Interoperation for Lazy and Eager Evaluation

- Matthews & Findler
  - Interoperation
  - Boundaries & natural embedding
  - Type safety and extensional equality
- Kinghorn
- Incompatible evaluation strategies

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

Set of terms, e

Set of terms, e

(1) $x \in e$

Set of terms, e

(1) $x \in \mathrm{e}$

(2) $M \in \mathrm{e} \Rightarrow \lambda x \,.\, M \in \mathrm{e}$

Set of terms, e

(1) $x \in \mathrm{e}$

(2) $M \in \mathrm{e} \Rightarrow \lambda x\,.\,M \in \mathrm{e}$

(3) $M, N \in \mathrm{e} \Rightarrow M\,N \in \mathrm{e}$

Set of terms, e

(1) $$x \in \mathrm{e}$$

(2) $$M \in \mathrm{e} \Rightarrow \lambda x \,.\, M \in \mathrm{e}$$

(3) $$M, N \in \mathrm{e} \Rightarrow M\,N \in \mathrm{e}$$

$$\mathrm{e} = x \mid \lambda x \,.\, \mathrm{e} \mid \mathrm{e}\ \mathrm{e}$$

Set of terms, e

(1) $x \in e$

(2) $M \in e \Rightarrow \lambda x \,.\, M \in e$

(3) $M, N \in e \Rightarrow M\, N \in e$

$$e = x \mid \lambda x \,.\, e \mid e\; e$$

$$v = \lambda x \,.\, e$$

(1) x
(2) $\lambda\ x\ .\ e$
(3) $e\ e$

z

by (1)

(1) x

(2) $\lambda\, x\,.\,e$

(3) e e

(1) x
(2) $\lambda\ x\ .\ e$
(3) e e

z
by (1)

$\lambda\ z\ .\ z$
by (1), (2)

z
by (1)

(1) x
(2) λ x . e
(3) e e

λ z . z
by (1), (2)

(λ z . z) (λ z . z)
by (1), (2), (3)

$$\lambda\, x\,.\,x\;x \equiv \lambda\, x\,.\,(x\;x) \neq (\lambda\, x\,.\,x)\;x$$

$$\lambda\, x\,.\, x\; x \equiv \lambda\, x\,.\,(x\; x) \neq (\lambda\, x\,.\, x)\; x$$

$$\lambda\, x\; x'\,.\, \mathrm{e} \equiv \lambda\, x\,.\,\lambda\, x'\,.\, \mathrm{e}$$

$$\lambda x\,.\,x\,x \equiv \lambda x\,.\,(x\,x) \neq (\lambda x\,.\,x)\,x$$

$$\lambda x\,x'\,.\,\mathrm{e} \equiv \lambda x\,.\,\lambda x'\,.\,\mathrm{e}$$

$$\mathrm{e}\ \mathrm{e}'\ \mathrm{e}'' = (\mathrm{e}\ \mathrm{e}')\ \mathrm{e}''$$

...y...

Free variable

...y... λ y . (...y...)

Free variable *Bound variable*

...y... λ y . (...y...)

Free variable *Bound variable*

y

Open term

$\ldots y \ldots$

Free variable

$\lambda\, y\, .\, (\ldots y \ldots)$

Bound variable

y

Open term

$\lambda\, y\, .\, y$

Closed term

$$term\,[\,expression\ argument\,/\,expression\ parameter\,] = term'$$

$$x\,[\,e\,/\,x\,] \;=\; e$$

term [*expression argument* / *expression parameter*] = *term′*

$$x\,[\,e\,/\,x\,] \;=\; e$$

$$x\,[\,e\,/\,x'\,] \;=\; x$$

term [*expression argument* / *expression parameter*] = *term′*

$$x\,[\,e\,/\,x\,] \;=\; e$$

$$x\,[\,e\,/\,x'\,] \;=\; x$$

$$(\lambda\, x\,.\,e)\,[\,e'\,/\,x\,] \;=\; \lambda\, x\,.\,e$$

term [expression argument / expression parameter] = term′

$$x\,[\,e\,/\,x\,] \;=\; e$$

$$x\,[\,e\,/\,x'\,] \;=\; x$$

$$(\lambda\, x\,.\,e)\,[\,e'\,/\,x\,] \;=\; \lambda\, x\,.\,e$$

$$(\lambda\, x\,.\,e)\,[\,e'\,/\,x'\,] \;=\; \lambda\, x\,.\,(e\,[\,e'\,/\,x'\,])$$

term [*expression argument* / *expression parameter*] = *term*′

$$x\,[\,e / x\,] \;=\; e$$

$$x\,[\,e / x'\,] \;=\; x$$

$$(\lambda\, x\,.\,e)\,[\,e' / x\,] \;=\; \lambda\, x\,.\,e$$

$$(\lambda\, x\,.\,e)\,[\,e' / x'\,] \;=\; \lambda\, x\,.\,(e\,[\,e' / x'\,])$$

$$(e\; e')\,[\,e'' / x\,] \;=\; (e\,[\,e'' / x\,])\;(e'\,[\,e'' / x\,])$$

Set of reductions, $\rightarrow$

Set of reductions, $\rightarrow$

$$(e, e') \in \rightarrow$$

$$e \rightarrow e'$$

Set of reductions, $\rightarrow$

$$(e, e') \in \rightarrow$$

$$e \rightarrow e'$$

$$e \rightarrow e'$$

$$e' \rightarrow e''$$

$$e \rightarrow e' \rightarrow e''$$

Set of reductions, $\rightarrow$

$$(e, e') \in \rightarrow$$

$$e \rightarrow e'$$

$$e \rightarrow e'$$

$$e' \rightarrow e''$$

$$e \rightarrow e' \rightarrow e''$$

$$(\lambda\, x\, .\, e)\; e' \rightarrow e\, [\, x \,/\, e'\,]$$

$$(\lambda\, x\, .\, \mathrm{e})\ \mathrm{e}' \rightarrow \mathrm{e}\ [\, x\, /\, \mathrm{e}'\,]$$

$$(\lambda\, x\, .\, e)\ e' \to e\,[\, x / e' \,]$$

$$e \to e' \Rightarrow e\ e'' \to e'\ e''$$

$$(\lambda\, x\, .\, e)\; e' \rightarrow e\, [\, x \,/\, e'\,]$$

$$e \rightarrow e' \Rightarrow e\; e'' \rightarrow e'\; e''$$

$$\frac{e \rightarrow e'}{e\; e'' \rightarrow e'\; e''}$$

$$(\lambda\, x\, .\, \mathrm{e})\ \mathrm{e}' \rightarrow \mathrm{e}\,[\, x \,/\, \mathrm{e}'\,]$$

$$\mathrm{e} \rightarrow \mathrm{e}' \Rightarrow \mathrm{e}\ \mathrm{e}'' \rightarrow \mathrm{e}'\ \mathrm{e}''$$

$$\frac{\mathrm{e} \rightarrow \mathrm{e}'}{\mathrm{e}\ \mathrm{e}'' \rightarrow \mathrm{e}'\ \mathrm{e}''}$$

$$\mathrm{e} \rightarrow \mathrm{e}' \Rightarrow v\ \mathrm{e} \rightarrow v\ \mathrm{e}'$$

$$(\lambda\, x\, .\, e)\ e' \rightarrow e\,[\,x\,/\,e'\,]$$

$$e \rightarrow e' \Rightarrow e\ e'' \rightarrow e'\ e''$$

$$\frac{e \rightarrow e'}{e\ e'' \rightarrow e'\ e''}$$

$$e \rightarrow e' \Rightarrow v\ e \rightarrow v\ e'$$

$$\frac{e \rightarrow e'}{v\ e \rightarrow v\ e'}$$

error condition → error

error condition $\rightarrow$ *error*

$$\frac{e \rightarrow e'}{e\ e'' \rightarrow e'\ e''}$$

$$\textit{error condition} \rightarrow \textit{error}$$

$$\frac{e \rightarrow e'}{e\ e'' \rightarrow e'\ e''}$$

$$\frac{e \rightarrow \textit{error}}{e\ e' \rightarrow \textit{error}}$$

$$\textit{error condition} \to \textit{error}$$

$$\frac{e \to e'}{e\ e'' \to e'\ e''}$$

$$\frac{e \to e'}{v\ e \to v\ e'}$$

$$\frac{e \to \textit{error}}{e\ e' \to \textit{error}}$$

$$\textit{error condition} \rightarrow \textit{error}$$

$$\frac{e \rightarrow e'}{e\ e'' \rightarrow e'\ e''} \qquad \frac{e \rightarrow \textit{error}}{e\ e' \rightarrow \textit{error}}$$

$$\frac{e \rightarrow e'}{v\ e \rightarrow v\ e'} \qquad \frac{e \rightarrow \textit{error}}{v\ e \rightarrow \textit{error}}$$

Set of evaluation contexts, E

Set of evaluation contexts, E

$$(\lambda\ x\ .\ \mathrm{e})\ \mathrm{e}' \rightarrow \mathrm{e}\ [x\ /\ \mathrm{e}']$$

Set of evaluation contexts, E

$$(\lambda\, x\,.\, \mathrm{e})\ \mathrm{e}' \rightarrow \mathrm{e}\ [x\,/\,\mathrm{e}']$$

$$E\ [\ (\lambda\, x\,.\, \mathrm{e})\ \mathrm{e}'\ ] \rightarrow E\ [\ \mathrm{e}\ [x\,/\,\mathrm{e}']\ ]$$

Set of evaluation contexts, E

$$(\lambda\, x\, .\, \mathrm{e})\ \mathrm{e}' \rightarrow \mathrm{e}\ [x\ /\ \mathrm{e}']$$

$$E\ [\ (\lambda\, x\, .\, \mathrm{e})\ \mathrm{e}'\ ] \rightarrow E\ [\ \mathrm{e}\ [x\ /\ \mathrm{e}']\ ]$$

$$E = [\ ]\ |\ E\ \mathrm{e}\ |\ v\ E$$

Set of evaluation contexts, E

$$(\lambda\, x\, .\, \mathrm{e})\ \mathrm{e}' \rightarrow \mathrm{e}\ [x\ /\ \mathrm{e}']$$

$$E\ [\ (\lambda\, x\, .\, \mathrm{e})\ \mathrm{e}'\ ] \rightarrow E\ [\ \mathrm{e}\ [x\ /\ \mathrm{e}']\ ]$$

$$E = [\ ]\ |\ E\ \mathrm{e}\ |\ v\ E$$

$$E' = \ldots[\ ]\ldots$$

Set of evaluation contexts, E

$$(\lambda\, x\,.\,\mathrm{e})\;\mathrm{e}' \rightarrow \mathrm{e}\,[x\,/\,\mathrm{e}']$$

$$E\,[\,(\lambda\, x\,.\,\mathrm{e})\;\mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}\,[x\,/\,\mathrm{e}']\,]$$

$$E = [\;] \mid E\;\mathrm{e} \mid v\;E$$

$$E' = \ldots[\;]\ldots$$

$$E'\,[\,\mathrm{e}\,] = \ldots\mathrm{e}\ldots$$

$$v \quad = \quad \lambda x\,.\,\mathrm{e} \mid \underline{n}$$

$$v \quad = \quad \lambda\, x \,.\, \mathrm{e} \mid \underline{n}$$

$$\mathrm{e} \quad = \quad x \mid v \mid \mathrm{e}\ \mathrm{e} \mid \textbf{+/-}\ \mathrm{e}\ \mathrm{e} \mid \textbf{if0}\ \mathrm{e}\ \mathrm{e}\ \mathrm{e} \mid \textbf{fun?}\ \mathrm{e}$$
$$\textbf{num?}\ \mathrm{e} \mid \textbf{wrong}\ \textit{string}$$

v = λ x . e | $\underline{n}$

e = x | v | e e | **+/-** e e | **if0** e e e | **fun?** e
num? e | **wrong** *string*

E = [ ] | E e | v E | **+/-** E e | **+/-** v E | **if0** E e e
fun? E | **num?** E

E [**wrong** *string*] → **Error:** *string*

$E\ [\ \textbf{wrong}\ \textit{string}\ ] \rightarrow \textbf{Error:}\ \textit{string}$

$E\ [\ \textbf{+}\ \underline{n}\ \underline{n'}\ ] \rightarrow E\ [\ \underline{n + n'}\ ]$

E [**wrong** *string*] → **Error:** *string*

$E\,[\,\mathbf{+}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{n + n'}\,]$

$E\,[\,\mathbf{-}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{\max(n - n', 0)}\,]$

E [**wrong** *string*] → **Error:** *string*

$E[\, \mathbf{+}\ \underline{n}\ \underline{n'} \,] \rightarrow E[\, \underline{n + n'} \,]$

$E[\, \mathbf{-}\ \underline{n}\ \underline{n'} \,] \rightarrow E[\, \underline{\mathrm{max}(n - n', 0)} \,]$

$E[\, \mathbf{if0}\ \underline{0}\ \mathrm{e}\ \mathrm{e}' \,] \rightarrow E[\, \mathrm{e} \,]$

E [**wrong** *string*] → **Error:** *string*

$E\,[\,\mathbf{+}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{n + n'}\,]$

$E\,[\,\mathbf{-}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{\mathrm{max}(n - n', 0)}\,]$

$E\,[\,\mathbf{if0}\ \underline{0}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}\,]$
$E\,[\,\mathbf{if0}\ \underline{n}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}'\,]$

E [**wrong** *string*] → **Error:** *string*

$E\,[\,\mathbf{+}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{n + n'}\,]$

$E\,[\,\mathbf{-}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{\max(n - n', 0)}\,]$

$E\,[\,\textbf{if0}\ \underline{0}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}\,]$
$E\,[\,\textbf{if0}\ \underline{n}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}'\,]$

$E\,[\,\textbf{fun?}\ (\lambda\, x\,.\,\mathrm{e})\,] \rightarrow E\,[\,\underline{0}\,]$

E [**wrong** *string*] → **Error:** *string*

$E[\,\mathbf{+}\ \underline{n}\ \underline{n'}\,] \rightarrow E[\,\underline{n + n'}\,]$

$E[\,\mathbf{-}\ \underline{n}\ \underline{n'}\,] \rightarrow E[\,\underline{\max(n - n', 0)}\,]$

$E[\,\mathbf{if0}\ \underline{0}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E[\,\mathrm{e}\,]$
$E[\,\mathbf{if0}\ \underline{n}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E[\,\mathrm{e}'\,]$

$E[\,\mathbf{fun?}\ (\lambda\, x\, .\, \mathrm{e})\,] \rightarrow E[\,\underline{0}\,]$
$E[\,\mathbf{fun?}\ v\,] \rightarrow E[\,\underline{1}\,]$

E [**wrong** *string*] → **Error:** *string*

$E\,[\,\mathbf{+}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{n + n'}\,]$

$E\,[\,\mathbf{-}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{\mathrm{max}(n - n', 0)}\,]$

$E\,[\,\mathbf{if0}\ \underline{0}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}\,]$
$E\,[\,\mathbf{if0}\ \underline{n}\ \mathrm{e}\ \mathrm{e}'\,] \rightarrow E\,[\,\mathrm{e}'\,]$

$E\,[\,\mathbf{fun?}\ (\lambda\, x\, .\, \mathrm{e})\,] \rightarrow E\,[\,\underline{0}\,]$
$E\,[\,\mathbf{fun?}\ v\,] \rightarrow E\,[\,\underline{1}\,]$

$E\,[\,\mathbf{num?}\ \underline{n}\,] \rightarrow E\,[\,\underline{0}\,]$

E [**wrong** *string*] → **Error:** *string*

$E\,[\,\mathbf{+}\ \underline{n}\ \underline{n'}\,] \to E\,[\,\underline{n + n'}\,]$

$E\,[\,\mathbf{-}\ \underline{n}\ \underline{n'}\,] \to E\,[\,\underline{\max(n - n', 0)}\,]$

$E\,[\,\mathbf{if0}\ \underline{0}\ \mathrm{e}\ \mathrm{e}'\,] \to E\,[\,\mathrm{e}\,]$
$E\,[\,\mathbf{if0}\ \underline{n}\ \mathrm{e}\ \mathrm{e}'\,] \to E\,[\,\mathrm{e}'\,]$

$E\,[\,\mathbf{fun?}\ (\lambda\, x\,.\,\mathrm{e})\,] \to E\,[\,\underline{0}\,]$
$E\,[\,\mathbf{fun?}\ v\,] \to E\,[\,\underline{1}\,]$

$E\,[\,\mathbf{num?}\ \underline{n}\,] \to E\,[\,\underline{0}\,]$
$E\,[\,\mathbf{num?}\ v\,] \to E\,[\,\underline{1}\,]$

Lambda calculus

Typing

Interoperation

Model

Laziness

Solution

Set of types, t

Set of types, t

$$t = \mathbf{N} \mid t \to t$$

Set of types, t

$$t = \mathbf{N} \mid t \rightarrow t$$

$$\lambda\, x : t \,.\, \mathrm{e}$$

Set of types, t

$$t = \mathbf{N} \mid t \to t$$

$$\lambda\, x : t\ .\ \mathrm{e}$$

$$t \to t \to t \equiv t \to (t \to t)$$

Set of judgments, $\vdash$

Set of judgments, $\vdash$

$$\mathrm{e} : t \equiv (\, \mathrm{e} \,,\, t \,)$$

Set of judgments, $\vdash$

$$e : t \equiv (e , t)$$

Γ *is* $x_n : t_n , \ldots , x_1 : t_1$

Set of judgments, $\vdash$

$$e : t \equiv (e , t)$$

Γ *is* $x_n : t_n , \ldots , x_l : t_l$

$$\lambda\, x_n : t_n . (\ldots \lambda\, x_l : t_l . e)$$

Set of judgments, $\vdash$

$$\mathrm{e} : t \equiv (\, \mathrm{e} \,, t \,)$$

$$\Gamma \textit{ is } x_n : t_n \,, \ldots \,, x_1 : t_1$$

$$\lambda\, x_n : t_n \,.\, (\ldots\, \lambda\, x_1 : t_1 \,.\, \mathrm{e})$$

$$(\, \Gamma \,, \mathrm{e} : t \,) \in \vdash$$

Set of judgments, $\vdash$

$$e : t \equiv (\, e \,, t \,)$$

Γ *is* $x_n : t_n \,, \ldots \,, x_1 : t_1$

$$\lambda\, x_n : t_n \,.\, (\ldots\, \lambda\, x_1 : t_1 \,.\, e)$$

$$(\, \Gamma \,, e : t \,) \in\, \vdash$$

$$\Gamma \vdash e : t \qquad \vdash e : t$$

Set of judgments, $\vdash$

$$e : t \equiv (e , t)$$

Γ *is* $x_n : t_n , \ldots , x_1 : t_1$

$$\lambda\, x_n : t_n \,.\, (\ldots\, \lambda\, x_1 : t_1 \,.\, e)$$

$$(\Gamma , e : t) \in \vdash$$

$$\Gamma \vdash e : t \qquad \vdash e : t$$

$$\Gamma \vdash t \qquad \vdash t$$

Number type

$\vdash \mathbf{N}$

Number type

$$\vdash \mathbf{N}$$

Function type

$$\frac{\Gamma \vdash t \;\text{—}\; \Gamma \vdash t'}{\Gamma \vdash t \to t'}$$

Number type

$$\vdash \mathbf{N}$$

Number

$$\vdash \underline{n} : \mathbf{N}$$

Function type

$$\frac{\Gamma \vdash t \quad \Gamma \vdash t'}{\Gamma \vdash t \to t'}$$

Number type

$$\vdash \mathbf{N}$$

Function type

$$\frac{\Gamma \vdash t \quad \Gamma \vdash t'}{\Gamma \vdash t \to t'}$$

Number

$$\vdash \underline{n} : \mathbf{N}$$

Variable

$$\Gamma, x : t \vdash x : \mathtt{t}$$

Number type

$$\vdash \mathbf{N}$$

Function type

$$\frac{\Gamma \vdash t \quad \Gamma \vdash t'}{\Gamma \vdash t \to t'}$$

Number

$$\vdash \underline{n} : \mathbf{N}$$

Variable

$$\Gamma, x : t \vdash x : t$$

Function

$$\frac{\Gamma, x : t \vdash e : t'}{\Gamma \vdash \lambda\, x : t \,.\, e : t \to t'}$$

Number type

$$\vdash \mathbf{N}$$

Function type

$$\frac{\Gamma \vdash t \quad \Gamma \vdash t'}{\Gamma \vdash t \to t'}$$

Number

$$\vdash \underline{n} : \mathbf{N}$$

Variable

$$\Gamma, x : t \vdash x : t$$

Function

$$\frac{\Gamma, x : t \vdash e : t'}{\Gamma \vdash \lambda\, x : t \,.\, e : t \to t'}$$

Application

$$\frac{\Gamma \vdash e : t \to t' \quad \Gamma \vdash e' : t}{\Gamma \vdash e\ e' : t'}$$

Arithmetic

$$\frac{\Gamma \vdash e : \mathbf{N} \quad \Gamma \vdash e' : \mathbf{N}}{\Gamma \vdash \mathbf{+/-}\ e\ e' : \mathbf{N}}$$

Arithmetic

$$\frac{\Gamma \vdash e : \mathbf{N} \quad \text{—} \quad \Gamma \vdash e' : \mathbf{N}}{\Gamma \vdash \mathbf{+/-}\ e\ e' : \mathbf{N}}$$

Condition

$$\frac{\Gamma \vdash e : \mathbf{N} \quad \text{—} \quad \Gamma \vdash e'/e'' : t}{\Gamma \vdash \mathbf{if0}\ e\ e'\ e'' : t}$$

Arithmetic

$$\frac{\Gamma \vdash e : \mathbf{N} \quad — \quad \Gamma \vdash e' : \mathbf{N}}{\Gamma \vdash \mathbf{+/\text{-}}\ e\ e' : \mathbf{N}}$$

Condition

$$\frac{\Gamma \vdash e : \mathbf{N} \quad — \quad \Gamma \vdash e'/e'' : t}{\Gamma \vdash \mathbf{if0}\ e\ e'\ e'' : t}$$

Error

$$\frac{\Gamma \vdash t}{\Gamma \vdash \mathbf{wrong}\ t\ \mathit{string} : t}$$

Type

$\Gamma \vdash \mathbf{T}$

Type

$\Gamma \vdash \mathbf{T}$

Number

$\Gamma \vdash \underline{n} : \mathbf{T}$

Type

$$\Gamma \vdash \mathbf{T}$$

Number

$$\Gamma \vdash \underline{n} : \mathbf{T}$$

Variable

$$\Gamma, x : \mathbf{T} \vdash x : \mathbf{T}$$

Type

$$\Gamma \vdash \mathbf{T}$$

Number

$$\Gamma \vdash \underline{n} : \mathbf{T}$$

Variable

$$\Gamma, x : \mathbf{T} \vdash x : \mathbf{T}$$

Application

$$\frac{\Gamma \vdash e : \mathbf{T} \quad \text{—} \quad \Gamma \vdash e' : \mathbf{T}}{\Gamma \vdash e\ e' : \mathbf{T}}$$

Type

$$\Gamma \vdash \mathbf{T}$$

Number

$$\Gamma \vdash \underline{n} : \mathbf{T}$$

Variable

$$\Gamma, x : \mathbf{T} \vdash x : \mathbf{T}$$

Application

$$\frac{\Gamma \vdash e : \mathbf{T} \quad — \quad \Gamma \vdash e' : \mathbf{T}}{\Gamma \vdash e\ e' : \mathbf{T}}$$

Function

$$\frac{\Gamma, x : \mathbf{T} \vdash e : \mathbf{T}}{\Gamma \vdash \lambda\, x\, .\, e : \mathbf{T}}$$

Type

$$\Gamma \vdash \mathbf{T}$$

Number

$$\Gamma \vdash \underline{n} : \mathbf{T}$$

Variable

$$\Gamma, x : \mathbf{T} \vdash x : \mathbf{T}$$

Application

$$\frac{\Gamma \vdash \mathrm{e} : \mathbf{T} \quad — \quad \Gamma \vdash \mathrm{e}' : \mathbf{T}}{\Gamma \vdash \mathrm{e}\ \mathrm{e}' : \mathbf{T}}$$

Function

$$\frac{\Gamma, x : \mathbf{T} \vdash \mathrm{e} : \mathbf{T}}{\Gamma \vdash \lambda\, x\, .\, \mathrm{e} : \mathbf{T}}$$

Arithmetic

$$\frac{\Gamma \vdash \mathrm{e} : \mathbf{T} \quad — \quad \Gamma \vdash \mathrm{e}' : \mathbf{T}}{\Gamma \vdash \mathbf{+/-}\ \mathrm{e}\ \mathrm{e}'}$$

Predicate

$$\frac{\Gamma \vdash e : \mathbf{T}}{\Gamma \vdash \textbf{fun?/num?}\ e : \mathbf{T}}$$

Predicate

$$\frac{\Gamma \vdash e : \mathbf{T}}{\Gamma \vdash \textbf{fun?/num?}\ e : \mathbf{T}}$$

Error

$$\Gamma \vdash \textbf{wrong}\ \textit{string} : \mathbf{T}$$

$\lambda x : \mathbf{N} . x$

$$\lambda x : \mathbf{N} . x$$

$$\lambda x : \mathbf{N} \rightarrow \mathbf{N} . x$$

$$\lambda x : \mathbf{N} . x$$

$$\lambda x : \mathbf{N} \to \mathbf{N} . x$$

$$\Lambda y . \lambda x : y . x$$

$$\lambda x : \mathbf{N} . x$$

$$\lambda x : \mathbf{N} \to \mathbf{N} . x$$

$$\Lambda y . \lambda x : y . x$$

$$(\Lambda y . \lambda x : y . x) \langle \mathbf{N} \rangle \to \lambda x : \mathbf{N} . x$$

$$\lambda x : \mathbf{N} . x$$

$$\lambda x : \mathbf{N} \to \mathbf{N} . x$$

$$\Lambda y . \lambda x : y . x$$

$$(\Lambda y . \lambda x : y . x) \langle \mathbf{N} \rangle \to \lambda x : \mathbf{N} . x$$

$$(\Lambda y . \lambda x : y . x) \langle \mathbf{N} \to \mathbf{N} \rangle \to \lambda x : \mathbf{N} \to \mathbf{N} . x$$

Type variables

y

Type variables

y

Type abstraction

$\Lambda\, y\, .\, e$

Type variables

y

Type abstraction

$\Lambda\, y\, .\, \mathrm{e}$

Type application

$\mathrm{e} \langle\, t\, \rangle$

Type variables

y

Type abstraction

$\Lambda\, y\, .\, e$

Type application

$e \langle\, t\, \rangle$

Universally-quantified / for-all types

$\forall\, y\, .\, t$

Type variables

y

Type abstraction

$\Lambda\, y\, .\, \mathrm{e}$

Type application

$\mathrm{e}\,\langle\, t\,\rangle$

Universally-quantified / for-all types

$\forall\, y\, .\, t$

Free & bound type variables

$\Lambda\, y\, .\, (\ldots\, y\, \ldots)$

$$\Lambda\, y\, y' \,.\, e \equiv \Lambda\, y \,.\, \Lambda\, y' \,.\, e$$

$$\Lambda\, y\, y' \,.\, \mathrm{e} \equiv \Lambda\, y \,.\, \Lambda\, y' \,.\, \mathrm{e}$$

$$\mathrm{e} \langle t \rangle \langle t' \rangle \equiv (\mathrm{e} \langle t \rangle) \langle t' \rangle$$

$$\Lambda\, y\, y' \,.\, e \equiv \Lambda\, y \,.\, \Lambda\, y' \,.\, e$$

$$e \langle t \rangle \langle t' \rangle \equiv (e \langle t \rangle) \langle t' \rangle$$

$$\Lambda\, y \,.\, e = \Lambda\, y' \,.\, e\, [\, y' / y \,]$$

term [type argument / type parameter] = term′

term [type argument / type parameter] = term′

$$x\,[\,t\,/\,y\,] \;=\; x$$

term [type argument / type parameter] = term′

$$x\,[\,t\,/\,y\,] \;=\; x$$

$$(\lambda\, x : t\, .\, \mathrm{e})\,[\,t\,/\,y\,] \;=\; \lambda\, x : t\,[\,t\,/\,y\,]\, .\, \mathrm{e}\,[\,t\,/\,y\,]$$

term [type argument / type parameter] = term′

$$x\,[\,t\,/\,y\,] \;=\; x$$

$$(\lambda\, x : t\,.\,\mathrm{e})\,[\,t\,/\,y\,] \;=\; \lambda\, x : t\,[\,t\,/\,y\,]\,.\,\mathrm{e}\,[\,t\,/\,y\,]$$

$$(\mathrm{e}\;\mathrm{e}')\,[\,t\,/\,y\,] \;=\; (\mathrm{e}\,[\,t\,/\,y\,])\;(\mathrm{e}'\,[\,t\,/\,y\,])$$

$$term\ [\ type\ argument\ /\ type\ parameter\ ] = term'$$

$$x\ [\ t\ /\ y\ ] \quad = \quad x$$

$$(\lambda\, x : t\ .\ \mathrm{e})\ [\ t\ /\ y\ ] \quad = \quad \lambda\, x : t\ [\ t\ /\ y\ ]\ .\ \mathrm{e}\ [\ t\ /\ y\ ]$$

$$(\mathrm{e}\ \mathrm{e}')\ [\ t\ /\ y\ ] \quad = \quad (\mathrm{e}\ [\ t\ /\ y\ ])\ (\mathrm{e}'\ [\ t\ /\ y\ ])$$

$$(\mathbf{+}/\mathbf{-}\ \mathrm{e}\ \mathrm{e}')\ [\ t\ /\ y\ ] \quad = \quad \mathbf{+}/\mathbf{-}\ (\mathrm{e}\ [\ t\ /\ y\ ])\ (\mathrm{e}'\ [\ t\ /\ y\ ])$$

term [type argument / type parameter] = term′

$$x\,[\,t\,/\,y\,] \;=\; x$$

$$(\lambda\, x : t\,.\,\mathrm{e})\,[\,t\,/\,y\,] \;=\; \lambda\, x : t\,[\,t\,/\,y\,]\,.\,\mathrm{e}\,[\,t\,/\,y\,]$$

$$(\mathrm{e}\ \mathrm{e}')\,[\,t\,/\,y\,] \;=\; (\mathrm{e}\,[\,t\,/\,y\,])\ (\mathrm{e}'\,[\,t\,/\,y\,])$$

$$(\mathbf{+/-}\ \mathrm{e}\ \mathrm{e}')\,[\,t\,/\,y\,] \;=\; \mathbf{+/-}\ (\mathrm{e}\,[\,t\,/\,y\,])\ (\mathrm{e}'\,[\,t\,/\,y\,])$$

$$(\mathbf{if0}\ \mathrm{e}\ \mathrm{e}'\ \mathrm{e}'')\,[\,t\,/\,y\,] \;=\; \mathbf{if0}\ (\mathrm{e}\,[\,t\,/\,y\,])\ (\mathrm{e}'\,[\,t\,/\,y\,])\ (\mathrm{e}''\,[\,t\,/\,y\,])$$

$$(\Lambda\, y\,.\,e)\,[\,t\,/\,y\,] \;=\; \Lambda\, y\,.\,e$$

$$(\Lambda y\,.\,e)\,[\,t\,/\,y\,] \;=\; \Lambda y\,.\,e$$

$$(\Lambda y\,.\,e)\,[\,t\,/\,y'\,] \;=\; \Lambda y\,.\,e\,[\,t\,/\,y'\,]$$

$$(\Lambda y . e)[t / y] = \Lambda y . e$$

$$(\Lambda y . e)[t / y'] = \Lambda y . e[t / y']$$

$$(e \langle t \rangle)[t' / y] = (e[t' / y]) \langle t[t' / y] \rangle$$

type [type argument / type parameter] = type′

type [type argument / type parameter] = type′

$$\mathbf{N}\,[\,t\,/\,y\,] \;=\; \mathbf{N}$$

$$\textit{type}\ [\ \textit{type argument}\ /\ \textit{type parameter}\ ] = \textit{type}'$$

$$\mathbf{N}\ [\ t\ /\ y\ ] \;=\; \mathbf{N}$$

$$(t \rightarrow t')\ [\ t\ /\ y\ ] \;=\; t\ [\ t\ /\ y\ ] \rightarrow t'\ [\ t\ /\ y\ ]$$

$$\textit{type}\ [\ \textit{type argument}\ /\ \textit{type parameter}\ ] = \textit{type}'$$

$$\mathbf{N}\,[\,t\,/\,y\,] \;=\; \mathbf{N}$$

$$(t \rightarrow t')\,[\,t\,/\,y\,] \;=\; t\,[\,t\,/\,y\,] \rightarrow t'\,[\,t\,/\,y\,]$$

$$y\,[\,t\,/\,y\,] \;=\; t$$

type [type argument / type parameter] = type′

$$\mathbf{N}\,[\,t\,/\,y\,] \;=\; \mathbf{N}$$

$$(t \rightarrow t')\,[\,t\,/\,y\,] \;=\; t\,[\,t\,/\,y\,] \rightarrow t'\,[\,t\,/\,y\,]$$

$$y\,[\,t\,/\,y\,] \;=\; t$$

$$y\,[\,t\,/\,y'\,] \;=\; y$$

type [type argument / type parameter] = type′

$$\mathbf{N}\,[\,t\,/\,y\,] \;=\; \mathbf{N}$$

$$(t \rightarrow t')\,[\,t\,/\,y\,] \;=\; t\,[\,t\,/\,y\,] \rightarrow t'\,[\,t\,/\,y\,]$$

$$y\,[\,t\,/\,y\,] \;=\; t$$

$$y\,[\,t\,/\,y'\,] \;=\; y$$

$$(\forall\, y\,.\,t)\,[\,t'\,/\,y\,] \;=\; \forall\, y\,.\,t$$

$$\textit{type}\ [\ \textit{type argument} / \textit{type parameter}\ ] = \textit{type}'$$

$$\mathbf{N}\ [\ t / y\ ] = \mathbf{N}$$

$$(t \to t')\ [\ t / y\ ] = t\ [\ t / y\ ] \to t'\ [\ t / y\ ]$$

$$y\ [\ t / y\ ] = t$$

$$y\ [\ t / y'\ ] = y$$

$$(\forall\, y \,.\, t)\ [\ t' / y\ ] = \forall\, y \,.\, t$$

$$(\forall\, y \,.\, t)\ [\ t' / y'\ ] = \forall\, y \,.\, t\ [\ t' / y'\ ]$$

$$E\,[\,(\Lambda\, y\,.\,e)\,\langle\, t\,\rangle\,] \to E\,[\,e\,[\,t\,/\,y\,]\,]$$

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

Scheme to ML

$$e_m = \cdots \mid \mathbf{ms}\ t_m\ e_s$$

ML to Scheme

$$e_s = \cdots \mid \mathbf{sm}\ t_m\ e_m$$

Scheme to ML

$$\frac{\Gamma \vdash_m t_m \quad \text{—} \quad \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ t_m\ e_s : t_m}$$

Scheme to ML

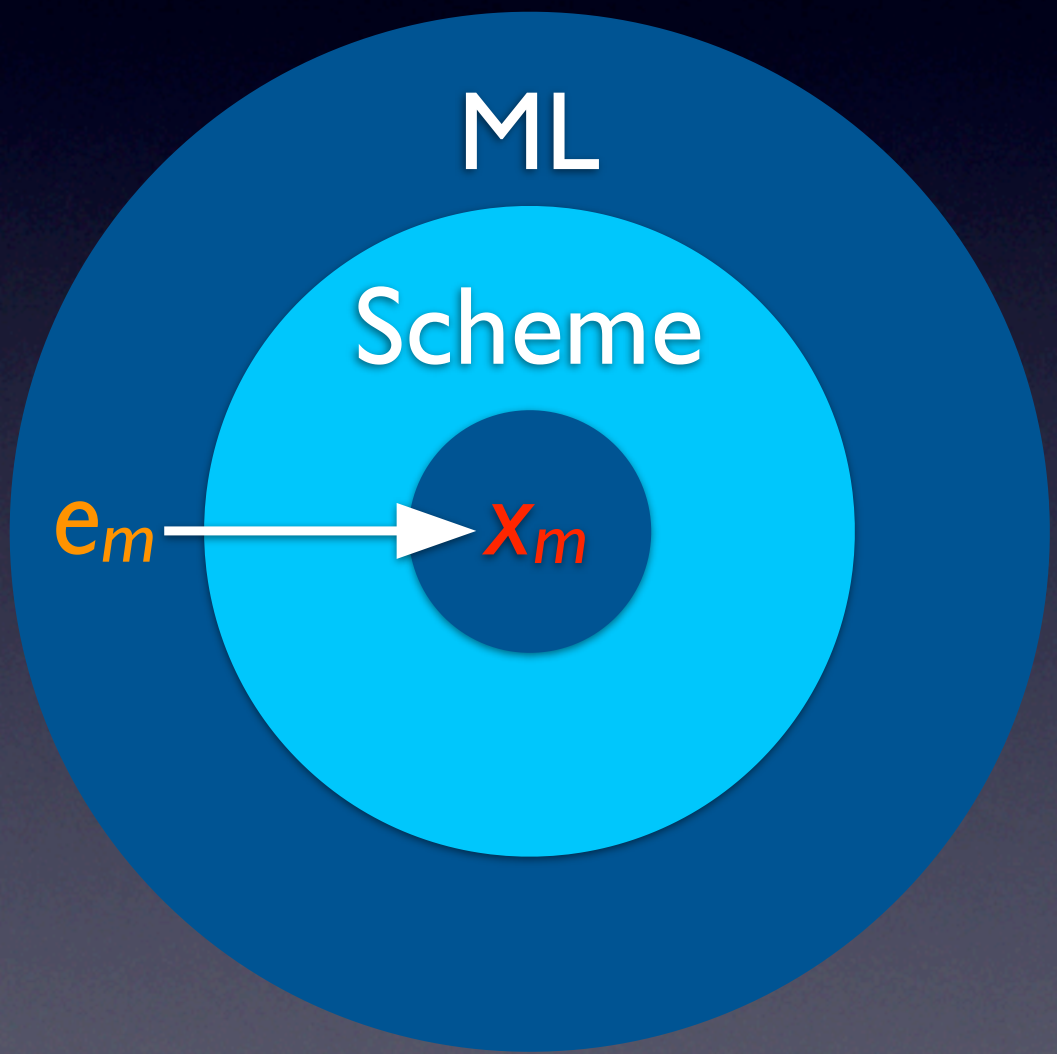
$$\frac{\Gamma \vdash_m t_m \quad \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ t_m\ e_s : t_m}$$

ML to Scheme

$$\frac{\Gamma \vdash_m t_m \quad \Gamma \vdash_m e_m : t_m' \quad t_m = t_m'}{\Gamma \vdash_s \mathbf{sm}\ t_m\ e_m : \mathbf{T}}$$

$$(\mathbf{ms}\ t_m\ (\mathbf{sm}\ t_m\ x_m))\ [\ e_m\ /\ x_m\ ]$$

$$(\mathbf{ms}\ t_m\ (\mathbf{sm}\ t_m\ x_m))\ [\ e_m\ /\ x_m\ ]$$



Boundary substitution

$$(\mathbf{ms}\ t_m\ e_s)\ [\ e_m\ /\ x_m\ ] = \mathbf{ms}\ t_m\ (e_s\ [\ e_m\ /\ x_m\ ])$$

Boundary substitution

$$(\mathbf{ms}\ t_m\ e_s)\ [\ e_m\ /\ x_m\ ] = \mathbf{ms}\ t_m\ (e_s\ [\ e_m\ /\ x_m\ ])$$

Foreign substitution

Boundary substitution

$$(\mathbf{ms}\ t_m\ e_s)\ [\ e_m\ /\ x_m\ ] = \mathbf{ms}\ t_m\ (e_s\ [\ e_m\ /\ x_m\ ])$$

Foreign substitution

$$(\ldots e_s \ldots e_s' \ldots)\ [\ e_m\ /\ x_m\ ] = \ldots e_s\ [\ e_m\ /\ x_m\ ] \ldots e_s'\ [\ e_m\ /\ x_m\ ] \ldots$$

Boundary substitution

$$(\mathbf{ms}\ t_m\ e_s)\ [\ e_m\ /\ x_m\ ] = \mathbf{ms}\ t_m\ (e_s\ [\ e_m\ /\ x_m\ ])$$

Foreign substitution

$$(\ldots e_s \ldots e_s' \ldots)\ [\ e_m\ /\ x_m\ ] = \ldots e_s\ [\ e_m\ /\ x_m\ ] \ldots e_s'\ [\ e_m\ /\ x_m\ ] \ldots$$

$$(\mathbf{sm}\ t_m\ e_m)\ [\ e_m'\ /\ x_m\ ] = \mathbf{sm}\ t_m\ (e_m\ [\ e_m'\ /\ x_m\ ])$$

Outermost evaluation context, $\mathcal{E}$

Outermost evaluation context, $\mathcal{E}$

$$\mathcal{E}\,[\,\mathbf{ms}\ \mathbf{N}\ \underline{n}\,]_m \rightarrow \mathcal{E}\,[\,\underline{n}\,]$$

Outermost evaluation context, $\mathscr{E}$

$$\mathscr{E}\,[\ \mathbf{ms\ N}\ \underline{n}\ ]_m \rightarrow \mathscr{E}\,[\ \underline{n}\ ]$$

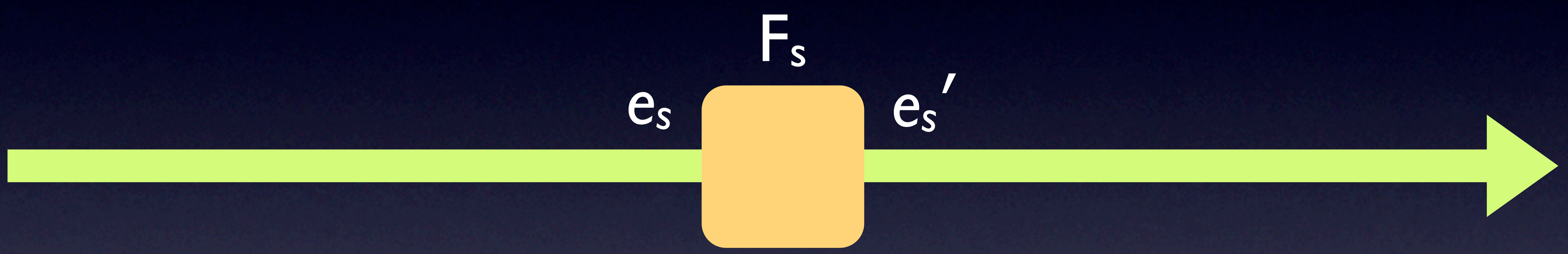
$$\mathscr{E}\,[\ \mathbf{ms\ N}\ v_s\ ]_m \rightarrow \mathscr{E}\,[\ \mathbf{wrong\ N}\ \text{“Not a number”}\ ]$$

Outermost evaluation context, $\mathcal{E}$

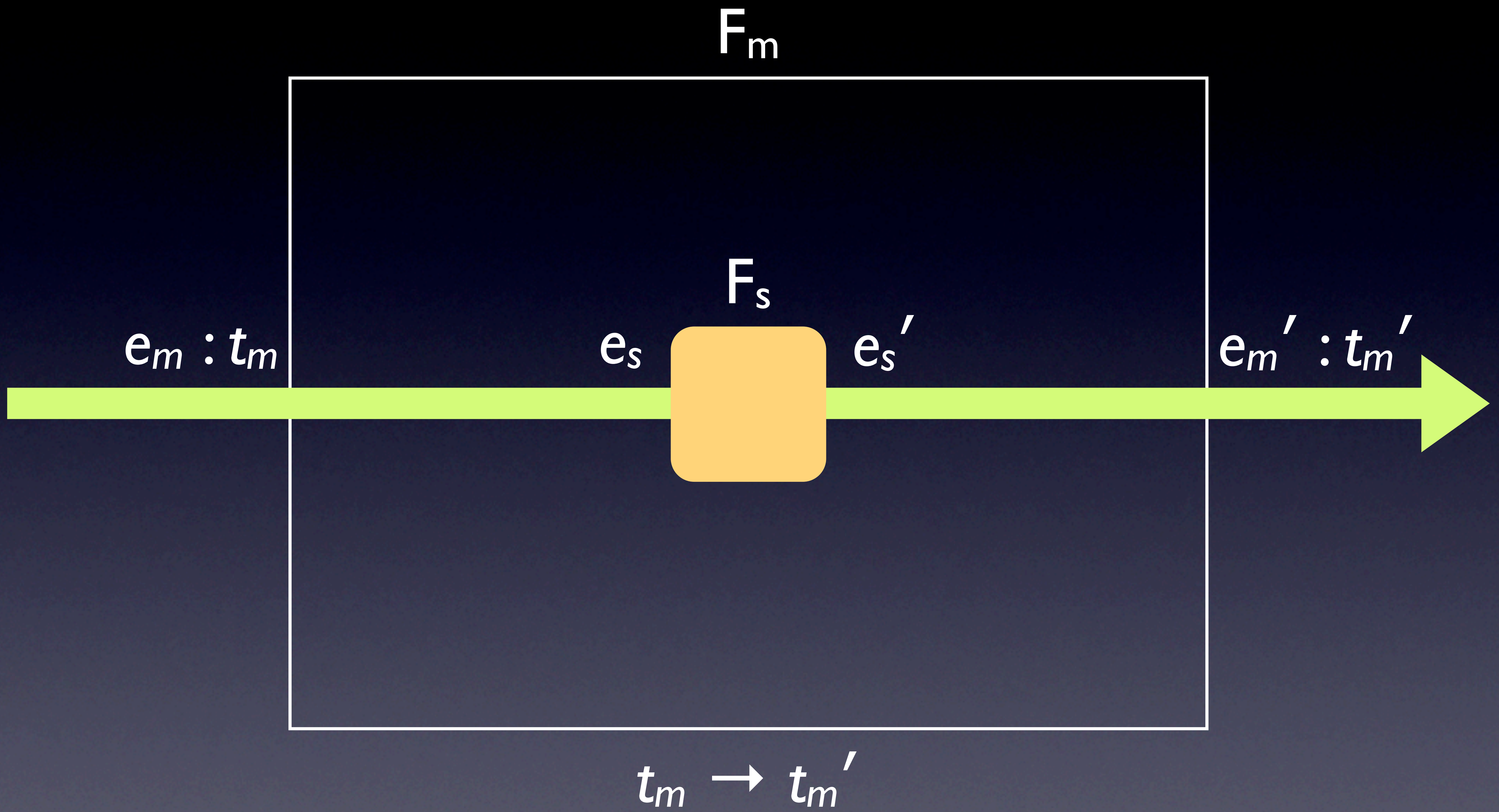
$$\mathcal{E}\,[\ \mathbf{ms\ N}\ \underline{n}\ ]_m \rightarrow \mathcal{E}\,[\ \underline{n}\ ]$$

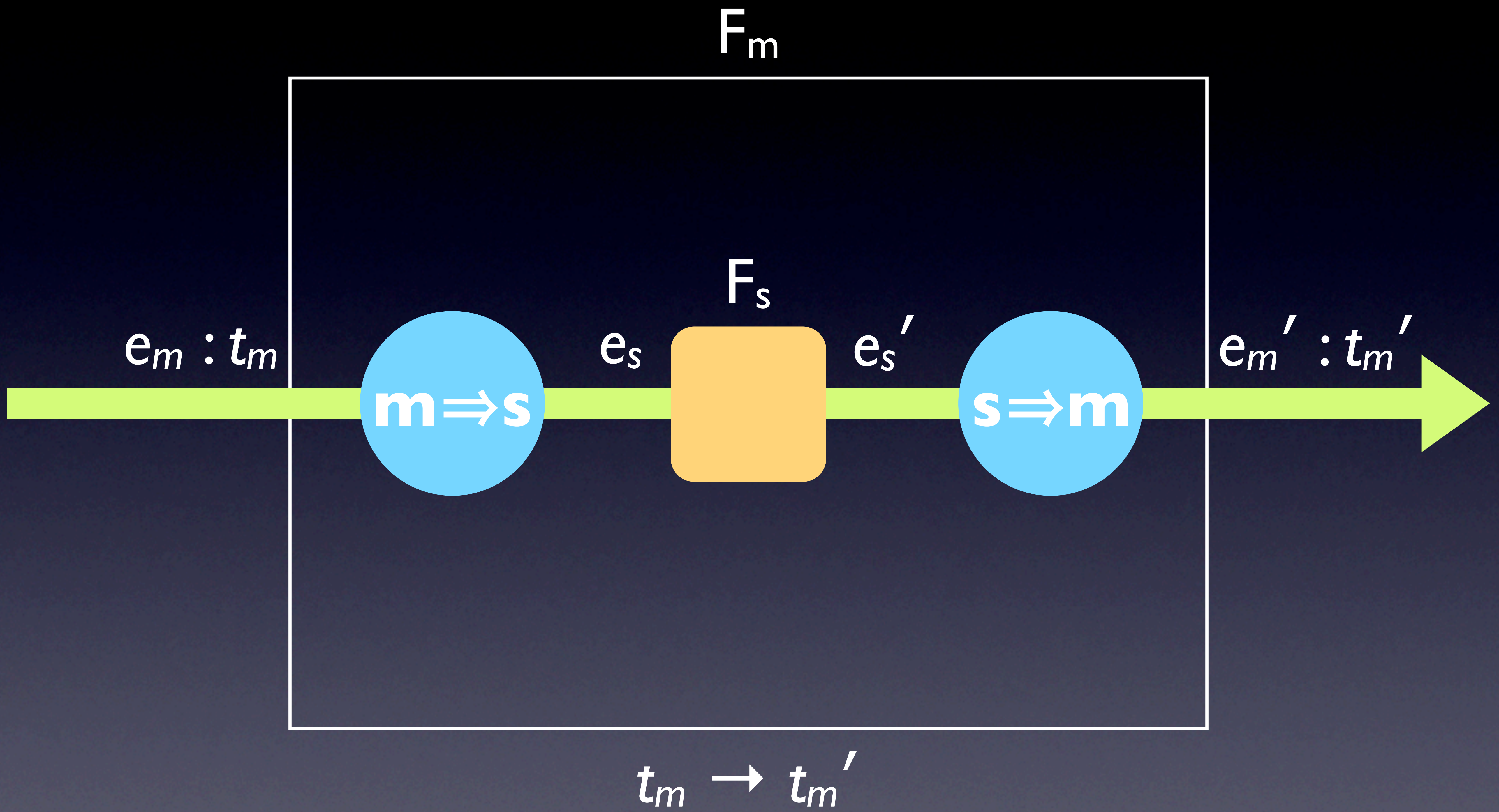
$$\mathcal{E}\,[\ \mathbf{ms\ N}\ v_s\ ]_m \rightarrow \mathcal{E}\,[\ \mathbf{wrong\ N}\ \text{“Not a number”}\ ]$$

$$\mathcal{E}\,[\ \mathbf{sm\ N}\ \underline{n}\ ]_s \rightarrow \mathcal{E}\,[\ \underline{n}\ ]$$



F_s
e_s
e_s'


F_m
F_s
$e_m : t_m$
e_s
e_s'
$e_m' : t_m'$
$t_m \rightarrow t_m'$


F_m
F_s
e_m : t_m
m⇒s
e_s
e_s′
s⇒m
e_m′ : t_m′
t_m → t_m′

ML to Scheme

sm $\equiv$ **sm** t_m x_m

ML to Scheme

$\mathrm{sm} \equiv \mathbf{sm}\ t_m\ x_m$

Scheme application

$\mathrm{app} \equiv F_s\ \mathrm{sm}$

ML to Scheme

$\mathrm{sm} \equiv \mathbf{sm}\ t_m\ x_m$

Scheme application

$\mathrm{app} \equiv \mathrm{F_s}\ \mathrm{sm}$

Scheme to ML

$\mathrm{ms} \equiv \mathbf{ms}\ t_m'\ \mathrm{app}$

ML to Scheme

$$\mathrm{sm} \equiv \mathbf{sm}\ t_m\ x_m$$

Scheme application

$$\mathrm{app} \equiv \mathrm{F_s}\ \mathrm{sm}$$

Scheme to ML

$$\mathrm{ms} \equiv \mathbf{ms}\ t_m{}'\ \mathrm{app}$$

Abstraction

$$\mathrm{F_m} \equiv \lambda\ x_m : t_m\ .\ \mathrm{ms}$$

ML to Scheme

$$\mathrm{sm} \equiv \mathbf{sm}\ t_m\ x_m$$

Scheme application

$$\mathrm{app} \equiv \mathrm{F_s}\ \mathrm{sm}$$

Scheme to ML

$$\mathrm{ms} \equiv \mathbf{ms}\ t_m{}'\ \mathrm{app}$$

Abstraction

$$\mathrm{F_m} \equiv \lambda\ x_m : t_m\ .\ \mathrm{ms}$$

$$\mathrm{F_m} \equiv \lambda\ x_m : t_m\ .\ \mathbf{ms}\ t_m{}'\ (\mathrm{F_s}\ (\mathbf{sm}\ t_m\ x_m))$$

$$\mathscr{E}[\,\mathbf{ms}\ (t_m \rightarrow t_m{}')\ (\lambda\ x_s\ .\ e_s)\ ]_m$$

$$\rightarrow$$

$$\mathscr{E}[\ \lambda\ x_m : t_m\ .\ \mathbf{ms}\ t_m{}'\ ((\lambda\ x_s\ .\ e_s)\ (\mathbf{sm}\ t_m\ x_m))\ ]$$

$$\mathcal{E}[\ \mathbf{ms}\ (t_m \to t_m{}')\ (\lambda\ x_s\ .\ e_s)\ ]_m$$

$$\to$$

$$\mathcal{E}[\ \lambda\ x_m : t_m\ .\ \mathbf{ms}\ t_m{}'\ ((\lambda\ x_s\ .\ e_s)\ (\mathbf{sm}\ t_m\ x_m))\ ]$$

$$\mathcal{E}[\ \mathbf{ms}\ (t_m \to t_m{}')\ v_s\ ]_m$$

$$\to$$

$$\mathcal{E}[\ \mathbf{wrong}\ (t_m \to t_m{}')\ \text{“Not a function”}\ ]$$

$$\mathcal{E}[\,\mathbf{ms}\ (t_m \to t_m')\ (\lambda\, x_s\, .\, e_s)\,]_m$$

$$\to$$

$$\mathcal{E}[\,\lambda\, x_m : t_m\, .\, \mathbf{ms}\ t_m'\ ((\lambda\, x_s\, .\, e_s)\ (\mathbf{sm}\ t_m\ x_m))\,]$$

$$\mathcal{E}[\,\mathbf{ms}\ (t_m \to t_m')\ v_s\,]_m$$

$$\to$$

$$\mathcal{E}[\,\mathbf{wrong}\ (t_m \to t_m')\ \text{“Not a function”}\,]$$

$$\mathcal{E}[\,\mathbf{sm}\ (t_m \to t_m')\ v_m\,]_s$$

$$\to$$

$$\mathcal{E}[\,\lambda\, x_s\, .\, \mathbf{sm}\ t_m'\ (v_m\ (\mathbf{ms}\ t_m\ x_s))\,]$$

$t_m = \cdots \mid \mathbf{L}$

$$t_m = \cdots \mid \mathbf{L} \qquad v_m = \cdots \mid \mathbf{ms}\ \mathbf{L}\ v_s$$

$t_m = \cdots \mid \mathbf{L}$ $\quad$ $v_m = \cdots \mid \mathbf{ms}\ \mathbf{L}\ v_s$ $\quad$ $\Gamma \vdash_m \mathbf{L}$

$t_m = \cdots \mid \mathbf{L} \qquad v_m = \cdots \mid \mathbf{ms}\ \mathbf{L}\ v_s \qquad \Gamma \vdash_m \mathbf{L}$

$$\mathcal{E}[\ \mathbf{sm}\ (\forall\ y_m\ .\ t_m)\ (\Lambda\ y_m'\ .\ e_m)\ ]_s$$

$$\rightarrow$$

$$\mathcal{E}[\ \mathbf{sm}\ t_m\ [\ \mathbf{L}\ /\ y_m\ ]\ e_m\ [\ \mathbf{L}\ /\ y_m'\ ]\ ]$$

$t_m = \cdots \mid \mathbf{L}$ $\qquad$ $v_m = \cdots \mid \mathbf{ms}\ \mathbf{L}\ v_s$ $\qquad$ $\Gamma \vdash_m \mathbf{L}$

$$\mathcal{E}\,[\ \mathbf{sm}\ (\forall\ y_m\ .\ t_m)\ (\Lambda\ y_m{}'\ .\ e_m)\ ]_s$$
$$\rightarrow$$
$$\mathcal{E}\,[\ \mathbf{sm}\ t_m\,[\ \mathbf{L}\ /\ y_m\ ]\ e_m\,[\ \mathbf{L}\ /\ y_m{}'\ ]\ ]$$

$$\mathcal{E}\,[\ \mathbf{sm}\ \mathbf{L}\ (\mathbf{ms}\ \mathbf{L}\ v_s)\ ]_s \rightarrow \mathcal{E}\,[\ v_s\ ]$$

$$\mathrm{id} \equiv \Lambda\, y\, .\, \mathbf{ms}\ (y \to y)\ (\lambda\, x\, .\, x)$$

$$\mathrm{id} \equiv \Lambda\, y\, .\, \mathbf{ms}\, (y \rightarrow y)\, (\lambda\, x\, .\, x)$$

$\mathrm{id} \langle\, \mathbf{N}\, \rangle$
behaves the same as
$\mathrm{id} \langle\, \mathbf{N} \rightarrow \mathbf{N}\, \rangle$

$$\text{id} \equiv \Lambda\, y\,.\, \mathbf{ms}\ (y \to y)\ (\lambda\, x\,.\, x)$$

$$\text{id}\ \langle\ \mathbf{N}\ \rangle$$

behaves the same as

$$\text{id}\ \langle\ \mathbf{N} \to \mathbf{N}\ \rangle$$

$$\text{id}_m \equiv \Lambda\, y\,.\, \mathbf{ms}\ (y \to y)\ (\lambda\, x\,.\, \mathbf{if0}\ (\mathbf{num?}\ x)\ x\ \underline{0})$$

$$\mathrm{id} \equiv \Lambda\, y\,.\, \mathbf{ms}\ (y \rightarrow y)\ (\lambda\, x\,.\, x)$$

$\mathrm{id}\ \langle\ \mathbf{N}\ \rangle$

behaves the same as

$\mathrm{id}\ \langle\ \mathbf{N} \rightarrow \mathbf{N}\ \rangle$

$$\mathrm{id}_m \equiv \Lambda\, y\,.\, \mathbf{ms}\ (y \rightarrow y)\ (\lambda\, x\,.\, \mathbf{if0}\ (\mathbf{num?}\ x)\ x\ \underline{0})$$

$\mathrm{id}_m\ \langle\ \mathbf{N}\ \rangle$

behaves differently than

$\mathrm{id}_m\ \langle\ \mathbf{N} \rightarrow \mathbf{N}\ \rangle$

sm t_m v_m

sm t_m v_m

num? (sm t_m v_m**)**

sm t_m v_m

num? (**sm** t_m v_m)

Set of conversion schemes, k

$\textbf{sm}\ t_m\ v_m$

$\textbf{num?}\ (\textbf{sm}\ t_m\ v_m)$

Set of conversion schemes, k

$$k_m = \textbf{L} \mid \textbf{N} \mid y_m \mid k_m \rightarrow k_m \mid \forall\, y_m\, .\, k_m \mid b \diamond t_m$$

$$\mathbf{sm}\ t_m\ v_m$$

$$\mathbf{num?}\ (\mathbf{sm}\ t_m\ v_m)$$

Set of conversion schemes, k

$$k_m = \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \rightarrow k_m \mid \forall\, y_m\, .\, k_m \mid b \diamond t_m$$

$$e_m = \cdots \mid \mathbf{ms}\ k_m\ e_s$$

$\mathbf{sm}\ t_m\ v_m$

$\mathbf{num?}\ (\mathbf{sm}\ t_m\ v_m)$

Set of conversion schemes, k

$$k_m = \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \rightarrow k_m \mid \forall\, y_m\,.\,k_m \mid b \diamond t_m$$

$$e_m = \cdots \mid \mathbf{ms}\ k_m\ e_s$$

$$e_s = \cdots \mid \mathbf{sm}\ k_m\ e_m$$

$$\mathbf{sm}\ t_m\ v_m$$

$$\mathbf{num?}\ (\mathbf{sm}\ t_m\ v_m)$$

Set of conversion schemes, k

$$k_m = \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \rightarrow k_m \mid \forall\, y_m\, .\, k_m \mid b \diamond t_m$$

$$e_m = \cdots \mid \mathbf{ms}\ k_m\ e_s$$

$$e_s = \cdots \mid \mathbf{sm}\ k_m\ e_m$$

$$v_s = \cdots \mid \mathbf{sm}\ k_m\ v_m$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{N} \rfloor = \mathbf{N}$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{N} \rfloor = \mathbf{N}$$

$$\lfloor y_m \rfloor = y_m$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{N} \rfloor = \mathbf{N}$$

$$\lfloor y_m \rfloor = y_m$$

$$\lfloor k_m \to k_m \rfloor = \lfloor k_m \rfloor \to \lfloor k_m \rfloor$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{N} \rfloor = \mathbf{N}$$

$$\lfloor y_m \rfloor = y_m$$

$$\lfloor k_m \rightarrow k_m \rfloor = \lfloor k_m \rfloor \rightarrow \lfloor k_m \rfloor$$

$$\lfloor \forall\, y_m \,.\, k_m \rfloor = \forall\, y_m \,.\, \lfloor k_m \rfloor$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{N} \rfloor = \mathbf{N}$$

$$\lfloor y_m \rfloor = y_m$$

$$\lfloor k_m \to k_m \rfloor = \lfloor k_m \rfloor \to \lfloor k_m \rfloor$$

$$\lfloor \forall\, y_m \,.\, k_m \rfloor = \forall\, y_m \,.\, \lfloor k_m \rfloor$$

$$\lfloor b \diamond t_m \rfloor = t_m$$

$$\frac{\Gamma \vdash_m \lfloor k_m \rfloor \longrightarrow \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ k_m\ e_s : \lfloor k_m \rfloor}$$

$$\frac{\Gamma \vdash_m \lfloor k_m \rfloor \quad \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ k_m\ e_s : \lfloor k_m \rfloor}$$

$$\mathscr{E}[\ (\Lambda\ y_m\ .\ e_m)\ \langle\ t_m\ \rangle\ ]_m \rightarrow \mathscr{E}[\ e_m\ [\ b \diamond t_m\ /\ y_m\ ]\ ]$$

$$\frac{\Gamma \vdash_m \lfloor k_m \rfloor \quad \Gamma \vdash_s \mathsf{e}_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ k_m\ \mathsf{e}_s : \lfloor k_m \rfloor}$$

$$\mathcal{E}[\ (\Lambda\ y_m\ .\ \mathsf{e}_m)\ \langle\ t_m\ \rangle\ ]_m \rightarrow \mathcal{E}[\ \mathsf{e}_m\ [\ b \diamond t_m\ /\ y_m\ ]\ ]$$

$$\mathcal{E}[\ \mathbf{ms}\ (b \diamond t_m)\ (\mathbf{sm}\ (b \diamond t_m)\ v_m)\ ]_m \rightarrow \mathcal{E}[\ v_m\ ]$$

$$\frac{\Gamma \vdash_m \lfloor k_m \rfloor \quad \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ k_m\ e_s : \lfloor k_m \rfloor}$$

$$\mathcal{E}[\ (\Lambda\ y_m\ .\ e_m)\ \langle\ t_m\ \rangle\ ]_m \rightarrow \mathcal{E}[\ e_m\ [\ b \diamond t_m\ /\ y_m\ ]\ ]$$

$$\mathcal{E}[\ \mathbf{ms}\ (b \diamond t_m)\ (\mathbf{sm}\ (b \diamond t_m)\ v_m)\ ]_m \rightarrow \mathcal{E}[\ v_m\ ]$$

$$\mathcal{E}[\ \mathbf{ms}\ (b \diamond t_m)\ v_s\ ]_m$$

$$\rightarrow$$

$$\mathcal{E}[\ \mathbf{wrong}\ \lfloor\ b \diamond t_m\ \rfloor\ \text{“Brand mismatch”}\ ]$$

$$(\lambda\, x_m : t_m \,.\, e_m)\,[\, b \diamond t_m' \,/\, y_m \,]$$

$$=$$

$$\lambda\, x_m : t_m\,[\, t_m' \,/\, y_m \,] \,.\, e_m\,[\, b \diamond t_m' \,/\, y_m \,]$$

$$(\lambda\ x_m : t_m\ .\ e_m)\ [\ b \diamond t_m{}'\ /\ y_m\ ]$$

$$=$$

$$\lambda\ x_m : t_m\ [\ t_m{}'\ /\ y_m\ ]\ .\ e_m\ [\ b \diamond t_m{}'\ /\ y_m\ ]$$

$$(\mathbf{ms}\ k_m\ e_s)\ [\ b \diamond t_m{}'\ /\ y_m\ ]$$

$$=$$

$$\mathbf{ms}\ k_m\ [\ t_m{}'\ /\ y_m\ ]\ e_s\ [\ b \diamond t_m{}'\ /\ y_m\ ]$$

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

ML & Scheme

- Numbers, arithmetic, conditions
- Functions, applications
- Errors
- Natural embedding
- Eager evaluation
- ‘m’ and ‘s’ subscripts

ML

- Statically typed
- Parametric polymorphism
- Fixed-point operations

$$t_m \;=\; \mathbf{N} \mid y_m \mid t_m \to t_m \mid \forall\, y_m \,.\, t_m$$

$$t_m \;=\; \mathbf{N} \mid y_m \mid t_m \to t_m \mid \forall\, y_m \,.\, t_m$$

$$k_m \;=\; \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \to k_m \mid \forall\, y_m \,.\, k_m \mid b \diamond t_m$$

$$t_m \;=\; \mathbf{N} \mid y_m \mid t_m \to t_m \mid \forall\, y_m \,.\, t_m$$

$$k_m \;=\; \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \to k_m \mid \forall\, y_m \,.\, k_m \mid b \diamond t_m$$

$$v_m \;=\; \lambda\, x_m : t_m \,.\, \mathrm{e}_m \mid \Lambda\, y_m \,.\, \mathrm{e}_m \mid \underline{n} \mid \mathbf{ms}\; \mathbf{L}\; v_s$$

$$t_m \;=\; \mathbf{N} \mid y_m \mid t_m \to t_m \mid \forall\, y_m \,.\, t_m$$

$$k_m \;=\; \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \to k_m \mid \forall\, y_m \,.\, k_m \mid b \diamond t_m$$

$$v_m \;=\; \lambda\, x_m : t_m \,.\, e_m \mid \Lambda\, y_m \,.\, e_m \mid \underline{n} \mid \mathbf{ms}\ \mathbf{L}\ v_s$$

$$e_m \;=\; \begin{array}{l} x_m \mid v_m \mid e_m\ e_m \mid e_m \langle\, t_m \,\rangle \mid \mathbf{fix}\ e_m \mid \mathbf{+/-}\ e_m\ e_m \\ \mathbf{if0}\ e_m\ e_m\ e_m \mid \mathbf{wrong}\ t_m\ \mathit{string} \mid \mathbf{ms}\ k_m\ e_s \end{array}$$

$$t_m \;=\; \mathbf{N} \mid y_m \mid t_m \to t_m \mid \forall\, y_m \,.\, t_m$$

$$k_m \;=\; \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \to k_m \mid \forall\, y_m \,.\, k_m \mid b \diamond t_m$$

$$v_m \;=\; \lambda\, x_m : t_m \,.\, e_m \mid \Lambda\, y_m \,.\, e_m \mid \underline{n} \mid \mathbf{ms}\ \mathbf{L}\ v_s$$

$$\begin{aligned} e_m \;=\; & x_m \mid v_m \mid e_m\ e_m \mid e_m \langle\, t_m \,\rangle \mid \mathbf{fix}\ e_m \mid \mathbf{+/-}\ e_m\ e_m \\ & \mathbf{if0}\ e_m\ e_m\ e_m \mid \mathbf{wrong}\ t_m\ \mathit{string} \mid \mathbf{ms}\ k_m\ e_s \end{aligned}$$

$$\begin{aligned} E_m \;=\; & [\ ]_m \mid E_m\ e_m \mid v_m\ E_m \mid E_m \langle\, t_m \,\rangle \mid \mathbf{fix}\ E_m \\ & \mathbf{+/-}\ E_m\ e_m \mid \mathbf{+/-}\ v_m\ E_m \mid \mathbf{if0}\ E_m\ e_m\ e_m \\ & \mathbf{ms}\ k_m\ E_s \end{aligned}$$

Scheme

- Dynamically typed
- Closed term typing
- Ad-hoc polymorphism

$$v_s \quad = \quad \lambda\, x_s \,.\, e_s \mid \underline{n} \mid \mathbf{sm}\ (b \diamond t_m)\ v_m$$

$$v_s \;=\; \lambda\, x_s\,.\, e_s \mid \underline{n} \mid \mathbf{sm}\ (b \diamond t_m)\ v_m$$

$$e_s \;=\; x_s \mid v_s \mid e_s\ e_s \mid \mathbf{+/-}\ e_s\ e_s \mid \mathbf{if0}\ e_s\ e_s\ e_s \mid \mathbf{fun?}\ e_s \mid \mathbf{num?}\ e_s \mid \mathbf{wrong}\ \mathit{string} \mid \mathbf{sm}\ k_m\ e_m$$

$$v_s \;=\; \lambda\, x_s\, .\, e_s \mid \underline{n} \mid \mathbf{sm}\, (b \diamond t_m)\, v_m$$

$$e_s \;=\; x_s \mid v_s \mid e_s\, e_s \mid \mathbf{+/\text{-}}\; e_s\, e_s \mid \mathbf{if0}\; e_s\, e_s\, e_s \mid \mathbf{fun?}\; e_s \mid \mathbf{num?}\; e_s \mid \mathbf{wrong}\; \textit{string} \mid \mathbf{sm}\; k_m\, e_m$$

$$E_s \;=\; [\;]_s \mid E_s\, e_s \mid v_s\, E_s \mid \mathbf{+/\text{-}}\; E_s\, e_s \mid \mathbf{+/\text{-}}\; v_s\, E_s \mid \mathbf{if0}\; E_s\, e_s\, e_s \mid \mathbf{fun?}\; E_s \mid \mathbf{num?}\; E_s \mid \mathbf{sm}\; k_m\, E_m$$

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

- Eager vs. lazy evaluation
- Introduce Haskell
- Incompatible evaluation strategies
  - Function behavior
  - Value conversion

$$t_h \;=\; \mathbf{N} \mid y_h \mid t_h \to t_h \mid \forall\, y_h\,.\,t_h$$

$$t_h \;=\; \mathbf{N} \mid y_h \mid t_h \to t_h \mid \forall\, y_h\,.\,t_h$$

$$k_h \;=\; \mathbf{L} \mid \mathbf{N} \mid y_h \mid k_h \to k_h \mid \forall\, y_h\,.\,k_h \mid b \diamond t_h$$

$$t_h \;=\; \mathbf{N} \mid y_h \mid t_h \to t_h \mid \forall\, y_h \,.\, t_h$$

$$k_h \;=\; \mathbf{L} \mid \mathbf{N} \mid y_h \mid k_h \to k_h \mid \forall\, y_h \,.\, k_h \mid b \diamond t_h$$

$$v_h \;=\; \lambda\, x_h : t_h \,.\, e_h \mid \Lambda\, y_h \,.\, e_h \mid \underline{n} \mid \mathbf{hs}\ \mathbf{L}\ v_s$$

$$t_h \;=\; \mathbf{N} \mid y_h \mid t_h \to t_h \mid \forall\, y_h \,.\, t_h$$

$$k_h \;=\; \mathbf{L} \mid \mathbf{N} \mid y_h \mid k_h \to k_h \mid \forall\, y_h \,.\, k_h \mid b \diamond t_h$$

$$v_h \;=\; \lambda\, x_h : t_h \,.\, e_h \mid \Lambda\, y_h \,.\, e_h \mid \underline{n} \mid \mathbf{hs}\ \mathbf{L}\ v_s$$

$$\begin{aligned} e_h \;=\; & x_h \mid v_h \mid e_h\ e_h \mid e_h \langle t_h \rangle \mid \mathbf{fix}\ e_h \mid \mathbf{+/-}\ e_h\ e_h \\ & \mathbf{if0}\ e_h\ e_h\ e_h \mid \mathbf{wrong}\ t_h\ \mathit{string} \end{aligned}$$

$$t_h \;=\; \mathbf{N} \mid y_h \mid t_h \to t_h \mid \forall\, y_h \,.\, t_h$$

$$k_h \;=\; \mathbf{L} \mid \mathbf{N} \mid y_h \mid k_h \to k_h \mid \forall\, y_h \,.\, k_h \mid b \diamond t_h$$

$$v_h \;=\; \lambda\, x_h : t_h \,.\, e_h \mid \Lambda\, y_h \,.\, e_h \mid \underline{n} \mid \mathbf{hs}\ \mathbf{L}\ v_s$$

$$e_h \;=\; \begin{array}{l} x_h \mid v_h \mid e_h\ e_h \mid e_h \langle\, t_h \,\rangle \mid \mathbf{fix}\ e_h \mid \mathbf{+/-}\ e_h\ e_h \\ \mathbf{if0}\ e_h\ e_h\ e_h \mid \mathbf{wrong}\ t_h\ string \end{array}$$

$$E_h \;=\; \begin{array}{l} [\ ]_h \mid E_h\ e_h \mid E_h \langle\, t_h \,\rangle \mid \mathbf{fix}\ E_h \mid \mathbf{+/-}\ E_h\ e_h \\ \mathbf{+/-}\ v_h\ E_h \mid \mathbf{if0}\ E_h\ e_h\ e_h \end{array}$$

$$e_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ e_m \mid \mathbf{hs}\ k_h\ e_s$$

$$e_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ e_m \mid \mathbf{hs}\ k_h\ e_s$$

$$e_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ e_h$$

$$e_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ e_m \mid \mathbf{hs}\ k_h\ e_s$$

$$e_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ e_h$$

$$e_s = \cdots \mid \mathbf{sh}\ k_h\ e_h$$

$$v_h = \cdots \mid \mathbf{hm}\ \mathbf{L}\ t_m\ v_m$$

$$v_h = \cdots \mid \mathbf{hm}\ \mathbf{L}\ t_m\ v_m$$

$$v_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ e_h$$

$$v_h = \cdots \mid \mathbf{hm}\ \mathbf{L}\ t_m\ v_m$$

$$v_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ e_h$$

$$v_s = \cdots \mid \mathbf{sh}\ k_h\ e_h$$

$$E_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ E_m \mid \mathbf{hs}\ k_h\ E_s$$

$$E_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ E_m \mid \mathbf{hs}\ k_h\ E_s$$

$$E_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ E_h$$

$$E_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ E_m \mid \mathbf{hs}\ k_h\ E_s$$

$$E_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ E_h$$

$$E_s = \cdots \mid \mathbf{sh}\ k_h\ E_h$$

$$\mathscr{E}\,[\ \mathbf{hm}\ (\forall\ y_h\ .\ t_h)\ (\forall\ y_m\ .\ t_m)\ (\Lambda\ y_m'\ .\ e_m)\ ]_h$$

$$\rightarrow$$

$$\mathscr{E}\,[\ \Lambda\ y_h\ .\ \mathbf{hm}\ t_h\ t_m\,[\ \mathbf{L}\ /\ y_m\ ]\ e_m\,[\ \mathbf{L}\ /\ y_m'\ ]\ ]$$

$$\mathscr{E}\,[\ \mathbf{hm}\ (\forall\ y_h\ .\ t_h)\ (\forall\ y_m\ .\ t_m)\ (\Lambda\ y_m{}'\ .\ e_m)\ ]_h$$

$$\rightarrow$$

$$\mathscr{E}\,[\ \Lambda\ y_h\ .\ \mathbf{hm}\ t_h\ t_m\,[\ \mathbf{L}\ /\ y_m\ ]\ e_m\,[\ \mathbf{L}\ /\ y_m{}'\ ]\ ]$$

$$\frac{\Gamma \vdash_h t_h \quad \text{—} \quad \Gamma \vdash_m t_m \quad \text{—} \quad \Gamma \vdash_m e_m : t_m{}' \quad \text{—} \quad t_m \doteq t_m{}'}{\Gamma \vdash_h \mathbf{hm}\ t_h\ t_m\ e_m : t_h}$$

$$x \doteq x$$

$$x \doteq y \Rightarrow y \doteq x$$

$$x \doteq y \wedge y \doteq z \Rightarrow x \doteq z$$

$$t_h \doteq \mathbf{L}$$

$$t_m \doteq \mathbf{L}$$

$$t_h = t_m \Rightarrow t_h \doteq t_m$$

$$K_s \equiv \lambda x \; y \,.\, x$$

$$\mathrm{K_s} \equiv \lambda\, x\ \ y\, .\, x$$

$$\mathrm{K_h} : \forall\, y_h\ y_h'\, .\, y_h \to y_h' \to y_h$$

$$\mathsf{K_s} \equiv \lambda\, x\ \ y\,.\,x$$

$$\mathsf{K_h} : \forall\, y_h\, y_h'\,.\,y_h \to y_h' \to y_h$$

$$\mathsf{K_h} \equiv \Lambda\, y_h\, y_h'\,.\,\mathbf{hs}\ (y_h \to y_h' \to y_h)\ \mathsf{K_s}$$

$$K_s \equiv \lambda\, x\;\; y\, .\, x$$

$$K_h : \forall\, y_h\, y_h{}'\, .\, y_h \to y_h{}' \to y_h$$

$$K_h \equiv \Lambda\, y_h\, y_h{}'\, .\, \mathbf{hs}\; (y_h \to y_h{}' \to y_h)\; K_s$$

$$K_{hn} \equiv K_h\, \langle \mathbf{N} \rangle\, \langle \mathbf{N} \rangle \to \mathbf{hs}\; (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\; K_s$$

$$K_s \equiv \lambda\, x\ \ y\, .\, x$$

$$K_h : \forall\, y_h\ y_h{}'\, .\, y_h \to y_h{}' \to y_h$$

$$K_h \equiv \Lambda\, y_h\ y_h{}'\, .\, \mathbf{hs}\ (y_h \to y_h{}' \to y_h)\ K_s$$

$$K_{hn} \equiv K_h\ \langle \mathbf{N} \rangle\ \langle \mathbf{N} \rangle \to \mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\ K_s$$

$$K_{hn}\ \underline{0}\ \Omega \to \underline{0}$$

$$K_s \equiv \lambda\, x\ \ y\, .\, x$$

$$K_h : \forall\, y_h\ y_h{}'\, .\, y_h \to y_h{}' \to y_h$$

$$K_h \equiv \Lambda\, y_h\ y_h{}'\, .\, \mathbf{hs}\ (y_h \to y_h{}' \to y_h)\ K_s$$

$$K_{hn} \equiv K_h\ \langle \mathbf{N} \rangle\ \langle \mathbf{N} \rangle \to \mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\ K_s$$

$$K_{hn}\ \underline{0}\ \Omega \to \underline{0}$$

$$K_{hn}\ \underline{0}\ \Omega \not\to$$

$$K_{hn}\ \underline{0}\ \Omega \equiv (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\ K_s}})\ \underline{0}\ \Omega$$

$$K_{hn}\ \underline{0}\ \Omega \equiv (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\ K_s}})\ \underline{0}\ \Omega$$

$$\to \underline{\underline{(\lambda\, x : \mathbf{N}\ .\ \mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ x)))\ \underline{0}}}\ \Omega$$

$$K_{hn}\ \underline{0}\ \Omega \equiv (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})}}\ \underline{\underline{K_s}})\ \underline{0}\ \Omega$$

$$\to \underline{\underline{(\lambda\, x : \mathbf{N}\, .\, \mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\underline{\underline{K_s}}\ (\mathbf{sh}\ \mathbf{N}\ x)))}}\ \underline{\underline{0}}\ \Omega$$

$$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\underline{\underline{\mathbf{sh}\ \mathbf{N}}}\ \underline{\underline{0}})))\ \Omega$$

$K_{hn}\ \underline{0}\ \Omega \equiv (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})}}\ \underline{\underline{K_s}})\ \underline{0}\ \Omega$

$\to \underline{\underline{(\lambda\, x : \mathbf{N}\ .\ \mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ x)))}}\ \underline{\underline{0}}\ \Omega$

$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\underline{\underline{\mathbf{sh}\ \mathbf{N}\ 0}})))\ \Omega$

$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\underline{\underline{K_s\ 0}}))\ \Omega$

$K_{hn}\ \underline{0}\ \Omega \equiv (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})}}\ \underline{\underline{K_s}})\ \underline{0}\ \Omega$

$\to \underline{\underline{(\lambda\, x : \mathbf{N}\, .\, \mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ x)))}}\ \underline{\underline{0}}\ \Omega$

$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\underline{\underline{\mathbf{sh}\ \mathbf{N}}}\ \underline{\underline{0}})))\ \Omega$

$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\underline{\underline{K_s}}\ \underline{\underline{0}}))\ \Omega$

$\to (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\lambda\, y\, .\, 0)}})\ \Omega$

$$K_{hn}\ 0\ \Omega \equiv (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\ K_s}})\ 0\ \Omega$$

$$\to \underline{\underline{(\lambda\, x : \mathbf{N}\, .\, \mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ x)))\ 0}}\ \Omega$$

$$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\underline{\underline{\mathbf{sh}\ \mathbf{N}\ 0}})))\ \Omega$$

$$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\underline{\underline{K_s\ 0}}))\ \Omega$$

$$\to (\underline{\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\lambda\, y\, .\, 0)}})\ \Omega$$

$$\to \underline{\underline{(\lambda\, x : \mathbf{N}\, .\, \mathbf{hs}\ \mathbf{N}\ ((\lambda\, y\, .\, 0)\ (\mathbf{sh}\ \mathbf{N}\ x)))\ \Omega}}$$

$\rightarrow$ **hs N** ((λ y . $\underline{0}$) ($\underline{\underline{\textbf{sh N } \Omega}}$))

$\rightarrow \mathbf{hs\ N}\ ((\lambda\ y\ .\ \underline{0})\ (\underline{\underline{\mathbf{sh\ N}\ \Omega}}))$

$\rightarrow \mathbf{hs\ N}\ ((\lambda\ y\ .\ \underline{0})\ (\underline{\underline{\mathbf{sh\ N}\ \Omega}}))$

$\rightarrow$ **hs N** $((\lambda\, y\, .\, \underline{0})\ (\underline{\underline{\mathbf{sh\ N}\ \Omega}}))$

$\rightarrow$ **hs N** $((\lambda\, y\, .\, \underline{0})\ (\underline{\underline{\mathbf{sh\ N}\ \Omega}}))$

$\rightarrow$ **hs N** $((\lambda\, y\, .\, \underline{0})\ (\underline{\underline{\mathbf{sh\ N}\ \Omega}}))$

$\vdots$

$$e_h = \cdots \mid \textbf{nil}\ t_h \mid \textbf{cons}\ e_h\ e_h \mid \textbf{hd}\ e_h \mid \textbf{tl}\ e_h \mid \textbf{null?}\ e_h$$

$$e_h = \cdots \mid \textbf{nil}\ t_h \mid \textbf{cons}\ e_h\ e_h \mid \textbf{hd}\ e_h \mid \textbf{tl}\ e_h \mid \textbf{null?}\ e_h$$

$$t_h = \cdots \mid \{\ t_h\ \}$$

$$e_h = \cdots \mid \textbf{nil}\ t_h \mid \textbf{cons}\ e_h\ e_h \mid \textbf{hd}\ e_h \mid \textbf{tl}\ e_h \mid \textbf{null?}\ e_h$$

$$t_h = \cdots \mid \{\ t_h\ \}$$

$$k_h = \cdots \mid \{\ k_h\ \}$$

$$e_h = \cdots \mid \mathbf{nil}\ t_h \mid \mathbf{cons}\ e_h\ e_h \mid \mathbf{hd}\ e_h \mid \mathbf{tl}\ e_h \mid \mathbf{null?}\ e_h$$

$$t_h = \cdots \mid \{\ t_h\ \}$$

$$k_h = \cdots \mid \{\ k_h\ \}$$

$$E_h = \cdots \mid \mathbf{hd}\ E_h \mid \mathbf{tl}\ E_h \mid \mathbf{null?}\ E_h$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\, t_h \,\}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \textbf{nil}\ t_h : \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \textbf{nil}\ t_h : \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \textbf{null?}\ e_h : \mathbf{N}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\, t_h \,\}}$$

$$\frac{\Gamma \vdash_h e_h : \{\, t_h \,\}}{\Gamma \vdash_h \mathbf{hd}\ e_h : t_h}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \mathbf{nil}\ t_h : \{\, t_h \,\}}$$

$$\frac{\Gamma \vdash_h e_h : \{\, t_h \,\}}{\Gamma \vdash_h \mathbf{null?}\ e_h : \mathbf{N}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \textbf{hd}\ e_h : t_h}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \textbf{nil}\ t_h : \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \textbf{tl}\ e_h : \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \textbf{null?}\ e_h : \mathbf{N}}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\, t_h \,\}}$$

$$\frac{\Gamma \vdash_h e_h : \{\, t_h \,\}}{\Gamma \vdash_h \mathbf{hd}\ e_h : t_h}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \mathbf{nil}\ t_h : \{\, t_h \,\}}$$

$$\frac{\Gamma \vdash_h e_h : \{\, t_h \,\}}{\Gamma \vdash_h \mathbf{tl}\ e_h : \{\, t_h \,\}}$$

$$\frac{\Gamma \vdash_h e_h : \{\, t_h \,\}}{\Gamma \vdash_h \mathbf{null?}\ e_h : \mathbf{N}}$$

$$\frac{\Gamma \vdash_h e_h : t_h \quad \Gamma \vdash_h e_h' : \{\, t_h \,\}}{\Gamma \vdash_h \mathbf{cons}\ e_h\ e_h' : \{\, t_h \,\}}$$

$\mathscr{E}$ [**hd** (**nil** t_h) $]_h \rightarrow \mathscr{E}$ [**wrong** t_h "Empty list"]

$\mathscr{E}$ [**hd** (**nil** t_h) $]_h \rightarrow \mathscr{E}$ [**wrong** t_h "Empty list"]

$\mathscr{E}$ [**hd** (**cons** e_h e_h') $]_h \rightarrow \mathscr{E}$ [e_h]

$\mathscr{E}$ [**hd** (**nil** t_h) $]_h \rightarrow \mathscr{E}$ [**wrong** t_h "Empty list"]

$\mathscr{E}$ [**hd** (**cons** e_h $e_h{}'$) $]_h \rightarrow \mathscr{E}$ [e_h]

$\mathscr{E}$ [**tl** (**nil** t_h) $]_h \rightarrow \mathscr{E}$ [**wrong** { t_h } "Empty list"]

$\mathscr{E}$ [**hd** (**nil** t_h)]$_h$ → $\mathscr{E}$ [**wrong** t_h "Empty list"]

$\mathscr{E}$ [**hd** (**cons** e_h e_h')]$_h$ → $\mathscr{E}$ [e_h]

$\mathscr{E}$ [**tl** (**nil** t_h)]$_h$ → $\mathscr{E}$ [**wrong** { t_h } "Empty list"]

$\mathscr{E}$ [**tl** (**cons** e_h e_h')]$_h$ → $\mathscr{E}$ [e_h']

$\mathscr{E}$ [**hd** (**nil** t_h)]$_h$ → $\mathscr{E}$ [**wrong** t_h "Empty list"]

$\mathscr{E}$ [**hd** (**cons** e_h e_h')]$_h$ → $\mathscr{E}$ [e_h]

$\mathscr{E}$ [**tl** (**nil** t_h)]$_h$ → $\mathscr{E}$ [**wrong** { t_h } "Empty list"]

$\mathscr{E}$ [**tl** (**cons** e_h e_h')]$_h$ → $\mathscr{E}$ [e_h']

$\mathscr{E}$ [**null?** (**nil** t_h)]$_h$ → $\mathscr{E}$ [$\underline{0}$]

$\mathcal{E}$ [**hd** (**nil** t_h) $]_h \rightarrow \mathcal{E}$ [**wrong** t_h "Empty list"]

$\mathcal{E}$ [**hd** (**cons** e_h e_h') $]_h \rightarrow \mathcal{E}$ [e_h]

$\mathcal{E}$ [**tl** (**nil** t_h) $]_h \rightarrow \mathcal{E}$ [**wrong** { t_h } "Empty list"]

$\mathcal{E}$ [**tl** (**cons** e_h e_h') $]_h \rightarrow \mathcal{E}$ [e_h']

$\mathcal{E}$ [**null?** (**nil** t_h) $]_h \rightarrow \mathcal{E}$ [$\underline{0}$]

$\mathcal{E}$ [**null?** (**cons** e_h e_h') $]_h \rightarrow \mathcal{E}$ [$\underline{1}$]

$$\mathscr{E}\,[\ \mathbf{hm}\ \{\ t_h\ \}\ \{\ t_m\ \}\ (\mathbf{nil}\ t_m{}')\ ]_h \rightarrow \mathscr{E}\,[\ \mathbf{nil}\ t_h\ ]$$

$$\mathcal{E}\,[\ \mathbf{hm}\ \{\ t_h\ \}\ \{\ t_m\ \}\ (\mathbf{nil}\ t_m{}')\ ]_h \rightarrow \mathcal{E}\,[\ \mathbf{nil}\ t_h\ ]$$

$$\mathcal{E}\,[\ \mathbf{hm}\ \{\ t_h\ \}\ \{\ t_m\ \}\ (\mathbf{cons}\ v_m\ v_m{}')\ ]_h$$

$$\rightarrow$$

$$\mathcal{E}\,[\ \mathbf{cons}\ (\mathbf{hm}\ t_h\ t_m\ v_m)\ (\mathbf{hm}\ \{\ t_h\ \}\ \{\ t_m\ \}\ v_m{}'\ ]\ ]$$

$$v_m = \cdots \mid \mathbf{nil}\ t_m \mid \mathbf{cons}\ v_m\ v_m$$

$$v_m = \cdots \mid \mathbf{nil}\ t_m \mid \mathbf{cons}\ v_m\ v_m$$

$$v_s = \cdots \mid \mathbf{nil} \mid \mathbf{cons}\ v_s\ v_s$$

$$v_m = \cdots \mid \textbf{nil}\ t_m \mid \textbf{cons}\ v_m\ v_m$$

$$v_s = \cdots \mid \textbf{nil} \mid \textbf{cons}\ v_s\ v_s$$

$$E_m = \cdots \mid \textbf{cons}\ E_m\ \mathrm{e}_\mathrm{m} \mid \textbf{cons}\ v_m\ E_m$$

$$v_m = \cdots \mid \mathbf{nil}\ t_m \mid \mathbf{cons}\ v_m\ v_m$$

$$v_s = \cdots \mid \mathbf{nil} \mid \mathbf{cons}\ v_s\ v_s$$

$$E_m = \cdots \mid \mathbf{cons}\ E_m\ \mathrm{e}_\mathrm{m} \mid \mathbf{cons}\ v_m\ E_m$$

$$E_s = \cdots \mid \mathbf{cons}\ E_s\ \mathrm{e}_\mathrm{s} \mid \mathbf{cons}\ v_s\ E_s$$

$$e_h = \cdots \mid \mathbf{fix}\ e_h$$

$$e_h = \cdots \mid \textbf{fix}\ e_h$$

$$\frac{\Gamma \vdash e_h : t_h \to t_h}{\Gamma \vdash \textbf{fix}\ e_h : t_h}$$

$$e_h = \cdots \mid \textbf{fix}\ e_h$$

$$\frac{\Gamma \vdash e_h : t_h \to t_h}{\Gamma \vdash \textbf{fix}\ e_h : t_h}$$

$$E\,[\ \textbf{fix}\ v_h\ ]_h \to E\,[\ v_h\ (\textbf{fix}\ v_h)\ ]$$

$$e_h = \cdots \mid \textbf{fix}\ e_h$$

$$\frac{\Gamma \vdash e_h : t_h \to t_h}{\Gamma \vdash \textbf{fix}\ e_h : t_h}$$

$$E[\ \textbf{fix}\ v_h\ ]_h \to E[\ v_h\ (\textbf{fix}\ v_h)\ ]$$

$$\text{zero} \equiv \lambda\, f : \mathbf{N} \to \mathbf{N}\,.\,\lambda\, n : \mathbf{N}\,.\ \textbf{if0}\ n\ \underline{0}\ (f\ (-\ n\ \underline{0}))$$

$$e_h = \cdots \mid \textbf{fix}\ e_h$$

$$\frac{\Gamma \vdash e_h : t_h \rightarrow t_h}{\Gamma \vdash \textbf{fix}\ e_h : t_h}$$

$$E\,[\ \textbf{fix}\ v_h\ ]_h \rightarrow E\,[\ v_h\ (\textbf{fix}\ v_h)\ ]$$

$$\text{zero} \equiv \lambda\, f : \mathbf{N} \rightarrow \mathbf{N}\ .\ \lambda\, n : \mathbf{N}\ .\ \textbf{if0}\ n\ \underline{0}\ (f\ (-\ n\ \underline{0}))$$

$$(\text{fix zero})\ \underline{7} \rightarrow \underline{0}$$

$$\text{zeroes}_h \equiv \mathbf{fix}\ (\lambda\ x : \{\ \mathbf{N}\ \}\ .\ \mathbf{cons}\ \underline{0}\ x)$$

$$\text{zeroes}_h \equiv \textbf{fix}\ (\lambda\ x : \{\ \mathbf{N}\ \}\ .\ \textbf{cons}\ \underline{0}\ x)$$

$$\text{zeroes}_h \rightarrow \textbf{cons}\ \underline{0}\ \text{zeroes}_h$$

$$\text{zeroes}_h \equiv \mathbf{fix}\ (\lambda\, x : \{\ \mathbf{N}\ \}\ .\ \mathbf{cons}\ \underline{0}\ x)$$

$$\text{zeroes}_h \rightarrow \mathbf{cons}\ \underline{0}\ \text{zeroes}_h$$

$$\text{zeroes}_m \equiv \mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \text{zeroes}_h$$

$$\text{zeroes}_h \equiv \textbf{fix}\ (\lambda\, x : \{\ \mathbf{N}\ \}\ .\ \textbf{cons}\ \underline{0}\ x)$$

$$\text{zeroes}_h \rightarrow \textbf{cons}\ \underline{0}\ \text{zeroes}_h$$

$$\text{zeroes}_m \equiv \textbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \text{zeroes}_h$$

$$\text{zeroes}_m \not\rightarrow$$

$\text{zeroes}_m \equiv$ **mh** $\{$ **N** $\}$ $\{$ **N** $\}$ $\underline{\underline{\text{zeroes}_h}}$ $\rightarrow$

$\mathrm{zeroes}_m \equiv \mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \underline{\underline{\mathrm{zeroes}_h}} \rightarrow$

$\underline{\underline{\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ (\mathbf{cons}\ 0\ \mathrm{zeroes}_h)}} \rightarrow$

$\mathrm{zeroes}_m \equiv \textbf{mh}\ \{\ \textbf{N}\ \}\ \{\ \textbf{N}\ \}\ \underline{\underline{\mathrm{zeroes}_h}} \rightarrow$

$\underline{\underline{\textbf{mh}\ \{\ \textbf{N}\ \}\ \{\ \textbf{N}\ \}\ (\textbf{cons}\ 0\ \mathrm{zeroes}_h)}} \rightarrow$

$\textbf{cons}\ (\underline{\underline{\textbf{mh}\ \textbf{N}\ \textbf{N}\ 0}})\ (\textbf{mh}\ \{\ \textbf{N}\ \}\ \{\ \textbf{N}\ \}\ \mathrm{zeroes}_h) \rightarrow$

$$\text{zeroes}_m \equiv \mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \underline{\underline{\text{zeroes}_h}} \rightarrow$$

$$\underline{\underline{\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ (\mathbf{cons}\ 0\ \text{zeroes}_h)}} \rightarrow$$

$$\mathbf{cons}\ (\underline{\underline{\mathbf{mh}\ \mathbf{N}\ \mathbf{N}\ 0}})\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \text{zeroes}_h) \rightarrow$$

$$\mathbf{cons}\ \underline{0}\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \text{zeroes}_h) =$$

$$\text{zeroes}_m \equiv \mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \underline{\underline{\text{zeroes}_h}} \rightarrow$$

$$\underline{\underline{\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ (\mathbf{cons}\ 0\ \text{zeroes}_h)}} \rightarrow$$

$$\mathbf{cons}\ (\underline{\underline{\mathbf{mh}\ \mathbf{N}\ \mathbf{N}\ 0}})\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \text{zeroes}_h) \rightarrow$$

$$\mathbf{cons}\ \underline{0}\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \text{zeroes}_h) =$$

$$\mathbf{cons}\ \underline{0}\ \text{zeroes}_m \not\rightarrow$$

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

Function conversion

$$\vdots$$

$$\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ \Omega))\ \underline{0}$$

$$\vdots$$

Function conversion

$$\vdots$$

$$\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\mathrm{K}_s\ (\mathbf{sh}\ \mathbf{N}\ \Omega))\ \underline{0} \qquad\qquad v_m\ E_m$$

$$\vdots$$

Function conversion

$\vdots$

$\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ \Omega))\ \underline{0}$ $\qquad v_m\ E_m$

$\vdots$

List construction conversion

$\vdots$

$\mathbf{cons}\ \underline{0}\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \mathrm{zeroes}_h)$

$\vdots$

Function conversion

$$\vdots$$

$$\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ \Omega))\ \underline{0} \qquad\qquad v_m\ E_m$$

$$\vdots$$

List construction conversion

$$\vdots$$

$$\mathbf{cons}\ \underline{0}\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \mathrm{zeroes}_h) \qquad\qquad \mathbf{cons}\ v_m\ E_m$$

$$\vdots$$

$E_m =$

$[\,]_m$

$E_m\ e_m$

$v_m\ E_m$

$E_m \langle t_m \rangle$

fix E_m

+/- $E_m\ e_m$

+/- $v_m\ E_m$

if0 $E_m\ e_m\ e_m$

cons $E_m\ e_m$

cons $v_m\ E_m$

hd E_m

tl E_m

null E_m

mh $t_m\ t_h\ E_h$

ms $k_s\ E_s$

$$
E_m = \begin{array}{c}
\vdots \\
v_m\ E_m \\
\mathbf{cons}\ E_m\ \mathrm{e}_m \\
\mathbf{cons}\ v_m\ E_m \\
\mathbf{mh}\ t_m\ t_h\ E_h \\
\vdots \\
E_m\ \mathrm{e}_m \\
v_m\ E_m \\
\vdots
\end{array}
$$

$$U_m = \begin{array}{c} \vdots \\ v_m\ U_m \\ \mathbf{cons}\ U_m\ e_m \\ \mathbf{cons}\ u_m\ U_m \\ \\ \vdots \\ F_m\ e_m \\ f_m\ U_m \\ \vdots \end{array}$$

$$F_m = U_m \mid \mathbf{mh}\ t_m\ t_h\ E_h$$

$$f_m = u_m \mid \mathbf{mh}\ t_m\ t_h\ E_h$$

$$u_m = \lambda\, x_m : t_m \,.\, e_m \mid \cdots$$

$$F_m = U_m \mid \textbf{mh}\ t_m\ t_h\ E_h$$

$$U_m =$$

$[\ ]_m$	$\textbf{if0}\ F_m\ e_m\ e_m$
$F_m\ e_m$	$\textbf{cons}\ U_m\ e_m$
$f_m\ U_m$	$\textbf{cons}\ u_m\ U_m$
$F_m \langle t_m \rangle$	$\textbf{hd}\ F_m$
$\textbf{fix}\ F_m$	$\textbf{tl}\ F_m$
$\textbf{+/-}\ F_m\ e_m$	$\textbf{null}\ F_m$
$\textbf{+/-}\ f_m\ F_m$	$\textbf{ms}\ k_s\ E_s$

$$\mathcal{E}[\,(\lambda\, x_m : t_m\,.\, e_m)\ u_m\,]_m \rightarrow \mathcal{E}[\, e_m\,[u_m / x_m]\,]$$

$$\mathcal{E}[\,\mathbf{hd}\ (\mathbf{cons}\ u_m\ u_m{}')\,]_m \rightarrow \mathcal{E}[\, u_m\,]$$

$$\mathcal{E}[\,\mathbf{tl}\ (\mathbf{cons}\ u_m\ u_m{}')\,]_m \rightarrow \mathcal{E}[\, u_m{}'\,]$$

$$\mathcal{E}[\,\mathbf{null}\ (\mathbf{cons}\ u_m\ u_m{}')\,]_m \rightarrow \mathcal{E}[\, \underline{\mathsf{I}}\,]$$

- Common expressions
- Incompatible strictness points
- Interoperation side effects
- Mirror non-strictness for embeddings

- Matthews & Findler
- Evaluation strategies
- Incompatible strictness points
- Forcing & deferring embedded evaluation

Questions