Example 11: A Cascade of Tanks

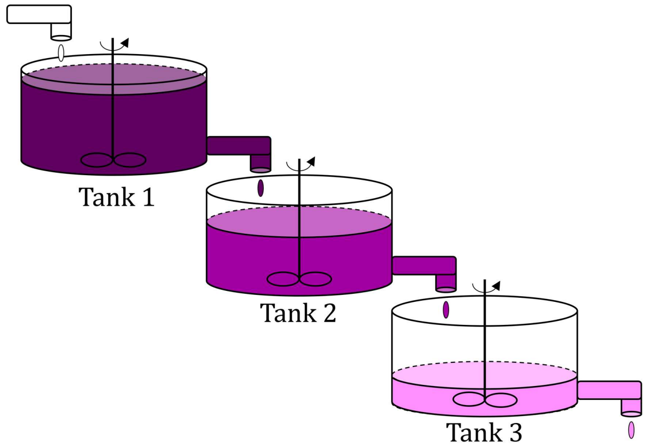

Figure 11. A Cascade of Tanks (All pictures' credit: A. Bae).

Consider a Brine cascade of three tanks of volumes [of liquid] V_1, V_2, V_3 with the same flow rate r for flowing to Tank-1 (with water containing no salt), from Tank-1 to Tank-2 and then to Tank-3.

Assume the salt concentration throughout each tank is uniform due to stirring in each tank.

We establish a homogeneous system to compute the salt concentration in each take at time t: $x_1(t), x_2(t), x_3(t)$.

According to chemical balance law that the concentration rate change in each tank is the difference between the input and output rates.

For convenience, we set $V_1 = 2,\ V_2 = 4,\ V_3 = 6$ and $r = 1$. We have the following DEs.

For Tank-1:

$$\frac{dx_1}{dt} = 0 - \frac{1}{2}x_1$$

For Tank-2:

$$\frac{dx_2}{dt} = \frac{1}{2}x_1 - \frac{1}{4}x_2$$

For Tank-3:

$$\frac{dx_3}{dt} = \frac{1}{4}x_2 - \frac{1}{6}x_3$$

If the initial concentrations are c_1, c_2 and c_3 for the three tanks respectively, we can form the following System of ODEs

$$\begin{cases} \dfrac{dx_1}{dt} = 0 - \dfrac{1}{2}x_1 \\ \dfrac{dx_2}{dt} = \dfrac{1}{2}x_1 - \dfrac{1}{4}x_2 \\ \dfrac{dx_3}{dt} = \dfrac{1}{4}x_2 - \dfrac{1}{6}x_3 \\ x_1(0) = c_1 \\ x_2(0) = c_2 \\ x_3(0) = c_3 \end{cases}$$

Solving the above linear System of DEs, we get the following solutions:

$$\begin{cases} x_1(t) = c_1 \exp\left(-\frac{t}{2}\right) \\ x_2(t) = -2c_1 \exp\left(-\frac{t}{2}\right) + (c_2 + 2c_1)\exp\left(-\frac{t}{4}\right) \\ x_3(t) = \frac{3}{2}c_1 \exp\left(-\frac{t}{2}\right) - (3c_2 + 6c_1)\exp\left(-\frac{t}{4}\right) \\ \qquad + \left(c_3 - \frac{3}{2}c_1 + 3c_2 + 6c_1\right)\exp\left(-\frac{t}{6}\right) \end{cases}$$

Let's examine a few cases:

Case 1: Three different initials and three different volumes all lead to the same ZERO. This means, when it last long enough, the **water down** effect is natural! Keep adding water to a tank of chemicals will eventually "wash away" all chemicals.

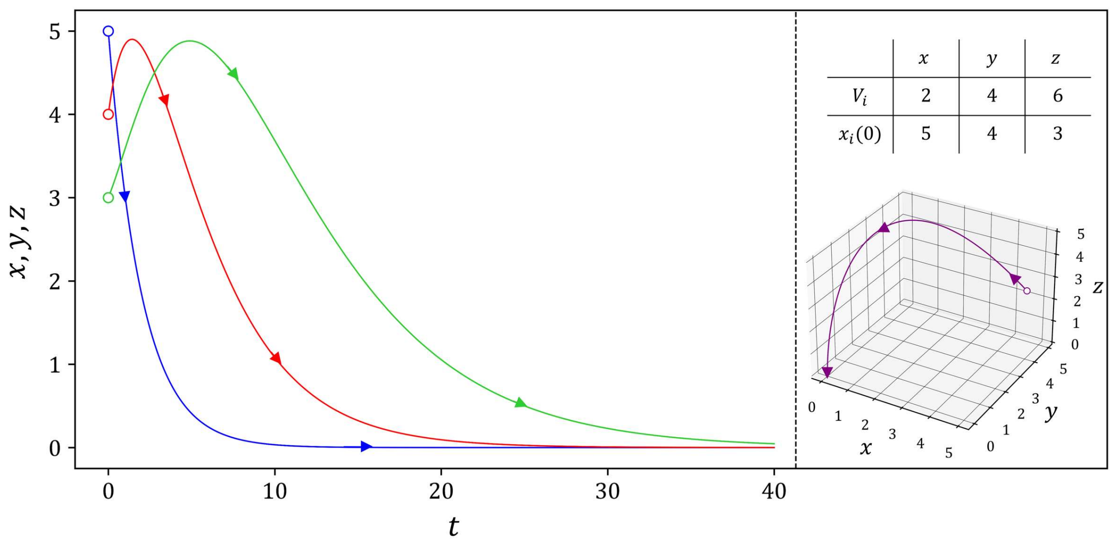

Figure 12. Solution to a cascade of tanks: all converge to zero.

Case 2: We propose to change the experiment: Instead of pump water to Tank-1, we pump back the last tank's solution, in this case Tank-3, back to Tank-1 and we wish to see how the concentrations will change.

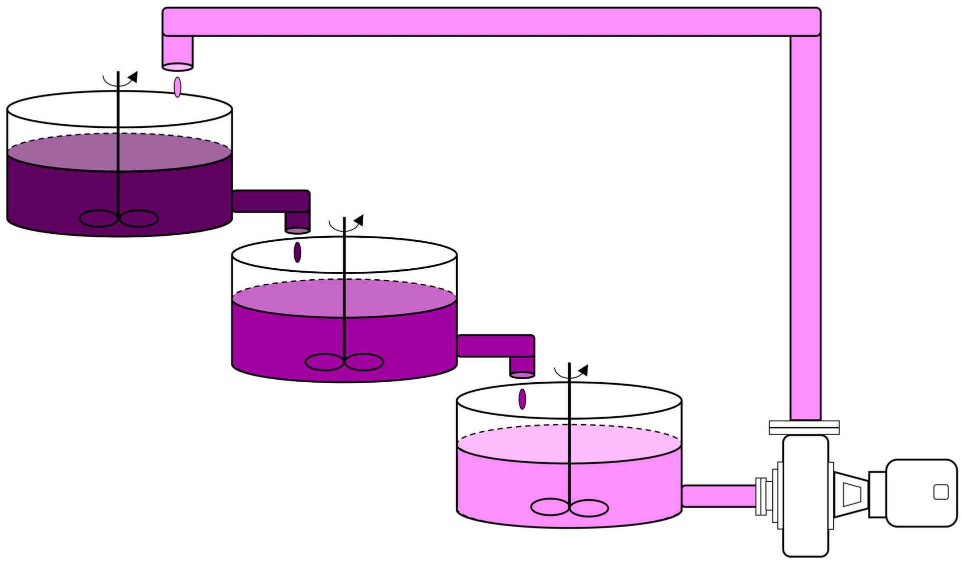

Figure 13. A Modified Cascade of Tanks (All pictures' credit: A. Bae).

Case 2: With pumping back, with the same volumes and different initials.

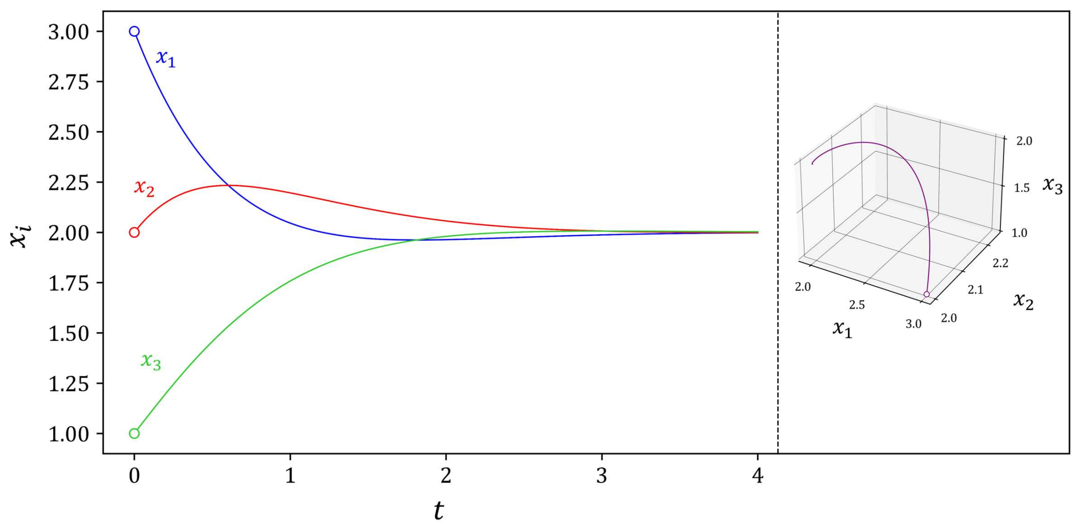

Figure 14. Concentrations with the same volumes and different initials.

Case 3: With pumping back, with different volumes and the same initials.

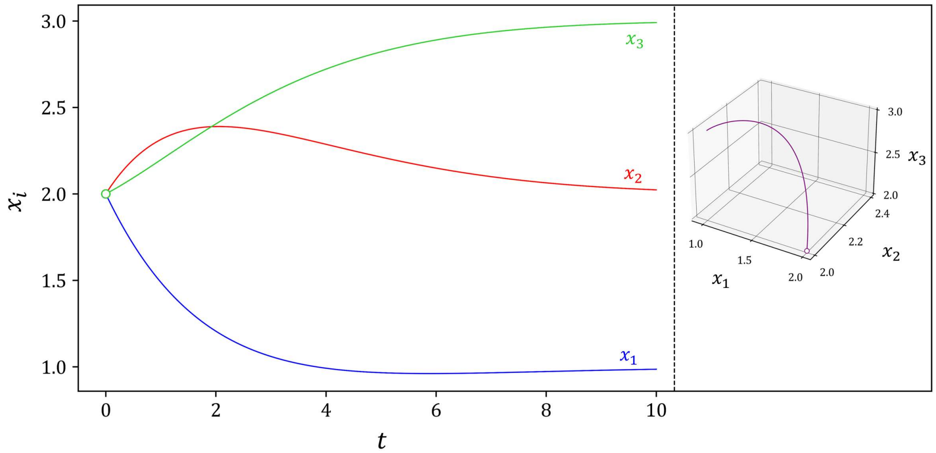

Figure 15. Concentrations with the same initials and different volumes initials.