벡터 내적

$u = \langle a_1, b_1 \rangle$, $v = \langle a_2, b_2 \rangle$ 라 했을 때 내적 $u \cdot v$ 를

$u \cdot v = a_1 a_2 + b_1 b_2$ => 벡터는 스칼라이다.

ex. $u = \langle 3, -2 \rangle$, $v = \langle 4, 5 \rangle$ $u \cdot v = 12 - 10 = 2$

$u = 2i + j$, $v = 5i - 6j$ $u = \langle 2, 1 \rangle$, $v = \langle 5, -6 \rangle$ $u \cdot v = 10 - 6 = 4$

벡터 내적 성질

1. $u \cdot v = v \cdot u$
2. $(\alpha u) \cdot v = \alpha(u \cdot v) = u \cdot (\alpha v)$
3. $(u+v) \cdot w = u \cdot w + v \cdot w$
4. $|u|^2 = u \cdot u$

θ가 u와 v의 사잇각이면 $u \cdot v = |u||v| \cos\theta$

ex. $u = \langle 2, 5 \rangle$, $v = \langle 4, -3 \rangle$ 에 대하여 u와 v가 이루는 각도는?

1) $u = \langle 3, -2 \rangle$, $v = \langle 4, 5 \rangle$ $\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{2}{\sqrt{13}\sqrt{41}} = \frac{2}{\sqrt{533}}$

2) $u = \langle 2, 1 \rangle$, $v = \langle 5, -6 \rangle$ $\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{72}{\sqrt{5}\sqrt{61}}$

ex) $u = \langle 2, -1, 1 \rangle$ $v = \langle 1, 1, 2 \rangle$ $u \cdot v$와 u와 v가 이루는 각도 θ 구하라

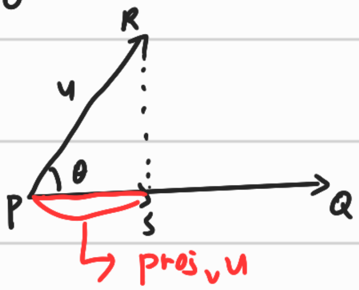
$u \cdot v = 2 - 1 + 2 = 3$ $|u| = |v| = \sqrt{6}$ $\cos\theta = \frac{3}{|u||v|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$ $\theta = 60^\circ$

ex) $2i + 2j - k$ 가 $5i - 4j + 2k$ 에 수직임을 보여라 → $u \cdot v = 0$

$u = (2, 2, -1)$ $v = (5, -4, 2)$

$u \cdot v = 10 - 8 - 2 = 0$

벡터 사영



v 위로의 u의 스칼라 사영 : $comp_v u = \frac{v \cdot u}{|v|}$

v 위로의 u의 벡터사영 : $proj_v u = \frac{v \cdot u}{|v|} \frac{v}{|v|} = \frac{v \cdot u}{|v|} v$

ex. $v = \langle -2, 3, 1 \rangle$ 위로의 $u = \langle 1, 1, 2 \rangle$ 의 스칼라사영과 벡터사영

$|v| = \sqrt{4+9+1} = \sqrt{14}$ $\quad comp_v u = \frac{v \cdot u}{|v|} = \frac{3}{\sqrt{14}}$

$proj_v u = \frac{v \cdot u}{|v|^2} = \frac{3}{(\sqrt{14})^2}(-2, 3, 1) = \frac{3}{14}(-2, 3, 1)$

ex. $u = \langle 2, -1, 3 \rangle$, $v = \langle 4, -1, 2 \rangle$ v 위로의 u 벡터사영과 v에 직교하는 u의 벡터성분

$proj_v u = \frac{v \cdot u}{|v|^2} v$ $\quad |v| = \sqrt{21}$ $\quad v \cdot u = 15$ $\quad proj_v u = \frac{15}{21}(4, -1, 2) = \frac{5}{7}(4, -1, 2)$

v에 직교하는 u의 벡터성분 $u - proj_v u = (2, -1, 3) - \frac{5}{7}(4, -1, 2)$

$= \left(-\frac{6}{7}, \frac{2}{7}, \frac{11}{7}\right)$

연습문제

1-1. $u = \langle 3, 1 \rangle$, $v = \langle 2, 4 \rangle$ $\quad u \cdot v = 3 \times 2 + 1 \times 4 = 10$

1-2 $u = \langle 2, -1, 3 \rangle$, $v = \langle 0, 2, 4 \rangle$ $\quad u \cdot v = 0 \cdot 2 + (-1) \cdot 2 + 3 \cdot 4 = 10$

2-1 $a = 3i - 2j$, $b = i + j$ $\quad \cos\theta = \frac{u \cdot v}{|u||v|}$ $\quad (3, -2)$ $\quad (1, 1)$ $\quad u \cdot v = 1$ $\quad |v| = \sqrt{13}$ $\quad |u| = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{26}}$

2-2 $a = 3i + j - 4k$, $b = -2i + 2j + k$ $\quad a = \langle 3, 1, -4 \rangle$ $\quad b = \langle -2, 2, 1 \rangle$

$a \cdot b = -8$ $\quad |a| = \sqrt{26}$ $\quad |b| = 3$

$\cos\theta = -\frac{8}{\sqrt{26}}$

3-1 $a = \langle 4, -1, 1 \rangle$, $b = \langle 2, 4, 4 \rangle$ $\quad$ 직교 조건 : $a \cdot b = 0$ $\quad a \cdot b = 8 - 4 + 4 \neq 0$

3-2 $a = 6i + 2j$ $b = -i + 3j$ $a = \langle 6, 2 \rangle$ $b = \langle -1, 3 \rangle$ $a \cdot b = -6 + 6 = 0$

4-1 $\langle 2, -1 \rangle$ 과 직교하는 벡터 $2x - y = 0$ $(t, 2t)$ $\therefore (1, 2)$

4-2 $6i + 2j - k$ 직교하는 벡터 $6x + 2y - z = 0$ $(0, 1, 2)$

5. $comp_v u$ 와 $proj_v u$

1) $u = \langle 2, 1 \rangle$ $v = \langle 3, 4 \rangle$ $comp_v u = \frac{u \cdot v}{|u||v|} \cdot |u|$ $u \cdot v = 14$
$|u| = \sqrt{5}$ $|v| = 5$

$\therefore comp_v u = \frac{14}{5}$ $proj_v u = \frac{u \cdot v}{|v|^2} v$

$\therefore proj_v u = \frac{14}{26} (3, 4)$

2) $u = 3i + j$, $v = 4i - 3j$ $u = \langle 3, 1 \rangle$, $v = \langle 4, -3 \rangle$ $u \cdot v = 9$
$|u| = \sqrt{10}$ $|v| = 5$

$comp_v u = \frac{9}{5}$ $proj_v u = \frac{9}{5} (4, -3)$

3) $u = \langle 2, 0, -2 \rangle$ $v = \langle 0, -3, 4 \rangle$ $u \cdot v = -8$ $|u| = 2\sqrt{2}$ $|v| = 5$

$comp_v u = -\frac{8}{5}$ $proj_v u = -\frac{8}{5} (0, -3, 4)$

6-1 $u = \langle 2, 3 \rangle$, $v = \langle 5, -7 \rangle$ $u \cdot v = 10 - 21 = -11$ $|u| = \sqrt{13}$ $|v| = \sqrt{74}$

$\cos\theta = -\frac{11}{\sqrt{961}}$

6-2 $u = \langle 1, -5, 4 \rangle$, $v = \langle 3, 3, 3 \rangle$ $\underline{u \cdot v = 0}$ $\therefore \cos\theta = 0$

7-1 $u = \langle 6, 2 \rangle$ $v = \langle 3, -9 \rangle$ $u \cdot v = 18 - 18 = 0$ $\therefore proj_v u = (0, 0)$

7-2 $u = \langle 3, 1, -7 \rangle$, $v = \langle 1, 0, 5 \rangle$ $u \cdot v = -32$
$|v| = \sqrt{26}$ $\therefore proj_v u = -\frac{32}{26} (1, 0, 5)$

8. 문제 7의 각각에서 v에 직교하는 u벡터의 성분은?

1. $u - proj_v u = \langle 6, 2 \rangle - \langle 0, 0 \rangle = (6, 2)$

2. $= \langle 3, 1, -7 \rangle - \frac{16}{13} (1, 0, 5)$

9. $|proj_v u|$ 구하라

9-1 $u = \langle 1, -2 \rangle, v = \langle -4, -3 \rangle$

$u \cdot v = -10$
$|v| = 5$

$-\frac{10}{25}(-4, -3) = -\frac{2}{5}(-4, -3)$

$|\text{proj}_v u| = 2$

9-2 $u = \langle 3, -2, 6 \rangle$ $v = \langle 1, 2, -7 \rangle$

$u \cdot v = -43$
$|v| = 54$

$\frac{-43}{54}(1, 2, -7)$

$|\text{proj}_v u| = 43$

10. $u = \langle 1, 0, 1 \rangle$ $v = \langle 0, 1, 1 \rangle$ u와 v의 모두 직교하는 단위벡터 구하라

$w = \langle x, y, z \rangle$

$u \cdot w = x + z = 0$
$u \cdot v = y + z = 0$

$x = y = -z$
$\underline{(1, 1, -1)}$.

$\frac{w}{|w|} = \frac{1}{\sqrt{3}}(1, 1, -1)$