1. (16 points) (Combining Factors)

   (a) Given the factors $P(A|C)$ and $P(B|A,C)$ what is the resulting factor after joining over C?

   ☐ $P(A,B,C)$ ☐ $P(A|B,C)$ $\surd$ $P(A,B|C)$ ☐ None of these.

   (b) Given the factors $P(A|B)$ and $P(B|C)$ and $P(C)$ which factor will be created after joining on C and summing out over C?

   ☐ $P(B,C)$ $\surd$ $P(B)$ ☐ $P(C)$ ☐ None of these

   (c) Given the factors $P(A|C)$ and $P(B|A,C)$ what is the resulting factor after joining over A and summing over A?

   ☐ $P(C)$ ☐ $P(B)$ ☐ $P(B,C)$ ☐ $P(A|C)$ $\surd$ $P(B|C)$

   ☐ None of the above.

   (d) Given the factors $P(C|A)$, $P(D|A,B,C)$, $P(B|A,C)$, what is the resulting factor after joining over C and summing over C?

   ☐ $P(D|A)$ ☐ $P(C,D|A)$ ☐ $P(B,C,D|A)$ $\surd$ $P(B,D|A)$ ☐ $P(C,B|A,D)P(A|D)$

   ☐ None of the above.

   Explanation.

   (a) $P(A|C)P(B|A,C) = P(A,B|C)$

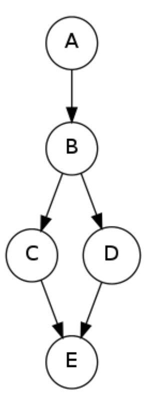
   (b) $P(B|C)P(C) = P(B,C)$. Summing over C, the result is $P(B)$.

   (c) $P(A|C)P(B|A,C) = P(A,B|C)$. Summing over A this becomes $P(B|C)$.

   (d) After joining over C, we take all the variables that appear to the left of the | symbol in any of the factors and put these on the left in the new factor. We condition on all the variables that were conditioned on in any of the factors but did not appear to the left of the | in any factors.

   In this case, we get $P(B,C,D|A)$ after joining, which then becomes $P(B,D|A)$ after summing over C.

2. (20 points) (Variable Elimination Tables) Assume the following Bayes Net and corresponding CPTs. In this exercise, we are given the query $P(C|e=1)$, and we will complete the tables for each factor generated during the elimination process.



After introducing evidence, we have the following probability tables.

A	$P(A)$
0	0.100
1	0.900

B	A	$P(B\|A)$
0	0	0.500
1	0	0.500
0	1	0.400
1	1	0.600

C	B	$P(C\|B)$
0	0	0.400
1	0	0.600
0	1	0.300
1	1	0.700

D	B	$P(D\|B)$
0	0	0.600
1	0	0.400
0	1	0.900
1	1	0.100

C	D	$P(e=1\|C,D)$
0	0	0.600
1	0	0.200
0	1	0.600
1	1	0.200

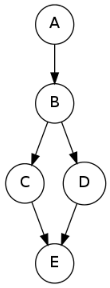
(a) Three steps are required for elimination, with the resulting factors listed below:
Step 1: eliminate A. We get the factor $f_1(B) = \sum_a P(a)P(B|a)$
Step 2: eliminate B. We get the factor $f_2(C,D) = \sum_b P(C|b)P(D|b)f_1(b)$
Step 3: eliminate D. We get the factor $f_3(C,e=1) = \sum_d P(e=1|C,d)f_2(C,d)$.
Fill in the missing quantities. (Some quantities are computed for you.)
$f_1(B=0) =$ ***.41***
$f_1(B=1) =$ ***.59***
$f_2(C=0,D=0) =$ ***.258*** $\quad f_2(C=0,D=1) = 0.083$
$f_2(C=1,D=0) =$ ***.519*** $\quad f_2(C=1,D=1) = 0.14$
$f_3(C=0,e=1) =$ ***.205*** $\quad f_3(C=1,e=1) = 0.132$

(b) After getting the final factor $f_3(C,e=1)$, a final renormalization step needs to be carried out to obtain the conditional probability $P(C|e=1)$. Fill in the final conditional probabilities below.
$P(C=0|e=1) =$ ***.608*** $\quad P(C=1|e=1) =$ ***.392***

Explanation.
Part (a): In order to eliminate a variable, you first perform a join over all of the tables involving that variable, and then sum out the variable. The join step is performed by multiplying all the conditional probability tables that contain the variable together. The summation is performed by summing over all possible values for the variable.
Step 1:
$f_1(B=0) \sum_A P(a)P(B=0|a) = (0.1)(0.5) + (0.9)(0.4) = 0.41$
$f_1(B=1) \sum_A P(a)P(B=1|a) = (0.1)(0.5) + (0.9)(0.6) = 0.59$
Step 2:
$f_2(C=0,D=0) = \sum_b P(C=0|b)P(D=0|b)f_1(b) = (0.4)(0.6)(0.41) + (0.3)(0.9)(0.59) = 0.258$
$f_2(C=1,D=0) = \sum_b P(C=1|b)P(D=0|b)f_1(b) = (0.6)(0.6)(0.41) + (0.7)(0.9)(0.59) = 0.519$
Step 3:
$f_3(C=0,e=1) = \sum_d P(e=1|C=0,d)f_2(C=0,d) = (0.6)(0.258) + (0.6)(0.083) = 0.205$
Part (b): From the definition of conditional probability, $P(C|e=1) = \frac{P(C,e=1)}{\sum_c P(c,e=1)}$, which is the same as renormalizing f_3.
$P(C=0|e=1) = \frac{0.205}{0.205+0.132} = 0.608$
$P(C=1|e=1) = \frac{0.132}{0.205+0.132} = 0.392$

3. (16 points) (Rejection Sampling) We will work with a Bayes' net of the following structure.

0.320	0.037	0.303	0.318	0.032	0.969	0.018	0.058	0.908	0.249



A	$P(A)$
0	0.200
1	0.800

B	A	$P(B\|A)$
0	0	0.800
1	0	0.200
0	1	0.400
1	1	0.600

C	B	$P(C\|B)$
0	0	0.600
1	0	0.400
0	1	0.400
1	1	0.600

D	B	$P(D\|B)$
0	0	0.800
1	0	0.200
0	1	0.600
1	1	0.400

E	C	D	$P(E\|C, D)$
0	0	0	0.800
1	0	0	0.200
0	1	0	0.600
1	1	0	0.400
0	0	1	0.400
1	0	1	0.600
0	1	1	0.400
1	1	1	0.600

In this question, we will perform rejection sampling to estimate $P(C = 1|B = 1, E = 1)$. Perform one round of rejection sampling, using the random samples given in the table below. Variables are sampled in the order A, B, C, D, E. In the boxes below, choose the value (0 or 1) that each variable gets assigned to. **Note**. *The sampling attempt should stop as soon as you discover that the sample will be rejected. In that case mark the assignment of that variable and write 'none' for the rest of the variables.*

When generating random samples, use as many values as needed from the table below, which we generated independently and uniformly at random from $[0, 1)$. Use numbers from left to right. To sample a binary variable W with probability $P(W = 0) = p$ and $P(W = 1) = 1 - p$ using a value a from the table, choose $W = 0$ if $a < p$ and $W = 1$ if $a \geq p$.

Enter either a 0 or 1 for each variable that you assign a value to. Upon rejecting a sample, enter its assigned value, and enter 'none' for the remaining variables. For example, if C gets rejected, fill in 'none' for D and E.

A: ___*1*___ **B:** ___*0*___ **C:** ___*none*___

D: ___*none*___ **E:** ___*none*___

Which variable will get rejected? ☐ A √ ***B*** ☐ C ☐ D ☐ E ☐ None is rejected

Explanation. In rejection sampling, you reject any sample for which the variables' values do not match the values of the evidence variables in what you are trying to estimate. In this case, any sample where $B \neq 1$ or $e \neq 1$ is rejected. Only B and E can ever be rejected, in this case B was rejected because its sampled value was 0

4. (16 points) (Estimating Probabilities from Samples) Below are a set of samples obtained by running rejection sampling for the Bayes' net from the previous question. Use them to estimate $P(C = 1|B = 1, E = 1)$. The estimation cannot be made whenever all samples were rejected. In this case, input -1 into the box below.

Sample 1

	0	1	rejected
A		x	
B	x		x
C			
D			
E			

Sample 2

	0	1	rejected
A	x		
B		x	
C	x		
D	x		
E		x	

Sample 3

	0	1	rejected
A		x	
B		x	
C		x	
D	x		
E		x	

Sample 4

	0	1	rejected
A		x	
B	x		x
C			
D			
E			

Sample 5

	0	1	rejected
A	x		
B		x	
C		x	
D	x		
E	x		x

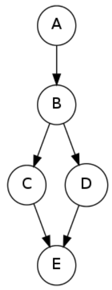
Estimation: ***0.5***

Explanation. The estimate for $P(C = 1|B = 1, E = 1)$ is equal to $\frac{\text{Number of samples where } C{=}1, B{=}1, E{=}1}{\text{Number of samples where } B{=}1 \text{ and } E{=}1}$. Because the samples were found using rejection sampling, any sample that doesn't have $B = 1$ and $E = 1$ was rejected, so this can be written as $\frac{\text{Number of unrejected samples where } C{=}1}{\text{Number of unrejected samples}}$.

In this case, there is 1 unrejected sample where $C = 1$, out of 2 unrejected samples. $\frac{1}{2} = 0.5$

5. (16 points) (Likelihood Weighting) We will work with a Bayes' net of the following structure.

0.249	0.052	0.299	0.773	0.715	0.550	0.703	0.105	0.236	0.153



A	$P(A)$
0	0.200
1	0.800

B	A	$P(B\|A)$
0	0	0.400
1	0	0.600
0	1	0.200
1	1	0.800

C	B	$P(C\|B)$
0	0	0.600
1	0	0.400
0	1	0.600
1	1	0.400

D	B	$P(D\|B)$
0	0	0.800
1	0	0.200
0	1	0.600
1	1	0.400

E	C	D	$P(E\|C,D)$
0	0	0	0.200
1	0	0	0.800
0	1	0	0.600
1	1	0	0.400
0	0	1	0.800
1	0	1	0.200
0	1	1	0.800
1	1	1	0.200

In this question, we will perform likelihood weighting to estimate $P(C = 1|B = 1, E = 1)$. Generate a sample and its weight, using the random samples given in the table below. Variables are sampled in the order A, B, C, D, E. In the table below, select the assignments to the variables you sampled.

When generating random samples, use as many values as needed from the table below, which we generated independently and uniformly at random from $[0, 1)$. Use numbers from left to right. To sample a binary variable W with probability $P(W = 0) = p$ and $P(W = 1) = 1 - p$ using a value a from the table, choose $W = 0$ if $a < p$ and $W = 1$ if $a \geq p$.

A: ___*1*___ **B:** ___*1*___ **C:** ___*0*___

D: ___*0*___ **E:** ___*1*___

What is the weight for the sample you obtained above? ___*0.64*___

Explanation. We sample A using the procedure described above with $p = .2$ to get $A = 1$. We are conditioning on B, so we set $B = 1$ without doing any sampling. We sample $C = 0$ and $D = 0$, then select $E = 1$ since we are conditioning on it as well.

For likelihood weighting, the evidence variables, in this case B and E, are fixed and the sample is given weight equal to the product of the probabilities of the evidence variables taking on those values given the sampled values for their parents.

In this case, the weight is equal to $P(B = 1|A = 1)P(E = 1|C = 0, D = 0) = 0.64$

6. (16 points) (Estimating Probabilities from Weighted Samples)

Below are a set of weighted samples obtained by running likelihood weighting for the Bayes' net from the previous question. Use them to estimate $P(C = 1|B = 1, E = 1)$. Input -1 in the box below if the estimation cannot be made.

Sample 1

	0	1
A		x
B		x
C		x
D		x
E		x

Weight = 0.64

Sample 2

	0	1
A		x
B		x
C		x
D		x
E		x

Weight = 0.64

Sample 3

	0	1
A		x
B		x
C	x	
D		x
E		x

Weight = 0.32

Sample 4

	0	1
A		x
B		x
C	x	
D	x	
E		x

Weight = 0.16

Sample 5

	0	1
A	x	
B		x
C		x
D		x
E		x

Weight = 0.48

Estimation: *0.7857142*

Explanation. The estimate obtained from likelihood sampling is equal to the weighted sum of samples in which $C = 1$ divided by the weighted sum of all the samples. In this case that is $\frac{1.76}{2.24} \approx 0.7857$.