Interoperation for Lazy and Eager Evaluation

- Matthews & Findler
  - Interoperation
  - Boundaries & natural embedding
  - Type safety and extensional equality
- Kinghorn
- Incompatible evaluation strategies

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

Set of terms, e

(1) $x \in e$

(2) $M \in e \Rightarrow \lambda\, x\, .\, M \in e$

(3) $M, N \in e \Rightarrow M\ N \in e$

$$e = x \mid \lambda\, x\, .\, e \mid e\ e$$

$$v = \lambda\, x\, .\, e$$

(1) x
(2) $\lambda\, x\, .\, e$
(3) $e\ e$

z
by (1)

$\lambda\, z\, .\, z$
by (1), (2)

$(\lambda\, z\, .\, z)\ (\lambda\, z\, .\, z)$
by (1), (2), (3)

$$\lambda\, x\, .\, x\, x \equiv \lambda\, x\, .\, (x\, x) \neq (\lambda\, x\, .\, x)\, x$$

$$\lambda\, x\, x'\, .\, e \equiv \lambda\, x\, .\, \lambda\, x'\, .\, e$$

$$e\ e'\ e'' = (e\ e')\ e''$$

$\ldots y \ldots$ — *Free variable*

$\lambda\, y\, .\, (\ldots y \ldots)$ — *Bound variable*

y — *Open term*

$\lambda\, y\, .\, y$ — *Closed term*

term [*expression argument* / *expression parameter*] = *term′*

$$x\, [\, e \,/\, x\,] \;=\; e$$

$$x\, [\, e \,/\, x'\,] \;=\; x$$

$$(\lambda\, x\, .\, e)\, [\, e' \,/\, x\,] \;=\; \lambda\, x\, .\, e$$

$$(\lambda\, x\, .\, e)\, [\, e' \,/\, x'\,] \;=\; \lambda\, x\, .\, (e\, [\, e' \,/\, x'\,])$$

$$(e\ e')\, [\, e'' \,/\, x\,] \;=\; (e\, [\, e'' \,/\, x\,])\ (e'\, [\, e'' \,/\, x\,])$$

Set of reductions, $\rightarrow$

$$(e, e') \in \rightarrow$$

$$e \rightarrow e'$$

$$e \rightarrow e'$$
$$e' \rightarrow e''$$
$$e \rightarrow e' \rightarrow e''$$

$$(\lambda\, x\, .\, e)\ e' \rightarrow e\,[\, x \,/\, e'\,]$$

$$(\lambda\, x\, .\, e)\ e' \rightarrow e\,[\, x \,/\, e'\,]$$

$$e \rightarrow e' \Rightarrow e\ e'' \rightarrow e'\ e''$$

$$\frac{e \rightarrow e'}{e\ e'' \rightarrow e'\ e''}$$

$$e \rightarrow e' \Rightarrow v\ e \rightarrow v\ e'$$

$$\frac{e \rightarrow e'}{v\ e \rightarrow v\ e'}$$

error condition $\rightarrow$ *error*

$$\frac{e \rightarrow e'}{e\ e'' \rightarrow e'\ e''} \qquad \frac{e \rightarrow \mathit{error}}{e\ e' \rightarrow \mathit{error}}$$

$$\frac{e \rightarrow e'}{v\ e \rightarrow v\ e'} \qquad \frac{e \rightarrow \mathit{error}}{v\ e \rightarrow \mathit{error}}$$

Set of evaluation contexts, E

$$(\lambda\, x\, .\, e)\ e' \rightarrow e\,[x \,/\, e']$$

$$E\,[\,(\lambda\, x\, .\, e)\ e'\,] \rightarrow E\,[\,e\,[x \,/\, e']\,]$$

$$E = [\,] \mid E\ e \mid v\ E$$

$$E' = \ldots[\,]\ldots$$

$$E'\,[\,e\,] = \ldots e \ldots$$

$$
\begin{aligned}
v &= \lambda\, x\,.\,\mathrm{e} \mid \underline{n} \\
\mathrm{e} &= x \mid v \mid \mathrm{e\ e} \mid \textbf{+/-}\ \mathrm{e\ e} \mid \textbf{if0}\ \mathrm{e\ e\ e} \mid \textbf{fun?}\ \mathrm{e} \\
&\quad \textbf{num?}\ \mathrm{e} \mid \textbf{wrong}\ \textit{string} \\
E &= [\,] \mid E\ \mathrm{e} \mid v\ E \mid \textbf{+/-}\ E\ \mathrm{e} \mid \textbf{+/-}\ v\ E \mid \textbf{if0}\ E\ \mathrm{e\ e} \\
&\quad \textbf{fun?}\ E \mid \textbf{num?}\ E
\end{aligned}
$$

E [**wrong** *string*] → **Error:** *string*

$E\,[\,\textbf{+}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{n + n'}\,]$

$E\,[\,\textbf{-}\ \underline{n}\ \underline{n'}\,] \rightarrow E\,[\,\underline{\max(n - n', 0)}\,]$

$E\,[\,\textbf{if0}\ \underline{0}\ \mathrm{e\ e'}\,] \rightarrow E\,[\,\mathrm{e}\,]$
$E\,[\,\textbf{if0}\ \underline{n}\ \mathrm{e\ e'}\,] \rightarrow E\,[\,\mathrm{e'}\,]$

$E\,[\,\textbf{fun?}\ (\lambda\, x\,.\,\mathrm{e})\,] \rightarrow E\,[\,\underline{0}\,]$
$E\,[\,\textbf{fun?}\ v\,] \rightarrow E\,[\,\underline{1}\,]$

$E\,[\,\textbf{num?}\ \underline{n}\,] \rightarrow E\,[\,\underline{0}\,]$
$E\,[\,\textbf{num?}\ v\,] \rightarrow E\,[\,\underline{1}\,]$

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Set of types, t

$t = \mathbf{N} \mid t \rightarrow t$

$\lambda\, x : t\,.\,\mathrm{e}$

$t \rightarrow t \rightarrow t \equiv t \rightarrow (t \rightarrow t)$

Set of judgments, $\vdash$

$\mathrm{e} : t \equiv (\mathrm{e}, t)$

Γ *is* $x_n : t_n, \ldots, x_l : t_l$

$\lambda x_n : t_n . (\ldots \lambda x_l : t_l . \mathrm{e})$

$(\Gamma, \mathrm{e} : t) \in \vdash$

$\Gamma \vdash \mathrm{e} : t \qquad \vdash \mathrm{e} : t$

$\Gamma \vdash t \qquad \vdash t$

Number type

$\vdash \mathbf{N}$

Function type

$$\frac{\Gamma \vdash t \;—\; \Gamma \vdash t'}{\Gamma \vdash t \rightarrow t'}$$

Number

$\vdash \underline{n} : \mathbf{N}$

Variable

$\Gamma, x : t \vdash x : t$

Function

$$\frac{\Gamma, x : t \vdash \mathrm{e} : t'}{\Gamma \vdash \lambda\, x : t . \mathrm{e} : t \rightarrow t'}$$

Application

$$\frac{\Gamma \vdash \mathrm{e} : t \rightarrow t' \;—\; \Gamma \vdash \mathrm{e}' : t}{\Gamma \vdash \mathrm{e}\ \mathrm{e}' : t'}$$

Arithmetic

$$\frac{\Gamma \vdash \mathrm{e} : \mathbf{N} \;—\; \Gamma \vdash \mathrm{e}' : \mathbf{N}}{\Gamma \vdash \textbf{+/-}\ \mathrm{e}\ \mathrm{e}' : \mathbf{N}}$$

Condition

$$\frac{\Gamma \vdash \mathrm{e} : \mathbf{N} \;—\; \Gamma \vdash \mathrm{e}'/\mathrm{e}'' : t}{\Gamma \vdash \textbf{if0}\ \mathrm{e}\ \mathrm{e}'\ \mathrm{e}'' : t}$$

Error

$$\frac{\Gamma \vdash t}{\Gamma \vdash \textbf{wrong}\ t\ \textit{string} : t}$$

Type

$\Gamma \vdash \mathbf{T}$

Number

$\Gamma \vdash \underline{n} : \mathbf{T}$

Variable

$\Gamma, x : \mathbf{T} \vdash x : \mathbf{T}$

Application

$$\frac{\Gamma \vdash \mathrm{e} : \mathbf{T} \;—\; \Gamma \vdash \mathrm{e}' : \mathbf{T}}{\Gamma \vdash \mathrm{e}\ \mathrm{e}' : \mathbf{T}}$$

Function

$$\frac{\Gamma, x : \mathbf{T} \vdash \mathrm{e} : \mathbf{T}}{\Gamma \vdash \lambda\, x . \mathrm{e} : \mathbf{T}}$$

Arithmetic

$$\frac{\Gamma \vdash \mathrm{e} : \mathbf{T} \;—\; \Gamma \vdash \mathrm{e}' : \mathbf{T}}{\Gamma \vdash \textbf{+/-}\ \mathrm{e}\ \mathrm{e}'}$$

Predicate

$$\frac{\Gamma \vdash e : \mathbf{T}}{\Gamma \vdash \textbf{fun?/num?}\ e : \mathbf{T}}$$

Error

$\Gamma \vdash$ **wrong** *string* : $\mathbf{T}$

$\lambda\, x : \mathbf{N} . x$

$\lambda\, x : \mathbf{N} \to \mathbf{N} . x$

$\Lambda\, y . \lambda\, x : y . x$

$(\Lambda\, y . \lambda\, x : y . x) \langle\, \mathbf{N}\, \rangle \to \lambda\, x : \mathbf{N} . x$

$(\Lambda\, y . \lambda\, x : y . x) \langle\, \mathbf{N} \to \mathbf{N}\, \rangle \to \lambda\, x : \mathbf{N} \to \mathbf{N} . x$

Type variables

y

Type abstraction

$\Lambda\, y . e$

Type application

$e \langle\, t\, \rangle$

Universally-quantified / for-all types

$\forall\, y . t$

Free & bound type variables

$\Lambda\, y . (\ldots\, y\, \ldots)$

$\Lambda\, y\, y' . e \equiv \Lambda\, y . \Lambda\, y' . e$

$e \langle\, t\, \rangle \langle\, t'\, \rangle \equiv (e \langle\, t\, \rangle) \langle\, t'\, \rangle$

$\Lambda\, y . e = \Lambda\, y' . e\, [\, y' / y\,]$

term [type argument / type parameter] = term′

$$x\,[\,t / y\,] = x$$

$$(\lambda\, x : t\,.\,e)\,[\,t / y\,] = \lambda\, x : t\,[\,t / y\,]\,.\,e\,[\,t / y\,]$$

$$(e\ e')\,[\,t / y\,] = (e\,[\,t / y\,])\ (e'\,[\,t / y\,])$$

$$(\mathbf{+/\text{-}}\ e\ e')\,[\,t / y\,] = \mathbf{+/\text{-}}\ (e\,[\,t / y\,])\ (e'\,[\,t / y\,])$$

$$(\mathbf{if0}\ e\ e'\ e'')\,[\,t / y\,] = \mathbf{if0}\ (e\,[\,t / y\,])\ (e'\,[\,t / y\,])\ (e''\,[\,t / y\,])$$

$$(\Lambda\, y\,.\,e)\,[\,t / y\,] = \Lambda\, y\,.\,e$$

$$(\Lambda\, y\,.\,e)\,[\,t / y'\,] = \Lambda\, y\,.\,e\,[\,t / y'\,]$$

$$(e\,\langle\, t\,\rangle)\,[\,t' / y\,] = (e\,[\,t' / y\,])\,\langle\, t\,[\,t' / y\,]\,\rangle$$

type [type argument / type parameter] = type′

$$\mathbf{N}\,[\,t / y\,] = \mathbf{N}$$

$$(t \to t')\,[\,t / y\,] = t\,[\,t / y\,] \to t'\,[\,t / y\,]$$

$$y\,[\,t / y\,] = t$$

$$y\,[\,t / y'\,] = y$$

$$(\forall\, y\,.\,t)\,[\,t' / y\,] = \forall\, y\,.\,t$$

$$(\forall\, y\,.\,t)\,[\,t' / y'\,] = \forall\, y\,.\,t\,[\,t' / y'\,]$$

$$E\,[\,(\Lambda\, y\,.\,e)\,\langle\, t\,\rangle\,] \to E\,[\,e\,[\,t / y\,]\,]$$

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Scheme to ML

$e_m = \cdots \mid \mathbf{ms}\ t_m\ e_s$

ML to Scheme

$e_s = \cdots \mid \mathbf{sm}\ t_m\ e_m$

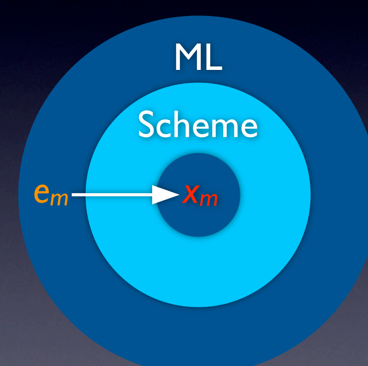
Scheme to ML

$$\frac{\Gamma \vdash_m t_m \quad \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ t_m\ e_s : t_m}$$

ML to Scheme

$$\frac{\Gamma \vdash_m t_m \quad \Gamma \vdash_m e_m : t_m' \quad t_m = t_m'}{\Gamma \vdash_s \mathbf{sm}\ t_m\ e_m : \mathbf{T}}$$

$(\mathbf{ms}\ t_m\ (\mathbf{sm}\ t_m\ x_m))\ [\ e_m\ /\ x_m\ ]$



Boundary substitution

$$(\mathbf{ms}\ t_m\ e_s)\ [\ e_m\ /\ x_m\ ] = \mathbf{ms}\ t_m\ (e_s\ [\ e_m\ /\ x_m\ ])$$

Foreign substitution

$$(\ldots e_s \ldots e_s' \ldots)\ [\ e_m\ /\ x_m\ ] = \ldots e_s\ [\ e_m\ /\ x_m\ ] \ldots e_s'\ [\ e_m\ /\ x_m\ ] \ldots$$

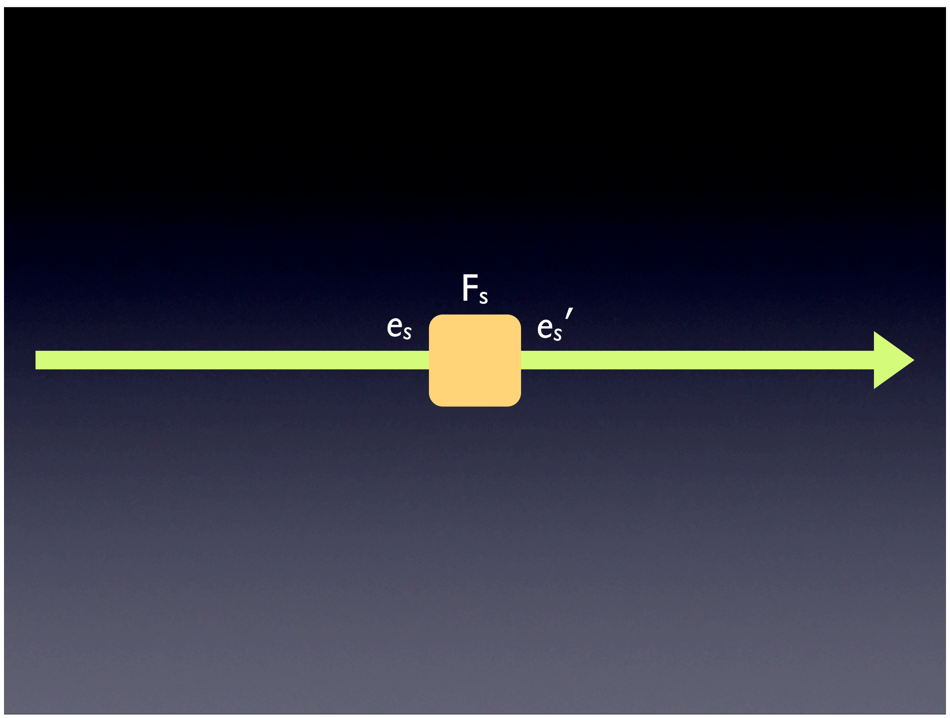
$$(\mathbf{sm}\ t_m\ e_m)\ [\ e_m'\ /\ x_m\ ] = \mathbf{sm}\ t_m\ (e_m\ [\ e_m'\ /\ x_m\ ])$$

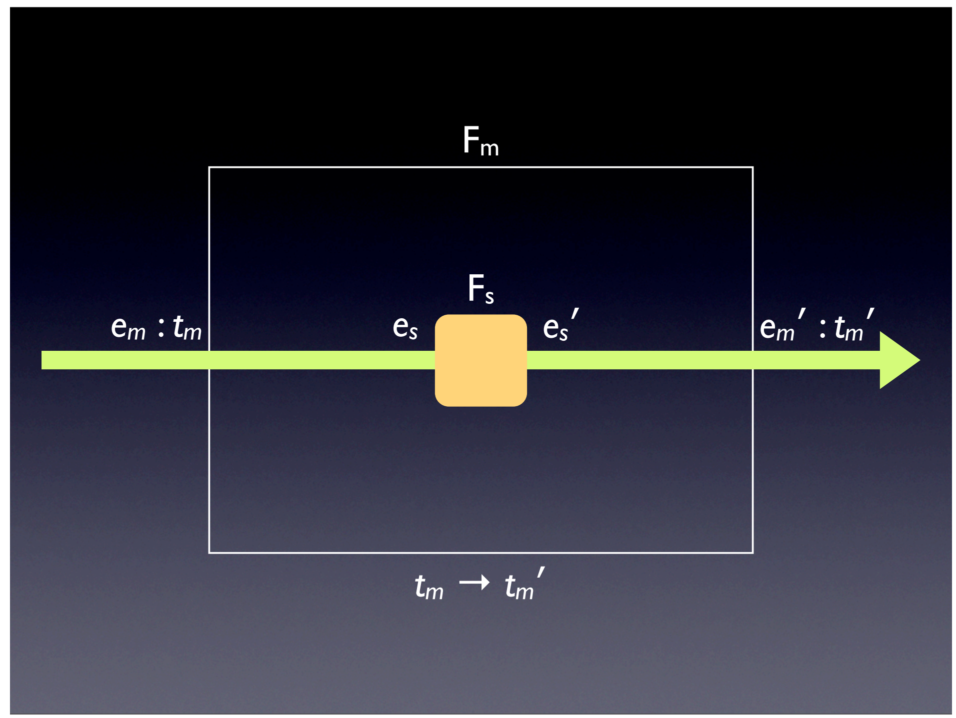
Outermost evaluation context, $\mathscr{E}$

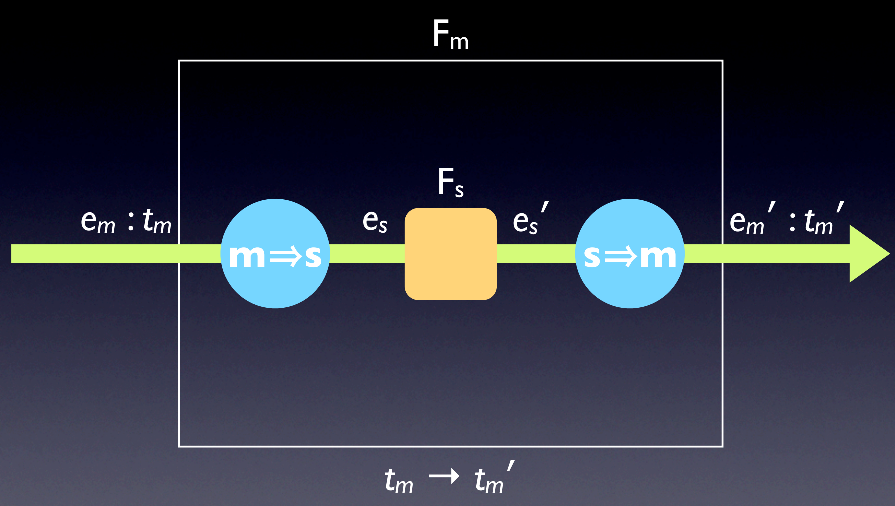
$$\mathscr{E}\ [\ \mathbf{ms}\ \mathbf{N}\ \underline{n}\ ]_m \rightarrow \mathscr{E}\ [\ \underline{n}\ ]$$

$$\mathscr{E}\ [\ \mathbf{ms}\ \mathbf{N}\ v_s\ ]_m \rightarrow \mathscr{E}\ [\ \mathbf{wrong}\ \mathbf{N}\ \text{“Not a number”}\ ]$$

$$\mathscr{E}\ [\ \mathbf{sm}\ \mathbf{N}\ \underline{n}\ ]_s \rightarrow \mathscr{E}\ [\ \underline{n}\ ]$$







ML to Scheme

$sm \equiv \mathbf{sm}\ t_m\ x_m$

Scheme application

$app \equiv F_s\ sm$

Scheme to ML

$ms \equiv \mathbf{ms}\ t_m'\ app$

Abstraction

$F_m \equiv \lambda\ x_m : t_m\ .\ ms$

$F_m \equiv \lambda\ x_m : t_m\ .\ \mathbf{ms}\ t_m'\ (F_s\ (\mathbf{sm}\ t_m\ x_m))$

$\mathscr{E}[\ \mathbf{ms}\ (t_m \to t_m')\ (\lambda\ x_s\ .\ e_s)\ ]_m$

$\to$

$\mathscr{E}[\ \lambda\ x_m : t_m\ .\ \mathbf{ms}\ t_m'\ ((\lambda\ x_s\ .\ e_s)\ (\mathbf{sm}\ t_m\ x_m))\ ]$

$\mathscr{E}[\ \mathbf{ms}\ (t_m \to t_m')\ v_s\ ]_m$

$\to$

$\mathscr{E}[\ \mathbf{wrong}\ (t_m \to t_m')$ "Not a function" $]$

$\mathscr{E}[\ \mathbf{sm}\ (t_m \to t_m')\ v_m\ ]_s$

$\to$

$\mathscr{E}[\ \lambda\ x_s\ .\ \mathbf{sm}\ t_m'\ (v_m\ (\mathbf{ms}\ t_m\ x_s))\ ]$

$t_m = \cdots \mid \mathbf{L}$ $\qquad v_m = \cdots \mid \mathbf{ms}\ \mathbf{L}\ v_s$ $\qquad \Gamma \vdash_m \mathbf{L}$

$\mathscr{E}[\ \mathbf{sm}\ (\forall\ y_m\ .\ t_m)\ (\Lambda\ y_m'\ .\ e_m)\ ]_s$

$\to$

$\mathscr{E}[\ \mathbf{sm}\ t_m\ [\ \mathbf{L}\ /\ y_m\ ]\ e_m\ [\ \mathbf{L}\ /\ y_m'\ ]\ ]$

$\mathscr{E}[\ \mathbf{sm}\ \mathbf{L}\ (\mathbf{ms}\ \mathbf{L}\ v_s)\ ]_s \to \mathscr{E}[\ v_s\ ]$

$$\mathrm{id} \equiv \Lambda\, y\,.\, \mathbf{ms}\ (y \to y)\ (\lambda\, x\,.\, x)$$

$\mathrm{id}\ \langle\, \mathbf{N}\, \rangle$
behaves the same as
$\mathrm{id}\ \langle\, \mathbf{N} \to \mathbf{N}\, \rangle$

$$\mathrm{id}_m \equiv \Lambda\, y\,.\, \mathbf{ms}\ (y \to y)\ (\lambda\, x\,.\, \mathbf{if0}\ (\mathbf{num?}\ x)\ x\ \underline{0})$$

$\mathrm{id}_m\ \langle\, \mathbf{N}\, \rangle$
behaves differently than
$\mathrm{id}_m\ \langle\, \mathbf{N} \to \mathbf{N}\, \rangle$

$$\mathbf{sm}\ t_m\ v_m$$

$$\mathbf{num?}\ (\mathbf{sm}\ t_m\ v_m)$$

Set of conversion schemes, k

$$k_m = \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \to k_m \mid \forall\, y_m\,.\, k_m \mid b \diamond t_m$$

$$e_m = \cdots \mid \mathbf{ms}\ k_m\ e_s$$

$$e_s = \cdots \mid \mathbf{sm}\ k_m\ e_m$$

$$v_s = \cdots \mid \mathbf{sm}\ k_m\ v_m$$

$$\lfloor \mathbf{L} \rfloor = \mathbf{L}$$

$$\lfloor \mathbf{N} \rfloor = \mathbf{N}$$

$$\lfloor y_m \rfloor = y_m$$

$$\lfloor k_m \to k_m \rfloor = \lfloor k_m \rfloor \to \lfloor k_m \rfloor$$

$$\lfloor \forall\, y_m\,.\, k_m \rfloor = \forall\, y_m\,.\, \lfloor k_m \rfloor$$

$$\lfloor b \diamond t_m \rfloor = t_m$$

$$\frac{\Gamma \vdash_m \lfloor k_m \rfloor \;\text{—}\; \Gamma \vdash_s e_s : \mathbf{T}}{\Gamma \vdash_m \mathbf{ms}\ k_m\ e_s : \lfloor k_m \rfloor}$$

$$\mathscr{E}\,[\, (\Lambda\, y_m\,.\, e_m)\ \langle\, t_m\, \rangle\,]_m \to \mathscr{E}\,[\, e_m\, [\, b \diamond t_m \,/\, y_m\,]\,]$$

$$\mathscr{E}\,[\, \mathbf{ms}\ (b \diamond t_m)\ (\mathbf{sm}\ (b \diamond t_m)\ v_m)\,]_m \to \mathscr{E}\,[\, v_m\,]$$

$$\mathscr{E}\,[\, \mathbf{ms}\ (b \diamond t_m)\ v_s\,]_m$$
$$\to$$
$$\mathscr{E}\,[\, \mathbf{wrong}\ \lfloor\, b \diamond t_m\, \rfloor\ \text{“Brand mismatch”}\,]$$

$$(\lambda\, x_m : t_m\, .\, e_m)\, [\, b \diamond t_m{}' \,/\, y_m \,] = \lambda\, x_m : t_m\, [\, t_m{}' \,/\, y_m \,]\, .\, e_m\, [\, b \diamond t_m{}' \,/\, y_m \,]$$

$$(\mathbf{ms}\; k_m\; e_s)\, [\, b \diamond t_m{}' \,/\, y_m \,] = \mathbf{ms}\; k_m\, [\, t_m{}' \,/\, y_m \,]\; e_s\, [\, b \diamond t_m{}' \,/\, y_m \,]$$

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ML & Scheme

- Numbers, arithmetic, conditions
- Functions, applications
- Errors
- Natural embedding
- Eager evaluation
- ‘m’ and ‘s’ subscripts

ML

- Statically typed
- Parametric polymorphism
- Fixed-point operations

$$t_m \;=\; \mathbf{N} \mid y_m \mid t_m \rightarrow t_m \mid \forall\, y_m \,.\, t_m$$

$$k_m \;=\; \mathbf{L} \mid \mathbf{N} \mid y_m \mid k_m \rightarrow k_m \mid \forall\, y_m \,.\, k_m \mid b \diamond t_m$$

$$v_m \;=\; \lambda\, x_m : t_m \,.\, e_m \mid \Lambda\, y_m \,.\, e_m \mid \underline{n} \mid \mathbf{ms}\ \mathbf{L}\ v_s$$

$$e_m \;=\; \begin{array}{l} x_m \mid v_m \mid e_m\ e_m \mid e_m \langle t_m \rangle \mid \mathbf{fix}\ e_m \mid \mathbf{+/\text{-}}\ e_m\ e_m \\ \mathbf{if0}\ e_m\ e_m\ e_m \mid \mathbf{wrong}\ t_m\ \mathit{string} \mid \mathbf{ms}\ k_m\ e_s \end{array}$$

$$E_m \;=\; \begin{array}{l} [\,]_m \mid E_m\ e_m \mid v_m\ E_m \mid E_m \langle t_m \rangle \mid \mathbf{fix}\ E_m \\ \mathbf{+/\text{-}}\ E_m\ e_m \mid \mathbf{+/\text{-}}\ v_m\ E_m \mid \mathbf{if0}\ E_m\ e_m\ e_m \\ \mathbf{ms}\ k_m\ E_s \end{array}$$

Scheme

- Dynamically typed
- Closed term typing
- Ad-hoc polymorphism

$$v_s \;=\; \lambda\, x_s \,.\, e_s \mid \underline{n} \mid \mathbf{sm}\ (b \diamond t_m)\ v_m$$

$$e_s \;=\; \begin{array}{l} x_s \mid v_s \mid e_s\ e_s \mid \mathbf{+/\text{-}}\ e_s\ e_s \mid \mathbf{if0}\ e_s\ e_s\ e_s \mid \mathbf{fun?}\ e_s \\ \mathbf{num?}\ e_s \mid \mathbf{wrong}\ \mathit{string} \mid \mathbf{sm}\ k_m\ e_m \end{array}$$

$$E_s \;=\; \begin{array}{l} [\,]_s \mid E_s\ e_s \mid v_s\ E_s \mid \mathbf{+/\text{-}}\ E_s\ e_s \mid \mathbf{+/\text{-}}\ v_s\ E_s \\ \mathbf{if0}\ E_s\ e_s\ e_s \mid \mathbf{fun?}\ E_s \mid \mathbf{num?}\ E_s \mid \mathbf{sm}\ k_m\ E_m \end{array}$$

Lambda calculus
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Solution

- Eager vs. lazy evaluation
- Introduce Haskell
- Incompatible evaluation strategies
  - Function behavior
  - Value conversion

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$$
\begin{aligned}
t_h &= \mathbf{N} \mid y_h \mid t_h \to t_h \mid \forall\, y_h \,.\, t_h \\
k_h &= \mathbf{L} \mid \mathbf{N} \mid y_h \mid k_h \to k_h \mid \forall\, y_h \,.\, k_h \mid b \diamond t_h \\
v_h &= \lambda\, x_h : t_h \,.\, e_h \mid \Lambda\, y_h \,.\, e_h \mid \underline{n} \mid \mathbf{hs}\ \mathbf{L}\ v_s \\
e_h &= x_h \mid v_h \mid e_h\ e_h \mid e_h \langle\, t_h \,\rangle \mid \mathbf{fix}\ e_h \mid \mathbf{+/\text{-}}\ e_h\ e_h \\
&\quad\ \mathbf{if0}\ e_h\ e_h\ e_h \mid \mathbf{wrong}\ t_h\ \mathit{string} \\
E_h &= [\ ]_h \mid E_h\ e_h \mid E_h \langle\, t_h \,\rangle \mid \mathbf{fix}\ E_h \mid \mathbf{+/\text{-}}\ E_h\ e_h \\
&\quad\ \mathbf{+/\text{-}}\ v_h\ E_h \mid \mathbf{if0}\ E_h\ e_h\ e_h
\end{aligned}
$$

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$$e_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ e_m \mid \mathbf{hs}\ k_h\ e_s$$

$$e_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ e_h$$

$$e_s = \cdots \mid \mathbf{sh}\ k_h\ e_h$$

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$$v_h = \cdots \mid \mathbf{hm}\ \mathbf{L}\ t_m\ v_m$$

$$v_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ e_h$$

$$v_s = \cdots \mid \mathbf{sh}\ k_h\ e_h$$

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$$E_h = \cdots \mid \mathbf{hm}\ t_h\ t_m\ E_m \mid \mathbf{hs}\ k_h\ E_s$$

$$E_m = \cdots \mid \mathbf{mh}\ t_m\ t_h\ E_h$$

$$E_s = \cdots \mid \mathbf{sh}\ k_h\ E_h$$

$$\mathscr{E}\,[\ \mathbf{hm}\ (\forall\ y_h\ .\ t_h)\ (\forall\ y_m\ .\ t_m)\ (\Lambda\ y_m{}'\ .\ e_m)\ ]_h$$
$$\rightarrow$$
$$\mathscr{E}\,[\ \Lambda\ y_h\ .\ \mathbf{hm}\ t_h\ t_m\ [\ \mathbf{L}\ /\ y_m\ ]\ e_m\ [\ \mathbf{L}\ /\ y_m{}'\ ]\ ]$$

$$\frac{\Gamma \vdash_h t_h \quad \Gamma \vdash_m t_m \quad \Gamma \vdash_m e_m : t_m{}' \quad t_m \doteq t_m{}'}{\Gamma \vdash_h \mathbf{hm}\ t_h\ t_m\ e_m : t_h}$$

$$x \doteq x$$
$$x \doteq y \Rightarrow y \doteq x$$
$$x \doteq y \wedge y \doteq z \Rightarrow x \doteq z$$
$$t_h \doteq \mathbf{L}$$
$$t_m \doteq \mathbf{L}$$
$$t_h = t_m \Rightarrow t_h \doteq t_m$$

$$\mathrm{K_s} \equiv \lambda\, x\ \ y\, .\, x$$

$$\mathrm{K_h} : \forall\ y_h\ y_h{}'\ .\ y_h \rightarrow y_h{}' \rightarrow y_h$$

$$\mathrm{K_h} \equiv \Lambda\ y_h\ y_h{}'\ .\ \mathbf{hs}\ (y_h \rightarrow y_h{}' \rightarrow y_h)\ \mathrm{K_s}$$

$$\mathrm{K_{hn}} \equiv \mathrm{K_h}\ \langle \mathbf{N} \rangle\ \langle \mathbf{N} \rangle \rightarrow \mathbf{hs}\ (\mathbf{N} \rightarrow \mathbf{N} \rightarrow \mathbf{N})\ \mathrm{K_s}$$

$$\mathrm{K_{hn}}\ \underline{0}\ \Omega \rightarrow \underline{0}$$

$$\mathrm{K_{hn}}\ \underline{0}\ \Omega \not\rightarrow$$

$$K_{hn}\ \underline{0}\ \underline{\Omega} \equiv \underline{(\mathbf{hs}\ (\mathbf{N} \to \mathbf{N} \to \mathbf{N})\ \underline{K_s})}\ \underline{0}\ \underline{\Omega}$$

$$\to \underline{(\lambda\, x : \mathbf{N} \,.\, \mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\underline{K_s}\ (\mathbf{sh}\ \mathbf{N}\ x)))}\ \underline{0}\ \Omega$$

$$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (K_s\ (\underline{\mathbf{sh}\ \mathbf{N}\ \underline{0}})))\ \Omega$$

$$\to (\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\underline{K_s}\ \underline{0}))\ \Omega$$

$$\to (\underline{\mathbf{hs}\ (\mathbf{N} \to \mathbf{N})\ (\lambda\, y \,.\, \underline{0})})\ \Omega$$

$$\to \underline{(\lambda\, x : \mathbf{N} \,.\, \mathbf{hs}\ \mathbf{N}\ ((\lambda\, y \,.\, \underline{0})\ (\mathbf{sh}\ \mathbf{N}\ x)))}\ \underline{\Omega}$$

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$$\to \mathbf{hs}\ \mathbf{N}\ ((\lambda\, y \,.\, \underline{0})\ (\underline{\mathbf{sh}\ \mathbf{N}\ \underline{\Omega}}))$$

$$\to \mathbf{hs}\ \mathbf{N}\ ((\lambda\, y \,.\, \underline{0})\ (\underline{\mathbf{sh}\ \mathbf{N}\ \underline{\Omega}}))$$

$$\to \mathbf{hs}\ \mathbf{N}\ ((\lambda\, y \,.\, \underline{0})\ (\underline{\mathbf{sh}\ \mathbf{N}\ \underline{\Omega}}))$$

$$\vdots$$

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$$e_h = \cdots \mid \mathbf{nil}\ t_h \mid \mathbf{cons}\ e_h\ e_h \mid \mathbf{hd}\ e_h \mid \mathbf{tl}\ e_h \mid \mathbf{null?}\ e_h$$

$$t_h = \cdots \mid \{\ t_h\ \}$$

$$k_h = \cdots \mid \{\ k_h\ \}$$

$$E_h = \cdots \mid \mathbf{hd}\ E_h \mid \mathbf{tl}\ E_h \mid \mathbf{null?}\ E_h$$

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$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \{\ t_h\ \}} \qquad \frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \mathbf{hd}\ e_h : t_h}$$

$$\frac{\Gamma \vdash_h t_h}{\Gamma \vdash_h \mathbf{nil}\ t_h : \{\ t_h\ \}} \qquad \frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \mathbf{tl}\ e_h : \{\ t_h\ \}}$$

$$\frac{\Gamma \vdash_h e_h : \{\ t_h\ \}}{\Gamma \vdash_h \mathbf{null?}\ e_h : \mathbf{N}} \qquad \frac{\Gamma \vdash_h e_h : t_h \quad \Gamma \vdash_h e_h' : \{\ t_h\ \}}{\Gamma \vdash_h \mathbf{cons}\ e_h\ e_h' : \{\ t_h\ \}}$$

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$\mathscr{E}$ [**hd** (**nil** t_h) $]_h \to \mathscr{E}$ [**wrong** t_h "Empty list"]

$\mathscr{E}$ [**hd** (**cons** e_h e_h') $]_h \to \mathscr{E}$ [e_h]

$\mathscr{E}$ [**tl** (**nil** t_h) $]_h \to \mathscr{E}$ [**wrong** { t_h } "Empty list"]

$\mathscr{E}$ [**tl** (**cons** e_h e_h') $]_h \to \mathscr{E}$ [e_h']

$\mathscr{E}$ [**null?** (**nil** t_h) $]_h \to \mathscr{E}$ [$\underline{0}$]

$\mathscr{E}$ [**null?** (**cons** e_h e_h') $]_h \to \mathscr{E}$ [$\underline{1}$]

$\mathscr{E}$ [**hm** { t_h } { t_m } (**nil** t_m') $]_h \to \mathscr{E}$ [**nil** t_h]

$\mathscr{E}$ [**hm** { t_h } { t_m } (**cons** v_m v_m') $]_h$

$\to$

$\mathscr{E}$ [**cons** (**hm** t_h t_m v_m) (**hm** { t_h } { t_m } v_m']]

$v_m = \cdots$ | **nil** t_m | **cons** v_m v_m

$v_s = \cdots$ | **nil** | **cons** v_s v_s

$E_m = \cdots$ | **cons** E_m e_m | **cons** v_m E_m

$E_s = \cdots$ | **cons** E_s e_s | **cons** v_s E_s

$e_h = \cdots$ | **fix** e_h

$$\frac{\Gamma \vdash e_h : t_h \to t_h}{\Gamma \vdash \textbf{fix}\ e_h : t_h}$$

E [**fix** v_h $]_h \to E$ [v_h (**fix** v_h)]

zero ≡ λ f : **N** → **N** . λ n : **N** . **if0** n $\underline{0}$ (f (- n $\underline{0}$))
(fix zero) $\underline{7} \to \underline{0}$

$\mathrm{zeroes}_h \equiv \mathbf{fix}\ (\lambda\, x : \{\ \mathbf{N}\ \}\ .\ \mathbf{cons}\ \underline{0}\ x)$

$\mathrm{zeroes}_h \rightarrow \mathbf{cons}\ \underline{0}\ \mathrm{zeroes}_h$

$\mathrm{zeroes}_m \equiv \mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \mathrm{zeroes}_h$

$\mathrm{zeroes}_m \not\rightarrow$

$\mathrm{zeroes}_m \equiv \mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \underline{\underline{\mathrm{zeroes}_h}} \rightarrow$

$\underline{\underline{\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ (\mathbf{cons}\ 0\ \mathrm{zeroes}_h)}} \rightarrow$

$\mathbf{cons}\ (\underline{\underline{\mathbf{mh}\ \mathbf{N}\ \mathbf{N}\ 0}})\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \mathrm{zeroes}_h) \rightarrow$

$\mathbf{cons}\ \underline{0}\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \mathrm{zeroes}_h) =$

$\mathbf{cons}\ \underline{0}\ \mathrm{zeroes}_m \not\rightarrow$

Lambda calculus
Typing
Interoperation
Model
Laziness
Solution

Function conversion

$\vdots$

$\mathbf{hs}\ (\mathbf{N} \rightarrow \mathbf{N})\ (K_s\ (\mathbf{sh}\ \mathbf{N}\ \Omega))\ \underline{0}$ $\qquad v_m\ E_m$

$\vdots$

List construction conversion

$\vdots$

$\mathbf{cons}\ \underline{0}\ (\mathbf{mh}\ \{\ \mathbf{N}\ \}\ \{\ \mathbf{N}\ \}\ \mathrm{zeroes}_h)$ $\qquad \mathbf{cons}\ v_m\ E_m$

$\vdots$

$$E_m =$$

$[\,]_m$	$\textbf{cons}\ E_m\ e_m$
$E_m\ e_m$	$\textbf{cons}\ v_m\ E_m$
$v_m\ E_m$	$\textbf{hd}\ E_m$
$E_m \langle t_m \rangle$	$\textbf{tl}\ E_m$
$\textbf{fix}\ E_m$	$\textbf{null}\ E_m$
$\textbf{+/-}\ E_m\ e_m$	$\textbf{mh}\ t_m\ t_h\ E_h$
$\textbf{+/-}\ v_m\ E_m$	$\textbf{ms}\ k_s\ E_s$
$\textbf{if0}\ E_m\ e_m\ e_m$	

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$$E_m = \begin{array}{c} \vdots \\ v_m\ E_m \\ \textbf{cons}\ E_m\ e_m \\ \textbf{cons}\ v_m\ E_m \\ \textbf{mh}\ t_m\ t_h\ E_h \\ \vdots \\ E_m\ e_m \\ v_m\ E_m \\ \vdots \end{array}$$

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$$U_m = \begin{array}{c} \vdots \\ v_m\ U_m \\ \textbf{cons}\ U_m\ e_m \\ \textbf{cons}\ u_m\ U_m \\ \vdots \\ F_m\ e_m \\ f_m\ U_m \\ \vdots \end{array}$$

$$F_m = U_m \mid \textbf{mh}\ t_m\ t_h\ E_h$$

$$f_m = u_m \mid \textbf{mh}\ t_m\ t_h\ E_h$$

$$u_m = \lambda\, x_m : t_m\, .\, e_m \mid \cdots$$

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$$F_m = U_m \mid \textbf{mh}\ t_m\ t_h\ E_h$$

$$U_m =$$

$[\,]_m$	$\textbf{if0}\ F_m\ e_m\ e_m$
$F_m\ e_m$	$\textbf{cons}\ U_m\ e_m$
$f_m\ U_m$	$\textbf{cons}\ u_m\ U_m$
$F_m \langle t_m \rangle$	$\textbf{hd}\ F_m$
$\textbf{fix}\ F_m$	$\textbf{tl}\ F_m$
$\textbf{+/-}\ F_m\ e_m$	$\textbf{null}\ F_m$
$\textbf{+/-}\ f_m\ F_m$	$\textbf{ms}\ k_s\ E_s$

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$$\mathcal{E}[\,(\lambda\, x_m : t_m \,.\, e_m)\; u_m\,]_m \rightarrow \mathcal{E}[\, e_m\, [u_m / x_m]\,]$$

$$\mathcal{E}[\,\mathbf{hd}\ (\mathbf{cons}\ u_m\ u_m')\,]_m \rightarrow \mathcal{E}[\, u_m\,]$$

$$\mathcal{E}[\,\mathbf{tl}\ (\mathbf{cons}\ u_m\ u_m')\,]_m \rightarrow \mathcal{E}[\, u_m'\,]$$

$$\mathcal{E}[\,\mathbf{null}\ (\mathbf{cons}\ u_m\ u_m')\,]_m \rightarrow \mathcal{E}[\, \underline{\perp}\,]$$

- Common expressions
- Incompatible strictness points
- Interoperation side effects
- Mirror non-strictness for embeddings

- Matthews & Findler
- Evaluation strategies
- Incompatible strictness points
- Forcing & deferring embedded evaluation

Questions