# 벡터 외적

$u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

$$u \times v = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

ex) $u = \langle 1, 2, -2 \rangle$, $v = \langle 3, 0, 1 \rangle$ $u \times v$ ?

$$u \times v = \left( \underbrace{\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}}_{=2}, \underbrace{-\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}}_{=-7}, \underbrace{\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}}_{=-6} \right) = (2, -7, -6)$$

## 벡터 외적 성질

1) $u \cdot (u \times v) = 0$ ($u \cdot v$ 와 $u$ 는 직교)
2) $v \cdot (u \times v) = 0$ ($u \cdot v$ 와 $v$ 는 직교)
3) $|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$ (라그랑주의 항등식)
4) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$ (벡터곱과 내적과의 관계)
5) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$ (벡터곱과 내적과의 관계)

ex) $u \times v$ 는 $u$ 와 $v$ 에 수직임을 보여라. $u = (1, 2, -2)$, $v = (3, 0, 1)$

$u \cdot (u \times v) = 0$

$v \cdot (u \times v) = 0$

$$u \times v = \left( \underbrace{\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}}_{=2}, \underbrace{-\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}}_{=-7}, \underbrace{\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}}_{-6} \right) = (2, -7, -6)$$

$(1, 2, -2) \cdot (2, -7, -6) = 2 - 14 + 12 = 0$

$(3, 0, 1) \cdot (2, -7, -6) = 6 - 6 = 0$

## 벡터 외적 성질 (2)

1) $u \times v = -(v \times u)$

2) $u \times (v + w) = (u \times v) + (u \times w)$

3) $k(u \times v) = (ku) \times v = u \times (kw)$

4) $u \times u = 0$

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

ex) $u = (1, 2, -2)$ $v = (3, 0, 1)$ $u \times v$ ?

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = 2i - 7j - 6k = (2, -7, -6)$$

벡터곱의 기하학적 의미 $|u \times v| = |u||v| \sin\theta$

$\sin\theta = \dfrac{|u \times v|}{|u||v|}$ $|u \times v|$ = 평행사변형 넓이

ex) 점 $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, $P_3(0, 4, 3)$으로 결정되는 삼각형의 넓이?

$\overrightarrow{P_1P_2} = \vec{P_2} - \vec{P_1} = (-3, -2, 2)$, $\overrightarrow{P_1P_3} = \vec{P_3} - \vec{P_1} = (-2, 2, 3)$

$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-3, -2, 2) \times (-2, 2, 3) = \begin{vmatrix} i & j & k \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = -10i + 5j - 10k$

$\therefore A = \frac{1}{2} |(-10, 5, -10)| = \frac{1}{2} \times 15 = \frac{15}{2}$

## 스칼라 3중적

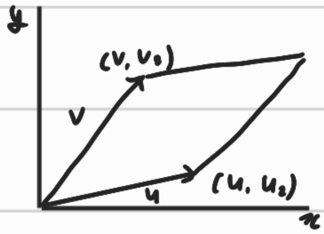
$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

ex) $u = 3i - 2j - 5k$ , $v = i + 4j - 4k$ , $w = 3j + 2k$ $\quad u\cdot(v\times w)$ ?

$$u\cdot(v\times w) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3(8+12) - (-4+15) = 60 - 11 = 49$$

1) 행렬식 $\det\begin{bmatrix} v_1 & v_2 \\ u_1 & u_2 \end{bmatrix}$ 의 절대값은 2차원 공간의 벡터 $u = (u_1, u_2)$ 와 $v = (v_1, v_2)$ 가

만드는 평행사변형의 넓이와 같다.




2) 행렬식 $\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$ 의 절대값은 평행육면체의 부피와 같다.

## 연습문제

1. $u \times v$ 계산

1-1 $u = (1, 2, -1)$ , $v = (1, 0, 2)$ $\quad u\times v = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 0 & 2 \end{vmatrix} = 4i - 3j - 2k = (4, -3, -2)$

1-2 $u = 2i - k$ , $v = 4j + k$ $\quad u\times v = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 4 & 1 \end{vmatrix} = 4i - 2j + 8k = (4, -2, 8)$

2-1 $u = (1, 0, 4)$ , $v = (1, -4, 2)$ 직교하는 단위벡터

$$u\times v = \begin{vmatrix} i & j & k \\ 1 & 0 & 4 \\ 1 & -4 & 2 \end{vmatrix} = 16i + 2j - 4k = (16, 2, -4) \quad \therefore \pm\frac{1}{16^2+4+16}(16, 2, -4)$$

2-2 $u = -2i + 3j - 3k$ , $v = 2i - k$

$u = (-2, 3, -3)$ , $v = (2, 0, -1)$ $\quad u\times v = \begin{vmatrix} i & j & k \\ -2 & 3 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -3i - 8j - 6k \quad \therefore \pm\frac{1}{9+64+36}(-3, -8, -6)$

3-1 $(2,3)$와 $(1,4)$를 이웃하는 두변으로하는 평행 사변형 넓이

$(2,3)\times(1,4) = \begin{vmatrix}2&3\\1&4\end{vmatrix} = 5$

3-2 $(2,1,0),(-1,2,0),(1,1,2)$ 부피

$\begin{vmatrix}2&1&0\\-1&2&0\\1&1&2\end{vmatrix} = 2\times(4+1)=10$

4. $u=(3,2,-1)$, $v=(0,2,-3)$, $w=(2,6,7)$

1) $v\times w = \begin{vmatrix}i&j&k\\0&2&-3\\2&6&7\end{vmatrix} = 32i+6j-4i = (32,6,-4)$

2) $(u\times v)\times w = (u\cdot w)v - (v\cdot w)u = 9(0,2,3)+9(3,2,-1) = (27,36,18)$

3) $u\times(v-2w) = (3,2,-1)\times(-4,-10,-17) = \begin{vmatrix}i&j&k\\3&2&-1\\-4&-10&-17\end{vmatrix} = -44i+55j-22k = (-44,55,-22)$

5. 벡터 u와 v에 모두 직교하는 벡터 구하라

5-1) $u=(-6,4,2)$, $v=(3,1,5)$ $u\times v = \begin{vmatrix}i&j&k\\-6&4&2\\3&1&5\end{vmatrix} = 18i+36j-18k = (18,36,-18)$

5-2) $u=(-2,1,5)$, $v=(3,0,-3)$ $u\times v = \begin{vmatrix}i&j&k\\-2&1&5\\3&0&-3\end{vmatrix} = -3i+9j+3k = (-3,9,3)$

6. u와 v가 만드는 평행 사변형 넓이

1) $u=(1,-1,2)$, $v=(0,3,1)$ $|u\times v| = \left\|\begin{matrix}i&j&k\\1&-1&2\\0&3&1\end{matrix}\right\| = -7i-j+3k = |(-7,-1,3)| = \sqrt{59}$

2) $u=(2,3,0)$, $v=(-1,2,-2)$ $|u\times v| = \left\|\begin{matrix}i&j&k\\2&3&0\\-1&2&-2\end{matrix}\right\| = |-6i+4j+7k| = \sqrt{101}$

3) $u=(3,-1,0)$, $v=(6,-2,8)$ $|u\times v| = \left\|\begin{matrix}i&j&k\\3&-1&4\\6&-2&8\end{matrix}\right\| = |0+0+0| = 0$

7. $u\cdot(v\times w)$를 구하여라

7-1 $u=(-1,2,4)$, $v=(3,4,-2)$, $w=(-1,2,5)$

$u\cdot(v\times w) = \begin{vmatrix}-1&2&4\\3&4&-2\\-1&2&5\end{vmatrix} = (-1)24+(-1)3(-10)+(-1)(-20) = -24+30+20 = 26$

7-2 $u=(3,-1,6)$, $v=(2,4,3)$, $w=(5,-1,2)$

$\begin{vmatrix}3&-1&6\\2&4&3\\5&-1&2\end{vmatrix} = 3\times11+(-11)+6\times(-22) = 33-11-132 = -110$

8. 평행육면체 부피

8-1) $u=(2,-6,2)$, $v=(0,4,-2)$, $w=(2,2,-4)$

$$\begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} = 2(-12) + 4\times 2 = -24+8 = -16$$

8-2) $u=(3,1,2)$, $v=(4,5,1)$, $w=(1,2,4)$

$$\begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 3\times 18 - 15 + 2\times 3 = 45$$