Single pixel imaging

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Abstract: This document introduce the necessary definitions and mathematical notions to understand the single pixel imaging techniques.

1. Introduction

The most basic setup for single pixel imaging (SPI) consists of projecting a series of patterns over a scene and collect or integrate the reflected light using a photodiode. Commonly, it is employed a spatial light modulator (SLM) to filter or reflect a pattern. Then, the photodiode integrates the interaction of the SLM pattern with scene to obtain a measure.

We denote as $\mathbf{f} \in \mathbb{R}^d$ the desire image of the scene with d the total number of pixels of the image, i.e. $d = \text{Height} \times \text{Width}$. Let $\mathbf{p} \in \mathbb{R}^d$ be a SLM pattern and let m be the obtained measurement by the photodiode. The measurement is a linear process that can be modeled as

$$m = \mathbf{p}^T \mathbf{f} \tag{1}$$

where $\mathbf{p}^T$ is the transpose of column vector $\mathbf{p}$.

To reconstruct an image we must to obtain several measurements using different SLM patterns. We define the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_k \end{bmatrix}^T \in \mathbb{R}^{k \times d}$ whose rows $\mathbf{p}_j \in \mathbb{R}^d$ are patterns for $j = 1, \ldots, k$, and let $\mathbf{m} \in \mathbb{R}^k$ be the corresponding measurements obtained as

$$\mathbf{m} = P\mathbf{f}. \tag{2}$$

The process of obtaining the vector of measurements is called *acquisition*. When performing real measurements it is considered the integration time Δt (in s) of the photodiode and number of photons per second N_0 (in ph/s) of the source light. If $\mathbf{f}$ and $\mathbf{p}$ are normalized, then a measurement

$$m = \Delta t N_0 \mathbf{p}^T \mathbf{f} \tag{3}$$

is given in units of number of photons.

Another component to be considered in real measurements is the *dark current* in the photodiode. The dark current is the measurement α (in ph/s) obtained when the source light is off or when $N_0 = 0$. This is caused by other source lights such as ambient light, and due to the nature of the semiconductor. Thus, real measurements can be modeled as

$$\mathbf{m} = (N_0 P\mathbf{f} + \alpha \mathbf{1}_k)\Delta t \tag{4}$$

where $\mathbf{1}_k \in \mathbb{R}^k$ with 1 at every position.

The process of obtaining the image $\mathbf{f}$ given the set of patterns P and measurements $\mathbf{m}$ is called *restoration*. Mathematically, the restoration consists of the problem of solve the linear system given in (4). For a simpler analysis, it will be considered the linear system given in (2) since the additive term $\Delta t \alpha \mathbf{1}_k$ can be measured by setting $N_0 = 0$, and the term $\Delta t N_0$ is just a scaling factor. There exists several methods to solve (2) such as compressive sensing, basis scan methods, non-iterative methods, linear iterative methods and non-linear iterative methods.

In the following section, we give a succinct review of these methods.

2. Restoration methods

2.1. Non-iterative methods

Non-iterative methods refers to the direct solution of the linear system given in (2) by matrix inversion. In general, when the number of pixels d and the number of patterns k are different, the matrix P has no inverse. In the special case when P is a square matrix then it is possible to solve the linear system as $\mathbf{f} = P^{-1}\mathbf{m}$. There are several examples of square matrices with a special structure that will be studied in the next sections.

Another approach to solve the restoration problem is the following: multiply on both sides of system (2) by P^T which becomes $P^T\mathbf{m} = P^T P\mathbf{f}$. Since $P^T P \in \mathbb{R}^{d \times d}$ is a square matrix then the desire image can be obtained as $\mathbf{f} = (P^T P)^{-1} P^T \mathbf{m}$. However, the matrix $P^T P$ is not full-rank when $k < d$ which means that has no inverse.

The latter formulation is equivalent to the problem of minimizing $\|P\mathbf{f}-\mathbf{m}\|^2$ when $k \geq d$. The solution that minimize this least-squares problem can be also be stated as

$$\mathbf{f}^{+} = P^{+}\mathbf{m} \tag{5}$$

where P^+ is the pseudo inverse matrix of P. If $P = U\Sigma V^T$ is the SVD decomposition of matrix P where $U \in \mathbb{R}^{k\times k}$ and $V \in \mathbb{R}^{d\times d}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{k\times d}$ is a rectangular diagonal matrix. Then, the pseudo-inverse of P is defined as

$$P^{+} = V\Sigma^{+}U^{T} \tag{6}$$

where Σ^+ is the pseudo-inverse of Σ, which is formed by replacing every non-zero diagonal entry by its reciprocal.

2.2. *Hadamard basis method*

The Hadamard transform has been used in image coding, communication systems and error correction codes, among others. In SPI as mentioned in the previous section, it is desired that the pattern matrix P has an inverse easy to compute. In particular, the Hadamard transform has the property that it is a symmetric matrix and is its own inverse.

A square matrix H_N of order N with elements -1 and +1 is called a Hadamard matrix if the following is satisfied

$$H_N H_N^T = N I_N \tag{7}$$

where I_N is the identity matrix of order N.

It is known that if a Hadamard matrix of order $N > 2$ exists, then $N \equiv 0 \pmod 4$. The latter statement is called the Hadamard-Paley conjecture which remains unproved. Hadamard matrices of order $N = 2^n$ where n is an integer are called Sylvester's matrices. In this case, if H is a Hadamard matrix of order N, then it is obtained a matrix of order $2N$ as

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}. \tag{8}$$

For instance, the Hadamard matrices of order 2, 4 and 8 are given by

$$H_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \quad H_4 = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}, \quad H_8 = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix} \tag{9}$$

where symbols + and - denote +1 and -1, respectively. For any positive $n \geq 1$, every entry of $H_N = [\mathrm{wal}_h(x,y)]$ with $x, y = 0, \ldots, N-1$, can be expressed as

$$\mathrm{wal}_h(x,y) = (-1)^{\sum_{i=0}^{n-1} x_i y_i} \tag{10}$$

with x_i, y_i are the i-th coefficients in the binary expansion of x, y using n bits.

The Hadamard method consists on constructing a Hadamard matrix whose rows are used as patterns, then the measurements are obtained as

$$\mathbf{m} = H_N \mathbf{f} \tag{11}$$

where the restored image $\mathbf{f}$ must has a dimension of a power of 2. For instance, if we wish to restore an image of dimension $d = 2^n \times 2^n = 2^{2n}$ then the order of the Hadamard matrix is $N = 2^{2n}$. The restored image after the projection of all basis patterns is obtained theoretically as $\mathbf{f} = \frac{1}{N} H_N^T \mathbf{m}$.

In real applications, the SLM is used to filter or reflect a pattern. A popular SLM technology is the DMD (Digital Micromirror Device) that are commonly incorporated in digital projectors. The DMD are configured to reflect binary patterns by moving each micromirror independently. This imposes an important limitation when implementing a SPI system based on Hadamard patterns, since there are negative values. Furthermore, this limitation will be called the positivity constraint of the patterns

To cope with the positivity constraint for Hadamard patterns, it is used a spliting strategy in which two matrix patterns are defined as

$$H_N = H_N^{+} - H_N^{-} \tag{12}$$

where H_N^+ and H_N^- are binary patterns such that the matrix $H_N^+ = (H_N + \mathbf{1}_N)/2$ is zero at each negative entry of H_N, the matrix $H_N^- = (\mathbf{1}_N - H_N)/2$ is zero (one) at each positive (negative) entry of H_N, and $\mathbf{1}_N$ is a matrix of ones of order N. Therefore, the measurements $\mathbf{m}_+$ and $\mathbf{m}_-$ acquired using the patterns H_N^+ and H_N^-, respectively, are subtracted to obtain the measurement $\hat{\mathbf{m}} = \mathbf{m}_+ - \mathbf{m}_-$. Using the measurement model given in (4),

$$\begin{aligned} \mathbf{m}_+ - \mathbf{m}_- &= (N_0 H_N^+ \mathbf{f} + \alpha \mathbf{1}_N)\Delta t - (N_0 H_N^- \mathbf{f} + \alpha \mathbf{1}_N)\Delta t && (13)\\ &= N_0 (H_N^+ - H_N^-)\mathbf{f}\Delta t && (14)\\ &= N_0 H_N \mathbf{f}\Delta t && (15)\\ &= \hat{\mathbf{m}} && (16) \end{aligned}$$

Notice that in the acquired measurement $\hat{\mathbf{m}}$ the dark current α cancels out. It has been proved that this splitting strategy is tolerant to the noise. Furthermore, the splitting method can be generalized for any given pattern matrix with negative values.

2.3. *The splitting method*

In the previous section, we have shown how the Hadamard method can be implemented by splitting the negative and positive values as two binary patterns. This strategy can be generalized to any pattern with negative and positive entries whose absolute values can be greater than one.

Let $P \in \mathbb{Z}^{k \times d}$ be a pattern matrix with integer entries and row representation given as $P = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_k \end{bmatrix}^T$. Then, each row pattern $\mathbf{p}_j$ can be splitted as

$$\mathbf{p}_j = \mathbf{p}_j^+ - \mathbf{p}_j^- \qquad (17)$$

where

$$\begin{aligned} \mathbf{p}_j^+ &= \max(\mathbf{0}_d, \mathbf{p}_j) && (18)\\ \mathbf{p}_j^- &= |\min(\mathbf{0}_d, \mathbf{p}_j)| && (19) \end{aligned}$$

with $\mathbf{0}_d$ be the zero vector of dimension d, and the max and min functions are applied entry wise.

For example, the matrix pattern $P \in \mathbb{Z}^{4\times 4}$ below is splitted as follows:

$$P = \begin{bmatrix} 9 & -3 & 4 & 5 \\ 4 & 3 & 1 & 2 \\ 10 & -5 & -2 & 1 \\ 6 & -4 & 2 & 2 \end{bmatrix}, \quad P^+ = \begin{bmatrix} 9 & 0 & 4 & 5 \\ 4 & 3 & 1 & 2 \\ 10 & 0 & 0 & 1 \\ 6 & 0 & 2 & 2 \end{bmatrix}, \quad P^- = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}. \qquad (20)$$

It can be noticed in the example above, that the positive patterns P^+ and P^- have integer values greater than one. These positive patterns cannot be projected since the SLM is only used for binary patterns. To cope with this restriction, any positive pattern $\mathbf{p} \in \mathbb{Z}_+^d \cup \{0\}$ will be decomposed as a weighted sum of m binary patterns $\mathbf{b}_i \in \{0,1\}^d$ for $i = 1, \ldots, m$. Then, the pattern $\mathbf{p}$ is expressed as

$$\mathbf{p} = \sum_{i=1}^{m} 2^{i-1} \mathbf{b}_i \qquad (21)$$

where each binary entry is obtained by $\mathbf{b}_i(j) = \varepsilon_i$ for $j = 1, \ldots, d$, with ε_i be the i-th bit of the binary expansion of $\mathbf{p}(j)$ using m bits. To define the number of bits to use, find the maximum value in $\mathbf{p}$ and then take the logarithm base 2; in symbols $m = \lceil \log_2 \max(\mathbf{p}) \rceil$.

For example, Fig. 1 shows a positive pattern of 4×4 with gray values between 0 and 15. Fig. 2 shows the binary decomposition of pattern in Fig. 1 with four bit planes $\mathbf{b}_1, \ldots, \mathbf{b}_4$ shown in Figs. 2(a)-(d), respectively.

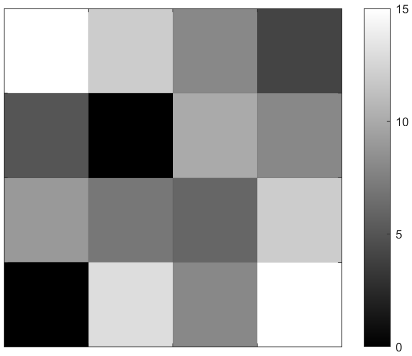
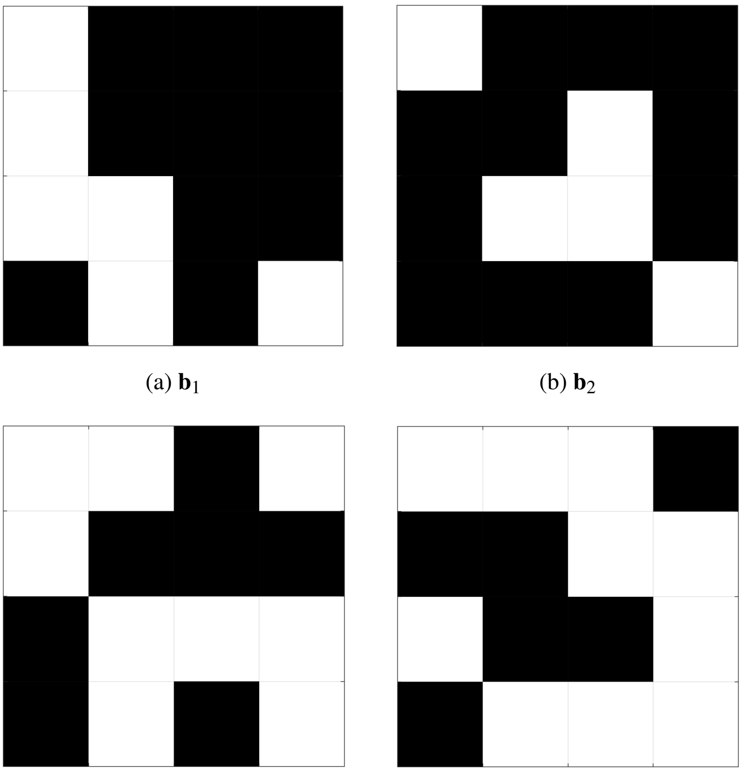


Fig. 1: Positive pattern of 4×4.



(a) $\mathbf{b}_1$

(b) $\mathbf{b}_2$

(c) $\mathbf{b}_3$

(d) $\mathbf{b}_4$

Fig. 2: Pattern decomposition with four bit planes of Fig. 1.