![\Delta]\!]^\bullet \bullet_3 [\![\Gamma]\!]^\circ) \\ [\![(\Delta \mathbin{-\!\!\bullet} \Gamma)]\!]^\bullet := ([\![\Delta]\!]^\circ \bullet_3 [\![\Gamma]\!]^\bullet) & [\![(\Delta \mathbin{-\!\!\bullet} \Gamma)]\!]^\circ := ([\![\Gamma]\!]^\circ \bullet_2 [\![\Delta]\!]^\bullet \end{array}$$

We proceed with adapting our display calculus from chapter 3, now making the explicit distinction between two phases in a derivation depending on whether invertible or non-invertible rules are applied. Compared to full focalization, however, we do not demand exhaustion of all available invertible inferences before a change of phase. Besides the incorporation of the focussing mechanism, derivations will now also be assumed to be term-labeled.

Definition 120. F.7.7 defines a weakly focussed display calculus for **LG**, differentiating between judgement forms $s \, : \, \Gamma \Rightarrow \Delta$, $\boxed{s : A} \Rightarrow \Gamma$ and $\Gamma \Rightarrow \boxed{s : A}$. The latter two together make up the *focussing phase*, while the first defines the *neutral phase.*

Lemma 84. We have the following implications:

1. If $s \, : \, \Gamma \Rightarrow \Delta$, also $[\![\Gamma]\!]^{\bullet}, [\![\Delta]\!]^{\circ} \vdash s : \bot$.
2. If $\Gamma \Rightarrow \boxed{s : A}$, also $[\![\Gamma]\!]^{\bullet} \vdash [\![A]\!] : s$ if $\epsilon(A) = +$, while $[\![\Gamma]\!]^{\bullet} \vdash [\![A]\!]^{\bot} : s$ if $\epsilon(A) = -$.
3. If $\boxed{s : A} \Rightarrow \Gamma$, also $[\![\Gamma]\!]^{\circ} \vdash [\![A]\!] : s$ if $\epsilon(A) = -$, while $[\![\Gamma]\!]^{\circ} \vdash [\![A]\!]^{\bot} : s$ if $\epsilon(A) = +$.

If Cut is not used in the source, then these results can be strengthened to the extent of using only Cut-free derivations in the target as well.

Proof. By straightforward induction. Note in particular that decisions and reactions are mapped, respectively, to applications of $({}^{\bot}L)$ and $({}^{\bot}R)$. $\square$

The next example briefly illustrates how focussing can block certain derivations, relying in particular on the (arbitrarily fixed) positive polarity of atoms.

Example 23. Consider the goal sequent $(np/n \bullet n) \bullet np\backslash s \vdash s$. Since we are now concerned only with provability, we omit mention of term labels. We can proceed by focussing on either $np\backslash s$ or on np/n.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{(not an axiom)}}{np/n \bullet n \Rightarrow \boxed{np}} \;\times \qquad \dfrac{\dfrac{\dfrac{}{s \Rightarrow \boxed{s}}\,Id}{s \Rightarrow s}\,D}{\boxed{s} \Rightarrow s}\,R}{\boxed{np\backslash s} \Rightarrow (np/n \bullet n) \multimap s}\,\backslash L}{np\backslash s \Rightarrow (np/n \bullet n) \multimap s}\,D}{\overline{(np/n \bullet n) \bullet np\backslash s \Rightarrow s}}\,dp
$$

Preorder laws

$$\frac{s\ :\ \Gamma \Rightarrow x : A \quad t\ :\ y : A \Rightarrow \Delta}{\begin{array}{c} s[\lambda y t/x]\ :\ \Gamma \Rightarrow \Delta \text{ if } \epsilon(A) = + \\ t[\lambda x s/y]\ :\ \Gamma \Rightarrow \Delta \text{ if } \epsilon(A) = - \end{array}} \circ \qquad \frac{}{\begin{array}{c} x : P \Rightarrow \boxed{x : P} \\ \boxed{x : N} \Rightarrow x : N \end{array}}\ Id$$

Display postulates (dp)

$$\begin{array}{c} s\ :\ \Delta \multimapdot \Gamma \Rightarrow \Theta \\ \hline\hline s\ :\ \Gamma \bullet \Delta \Rightarrow \Theta \\ \hline\hline s\ :\ \Delta \Rightarrow \Gamma \multimapdot \Theta \end{array} \qquad \begin{array}{c} s\ :\ \Gamma \multimapdotinv \Theta \Rightarrow \Delta \\ \hline\hline s\ :\ \Gamma \bullet \Delta \Rightarrow \Theta \\ \hline\hline s\ :\ \Gamma \Rightarrow \Theta \multimapdotinv \Delta \end{array} \qquad \begin{array}{c} s\ :\ \langle \Gamma \rangle \Rightarrow \Delta \\ \hline\hline s\ :\ \Gamma \Rightarrow \langle \Delta \rangle \end{array}$$

Decisions and reactions

$$\frac{\Gamma \Rightarrow \boxed{s : P}}{(x\ s)\ :\ \Gamma \Rightarrow x : P}\ D \quad \frac{\boxed{s : N} \Rightarrow \Gamma}{(x\ s)\ :\ x : N \Rightarrow \Gamma}\ D \quad \frac{s\ :\ \Gamma \Rightarrow x : N}{\Gamma \Rightarrow \boxed{\lambda x s : N}}\ R \quad \frac{s\ :\ x : P \Rightarrow \Delta}{\boxed{\lambda x s : P} \Rightarrow \Delta}\ R$$

Focussing phase

$$\frac{\Gamma \Rightarrow \boxed{s : A} \quad \Delta \Rightarrow \boxed{t : B}}{\Gamma \bullet \Delta \Rightarrow \boxed{(s \otimes_1 t) : A \otimes B}}\ \otimes R \qquad \frac{\boxed{s : A} \Rightarrow \Gamma \quad \boxed{t : B} \Rightarrow \Delta}{\boxed{(t \otimes_1 s) : A \oplus B} \Rightarrow \Gamma \bullet \Delta}\ \oplus L$$

$$\frac{\Delta \Rightarrow \boxed{t : B} \quad \boxed{s : A} \Rightarrow \Gamma}{\boxed{(s \otimes_2 t) : B \backslash A} \Rightarrow \Delta \multimapdot \Gamma}\ \backslash L \qquad \frac{\Delta \Rightarrow \boxed{t : B} \quad \boxed{s : A} \Rightarrow \Gamma}{\boxed{(t \otimes_3 s) : A / B} \Rightarrow \Gamma \multimapdotinv \Delta}\ / L$$

$$\frac{B \Rightarrow \boxed{s : \Delta} \quad \Gamma \Rightarrow \boxed{t : A}}{\Gamma \multimapdotinv \Delta \Rightarrow \boxed{(s \otimes_2 t) : A \oslash B}}\ \oslash R \qquad \frac{B \Rightarrow \boxed{t : \Delta} \quad \Gamma \Rightarrow \boxed{s : A}}{\Delta \multimapdot \Gamma \Rightarrow \boxed{(t \otimes_3 s) : B \obslash A}}\ \obslash R$$

$$\frac{\Gamma \Rightarrow \boxed{s : A}}{\langle \Gamma \rangle \Rightarrow \boxed{\Diamond s : \Diamond A}}\ \Diamond R \qquad \frac{\boxed{s : A} \Rightarrow \Gamma}{\boxed{\Diamond s : \Box A} \Rightarrow \langle \Gamma \rangle}\ \Box L$$

Neutral phase

$$\frac{s\ :\ y : A \bullet z : B \Rightarrow \Gamma}{\langle x \mid (y \otimes_1 z).s \rangle\ :\ x : A \otimes B \Rightarrow \Gamma}\ \otimes L \qquad \frac{s\ :\ \Gamma \Rightarrow y : A \bullet z : B}{\langle x \mid (z \otimes_1 y).s \rangle\ :\ \Gamma \Rightarrow x : A \oplus B}\ \oplus R$$

$$\frac{s\ :\ \Gamma \Rightarrow z : B \multimapdot y : A}{\langle x \mid (y \otimes_2 z).s \rangle\ :\ \Gamma \Rightarrow x : B \backslash A}\ \backslash R \qquad \frac{s\ :\ \Gamma \Rightarrow y : A \multimapdotinv z : B}{\langle x \mid (z \otimes_3 y).s \rangle\ :\ \Gamma \Rightarrow x : A / B}\ / R$$

$$\frac{s\ :\ y : B \multimapdotinv z : A \Rightarrow \Gamma}{\langle x \mid (y \otimes_2 z).s \rangle\ :\ x : B \oslash A \Rightarrow \Gamma}\ \oslash L \qquad \frac{s\ :\ z : B \multimapdot y : A \Rightarrow \Gamma}{\langle x \mid (z \otimes_3 y).s \rangle\ :\ x : B \obslash A \Rightarrow \Gamma}\ \obslash L$$

$$\frac{s\ :\ \langle y : A \rangle \Rightarrow \Gamma}{\langle x \mid \Diamond y.s \rangle\ :\ x : \Diamond A \Rightarrow \Gamma}\ \Diamond L \qquad \frac{s\ :\ \Gamma \Rightarrow \langle y : A \rangle}{\langle x \mid \Diamond y.s \rangle\ :\ \Gamma \Rightarrow x : \Box A}\ \Box R$$

Figure 7.7: Weak focalization.

If instead we choose to focus first on np/n, the derivation goes through:

$$
\dfrac{
  \dfrac{}{n \Rightarrow \boxed{n}}\,Id
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{np \Rightarrow \boxed{np}}\,Id
          \qquad
          \dfrac{\dfrac{\dfrac{}{s \Rightarrow \boxed{s}}\,Id}{s \vdash s}\,D}{\boxed{s} \Rightarrow s}\,R
        }{\boxed{np\backslash s} \Rightarrow np \multimap s}\,\backslash L
      }{np\backslash s \Rightarrow np \multimap s}\,D
    }{np \Rightarrow s \multimapinv np\backslash s}\,dp
  }{\boxed{np} \Rightarrow s \multimapinv np\backslash s}\,R
}{
  \dfrac{
    \boxed{np/n} \Rightarrow (s \multimapinv np\backslash s) \multimapinv n
  }{
    \dfrac{np/n \Rightarrow (s \multimapinv np\backslash s) \multimapinv n}{(np/n \bullet n) \bullet np\backslash s \Rightarrow s}\,dp
  }\,D
}\,/L
$$

Had we assigned atoms negative polarity, the converse situation would have arisen.[4]

Continuation-passing style

We next motivate the use of $[\![\cdot]\!]$ as a continuation-passing style translation, showing how it induces an evaluation strategy for eliminating principal Cuts. Consider, to this end, the following case from the unlabeled display calculus of chapter 3:

$$
\dfrac{
  \dfrac{\Gamma \bullet B \Rightarrow A}{\Gamma \Rightarrow A/B}\,/R
  \qquad
  \dfrac{\Delta \Rightarrow B \quad A \Rightarrow \Theta}{A/B \Rightarrow \Theta \multimapinv \Delta}\,/L
}{\Gamma \Rightarrow \Theta \multimapinv \Delta}\,\circ
$$

We find that there exist two ways of reducing it, depending on whether we first apply a Cut on A and then on B, or the other way around.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \Delta \Rightarrow B
      \quad
      \dfrac{\Gamma \Rightarrow A \multimapinv B}{B \Rightarrow \Gamma \multimap A}\,dp
    }{\Delta \Rightarrow \Gamma \multimap A}\,\circ
  }{\Gamma \bullet \Delta \Rightarrow A}\,dp
  \qquad
  A \Rightarrow \Theta
}{
  \dfrac{\Gamma \bullet \Delta \Rightarrow \Theta}{\Gamma \Rightarrow \Theta \multimapinv \Delta}\,dp
}\,\circ
\qquad
\dfrac{
  \Delta \Rightarrow B
  \quad
  \dfrac{
    \dfrac{
      \dfrac{\Gamma \Rightarrow A \multimapinv B}{\Gamma \bullet B \Rightarrow A}\,dp
      \quad
      A \Rightarrow \Theta
    }{\Gamma \bullet B \Rightarrow \Theta}\,\circ
  }{B \Rightarrow \Gamma \multimap \Theta}\,dp
}{
  \dfrac{\Delta \Rightarrow \Gamma \multimap \Theta}{\Gamma \Rightarrow \Theta \multimapinv \Delta}\,dp
}\,\circ
$$

[4]While our initial attempt at demonstrating $(np/n \bullet n) \bullet np\backslash s \Rightarrow s$ does not have any formal status, its depiction in tree format serving only an expository purpose, it should be noted that Girard's [2001] Ludics provide a proof theory for such failed derivations, containing a special rule (the *daemon*) to signal the need for backtracking.

Since these contractions further reduce to derivations differing only up to trivial rule permutations, this observation seems irrelevant for the identification of proofs. We say Cut is *associative*.[5] Similar results fail for classical Cut elimination, where the use of weakening in one of the premises may counteract confluency [see Danos et al., 1997]. The recovery of computational content requires a strategy for dealing with such critical pairs, which may be explicated using a double negation translation back into intuitionistic logic. Adapting such tools to **LG**, as we shall do presently, leads to the non-determinism of classical Cut elimination tagging along for the ride. To see this, we compute the terms corresponding to the above two contractions. Assume $\epsilon(A) = -$ and $\epsilon(B) = +$, writing Q for B and M for A. We then have

$$\begin{array}{l} t_1 \ : \ x:M \Rightarrow \Theta \\ t_2 \ : \ \Delta \Rightarrow y:Q \\ s \ : \ \Gamma \Rightarrow u:M \mathbin{\bullet\!\!-} v:Q \end{array}$$

First applying Cut on Q (B) and then on M (A) gives

$$\dfrac{\dfrac{\dfrac{t_2 \ : \ \Delta \Rightarrow y:Q \qquad \dfrac{s \ : \ \Gamma \Rightarrow u:M \mathbin{\bullet\!\!-} v:Q}{s \ : \ v:Q \Rightarrow \Gamma \mathbin{-\!\!\bullet} u:M}\,dp}{\dfrac{t_2[\lambda vs/y] \ : \ \Delta \Rightarrow \Gamma \mathbin{-\!\!\bullet} u:M}{t_2[\lambda vs/y] \ : \ \Gamma \bullet \Delta \Rightarrow u:M}\,dp}\circ \qquad t_1 \ : \ x:M \Rightarrow \Theta}{t_1[\lambda u t_2[\lambda vs/y]/x] \ : \ \Gamma \bullet \Delta \Rightarrow \Theta}\circ}{t_1[\lambda u t_2[\lambda vs/y]/x] \ : \ \Gamma \Rightarrow \Theta \mathbin{\bullet\!\!-} \Delta}\,dp$$

while first cutting on M and then on Q leads to

$$\dfrac{\dfrac{t_2 \ : \ \Delta \Rightarrow y:Q \qquad \dfrac{\dfrac{\dfrac{s \ : \ \Gamma \Rightarrow u:M \mathbin{\bullet\!\!-} v:Q}{s \ : \ \Gamma \bullet v:Q \Rightarrow u:M}\,dp \qquad t_1 \ : \ x:M \Rightarrow \Theta}{t_1[\lambda us/x] \ : \ \Gamma \bullet v:Q \Rightarrow \Theta}\circ}{t_1[\lambda us/x] \ : \ v:Q \Rightarrow \Gamma \mathbin{-\!\!\bullet} \Theta}\,dp}{t_2[\lambda v t_1[\lambda us/x]/y] \ : \ \Delta \Rightarrow \Gamma \mathbin{-\!\!\bullet} \Theta}\circ}{t_2[\lambda v t_1[\lambda us/x]/y] \ : \ \Gamma \Rightarrow \Theta \mathbin{\bullet\!\!-} \Delta}\,dp$$

[5] Terminology derives from category theory [Saunders Mac Lane, 1998]. Using the logical interpretation employed by Lambek and Scott [1988], suppose we have deductions f of B from A, g of C from B and h of D from C (notation: $f : A \to B$, $g : B \to C$ and $h : C \to D$). We then have two deductions of D from A, depending on whether we first compose f and g (notation: $g \cdot f$), or g and h ($h \cdot g$):

$$\dfrac{\dfrac{f : A \to B \quad g : B \to C}{g \cdot f : A \to C} \qquad h : C \to D}{h \cdot (g \cdot f) : A \to D} \qquad \dfrac{f : A \to B \qquad \dfrac{g : B \to C \quad h : C \to D}{h \cdot g}}{(h \cdot g) \cdot f : A \to D}$$

Category theory is concerned with equations between arrows, among which associativity of composition, i.e., $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

and these need not be $\beta\eta$-equivalent. Double negation translations exist, however, in order to tame nondeterminism, not to point it out. And indeed, the focussing phase restores confluency, the above principal Cut now reformulable as follows:

$$\dfrac{\dfrac{s\ :\ \Gamma \Rightarrow u : M \leftharpoondown v : Q}{\langle w \mid (v \otimes_3 u).s\rangle\ :\ \Gamma \Rightarrow w : M/Q}\ /R \qquad \dfrac{\dfrac{\Delta \Rightarrow \boxed{t_2 : Q} \quad \boxed{t_1 : M} \Rightarrow \Theta}{\boxed{(t_2 \otimes_3 t_1) : M/Q} \Rightarrow \Theta \leftharpoondown \Delta}\ /L}{(z\ (t_2 \otimes_3 t_1))\ :\ z : M/Q \Rightarrow \Theta \leftharpoondown \Delta}\ D}{(\lambda w\langle w \mid (v \otimes_3 u).s\rangle\ (t_2 \otimes_3 t_1))\ :\ \Gamma \Rightarrow \Theta \leftharpoondown \Delta}\ \circ$$

Its two possible immediate reductions derive terms $(\lambda u(\lambda v s\, t_2)\, t_1)$ and $(\lambda v(\lambda u s\, t_1)\, t_2)$, both clearly $\beta\eta$-equivalent with $(\lambda w\langle w \mid (v \otimes_3 u).s\rangle\ (t_2 \otimes_3 t_1))$: determinism restored. In particular, a principal Cut can no longer be created without use of the focussing phase, thus preventing situations such as discussed above from arising.

With the above short introduction, we now define our CPS translation by using $[\![\cdot]\!]$ to find a term assignment for derivations inside chapter 3's unfocussed display calculus.

Definition 121. F.7.8 renders the Andreoli/Girard translation in continuation-passing style. For reasons of space, we concentrate only on a select number of connectives in showing how to interpret the logical introductions, taking these as typical cases.

Note the interpretation of structures remains unchanged from D.119, the difference with F.7.7 being that we only use the neutral phase. Furthermore, in each case we considered all possible values under ϵ of the active and main formulas involved.

As a final observation, note that we have, for each binary connective, a choice of polarity for its two arguments such that a situation akin to that presented in the introductory remarks to this subsection arises. Particularly, when the arguments' polarities coincide with those found in the focussing phase in F.7.7, i.e., + if upward monotone while − otherwise, we find that we have two natural ways of proceeding in finding a term assignment. In F.7.8, this situation is recognized in $(/L)$, where we might as well have used the following term instead for $\epsilon(A) = -$ and $\epsilon(B) = +$:

$$t_2[\lambda w t_1[\lambda v(x\ (w \otimes_3 v))/z]/y]$$

Preorder laws

$$\frac{s\ :\ \Gamma \Rightarrow x:A \quad t\ :\ y:A \Rightarrow \Delta}{\begin{array}{r} s[\lambda y t/x]\ :\ \Gamma \Rightarrow \Delta \text{ if } \epsilon(A) = + \\ t[\lambda x s/y]\ :\ \Gamma \Rightarrow \Delta \text{ if } \epsilon(A) = - \end{array}} \circ \qquad \frac{}{\begin{array}{l} (y\ x)\ :\ x:P \Rightarrow y:P \\ (x\ y)\ :\ x:N \Rightarrow y:N \end{array}} \mathit{Id}$$

Logical rules $(/, \Box)$

$$\frac{s\ :\ y:A \bullet z:B \Rightarrow \Gamma}{\langle x \mid (y \otimes_1 z).s\rangle\ :\ x:A \otimes B \Rightarrow \Gamma} /R$$

$$\frac{t_2\ :\ \Delta \Rightarrow y:Q \quad t_1\ :\ z:P \Rightarrow \Gamma}{\begin{array}{r} t_2[\lambda w(x\ (w \otimes_3 \lambda z t_1))/y]\ :\ x:A/B \Rightarrow \Gamma \bullet\!\!- \Delta \text{ if } \epsilon(A) = +,\ \epsilon(B) = + \\ t_1[\lambda v t_2[\lambda w(x\ (w \otimes_3 v))/y]/z]\ :\ x:A/B \Rightarrow \Gamma \bullet\!\!- \Delta \text{ if } \epsilon(A) = -,\ \epsilon(B) = + \\ (x\ (\lambda y t_2 \otimes_3 \lambda z t_1))\ :\ x:A/B \Rightarrow \Gamma \bullet\!\!- \Delta \text{ if } \epsilon(A) = +, \epsilon(B) = - \\ t_1[\lambda v(x\ (\lambda y t_2 \otimes_3 v))]\ :\ x:A/B \Rightarrow \Gamma \bullet\!\!- \Delta \text{ if } \epsilon(A) = -, \epsilon(B) = - \end{array}} /L$$

$$\frac{s\ :\ \Gamma \Rightarrow \langle y:A\rangle}{\langle x \mid \Diamond y.s\rangle\ :\ \Gamma \Rightarrow x:\Box A} \Box R$$

$$\frac{s\ :\ y:A \Rightarrow \Gamma}{\begin{array}{r} (x\ \Diamond \lambda y s)\ :\ x:\Box A \Rightarrow \langle\Gamma\rangle \text{ if } \epsilon(A) = + \\ s[\lambda z(x\ \Diamond z)/y]\ :\ x:\Box A \Rightarrow \langle\Gamma\rangle \text{ if } \epsilon(A) = - \end{array}} \Box L$$

Figure 7.8: Andreoli/Girard rendered continuation-passing style.

7.6.2 Call-by-name and Call-by-value

We next present alternative translations $\lfloor A \rfloor$ and $\lceil A \rceil$, adapting, respectively, the *call-by-name* and *call-by-value* interpretations from Bernardi and Moortgat [2007], inspired by Curien and Herbelin [2000]. Roughly, $\lceil A \rceil$ expresses all of A's negative subformulas positively using their De Morgan equivalents, while $\lfloor A \rfloor$ replaces all positive subformulas by De Morgan equivalents of their duals. In particular, $\lceil A \rceil^+ = A$ and $\lfloor A \rfloor^- = A$. In contrast with $[\![A]\!]$, the polarities of subformulas are not taken into account, thus resulting in a less efficient translation possibly containing iterated negations in its image.

Definition 122. We define, by induction, the call-by-name (CBN) and call-by-value

(CBV) translations $\lfloor A \rfloor$ and $\lceil A \rceil$ respectively, as follows. First, for CBV:

$$\begin{array}{rclrcl}
\lceil p \rceil & := & p & & & \\
\lceil A \otimes B \rceil & := & \lceil A \rceil \otimes_1 \lceil B \rceil & \lceil A \oplus B \rceil & := & (\lceil B \rceil^{\perp} \otimes_1 \lceil A \rceil^{\perp})^{\perp} \\
\lceil A \oslash B \rceil & := & \lceil A \rceil \otimes_2 \lceil B \rceil^{\perp} & \lceil B \backslash A \rceil & := & (\lceil A \rceil^{\perp} \otimes_2 \lceil B \rceil)^{\perp} \\
\lceil B \obslash A \rceil & := & \lceil B \rceil^{\perp} \otimes_3 \lceil A \rceil & \lceil A / B \rceil & := & (\lceil B \rceil \otimes_3 \lceil A \rceil^{\perp})^{\perp} \\
\lceil \Diamond A \rceil & := & \Diamond \lceil A \rceil & \lceil \Box A \rceil & := & (\Diamond \lceil A \rceil^{\perp})^{\perp}
\end{array}$$

while for CBN:

$$\begin{array}{rclrcl}
\lfloor p \rfloor & := & p^{\perp} & & & \\
\lfloor A \otimes B \rfloor & := & (\lfloor A \rfloor^{\perp} \otimes_1 \lfloor B \rfloor^{\perp})^{\perp} & \lfloor A \oplus B \rfloor & := & \lfloor B \rfloor \otimes_1 \lfloor A \rfloor \\
\lfloor A \oslash B \rfloor & := & (\lfloor A \rfloor^{\perp} \otimes_2 \lfloor B \rfloor)^{\perp} & \lfloor B \backslash A \rfloor & := & \lfloor A \rfloor \otimes_2 \lfloor B \rfloor^{\perp} \\
\lfloor B \obslash A \rfloor & := & (\lfloor B \rfloor^{\perp} \otimes_3 \lfloor A \rfloor^{\perp})^{\perp} & \lfloor A / B \rfloor & := & \lfloor B \rfloor^{\perp} \otimes_3 \lfloor A \rfloor \\
\lfloor \Diamond A \rfloor & := & (\Diamond \lfloor A \rfloor^{\perp})^{\perp} & \lfloor \Box A \rfloor & := & \Diamond \lfloor A \rfloor
\end{array}$$

A straightforward induction proves:

Lemma 85. For any formula A, $A = \lceil A \rceil^{+} = \lfloor A \rfloor^{-}$.

Definition 123. We again extend the maps $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ to the level of structures. Starting with call-by-value, we have the mutually defined maps $\lceil \cdot \rceil^{\bullet}$ and $\lceil \cdot \rceil^{\circ}$, acting on in- and output structures respectively:

$$\begin{array}{rr}
\lceil x : A \rceil^{\bullet} := x : \lceil A \rceil & \lceil x : A \rceil^{\circ} := x : \lceil A \rceil^{\perp} \\
\lceil (\Gamma \bullet \Delta) \rceil^{\bullet} := (\lceil \Gamma \rceil^{\bullet} \bullet_1 \lceil \Delta \rceil^{\bullet}) & \lceil (\Gamma \bullet \Delta) \rceil^{\circ} := (\lceil \Delta \rceil^{\circ} \bullet_1 \lceil \Gamma \rceil^{\circ}) \\
\lceil (\Gamma \leftharpoondown \Delta) \rceil^{\bullet} := (\lceil \Gamma \rceil^{\bullet} \bullet_2 \lceil \Delta \rceil^{\circ}) & \lceil (\Gamma \leftharpoondown \Delta) \rceil^{\circ} := (\lceil \Delta \rceil^{\bullet} \bullet_3 \lceil \Gamma \rceil^{\circ}) \\
\lceil (\Delta \rightharpoondown \Gamma) \rceil^{\bullet} := (\lceil \Delta \rceil^{\circ} \bullet_3 \lceil \Gamma \rceil^{\bullet}) & \lceil (\Delta \rightharpoondown \Gamma) \rceil^{\circ} := (\lceil \Gamma \rceil^{\circ} \bullet_2 \lceil \Delta \rceil^{\bullet})
\end{array}$$

Similarly, for call-by-name:

$$\begin{array}{rr}
\lfloor x : A \rfloor^{\bullet} := x : \lfloor A \rfloor^{\perp} & \lfloor x : A \rfloor^{\circ} := x : \lfloor A \rfloor \\
\lfloor (\Gamma \bullet \Delta) \rfloor^{\bullet} := (\lfloor \Gamma \rfloor^{\bullet} \bullet_1 \lfloor \Delta \rfloor^{\bullet}) & \lfloor (\Gamma \bullet \Delta) \rfloor^{\circ} := (\lfloor \Delta \rfloor^{\circ} \bullet_1 \lfloor \Gamma \rfloor^{\circ}) \\
\lfloor (\Gamma \leftharpoondown \Delta) \rfloor^{\bullet} := (\lfloor \Gamma \rfloor^{\bullet} \bullet_2 \lfloor \Delta \rfloor^{\circ}) & \lfloor (\Gamma \leftharpoondown \Delta) \rfloor^{\circ} := (\lfloor \Delta \rfloor^{\bullet} \bullet_3 \lfloor \Gamma \rfloor^{\circ}) \\
\lfloor (\Delta \rightharpoondown \Gamma) \rfloor^{\bullet} := (\lfloor \Delta \rfloor^{\circ} \bullet_3 \lfloor \Gamma \rfloor^{\bullet}) & \lfloor (\Delta \rightharpoondown \Gamma) \rfloor^{\circ} := (\lfloor \Gamma \rfloor^{\circ} \bullet_2 \lfloor \Delta \rfloor^{\bullet})
\end{array}$$

Definition 124. Figures 7.9 and 7.10 provide display calculi for **LG** for deriving normal forms under the CBV and CBN translations respectively. Three judgement forms are used, the neutral phase $s : \Gamma \Rightarrow \Delta$ being shared by both figures, while CBN additionally has a left focussing phase $\boxed{s : A} \Rightarrow \Delta$, its right-sided counterpart $\Gamma \Rightarrow \boxed{s : A}$ being used in CBV instead.

Preorder laws.

$$\frac{s \,:\, \Gamma \Rightarrow x : A \quad t \,:\, \Delta \Rightarrow y : B}{s[\lambda y t / x] \,:\, \Gamma \Rightarrow \Delta}\ \circ \qquad \frac{}{x : A \Rightarrow \boxed{x : A}}\ Id$$

Decisions

$$\frac{\Gamma \Rightarrow \boxed{s : A}}{(x\ s) \,:\, \Gamma \Rightarrow x : A}\ D$$

Right inferences

$$\frac{\Gamma \Rightarrow \boxed{s : A} \quad \Delta \Rightarrow \boxed{t : B}}{\Gamma \bullet \Delta \Rightarrow \boxed{(s \otimes_1 t) : A \otimes B}}\ \otimes R \qquad \frac{s \,:\, \Gamma \Rightarrow x : A \bullet y : B}{\Gamma \Rightarrow \boxed{\lambda(y \otimes_1 x) s : A \oplus B}}\ \oplus R$$

$$\frac{s \,:\, \Gamma \Rightarrow y : B \multimap x : A}{\Gamma \Rightarrow \boxed{\lambda(x \otimes_2 y) s : B \backslash A}}\ \backslash R \qquad \frac{s \,:\, \Gamma \Rightarrow x : A \multimapinv y : B}{\Gamma \Rightarrow \boxed{\lambda(y \otimes_3 x) s : A / B}}\ / R$$

$$\frac{t \,:\, x : B \Rightarrow \Delta \quad \Gamma \Rightarrow \boxed{s : A}}{\Gamma \multimapinv \Delta \Rightarrow \boxed{(s \otimes_2 \lambda x t) : A \oslash B}}\ \oslash R \qquad \frac{t \,:\, x : B \Rightarrow \Delta \quad \Gamma \Rightarrow \boxed{s : A}}{\Delta \multimap \Gamma \Rightarrow \boxed{(\lambda x t \otimes_3 s) : B \obslash A}}\ \obslash R$$

$$\frac{\Gamma \Rightarrow \boxed{s : A}}{\langle \Gamma \rangle \Rightarrow \boxed{\Diamond s : \Diamond A}}\ \Diamond R \qquad \frac{s \,:\, \Gamma \Rightarrow \langle x : A \rangle}{\Gamma \Rightarrow \boxed{\lambda \Diamond x s : \Box A}}\ \Box R$$

Left inferences

$$\frac{s \,:\, y : A \bullet z : B \Rightarrow \Gamma}{\langle x \mid (y \otimes_1 z).s \rangle \,:\, x : A \otimes B \Rightarrow \Gamma}\ \otimes L \qquad \frac{s \,:\, y : A \Rightarrow \Gamma \quad t \,:\, z : B \Rightarrow \Delta}{(x\ (\lambda z t \otimes_1 \lambda y s)) \,:\, x : A \oplus B \Rightarrow \Gamma \bullet \Delta}\ \oplus L$$

$$\frac{\Delta \Rightarrow \boxed{t : B} \quad s \,:\, y : A \Rightarrow \Gamma}{(x\ (\lambda y s \otimes_2 t)) \,:\, x : B \backslash A \Rightarrow \Delta \multimap \Gamma}\ \backslash L \qquad \frac{\Delta \Rightarrow \boxed{t : B} \quad s \,:\, y : A \Rightarrow \Gamma}{(x\ (t \otimes_3 \lambda y s)) \,:\, x : A / B \Rightarrow \Gamma \multimapinv \Delta}\ / L$$

$$\frac{s \,:\, y : A \multimapinv z : B \Rightarrow \Gamma}{\langle x \mid (y \otimes_2 z).s \rangle \,:\, x : A \oslash B \Rightarrow \Gamma}\ \oslash L \qquad \frac{s \,:\, z : B \multimap y : A \Rightarrow \Gamma}{\langle x \mid (z \otimes_3 y).s \rangle \,:\, x : B \obslash A \Rightarrow \Gamma}\ \obslash L$$

$$\frac{s \,:\, \langle y : A \rangle \Rightarrow \Gamma}{\langle x \mid \Diamond y.s \rangle \,:\, x : \Diamond A \Rightarrow \Gamma}\ \Diamond L \qquad \frac{s \,:\, y : A \Rightarrow \Gamma}{(x\ \Diamond \lambda y s) \,:\, x : \Box A \Rightarrow \langle \Gamma \rangle}\ \Box L$$

Figure 7.9: **LGQ.** (Call-by-value) Display postulates remain unchanged from F.7.7.

Preorder laws.

$$\frac{s\ :\ \Gamma \Rightarrow x : A \quad t\ :\ \Delta \Rightarrow y : B}{t[\lambda xs/y]\ :\ \Gamma \Rightarrow \Delta}\ \circ \qquad \frac{}{\boxed{\lambda y(x\ y) : A} \Rightarrow x : A}\ \mathit{Id}$$

Decisions

$$\frac{\boxed{s : A} \Rightarrow \Gamma}{(x\ s)\ :\ x : A \Rightarrow \Gamma}\ D$$

Right inferences

$$\frac{s\ :\ \Gamma \Rightarrow y : A \quad t\ :\ \Delta \Rightarrow z : B}{(x\ (\lambda ys \otimes_1 \lambda zt))\ :\ \Gamma \bullet \Delta \Rightarrow x : A \otimes B}\ \otimes R \qquad \frac{s\ :\ \Gamma \Rightarrow y : A \bullet z : B}{\langle x \mid (z \otimes_1 y).s\rangle\ :\ \Gamma \Rightarrow x : A \oplus B}\ \oplus R$$

$$\frac{s\ :\ \Gamma \Rightarrow z : B \mathbin{-\!\bullet} y : A}{\langle x \mid (y \otimes_2 z).s\rangle\ :\ \Gamma \Rightarrow x : B\backslash A}\ \backslash R \qquad \frac{s\ :\ \Gamma \Rightarrow y : A \mathbin{\bullet\!-} z : B}{\langle x \mid (z \otimes_3 y).s\rangle\ :\ \Gamma \Rightarrow x : A/B}\ /R$$

$$\frac{\boxed{t : B} \Rightarrow \Delta \quad s\ :\ \Gamma \Rightarrow y : A}{(x\ (\lambda ys \otimes_2 t))\ :\ \Gamma \mathbin{\bullet\!-} \Delta \Rightarrow x : A \oslash B}\ \oslash R \qquad \frac{\boxed{t : B} \Rightarrow \Delta \quad s\ :\ \Gamma \Rightarrow y : A}{(x\ (t \otimes_3 \lambda ys))\ :\ \Delta \mathbin{-\!\bullet} \Gamma \Rightarrow x : B \obslash A}\ \obslash R$$

$$\frac{s\ :\ \Gamma \Rightarrow y : A}{(x\ \Diamond \lambda ys)\ :\ \langle\Gamma\rangle \Rightarrow x : \Diamond A}\ \Diamond R \qquad \frac{s\ :\ \Gamma \Rightarrow \langle y : A\rangle}{\langle x \mid \Diamond y.s\rangle\ :\ \Gamma \Rightarrow x : \Box A}\ \Box R$$

Left inferences

$$\frac{s\ :\ x : A \bullet y : A \Rightarrow \Gamma}{\boxed{\lambda(x \otimes_1 y)s : A \otimes B} \Rightarrow \Gamma}\ \otimes L \qquad \frac{\boxed{s : A} \Rightarrow \Gamma \quad \boxed{t : B} \Rightarrow \Delta}{\boxed{(t \otimes_1 s) : A \oplus B} \Rightarrow \Gamma \bullet \Delta}\ \oplus L$$

$$\frac{t\ :\ \Delta \Rightarrow x : B \quad \boxed{s : A} \Rightarrow \Gamma}{\boxed{(s \otimes_2 \lambda xt) : B\backslash A} \Rightarrow \Delta \mathbin{-\!\bullet} \Gamma}\ \backslash L \qquad \frac{t\ :\ \Delta \Rightarrow x : B \quad \boxed{s : A} \Rightarrow \Gamma}{\boxed{(\lambda xt \otimes_3 s) : A/B} \Rightarrow \Gamma \mathbin{\bullet\!-} \Delta}\ /L$$

$$\frac{s\ :\ x : A \mathbin{\bullet\!-} y : B \Rightarrow \Gamma}{\boxed{\lambda(x \otimes_2 y)s : A \oslash B} \Rightarrow \Gamma}\ \oslash L \qquad \frac{s\ :\ y : B \mathbin{-\!\bullet} x : A \Rightarrow \Gamma}{\boxed{\lambda(y \otimes_3 x)s : B \obslash A} \Rightarrow \Gamma}\ \obslash L$$

$$\frac{s\ :\ \langle x : A\rangle \Rightarrow \Gamma}{\boxed{\lambda \Diamond xs : \Diamond A} \Rightarrow \Gamma}\ \Diamond L \qquad \frac{\boxed{s : A} \Rightarrow \Gamma}{\boxed{\Diamond s : \Box A} \Rightarrow \langle\Gamma\rangle}\ \Box L$$

Figure 7.10: **LGT**. (Call-by-name) Display postulates remain unchanged from F.7.7.

Bernardi and Moortgat [2007] did not provide sequent calculi deriving only normal forms for their translations, rather presenting them in continuation-passing style. We repeat this exercise in Figures 7.11 and 7.12, though extending upon their work in additionally providing interpretations for the tensor and par as well.

Remark 15. In comparing the CBN/CBV translations $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ with their polarized counterpart $[\![\cdot]\!]$ at the level of lexical semantics, the latter often amounts to the more economic one. Strictly speaking, we will assume our lexical semantics to be defined in ordinary typed λ-calculus, similar to the two-step approach taken in chapter 2. To this end, we roughly translate type constructors $\otimes_i$ ($i \in \{1,2,3\}$) into ordinary products $\times$ while negations become function types with result t, and use the mechanism of paired abstraction $\lambda\langle x,y\rangle s$ for abbreviating $\lambda zs[\pi^1(z)/x, \pi^2(z)/y]$. Assuming basic types *np* and s translate to entities e and truth values t respectively, consider for example a ditransitive verb like *offered*, categorized $((np\backslash s)/np)/np$ (abbreviated *dtv*), receiving denotations of types $\lfloor dtv\rfloor^\perp$ (CBN), $\lceil dtv\rceil$ (CBV) and $[\![dtv]\!]^\perp$ (polarized). We then have, using a constant $\textsc{offered}$ of type $e \to e \to t$:

$$
\begin{array}{rl}
\text{CBN:} & \lambda\langle Z,\langle Y,\langle q,X\rangle\rangle\rangle(Z\ \lambda z(Y\ \lambda y(X\ \lambda x(q\ (((\textsc{offered}\ z)\ y)\ x))))) \\
\text{CBV:} & \lambda\langle z,Y\rangle(Y\ \lambda\langle y,X\rangle(X\ \lambda\langle q,x\rangle(q\ (((\textsc{offered}\ z)\ y)\ x)))) \\
\text{polarized:} & \lambda\langle z,\langle y,\langle q,x\rangle\rangle\rangle(q\ (((\textsc{offered}\ z)\ y)\ x))
\end{array}
$$

7.7 Linguistic applications

We consider the linguistic applications of the foregoing proof-theoretic investigations. Roughly, the previous sections lay the foundations for a compositional semantics, describing how syntactic derivations parameterize the interpretation of the phrases they construct over those of the words found therein. Moreover, our proposals were shown suitable for use as the target language of continuation-passing style translations, the latter's applicability to natural language semantics having been extensively argued for in recent years. Crucially, however, the degree of resource sensitivity typical of traditional Lambek calculi is retained, making ours a viable platform for conducting both syntactic and semantic investigation.

We will draw from data involving non-local scope construal, counted among the 'covert' types of discontinuity within the generative literature. They have figured prominently in the rise of Montague's Universal Grammar framework, particularly through their treatment in the seminal paper [Montague, 1973]. Our analysis takes inspiration from Partee and Rooth's [1983] and Hendriks' [1993] accounts of *type-shifting*, envisioning a flexible correspondence between the syntactic and the semantic components of a grammar by relaxing compositionality to a relation. Restated,

Preorder laws

$$\frac{s \;:\; \Gamma \Rightarrow x : A \quad t \;:\; \Delta \Rightarrow y : B}{s[\lambda y t / x] \;:\; \Gamma \Rightarrow \Delta}\;\circ \qquad\qquad \frac{}{(y\ x) \;:\; x : A \Rightarrow y : A}\;\mathit{Id}$$

(Co)residuated families

$$\frac{s \;:\; y : A \bullet z : B \Rightarrow \Gamma}{\langle x \mid (y \otimes_1 z).s\rangle \;:\; x : A \otimes B \Rightarrow \Gamma}\;\otimes L \qquad \frac{s \;:\; \Gamma \Rightarrow y : A \bullet z : B}{(x\ \lambda(z \otimes_1 y) s) \;:\; \Gamma \Rightarrow x : A \oplus B}\;\oplus R$$

$$\frac{s \;:\; \Gamma \Rightarrow y : A \bullet\!\!- y : B}{(x\ \lambda(z \otimes_3 y) s) \;:\; \Gamma \Rightarrow x : A / B}\;/R \qquad \frac{s \;:\; z : B -\!\!\bullet y : A \Rightarrow \Gamma}{\langle x \mid (z \otimes_3 y).s\rangle \;:\; x : B \obslash A \Rightarrow \Gamma}\;\obslash L$$

$$\frac{s \;:\; \Gamma \Rightarrow y : A \quad t \;:\; \Delta \Rightarrow z : B}{s[\lambda u t[\lambda v (x\ (u \otimes_1 v))/z]/y] \;:\; \Gamma \bullet \Delta \Rightarrow x : A \otimes B}\;\otimes R$$

$$\frac{t \;:\; \Delta \Rightarrow z : B \quad s \;:\; y : A \Rightarrow \Gamma}{t[\lambda u (x\ (t \otimes_3 \lambda y s))/z] \;:\; x : B \backslash A \Rightarrow \Delta -\!\!\bullet \Gamma}\;/L$$

$$\frac{s \;:\; y : A \Rightarrow \Gamma \quad t \;:\; z : B \Rightarrow \Delta}{(x\ (\lambda z t \otimes_1 \lambda y s)) \;:\; x : A \oplus B \Rightarrow \Gamma \bullet \Delta}\;\oplus L$$

$$\frac{t \;:\; y : B \Rightarrow \Delta \quad s \;:\; \Gamma \Rightarrow z : A}{s[\lambda v (x\ (\lambda y t \otimes_3 v))/z] \;:\; \Delta -\!\!\bullet \Gamma \Rightarrow x : B \obslash A}\;\obslash R$$

Modalities

$$\frac{s \;:\; \langle y : A\rangle \Rightarrow \Gamma}{\langle x \mid \Diamond y.s\rangle \;:\; x : \Diamond A \Rightarrow \Gamma}\;\Diamond L \qquad \frac{s \;:\; \Gamma \Rightarrow y : A}{s[\lambda u (x\ \Diamond u)/y] \;:\; \langle \Gamma\rangle \Rightarrow x : \Diamond A}\;\Diamond R$$

$$\frac{s \;:\; \Gamma \Rightarrow \langle y : A\rangle}{(x\ \lambda \Diamond y s) \;:\; \Gamma \Rightarrow x : \Box A}\;\Box R \qquad \frac{s \;:\; y : A \Rightarrow \Gamma}{(x\ \Diamond \lambda y s) \;:\; x : \Box A \Rightarrow \langle \Gamma\rangle}\;\Box L$$

Figure 7.11: Call-by-value rendered continuation-passing style. For reasons of space, we have omitted the rules for $\backslash$ and $\oslash$, being similar to those for $/$ and $\obslash$.

Preorder laws

$$\frac{s\ :\ \Gamma \Rightarrow x:A \quad t\ :\ \Delta \Rightarrow y:B}{t[\lambda x s/y]\ :\ \Gamma \Rightarrow \Delta}\ \circ \qquad \frac{}{(x\ y)\ :\ x:A \Rightarrow y:A}\ \mathit{Id}$$

(Co)residuated families

$$\frac{s\ :\ y:A \bullet z:B \Rightarrow \Gamma}{(x\ \lambda(y \otimes_1 z)s)\ :\ x:A \otimes B \Rightarrow \Gamma}\ \otimes L \qquad \frac{s\ :\ \Gamma \Rightarrow y:A \bullet z:B}{\langle x \mid (z \otimes_1 y).s\rangle\ :\ \Gamma \Rightarrow x:A \oplus B}\ \oplus R$$

$$\frac{s\ :\ \Gamma \Rightarrow y:A \leftharpoondown\!\!\bullet\ z:B}{\langle x \mid (z \otimes_3 y).s\rangle\ :\ \Gamma \Rightarrow x:A/B}\ /R \qquad \frac{s\ :\ z:B \multimap\!\!\bullet\ y:A \Rightarrow \Gamma}{(x\ \lambda(z \otimes_3 y)s)\ :\ x:B \oslash A \Rightarrow \Gamma}\ \oslash L$$

$$\frac{s\ :\ \Gamma \Rightarrow y:A \quad t\ :\ \Delta \Rightarrow z:B}{(x\ (\lambda y s \otimes_1 \lambda z t))\ :\ \Gamma \bullet \Delta \Rightarrow x:A \otimes B}\ \otimes R$$

$$\frac{t\ :\ \Delta \Rightarrow y:B \quad s\ :\ z:A \Rightarrow \Gamma}{s[\lambda v(x\ (\lambda y t \otimes_3 v))/z]\ :\ x:A/B \Rightarrow \Gamma \leftharpoondown\!\!\bullet\ \Delta}\ /L$$

$$\frac{s\ :\ y:A \Rightarrow \Gamma \quad t\ :\ z:B \Rightarrow \Delta}{s[\lambda u t[\lambda v(x\ (v \otimes_1 u))/z]/y]\ :\ x:A \oplus B \Rightarrow \Gamma \bullet \Delta}\ \oplus L$$

$$\frac{t\ :\ z:B \Rightarrow \Delta \quad s\ :\ \Gamma \Rightarrow y:A}{t[\lambda v(x\ (v \otimes_{)} \lambda y s)/z]\ :\ \Delta \multimap\!\!\bullet\ \Gamma \Rightarrow x:B \oslash A}\ \oslash R$$

Modalities

$$\frac{s\ :\ \langle y:A\rangle \Rightarrow \Gamma}{(x\ \lambda \diamond y s)\ :\ x:\diamond A \Rightarrow \Gamma}\ \diamond L \qquad \frac{s\ :\ \Gamma \Rightarrow y:A}{x:\diamond A\ :\ (x\ \diamond \lambda y s) \Rightarrow \langle\Gamma\rangle}\ \diamond R$$

$$\frac{s\ :\ \Gamma \Rightarrow \langle y:A\rangle}{x:\Box A\ :\ \langle x \mid\ \diamond y.s\rangle \Rightarrow \Gamma}\ \Box R \qquad \frac{s\ :\ y:A \Rightarrow \Gamma}{s[\lambda u(x\ \diamond u)/y]\ :\ x:\Box A \Rightarrow \langle\Gamma\rangle}\ \Box L$$

Figure 7.12: Call-by-name rendered continuation-passing style. For reasons of space, we have omitted the rules for $\backslash$ and $\oslash$, being similar to those for / and $\oslash$.

any given derivation has a possibly non-singleton set of interpretations. The latter is considered closed under a small number of *type-shifting rules*, being derived rules of inference in the target logic for semantic interpretation. We show that syntactic counterparts for said rules are already derivable within our term language for $\mathbf{LG}_\varnothing$, restoring the strict correspondence between syntax and semantics, while moreover no recourse to any Grishin interactions need be made.

The further contents of this section are organized as follows. In §7.1 we first make some general remarks on the comparison between the lexical and derivational semantics, particularly with regard to their differences in resource sensitivity. We next motivate our approach by example, demonstrating various cases of scopal ambiguities. A more precise characterization of expressivity is attempted in §7.2 through the comparison with type-shifting. Finally, §7.3 discusses some of the shortcomings of our analyses. Like in chapter 2, we assume a two-step approach to semantic interpretation, separating the derivational from the lexical semantics. That said, we will be a little more sloppy in its execution this time around.

7.7.1 Data analyzed

We survey several traditional sample sentences exhibiting various cases of scopal ambiguities. In providing their analyses, we admit ourselves some leverage in the strictness of the resource management regime adopted thus far: while the 'derivational' semantics of §4 reflects that of **LG**, the lexical semantics, constituting our means of referring to the world around us, need not be so restricted.

Put in more precise terms, in filling the lexical gap left by previous sections, we permit access to the full simply-typed λ-calculus, augmented by the logical constants of first-order predicate logic. As usual, we adopt base types e and t, interpreting, respectively, a fixed set of 'entities', or discourse referents, and the Boolean truth values. Complex types are either implications $\phi \to \tau$ or products $\phi \times \tau$, the latter allowing the formation of pairs $\langle s, t\rangle$ and the left- and right projections $\pi_1(s)$ and $\pi_2(s)$. The derivational types and terms carry straight over through the following homomorphisms. First, at the level of types, we systematically replace s, *np* and n with t, e and $\neg e$ (abbreviating $e \to t$), while $\otimes_i$ is exchanged for $\times$, negations $\cdot^\perp$ become implications $\cdot \to t$ and occurrences of $\Diamond$ are dropped. At the level of terms, case analyses $\langle t \mid (y \otimes_i z).s\rangle$ and $\langle t \mid \Diamond y.s\rangle$ become substitutions $s[\pi_1(t)/y][\pi_2(t)/z]$ and $s[t/y]$, while $(s \otimes_i t)$ and $\Diamond s$ are replaced with $\langle s, t\rangle$ and s. Note the latter specification assumes that the consequent Σ of a sequent is translated to the type t of truth values if empty. Naturally, we can make these remarks further precise, although doing so would make for a trivial exercise.

Word	Term	Category
Alice	$\textsc{alice}$	np
everyone	$\lambda Q \forall x((\textsc{person}\ x) \supset (Q\ x))$	$\Diamond \Box np$
a	$\lambda \langle P, Q \rangle \exists x((P\ x) \wedge (Q\ x))$	np/n
unicorn	$\textsc{unicorn}$	n
yawn(ed)	$\lambda \langle q, x \rangle (q\ (\textsc{yawn}\ x))$	$np \backslash s$
found	$\lambda \langle y, \langle q, x \rangle \rangle (q\ ((\textsc{find}\ y)\ x))$	$(np \backslash s)/np$
sought	$\lambda \langle Y, \langle q, x \rangle \rangle (q\ ((\textsc{seek}\ Y)\ x))$	$(np \backslash s)/\Box \Diamond np$
thinks	$\lambda \langle Q, \langle q, x \rangle \rangle (q\ ((\textsc{think}\ (Q\ \lambda pp))\ x))$	$(np \backslash s)/\Box \Diamond s$
heard	$\lambda \langle p, \langle q, x \rangle \rangle (q\ ((\textsc{hear}\ p)\ x))$	$(np \backslash s)/s$
and	$\lambda \langle R, \langle \langle Y, \langle q, x \rangle \rangle, S \rangle \rangle$ $((S\ \langle Y, \langle q, x \rangle \rangle) \wedge (R\ \langle Y, \langle q, x \rangle \rangle))$	$(tv \backslash tv)/tv$

Figure 7.13: Sample lexicon, where *tv* abbreviates $(np \backslash s)/\Box \Diamond np$.

The analyses to come are summarized by the lexicon of Figure 7.13. We note:

1. While, by virtue of its resource sensitivity, we could formulate the syntactic component of our analysis immediately inside the type language (as was done, for instance, by Bastenhof [2011] and Bastenhof [2012b]), we instead choose to use the (weakly) focussed variation on **LG** of §6.1 for this purpose. Assuming words are inserted as hypotheses, we obtain the following types:

Word	Type	Word	Type
Alice	np	found	$(np \otimes_3 (s^\perp \otimes_2 np))^\perp$
everyone	$\Diamond(\Diamond np^\perp)^\perp$	sought	$(\Diamond(\Diamond np^\perp)^\perp \otimes_3 (s^\perp \otimes_2 np))^\perp$
a	$(n \otimes_3 np^\perp)^\perp$	thinks	$(\Diamond(\Diamond s^\perp)^\perp \otimes_3 (s^\perp \otimes_2 np))^\perp$
unicorn	n	heard	$(s \otimes_3 (s^\perp \otimes_2 np))^\perp$
yawn(ed)	$(s^\perp \otimes_2 np)^\perp$	and	$(tv^\perp \otimes_3 (tv \otimes_2 tv^\perp))^\perp$

   overloading the abbreviation *tv*, now used to denote $(\Diamond(\Diamond np^\perp)^\perp \otimes_3 (s^\perp \otimes_2 np))^\perp$. Note in particular that occurrences of $\Diamond \Box$ in front of a positive formula are translated as $\Diamond(\cdot^\perp)^\perp$. When working directly inside the type language for the syntactic analysis, we could simplify as $\cdot^{\perp\perp}$.

2. In defining lexical denotations, have used constants $\textsc{alice}$ (of type e), $\textsc{person}$, $\textsc{unicorn}$, $\textsc{yawn}$ $(e \to t)$, $\textsc{find}$ $(e \to (e \to t))$, $\textsc{seek}$ $(((e \to t) \to t) \to (e \to t))$, $\textsc{think}$ and $\textsc{hear}$ $(t \to (e \to t))$.

We next consider object-wide and embedded scope readings, scope sieves, coordination and intensionality.

Object-wide scope

As a warm-up, consider the familiar linear and inverse scope readings (1a), (1b):

1. Everyone found a unicorn.

1a. For every person x, there exists some unicorn y s.t. x found y.

1b. There exists some unicorn y s.t. for every person x, x found y.

The quantified noun phrases are assigned formulas reflecting their interpretation as second-order properties through use of $\Diamond\,\Box$, inserting double negations at the level of types. Using the entries found in F.7.13, we can construct the derivations in Figures 7.14 and 7.15. In simply typed λ-calculus:

$$(x\ \lambda a(u\ \langle v, \lambda b(y\ \langle b, \langle \lambda w(z\ w), a\rangle\rangle)\rangle))$$
$$(u\ \langle v, \lambda b(x\ \lambda a(y\ \langle b, \langle \lambda w(z\ w), a\rangle\rangle))\rangle)$$

Replacing the free variables with lexical denotations and reducing the resulting terms provides us with the desired interpretations:

$$\forall x((\textsc{person}\ x) \supset \exists y((\textsc{unicorn}\ y) \wedge (z\ ((\textsc{find}\ y)\ x)))) \text{ (1a)}$$
$$\exists y((\textsc{unicorn}\ y) \wedge \forall x((\textsc{person}\ x) \supset (z\ ((\textsc{find}\ y)\ x)))) \text{ (1b)}$$

Embedded scope

We illustrate non-local scope construal from inside a complement clause:

2. Alice thinks a unicorn yawned.

2a. Alice thinks there exists a unicorn y s.t. y yawned.

2b. For some unicorn y, Alice thinks y yawned.

By having the verb select for a clausal complement of type $\Box\Diamond\, s$ instead of s, the deductive machinery is given enough freedom to derive both (2a) and (2b). Using the entries in F.7.13, we derive three terms, as shown in Figures 7.16, 7.17 and 7.18:

$$(t\ \langle \lambda q(u\ \langle v, \lambda y(w\ \langle \lambda p(q\ p), y\rangle)\rangle), \langle \lambda z'(z\ z'), a\rangle\rangle)$$
$$(u\ \langle v, \lambda y(t\ \langle \lambda q(w\ \langle \lambda p(q\ p), y\rangle), \langle \lambda z'(z\ z'), a\rangle\rangle)\rangle)$$
$$(u\ \langle v, \lambda y(w\ \langle \lambda p(t\ \langle \lambda q(q\ p), \langle \lambda z'(z\ z'), a\rangle\rangle), y\rangle)\rangle)$$

collapsing into the two desired readings after lexical insertion:

$$(z\ ((\textsc{think}\ \exists y((\textsc{unicorn}\ y) \wedge (\textsc{yawn}\ y)))\ \textsc{alice}))\ (2a)$$
$$\exists y((\textsc{unicorn}\ y) \wedge (z\ ((\textsc{think}\ (\textsc{yawn}\ y))\ \textsc{alice})))\ (2b)$$

$$\frac{}{w : s \Rightarrow \boxed{w : s}}\ Id$$

$$\frac{w : s \Rightarrow \boxed{w : s}}{(z\ w)\ :\ w : s \Rightarrow z : s}\ D$$

$$\frac{(z\ w)\ :\ w : s \Rightarrow z : s}{\boxed{\lambda w(z\ w) : s} \Rightarrow z : s}\ R$$

$$\frac{}{a : np \Rightarrow \boxed{a : np}}\ Id$$

$$\frac{a : np \Rightarrow \boxed{a : np} \qquad \boxed{\lambda w(z\ w) : s} \Rightarrow z : s}{\boxed{(\lambda w(z\ w) \otimes_2 a) : np\backslash s} \Rightarrow a : np \multimap z : s}\ \backslash L$$

$$\frac{}{b : np \Rightarrow \boxed{b : np}}\ Id$$

$$\frac{b : np \Rightarrow \boxed{b : np} \qquad \boxed{(\lambda w(z\ w) \otimes_2 a) : np\backslash s} \Rightarrow a : np \multimap z : s}{\boxed{(b \otimes_3 (\lambda w(z\ w) \otimes_2 a)) : (np\backslash s)/np} \Rightarrow (a : np \multimap z : s) \multimapinv b : np}\ /L$$

$$\frac{\boxed{(b \otimes_3 (\lambda w(z\ w) \otimes_2 a)) : (np\backslash s)/np} \Rightarrow (a : np \multimap z : s) \multimapinv b : np}{(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))\ :\ y : (np\backslash s)/np \Rightarrow (a : np \multimap z : s) \multimapinv b : np}\ D$$

$$\frac{(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))\ :\ y : (np\backslash s)/np \Rightarrow (a : np \multimap z : s) \multimapinv b : np}{(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))\ :\ b : np \Rightarrow y : (np\backslash s)/np \multimap (a : np \multimap z : s)}\ dp$$

$$\frac{(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))\ :\ b : np \Rightarrow y : (np\backslash s)/np \multimap (a : np \multimap z : s)}{\boxed{\lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))) : np} \Rightarrow y : (np\backslash s)/np \multimap (a : np \multimap z : s)}\ R$$

$$\frac{}{v : n \Rightarrow \boxed{v : n}}\ Id$$

$$\frac{v : n \Rightarrow \boxed{v : n} \qquad \boxed{\lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))) : np} \Rightarrow y : (np\backslash s)/np \multimap (a : np \multimap z : s)}{\boxed{(v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))) : np/n} \Rightarrow (y : (np\backslash s)/np \multimap (a : np \multimap z : s)) \multimapinv v : n}\ /L$$

$$\frac{\boxed{(v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))) : np/n} \Rightarrow (y : (np\backslash s)/np \multimap (a : np \multimap z : s)) \multimapinv v : n}{(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))\ :\ u : np/n \Rightarrow (y : (np\backslash s)/np \multimap (a : np \multimap z : s)) \multimapinv v : n}\ D$$

$$\frac{(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))\ :\ u : np/n \Rightarrow (y : (np\backslash s)/np \multimap (a : np \multimap z : s)) \multimapinv v : n}{(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))\ :\ a : np \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}\ dp$$

$$\frac{(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))\ :\ a : np \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}{\boxed{\lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))) : np} \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}\ R$$

$$\frac{\boxed{\lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))) : np} \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}{\boxed{\diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))) : \Box\, np} \Rightarrow \langle z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))\rangle}\ \Box L$$

$$\frac{\boxed{\diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))) : \Box\, np} \Rightarrow \langle z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))\rangle}{(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))))\ :\ x' : \Box\, np \Rightarrow \langle z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))\rangle}\ D$$

$$\frac{(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))))\ :\ x' : \Box\, np \Rightarrow \langle z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))\rangle}{(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))))\ :\ \langle x' : \Box\, np\rangle \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}\ dp$$

$$\frac{(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))))\ :\ \langle x' : \Box\, np\rangle \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}{\langle x \mid\ \diamond\, x'.(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))))\rangle\ :\ x : \diamond\Box\, np \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}\ \diamond L$$

$$\frac{\langle x \mid\ \diamond\, x'.(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))))\rangle\ :\ x : \diamond\Box\, np \Rightarrow z : s \multimapinv (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n))}{\langle x \mid\ \diamond\, x'.(x'\ \diamond\ \lambda a(u\ (v \otimes_3 \lambda b(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))))))\rangle\ :\ x : \diamond\Box\, np \bullet (y : (np\backslash s)/np \bullet (u : np/n \bullet v : n)) \Rightarrow z : s}\ dp$$

Figure 7.14: Deriving the first of two scopal readings for “Everyone found a unicorn.”

$$\dfrac{}{w : s \Rightarrow \boxed{w : s}}\ \textit{Id}$$

$$\dfrac{w : s \Rightarrow \boxed{w : s}}{(z\ w)\ :\ w : s \Rightarrow z : s}\ D$$

$$\dfrac{(z\ w)\ :\ w : s \Rightarrow z : s}{\boxed{\lambda w(z\ w) : s} \Rightarrow z : s}\ R$$

$$\dfrac{}{a : \mathbf{np} \Rightarrow \boxed{a : \mathbf{np}}}\ \textit{Id}$$

$$\dfrac{a : \mathbf{np} \Rightarrow \boxed{a : \mathbf{np}} \qquad \boxed{\lambda w(z\ w) : s} \Rightarrow z : s}{\boxed{(\lambda w(z\ w) \otimes_2 a) : \mathbf{np}\backslash s} \Rightarrow a : \mathbf{np} \mathbin{-\!\!\bullet} z : s}\ \backslash L$$

$$\dfrac{}{b : \mathbf{np} \Rightarrow \boxed{b : \mathbf{np}}}\ \textit{Id}$$

$$\dfrac{b : \mathbf{np} \Rightarrow \boxed{b : \mathbf{np}} \qquad \boxed{(\lambda w(z\ w) \otimes_2 a) : \mathbf{np}\backslash s} \Rightarrow a : \mathbf{np} \mathbin{-\!\!\bullet} z : s}{\boxed{(b \otimes_3 (\lambda w(z\ w) \otimes_2 a)) : (\mathbf{np}\backslash s)/\mathbf{np}} \Rightarrow (a : \mathbf{np} \mathbin{-\!\!\bullet} z : s) \mathbin{\bullet\!\!-} b : \mathbf{np}}\ /L$$

$$\dfrac{}{(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))\ :\ y : (\mathbf{np}\backslash s)/\mathbf{np} \Rightarrow (a : \mathbf{np} \mathbin{-\!\!\bullet} z : s) \mathbin{\bullet\!\!-} b : \mathbf{np}}\ D$$

$$\dfrac{}{(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a)))\ :\ a : \mathbf{np} \Rightarrow z : s \mathbin{\bullet\!\!-} (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet b : \mathbf{np})}\ dp$$

$$\dfrac{}{\boxed{\lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))) : \mathbf{np}} \Rightarrow z : s \mathbin{\bullet\!\!-} (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet b : \mathbf{np})}\ R$$

$$\dfrac{}{\boxed{\Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))) : \Box\, \mathbf{np}} \Rightarrow \langle z : s \mathbin{\bullet\!\!-} (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet b : \mathbf{np})\rangle}\ \Box L$$

$$\dfrac{}{(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\ :\ x' : \Box\, \mathbf{np} \Rightarrow \langle z : s \mathbin{\bullet\!\!-} (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet b : \mathbf{np})\rangle}\ D$$

$$\dfrac{}{(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\ :\ \langle x' : \Box\, \mathbf{np}\rangle \Rightarrow z : s \mathbin{\bullet\!\!-} (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet b : \mathbf{np})}\ dp$$

$$\dfrac{}{\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle\ :\ x : \Diamond\, \Box\, \mathbf{np} \Rightarrow z : s \mathbin{\bullet\!\!-} (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet b : \mathbf{np})}\ \Diamond L$$

$$\dfrac{}{\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle\ :\ b : \mathbf{np} \Rightarrow y : (\mathbf{np}\backslash s)/\mathbf{np} \mathbin{-\!\!\bullet} (x : \Diamond\, \Box\, \mathbf{np} \mathbin{-\!\!\bullet} z : s)}\ dp$$

$$\dfrac{}{\boxed{\lambda b\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle : \mathbf{np}} \Rightarrow y : (\mathbf{np}\backslash s)/\mathbf{np} \mathbin{-\!\!\bullet} (x : \Diamond\, \Box\, \mathbf{np} \mathbin{-\!\!\bullet} z : s)}\ R$$

$$\dfrac{}{v : n \Rightarrow \boxed{v : n}}\ \textit{Id}$$

$$\dfrac{v : n \Rightarrow \boxed{v : n} \qquad \boxed{\lambda b\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle : \mathbf{np}} \Rightarrow y : (\mathbf{np}\backslash s)/\mathbf{np} \mathbin{-\!\!\bullet} (x : \Diamond\, \Box\, \mathbf{np} \mathbin{-\!\!\bullet} z : s)}{\boxed{(v \otimes_3 \lambda b\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle) : \mathbf{np}/n} \Rightarrow (y : (\mathbf{np}\backslash s)/\mathbf{np} \mathbin{-\!\!\bullet} (x : \Diamond\, \Box\, \mathbf{np} \mathbin{-\!\!\bullet} z : s)) \mathbin{\bullet\!\!-} v : n}\ /L$$

$$\dfrac{}{(u\ (v \otimes_3 \lambda b\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle))\ :\ u : \mathbf{np}/n \Rightarrow (y : (\mathbf{np}\backslash s)/\mathbf{np} \mathbin{-\!\!\bullet} (x : \Diamond\, \Box\, \mathbf{np} \mathbin{-\!\!\bullet} z : s)) \mathbin{\bullet\!\!-} v : n}\ D$$

$$\dfrac{}{(u\ (v \otimes_3 \lambda b\langle x \mid\ \Diamond\, x'.(x'\ \Diamond\, \lambda a(y\ (b \otimes_3 (\lambda w(z\ w) \otimes_2 a))))\rangle))\ :\ x : \Diamond\, \Box\, \mathbf{np} \bullet (y : (\mathbf{np}\backslash s)/\mathbf{np} \bullet (u : \mathbf{np}/n \bullet v : n)) \Rightarrow z : s}\ dp$$

Figure 7.15: Deriving the second of two scopal readings for “Everyone found a unicorn.”

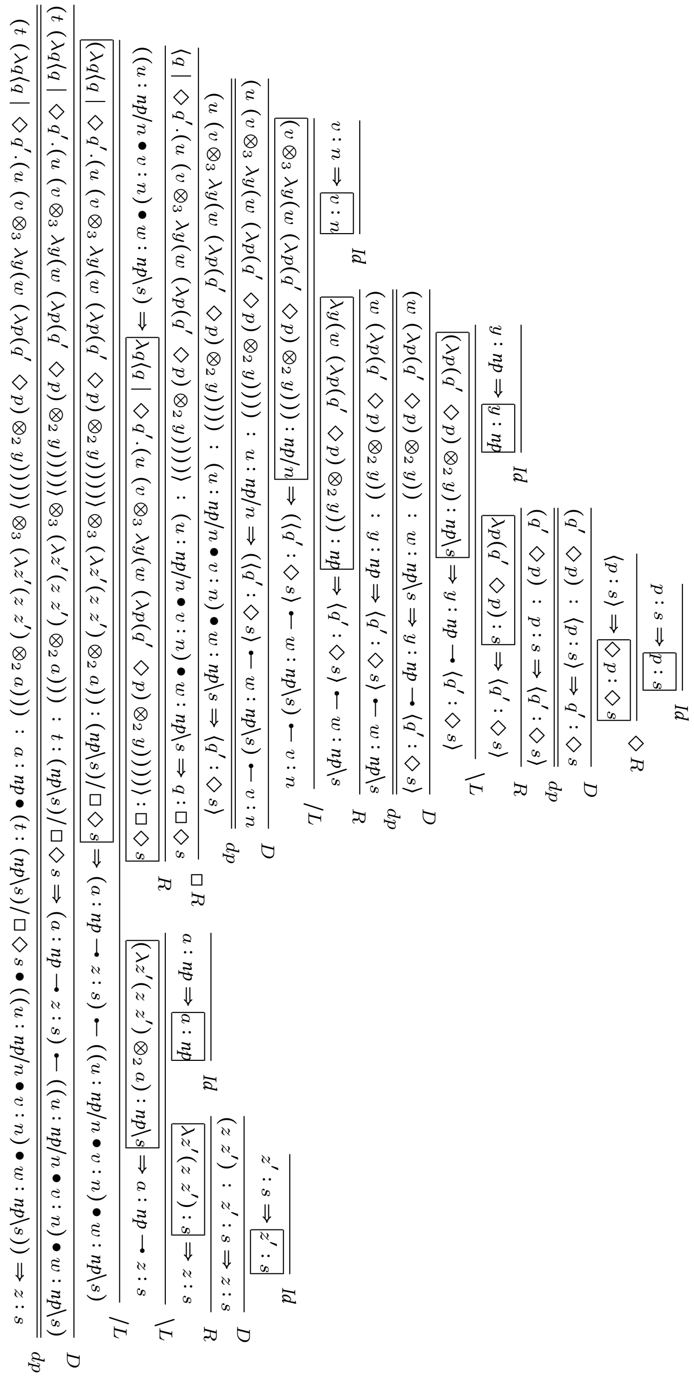


Figure 7.16: "Alice thinks a unicorn yawned." (1)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$y : np \Rightarrow \boxed{y : np}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$p : s \Rightarrow \boxed{p : s}$}
\RightLabel{$\Diamond R$}
\UnaryInfC{$\langle p : s\rangle \Rightarrow \boxed{\Diamond p : \Diamond s}$}
\RightLabel{D}
\UnaryInfC{$(q' \ \Diamond p) \ : \ \langle p : s\rangle \Rightarrow q' : \Diamond s$}
\RightLabel{dp}
\doubleLine
\UnaryInfC{$(q' \ \Diamond p) \ : \ p : s \Rightarrow \langle q' : \Diamond s\rangle$}
\RightLabel{R}
\UnaryInfC{$\boxed{\lambda p(q' \ \Diamond p) : s} \Rightarrow \langle q' : \Diamond s\rangle$}
\RightLabel{$\backslash L$}
\BinaryInfC{$\boxed{(\lambda p(q' \ \Diamond p) \otimes_2 y) : np\backslash s} \Rightarrow y : np \multimap \langle q' : \Diamond s\rangle$}
\RightLabel{D}
\UnaryInfC{$(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y)) \ : \ w : np\backslash s \Rightarrow y : np \multimap \langle q' : \Diamond s\rangle$}
\RightLabel{dp}
\doubleLine
\UnaryInfC{$(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y)) \ : \ y : np \bullet w : np\backslash s \Rightarrow \langle q' : \Diamond s\rangle$}
\RightLabel{$\Box R$}
\UnaryInfC{$\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \ : \ y : np \bullet w : np\backslash s \Rightarrow q : \Box\Diamond s$}
\RightLabel{R}
\UnaryInfC{$y : np \bullet w : np\backslash s \Rightarrow \boxed{\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle : \Box\Diamond s}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$a : np \Rightarrow \boxed{a : np}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$z' : s \Rightarrow \boxed{z' : s}$}
\RightLabel{D}
\UnaryInfC{$(z \ z') \ : \ z' : s \Rightarrow z : s$}
\RightLabel{R}
\UnaryInfC{$\boxed{\lambda z'(z \ z') : s} \Rightarrow z : s$}
\RightLabel{$\backslash L$}
\BinaryInfC{$\boxed{(\lambda z'(z \ z') \otimes_2 a) : np\backslash s} \Rightarrow a : np \multimap z : s$}
\RightLabel{$/L$}
\BinaryInfC{$\boxed{(\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a)) : (np\backslash s)/\Box\Diamond s} \Rightarrow (a : np \multimap z : s) \multimapinv (y : np \bullet w : np\backslash s)$}
\RightLabel{D}
\UnaryInfC{$(t \ (\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \ : \ t : (np\backslash s)/\Box\Diamond s \Rightarrow (a : np \multimap z : s) \multimapinv (y : np \bullet w : np\backslash s)$}
\RightLabel{dp}
\doubleLine
\UnaryInfC{$(t \ (\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \ : \ y : np \Rightarrow (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s$}
\RightLabel{R}
\UnaryInfC{$\boxed{\lambda y(t \ (\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) : np} \Rightarrow (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$v : n \Rightarrow \boxed{v : n}$}
\RightLabel{$/L$}
\BinaryInfC{$\boxed{(v \otimes_3 \lambda y(t \ (\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a)))) : np/n} \Rightarrow ((t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s) \multimapinv v : n$}
\RightLabel{D}
\UnaryInfC{$(u \ (v \otimes_3 \lambda y(t \ (\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))))) \ : \ u : np/n \Rightarrow ((t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s) \multimapinv v : n$}
\RightLabel{dp}
\doubleLine
\UnaryInfC{$(u \ (v \otimes_3 \lambda y(t \ (\lambda q\langle q \mid \ \Diamond q'.(w \ (\lambda p(q' \ \Diamond p) \otimes_2 y))\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))))) \ : \ a : np \bullet (t : (np\backslash s)/\Box\Diamond s \bullet ((u : np/n \bullet v : n) \bullet w : np\backslash s)) \Rightarrow z : s$}
\end{prooftree}
$$

Figure 7.17: “Alice thinks a unicorn yawned.” (2)

$$\dfrac{}{p : s \Rightarrow \boxed{p : s}}\ Id$$

$$\dfrac{p : s \Rightarrow \boxed{p : s}}{\langle p : s\rangle \Rightarrow \boxed{\Diamond p : \Diamond s}}\ \Diamond R$$

$$\dfrac{\langle p : s\rangle \Rightarrow \boxed{\Diamond p : \Diamond s}}{(q' \ \Diamond p) \ : \ \langle p : s\rangle \Rightarrow q' : \Diamond s}\ D$$

$$\dfrac{(q' \ \Diamond p) \ : \ \langle p : s\rangle \Rightarrow q' : \Diamond s}{(q' \ \Diamond p) \ : \ p : s \Rightarrow \langle q' : \Diamond s\rangle}\ dp$$

$$\dfrac{(q' \ \Diamond p) \ : \ p : s \Rightarrow \langle q' : \Diamond s\rangle}{\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \ : \ p : s \Rightarrow q : \Box\Diamond s}\ \Box R$$

$$\dfrac{\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \ : \ p : s \Rightarrow q : \Box\Diamond s}{p : s \Rightarrow \boxed{\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle : \Box\Diamond s}}\ R$$

$$\dfrac{}{a : np \Rightarrow \boxed{a : np}}\ Id$$

$$\dfrac{}{z' : s \Rightarrow \boxed{z' : s}}\ Id$$

$$\dfrac{z' : s \Rightarrow \boxed{z' : s}}{(z \ z') \ : \ z' : s \Rightarrow z : s}\ D$$

$$\dfrac{(z \ z') \ : \ z' : s \Rightarrow z : s}{\boxed{\lambda z'(z \ z') : s} \Rightarrow z : s}\ R$$

$$\dfrac{a : np \Rightarrow \boxed{a : np} \qquad \boxed{\lambda z'(z \ z') : s} \Rightarrow z : s}{\boxed{(\lambda z'(z \ z') \otimes_2 a) : np\backslash s} \Rightarrow a : np \multimap z : s}\ \backslash L$$

$$\dfrac{p : s \Rightarrow \boxed{\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle : \Box\Diamond s} \qquad \boxed{(\lambda z'(z \ z') \otimes_2 a) : np\backslash s} \Rightarrow a : np \multimap z : s}{\boxed{(\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a)) : (np\backslash s)/\Box\Diamond s} \Rightarrow (a : np \multimap z : s) \multimapinv p : s}\ /L$$

$$\dfrac{\boxed{(\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a)) : (np\backslash s)/\Box\Diamond s} \Rightarrow (a : np \multimap z : s) \multimapinv p : s}{(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \ : \ t : (np\backslash s)/\Box\Diamond s \Rightarrow (a : np \multimap z : s) \multimapinv p : s}\ D$$

$$\dfrac{(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \ : \ t : (np\backslash s)/\Box\Diamond s \Rightarrow (a : np \multimap z : s) \multimapinv p : s}{(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \ : \ p : s \Rightarrow t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)}\ dp$$

$$\dfrac{(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \ : \ p : s \Rightarrow t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)}{\boxed{\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) : s} \Rightarrow t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)}\ R$$

$$\dfrac{}{y : np \Rightarrow \boxed{y : np}}\ Id$$

$$\dfrac{y : np \Rightarrow \boxed{y : np} \qquad \boxed{\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) : s} \Rightarrow t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)}{\boxed{(\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y) : np\backslash s} \Rightarrow y : np \multimap (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s))}\ \backslash L$$

$$\dfrac{\boxed{(\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y) : np\backslash s} \Rightarrow y : np \multimap (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s))}{(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)) \ : \ w : np\backslash s \Rightarrow y : np \multimap (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s))}\ D$$

$$\dfrac{(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)) \ : \ w : np\backslash s \Rightarrow y : np \multimap (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s))}{(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)) \ : \ y : np \Rightarrow (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s}\ dp$$

$$\dfrac{(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)) \ : \ y : np \Rightarrow (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s}{\boxed{\lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)) : np} \Rightarrow (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s}\ R$$

$$\dfrac{}{v : n \Rightarrow \boxed{v : n}}\ Id$$

$$\dfrac{v : n \Rightarrow \boxed{v : n} \qquad \boxed{\lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)) : np} \Rightarrow (t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s}{\boxed{(v \otimes_3 \lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y))) : np/n} \Rightarrow ((t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s) \multimapinv v : n}\ /L$$

$$\dfrac{\boxed{(v \otimes_3 \lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y))) : np/n} \Rightarrow ((t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s) \multimapinv v : n}{(u \ (v \otimes_3 \lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)))) \ : \ u : np/n \Rightarrow ((t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s) \multimapinv v : n}\ D$$

$$\dfrac{(u \ (v \otimes_3 \lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)))) \ : \ u : np/n \Rightarrow ((t : (np\backslash s)/\Box\Diamond s \multimap (a : np \multimap z : s)) \multimapinv w : np\backslash s) \multimapinv v : n}{(u \ (v \otimes_3 \lambda y(w \ (\lambda p(t \ (\lambda q\langle q \mid \Diamond q'.(q' \ \Diamond p)\rangle \otimes_3 (\lambda z'(z \ z') \otimes_2 a))) \otimes_2 y)))) \ : \ a : np \bullet (t : (np\backslash s)/\Box\Diamond s \bullet ((u : np/n \bullet v : n) \bullet w : np\backslash s)) \Rightarrow z : s}\ dp$$

Figure 7.18: “Alice thinks a unicorn yawned.” (3)

Scope sieves

We next consider scope sieves. Here, non-local scope is enforced, as typically observed with perception verbs [see Hendriks, 1993, p.108]:

3. Alice heard a unicorn yawn.

The desired result obtains by having the matrix verb select for s instead of $\Box \Diamond s$. F.7.19 provides the derivation, resulting in the following simply typed λ-term:

$$(u \, \langle v, \lambda y(w \, \langle \lambda q(h \, \langle q, \langle \lambda z'(z \, z'), a \rangle \rangle), y \rangle) \rangle)$$

the desired result following after lexical insertion:

$$\exists y((\textsc{unicorn} \; y) \wedge (z \; ((\textsc{hear} \; y) \; \textsc{alice})))$$

Ignoring term labeling, the following partial derivation illustrates how subcategorization for s rather than $\Box \Diamond s$ accounts for the unavailability of the local reading:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(np/n \bullet n) \bullet np\backslash s \Rightarrow \boxed{s}}\times \qquad \dfrac{\dfrac{}{np \Rightarrow \boxed{np}}\mathit{Id} \qquad \dfrac{\dfrac{\dfrac{}{s \vdash \boxed{s}}\mathit{Id}}{s \vdash s}D}{\boxed{s} \Rightarrow s}R}{\boxed{np\backslash s} \Rightarrow np \multimap s}\backslash L}{\boxed{(np\backslash s)/s} \Rightarrow (np \multimap s) \multimapinv ((np/n \bullet n) \bullet}/L}{(np\backslash s)/s \Rightarrow (np \multimap s) \multimapinv ((np/n \bullet n) \bullet np\backslash s)}D}{np \bullet ((np\backslash s)/s \bullet ((np/n \bullet n) \bullet np\backslash s)) \Rightarrow s}dp$$

On the other hand, it does not seem clear how to enforce the local reading (e.g., as needed for scope islands), unless when the polarities of atoms are switched from positive to negative. The reader is referred to §5 for further discussion.

Coordination and intensionality

We conclude with intensionality and coordination, treated together to show their interplay. Our treatment of intensionality is simplified w.r.t. our choice of semantic type for interpreting sentences, although this can be easily remedied if the reader so desires. Furthermore, *found* is treated extensionally, in line with Gamut [1991].

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$v : n \Rightarrow \boxed{v : n}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$y : \mathit{np} \Rightarrow \boxed{y : \mathit{np}}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$q : s \Rightarrow \boxed{q : s}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$a : \mathit{np} \Rightarrow \boxed{a : \mathit{np}}$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$z' : s \Rightarrow \boxed{z' : s}$}
\RightLabel{D}
\UnaryInfC{$(z\ z')\ :\ z' : s \Rightarrow z : s$}
\RightLabel{R}
\UnaryInfC{$\boxed{\lambda z'(z\ z') : s} \Rightarrow z : s$}
\RightLabel{$\backslash L$}
\BinaryInfC{$\boxed{(\lambda z'(z\ z') \otimes_2 a) : \mathit{np}\backslash s} \Rightarrow a : \mathit{np} \mathbin{-\!\bullet} z : s$}
\RightLabel{$/L$}
\BinaryInfC{$\boxed{(q \otimes_3 (\lambda z'(z\ z') \otimes_2 a)) : (\mathit{np}\backslash s)/s} \Rightarrow (a : \mathit{np} \mathbin{-\!\bullet} z : s) \mathbin{\bullet\!-} q : s$}
\RightLabel{D}
\UnaryInfC{$(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a)))\ :\ h : (\mathit{np}\backslash s)/s \Rightarrow (a : \mathit{np} \mathbin{-\!\bullet} z : s) \mathbin{\bullet\!-} q : s$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a)))\ :\ q : s \Rightarrow h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s)$}
\RightLabel{R}
\UnaryInfC{$\boxed{\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) : s} \Rightarrow h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s)$}
\RightLabel{$\backslash L$}
\BinaryInfC{$\boxed{(\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y) : \mathit{np}\backslash s} \Rightarrow y : \mathit{np} \mathbin{-\!\bullet} (h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s))$}
\RightLabel{D}
\UnaryInfC{$(w\ (\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y))\ :\ w : \mathit{np}\backslash s \Rightarrow y : \mathit{np} \mathbin{-\!\bullet} (h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s))$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$(w\ (\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y))\ :\ y : \mathit{np} \Rightarrow (h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s)) \mathbin{\bullet\!-} w : \mathit{np}\backslash s$}
\RightLabel{R}
\UnaryInfC{$\boxed{\lambda y(w\ (\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y)) : \mathit{np}} \Rightarrow (h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s)) \mathbin{\bullet\!-} w : \mathit{np}\backslash s$}
\RightLabel{$/L$}
\BinaryInfC{$\boxed{(v \otimes_3 \lambda y(w\ (\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y))) : \mathit{np}/n} \Rightarrow ((h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s)) \mathbin{\bullet\!-} w : \mathit{np}\backslash s) \mathbin{\bullet\!-} v : n$}
\RightLabel{D}
\UnaryInfC{$\boxed{(u\ (v \otimes_3 \lambda y(w\ (\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y)))) : u : \mathit{np}/n} \Rightarrow ((h : (\mathit{np}\backslash s)/s \mathbin{-\!\bullet} (a : \mathit{np} \mathbin{-\!\bullet} z : s)) \mathbin{\bullet\!-} w : \mathit{np}\backslash s) \mathbin{\bullet\!-} v : n$}
\doubleLine
\RightLabel{dp}
\UnaryInfC{$(u\ (v \otimes_3 \lambda y(w\ (\lambda q(h\ (q \otimes_3 (\lambda z'(z\ z') \otimes_2 a))) \otimes_2 y))))\ :\ a : \mathit{np} \bullet (h : (\mathit{np}\backslash s)/s \bullet ((u : \mathit{np}/n \bullet v : n) \bullet w : \mathit{np}\backslash s)) \Rightarrow z : s$}
\end{prooftree}
$$

Figure 7.19: “Alice heard a unicorn yawn.”

4. Alice sought and found a unicorn.

4a. Alice sought a unicorn, and there exists a unicorn y s.t. Alice found y.

4b. There exists a unicorn y s.t. Alice sought y and Alice found y.

For reasons of space, we show only the end result after lexical insertion:

$$(z\ ((\textsc{seek}\ \lambda P \exists y((\textsc{unicorn}\ y) \wedge (P\ y)))\ \textsc{alice})) \wedge \exists y((\textsc{unicorn}\ y) \wedge (z\ ((\textsc{find}\ y)\ \textsc{alice})))\ (4a)$$
$$\exists y((\textsc{unicorn}\ y) \wedge (z\ ((\textsc{seek}\ \lambda P(P\ y))\ \textsc{alice})) \wedge (z\ ((\textsc{find}\ y)\ \textsc{alice})))\ (4b)$$

7.7.2 Type-shifting

Having described $\mathbf{LG}_{\varnothing}$'s semantic expressivity by empirical illustration, we now compare our treatment of scopal ambiguities with a more well-understood solution. Hendriks [1993], generalizing [Partee and Rooth, 1983], proposed a flexible mapping between syntax and semantics by closing the readings associated with a given syntactic derivation under the following derived semantic inference rules, referred to collectively by *type shifting*:

1. **Value Raising**. Assuming right-associative bracketing, terms of type $\sigma_1 \to \cdots \to \sigma_n \to \sigma$ may be lifted to the type $\sigma_1 \to \cdots \to (\sigma \to \tau') \to \tau'$, for any τ'. E.g., a proper name 'Alice' with interpretation of type e undergoes value raising to derive a term of the type $(e \to t) \to t$ for generalized quantifiers.

2. **Argument Raising**. Inhabitants of $\sigma_1 \to \cdots \to \sigma_n \to \tau$ may transition into $\sigma_1 \to \cdots \to ((\sigma_i \to \tau') \to \tau') \to \cdots \to \sigma_n \to \tau$. Typically applies to the subject and object positions of transitive verbs. E.g., $e \to (e \to t)$ undergoes argument raising twice to derive the type of a binary relation on generalized quantifiers. Depending on which of the arguments is raised first, we get different readings.

3. **Argument Lowering**. Inhabitants of $\sigma_1 \to \cdots \to \sigma_n \to \tau$ derive from terms in $\sigma_1 \to \cdots \to ((\sigma_i \to \tau') \to \tau') \to \cdots \to \sigma_n \to \tau$. E.g., lowering and subsequent raising of an intensional verb's object position derives a de re reading.

By applying the right combinations of these rules, all combinatorially available scopal readings for a given sentence can be derived, as proved by Hendriks. What is important to note is that all rules involved are derivable within λ-calculus. In contrast, analogous rules for the traditional incarnations of type-logical grammar necessitate full associativity and commutativity, leading to serious overgeneration. Consequently, type-shifting seemed exclusive to the realm of semantics, necessitating the relaxation of compositionality if it was to be made any use of.

We claim the existence of derivable rules of inference within the term language for $\mathbf{LG}_\varnothing$ mapping to Hendriks' type-shifting schemas under semantic interpretation. Consequently, the strict correspondence between syntax and semantics can be restored, while preserving semantic expressivity. One restriction, however, applies: the types τ' in the above explanation of type-shifting always are to be t. In practice, sufficient generality is retained to account for scoping within clausal domains.

Theorem 7.7.1. The following rules of *Value Raising* (VR), *Argument Raising* (AR) and *Argument Lowering* (AL) are derivable within $\mathbf{LG}_\varnothing$:

$$\begin{array}{rcll}
\Gamma \vdash s : A & \Rightarrow & \Gamma \vdash \lambda x(x\ s) : A^{\bot\bot} & (VR)\\
\Gamma \vdash s : (A \otimes_3 B)^\bot & \Rightarrow & \Gamma \vdash \lambda\langle x, v\rangle(x\ \lambda u(s\ \langle u, v\rangle)) : (A^{\bot\bot} \otimes_3 B)^\bot & (AR^3)\\
\Gamma \vdash s : (A \otimes_2 B)^\bot & \Rightarrow & \Gamma \vdash \lambda\langle v, x\rangle(x\ \lambda u(s\ \langle v, u\rangle)) : (A \otimes_2 B^{\bot\bot})^\bot & (AR^2)\\
\Gamma \vdash s : (A^{\bot\bot} \otimes_3 B)^\bot & \Rightarrow & \Gamma \vdash \lambda\langle u, v\rangle(s\ \langle \lambda x(x\ u), v\rangle) : (A \otimes_3 B)^\bot & (AL^3)\\
\Gamma \vdash s : (A \otimes_2 B^{\bot\bot})^\bot & \Rightarrow & \Gamma \vdash \lambda\langle v, u\rangle(s\ \langle v, \lambda x(x\ u)\rangle) : (A \otimes_2 B)^\bot & (AL^2)
\end{array}$$

Example 24. We illustrate T.7.7.1 with the assignment of the word 'found' with a type $(np \otimes_3 (s^\bot \otimes_2 np))^\bot$ and term $\lambda\langle y, \langle q, x\rangle\rangle(q\ ((\textsc{find}\ y)\ x))$. Note the latter is of type $\neg(e \times (\neg t \times e))$, which by uncurrying amounts to $e \to (e \to \neg\neg t)$. By (AR^3),

$$\text{found} \vdash \lambda\langle Y, \langle q, x\rangle\rangle(Y\ \lambda y(q\ ((\textsc{find}\ y)\ x))) : (np^{\bot\bot} \otimes_3 (s^\bot \otimes_2 np))^\bot$$

which is of semantic type $\neg(\neg\neg e \times (\neg t \times e))$, i.e., $\neg\neg e \to (e \to \neg\neg t)$.

We still lack the means to target other positions besides the direct object for Argument Raising (or -Lowering), a possibility rendered available in Hendriks' presentation by allowing for any number of arguments preceding the one targeted by type-shifting. The following result allows any intervening arguments to be stripped off one by one, and to be added back after the desired shift has been applied.

Lemma 86. The following are derivable rules of inference:

$$\frac{\Gamma \bullet_1 x : A \vdash s : B^\bot}{\Gamma \vdash \lambda\langle x, y\rangle(s\ y) : (A \otimes_3 B)^\bot} <
\qquad
\frac{x : B \bullet_1 \Gamma \vdash s : A^\bot}{\Gamma \vdash \lambda\langle y, x\rangle(s\ y) : (A \otimes_2 B)^\bot} >$$

$$\frac{\Gamma \vdash s : (A \otimes_3 B)^\bot}{\Gamma \bullet_1 x : A \vdash \lambda y(s\ \langle x, y\rangle) : B^\bot} <'
\qquad
\frac{\Gamma \vdash s : (A \otimes_2 B)^\bot}{x : B \bullet_1 \Gamma \vdash \lambda y(s\ \langle y, x\rangle) : A^\bot} >'$$

Example 25. Continuing where we left off in E.24, (AR^2) combines with $(<)$ and $(<')$ to allow for Argument Raising of the subject position, resulting in

$$\text{found} \vdash \lambda\langle Y, \langle q, X\rangle\rangle(X\ \lambda x(Y\ \lambda y(q\ ((\textsc{find}\ y)\ x)))) : (np^{\bot\bot} \otimes_3 (s^\bot \otimes_2 np^{\bot\bot}))^\bot$$

the term involved being of type $\neg\neg e \to (\neg\neg e \to \neg\neg t)$ after uncurrying. Note we could also have applied (AR^2) first, followed by (AR^3) to derive

$$\text{found} \vdash \lambda\langle Y, \langle q, X\rangle\rangle(Y\ \lambda y(X\ \lambda x(q\ ((\textsc{find}\ y)\ x)))) : (\mathit{np}^{\perp\perp} \otimes_3 (s^{\perp} \otimes_2 \mathit{np}^{\perp\perp}))^{\perp}$$

The latter allows for the derivation of object-wide scope readings, whereas raising the subject after the object favors subject-wide scope.

7.7.3 Evaluation

We briefly discuss, first, a point of critique concerning our capacity to block scopal readings that, while combinatorially possible, are unrealized by linguistic reality. Second, we draw attention to a curious discrepancy with Hendriks' account of verbs taking clausal complements, and, finally, make a brief comparison with a proposal for the categorial analysis of scope closely related to ours.

Blocking scopal readings

While we could ensure derivability of all combinatorially available scopal readings for the various examples we considered, this also constitutes the main limitation of our approach: situations where linguistic reality excludes certain readings are not so easily accounted for. While we had a small success with the analysis of scope sieves, scope islands fall outside our coverage. The reasons go deep: through the display postulates, any formula may be isolated as the whole of one of the sequents' components, which may subsequently be abstracted over through $({}^{\perp}I)$. In other words, accounting for scope islands means to restrict the display postulates, which constitute an essential ingredient for the notion of structure adopted by **LG**.

On verbs taking clausal complements

Our analysis of verbs taking clausal complements is host to an interesting curiosity, setting it apart from Hendriks' account. One might first expect the category $(\mathit{np}\backslash s)/\mathit{np}$ assigned to *heard* in §7.1 to provide at least the local reading, with type-shifting required for non-local scope construal. Such a situation would parallel Hendriks [1993], which starts from a minimal semantic type assignment. Instead, we find that only the non-local reading is derived, whereas a local reading necessitates the prefixing of the s argument by $\Box\Diamond$.

(Co)negations

Another account of scopal ambiguities similar to ours was recently put forward by Moortgat [2010], also working within **LG**. There, scopal ambiguities were accounted for by adding minimal (co)negations to the logical vocabulary, used similarly to our shifts. We note, however, that Moortgat's CPS translation can be factorized through our term language for **LG**, showing that his minimal (co)negations serve to enforce the appearance of linear negations (i.e., polarity shifts) in the target, much similar to our use of $\diamond$ and $\square$.

7.8 Related Topics

7.8.1 Synthetic inference rules

Our account of strong focalization makes essential use of so-called 'synthetic' inference rules, abbreviating successive left introductions of $\otimes_i$ and $\diamond$ into a single inference step. Since [Girard, 2000, Andreoli, 2001], the literature on focussed proof search has become home to various implementations of such synthetic rules, mostly concerning classical (linear) logic. The current account borrows a bit from everything, but is perhaps most similar to that of Zeilberger [2008] in its treatment of non-invertible inferences, while more strongly resembling [Andreoli, 2001] for the invertible rules. It should be noted, however, that Zeilberger's work stresses a higher order interpretation of focussed proofs through the use of Martin-Löfs generalized inductive definitions, and proves normalization accordingly.

7.8.2 Focussing as a semantics of proofs

Following Andreoli, we have explained focussing as a method of streamlining Cut-free backward-chaining proof search. Around the same time as [Andreoli, 1992], however, Girard [1991] independently published on a similar sequent calculus (weakly focalized, by current terminology) for classical logic, with the aim of restoring the Church-Rosser property for Cut elimination, bypassing Lafont's critical pairs. In particular, Girard's results inspired a novel translation into intuitionistic logic (adapted to **LG** in §2), achieving parsimony by making the introduction of double negations contingent upon the polarity of the formula being translated. Focussed derivations thus seem particularly suited to serve as a constructive theory of (classical) proofs, a theme further pursued by Zeilberger [2008].

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Samenvatting in het Nederlands

Onderwerp van dit proefschrift is de categoriale typenlogica (CTL); een discipline op het grensvlak van de taalkunde en de wiskundige logica. Zij behelst grofweg een identificatie tussen, in de eerste plaats, syntactische categorieën en logische formules, alsmede tussen grammaticaliteit en bewijsbaarheid. Meer algemeen wordt gesproken van een tak van de categoriale grammatica, zoals voortgekomen uit de werken van Ajdukiewicz en Bar-Hillel. Waar laatstgenoemde nog primair een interpretatie in de aritmetica voorzag, bood Lambek in de late jaren '50 en vroege jaren '60 een logische herconceptualisering van de betrokken concepten, daarmee de geboorte inluidend van CTL.

Naar aanleiding van Lambeks werk uitte Chomsky het vermoeden dat er qua expressieve kracht geen winst werd geboekt op context-vrije herschrijfgrammatica's, hetgeen met de tijd werd bevestigd door Pentus. Ten gevolge hiervan mogen we heden ten dage spreken van een CTL *landschap*, zijnde verdeeld over verschillende voorstellen aangaande de juiste aanpak van genoemde beperking. Het merendeel draagt echter de erfenis van een asymmetrie aanwezig in Lambeks oorspronkelijke voorstel: een veelvoud van hypotheses (de categoriën toegekend aan woorden) wordt afgezet tegen een unieke conclusie (de categorie toegekend aan de samenstellende expressie). Dit proefschrift betreft een logisch en linguïstisch onderzoek naar de gelijktrekking tussen de status van hypothese en conclusie in CTL.

De vroege jaren '90 zagen een herstel van symmetrie in onafhankelijk werk van Abrusci, Lambek en Yetter, geïnspireerd door het succes van Girards (klassieke) lineaire logica. Reeds anticiperend op dit gedachtengoed was echter een voorstel van Grishin tot de dualisering van het logisch vocabulaire zoals aanwezig in Lambeks grondleggend werk aan CTL. Uniek aan zijn ideeën bleek de systematische verkenning van mogelijkheden tot interactie tussen de twee families van connectieven. Onder de noemer *Lambek-Grishin calculus* (**LG**) werd Grishins werk in de laatste jaren weer opgepakt door Moortgat en consorten, met een nadruk op de taalkundige toepassingen van Grishins interactie principes. Gegeven haar relatieve jeugd, echter, betreffen zowel de wiskundige als linguïstische eigenschappen van **LG** nog goeddeels onontgonnen gebied. De contributies van dit proefschrift betreffen voornamelijk **LG**, en kunnen als volgt worden opgesomd.

1. Wij beschouwen calculi nauw gerelateerd aan **LG**, in het bijzonder de klassiek non associatieve Lambek calculus van De Groote en Lamarche, alsmede een non associatieve variant van Lambeks bilineaire logica. Een sequentencalculus voor laatstgenoemde wordt gedefinieerd, in vergezelling van een modeltheoretisch bewijs van toelaatbaarheid van de snede regel.

2. Op het gebied van expressiviteit tonen wij aan dat in afwezigheid van enige interactie principes het (zwakke) uitdrukkingsvermogen van LG overeenkomt met dat van context-vrije herschrijfgrammatica's. Dit resultaat complementeert eerdere bevindingen van Moot en Melissen betreffende de expressieve ondergrens in het bijzijn van een specifieke keuze voor interacties. Ter verkrijging van dit resultaat definiëren wij bovendien een gelabeld afleidingsformaat in de stijl van Negri en Kurtonina, informatie incorporerend over Kurtonina en Moortgats relationele modellen voor **LG**

3. Voortbouwend op werk van Moortgat en Pentus tonen wij gezondheid en volledigheid van de kleinste equivalentierelatie die afleidbaarheid bevat t.o.v. een notie van model geïnspireerd door Cockett en Seely's zwak distributieve categoriën. Linguïstische toepassingen worden gevonden in de terugdringing van lexicale ambiguïteit.

4. Als laatste thema bespreken wij formele semantiek in de Montagoviaanse traditie. Traditionele incarnaties van CTL delen hun asymmetrie met dat van de intuïtionistische logica, overeenkomend met het extensionele fragment van Montagues λ-calculus. De symmetrie inherent aan **LG** gaat echter ten koste van deze directe correspondentie. In voorgaand werk bieden Bernardi en Moortgat ter oplossing een aanpassing van de dubbele negatie vertalingen van Cockett en Seely. Wij bouwen voort op deze resultaten met een derde translatie, overeenkomend met een aanpassing van Girards gepolariseerde interpretatie van klassieke logica, en met het expliciet maken van een correspondentie van deze translaties met zoekstrategieën voor de automatische constructie van snede-vrije bewijzen.

Curriculum Vitae

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Education

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Publications

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