An introduction to single pixel imaging

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Introduction

The *single pixel imaging* (SPI) technique can obtain images by using only a point detector without spatial resolution, having special applications at non-visible wavelengths.

The basic setup for SPI consists of projecting a series of patterns over a scene and integrate the reflected light using a photodiode. It is employed a spatial light modulator (SLM) to filter a pattern. Commonly, the digital micromirror device (DMD) of a digital projector is used as SLM.

Computational imaging configurations for SPI

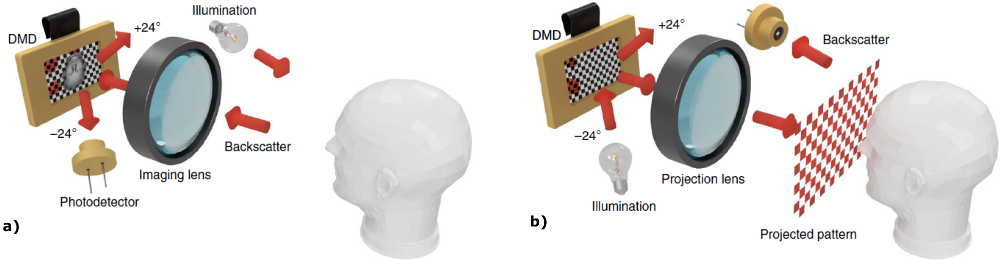


Figure: a) direct filtering and b) indirect filtering.

- Edgar, M.P., Gibson, G.M. Padgett, M.J. Principles and prospects for single-pixel imaging. Nature Photon 13, 13–20 (2019).

Background

The measurement is a linear process that can be modeled as

$$m = \mathbf{p}^T\mathbf{f} \tag{1}$$

where $\mathbf{p} \in \mathbb{R}^d$ is a SLM pattern, $\mathbf{f} \in \mathbb{R}^d$ is the desire image of the scene with d the total number of pixels of the image, i.e. $d =$ Height $\times$ Width, and m is the obtained measurement by the photodiode.

To reconstruct an image we must to obtain several measurements using different SLM patterns.

We define the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_k \end{bmatrix}^T \in \mathbb{R}^{k \times d}$ whose rows $\mathbf{p}_j \in \mathbb{R}^d$ are patterns for $j = 1, \ldots, k$, and let $\mathbf{m} \in \mathbb{R}^k$ be the corresponding measurements obtained as

$$\mathbf{m} = P\mathbf{f}. \tag{2}$$

The process of obtaining the vector of measurements $\mathbf{m}$ is called *acquisition*.

The process of obtaining the image $\mathbf{f}$ given the set of patterns P and measurements $\mathbf{m}$ is called *restoration*.

Mathematically, the restoration problem consists in solving the linear system $\mathbf{m} = P\mathbf{f}$. There exists several restoration methods such as compressive sensing, basis scan methods, non-iterative methods, linear iterative methods and non-linear iterative methods.

- Liheng Bian, Jinli Suo, Qionghai Dai, and Feng Chen, "Experimental comparison of single-pixel imaging algorithms," J. Opt. Soc. Am. A 35, 78-87 (2018)
- Graham M. Gibson, Steven D. Johnson, and Miles J. Padgett, "Single-pixel imaging 12 years on: a review," Opt. Express 28, 28190-28208 (2020)

Restoration methods

In general, when the number of pixels d and the number of patterns k are different, the matrix P has no inverse. Otherwise, the solution to the linear system is

$$\mathbf{f} = P^{-1}\mathbf{m}.$$

There are several examples of square matrices with a special structure that will be studied in the next slides.

The *least squares approach*: multiply on both sides of system $\mathbf{m} = P\mathbf{f}$ by P^T which becomes

$$P^T\mathbf{m} = P^T P\mathbf{f}.$$

Since $P^T P \in \mathbb{R}^{d\times d}$ is a square matrix then the desire image can be obtained as

$$\mathbf{f} = (P^T P)^{-1} P^T\mathbf{m}.$$

When $k < d$, the matrix $P^T P$ is not full-rank which means that has no inverse.

The latter formulation is equivalent to the problem of minimizing $\|P\mathbf{f} - \mathbf{m}\|^2$ when $k \geq d$.

The solution that minimize this least-squares problem can be also be stated as

$$\mathbf{f}^{+} = P^{+}\mathbf{m} \tag{3}$$

where P^{+} is the pseudo inverse matrix of P. If $P = U\Sigma V^{T}$ is the SVD decomposition of matrix P where $U \in \mathbb{R}^{k\times k}$ and $V \in \mathbb{R}^{d\times d}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{k\times d}$ is a rectangular diagonal matrix. Then, the pseudo-inverse of P is defined as

$$P^{+} = V\Sigma^{+}U^{T} \tag{4}$$

where Σ^{+} is the pseudo-inverse of Σ, which is formed by replacing every non-zero diagonal entry by its reciprocal.

The Hadamard basis method

A square matrix H_N of order N with elements -1 and +1 is called a Hadamard matrix if the following is satisfied

$$H_N H_N^T = N I_N \tag{5}$$

where I_N is the identity matrix of order N.

Hadamard matrices of order $N = 2^n$ where n is an integer are called Sylvester's matrices. In this case, if H is a Hadamard matrix of order N, then it is obtained a matrix of order $2N$ as

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}.$$

Examples of Hadamard matrices of order 2, 4 and 8 are given as follow:

$$H_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \quad H_4 = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}, \quad H_8 = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}$$

where symbols + and - denote +1 and -1, respectively.

The Hadamard method consists on constructing a Hadamard matrix whose rows are used as patterns, then the measurements are obtained as

$$\mathbf{m} = H_N \mathbf{f}$$

where the restored image $\mathbf{f}$ has a dimension of a power of 2.

For instance, if we wish to restore an image of dimension $d = 2^n \times 2^n = 2^{2n}$ then the order of the Hadamard matrix must be $N = 2^{2n}$. The restored image after the projection of all basis patterns is obtained theoretically as

$$\mathbf{f} = \frac{1}{N} H_N^T \mathbf{m}.$$

Sampling using Hadamard patterns

Projecting all Hadamard patterns can be an exhaustive process which depends on the size of the reconstructed image. Several sampling methods only select the most significant Hadamard patterns and discard all others.

Other approaches order the rows of the Hadamard matrix from highest to lowest importance. Thus, given an ordered Hadamard matrix it can be used to sample a scene by projecting a percentage of the patterns.

Common ordering of Hadamard matrices: 1) natural Hadamard, Walsh Paley, Cal Sal, Walsh system and normalized Haar.

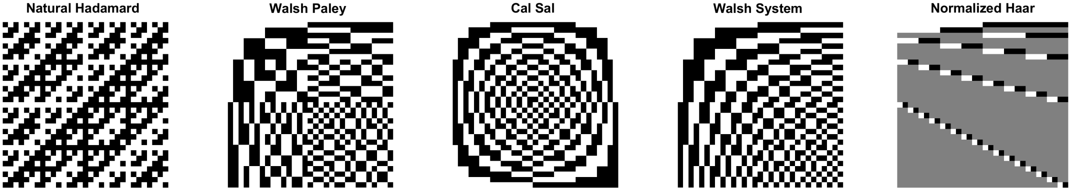


Figure: Hadamard matrices of size 32×32.

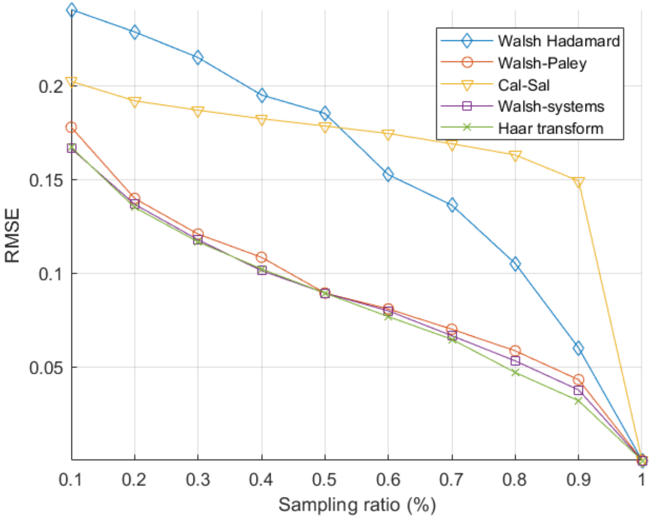


Figure: left) original image and right) sampling performance given in RMSE.


Hadamard
Walsh Paley
Cal Sal
Walsh system
Haar transform
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%

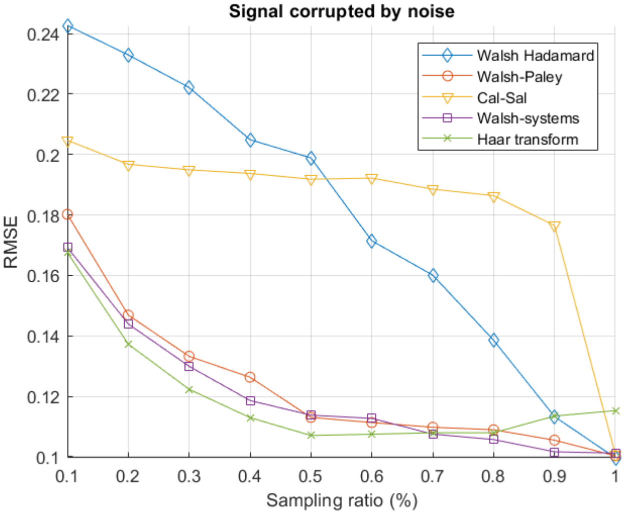


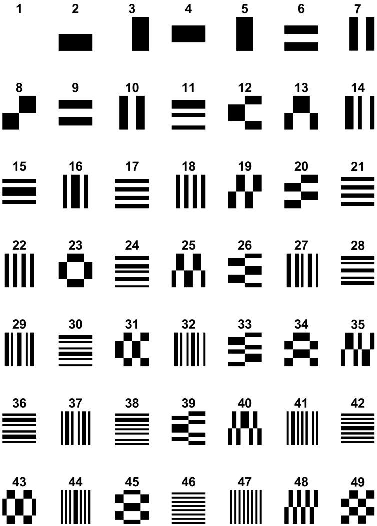
Figure: left) original image and right) sampling performance given in RMSE with added noise snr=45.


Hadamard
Walsh Paley
Cal Sal
Walsh system
Haar transform
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%

Several authors assert that the most significant patterns in the Hadamard matrix are those whose white pixels form biggest regions.

However, to count the number of white regions in each pattern of the Hadamard matrix is an expensive task. *An important problem is the to reduce the complexity of the region counting process*.

- Sun, MJ., Meng, LT., Edgar, M.P. et al. A Russian Dolls ordering of the Hadamard basis for compressive single-pixel imaging. Sci Rep 7, 3464 (2017).
- Yu, W.-K.; Liu, Y.-M. Single-Pixel Imaging with Origami Pattern Construction. Sensors 2019, 19, 5135.
- Yu, W.-K. Super Sub-Nyquist Single-Pixel Imaging by Means of Cake-Cutting Hadamard Basis Sort. Sensors 2019, 19, 4122.



The first 49 rows of the Hadamard matrix of size 4096×4096.
Here the *cake-cutting* method was used which consists of sorting the rows in ascending order with respect to the number of blocks.

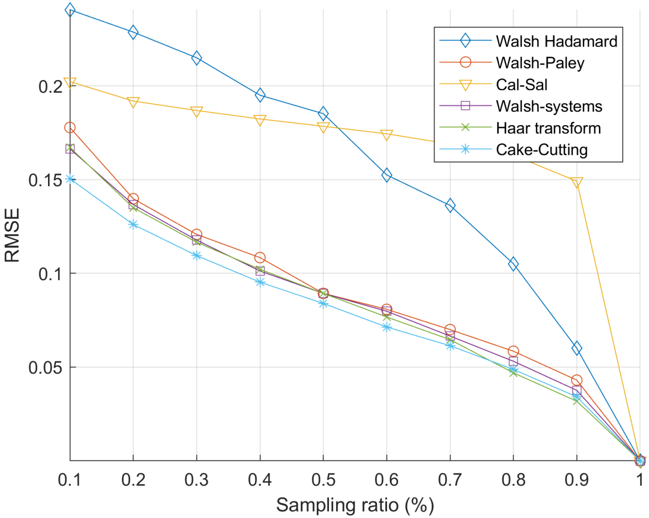


Figure: left) original image and right) sampling performance given in RMSE.

It can be notice that sampling with Hadamard transform omit the non-projected patterns by making zero the corresponding measurement.

A research problem consists in predicting those measurements whose patterns were not projected.