Example 10: Predator–prey differential equations

Figure 4. Lynx chases hare.

The **recorded** population changes for ~90 years:

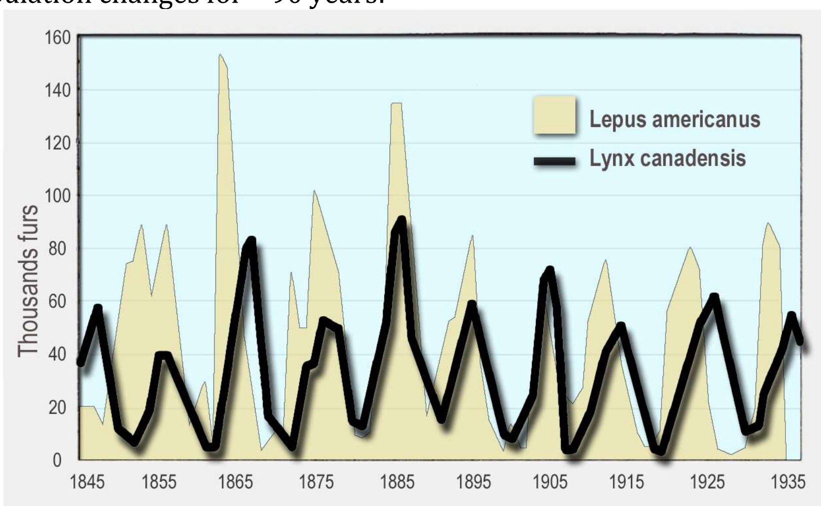


Figure 5. The running counts of lynx and hare for ~90 years.

L-V's assumptions: The L-V model makes several assumptions about the environment and evolution of the predator and prey populations:

- The prey always finds ample food.
- The food supply of the predator depends entirely on the prey.
- The rate of population change is proportional to its size.
- The environment & genetics remain unchanged.

The L-V model:

$$\begin{cases} \dfrac{dx}{dt} = x(\alpha_x - \beta_x y) \\ \dfrac{dy}{dt} = -y(\alpha_y - \beta_y x) \end{cases}$$

where

- x=Prey population
- y=Predator population
- t=time

- $\alpha_x, \alpha_y, \beta_x, \beta_y$ = parameters representing the interaction of the two species

A typical solution of this system:

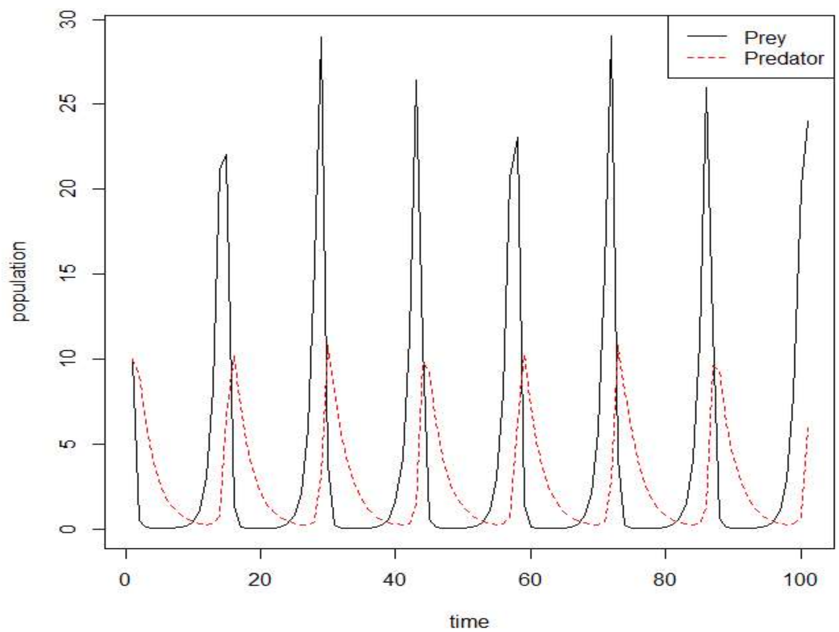


Figure 6. A typical solution of the L-V model.

Analytically, we can manipulate this system of DEs a little to gain insights,

$$\frac{dy}{dx} = -\frac{y(\alpha_y - \beta_y x)}{x(\alpha_x - \beta_x y)}$$

Or this 1st order DE can be transformed as

$$\frac{\alpha_x - \beta_x y}{y} dy + \frac{\alpha_y - \beta_y x}{x} dx = 0$$

Or

$$\left(\frac{\alpha_x}{y} - \beta_x\right) dy + \left(\frac{\alpha_y}{x} - \beta_y\right) dx = 0$$

Integrating the above, we get

$$(\alpha_x \ln y - \beta_x y) + (\alpha_y \ln x - \beta_y x) = c$$

Or

$$(\alpha_x \ln y - \beta_x y) + (\alpha_y \ln x - \beta_y x) = c$$

leading to an elegant form of the GS of the population DEs

$$\alpha_x \ln y + \alpha_y \ln x = \beta_x y + \beta_y x + c$$

Setting the IC's: $x(= 0) = y(t = 0) = P_0$, we have

$$c = (\alpha_x + \alpha_y) \ln P_0 - (\beta_x + \beta_y) P_0$$

Thus, the PS is

$$\alpha_x \ln \frac{y}{P_0} + \alpha_y \ln \frac{x}{P_0} = \beta_x (y - P_0) + \beta_y (x - P_0)$$

This is the L-V system's solution in phase space.

A summary of typical solutions

Parameter combinations for all 4 (or 5) cases:

Table 1. Parameters used for numerical experiments of L-V model.

Cases	α_x	α_y	β_x	β_y
A	1.00	0.99	0.01	0.01
B	1.00	1.00	0.03	0.03
C	1.00	1.00	1.00	1.00
D	1.00	1.00	0.01	0.01
E	??	??	??	??
Initial conditions for all cases P_0				
$x(t=0)$	100			
$y(t=0)$	100			

Solution A: Looking up the parameters in the table, one found

Case	α_x	α_y	β_x	β_y
A	1.00	0.99	0.01	0.01

Using the above analytical solution, for this special case, we get

$$\alpha_x \ln\frac{y}{P_0} + \alpha_y \ln\frac{x}{P_0} = \beta_x(y - P_0) + \beta_y(x - P_0)$$

we get, for this special case,

$$\ln\frac{y}{100} + \frac{99}{100}\ln\frac{x}{100} = \frac{1}{100}(x+y) - 2$$

Very important to notice that **both populations are quite stable** (minor changes from $P_0 = 100$)

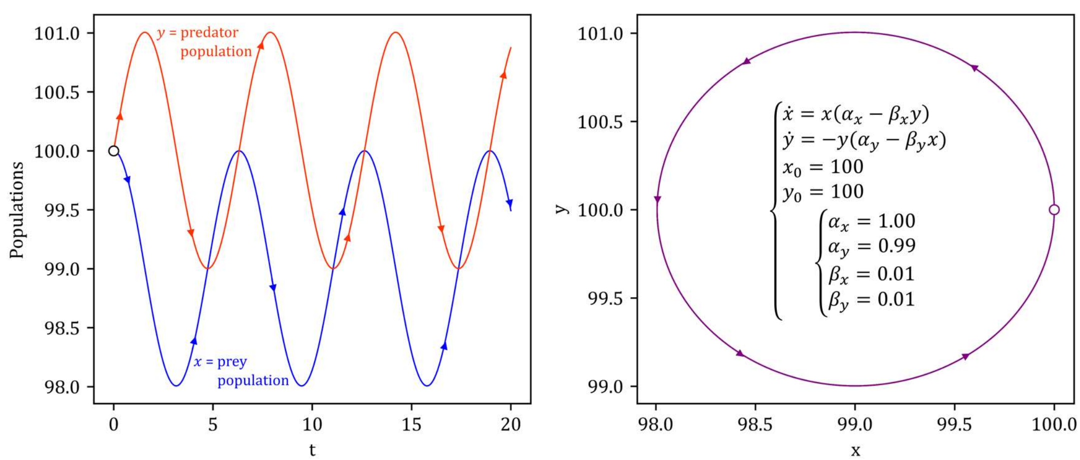


Figure 7. L-V model solution for parameter set A.

Solution B: Looking up the parameters in the table, one found,

Case	α_x	α_y	β_x	β_y
B	1.00	1.00	0.03	0.03

Using the above analytical solution,

$$\alpha_x \ln\frac{y}{P_0} + \alpha_y \ln\frac{x}{P_0} = \beta_x(y - P_0) + \beta_y(x - P_0)$$

for this special case, we get

$$\ln\frac{y}{100} + \ln\frac{x}{100} = \frac{3}{100}(x + y) - 6$$

Very important to notice that **both populations change dramatically** (both almost vanish)

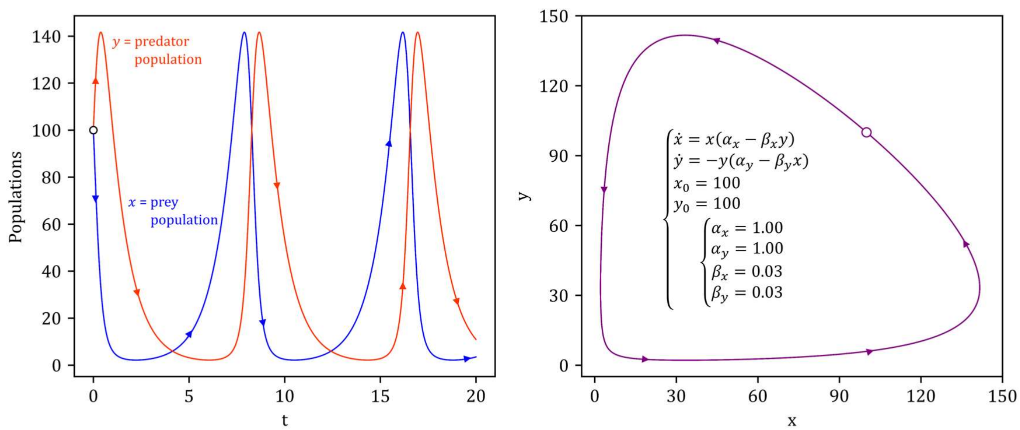


Figure 8. L-V model solution for parameter set B.

Solution C: Looking up the parameters in the table, one found,

Case	α_x	α_y	β_x	β_y
C	1.00	1.00	1.00	1.00

Using the above analytical solution,

$$\alpha_x \ln \frac{y}{P_0} + \alpha_y \ln \frac{x}{P_0} = \beta_x (y - P_0) + \beta_y (x - P_0)$$

for this special case, we get,

$$\ln \frac{y}{100} + \ln \frac{x}{100} = x + y - 200$$

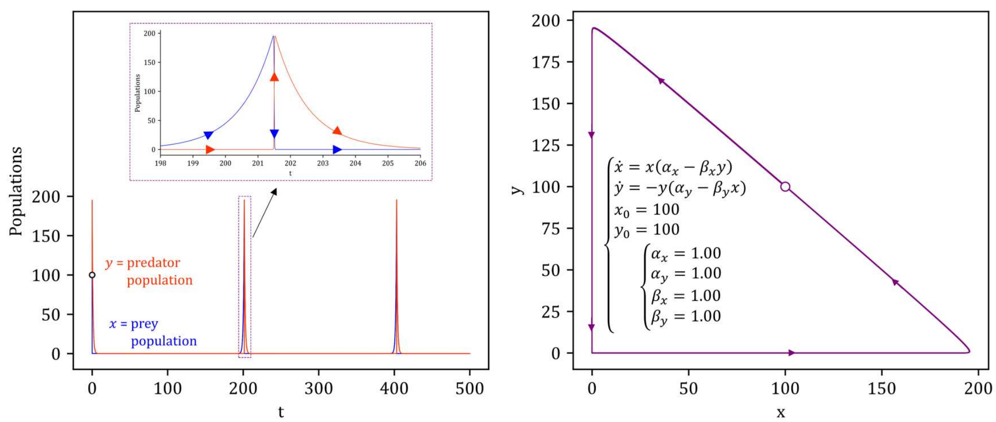


Figure 9. L-V model solution for parameter set C.

One note of **caution**:

(1) In the above Solutions B and C, when the population counts become crazily low, both species likely go extinct. End of story.

(2) Recovering from the apparent extinction, in these extreme situations, as predicted by the L-V model, is merely a numerical fantasy, epitomizing the weaknesses of most numerical models (and any models, in general). No surprise!

Solution D: Looking up the parameters in the table, one found,

Case	α_x	α_y	β_x	β_y
C	1.00	1.00	0.01	0.01

Using the above analytical solution,

$$\alpha_x \ln\frac{y}{P_0} + \alpha_y \ln\frac{x}{P_0} = \beta_x(y - P_0) + \beta_y(x - P_0)$$

for this special case, we get

$$\ln\frac{y}{100} + \ln\frac{x}{100} = \frac{1}{100}(x + y) - 2$$

This equation has only one solution $x = y = 100$ regardless of t. Thus, we see only one dot in the phase diagram (right). This is a crazy situation: It appears to lock up.

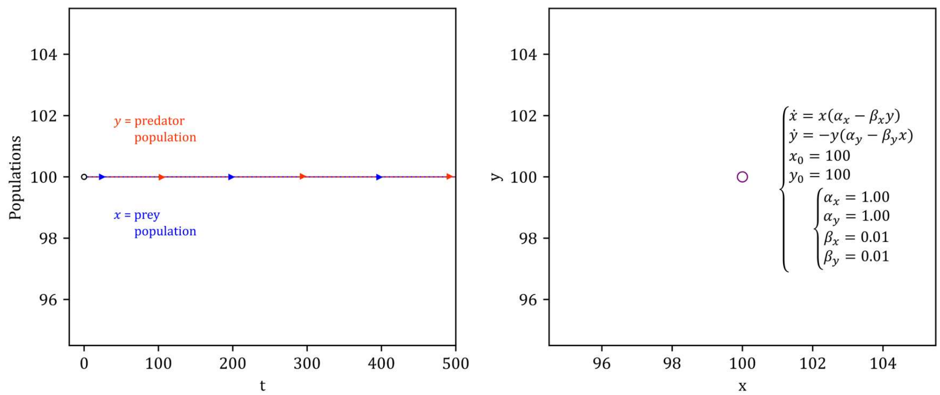
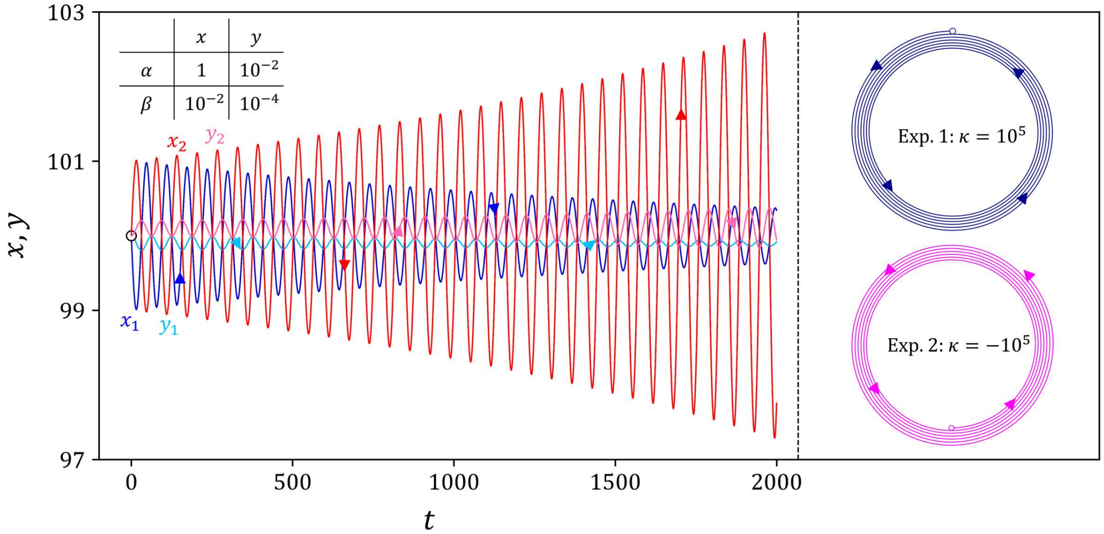


Figure 10. L-V model solution for parameter set D.

I have two more cases to show here:

Modified Case E (See parameters in picture)



Modified Case F (See parameters in picture)