PID Control for Quadcopter Stabilization

Andrew Gibiansky
andrew.gibiansky@gmail.com

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System

Suppose we have some system with the transfer function

$$G(s) = \frac{1}{a_0 s^2 + a_1 s}.$$

We would like this system to follow some desired trajectory θ. However, we have a disturbance (due to modeling error or external factors) summarized by a term $D(s)$. In addition, this system is clearly not asymptotically stable.

PD Control

We can drive the difference between the true trajectory and the desired trajectory to zero. The open loop system would be:

$$\theta_t(s) = \theta(s)G(s).$$

In order to drive the error to zero, we introduce a PD compensator:

$$F(s) = K_p + K_d s.$$

The new input is given by

$$U(s) = F(s)(\theta_t(s) - \theta(s)).$$

Then, the output is

$$\theta_t(s) = (F(s)(\theta_t(s) - \theta(s)) + D(s))G(s) \implies$$

$$\theta_t(s) = \frac{K_p\theta(s) - D(s) + K_d\theta(s)s}{K_p + K_d s - a_1 s - a_0 s^2}$$

Final Value Theorem

Theorem: Suppose $F(s)$ is the unilateral Laplace transform of $f(t)$, and $f(t)$ converges to some value as $t \to \infty$. Then,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s).$$

PD Control Results

Suppose we have a step trajectory

$$\theta(s) = \mathcal{L}\{T\} = \frac{T}{s}$$

and a similar constant disturbance term

$$D(s) = \mathcal{L}\{D\} = \frac{D}{s}$$

Then, the tracking error is

$$\begin{aligned} E(s) &= \theta(s) - \theta_t(s) \\ &= \frac{a_0 T s^2 + T a_1 s - D}{s(K_p + K_d s - a_1 s - a_0 s^2)} \end{aligned}$$

Steady State Error

By the final value theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = -\frac{D}{K_p}.$$

Steady state error!

In physical systems, we cannot make the gain K_p arbitrarily large, so this becomes a real problem. A PD controller cannot accurately track a trajectory with a step in it.
(The error comes from the fact that eventually the system has zero derivative error, so K_d is ineffective, and the constant disturbance balances out the proportional gain K_p.)

PID Control

Add an integral term:

$$F(s) = K_p + K_d s + K_i / s.$$

This integral term ensures that if we remain in a steady state error for long enough, the build-up will overcome any constant disturbance. The error becomes:

$$E(s) = \frac{Ta_0 s^2 + Ta_1 s - D}{K_i + K_p s + K_d s^2 - a_0 s^3 - a_1 s^2}$$

Final value theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = 0.$$

General Case: PD Control

Suppose we have a system with transfer function $H(s)/s$, such that $H(0) \neq 0$. Then, using a PD controller where $F(s) = K_p + K_d s$ we have

$$E(s) = (F(s)E(s) + D(s))H(s)/s - \theta(s)$$

$$E(s) = \frac{F(s)H(s)/s + D(s)H(s)/s - \theta(s)}{1 - F(s)H(s)/s}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{F(s)H(s) + D(s)H(s) - s\theta(s)}{s - F(s)H(s)}$$

$$= \lim_{s\to 0} \frac{sF(s)H(s) + s(D/s)H(s) - s^2\theta(s)}{s - F(s)H(s)} = -\frac{D}{K_p}$$

We have the steady state error shown before.

General Case: PID Control

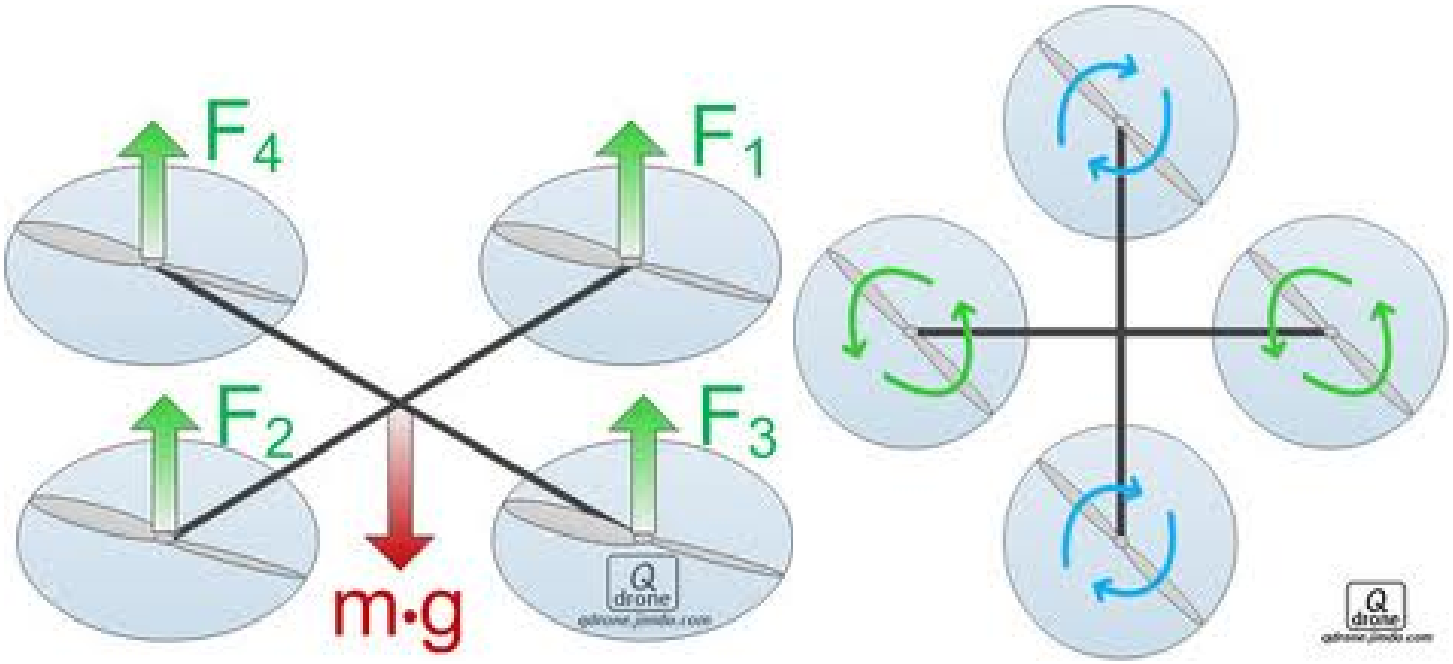
With a PID control with $F(s) = K_p + K_d s + K_i/s$, we obtain the same form of the result:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sF(s)H(s) + s(D/s)H(s) - s^2\theta(s)}{s - F(s)H(s)}$$
$$= \lim_{s\to 0} \frac{D}{-K_p + K_d s + K_i/s} = 0$$

Thus, the result we found before generalize for any transfer function of the form $H(s)/s$ where $H(0) \neq 0$.

Quadcopters

Quadcopter Thrusts



Degrees of freedom provided by extra motors.

Coordinate Systems

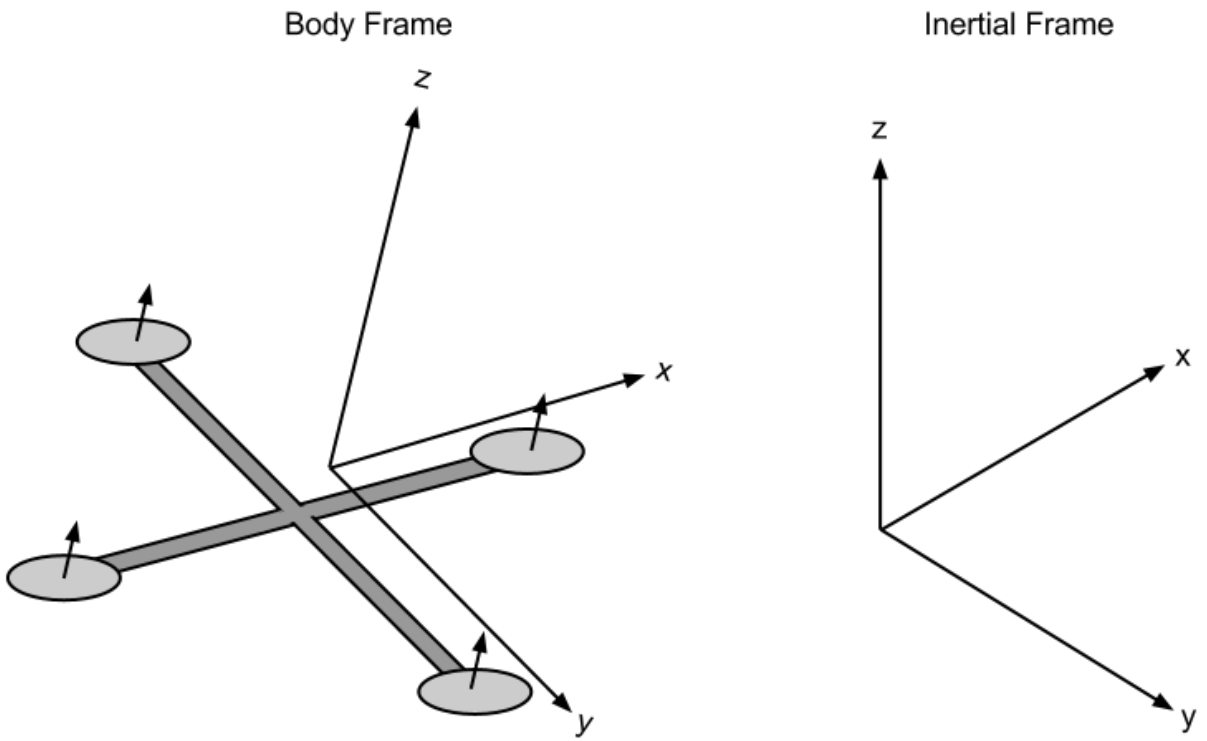


Figure: Quadcopter Body Frame and Inertial Frame

Forces and Torques

- Thrust force:

$$T_B = \sum_{i=1}^{4} T_i = k \begin{bmatrix} 0 \\ 0 \\ \sum {\omega_i}^2 \end{bmatrix}.$$

- Drag force:

$$F_D = \begin{bmatrix} -k_d \dot{x} \\ -k_d \dot{y} \\ -k_d \dot{z} \end{bmatrix}$$

- Motor torques:

$$\tau_B = \begin{bmatrix} Lk({\omega_1}^2 - {\omega_3}^2) \\ Lk({\omega_2}^2 - {\omega_4}^2) \\ b\left({\omega_1}^2 - {\omega_2}^2 + {\omega_3}^2 - {\omega_4}^2\right) \end{bmatrix}$$

Equations of Motion: Linear

- Inertial Frame:

$$m\ddot{x} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + RT_B + F_D$$

- Body Frame:

$$m\ddot{x} = R^{-1} \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + T_B + R^{-1} F_D - \omega \times (m\dot{x})$$

Equations of Motion: Rotational

- Euler's Equations (Rigid Body Dynamics):

$$I\dot{\omega} + \omega \times (I\omega) = \tau$$

- Body Frame Dynamics:

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

$$\dot{\omega} = \begin{bmatrix} \tau_\phi {I_{xx}}^{-1} \\ \tau_\theta {I_{yy}}^{-1} \\ \tau_\psi {I_{zz}}^{-1} \end{bmatrix} - \begin{bmatrix} \frac{I_{yy}-I_{zz}}{I_{xx}} \omega_y \omega_z \\ \frac{I_{zz}-I_{xx}}{I_{yy}} \omega_x \omega_z \\ \frac{I_{xx}-I_{yy}}{I_{zz}} \omega_x \omega_y \end{bmatrix}$$

Simulation

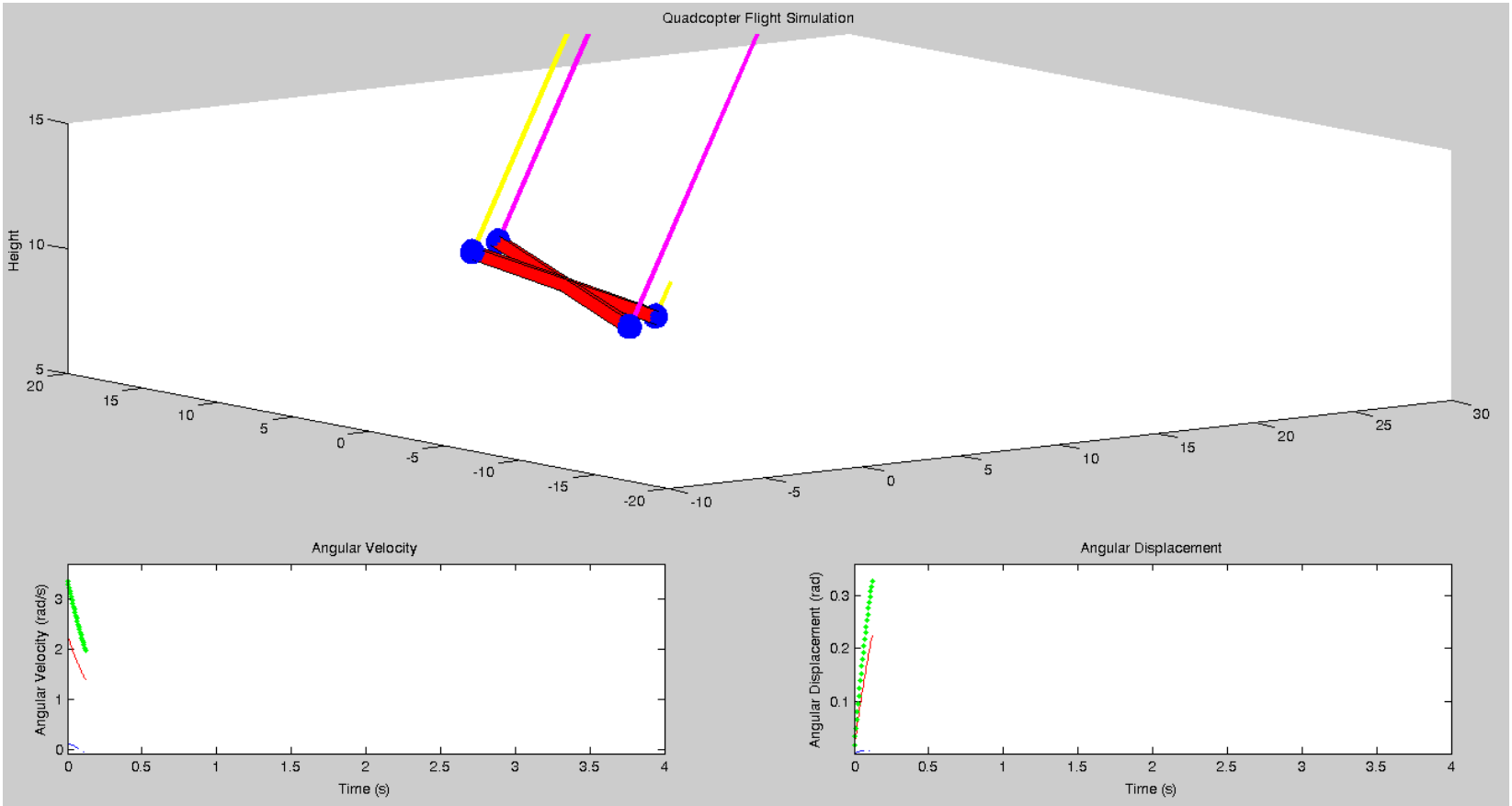
```
% Simulation times, in seconds.
start_time = 0; end_time = 10; dt = 0.005;
times = start_time:dt:end_time;
N = numel(times);

% Initial simulation state.
x = [0; 0; 10]; xdot = zeros(3, 1); theta = zeros(3, 1);

% Simulate some disturbance in the angular velocity.
% The magnitude of the deviation is in radians / second.
deviation = 100;
thetadot = deg2rad(2 * deviation * rand(3, 1) - deviation);

% Step through the simulation, updating the state.
for t = times
    i = input(t);                                          % Get controller input
    omega = thetadot2omega(thetadot, theta);               % Convert (φ̇, θ̇, ψ̇) → ω
    a = acceleration(i, theta, xdot, m, g, k, kd);         % Compute linear acceleration
    omegadot = angular_acceleration(i, omega, I, L, b, k); % Compute angular acceleration
    omega = omega + dt * omegadot;                         % ω = ω + dt × ω̇
    thetadot = omega2thetadot(omega, theta);               % Convert ω → (φ̇, θ̇, ψ̇)
    theta = theta + dt * thetadot;                         % θ⃗ = θ⃗ + dt × (φ̇, θ̇, ψ̇)
    xdot = xdot + dt * a;                                  % ẋ = ẋ + dt × a
    x = x + dt * xdot;                                     % x = x + dt × ẋ
end
```

Visualization



YouTube Movie

PD Control

- Control signal should be turned into a torque:

$$\tau = Iu(t) = I\dot{\omega}$$

- We can set torques:

$$\begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} -I_{xx}\left(K_d\dot{\phi} + K_p\int_0^T \dot{\phi}\,\mathrm{d}t\right) \\ -I_{yy}\left(K_d\dot{\theta} + K_p\int_0^T \dot{\theta}\,\mathrm{d}t\right) \\ -I_{zz}\left(K_d\dot{\psi} + K_p\int_0^T \dot{\psi}\,\mathrm{d}t\right) \end{bmatrix}$$

- Our electronic gyro outputs angular velocities, so we integrate them to get the angle. This integral is multiplied by the *proportional* gain.

Motor Angular Velocities

- We do not actually set torques, we set voltages over the motors. These correspond directly to the motor angular velocities ω_i.
- We can solve for the inputs $\gamma_i = {\omega_i}^2$:

$$\tau_B = \begin{bmatrix} Lk(\gamma_1 - \gamma_3) \\ Lk(\gamma_2 - \gamma_4) \\ b(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4) \end{bmatrix} = \begin{bmatrix} -I_{xx}\left(K_d\dot{\phi} + K_p\int_0^T \dot{\phi}\,\mathrm{d}t\right) \\ -I_{yy}\left(K_d\dot{\theta} + K_p\int_0^T \dot{\theta}\,\mathrm{d}t\right) \\ -I_{zz}\left(K_d\dot{\psi} + K_p\int_0^T \dot{\psi}\,\mathrm{d}t\right) \end{bmatrix}$$

- For the last constraint, choose one to keep the quadcopter aloft:

$$T = \frac{mg}{\cos\theta\cos\phi}$$

PD Control

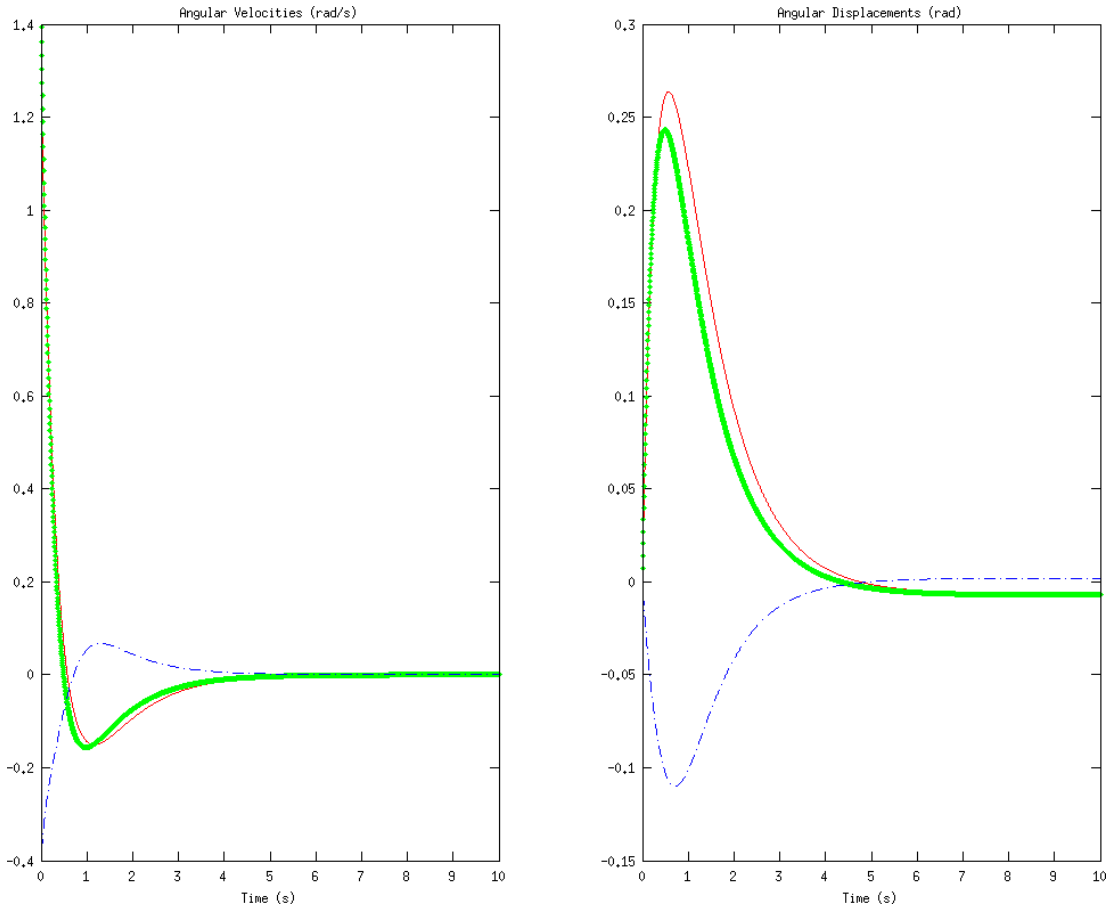


Figure: Left: Angular velocities. Right: angular displacements (ϕ, θ, ψ). Note that there is some small steady-state error (approximately 0.3°).

- We can do better by adding an integral term to make this a PID controller.

$$e_\phi = K_d\dot{\phi} + K_p\int_0^T \dot{\phi}\,\mathrm{d}t + K_i\int_0^T\int_0^T \dot{\phi}\,\mathrm{d}t\,\mathrm{d}t$$

$$e_\theta = K_d\dot{\theta} + K_p\int_0^T \dot{\theta}\,\mathrm{d}t + K_i\int_0^T\int_0^T \dot{\theta}\,\mathrm{d}t\,\mathrm{d}t$$

$$e_\psi = K_d\dot{\psi} + K_p\int_0^T \dot{\psi}\,\mathrm{d}t + K_i\int_0^T\int_0^T \dot{\psi}\,\mathrm{d}t\,\mathrm{d}t$$

- To avoid integral wind-up, only start integrating the double-integral once the angle deviation is relatively small.

Integral Windup

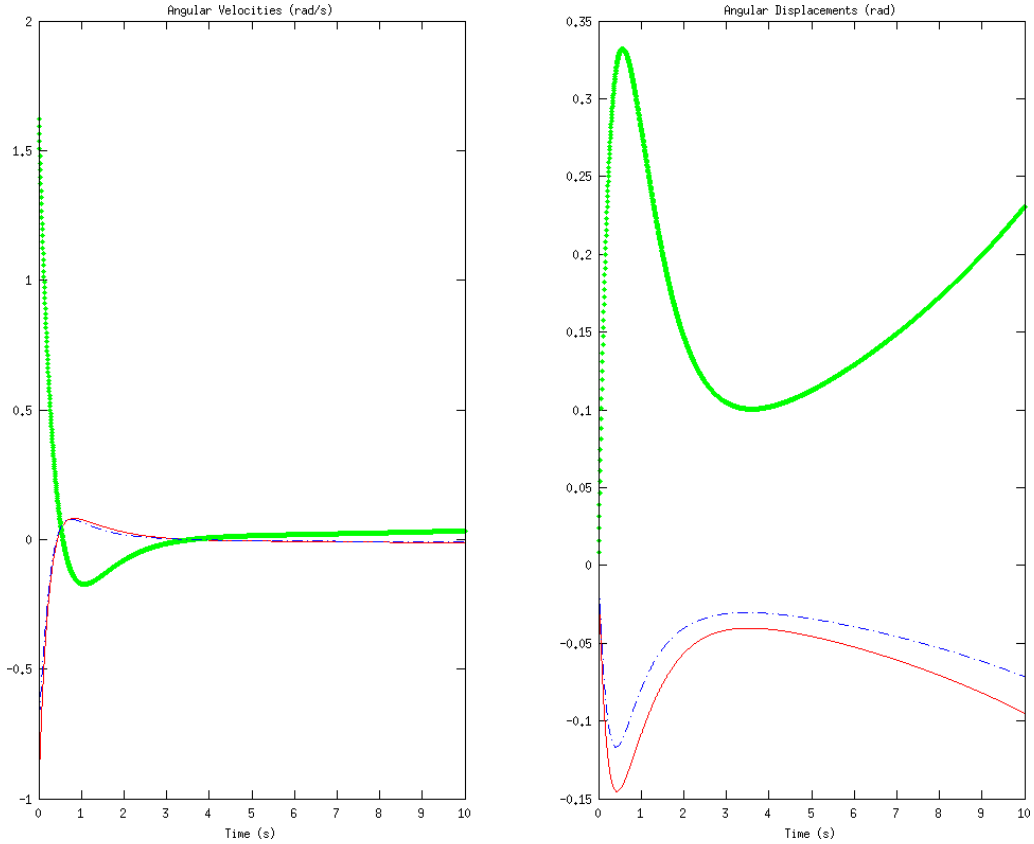


Figure: In some cases, integral wind-up can cause lengthy oscillations instead of settling. In other cases, wind-up may actually cause the system to become unstable, instead of taking longer to reach a steady state.

PID Control

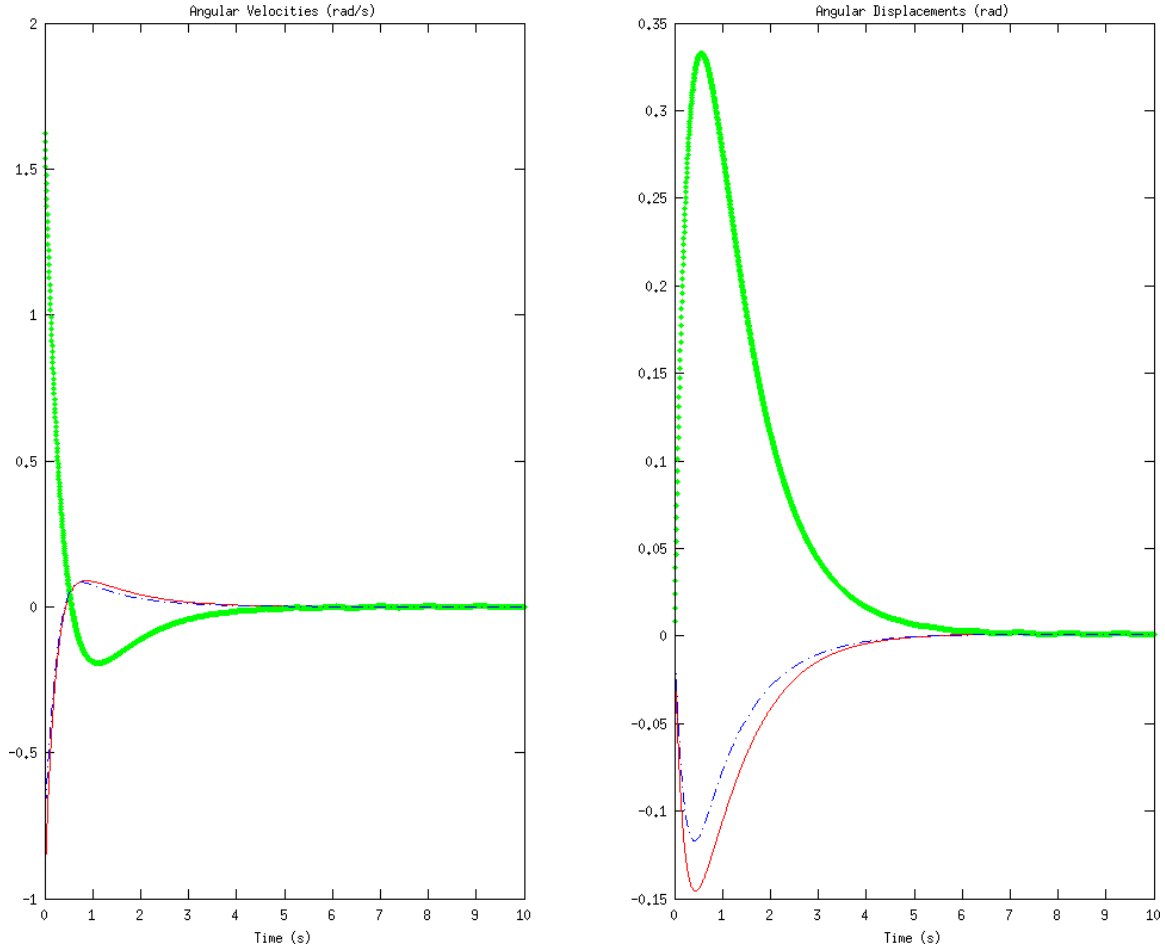


Figure: With a properly implemented PID, we achieve an error of approximately 0.06° after 10 seconds.

Automatic Gain Tuning

- Tuning the PID gains (K_i, K_p, and K_d) can be difficult.
- Quality of results depends on gain values and the ratios of the different gain values.
- “Best” gains might be different for different modes of operation.
- Requires expert intuition and a lot of time to tune them.
- We would like to do this automatically.

Extremum Seeking

- Define a cost function, defining the quality of a set of parameters:

$$J(\vec{\theta}) = \frac{1}{t_f - t_o} \int_{t_0}^{t_f} e(t, \vec{\theta})^2 \, dt$$

- Use gradient descent to minimize this cost function in parameter-space:

$$\vec{\theta}(k+1) = \vec{\theta}(k) - \alpha \nabla J(\vec{\theta})$$

- Approximate gradient numerically:

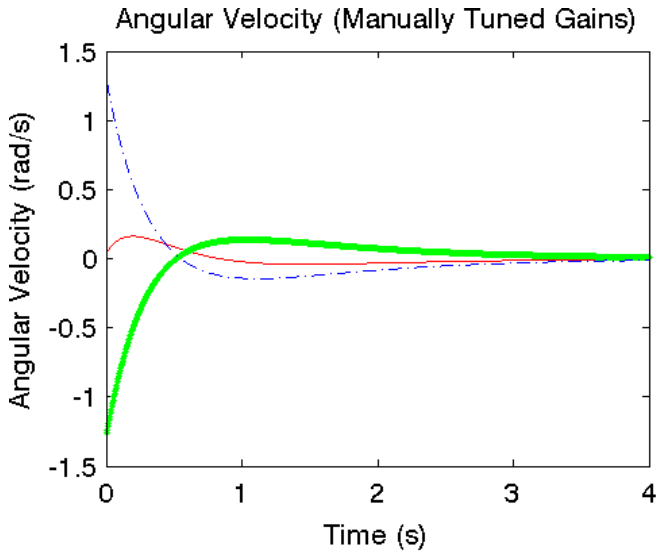
$$\nabla J(\vec{\theta}) = \left(\frac{\partial}{\partial K_p} J(\vec{\theta}), \frac{\partial}{\partial K_i} J(\vec{\theta}), \frac{\partial}{\partial K_d} J(\vec{\theta}) \right).$$

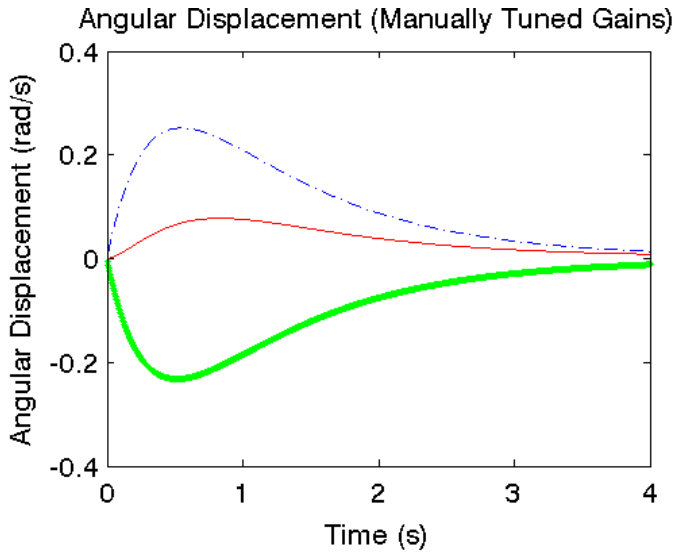
$$\frac{\partial}{\partial K} J(\vec{\theta}) \approx \frac{J(\vec{\theta} + \delta \cdot \hat{u}_K) - J(\vec{\theta} - \delta \cdot \hat{u}_K)}{2\delta}$$

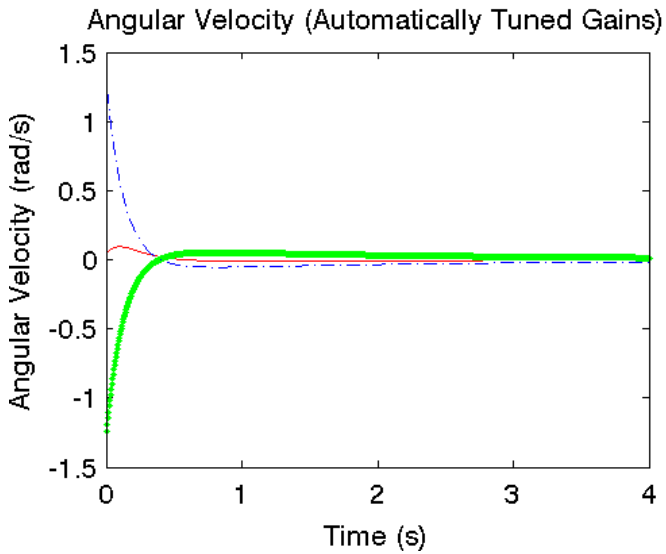
Gradient Descent Tricks

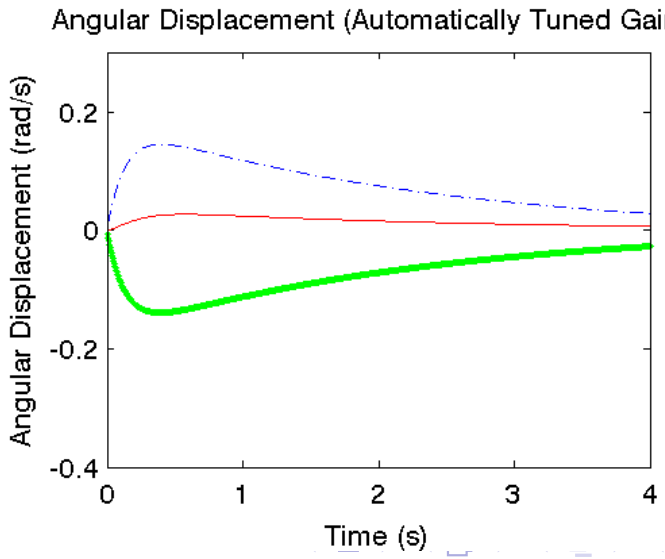
- Adjust step size α as we go along, to become more precise as time goes on.
- Computing $e(t, \vec{\theta})$ means running a simulation with some random initial disturbance. Use many simulations, take the average. As we go along, average more simulations.
- Repeat many times to produce many local minima, then choose the best one and call it the "global minimum".
- Automatically choose a time to stop iterating (when we're no longer improving our average cost).

Manual Gains vs. Automatically Tuned Gains









Questions?

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