단위벡터 ($u = \frac{v}{|v|}$) 만으로만 이루어진 직교 ($u_1 \cdot u_2 = 0$) 집합을 정규직교집합

ex) $v_1 = \langle 1, 0, 2 \rangle$, $v_2 = \langle -2, 0, 1 \rangle$, $v_3 = \langle 0, 1, 0 \rangle$ 일때 $\{v_1, v_2, v_3\}$ 에서 정규직교집합?

$u_1 = \frac{v}{|v_1|} = \langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \rangle$ $u_2 = (\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$, $u_3 = (0, 1, 0)$

$v_3 = u_3$ $\therefore \{u_1, u_2, u_3\}$

ex) $S = \{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle, \langle -\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \rangle, \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle\}$ 정규직교집합 증명

$u_1 \cdot u_2 = \left(\frac{1}{\sqrt{2}} \times -\frac{\sqrt{2}}{6}\right) + \frac{1}{6}) = 0$ $u_2 \cdot u_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$

$u_1 \cdot u_3 = \left(\frac{1}{\sqrt{2}} \cdot \frac{2}{3}\right) + \left(\frac{1}{\sqrt{2}} \cdot -\frac{2}{3}\right) = 0$

$S = \{v_1, v_2, \cdots v_n\}$ 이 내적공간 V에 대한 정규직교기저 이여 v가 V내의 임의의 벡터일 때

$v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \cdots + \langle v, v_n \rangle v_n$

ex) $v_1 = \langle 0, 1, 0 \rangle$, $v_2 = \langle -\frac{4}{5}, 0, \frac{3}{5} \rangle$, $v_3 = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle$ 일때 정규직교집합.

$v = \langle 1, 1, 1 \rangle$를 S 내의 벡터들의 1차 결합으로 표시

$\langle v, v_1 \rangle = 1$ $\langle v, v_2 \rangle = -\frac{1}{5}$ $\langle v, v_3 \rangle = \frac{7}{5}$

$v = v_1 - \frac{1}{5} v_2 + \frac{7}{5} v_3 = \langle 0, 1, 0 \rangle - \frac{1}{5} \langle -\frac{4}{5}, 0, \frac{3}{5} \rangle + \frac{7}{5} \langle \frac{3}{5}, 0, \frac{4}{5} \rangle$

Gram-schmidt 정규직교화 과정

$B = \{v_1, v_2 \cdots v_n\}$, $B' = \{w_1, w_2 \cdots w_n\}$

$w_1 = v_1$

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$ $\Rightarrow proj_v w$

$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$

$\therefore B' \Rightarrow$ 직교기저

ex) R^3 에서 두 벡터 $v_1 = \langle 0,1,0 \rangle$, $v_2 = \langle 1,1,1 \rangle$ 은 한 평면 생성.

부분공간에 대한 정규직교기저 구하라

$w_1 = v_1 = \langle 0,1,0 \rangle$

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \langle 1,1,1 \rangle - \frac{1}{1} \langle 0,1,0 \rangle = \langle 1,0,1 \rangle$

$u_1 = \frac{w_1}{|w_1|} = \langle 0,1,0 \rangle \quad u_2 = \frac{w_2}{|w_2|} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

ex) 해공간에대한 정규직교기저 구하라

$\begin{bmatrix} 1 & 1 & 0 & 7 \\ 2 & 1 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 7 \\ 0 & 1 & -2 & 8 \end{bmatrix}$ $\quad x_3 = s, \ x_4 = t \quad x_1 + 2s - 8t + 7t = -2s + t$

$x_2 = 2s - 8t$

$\begin{bmatrix} -2s+t \\ 2s-8t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -8 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \therefore B = \{ \langle 1\ -8\ 0\ 1 \rangle, \langle -2\ 2\ 1\ 0 \rangle \}$

$-2 \ -16 + 1$

$w_1 = v_1 = \langle -2\ 2\ 1\ 0 \rangle$

$w_2 = v_2 - \frac{\langle v_2\ w_1 \rangle}{\langle w_1\ w_1 \rangle} w_1 = \langle 1\ -8\ 0\ 1 \rangle - \frac{(-18)}{9} \langle -2\ 2\ 1\ 0 \rangle$

$= \langle 1\ -8\ 0\ 1 \rangle + \langle -4\ 4\ 2\ 0 \rangle = \langle -3\ -4\ 2\ 1 \rangle$

$\therefore u_1 = \frac{1}{3} \langle -2\ 2\ 1\ 0 \rangle \quad u_2 = \frac{1}{\sqrt{30}} \langle -3\ -4\ 2\ 1 \rangle$

연습문제

2-3 정규직교 기저로 변환해라

$B = \{ \langle 4, -3, 0 \rangle, \langle 1, 2, 0 \rangle, \langle 0, 0, 4 \rangle \}$

$v_1 = \langle 4\ -3\ 0 \rangle \quad v_2 = \langle 1\ 2\ 0 \rangle \quad v_3 = \langle 0\ 0\ 4 \rangle$

$w_1 = v_1 = \langle 4\ -3\ 0 \rangle$

$w_2 = v_2 - \frac{\langle w_1\ v_2 \rangle}{\langle w_1\ w_1 \rangle} w_1 = \langle 1\ 2\ 0 \rangle - \frac{-2}{25} \langle 4\ -3\ 0 \rangle = \langle \frac{33}{25}\ \frac{44}{25}\ 0 \rangle$

$V_1 \cdot V_3 = 0 \quad V_2 \cdot V_3 = 0$ 이므로 이미 직교 기저

$\therefore u_1 = \frac{1}{5}\langle 4\ {-3}\ 0\rangle \quad u_2 = \frac{v_2}{\|v_2\|} = \frac{5}{11}\langle \frac{33}{25}\ \frac{44}{25}\ 0\rangle = \langle \frac{3}{5}\ \frac{4}{5}\ 0\rangle$

4-1 $2x_1 + 2x_2 - 6x_3 + 4x_4 = 0$
$x_1 + 2x_2 - 3x_3 + 4x_4 = 0$
$x_1 + x_2 - 3x_3 + 2x_4 = 0$

$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 1 & 2 & -3 & 2 \\ 1 & 1 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x_3 = t \quad x_4 = s \quad x_2 = 0 \quad x_1 - 3t + 2s = 0 \quad x_1 = 3t - 2s$

$\begin{bmatrix} 3t-2s \\ 0 \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

1. 정규 직교 인지 판단

1-1 $\{\langle -4\ 6\rangle, \langle 5\ 0\rangle\}$ 정규직교X

1-2 $\{\langle \frac{3}{5}\ \frac{4}{5}\rangle, \langle -\frac{4}{5}\ \frac{3}{5}\rangle\}$ $-\frac{12}{25} + \frac{12}{25}$ 정규직교

1-3 $\{\langle 4\ {-1}\ 1\rangle, \langle -1\ 0\ 4\rangle, \langle -4\ {-17}, -1\rangle\}$

$16 + 0 + 4 = 12$ 정규직교X

1-4 $\{\langle \frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\rangle, \langle 0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\rangle, \langle \frac{\sqrt{5}}{5}, 0, \frac{1}{2}\rangle\}$

$0 + 0 + x$ $\therefore$ 정규 직교 X

1-5 $\{\langle \frac{\sqrt{2}}{2}\ 0\ 0\ \frac{\sqrt{2}}{2}\rangle, \langle 0\ \frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\ 0\rangle, \langle -\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\rangle\}$ 정규직교

2 정규직교기저로 변환

2-1 $B = \{\langle 3\ 4\rangle, \langle 1\ 0\rangle\}$

$w_1 = v_1 = \langle 3\ 4\rangle$
$w_2 = v_2 - \frac{\langle v_2\ w_1\rangle}{\langle w_1\ w_1\rangle} w_1 = \langle 1\ 0\rangle - \frac{3}{5}\langle 3\ 4\rangle$
$= \langle 1\ 0\rangle - \langle \frac{9}{5}\ \frac{12}{5}\rangle = \langle -\frac{4}{5}\ -\frac{12}{5}\rangle$

2-2 $B = \{\langle 1\ {-2}\ 2\rangle\ \langle 2\ 2\ 1\rangle\ \langle 2\ {-1}\ {-2}\rangle\}$

$u \cdot v = 0 \quad u \cdot w = 0 \quad v \cdot w = 0 \quad \therefore \langle 1\ {-2}\ 2\rangle, \langle 2\ 2\ 1\rangle, \langle 2\ {-1}\ {-2}\rangle$

3. 정규직교화 변환

3-1 $\{\langle -8\ 3\ 5\rangle\} \rightarrow \frac{1}{\sqrt{64+9+25}}\langle -8\ 3\ 5\rangle$

3-3 $B=\{\langle 1\ 2\ -1\ 0\rangle, \langle 2\ 2\ 0\ 1\rangle\}$

$w_1 = v_1 = \langle 1\ 2\ -1\ 0\rangle$

$w_2 = v_2 - \frac{\langle v_2\ w_1\rangle}{\langle w_1\ w_1\rangle} w_1 = \langle 2\ 2\ 0\ 1\rangle - \frac{6}{6}\langle 1\ 2\ -1\ 0\rangle$

$= \langle 1\ 0\ 1\ 1\rangle$

$u_1 = \frac{1}{\sqrt{6}}\langle 1\ 2\ -1\ 0\rangle$

$u_2 = \frac{1}{\sqrt{3}}\langle 1\ 0\ 1\ 1\rangle$

4-2 $x_1 + x_2 - x_3 - x_4 = 0$

$2x_1 + x_2 - 2x_3 - 2x_4 = 0$

$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $\quad x_3 = s,\ x_4 = t \quad x_2 = 0,\ x_1 = t+s \quad s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$v_1 = \langle 1\ 0\ 1\ 0\rangle \quad v_2 = \langle 1\ 0\ 0\ 1\rangle$

$w_1 = v_1 = \langle 1\ 0\ 1\ 0\rangle \rightarrow u_1 = \frac{1}{\sqrt{2}}\langle 1\ 0\ 1\ 0\rangle$

$w_2 = v_2 - \frac{\langle v_2\ w_1\rangle}{\langle w_1\ w_1\rangle} w_1 = \langle 1\ 0\ 0\ 1\rangle - \frac{1}{2}\langle 1\ 0\ 1\ 0\rangle = \langle \frac{1}{2}\ 0\ -\frac{1}{2}\ 1\rangle$

$\downarrow$

$\frac{\sqrt{6}}{3}\langle \frac{1}{2}\ 0\ -\frac{1}{2}\ 1\rangle \leftarrow \frac{1}{\sqrt{\frac{3}{2}}}\langle \frac{1}{2}\ 0\ -\frac{1}{2}\ 1\rangle$