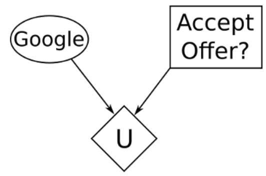
1. (21 points) DECISIONS

   You've been job hunting, and you've narrowed your options to two companies: Acme and Google. You already have an offer from Acme, but it expires today, and you are still waiting for a response from Google. You are faced with the dilemma of whether or not to accept the offer from Acme, which is modeled by the following decision diagram:



   Your prior belief about whether Google will hire you and utility over possible outcomes are as follows:

Google outcome	P(Google outcome)
hired	0.25
not hired	0.75

Action	Google outcome	U
accept Acme offer	hired	2000
accept Acme offer	not hired	8000
reject Acme offer	hired	10000
reject Acme offer	not hired	0

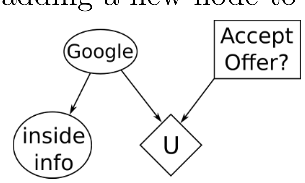
   (a) What is the expected utility of each action? (Note: throughout this problem answers will be evaluated to whole-number precision, so your answer should differ by no more than 1 from the exact answer.)

   Action: accept Acme offer = ***6500.0***

   Action: reject Acme offer = ***2500.0***

   Which action should you take? ☐ reject √ ***accept***

   (b) Suddenly, the phone rings. It's your uncle, who works at Google. Your uncle tells you he has some inside information about the status of your application. Your uncle won't tell you what the information is yet, but he might be willing to divulge it for the right price. You model the new situation by adding a new node to your decision diagram:



   You create a CPT to model the relationship between the inside information and Google's future hiring decision:

info	Google outcome	P(info | Google outcome)
good news	hired	0.7
bad news	hired	0.3
good news	not hired	0.1
bad news	not hired	0.9

   We'll help grind through the probabilistic inference. The resulting distributions are:

info	P(info)
good news	0.25
bad news	0.75

Google outcome	info	P(Google outcome | info)
hired	good news	0.7
not hired	good news	0.3
hired	bad news	0.1
not hired	bad news	0.9

Fill in the expected utilities for each action, for each possible type of information we could be given:

EU(accept Acme offer | good news) = ***3800.0***

EU(reject Acme offer | good news) = ***7000.0***

EU(accept Acme offer | bad news) = ***7400.0***

EU(reject Acme offer | bad news) = ***1000.0***

What is the maximum expected utility for each type of information we could be given?

MEU(good news) = ***7000.0***

MEU(bad news) = ***7400.0***

If we are given the inside information, what is the expected value of MEU?

Answer. ***7300.0***

What is the value of perfect information of the random variable Inside Info?

Answer. ***800.0***

Explanation.
Part (a) EU(accept) = P(hired)U(accept, hired) + P(not hired)U(accept, not hired) = 6500
EU(decline) = P(hired)U(decline, hired) + P(not hired)U(decline, not hired) = 2500
You should take the "accept" action, because it gives more expected utility.
Part (b) A is your action, G is Google outcome, and I is info.
Use the following equation to calculate EU(a | i).
$EU(a \mid i) = \sum_G P(g \mid i)U(a, g)$
For example, here is how you would calculate EU(accept | good news).
EU(accept | good news) = P(hired | good news)U(accept, hired) + P(not hired | good news)U(accept, not hired) = 3800
Use the following equation to calculate MEU(i).
$MEU(i) = max_a EU(a \mid i)$
For example, here is how you would calculate MEU(good news).
MEU(good news) = max(EU(accept | good news), EU(reject | good news)) = 7000
The expected MEU of I is:
MEU(I) = P(good news)MEU(good news) + P(bad news)MEU(bad news) = 7300
VPI(I) = MEU(I) - MEU(∅) = 7300 - 6500 = 800

2. (16 points) Value of Perfect Information

Consider the value of perfect information (VPI) of observing some node in an arbitrary decision network. Which of the following are true statements?

- □ VPI is guaranteed to be positive (> 0).
- √ ***VPI is guaranteed to be nonnegative (≥ 0).***
- □ VPI is guaranteed to be nonzero.
- √ ***The MEU after observing a node could potentially be less than the MEU before observing that node.***
- □ For any two nodes X and Y, $\text{VPI}(X) + \text{VPI}(Y) \geq \text{VPI}(X, Y)$. That is, the sum of individual VPI's for two nodes is always greater than or equal to the VPI of observing both nodes.
- √ ***VPI is guaranteed to be exactly zero for any node that is conditionally independent (given the evidence so far) of all parents of the utility node.***

Explanation.

Option 1: False. VPI is only guaranteed to be non-negative.

Option 2: True. Refer to lecture video.

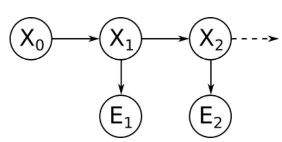
Option 3: False.

Option 4: True. It is only guaranteed that the expected value of information is non-negative. It could be the case that you observe very bad news, which would mean that your MEU goes down.

Option 5: False. Observing both X and Y as evidence can give more utility than the sum of observing both of them individually. For example, suppose you have two fair coins and you have a choice that gives you positive utility when they land on the same value. Knowing either coin individually doesn't help you since the probability of receiving utility is still .5, but knowing both does.

Option 6: True. If the node is conditionally independent of all parents of the utility node, observing that node would tell you nothing about the utility node.

3. (21 points) HMMs, Part I. Consider the HMM shown below.



The prior probability $P(X_0)$, dynamics model $P(X_{t+1} \mid X_t)$, and sensor model $P(E_t \mid X_t)$ are as follows:

X_0	$P(X_0)$
0	0.15
1	0.85

X_{t+1}	X_t	$P(X_{t+1}\|X_t)$
0	0	0.6
1	0	0.4
0	1	0.9
1	1	0.1

E_t	X_t	$P(E_t\|X_t)$
a	0	0.8
b	0	0.15
c	0	0.05
a	1	0.35
b	1	0.05
c	1	0.6

We perform a first dynamics update, and fill in the resulting belief distribution $B'(X_1)$.

X_1	$B'(X_1)$
0	0.855
1	0.145

We incorporate the evidence $E_1 = c$. We fill in the evidence-weighted distribution $P(E_1 = c \mid X_1)B'(X_1)$, and the (normalized) belief distribution $B(X_1)$.

X_1	$P(E_1 = c \mid X_1)B'(X_1)$
0	0.04275
1	0.087

X_1	$B(X_1)$
0	0.329479768786
1	0.670520231214

You get to perform the second dynamics update. Fill in the resulting belief distribution $B'(X_2)$.

$B'(X_2 = 0) =$ ***0.80115***

$B'(X_2 = 1) =$ ***0.19884***

Now incorporate the evidence $E_2 = c$. Fill in the evidence-weighted distribution $P(E_2 = c \mid X_2)B'(X_2)$, and the (normalized) belief distribution $B(X_2)$.

$P(E_2 = c \mid X_2)B'(X_2)$ when $X_2 = 0$ is ***0.04005***

$P(E_2 = c \mid X_2)B'(X_2)$ when $X_2 = 1$ is ***0.11930***

$B(X_2 = 0) =$ ***0.25136***

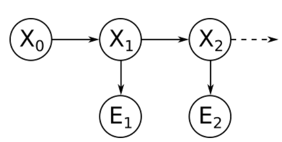
$B(X_2 = 1) =$ ***0.74863***

Explanation. To perform the time update:

$B'(X_2) = P(X_2 \mid X_1 = 0)B(X_1 = 0) + P(X_2 \mid X_1 = 1)B(X_1 = 1)$

To perform the first step of the evidence update, use exactly the equation that is listed. To calculate $B(X_2)$, normalize the numbers you calculated in the first step of the evidence update.

4. (21 points) HMMs, Part II. Consider the same HMM, but with different probabilities.



The prior probability $P(X_0)$, dynamics model $P(X_{t+1} \mid X_t)$, and sensor model $P(E_t \mid X_t)$ are as follows:

X_0	$P(X_0)$
0	0.2
1	0.8

X_{t+1}	X_t	$P(X_{t+1} \mid X_t)$
0	0	0.3
1	0	0.7
0	1	0.05
1	1	0.95

E_t	X_t	$P(E_t \mid X_t)$
a	0	0.3
b	0	0.15
c	0	0.55
a	1	0.1
b	1	0.45
c	1	0.45

In this question we'll assume the sensor is broken and we get no more evidence readings after E_2. We are forced to rely on dynamics updates only going forward. In the limit as $t \to \infty$, our belief about X_t should converge to a stationary distribution $\tilde{B}(X_\infty)$ defined as follows:

$\tilde{B}(X_\infty) := \lim_{t\to\infty} P(X_t \mid E_1, E_2)$

(a) Recall that the stationary distribution satisfies the equation
$\tilde{B}(X_\infty) = \sum_{X_\infty} P(X_{t+1} \mid X_t)\tilde{B}(X_\infty)$
for all values in the domain of X.

In the case of this problem, we can write these relations as a set of linear equations of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tilde{B}(X_\infty = 0) \\ \tilde{B}(X_\infty = 1) \end{bmatrix} = \begin{bmatrix} \tilde{B}(X_\infty = 0) \\ \tilde{B}(X_\infty = 1) \end{bmatrix}$$

In the spaces below, fill in the coefficients of the linear system.

$a =$ ***0.3***

$b =$ ***0.05***

$c =$ ***0.7***

$d =$ ***0.95***

(b) The system you have written has many solutions (consider (0,0), for example), but to get a probability distribution we want the solution that sums to one. Fill in your solution below.

$\tilde{B}(X_\infty = 0) =$ ***0.06666***

$\tilde{B}(X_\infty = 1) =$ ***0.93333***

Explanation.

Part (a) In this problem, we will evaluate this HMM using only time updates. Because we only consider time updates, to update our beliefs from one step to the next, we will use the following equation.

$B(X_{t+1}) = P(X_{t+1} \mid X_t = 0)B(X_t = 0) + P(X_{t+1} \mid X_t = 1)B(X_t = 1)$

After infinite time-steps, our equation will look like this, where X' refers to the previous time step.

$B(X_\infty = 0) = P(X_\infty = 0 \mid X'_\infty = 0)B(X'_\infty = 0) + P(X_\infty = 0 \mid X'_\infty = 1)B(X'_\infty = 1)$

$B(X_\infty = 1) = P(X_\infty = 1 \mid X'_\infty = 0)B(X'_\infty = 0) + P(X_\infty = 1 \mid X'_\infty = 1)B(X'_\infty = 1)$

You should notice that $B(X_\infty)$ is a linear combination of $B(X'_\infty)$, where the coefficients (a, b, c, d) converge to the time step update probabilities.

$B(X_\infty = 0) = a \times B(X'_\infty = 0) + b \times B(X'_\infty = 1)$

$B(X_\infty = 1) = c \times B(X'_\infty = 0) + d \times B(X'_\infty = 1)$

$a = P(X_{t+1} = 0 \mid X_t = 0)$

$b = P(X_{t+1} = 0 \mid X_t = 1)$

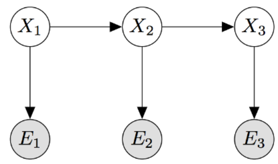
$c = P(X_{t+1} = 1 \mid X_t = 0)$

$d = P(X_{t+1} = 1 \mid X_t = 1)$

Part (b) To find $B(X_\infty = 0)$ and $B(X_\infty = 1)$, we use the above equations and an additional constraint that the two must sum to 1, because it is a probability distribution.

$B(X_\infty = 0) + B(X_\infty = 1) = 1$

5. (21 points) Modified HMM Update Equations. Consider the HMM graph structure shown below



Recall the Forward algorithm is a two step iterative algorithm used to approximate the probability distribution, $P(X_t \mid e_1, \ldots, e_t)$.

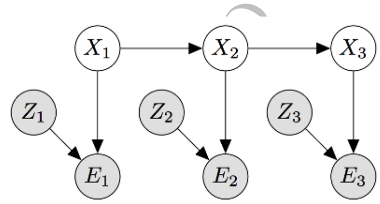
The two steps of the algorithm are as follows:

Elapse Time: $P(X_t \mid e_{1\ldots t-1}) = \Sigma_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1})$

Observe: $P(X_t \mid e_{1\ldots t}) = \frac{P(e_t|X_t)P(X_t|e_{1\ldots t-1})}{\sum_{x_t} P(e_t|x_t)P(x_t|e_{1\ldots t-1})}$

For this problem we will consider modifying the forward algorithm as the HMM graph structure changes. Our goal will continue to be to create an iterative algorithm which is able to compute the distribution of states, X_t, given all available evidence from time 0 to time t.

(a) Consider the graph below where new observed variables, Z_i, are introduced and influence the evidence.



What will the modified elapse time update be?

$P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) =$

☐ $\sum_{x_{t-1}} P(X_t \mid z_{1\ldots t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$
√ $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$
☐ $\sum_{x_{t-1}} P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) P(x_{t-1} \mid x_{t-1}, z_{1\ldots t-1})$
☐ $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1})$ (no change)

What will the modified observed update be?

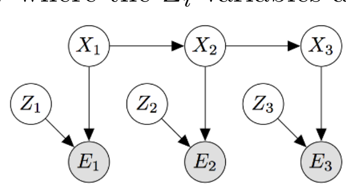
$P(X_t \mid e_{1\ldots t}, z_{1\ldots t}) =$

☐ $\frac{P(e_t|X_t,z_t)P(X_t|e_{1\ldots t-1},z_{1\ldots t-1})}{\sum_{x_t,z_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$ √ $\frac{P(e_t|X_t,z_t)P(X_t|e_{1\ldots t-1},z_{1\ldots t-1})}{\sum_{x_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$

☐ $\frac{P(e_t|X_t)P(X_t|e_{1\ldots t-1})}{\sum_{z_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$ ☐ $\frac{P(e_t|X_t,z_t)P(X_t|e_{1\ldots t-1},z_{1\ldots t-1})}{\sum_{z_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$

☐ $\frac{P(e_t|X_t)P(X_t|e_{1\ldots t-1})}{\sum_{x_t} P(e_t|x_t)P(x_t|e_{1\ldots t-1})}$ (no change)

(b) Next, consider the graph below where the Z_i variables are unobserved.



What will the modified elapse time update be?

$P(X_t \mid e_{1\ldots t-1}) =$

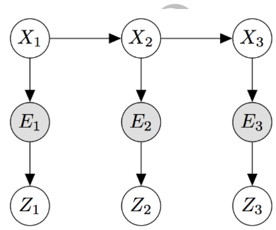
- ☐ $\sum_{x_{t-1}} P(X_t \mid z_{1\ldots t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$
- ☐ $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$
- ☐ $\sum_{x_{t-1}} P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) P(x_{t-1} \mid x_{t-1}, z_{1\ldots t-1})$
- √ $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1})$ ***(no change)***

What will the modified observed update be?

$P(X_t \mid e_{1\ldots t}) =$

- ☐ $\frac{P(X_t|e_{1\ldots t-1})P(z_t)P(e_t|X_t,z_t)}{\sum_{z_t} P(x_t|e_{1\ldots t-1})P(e_t|x_t,z_t)P(z_t)}$
- ☐ $\frac{P(X_t|e_{1\ldots t-1})\sum_{z_t} P(z_t)P(e_t|X_t,z_t)}{P(x_t|e_{1\ldots t-1})\sum_{z_t} P(e_t|x_t,z_t)P(z_t)}$
- ☐ $\frac{P(X_t|e_{1\ldots t-1})P(z_t)P(e_t|X_t,z_t)}{\sum_{x_t} P(x_t|e_{1\ldots t-1})P(e_t|x_t,z_t)P(z_t)}$
- √ $\frac{P(X_t|e_{1\ldots t-1})\sum_{z_t} P(z_t)P(e_t|X_t,z_t)}{\sum_{x_t} P(x_t|e_{1\ldots t-1})\sum_{z_t} P(e_t|x_t,z_t)P(z_t)}$
- ☐ $\frac{P(e_t|X_t)P(X_t|e_{1\ldots t-1})}{\sum_{x_t} P(e_t|x_t)P(x_t|e_{1\ldots t-1})}$ (no change)

(c) Finally, consider a graph where the newly introduced variables are unobserved and influenced by the evidence nodes.



What will the modified elapse time update be?

$P(X_t \mid e_{1\ldots t-1}) =$

- ☐ $\sum_{x_{t-1}} P(X_t \mid z_{1\ldots t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$
- ☐ $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$
- ☐ $\sum_{x_{t-1}} P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) P(x_{t-1} \mid x_{t-1}, z_{1\ldots t-1})$
- √ $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1})$ ***(no change)***

What will the modified observed update be?

$P(X_t \mid e_{1\ldots t}) =$

- ☐ $\frac{P(e_t|X_t,z_t)P(X_t|e_{1\ldots t-1},z_{1\ldots t-1})}{\sum_{z_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$
- ☐ $\frac{P(e_t|X_t,z_t)P(X_t|e_{1\ldots t-1},z_{1\ldots t-1})}{\sum_{x_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$
- ☐ $\frac{P(e_t|X_t)P(X_t|e_{1\ldots t-1})}{\sum_{x_t,z_t} P(e_t|x_t,z_t)P(x_t|e_{1\ldots t-1},z_{1\ldots t-1})}$
- √ $\frac{P(e_t|X_t)P(X_t|e_{1\ldots t-1})}{\sum_{x_t} P(e_t|x_t)P(x_t|e_{1\ldots t-1})}$ ***(no change)***

<u>Explanation.</u>

Part (a) To determine the elapse time update $P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1})$ first note that:
$P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) = \sum_{x_{t-1}} P(X_t, x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1}) =$
$\sum_{x_{t-1}} P(X_t \mid x_{t-1}, e_{1\ldots t-1}, z_{1\ldots t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$ because $P(a, b) = P(a \mid b) * P(b)$
which equals $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1}, z_{1\ldots t-1})$ because X_t is independent of $e_{1\ldots t-1}, z_{1\ldots t-1}$ given x_{t-1}

To determine the observation update $P(X_t \mid e_{1\ldots t}, z_{1\ldots t})$:
note that $P(X_t \mid e_{1\ldots t}, z_{1\ldots t}) = \frac{P(X_t, e_t, z_t | e_{1\ldots t-1}, z_{1\ldots t-1})}{P(e_t, z_t | e_{1\ldots t-1}, z_{1\ldots t-1})}$ by the definition of conditional probability
Looking only at the numerator:
$P(X_t, e_t, z_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) =$
$P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) P(z_t \mid X_t, e_{1\ldots t-1}, z_{1\ldots t-1}) P(e_t \mid X_t, z_t, e_{1\ldots t-1}, z_{1\ldots t-1})$
Then, $P(X_t \mid e_{1\ldots t-1}, z_{1\ldots t-1}) P(z_t) P(e_t \mid X_t, z_t)$
Due to normalization, we divide the numerator by the sum across all rows in the above expression, so we get:
$\frac{P(X_t|e_{1\ldots t-1}, z_{1\ldots t-1}) P(z_t) P(e_t|X_t, z_t)}{P(z_t) \sum_{x_t} P(x_t|e_{1\ldots t-1}, z_{1\ldots t-1}) P(e_t|x_t, z_t)}$
The $P(z_t)$ terms cancel, so we are left with:
$\frac{P(X_t|e_{1\ldots t-1}, z_{1\ldots t-1}) P(e_t|X_t, z_t)}{\sum_{x_t} P(x_t|e_{1\ldots t-1}, z_{1\ldots t-1}) P(e_t|x_t, z_t)}$

Part (b) To determine, $P(X_t \mid e_{1\ldots t-1})$, first rewrite as:
$\sum_{x_{t-1}} P(X_t, x_{t-1} \mid e_{1\ldots t-1})$. Note that since z_t is not observed, the variable z_t will not occur in the evidence of the belief for X_t or X_{t-1}. The term can then just be simplified to $\sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1\ldots t-1})$, which is the same as in a normal HMM.
To determine the observation update $P(X_t \mid e_{1\ldots t})$, first note that:
$P(X_t \mid e_{1\ldots t}) = \frac{P(X_t, e_t|e_{1\ldots t-1})}{\sum_{x_t} P(x_t, e_t|e_{1\ldots t-1})}$.
Looking at the numerator, note that:
$P(X_t, e_t \mid e_{1\ldots t-1}) = P(e_t \mid X_t, e_{1\ldots t-1}) P(X_t \mid e_{1\ldots t-1})$
$= P(X_t \mid e_{1\ldots t-1}) \sum_{z_t} P(z_t) P(e_t \mid X_t, e_{1\ldots t-1}, z_t)$
Since e_t is independent of $e_{1\ldots t-1}$ given X_t this just simplifies to
$P(X_t \mid e_{1\ldots t-1}) \sum_{z_t} P(z_t) P(e_t \mid X_t, z_t)$
Now, we can rewrite the final observation update as:
$\frac{P(X_t|e_{1\ldots t-1}) \sum_{z_t} P(z_t) P(e_t|X_t, z_t)}{\sum_{x_t} P(x_t|e_{1\ldots t-1}) \sum_{z_t} P(e_t|x_t, z_t) P(z_t)}$

Part (c) For both updates, nothing changes since Z_t is independent of X_t given E_t and the conditional probability tables for E_t do not depend on Z_t.