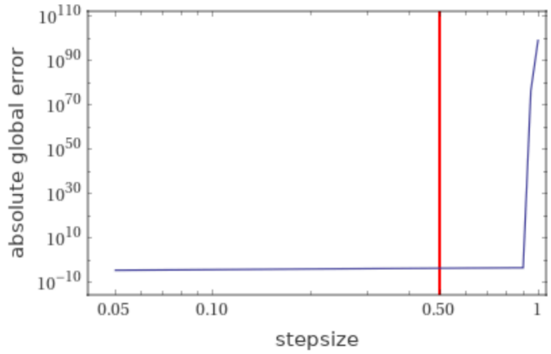

Figure 36. Figure 33. |GTE| vs. h for Forward Euler.

We noticed sudden failure at h~0.9. This is, of course, related to the particular structure of the IVP.

Example 7: Solving initial value problem

$$\begin{cases} e^{-x'} = x' + x^3 - 3e^{-t^3} \\ x(t=0) = 1 \end{cases}$$

We solve this IVP by three different methods, the forward Euler (RK1), the Heun's (RK2), and RK4.

This equation is quite crazy… and we need to write programs to solve for the x', numerically, at each step when a slope is needed. It's more complex than the implicit method where we need to solve for the actual function values.

Let me explain the steps for RK1 (forward Euler method) in details, assuming step size $h = 0.1$

Step n	t	x_n	$k_n = x_n' =$ Root of $e^{-x'} = x' + x^3 - 3e^{-t^3}$	$x_{n+1} = x_n + hk_n$
0	0.0	1.00000	2.12003	1.00000+0.1*2.12003=1.21200
1	0.1	1.21200	1.45097	1.21200+0.1*1.45097=1.35710
…	…	…	…	…

Let me explain the steps for RK2 (Heun's method) in details, assuming step size $h = 0.1$. The following outlines enough steps for this method that can be generalized to other explicit methods:

Step 0: Initializing

Set $t = 0, x_0 = 1, h = 0.1$

Step 1: Computing the average slope (needing the slope at $t = 0$ and $t = 0 + h$)

For slope at $t = 0$, solve (Newton's method, e.g.)

$$e^{-x'} = x' + x^3 - 3e^{-t^3}$$

The root of this algebraic equation is the slope at $t = 0$:

$$k_0 = x_0'$$

Next, use RK1 to compute the function value at $t = 0 + h$. This is a temporary solution for computing the slope only

$$\tilde{x}_1 = x_0 + hk_0$$

For slope at $t = 0 + h$, solve (Newton's method, e.g.) using $x = \tilde{x}_1$ and $t = 0 + h$

$$e^{-x'} = x' + x^3 - 3e^{-t^3}$$

The root of this algebraic equation is the slope at $t = 0 + h$:

$$\tilde{k}_1 = \tilde{x}_1'$$

Finally, the average slope is

$$\bar{k}_0 = \frac{1}{2}(k_0 + \tilde{k}_1)$$

Step 2: Advance to the next step, truly!

$$x_1 = x_0 + h\bar{k}_0$$

You may loop back to compute,

$$x_2, x_3, x_4, \ldots$$

The solutions (and their errors) using three different methods:

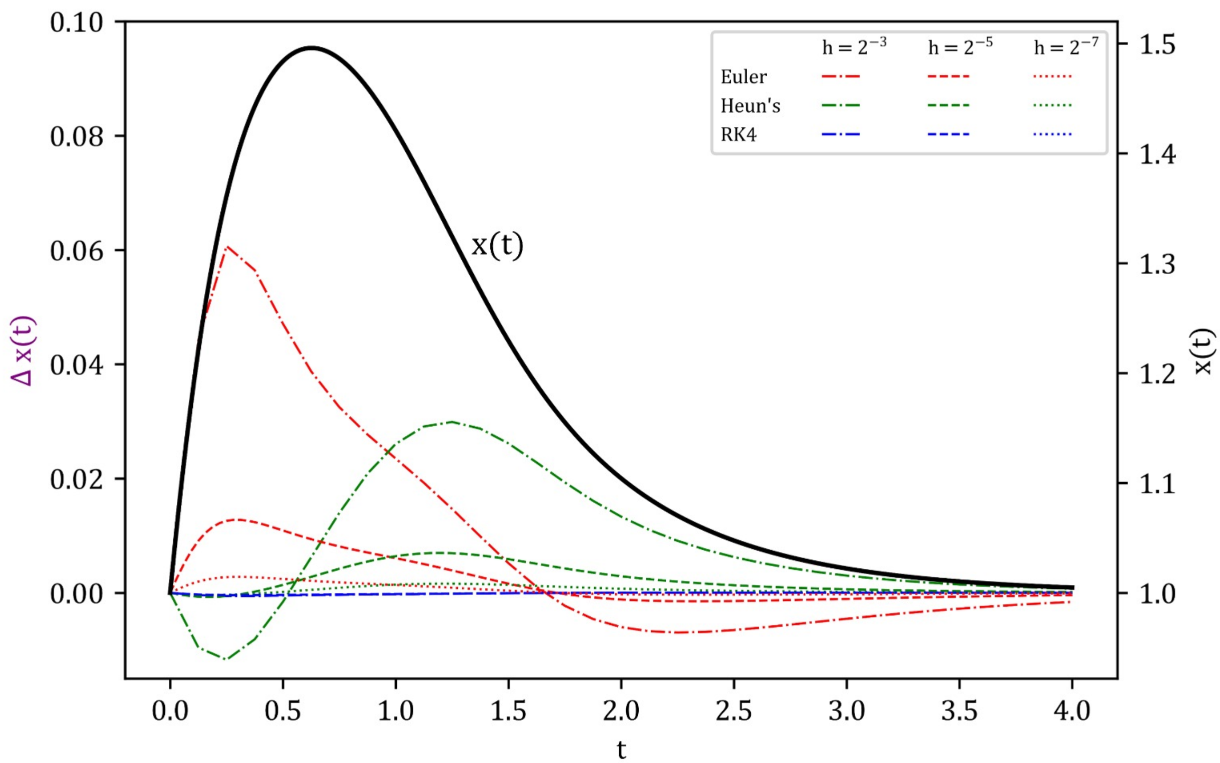

Figure 37. The exact solution (solid dark curve) with the right vertical axis; the 9 dashed curves are in three color groups with each color for one one method: Euler (red), Heun's (green) and RK4 (blue) and for each method, we have three mesh sizes indicated by the different dashed lines.