n 차 정방행렬 A가 n 개의 1차 독립인 고유 벡터를 가지면

$A = PDP^{-1}$ 이다. P= 고유벡터 , D= 대각행렬 $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

ex) $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 일 때 $A = PDP^{-1}$ 가 되는 D랑 P 구하라

$\begin{bmatrix} \lambda-3 & -2 & 0 \\ -2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix}$ $(\lambda-1)\left((\lambda-3)^2-4\right) = (\lambda-1)(\lambda^2-6\lambda+5) = (\lambda-1)(\lambda-1)(\lambda-5)$ $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 5$

$\lambda = 1$ 일 때 $\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_3 = t$ $x_2 = s$ $x_1 = -s$ $s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\therefore \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 5$ 일 때 $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_3 = 0$ $x_1 = t$ $x_2 = t$ $\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

ex) $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ A^{99} ?

$A = PDP^{-1}$ $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$

$\therefore A^{99} = PD^{99}P^{-1}$

$\begin{vmatrix} \lambda-3 & -2 & 0 \\ -2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\left((\lambda-3)^2-4\right) = (\lambda-1)(\lambda^2-6\lambda+5) = (\lambda-1)^2(\lambda-5)$

$\lambda = 1$ $\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\lambda = 5$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5^{99} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5^{99}+1 & 5^{99}-1 & 0 \\ 5^{99}-1 & 5^{99}+1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

ex) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ A^{915} 구하라

$\begin{vmatrix} \lambda-1 & -2 \\ -4 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5)$

$\lambda = -1$ $\begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x_2 = t$ $x_1 = -t$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 5$ $\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $x_2 = t$ $x_1 = \frac{t}{2}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$ $P = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$

$P^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ $\frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5^{915} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5^{915}-2 & 5^{915}+1 \\ 2 \cdot 5^{915}+2 & 2 \cdot 5^{915}-1 \end{bmatrix}$

ex) $A=\begin{bmatrix}1&2\\4&3\end{bmatrix}$ $4A^{15}+2A^{7}+I$?

$4PD^{15}P^{-1}+2PD^{7}P^{-1}+PP^{-1}$

$=P(4D^{15}+2D^{7}+I)P^{-1}$ $D=\begin{bmatrix}-1&0\\0&5\end{bmatrix}$ $P=\begin{bmatrix}-1&1\\1&2\end{bmatrix}$ $P^{-1}=\frac{1}{3}\begin{bmatrix}-2&1\\1&1\end{bmatrix}$

$P\left(4\begin{pmatrix}-1&0\\0&5^{15}\end{pmatrix}+2\begin{pmatrix}-1&0\\0&5^{7}\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)P^{-1}$

$=P\begin{bmatrix}-4-7&0\\0&4\cdot5^{15}+2\cdot5^{7}+1\end{bmatrix}P^{-1}$ $=\frac{1}{3}\begin{bmatrix}4\cdot5^{15}+2\cdot5^{7}-9 & 4\cdot5^{15}+2\cdot5^{7}+6\\ 8\cdot5^{15}+4\cdot5^{7}+12 & 8\cdot5^{15}+4\cdot5^{7}-3\end{bmatrix}$

ex) $A=PDP^{-1}$ 성립하는 대각행렬 D와 정칙행렬 P 구하라

1) $A=\begin{bmatrix}1&0&0\\4&3&2\\4&2&3\end{bmatrix}$

$\begin{bmatrix}\lambda-1&0&0\\-4&\lambda-3&-2\\-4&-2&\lambda-3\end{bmatrix}\rightarrow(\lambda-1)\left((\lambda-3)^2-4\right)=(\lambda-1)(\lambda^2-6\lambda+5)=(\lambda-1)^2(\lambda-5)$

$\lambda=1$ $\begin{bmatrix}0&0&0\\-4&-2&-2\\-4&-2&-2\end{bmatrix}\rightarrow\begin{bmatrix}2&1&1\\0&0&0\\0&0&0\end{bmatrix}$ $x_2=s$, $x_3=t$ $2x_1=-s-t$, $x_1=-\frac{s+t}{2}$ $s\begin{bmatrix}-\frac{1}{2}\\1\\0\end{bmatrix}+t\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}\Rightarrow\begin{bmatrix}-1\\2\\0\end{bmatrix}\begin{bmatrix}-1\\0\\2\end{bmatrix}$

$\lambda=5$ $\begin{bmatrix}4&0&0\\-4&2&-2\\-4&-2&2\end{bmatrix}\rightarrow\begin{bmatrix}4&0&0\\0&4&-4\\-4&-2&2\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0\\0&1&-1\\-2&-1&1\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}$ $x_3=t$, $x_2=t$ $x_1=0$ $\begin{bmatrix}0\\1\\1\end{bmatrix}$

연습문제

1. 대각행렬 D와 정칙 P가 있는지 확인해라

1-1 $A=\begin{bmatrix}19&-9&-6\\25&-11&-9\\17&-9&-4\end{bmatrix}$ X

1-2 $A=\begin{bmatrix}-1&4&-2\\-3&4&0\\-3&1&3\end{bmatrix}$

$\begin{vmatrix}\lambda+1&-4&2\\3&\lambda-4&0\\3&-1&\lambda-3\end{vmatrix}=2(3-3(\lambda-4))+(\lambda-3)(\lambda^2-3\lambda+8)=-6\lambda+18+\lambda^3-3\lambda^2+8\lambda-3\lambda^2+9\lambda-24$

$=\lambda^3-6\lambda^2+11\lambda-6$

$=(\lambda-1)(\lambda-2)(\lambda-3)$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 11 & -6\\ & & 2 & -8 & 6\\ \hline 1 & 1 & -4 & 3 & 0\\ & & 1 & -3 & \\ \hline & 1 & -3 & 0 & \end{array}$$

∴ 존재

1-4 $\begin{bmatrix}0&0&0\\0&0&0\\3&0&1\end{bmatrix}$ $\begin{vmatrix}\lambda&0&0\\0&\lambda&0\\-3&0&\lambda-1\end{vmatrix}=(\lambda-1)\lambda^2=0$ $\lambda=0$ or 1 ∴ 존재

2. $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

2-1 A^{27} ?

$\begin{bmatrix} \lambda & -1 \\ 2 & \lambda-3 \end{bmatrix}$ $\lambda^2 - 3\lambda + 2 = 0$ $(\lambda-1)(\lambda-2) = 0$ $\lambda = 1$ or 2

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda=2$ $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$ $x_1 = \frac{t}{2}$, $x_2 = t$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{27} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2^{27} \\ 1 & 2^{28} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2^{27}+2 & 2^{27}-1 \\ 2-2^{28} & 2^{28}-1 \end{bmatrix}$

2-2 $A^{17} - 3A^5 + 2A^2 + I$ 계산

$PD^{17}P^{-1} - 3PD^5P^{-1} + 2PDP^{-1} + PP^{-1}$

$= P(D^{17} - 3D^5 + 2D + I)P^{-1} = P\left(\begin{bmatrix} 1 & 0 \\ 0 & 2^{17} \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 2^5 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)P^{-1}$

$P\left(\begin{bmatrix} 1 & 0 \\ 0 & 2^{17} - 3 \cdot 2^5 + 5 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 89 - 2^{17} & -88 + 2^{17} \\ 176 - 2^{18} & -175 + 2^{18} \end{bmatrix}$

$2^{17} - 3 \cdot 2^5 + 4 + 1$