1. (14 points) PROBABILITY, PART I

Below is a table listing the probabilities of three binary random variables. Fill in the correct values for each marginal or conditional probability below.

X_0	X_1	X_2	$P(X_0, X_1, X_2)$
0	0	0	0.160
1	0	0	0.100
0	1	0	0.120
1	1	0	0.040
0	0	1	0.180
1	0	1	0.200
0	1	1	0.120
1	1	1	0.080

(a) $P(X_0 = 1, X_1 = 0, X_2 = 1)$

Explanation. (can be read off the table)

(a) ***.2***

(b) $P(X_0 = 0, X_1 = 1)$

Explanation. Compute probability of all configurations of variables X_0, X_1, X_2 that have $X_0 = 0, X_1 = 1$: $P(X_0 = 0, X_1 = 1) = P(X_0 = 0, X_1 = 1, X_2 = 0) + P(X_0 = 0, X_1 = 1, X_2 = 1)$

(b) ***.24***

(c) $P(X_2 = 0)$

Explanation. Compute probability of all configurations of X_0, X_1, X_2 that have $X_2 = 0$: $P(X_2 = 0) = P(X_0 = 0, X_1 = 0, X_2 = 0) + P(X_0 = 0, X_1 = 1, X_2 = 0) + P(X_0 = 1, X_1 = 0, X_2 = 0) + P(X_0 = 1, X_1 = 1, X_2 = 0)$.

(c) ***.42***

(d) $P(X_1 = 0 \mid X_0 = 1)$

Explanation. Use definition of conditional probability to compute $P(X_1 = 0 \mid X_0 = 1) = \frac{P(X_0=1, X_1=0)}{P(X_0=1)}$; calculate $P(X_0 = 1, X_1 = 0)$ and $P(X_0 = 1)$ as in the previous parts.

(d) ***.71428***

(e) $P(X_0 = 1, X_1 = 0 \mid X_2 = 1)$

Explanation. $P(X_0 = 1, X_1 = 0 \mid X_2 = 1) = \frac{P(X_0=1, X_1=0, X_2=1)}{P(X_2=1)}$.

(e) ***.34482***

(f) $P(X_0 = 1 \mid X_1 = 0, X_2 = 1)$

Explanation. $P(X_0 = 1 \mid X_1 = 0, X_2 = 1) = \frac{P(X_0=1, X_1=0, X_2=1)}{P(X_1=0, X_2=1)}$.

(f) ***.52631***

2. (14 points) PROBABILITY, PART II

Given are the prior distribution $P(X)$ and two conditional distributions $P(Y \mid X)$ and $P(Z \mid Y)$ shown below. Also, assume Z is independent of X given Y. All variables are binary (0-1 variables).

Compute the following joint distributions based on the chain rule.

X	$P(X)$
0	0.500
1	0.500

Y	X	$P(Y\|X)$
0	0	0.600
1	0	0.400
0	1	0.900
1	1	0.100

Z	Y	$P(Z\|Y)$
0	0	0.100
1	0	0.900
0	1	0.700
1	1	0.300

(a) $P(X = 0, Y = 0)$

(a) ***0.3***

(b) $P(X = 0, Y = 0, Z = 0)$

(b) ***.03***

(c) $P(X = 0, Y = 1)$

(c) ***0.2***

(d) $P(X = 1, Y = 0, Z = 1)$

(d) ***.405***

(e) $P(X = 1, Y = 0)$

(e) ***.45***

(f) $P(X = 1, Y = 1, Z = 0)$

(f) ***.035***

(g) $P(X = 1, Y = 1)$

(g) ***0.05***

(h) $P(X = 1, Y = 1, Z = 1)$

(h) ***.015***

Explanation. In general, from the chain rule, we have $P(X, Y) = P(X) \times P(Y \mid X)$ and $P(X, Y, Z) = P(X)P(Y \mid X)P(Z \mid X, Y)$. We are given that Z is independent of X given Y, hence we can simplify $P(X, Y, Z) = P(X)P(Y \mid X)P(Z \mid Y)$. We computed $P(X)P(Y \mid X) = P(X, Y)$ in the first part, and can re-use that result in the second part: $P(X, Y, Z) = P(X) \times P(Y \mid X) \times P(Z \mid Y) = P(X, Y) \times P(Z \mid Y)$.

3. (14 points) PROBABILITY, PART III

For each of the following four subparts, you are given three joint probability distribution tables. For each distribution, please identify if the given independence / conditional independence assumption is true or false.

For your convenience, we have also provided some marginal and conditional probability distribution tables that could assist you in solving this problem.

(a) X is independent of Y. $\checkmark$ ***True*** ☐ False

X	Y	$P(X,Y)$
0	0	0.240
1	0	0.160
0	1	0.360
1	1	0.240

X	$P(X)$
0	0.600
1	0.400

Y	$P(Y)$
0	0.400
1	0.600

(b) X is independent of Y. $\checkmark$ ***True*** ☐ False

X	Y	$P(X,Y)$
0	0	0.540
1	0	0.360
0	1	0.060
1	1	0.040

X	$P(X)$
0	0.600
1	0.400

X	Y	$P(X\|Y)$
0	0	0.600
1	0	0.400
0	1	0.600
1	1	0.400

<u>Explanation.</u>

Parts (a) and (b): Two variables X, Y are independent if $P(X,Y) = P(X)P(Y)$ or equivalently, $P(X|Y) = \frac{P(X,Y)}{P(Y)} = P(X)$. The way to solve this problem is to see if $P(X,Y) = P(X)P(Y)$ or $P(X) = P(X|Y)$ for all combinations of X, Y.

(c) X is independent of Y given Z. ☐ True ✓ ***False***

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.280
1	0	0	0.070
0	1	0	0.210
1	1	0	0.140
0	0	1	0.060
1	0	1	0.060
0	1	1	0.030
1	1	1	0.150

X	Z	$P(X\|Z)$
0	0	0.700
1	0	0.300
0	1	0.300
1	1	0.700

Y	Z	$P(Y\|Z)$
0	0	0.500
1	0	0.500
0	1	0.400
1	1	0.600

X	Y	Z	$P(X,Y\|Z)$
0	0	0	0.400
1	0	0	0.100
0	1	0	0.300
1	1	0	0.200
0	0	1	0.200
1	0	1	0.200
0	1	1	0.100
1	1	1	0.500

Explanation. Two variables X, Y are conditionally independent given Z if $P(X,Y|Z) = P(X|Z)P(Y|Z)$ or equivalently, $P(X|Y,Z) = \frac{P(X,Y|Z)}{P(Y|Z)} = P(X|Z)$.

(d) X is independent of Y given Z. $\surd$ ***True*** ☐ False

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.140
1	0	0	0.140
0	1	0	0.060
1	1	0	0.060
0	0	1	0.048
1	0	1	0.192
0	1	1	0.072
1	1	1	0.288

X	Z	$P(X\|Z)$
0	0	0.500
1	0	0.500
0	1	0.200
1	1	0.800

Y	Z	$P(Y\|Z)$
0	0	0.700
1	0	0.300
0	1	0.400
1	1	0.600

X	Y	Z	$P(X,Y\|Z)$
0	0	0	0.350
1	0	0	0.350
0	1	0	0.150
1	1	0	0.150
0	0	1	0.080
1	0	1	0.320
0	1	1	0.120
1	1	1	0.480

Explanation. Two variables X, Y are conditionally independent given Z if $P(X,Y|Z) = P(X|Z)P(Y|Z)$ or equivalently, $P(X|Y,Z) = \frac{P(X,Y|Z)}{P(Y|Z)} = P(X|Z)$.

4. (16 points) Chain Rule

Select all expressions that are equivalent to the specified probability using the given independence assumptions.

(a) Given no independence assumptions, $P(A, B \mid C) =$

- ☐ $\frac{P(C|A)P(A|B)P(B)}{P(C)}$
- ☐ $\frac{P(B,C|A)P(A)}{P(B,C)}$
- √ $P(A \mid B, C)P(B \mid C)$
- ☐ $\frac{P(A|C)P(B,C)}{P(C)}$

Explanation. The first choice is only true if C is independent of B given A. The second is $P(A \mid B, C)$. The third choice is correct. The fourth is only true if A is independent of B given C.

(b) Given that A is independent of B given C, $P(A, B \mid C) =$

- ☐ $\frac{P(C|A)P(A|B)P(B)}{P(C)}$
- ☐ $\frac{P(B,C|A)P(A)}{P(B,C)}$
- √ $P(A \mid B, C)P(B \mid C)$
- √ $\frac{P(A|C)P(B,C)}{P(C)}$

Explanation. The third choice is the same as above, so it is still true with the extra conditions. The fourth choice is true because $P(A \mid C) = P(A \mid B, C)$ when A is independent of B given C, so we end up with $P(A, B, C)/P(C)$.

(c) Given no independence assumptions, $P(A \mid B, C) =$

- ☐ $\frac{P(C|A)P(A|B)P(B)}{P(C)}$
- √ $\frac{P(B,C|A)P(A)}{P(B,C)}$
- ☐ $\frac{P(A|C)P(C|B)P(B)}{P(B,C)}$
- √ $\frac{P(C|A,B)P(B|A)P(A)}{P(B|C)P(C)}$

Explanation. Both the second and third choices are ways of writing $P(A, B, C)/P(B, C)$.

(d) Given that A is independent of B given C, $P(A \mid B, C) =$

- ☐ $\frac{P(C|A)P(A|B)P(B)}{P(C)}$
- √ $\frac{P(B,C|A)P(A)}{P(B,C)}$
- √ $\frac{P(A|C)P(C|B)P(B)}{P(B,C)}$
- √ $\frac{P(C|A,B)P(B|A)P(A)}{P(B|C)P(C)}$

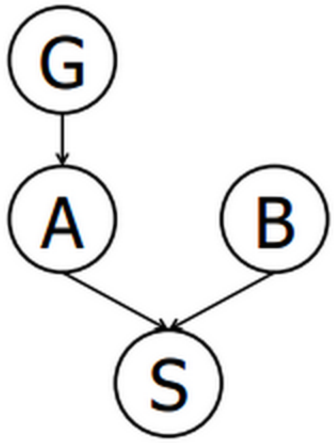
Explanation. In addition to the two cases that are always true, the third option is true with these conditions because $P(A|C) = P(A|C, B)$ when A is independent of B given C.

5. (16 points) BAYES' NETS AND PROBABILITY

Suppose that a patient can have a symptom (S) that can be caused by two different diseases (A and B). It is known that the variation of gene G plays a big role in the manifestation of disease A. The Bayes' Net and corresponding probability tables for this situation are shown below.

G	$P(G)$
g	0.10
$\neg g$	0.90

G	A	$P(A \mid G)$
g	a	1.00
g	$\neg a$	0.00
$\neg g$	a	0.10
$\neg g$	$\neg a$	0.90



B	$P(B)$
b	0.40
$\neg b$	0.60

A	B	S	$P(S \mid A, B)$
a	b	s	1.00
a	b	$\neg s$	0.00
a	$\neg b$	s	0.90
a	$\neg b$	$\neg s$	0.10
$\neg a$	b	s	0.80
$\neg a$	b	$\neg s$	0.20
$\neg a$	$\neg b$	s	0.10
$\neg a$	$\neg b$	$\neg s$	0.90

(a) P(g, a, b, s) = ***.04***

(b) Probability patient has disease A = ***0.19***

(c) Prob. patient has disease A given they have disease B = ***0.19***

(d) Prob. patient has disease A given they have symptom S and disease B = ***0.2267***

(e) Prob. patient has disease carrying variation G given they have disease A = ***0.5263***

(f) Prob. patient has disease carrying variation G given they have disease B = ***0.1***

<u>Explanation.</u>

(a) $P(g, a, b, s) = P(g)P(a|g)P(b)P(s|b, a) = (0.1)(1)(0.4)(1) = 0.04$

(b) $P(a) = P(a|g)P(g) + P(a|g)P(g) = (1.0)(0.1) + (0.1)(0.9) = 0.19$

(c) $P(a|b) = P(a) = 0.19$. Reason: A is independent of B (inferred from the Bayes' net).

(d) $P(a|s, b) = \frac{P(a,b,s)}{P(a,b,s)+P(a,b,s)} = \frac{P(a)P(b)P(s|a,b)}{P(a)P(b)P(s|a,b)+P(a)P(b)P(s|a,b)} = \frac{(0.19)(0.4)(1)}{(0.19)(0.4)1+(0.81)(0.4)(0.8)} = 0.2267$

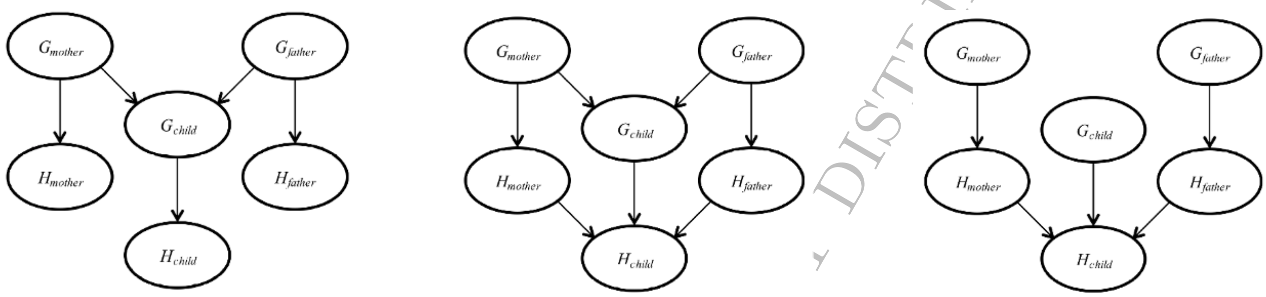
(e) $P(g|a) = \frac{P(g)P(a|g)}{P(g)P(a|g)+P(g)P(a|g)} = \frac{(0.1)1}{(0.1)1+(0.9)(0.1)} = \frac{0.1}{0.1+0.09} = 0.5263$.

(f) $P(g|b) = P(g) = 0.1$ Reason: G is independent of B (inferred from the Bayes' net).

6. (14 points) Bayes' Nets Independence

Let H_x be a random variable denoting the handedness of an individual x, with possible values l or r. A common hypothesis is that left- or right-handedness is inherited by a simple mechanism; that is, perhaps there is a gene G_x, also with values l or r, and perhaps actual handedness turns out mostly the same (with some probability s) as the gene an individual possesses. Furthermore, perhaps the gene itself is equally likely to be inherited from either of an individual's parents, with a small nonzero probability m of a random mutation flipping the handedness.

The following images are possible models involving the genes G and handednesses H.



(a) (b) (c)

i. Which of the three networks above claim that
$P(G_{father}, G_{mother}, G_{child}) = P(G_{father})P(G_{mother})P(G_{child})$?
□ (a) □ (b) √ ***(c)***

Explanation. Only (c) indicates that the three Gs are marginally independent.

ii. Which of the three networks make independence claims that are consistent with the hypothesis about the inheritance of handedness?
√ ***(a)*** √ ***(b)*** □ (c)

Explanation. Both (a) and (b) show that G_{child} is marginally dependent on the genes of the parent and that handedness of each individual is dependent on their genes.

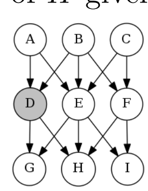
iii. Which of the three networks is the best description of the hypothesis?
√ ***(a)*** □ (b) □ (c)

Explanation. In addition to the independence claims discussed in part 2, (a) also indicates that handedness of the child is independent of the handedness of their parents given the child's gene G_{child}.

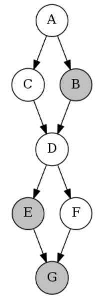
7. (12 points) D-Separation.

Given several graphical models, shown below, each associated with an independence (or conditional independence) assertion; specify whether the assertion is true or false.

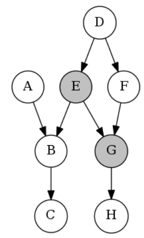
i. It is guaranteed that G is independent of H given D. □ True √ ***False***



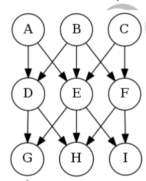
ii. It is guaranteed that A is independent of D given E, B, G. ☐ True √ ***False***



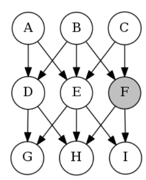
iii. It is guaranteed that H is independent of B given G, E. √ ***True*** ☐ False



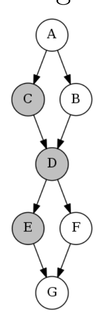
iv. It is guaranteed that A is independent of C. √ ***True*** ☐ False



v. It is guaranteed that D is independent of C given F. ☐ True √ ***False***



vi. It is guaranteed that G is independent of B given C, E, D. √ ***True*** ☐ False



Explanation.

i. For this question, we use the d-separation rules to decide whether there is an active path between the nodes in question. There is an active path from G to H through E, so they are not guaranteed to be independent.

ii. There is an active path from A to D through C.

iii. The observation at E blocks all paths through it and the observation at G also blocks paths to H. Since there are no active paths, H and B are independent given these observations.

iv. A and C are only connected through paths with v-structures, but v-structures are blocked with no observations.

v. The path $D - B - F - C$ is active, so there is no independence guarantee.

vi. The observation at D blocks paths between the upper and lower sections of the graph, so there are no active paths.