역행렬이 존재하는 행렬을 **정칙행렬**이라고 함

예1) $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{(-1\cdot 1)-(2\cdot -1)}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = B$$

예2) $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$ 역행렬은?

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_1 - R_2 \to R_2 \quad \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$6R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \quad 4R_2 - R_3 \to R_3 \quad \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 4 & 1 \end{array}\right]$$

$$-R_3 \to R_3 \quad \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_2 + R_3 \to R_2 \quad \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad \therefore A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

A가 가역이고 k는 양의 정수이고 c는 스칼라라고 할 때

1) $(A^{-1})^{-1} = A$ 2) $(A^k)^{-1} = (A^{-1})^k$

3) $(cA)^{-1} = \frac{1}{c}A^{-1}$ 4) $(A^T)^{-1} = (A^{-1})^T$

A, B 가 정칙행렬일 때 $(AB)^{-1} = B^{-1}A^{-1}$

C 가 정칙행렬 이면 $AC = BC$ 일때 $A = B$, $CA = CB$ 일때 $A = B$

A가 정칙이면 $Ax = b$는 유일해 $x = A^{-1}b$ 가 짐

1. 역행렬 구하라

1) $\begin{bmatrix}1&3\\2&7\end{bmatrix} \Rightarrow \begin{bmatrix}7&-3\\-2&1\end{bmatrix}$ 2) $\begin{bmatrix}-7&33\\4&-19\end{bmatrix} \Rightarrow \begin{bmatrix}-19&-33\\-4&-7\end{bmatrix}$

3) $\begin{bmatrix}1&1&1\\3&5&4\\3&6&5\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\3&5&4&0&1&0\\3&6&5&0&0&1\end{array}\right] \xrightarrow{R_2-R_3\to R_3} \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\3&5&4&0&1&0\\0&-1&-1&0&1&-1\end{array}\right] \xrightarrow{3R_1-R_2\to R_2} \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-2&-1&3&-1&0\\0&-1&-1&0&1&-1\end{array}\right]$

$-R_2\to R_2,\ -R_3\to R_3$ $\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&2&1&-3&1&0\\0&1&1&0&-1&1\end{array}\right] \xrightarrow{R_2-R_3\to R_2} \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-3&2&-1\\0&1&1&0&-1&1\end{array}\right]$

$R_3-R_2\to R_3$ $\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-3&2&-1\\0&0&1&3&-3&2\end{array}\right] \xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|ccc}1&0&1&4&-2&1\\0&1&0&-3&2&-1\\0&0&1&3&-3&2\end{array}\right] \xrightarrow{R_1-R_3\to R_1} \left[\begin{array}{ccc|ccc}1&0&0&1&1&-1\\0&1&0&-3&2&-1\\0&0&1&3&-3&2\end{array}\right]$ $\therefore \begin{bmatrix}1&1&-1\\-3&2&-1\\3&-3&2\end{bmatrix}$

4) $\begin{bmatrix}1&2&-1\\3&7&-10\\7&16&21\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\3&7&-10&0&1&0\\7&16&21&0&0&1\end{array}\right] \xrightarrow{3R_1-R_2\to R_2} \left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-1&7&3&-1&0\\7&16&21&0&0&1\end{array}\right] \xrightarrow{7R_1-R_3\to R_3} \left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-1&7&3&-1&0\\0&-2&-28&7&0&-1\end{array}\right]$

$2R_2-R_3\to R_3$ $\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-1&7&3&-1&0\\0&0&42&-1&-2&1\end{array}\right] \Rightarrow$ 역행렬 X

5) $\begin{bmatrix}1&1&2\\3&1&0\\-2&0&3\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\3&1&0&0&1&0\\-2&0&3&0&0&1\end{array}\right] \xrightarrow{3R_1-R_2\to R_2} \left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&2&6&3&-1&0\\-2&0&3&0&0&1\end{array}\right] \xrightarrow{2R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&2&6&3&-1&0\\0&2&7&2&0&1\end{array}\right]$

$R_3-R_2\to R_3$ $\left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&2&6&3&-1&0\\0&0&1&-1&1&1\end{array}\right] \xrightarrow{\frac{R_2}{2}\to R_2} \left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&3&\frac{3}{2}&-\frac{1}{2}&0\\0&0&1&-1&1&1\end{array}\right] \xrightarrow{3R_3-R_2\to R_2} \left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&0&-\frac{3}{2}&\frac{7}{2}&0\\0&0&1&-1&1&1\end{array}\right]$

$R_1-R_2\to R_1$ $\left[\begin{array}{ccc|ccc}1&0&2&\frac{11}{2}&-\frac{7}{2}&0\\0&1&0&-\frac{3}{2}&\frac{7}{2}&0\\0&0&1&-1&1&1\end{array}\right] \xrightarrow{R_1-2R_3\to R_1} \left[\begin{array}{ccc|ccc}1&0&0&\frac{15}{2}&-\frac{11}{2}&-2\\0&1&0&-\frac{3}{2}&\frac{7}{2}&0\\0&0&1&-1&1&1\end{array}\right]$ $\therefore \begin{bmatrix}\frac{15}{2}&-\frac{11}{2}&0\\-\frac{3}{2}&\frac{7}{2}&0\\-1&1&1\end{bmatrix}$

2. $-x+y=4$, $-2x+y=0$ $\Rightarrow \begin{bmatrix}-1&1\\-2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1&1\\-2&1\end{bmatrix}^{-1}\begin{bmatrix}4\\0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1&-1\\2&-1\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}=\begin{bmatrix}4\\8\end{bmatrix}$ $\therefore x=4\ y=8$

2-2 $-x+y=0$, $-2x+y=0$ $\Rightarrow \begin{bmatrix}-1&1\\-2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&-1\\2&-1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$ $x=0,\ y=0$

2-3 $\begin{bmatrix}3&2&2\\2&2&2\\-4&4&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\5\\2\end{bmatrix}$ $\left[\begin{array}{ccc|ccc}3&2&2&1&0&0\\2&2&2&0&1&0\\-4&4&3&0&0&1\end{array}\right] \xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\2&2&2&0&1&0\\-4&4&3&0&0&1\end{array}\right]$

$2R_2+R_3\to R_3$ $\left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\2&2&2&0&1&0\\0&8&7&0&2&1\end{array}\right] \xrightarrow{\frac{R_2}{2}\to R_2} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\1&1&1&0&\frac{1}{2}&0\\0&8&7&0&2&1\end{array}\right] \xrightarrow{R_1-R_2\to R_2} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&1&1&-\frac{3}{2}&0\\0&8&7&0&2&1\end{array}\right]$

$8R_2-R_3\to R_3$ $\left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&1&1&-\frac{3}{2}&0\\0&0&1&8&-14&-1\end{array}\right] \xrightarrow{R_2-R_3\to R_2} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&7&\frac{31}{2}&-1\\0&0&1&8&-14&-1\end{array}\right]$ $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&-1&0\\7&\frac{31}{2}&-1\\8&-14&-1\end{bmatrix}\begin{bmatrix}0\\5\\2\end{bmatrix}$ $x=-5$, $y=75.5$, $z=-72$

2-4 $\begin{bmatrix}3&2&2\\2&2&2\\-4&4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\left[\begin{array}{ccc|ccc}3&2&2&1&0&0\\2&2&2&0&1&0\\-4&4&3&0&0&1\end{array}\right] \xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\2&2&2&0&1&0\\-4&4&3&0&0&1\end{array}\right] \xrightarrow{2R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\2&2&2&0&1&0\\0&8&7&0&2&1\end{array}\right]$

$R_2-2R_1\to R_2$ $\left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&2&2&-2&3&0\\0&8&7&0&2&1\end{array}\right] \xrightarrow{4R_2-R_3\to R_3} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&2&2&-2&3&0\\0&0&1&-8&10&-1\end{array}\right] \xrightarrow{\frac{R_2}{2}} \left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&1&-1&\frac{5}{2}&0\\0&0&1&-8&-10&-1\end{array}\right]$

$R_2-R_3\to R_2$ $\left[\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&-1&\frac{5}{2}&0\\0&0&1&7&\frac{5}{2}&1\end{array}\right]$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&-1&0\\-1&\frac{5}{2}&0\\7&\frac{5}{2}&1\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}$ $x_1=x_2=x_3=0$

3. $A = A^{-1}$ 인 x를 구하라 $A = \begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{-9+2x}\begin{bmatrix} -3 & -x \\ 2 & 3 \end{bmatrix}$ $\quad -9+2x = -1 \quad 2x=8 \quad \underline{x=4}$

4. A 구하라 $(2A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\frac{1}{2}A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \frac{1}{2}A^{-1}\cdot A = \frac{1}{2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ $\quad A = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$ $\quad A = \frac{1}{2}\cdot -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$A = -\frac{1}{4}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

5. $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ 정칙인가? A^{-1}?

$\sin^2\theta + \cos^2\theta = 1$ 이므로 정칙 $\quad A^{-1} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

6. $A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$ 어떤 조건에서 정칙? A가 가역이면 A^{-1}?

$a_{11}\cdot a_{22}\cdots a_{nn} \neq 0$ 일때 정칙 $\quad A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 & 0 \\ \cdots & & \cdots & \\ 0 & 0 & 0 & \frac{1}{a_{nn}} \end{bmatrix}$

7-1 $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ A^{-1}? $\quad A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

7-2 $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ A^{-1}? $\quad A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

7-3 $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ A^{-1}? $\quad A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ A^{-3}? $A^2 - 2A + I$?

$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$ $\quad A^{-2} = \frac{1}{9}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ -8 & 1 \end{bmatrix}$ $\quad A^{-3} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ -8 & 1 \end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$

$= \underline{\frac{1}{27}\begin{bmatrix} 27 & 0 \\ -26 & 1 \end{bmatrix}} = A^{-3}$

$A^2 - 2A + I = (A-I)^2 = \left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^2 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}$

9. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ A^{-1}?

$\frac{1}{\sin^2\theta + \cos^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

10-1 $x + y + 2z = -1$

$x + 2y + 3z = -5$

$3x + y + z = 3$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & -1 \\ 1 & 2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right] \quad \begin{array}{c} R_2 - R_1 \\ \to R_2 \end{array} \left[\begin{array}{ccc:c} 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & -4 \\ 3 & 1 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} 3R_1 - R_3 \\ \to R_3 \end{array} \left[\begin{array}{ccc:c} 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & -4 \\ 0 & 2 & 5 & 6 \end{array}\right] \quad \begin{array}{c} 2R_2 - R_3 \\ \to R_3 \end{array} \left[\begin{array}{ccc:c} 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$y - 2 = -4$

$z = 2$ $y = -2$

$x - 2 + 4 = -1$ $x = -3$

$\therefore x = -3$ $y = -2$

$z = 2$