(10pts) 1. (Probability) Use the table to calculate the following values. (You need not simplify answers.)

X_1	X_2	X_3	$\Pr(X_1, X_2, X_3)$
0	0	0	0.05
1	0	0	0.1
0	1	0	0.4
1	1	0	0.1
0	0	1	0.1
1	0	1	0.05
0	1	1	0.2
1	1	1	0.0

(a) $\Pr(X_1 = 1, X_2 = 0) =$ ***0.15***

(b) $\Pr(X_3 = 0) =$ ***0.65***

(c) $\Pr(X_2 = 1 \mid X_3 = 1) =$ ***0.2/0.35***

(d) $\Pr(X_1 = 0 \mid X_2 = 1, X_3 = 1) =$ ***1***

(e) $\Pr(X_1 = 0, X_2 = 1 \mid X_3 = 1) =$ ***0.2/0.35***

2. (Pacman with Feature-Based Q-Learning) Suppose you want to use a Q-learning agent for Pacman, but the state size for a large grid is too massive to hold in memory, so you decide to switch to a feature-based representation of Pacman's state.

Consider the two features:

- f_g = the number of ghosts within 1 step of Pacman
- f_p = the number of food pellets within 1 step of Pacman

Hints. (1) These features depend only on the state, not the actions you take.
(2) You are guaranteed at least partial credit if you write down the correct formula for each part of this problem before carrying out the computations.

(2pts) (a) What are the values of the two features for the game state of the Pacman board shown above.

$f_g =$ ___2___ $f_p =$ ___1___

(4pts) (b) In Q Learning you train off of a few episodes so your weights begin to take on values. Assume that right now $w_g = 100$ and $w_p = -10$. Calculate the Q value for the game state shown above.

Answer: First of all, the Q value will not depend on what action is taken, because the features we extract do not depend on the action, only the state. $Q(s,a) = w_g \cdot f_g + w_p \cdot f_p = 100 \cdot 2 + -10 \cdot 1 = 190$.

(4pts) (c) After receiving an *episode* (start state s, action a, end state s', and reward r), you update your values. The start state of the episode is the state above (for which you already calculated the feature values and the expected Q value). The next state has feature values $f_g = 0$ and $f_p = 2$ and the reward is 50. Assuming a discount of $\gamma = 0.5$, calculate the new estimate of the Q value for s based on this episode.

Answer: $Q_{\text{new}}(s,a) = R(s,a,s') + \gamma \cdot \max_{a'} Q(s',a') = 50 + 0.5 \cdot (100 \cdot 0 + (-10) \cdot 2) = 40$

(4pts) (d) Now use the difference $Q_{\text{new}}(s,a) - Q(s,a)$, and learning rate $\alpha = 0.5$, to update the weights for each feature.

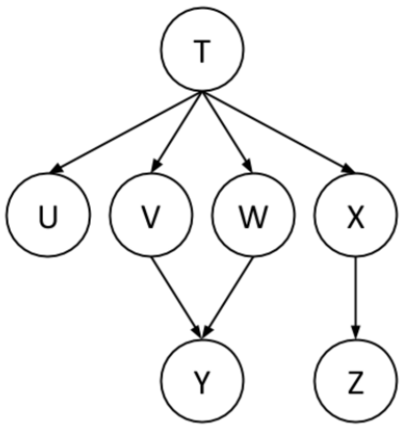
Answer:

$$w_g = w_g + \alpha \cdot (Q_{\text{new}}(s,a) - Q(s,a)) \cdot f_g(s,a) = 100 + 0.5 \cdot (40 - 190) \cdot 2 = -50$$

$$w_p = w_p + \alpha \cdot (Q_{\text{new}}(s,a) - Q(s,a)) \cdot f_p(s,a) = -10 + 0.5 \cdot (40 - 190) \cdot 1 = -85$$

Note that now the weight on ghosts is negative, which makes sense (ghosts should indeed be avoided). Although the weight on food pellets is now also negative, the difference between the two weights is now much lower.

3. (D-Separation) Indicate whether each of the following conditional independence relationships is guaranteed to be true in the Bayes Net below. If the independence relationship does not hold, identify all active (d-connected) paths in the graph; write A-C-D to denote an active path from A to C to D.



(2pts) (a) U $\perp\!\!\!\perp$ X

☐ True √ ***False, because of the active path(s):*** ***U-T-X***

(2pts) (b) U $\perp\!\!\!\perp$ X | T

√ ***True*** ☐ False, because of the active path(s): ***N/A***

(2pts) (c) V $\perp\!\!\!\perp$ W | Y

☐ True √ ***False, because of the active path(s):*** ***V-T-W and V-Y-W***

(2pts) (d) V $\perp\!\!\!\perp$ W | T

√ ***True*** ☐ False, because of the active path(s): ***N/A***

(1pts) (e) T $\perp\!\!\!\perp$ Y | V

☐ True √ ***False, because of the active path(s):*** ***T-W-Y***

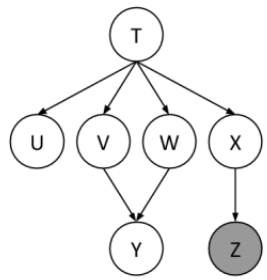
(1pts) (f) Y $\perp\!\!\!\perp$ Z | W

☐ True √ ***False, because of the active path(s):*** ***Y-V-T-X-Z***

(1pts) (g) Y $\perp\!\!\!\perp$ Z | T

√ ***True*** ☐ False, because of the active path(s): ***N/A***

4. (Variable Elimination) Using the same Bayes Net as in the previous problem (and shown below), we want to compute Pr(Y | +z). All variables have binary domains. Assume we run variable elimination to compute the answer to this query, with the following variable elimination ordering: X, T, U, V, W.



After inserting evidence we have the following factors to start with:
Pr(T), Pr(U | T), Pr(V | T), Pr(W | T), Pr(X | T), Pr(Y | V, W), Pr(+z | X)

(0pts) (a) (example) When eliminating X we generate a new factor f_1 as follows:

$f_1(+z \mid T) = \Sigma_x \Pr(x \mid T) \Pr(+z \mid x)$

leaving the factors Pr(T), Pr(U | T), Pr(V | T), Pr(W | T), Pr(Y | V, W), $f_1(+z \mid T)$.

(4pts) (b) Eliminating T we generate a new factor f_2 as follows (express f_2 as a sum, Σ_t):

$f_2(U, V, W, +z) = \Sigma_t \Pr(t) \Pr(U \mid t) \Pr(V \mid t) \Pr(W \mid t) f_1(+z \mid t)$

leaving the two factors ***Pr(Y | V, W)*** and ***f_2(U, V, W, +z)***.

(4pts) (c) When eliminating U we generate a new factor f_3 as follows:

$f_3(V, W, +z) = \Sigma_u f_2(u, V, W, +z)$

leaving the two factors ***Pr(Y | V, W)*** and ***f_3(V, W, +z)***.

(3pts) (d) When eliminating V we generate a new factor f_4 as follows:

$f_4(W, Y, +z) = \Sigma_v f_3(v, W, +z) \Pr(Y \mid v, W)$ leaving the factor ***f_4(V, W, +z)***.

(3pts) (e) When eliminating W we generate a new factor f_5 as follows:

$f_5(Y, +z) = \Sigma_w f_4(w, Y, +z)$ leaving the factor ***f_5(Y, +z)***.

4. (continued)

(3pts) (f) How would we obtain $\Pr(Y \mid +z)$ from the factor left above?

Answer: Normalize $f_5(Y, +z)$ to obtain

$$\Pr(y \mid +z) = \frac{f_5(y, +z)}{\Sigma_{y'} f_5(y', +z)}$$

EXTRA CREDIT

(1pt bonus) (g) What is the largest factor generated during the above process?

Answer. ***f_2(U, V, W, +z)***

(1pt bonus) (h) How many unconditioned variables does it have?

Answer. ***3***

(1pt bonus) (i) How many probability entries will its distribution table have?

Answer. ***$2^3 = 8$***

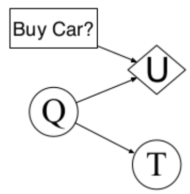
Explanation. The largest factor, f_2(U, V, W, +z), will have $2^3 = 8$ probability entries; U, V, W are binary variables, and we only need to store the probability for +z for each possible setting of these variables.

(1pt bonus) (j) Does there exist a better elimination ordering (generating smaller largest factors)?

□ No √ ***Yes, one such ordering is X, U, T, V, W.***

Explanation. All factors generated with the ordering X, U, T, V, W contain at most 2 unconditioned variables, so the tables will have at most $2^2 = 4$ probability entries (as all variables are binary).

5. (Decision Networks and VPI) A used car buyer is deciding whether to carry out test of a car and then, depending on the outcome, decide which car to buy. Assume you (the buyer) are deciding whether to buy car c and that there is time to carry one test which costs \$50 and which will tell you whether the car is in good shape (quality Q = +q) or bad shape (quality Q = -q). The test will have one of two outcomes: pass (T = pass) or fail (T = fail). Car c costs \$1,500, and its market value is \$2,000 if it is already in good shape, and, if it's in bad shape, you would spend \$700 in repairs to get it into good shape. You estimate that c has 70% chance of being in good shape. The Decision Network is shown below.



(2pts) (a) Calculate the expected utility (or net gain) from buying car c, given no test information.

> **Answer:** EU(buy) = Pr(Q = +q) · U(+q, buy) + Pr(Q = -q) · U(-q, buy) = 0.7 · 500 + 0.3 · (-200) = 290

(2pts) (b) Suppose Pr(T = pass | Q = +q) = 0.9 and Pr(T = pass | Q = -q) = 0.2. Calculate the probability that the car will pass or fail. [*Hint.* Sum Pr(T = pass, q) over q.]

> **Answer:**
>
> $$\begin{aligned}\Pr(\mathrm{T} = \text{pass}) &= \Sigma_{\mathrm{q}} \Pr(\mathrm{T} = \text{pass}, \mathrm{Q} = \mathrm{q}) \\ &= \Pr(\mathrm{T} = \text{pass} \mid \mathrm{Q} = +\mathrm{q})\Pr(\mathrm{Q} = +\mathrm{q}) + \Pr(\mathrm{T} = \text{pass} \mid \mathrm{Q} = -\mathrm{q})\Pr(\mathrm{Q} = -\mathrm{q}) \\ &= 0.69\end{aligned}$$
>
> $$\Pr(\mathrm{T} = \text{fail}) = 1 - \Pr(\mathrm{T} = \text{pass}) = 0.31$$

(4pts) (c) Calculate the probability the car is in good shape given each test outcome.

> **Answer:**
>
> $$\begin{aligned}\Pr(\mathrm{Q} = +\mathrm{q} \mid \mathrm{T} = \text{pass}) &= \frac{\Pr(\mathrm{T} = \text{pass} \mid \mathrm{Q} = +\mathrm{q})\Pr(\mathrm{Q} = +\mathrm{q})}{\Pr(\mathrm{T} = \text{pass})} \\ &= \frac{0.9 \cdot 0.7}{0.69} = \frac{21}{23} \approx 0.91\end{aligned}$$

$$\Pr(Q = +q \mid T = \text{fail}) = \frac{\Pr(T = \text{fail} \mid Q = +q)\Pr(Q = +q)}{\Pr(T = \text{fail})}$$
$$= \frac{0.1 \cdot 0.7}{0.31} = \frac{7}{31} \approx 0.22$$

(4pts) (d) Calculate the expected utility of each purchasing decision for each outcome of the test. ("¬ buy" denotes the decision "don't buy")

Answer:

$$\text{EU}(\text{buy} \mid T = \text{pass}) = \Pr(Q = +q \mid T = \text{pass})U(+q, \text{buy}) + \Pr(Q = -q \mid T = \text{pass})U(-q, \text{buy})$$
$$\approx 0.91 \cdot 500 + 0.09 \cdot (-200) \approx 437$$

$$\text{EU}(\text{buy} \mid T = \text{fail}) = \Pr(Q = +q \mid T = \text{fail})U(+q, \text{buy}) + \Pr(Q = -q \mid T = \text{fail})U(-q, \text{buy})$$
$$\approx 0.22 \cdot 500 + 0.78 \cdot (-200) = -46$$

$$\text{EU}(\neg\text{buy} \mid T = \text{pass}) = 0, \quad \text{EU}(\neg\text{buy} \mid T = \text{fail}) = 0$$

(3pts) (e) Use your previous answer to determine the max expected utility for each outcome and identify the optimal decision in each case.

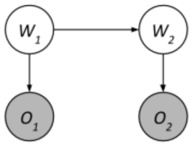
MEU(T = pass) = ***437*** with decision √ ***buy*** ☐ ¬ buy

MEU(T = fail) = ***0*** with decision ☐ buy √ ¬ ***buy***

(3pts) (f) Calculate the value of perfect information of the test and determine whether you should pay to have the test done.

Answer: VPI(T) = (Σ_t Pr(T = t) MEU (T = t)) - MEU(∅) = 0.69 · 437 + 0.31 · 0 - 290 ≈ 11.53
so you shouldn't pay for the test since it costs $50.

6. (HMM) For the following Hidden Markov Model, use the forward algorithm to compute $\Pr(W_2 \mid O_1 = A, O_2 = B)$ by following the steps below.



W_1	$\Pr(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$\Pr(W_{t+1} \mid W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$\Pr(O_t \mid W_t)$
0	A	0.9
0	B	0.1
1	A	0.5
1	B	0.5

(5pts) (a) Compute $\Pr(W_1 = w, O_1 = A)$ for each $w \in \{0, 1\}$.

Answer: $\Pr(W_1, O_1 = A) = \Pr(W_1)\Pr(O_1 = A \mid W_1)$
$\Pr(W_1 = 0, O_1 = A) = (0.3)(0.9) = 0.27$
$\Pr(W_1 = 1, O_1 = A) = (0.7)(0.5) = 0.35$

(5pts) (b) Compute $\Pr(W_2 = w, O_1 = A)$ for $w \in \{0, 1\}$. [*Hint.* Sum $\Pr(x_1, W_2, O_1 = A)$ over x_1, rewriting summands using independence assumptions of the model.]

Answer: $\Pr(W_2, O_1{=}\,A) = \Sigma_{x_1} \Pr(x_1, O_1{=}\,A)\Pr(W_2 \mid x_1)$
$\Pr(W_2 = 0, O_1 = A) = (0.27)(0.4) + (0.35)(0.8) = 0.388$
$\Pr(W_2 = 1, O_1 = A) = (0.27)(0.6) + (0.35)(0.2) = 0.232$

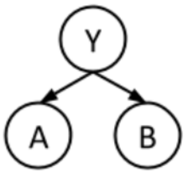
(3pts) (c) Compute $\Pr(W_2 = w, O_1 = A, O_2 = B)$ for each $w \in \{0, 1\}$.

Answer: $\Pr(W_2, O_1{=}\,A, O_2 = B) = \Pr(W_2, O_1 = A)\Pr(O_2 = B \mid W_2)$
$\Pr(W_2 = 0, O_1 = A, O_2 = B) = (0.388)(0.1) = 0.0388$
$\Pr(W_2 = 1, O_1 = A, O_2 = B) = (0.232)(0.5) = 0.116$

(3pts) (d) Compute $\Pr(W_2 = w \mid O_1 = A, O_2 = B)$ for each $w \in \{0, 1\}$. [*Hint.* Normalize.]

Answer: Normalizing the distribution above, we have
$\Pr(W_2 = 0 \mid O_1 = A, O_2 = B) = 0.0388/(0.0388 + 0.116) \approx 0.25$
$\Pr(W_2 = 1 \mid O_1 = A, O_2 = B) = 0.116/(0.0388 + 0.116) \approx 0.75$

7. (Naive Bayes) In this question, you will train a Naive Bayes classifier to predict class labels Y as a function of input features A and B. The variables Y, A, B are binary; i.e., the domain is $\{0, 1\}$ in each case. You are given 10 training points from which you will estimate the distribution.



A	1	1	1	1	0	1	0	1	1	1
B	1	0	0	1	1	1	1	0	1	1
Y	1	1	0	0	0	1	1	0	0	0

(10pts) (a) Given the data in the table above, compute the maximum likelihood estimates for Pr(Y), Pr(A | Y), and Pr(B | Y) and write the results (as fractions, not decimals) in the tables below.

Y	Pr(Y)
0	3/5
1	2/5

A	Y	Pr(A | Y)
0	0	1/6
1	0	5/6
0	1	1/4
1	1	3/4

B	Y	Pr(B | Y)
0	0	1/3
1	0	2/3
0	1	1/4
1	1	3/4

(4pts) (b) Consider a new data point (A = 1, B = 1). What label would this classifier assign to this sample? Write you answer on the line at bottom of the page.
[*Hint*. Compute Pr(Y = y, A = 1, B = 1) for each $y \in \{0, 1\}$.]

Answer:

$$\begin{aligned} P(Y=0, A=1, B=1) &= P(Y=0)P(A=1 \mid Y=0)P(B=1 \mid Y=0) \\ &= (3/5)(5/6)(2/3) \\ &= 1/3 \end{aligned}$$

$$\begin{aligned} P(Y=1, A=1, B=1) &= P(Y=1)P(A=1 \mid Y=1)P(B=1 \mid Y=1) \\ &= (2/5)(3/4)(3/4) \\ &= 9/40 \end{aligned}$$

Classifier predicts label Y = ***0***.

(4pt bonus) (c) Recompute the distribution Pr(A | Y) using Laplace Smoothing with k = 2.

A	Y	Pr(A | Y)
0	0	3/10
1	0	7/10
0	1	3/8
1	1	5/8