THE GALERKIN FINITE ELEMENT METHOD FOR ADVECTION DIFFUSION EQUATION

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CONTENT

INTRODUCTION

- The advection-diffusion equation is generally used to describe:
  - Mass
  - Heat
  - Velocity and Vorticity
- The equation has been used as a model equation in many engineering and chemistry problems such as:
  - Spread of pollutants in rivers and streams
  - Thermal pollution in river systems
  - Flow in porous media
  - Dispersion of tracers in porous media
  - Dispersion of dissolved material in estuaries and coastal area
  - Intrusion of salt water into fresh water aquifers
  - Spread of solute in a liquid flowing through a tube
  - Contaminant dispersion in shallow lakes
  - Model water transport in soils

TARGET PROBLEM

Consider the one-dimensional advection –diffusion equation

$$\frac{\partial u}{\partial t}+\alpha\frac{\partial u}{\partial x}-\lambda\frac{\partial^2 u}{\partial x^2}=0, a\le x\le b \qquad (1)$$

With the initial condition

$$U(x,0)=\varphi(x) \qquad (2)$$

and the boundary condition (Dirichlet boundary condition)

$$U(Lx,t)=L(t), U(Rx,t)=R(t), t>0 \qquad (3)$$

In a restricted solution domain over an space/time interval $[a,b]\times[0,T]$

In EQ.(1), α and λ denote the steady uniform velocity abd the constant diffusion coefficient respectively

WEAK FROMULATION OF THE PROBLEM

The weak formulation of the given problem (1) over a typical element (x_a, x_b) is given by

$$\int_{x_a}^{x_b} w(x)\left\{\frac{\partial U}{\partial t} - \lambda\frac{\partial^2 U}{\partial x^2} + \alpha\frac{\partial U}{\partial x}\right\}dx = 0 \qquad (4)$$

Where w(x) are arbitrary test functions and may be viewed as the variation in U(x). We choose the test function w(a) =w(b) = 0 to reduce the order of integration, then we arrive at the following of equations

$$\int_{x_a}^{x_b} w(x)\{U_{\mathrm{t}} + \lambda w_x U_x + \alpha w(x) U_x\}dx = 0 \qquad (5)$$

FINITE ELEMENT FORMULATION OF THE PROBLEM

The finite-element model can be obtained form the Eq.(5) by substituting finite element approximations in the form

$$U(x,t) = U_h^n = \sum_{j=1}^{N} u_j \phi_j \qquad (6)$$

Subtituting w=ϕ_i and (6) in equation (5) to obtain the ith equation of this system, we have

$$\int_{x_a}^{x_b} \left[\phi_i \left(\sum_{j=1}^{N} \frac{du_j}{dt} \phi_j \right) + \alpha \phi_i \left(\sum_{j=1}^{N} u_j \frac{d\phi_j}{dx} \right) + \lambda \frac{d\phi_i}{dx} \left(\sum_{j=1}^{N} u_j \frac{d\phi_j}{dx} \right) \right] dx = 0 \qquad (7)$$

$$\sum_{j=1}^{N} \left[\left(\int_{x_a}^{x_b} \phi_i \phi_j dx \right) \frac{du_j}{dt} + \alpha \left(\int_{x_a}^{x_b} \phi_i \frac{d\phi_j}{dx} dx \right) u_j + \lambda \left(\int_{x_a}^{x_b} \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx \right) u_j \right] = 0 \qquad (8)$$

FINITE ELEMENT DISCRETIZATION

Assume that we have a uniform partition of [a,b] into M element with mesh size $h = \frac{b-a}{M}$.

Total time T is divided into N element, $t = \frac{T}{N}$

The time direction is discretized by Crank-Nicolson method, then we can obtain

$$\sum_{t=1}^{N} \sum_{i=1}^{M} \left(u_i^n \int_{\Omega i} \phi_i \phi_j dx - u_i^{n-1} \int_{\Omega i} \phi_i \phi_j dx \right)$$

$$= \sum_{t=1}^{N} \left(-\frac{\Delta t}{2} \sum_{i=1}^{M} u_i^n \int_{\Omega i} \phi_i' \phi_j' dx - \frac{\Delta t}{2} \sum_{i=1}^{M} u_i^n \int_{\Omega i} \phi_i \phi_j' dx \right) + \tag{9}$$

$$\sum_{t=1}^{N} \left(-\frac{\Delta t}{2} \sum_{i=1}^{M} u_i^{n-1} \int_{\Omega i} \phi_i' \phi_j' dx - \frac{\Delta t}{2} \sum_{i=1}^{M} u_i^{n-1} \int_{\Omega i} \phi_i \phi_j' dx \right)$$

$$\sum_{t=1}^{N} \sum_{i=1}^{M} \left(\int_{\Omega i} \phi_i \phi_j dx + \frac{\Delta t}{2} \int_{\Omega i} \phi_i' \phi_j' dx + \frac{\Delta t}{2} \int_{\Omega i} \phi_i \phi_j' dx \right) u_i^n \tag{10}$$

$$= \sum_{t=1}^{N} \sum_{i=1}^{M} \left(\int_{\Omega i} \phi_i \phi_j dx - \frac{\Delta t}{2} \int_{\Omega i} \phi_i' \phi_j' dx - \frac{\Delta t}{2} \int_{\Omega i} \phi_i \phi_j' dx \right) u_i^{n-1}$$

DISCRETIZATION FORMULATION

$$\sum_{t=1}^{N}\sum_{i=1}^{M}\left(\frac{\Delta t}{2}A+\frac{\Delta t}{2}B+C\right)u_i^n=\sum_{t=1}^{N}\sum_{i=1}^{M}\left(C-\frac{\Delta t}{2}A-\frac{\Delta t}{2}B\right)u_i^{n-1} \qquad (11)$$

$$A(i,i)=\sum_{i=1}^{M}\int_{\Omega i}\phi_i^{'}(x)\phi_i^{'}(x)dx$$

$$=\int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^{'}\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^{'}dx+\int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^{'}\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^{'}dx$$

$$=\int_0^1\frac{1}{h_i^{\ 2}}d\left((x_i-x_{i-1})\xi+x_{i-1}\right)+\int_0^1\frac{1}{h_{i+1}^{\ \ 2}}d\left((x_{i+1}-x_i)\xi+x_i\right)$$

$$=\int_0^1\frac{1}{h_i}d(\xi)+\int_0^1\frac{1}{h_{i+1}}d(\xi)$$

$$=\frac{2}{h}$$

$$\xi=\frac{x-x_{i-1}}{x_i-x_{i-1}}$$

$$x=(x_i-x_{i-1})\xi+x_{i-1}$$

$$h_i=x_i-x_{i-1}$$

$$h_{i+1}=x_{i+1}-x_i$$

$$\frac{x_{i+1}-x}{x_{i+1}-x_i}=1-\frac{x-x_i}{x_{i+1}-x_i}=1-\xi$$

DISCRETIZATION FORMULATION

$$A_{(\mathrm{i},i-1)}=\sum_{i=1}^{M}\int_{\Omega i}\phi_i^{'}(x)\phi_{i\text{-}1}^{'}(x)dx$$

$$=\int_{x_1}^{x_2}\left(\frac{x_2-x}{x_2-x_1}\right)^{'}\left(\frac{x-x_1}{x_2-x_1}\right)^{'}dx$$

$$=\int_0^1\frac{-1}{h_1^{\ 2}}d\left(\left(x_2-x_1\right)\xi+x_1\right)$$

$$=\frac{-1}{h}$$

$$A_{(\mathrm{i\text{-}1},i)}=\sum_{i=1}^{M}\int_{\Omega i}\phi_i^{'}(x)\phi_{i\text{-}1}^{'}(x)dx$$

$$=\int_{x_M}^{x_{M+1}}\left(\frac{x_{M+1}-x}{x_{M+1}-x_M}\right)^{'}\left(\frac{x-x_M}{x_{M+1}-x_M}\right)^{'}dx$$

$$=\int_0^1\frac{-1}{h^{\ 2}}d\left(h\xi+x_M\right)$$

$$=\frac{-1}{h}$$

DISCRETIZATION FORMULATION

$$B_{(i,i)} = \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)' dx + \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)' dx$$

$$= \int_0^1 \frac{1}{h_i} \xi d\left((x_i - x_{i-1})\xi + x_{i-1}\right) + \int_0^1 \frac{-1}{h_{i+1}}(1-\xi) d\left((x_i - x_{i-1})\xi + x_{i-1}\right)$$

$$= \int_0^1 \xi d(\xi) - \int_0^1 (1-\xi) d(\xi)$$

$$= 0$$

$$B_{(i,i-1)} = \int_1^2 \phi_i \phi_{i-1}' dx$$

$$= \int_1^2 \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)' dx$$

$$= \int_0^1 (\xi) \frac{1}{h_1} d(h_1 \xi + x_1)$$

$$= \frac{1}{2}$$

$$B_{(i-1,i)} = \int_M^{M+1} \phi_{i-1} \phi_{i-1}' dx$$

$$= \int_{M-1}^{M} \left(\frac{x_{M+1} - x}{x_{M+1} - x_M}\right)\left(\frac{x_{M+1} - x}{x_{M+1} - x_M}\right)' dx$$

$$= \int_0^1 (1-\xi) \frac{-1}{h_M} d(h_1 \xi + x_1)$$

$$= -\frac{1}{2}$$

DISCRETIZATION FORMULATION

$$C_{(i,i)} = \int_{\Omega} (\phi_i)^2 dx = \sum_{i=1}^{M} \int_{\Omega i} \left(\phi_i^{\,2}\right) dx$$

$$= \int_{x_{i-1}}^{x_i} (\phi_i)^2 dx + \int_{x_i}^{x_{i+1}} (\phi_i)^2 dx$$

$$= (x_i - x_{i-1}) \int_0^1 \xi^2 d\xi + (x_{i+1} - x_i) \int_0^1 (1-\xi)^2 d\xi$$

$$= \frac{1}{3} h_i + \frac{1}{3} h_{i+1}$$

$$C_{(i,i-1)} = \int_{\Omega} (\phi_i \phi_{i-1}) dx = \sum_{i=1}^{M} \int_{\Omega i} (\phi_i \phi_{\mathrm{i}-1}) dx$$

$$= \int_{x_1}^{x_2} (\phi_2 \phi_1) dx$$

$$= (x_2 - x_1) \int_0^1 \xi (1-\xi) d\xi$$

$$= \frac{1}{6} h_1$$

$$C_{(i-1,i)} = \int_{\Omega} (\phi_i \phi_{i-1}) dx = \sum_{i=1}^{M} \int_{\Omega i} (\phi_i \phi_{\mathrm{i}-1}) dx$$

$$= \int_{x_M}^{x_{M+1}} (\phi_{M+1} \phi_M) dx$$

$$= (x_{M+1} - x_M) \int_0^1 \xi (1-\xi) d\xi$$

$$= \frac{1}{6} h$$

TEST PROBLEMS

For the test problems, accuracy of the proposed algorithms is worked out by measuring error norm $L_\propto$, L_2 and Ch^2

$L_\propto = \max |u_m$-$U_m|$, and the order of convergence is computed by the formula

$$\text{convergent } rate = \frac{\log |(L_\propto)_{hi}/((L_\propto)_{hi+1}|}{\log |\frac{h_i}{h_{i+1}}|}$$

Where $(L_\propto)_{hi}$ is the error norm $L_\propto$]

$L_2 = \sqrt{}((u_1 - U_1)^2 + \ldots (u_M - U_M)^2)$

$$\text{convergent } rate = \frac{\log |(L_2)_{hi}/((L_2)_{hi+1}|}{\log |\frac{h_i}{h_{i+1}}|}$$

$||u_m - U_m|| \cong Ch^2$

$$\log(u_m - U_m) = \log C + 2\log h$$

FIRST TEST PROBLEM

The exact solution of advection-diffusion is

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x - \widetilde{x}_0 - \alpha t)^2}{\lambda(4t+1)} \right) \qquad (12)$$

This solution corresponds to a wave of magnitude $\frac{1}{\sqrt{1+4t}}$, initially centered on the position x0 propagating towards the right across the interval [a,b] over the time T with a steady velocity α. In this solution, we set the time t=5 with α =0.8m/s, $\lambda = \frac{0.005m2}{s}$ and 0$\leq x \leq 9$.

$$L_X = 0, R_x = 9, M = N = 100, L_t = 0, R_t = 5$$

$initial\ condition$: $@(x)1/sqrt(4 * t0 + 1) * exp(-(x - x0 - beta * t0).\hat{}2/(alpha * (4 * t0 + 1)))$

Boundary condition

U(0,t)=L(t)=@(t)1./sqrt(4*t+1)*exp(-(Lx-x0-beta*t).^2/(alpha*(4*t+1)))
U(5,t)= R(t)=@(t)1./sqrt(4*t+1)*exp(-(Rx-x0-beta*t).^2/(alpha*(4*t+1)))

RESULTS

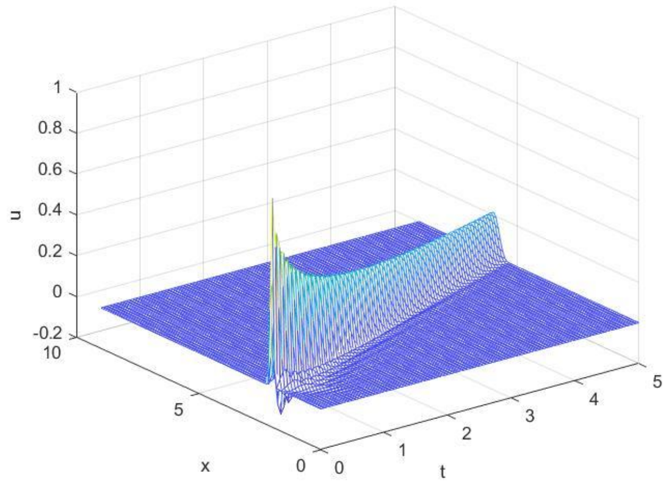


Figure 1. Wave profiles

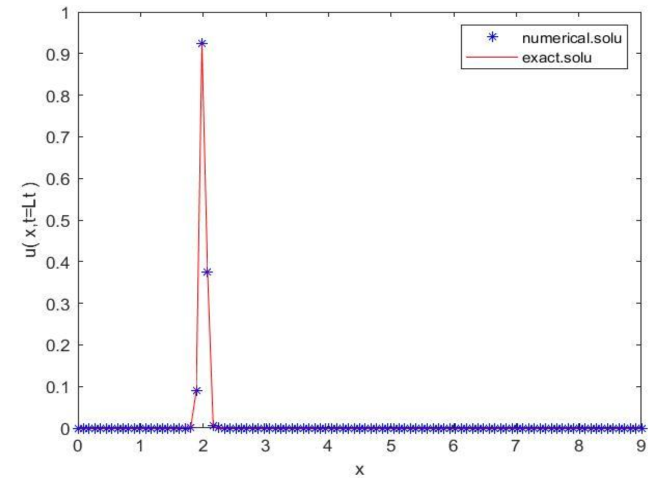


Figure2. Comparation between the exact solution and the numerical solution at t=0s

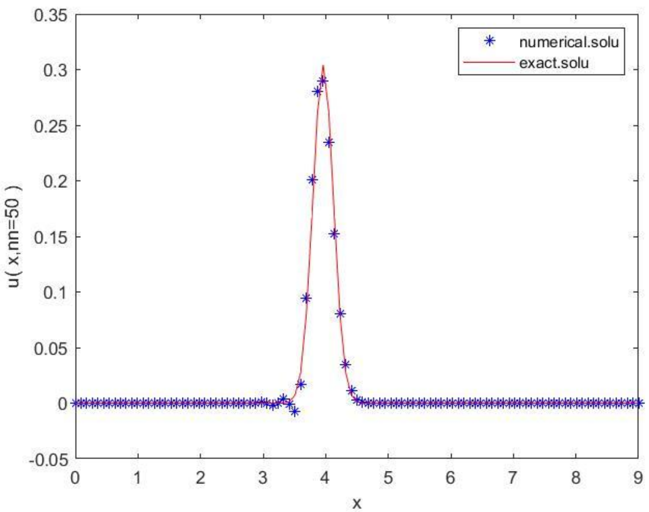


Figure3. Comparation between the exact solution and the numerical solution at t=2.5s

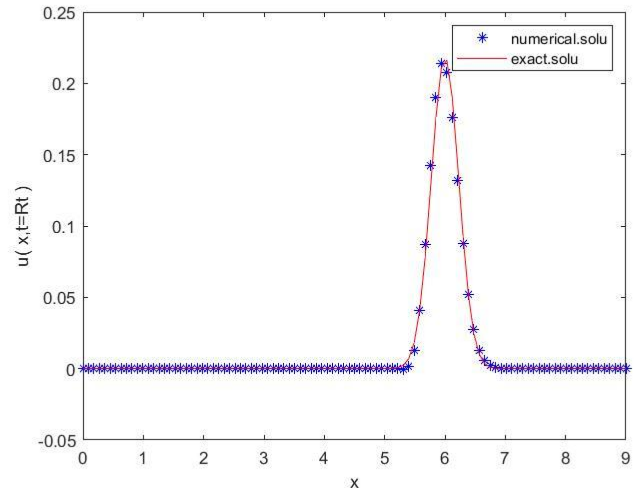


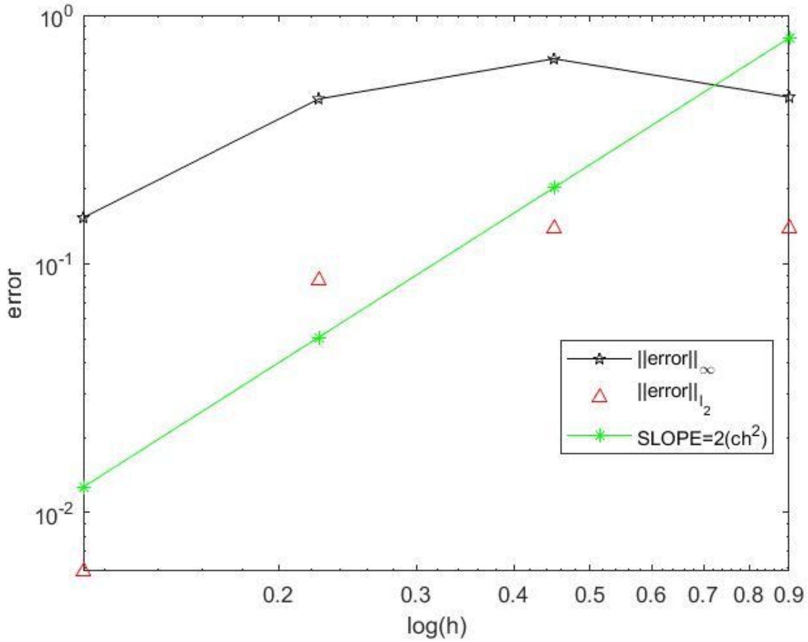
Figure 4. Comparation between the exact solution and the numerical solution at t=5s

ERROR AND RATE OF CONVERGENCE

- Error norm $L_\propto$, L_2 and order of convergence rate are analyzed at time=5.

X	h	Max_error	Convergent rate	L2_error	Convergent rate
0.9 1.8 3.6 7.2	0.9	0.4679	-0.5134	0.1389	-0.0003
0.9 1.8 3.6 7.2	0.45	0.6679	0.5372	0.1390	0.6931
0.9 1.8 3.6 7.2	0.225	0.4603	1.5865	0.0860	3.8874
0.9 1.8 3.6 7.2	0.1125	0.1533		0.0058	

ERROR AND RATE OF CONVERGENCE



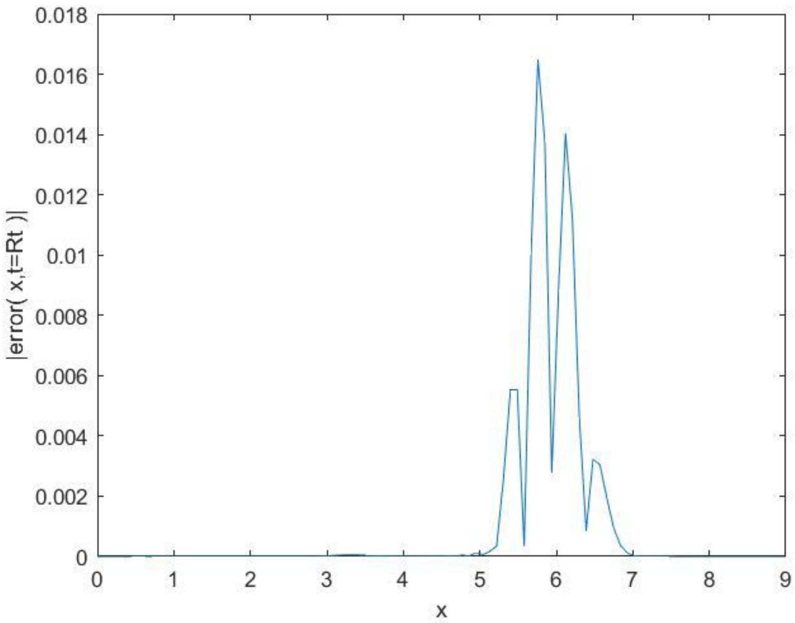


Figure 5. Error norm $L_\propto$, $L_{2,}$ ch^2and order of convergence rate are analyzed at time=5.

Figure 6. Absolute error at time t=5.

SECOND TEST PROBLEM

The exact solution of advection-diffusion is

$$U(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(\frac{-(x - 0.5 - t)^2}{0.00125 + 0.04t} \right) \qquad (13)$$

This analytical solution of the EQ (13) in the region bounded by $0 \leq x \leq 1$. In this solution, we set the time t=2 with α =1m/s, $\lambda = \frac{0.01m2}{s}$

$$L_X = 0, R_x = 1, M = N = 100, L_t = 0, R_t = 2$$

$$initial\ condition: @(x)exp(-(x - 0.5).\hat{}2/0.00125) * 0.025/sqrt(0.000625)$$

Boundary condition

U(0,t)=L(t)=@(t) exp(-(-0.5-t).^2./(0.00125+0.04*t))*0.025/sqrt(0.000625+0.02*t)
U(1,t)= R(t)=@(t) exp(-(0.5-t).^2./(0.00125+0.04*t))*0.025/sqrt(0.000625+0.02*t)

RESULTS

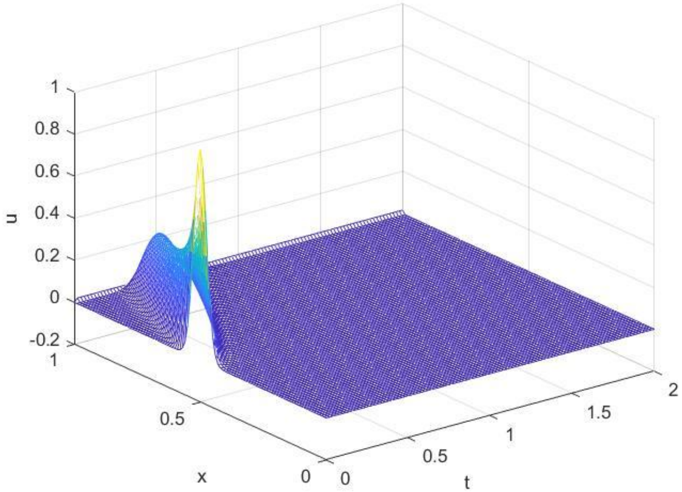


Figure 7. Wave profiles

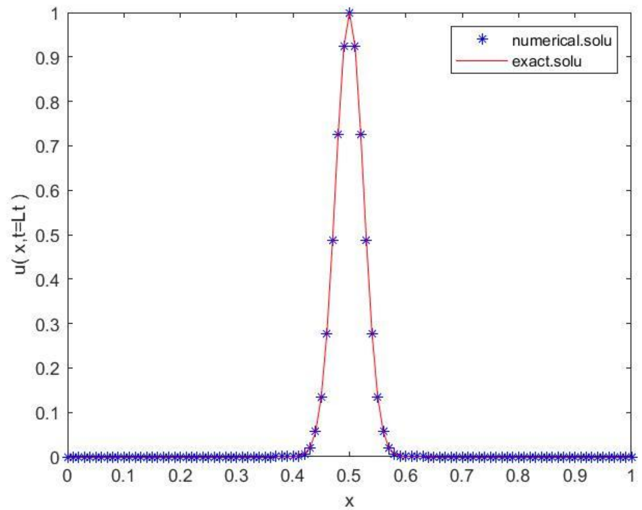


Figure8. Comparation between the exact solution and the numerical solution at t=0s

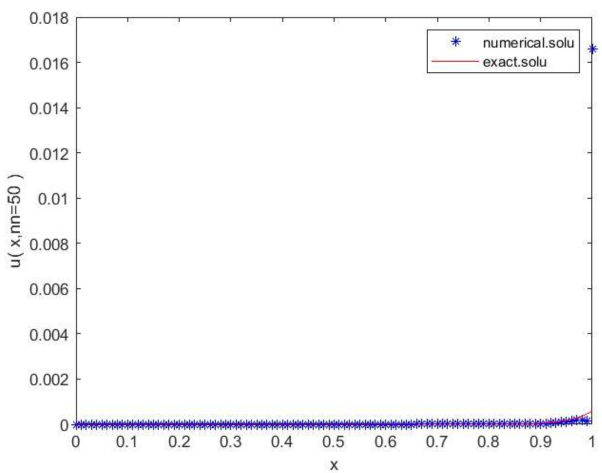


Figure9. Comparation between the exact solution and the numerical solution at t=1s

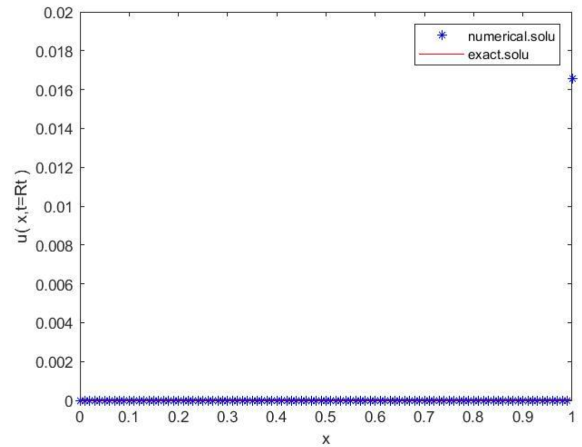


Figure 10. Comparation between the exact solution and the numerical solution at t=2s

ERROR AND RATE OF CONVERGENCE

- Error norm $L_\propto$, L_2 and order of convergence rate are analyzed at time=2.

X	h	Max_error	Convergent rate	L2_error	Convergent rate
0.1 0.2 0.4 0.8	0.1000	0.4004	0.8189	0.0034	0.6592
0.1 0.2 0.4 0.8	0.0500	0.2270	0.0002	0.0022	0.4973
0.1 0.2 0.4 0.8	0.0250	0.2269	0.0000	0.0015	0.4993
0.1 0.2 0.4 0.8	0.0125	0.2269		0.0011	

ERROR AND RATE OF CONVERGENCE

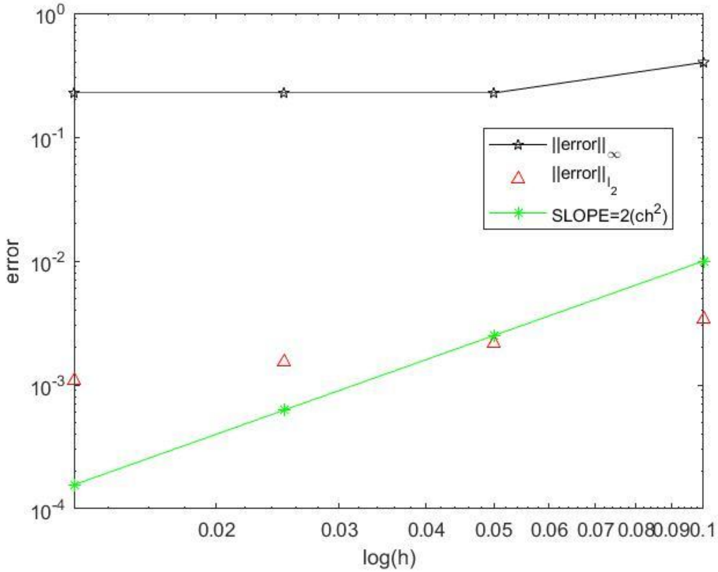


Figure 11. Error norm $L_{\propto}$, $L_{2,}$ ch^2and order of convergence rate are analyzed at t=2.

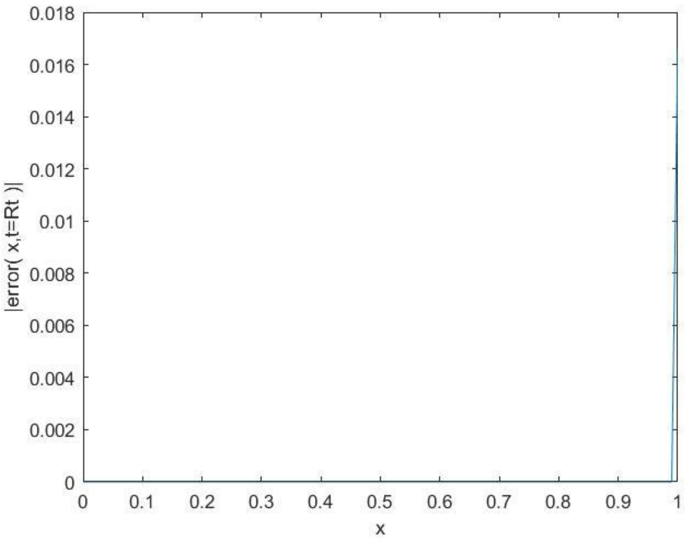


Figure 12. Absolute error at time t=2.

THIRD TEST PROBLEM

$L_X = 0, R_x = 2, M = N = 100, L_t = 0, R_t = 1$

$initial\ condition$: $@(x)$ exp$((x-2).^2)$

$Boundary\ condition$: $U(0,t) = L(t) = @(t)$ 0, $U(1,t) = R(t) = @(t)$ 0

Crank-Nicolson for time discretization

$$Test1\text{: } \alpha = 1m/s,\ \lambda = \frac{0m2}{s}$$

$$Test2\text{: } \alpha = 0m/s,\ \lambda = \frac{0.05m2}{s}$$

$$Test3\text{: } \alpha = 1m/s,\ \lambda = \frac{0.05m2}{s}$$

Backward Euler for time discretization

$$Test4\text{: } \alpha = 1m/s,\ \lambda = \frac{0m2}{s}$$

$$Test5\text{: } \alpha = 0m/s,\ \lambda = \frac{0.05m2}{s}$$

$$Test6\text{: } \alpha = 1m/s,\ \lambda = \frac{0.05m2}{s}$$

RESULTS

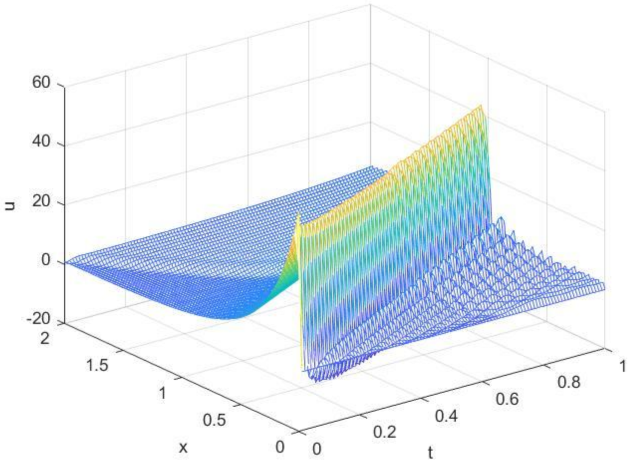


Figure 13.Numerical solution of Test1 for$\alpha = \frac{1m}{s}$, $\lambda = \frac{0m2}{s}$

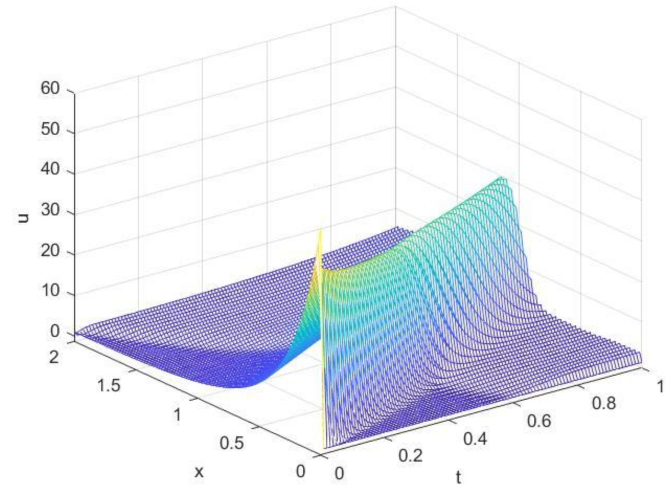


Figure14. Numerical solution of Test4 for$\alpha = \frac{1m}{s}$, $\lambda = \frac{0m2}{s}$

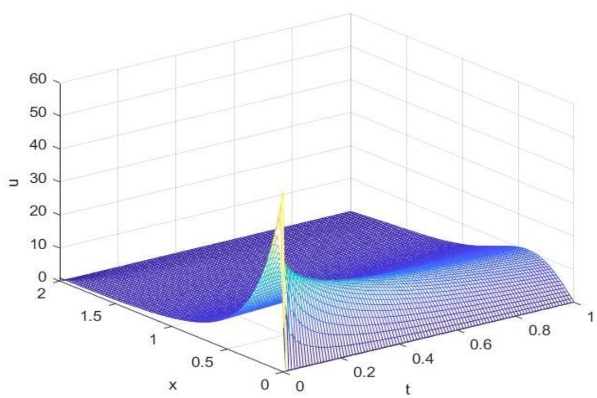


Figure15. Numerical solution of Test2 for$\alpha = \frac{0m}{s}$, $\lambda = \frac{0.05m2}{s}$

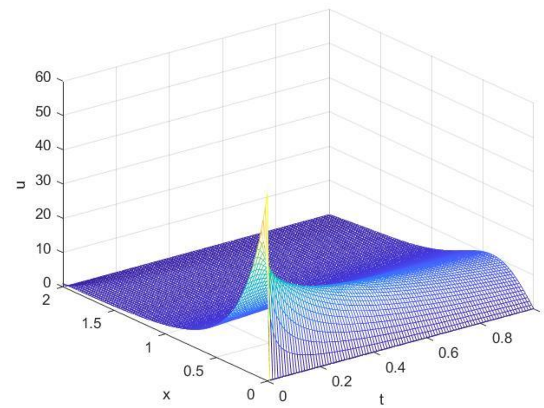


Figure 16. Numerical solution of Test5 for$\alpha = \frac{0m}{s}$, $\lambda = \frac{0.05m2}{s}$

RESULTS

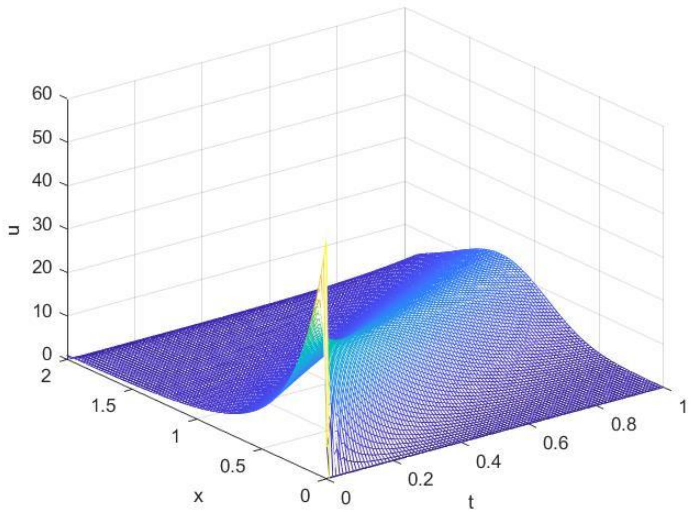


Figure17. Numerical solution of Test3 for$\alpha = \frac{1m}{s}$, $\lambda = \frac{0.05m2}{s}$

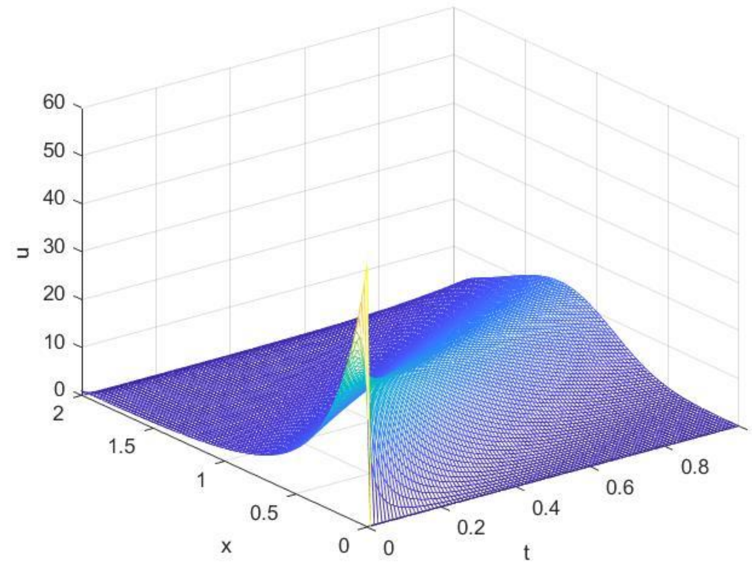


Figure18. Numerical solution of Test6 for$\alpha = \frac{1m}{s}$, $\lambda = \frac{0.05m2}{s}$

Crank-Nicolson scheme: the solution oscillates for the Test1
Backward Euler scheme: the solution is stable

CONCLUSION

- In this report, Galerkin-finite element method is proposed to find the solutions of advection-diffusion equation and the Crank-Nicolson scheme and backward euler for time discretization are compared for the solutions. In the solution procedure, the first step is to make weak formulation and then develop the finite element formulation. As test problem, two different solutions are chosen. Maximum errors norm $L_\propto$, L_2 and $Ch^2$are calculated and found that errors are small. A comparison of numerical and analytical solutions is made.

FUTURE WORK

- To apply the standard k-epsilon model for sediment free case,based on my previous flood modeling and turbidity current modeling.
- Set up the model in TELEMAC-3D to test the model’s sensitivity
- Finish the assignment report!

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Thank you