

고유치

$$Ax = \lambda x$$

스칼라 λ 가 고유치, x 는 고유 벡터ex) 고유치와 고유 벡터 구하라 $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4 = 0 \quad \lambda = 4 \text{ or } -1$$

$$\begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad 3x_1 - 2x_2 = 0 \quad x_1 = \frac{2}{3}x_2 \quad x_1 = \begin{bmatrix} \frac{2}{3}t \\ t \end{bmatrix} \sim x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ -3 & -3 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ex) $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ 에 대응하는 고유 벡터 구하라

$$\begin{bmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{bmatrix} \rightarrow \lambda^2 + 3\lambda - 10 + 12 = \lambda^2 + 3\lambda + 2 \quad \lambda = -1 \text{ or } \lambda = -2$$

$$\lambda = -1 \text{ 일 때 } \begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \quad x_1 = 4x_2 \quad \begin{bmatrix} 4t \\ t \end{bmatrix} \quad \therefore x_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\lambda = -2 \text{ 일 때 } \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \quad x_1 = 3x_2 \quad x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

ex) $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ 고유치와 고유 벡터

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{bmatrix}$$

$$(\lambda^2 + \lambda - 20 + 18) \times (\lambda - 1) + (-3)(-3\lambda + 12 - 18) + 3(-18 + 6(\lambda + 5))$$

$$\Rightarrow (\lambda^2 + \lambda - 2)(\lambda - 1) + 9\lambda + 18 + 18\lambda + 36$$

$$\frac{\lambda^3 + \lambda^2 - 2\lambda - \lambda^2}{-\lambda + 2} + 27\lambda + 54$$

$$= \lambda^3 - 3\lambda + 27\lambda + 54 = \lambda^3 + 24\lambda + 54$$

$$\Rightarrow (\lambda + 2)^2 (\lambda - 4) = 0$$

$$\textcircled{1} \lambda = -2 \quad \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -6 & 6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 - x_2 + x_3 = 0 \quad \begin{matrix} x_2 = s \\ x_3 = t \\ x_1 = s - t \end{matrix} \quad s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\textcircled{2} \lambda = 4 \quad \begin{bmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 & -3 \\ 6 & -1 & 0 \\ -6 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_2 = t \\ x_1 = t \\ x_3 = 2t \end{matrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

연습문제

$$1-1 \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} \rightarrow (\lambda^2 - 1) = 0 \quad \lambda = 1 \text{ or } \lambda = -1$$

$$\lambda = 1 \quad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad x_2 = t \quad x_1 = 0 \quad \therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad x_2 = t \quad x_1 = 0 \quad \therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$1-2 \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0 \quad \lambda(\lambda - 2) = 0 \quad \lambda = 0 \text{ or } 2$$

$$\lambda = 0 \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 2 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$2-1 \quad A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} \rightarrow \lambda^2 - 11\lambda + 28 - 4 = \lambda^2 - 11\lambda + 24 = 0 \quad \lambda = 3 \text{ or } 8$$

$$\lambda = 3 \quad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \quad 2x_1 = x_2 \quad x_2 = t \quad x_1 = \frac{1}{2}t \quad t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \therefore (a)$$

$$\lambda = 8 \quad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad x_2 = t \quad x_1 = 2t \quad t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \therefore (b)$$

$$2-2 \quad \begin{bmatrix} -1 & -1 & -1 \\ -2 & 0 & -2 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & -1 & -1 \\ -2 & -\lambda & -2 \\ 3 & 3 & 1-\lambda \end{vmatrix} = -(\lambda+1)(-\lambda(-\lambda)-6) + (-1)(-1)(-2+2\lambda+6) + 6+3\lambda$$

$$= -(\lambda+1)(\lambda^2 - \lambda - 6) + 2\lambda + 4 + 6 + 3\lambda$$

$$= -(\lambda^3 - \lambda^2 - 6\lambda + \lambda^2 - \lambda - 6) + 5\lambda + 10$$

$$= -\lambda^3 + 7\lambda + 6 + 5\lambda + 10$$

$$= -\lambda^3 + 12\lambda + 16 = \lambda^3 - 12\lambda - 16$$

$$\begin{vmatrix} 1 & 0 & -12 & -16 \\ -2 & -4 & 16 \\ 1 & 2 & -8 & 0 \end{vmatrix}$$

$$(\lambda^2 + 2\lambda - 8)(\lambda + 2) = (\lambda - 2)(\lambda + 4)(\lambda + 2)$$

$$\lambda=2 \quad \begin{bmatrix} -3 & -1 & 1 \\ -2 & -2 & -2 \\ 3 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \\ 3 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -5 \end{bmatrix} \dots$$

$$3-1 \quad \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda-6 & 3 \\ 2 & \lambda-1 \end{bmatrix} \quad \lambda^2 - 7\lambda + 6 = 0 \quad \lambda(\lambda-7) = 0 \quad \lambda = 0 \text{ or } 7$$

$$\lambda=0 \quad \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \quad x_2 = t \quad x_1 = \frac{t}{2} \quad x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda=7 \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad x_2 = t \quad x_1 = -3t \quad x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$3-2 \quad \begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-1 & \frac{3}{2} \\ -\frac{1}{2} & \lambda+1 \end{vmatrix} \Rightarrow \lambda^2 - 1 + \frac{3}{4} = \lambda^2 - \frac{1}{4} = (\lambda + \frac{1}{2})(\lambda - \frac{1}{2}) \quad \lambda = -\frac{1}{2} \text{ or } \frac{1}{2}$$

$$\lambda = \frac{1}{2} \quad \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \quad \therefore \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\lambda = -\frac{1}{2} \quad \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$3-3 \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{vmatrix} \lambda-2 & 0 & -1 \\ 0 & \lambda-3 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-2)(\lambda-3)(\lambda-1) = (\lambda-2)(\lambda^2 - 4\lambda + 3) = \lambda^3 - 4\lambda^2 + 3\lambda - 2\lambda^2 + 8\lambda - 6 = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$$(\lambda-3)(\lambda-1)(\lambda-2)$$

$$\begin{array}{r} 1 \quad -6 \quad 11 \quad -6 \\ 2 \quad \underline{2 \quad -8 \quad 6} \\ 1 \quad -4 \quad 3 \quad 0 \\ 1 \quad \underline{1 \quad -3} \\ 1 \quad -3 \quad 0 \end{array}$$

$$\lambda=1 \quad \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 = t \quad x_2 = -2t \quad x_1 = -t \quad \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda=2 \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = t \quad x_2 = 0 \quad x_3 = 0 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda=3 \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = x_3 = 0 \quad x_2 = t \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$3-4 \quad \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-1 & -2 & 2 \\ 2 & \lambda-5 & 2 \\ 6 & -6 & \lambda+3 \end{vmatrix} \rightarrow (\lambda+3)(\lambda-3)^2 = 0$$

$$\lambda=3 \quad \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 = s \quad x_1 = 5-t \quad x_2 = t \quad s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\lambda = -3 \quad \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 1 \\ 1 & -4 & 1 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{4}{3}t + t = 0 \quad x_3 = t \quad x_2 = \frac{t}{3}$$

$$x_1 = \frac{t}{3} \quad \therefore \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$4-1 \quad A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & 1 \\ -3 & \lambda + 2 \end{bmatrix} \quad \lambda^2 - 4 + 3 = (\lambda - 1)(\lambda + 1)$$

$$\lambda = 1 \quad \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1 \quad \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = t \\ -3t = x_2 \\ t \quad x_3 = 3 \end{matrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$4-2 \quad \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & -1 \\ -3 & \lambda + 2 \end{bmatrix} \rightarrow \lambda^2 - 4 - 3 = \lambda^2 - 7 = 0 \quad \lambda = \pm \sqrt{7}$$

$$4-3 \quad \begin{bmatrix} 2t & t \\ t & 2t \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2t & -t \\ -t & \lambda - 2t \end{bmatrix} \rightarrow \lambda^2 - 4t + 4t^2 - t^2 = (\lambda - t)(\lambda - 3t) \quad \lambda = t \text{ or } 3t$$

$$\lambda = t \quad \begin{bmatrix} -t & -t \\ -t & -t \end{bmatrix} \rightarrow \begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -x_2 \\ t = -t \end{matrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 3t \quad \begin{bmatrix} t & -t \\ -t & t \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$4-4 \quad \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 3 & -1 & -2 \\ 0 & \lambda & 0 \\ -1 & -1 & \lambda - 2 \end{bmatrix} \quad \lambda((\lambda - 3)(\lambda - 2) - 2) = \lambda(\lambda^2 - 5\lambda + 4) = \lambda(\lambda - 1)(\lambda - 4)$$

$$\lambda = 4 \quad \begin{bmatrix} 1 & -1 & -2 \\ 0 & 4 & 0 \\ -1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix} \quad x_1 = 2t \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 1 \quad \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_3 = 1 \\ x_2 = 0 \end{matrix} \quad x_1 = -1 \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 0 \quad \begin{bmatrix} -3 & -1 & -2 \\ 0 & 0 & 0 \\ -1 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = 0 \\ x_3 = t \end{matrix} \quad x_2 = -2t \quad \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$