Method.4: Heun's method

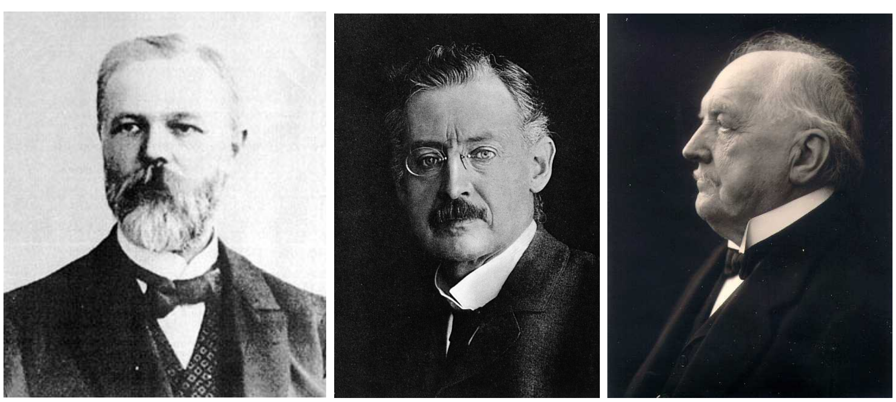
(AKA: Modified Euler's method, Explicit trapezoidal rule)

A little history: Over the last ~100 years, these methods got successive improvements…Please read a more detailed history of Heun's and many similar methods. The following table is a brief chronological record of the Runge-Kutta methods:

Table 1. chronological record of the Runge-Kutta methods

p	s	Author	Year	
2	2	Runge	1895	
3	3	Heun	1900	
4	4	Kutta	1901	
5	6	Kutta	1901	
5	6	Nyström	1925	(correction to Kutta)
6	8	Huťa	1956	
6	7	Butcher	1964	
7	9	Butcher		(known since approximately 1968)
8	11	Curtis	1970	
8	11	Cooper and Verner	1972	(announced 1969 in J.H. Verner's thesis)
10	18	Curtis	1975	
10	17	Hairer	1978	

The key developers include Heun, Runge, and Kutta.



Karl Heun (1859-1929), Carl Runge (1856-1927), Martin Kutta (1867-1944)

Mathematical form (also, a 2-step method):

$$
\begin{aligned}
\tilde{y}_{n+1} &= y_n + h\,f(x_n, y_n) \\
y_{n+1} &= y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})\big) \\
&= y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_{n+1}, y_n + h\,f(x_n, y_n))\big)
\end{aligned}
$$

Thus,

$$y_{n+1} = y_n + h * \frac{1}{2}\left(\underbrace{f(x_n, y_n)}_{y_n'} + \underbrace{f(x_{n+1}, y_n + h\, f(x_n, y_n))}_{y_{n+1}'}\right)$$

This method is quite important because it uses slope of a predicted future point. The following is an illustration of the algorithm:

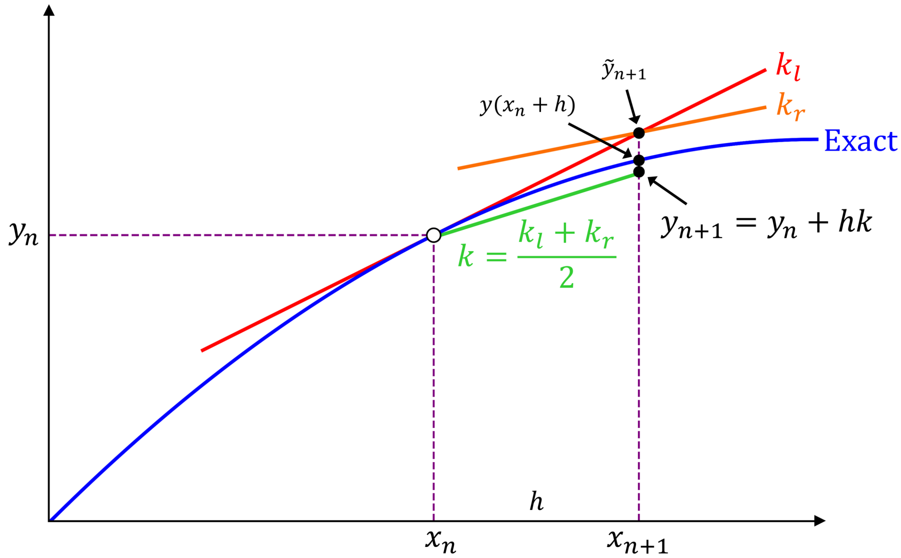


Figure 16. Heun's method.

After some basic analysis, comparing with Taylor series, we get the following LTE

$$\mathrm{LTE}(h) = O(h^3)$$

Thus, Heun's method is **order-2** method.

Recall, Euler's method is of **Order-1**.

Another form of the Heun's method (Improved Euler method):

$$y_{n+1} = y_n + h * \frac{1}{4}\left(\underbrace{f(x_n, y_n)}_{y_n'} + 3\,\underbrace{f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}h\, f(x_n, y_n)\right)}_{{y'}_{n+\frac{2}{3}}}\right)$$