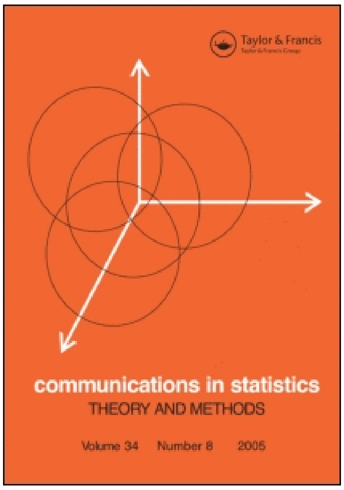

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INFERENCE FOR STRESS-STRENGTH MODELS BASED ON WEINMAN MULTIVARIATE EXPONENTIAL SAMPLES

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INFERENCE FOR STRESS-STRENGTH MODELS BASED ON WEINMAN MULTIVARIATE EXPONENTIAL SAMPLES

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ABSTRACT

The analysis of stress-strength systems is connected to the probability $P(X < Y)$, where X represents the stress subject to an object and Y is its strength. We consider estimation of $P(X < Y)$ when the underlying data consists of two samples of order statistics from Weinman multivariate exponential distributions with a common location parameter. Maximum likelihood estimators and uniform minimum variance unbiased estimators of $P(X < Y)$ are presented, when the location parameter is assumed to be known and unknown, respectively. Moreover, some distributional properties, a confidence interval and asymptotic results are established. The results can be applied to various data set-ups based on exponential distributions, e.g., ordinary order statistics, progressive type II censored order statistics, sequential order statistics and record values.

Key Words: Binomial mixtures; Order statistics; Progressive type II censoring; Stress-strength system; Weinman multivariate exponential distribution.

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1. INTRODUCTION

Stress-strength models have been extensively investigated in the statistical literature (cf. (1)). The quantity of interest in these models is the probability $P(X < Y)$, where X represents some stress subjected to an object and Y corresponds to the strength of this object. Especially, the case of underlying exponential distributions $\mathrm{Exp}(\mu, \vartheta)$ with density function

$$f(z) = \vartheta^{-1}\, \mathrm{e}^{-(z-\mu)/\vartheta}, \quad z \geq \mu,$$

has received great attention, because it plays an important role in life testing (2). Given independent exponentially distributed random variables $X \sim \mathrm{Exp}(\mu, \vartheta_1)$ and $Y \sim \mathrm{Exp}(\mu, \vartheta_2)$ with the same location parameter $\mu \in \mathbf{R}$ and scale parameters $\vartheta_1, \vartheta_2 > 0$, statistical inference is concerned with the ratio

$$P = P(X < Y) = \frac{\vartheta_2}{\vartheta_1 + \vartheta_2}.$$

Many papers deal with the i.i.d. model, i.e., the estimation procedures are based on two independent i.i.d. samples $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$. Various estimation concepts have been taken into account, e.g., maximum likelihood estimation (cf. (2),(3),(4)), minimum variance estimation (cf. (5), (6), (7), (8)) and Bayesian estimation (cf. (9), (10)). (11) compares the maximum likelihood estimator (MLE) and the uniform minimum variance unbiased estimator (UMVUE) of P in the independent one-parameter exponential model, i.e., $\mu = 0$. For results which take dependence among $(X_1, \ldots, X_{n_1})$ and $(Y_1, \ldots, Y_{n_2})$ into consideration we refer to (2) and (1).

In a recent paper, (12) were concerned with minimum variance unbiased estimation of the probability $P(X < Y)$ where $X \sim \mathrm{Exp}(\mu, \vartheta_1)$ and $Y \sim \mathrm{Exp}(\mu, \vartheta_2)$ are independent random variables and sampling is from possibly type II censored Weinman multivariate exponential distributions. The density function of order statistics from a Weinman multivariate exponential distribution with parameters μ, ϑ and $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is given by (cf. (13), p. 52)

$$f(y_1, \ldots, y_n) = \frac{n!}{\vartheta^n} \prod_{i=1}^{n} \alpha_i \exp\left\{ -\frac{\alpha_i}{\vartheta}(n-i+1)(y_i - y_{i-1}) \right\},$$

where $\mu = y_0 \leq y_1 \leq \cdots \leq y_n$, $\alpha_1, \ldots, \alpha_n > 0$. For brevity, such a distribution is denoted by $\mathrm{WME}_n(\mu, \vartheta, \tilde{\alpha})$. As pointed out by (13) such an approach can be viewed as a unifying one, because several models of ordered random variables are included in the distributional sense. Choosing particular values for $\tilde{\alpha}$, well-known models arise, e.g., $\alpha_i = 1$, $1 \leq i \leq n$, corresponds to the case of ordinary order statistics whereas $\alpha_i = 1/(n-i+1)$, $1 \leq i \leq n$, corresponds to the model of record values. Another important particular model is a sampling scheme called progressive type II censoring. For details concerning this model as well as references we refer to (14) and (15). Hence, our distributional assumption leads to a variety of models. If we choose particular values for the model parameters $\tilde{\alpha}$ and $\tilde{\beta}$, the estimation can be viewed as a procedure which is based on ordinary order statistics, record values or progressive type II censored data, respectively. From this point of view, (12) point out that their results include results obtained for one- and two-parameter exponential distributions, e.g., (5) (known location, uncensored sample), (6) (known location, censored sample) and (7), (8) (unknown location, uncensored sample). Additionally, the case of an unknown location parameter and a censored sample is covered in the resulting estimators. Moreover, the results can be applied to compare so-called sequential k-out-of-n systems (cf. (12), (16), (17)).

In this paper, we deal with estimation of $P(X < Y)$ in the following situation: Suppose that two independent samples $X_*^{(1)}, \ldots, X_*^{(n_1)} \sim \mathrm{WME}_{n_1}(\mu, \vartheta_1, \tilde{\alpha})$ and $Y_*^{(1)}, \ldots, Y_*^{(n_2)} \sim \mathrm{WME}_{n_2}(\mu, \vartheta_2, \tilde{\beta})$ with censoring numbers r_1 and r_2, respectively, are given. The model parameters $\tilde{\alpha}$ and $\tilde{\beta}$ are assumed to be known. The location parameter of both distributions is given by μ, whereas the scale parameters may be different. In Section 2 we calculate the MLE of P in the preceding situation. We show that its distribution is a mixture of generalized three-parameter beta distributions. Based on this property explicit expression of the moments of the MLE are given in Section 3. In particular, a finite sum representation of the expectation is established. In Section 4 we propose an approximate confidence interval for P, which is based on the UMVUE of the ratio $n_1\alpha_1\vartheta_2/(n_1\alpha_1\vartheta_2 + n_2\beta_1\vartheta_1)$. In Section 5 these results are extended to the more sample case. The estimation procedure is based on s_1 independent samples $X_{*i}^{(1)}, \ldots, X_{*i}^{(n_{1i})}$ and s_2 independent samples $Y_{*j}^{(1)}, \ldots, Y_{*j}^{(n_{2j})}$, which are possibly type II censored with numbers r_{1i} and r_{2j}, respectively. We present the MLE and the UMVUE of P in this more general situation and establish asymptotic results w.r.t. an increasing number of samples, e.g., consistency, asymptotic distribution and the asymptotic equivalence of the MLE and the UMVUE. The asymptotic results are inspired by the work of (7). Although our sampling situation is different from that of (7), their set-up can be embedded in our model. Our approach includes the asymptotics presented in (7) by choosing particular values for $\tilde{\alpha}$ and $\tilde{\beta}$.

2. ESTIMATORS OF $P(X < Y)$ AND THEIR DISTRIBUTION

Before turning to the mentioned problem we calculate the simultaneous MLEs of $(\vartheta_1, \vartheta_2)$, if μ is known, and of $(\mu, \vartheta_1, \vartheta_2)$, if μ is supposed to be unknown. For brevity, let $\gamma_{1j} = (n_1 - j + 1)\alpha_j$, $1 \leq j \leq n_1$, and $\gamma_{2j} = (n_2 - j + 1)\beta_j$, $1 \leq j \leq n_2$.

Theorem 2.1. *Let* $Z = \min\{X_*^{(1)}, Y_*^{(1)}\}$, $W_1 = \frac{1}{\gamma_{11}} \sum_{j=1}^{r_1} \gamma_{1j}(X_*^{(j)} - X_*^{(j-1)})$, $X_*^{(0)} = 0$ *and* $W_2 = \frac{1}{\gamma_{21}} \sum_{j=1}^{r_2} \gamma_{2j}(Y_*^{(j)} - Y_*^{(j-1)})$, $Y_*^{(0)} = 0$.

1. *If the location parameter* μ *is assumed to be known, the MLE of* $(\vartheta_1, \vartheta_2)$ *is given by*
$$\left(\vartheta_1^*, \vartheta_2^*\right) = \left(\frac{\gamma_{11}}{r_1}(W_1 - \mu), \frac{\gamma_{21}}{r_2}(W_2 - \mu)\right).$$
2. *If the location parameter* μ *is assumed to be unknown, the MLE of* $(\mu, \vartheta_1, \vartheta_2)$ *is given by*
$$(\hat{\mu}, \hat{\vartheta}_1, \hat{\vartheta}_2) = \left(Z, \frac{\gamma_{11}}{r_1}(W_1 - Z), \frac{\gamma_{21}}{r_2}(W_2 - Z)\right).$$

Proof: We omit the proof of the first assertion, because it proceeds similar to the presented one. In the case of an unknown location parameter the likelihood function is given by

$$\begin{aligned}
&L(\mu, \vartheta_1, \vartheta_2; x_i, 1 \leq i \leq r_1, y_j, 1 \leq j \leq r_2)\\
&\quad= \left(\prod_{j=1}^{2} \frac{n_j!}{(n_j - r_j)!}\right)\left(\prod_{j=1}^{r_1} \alpha_j \prod_{i=1}^{r_2} \beta_i\right) \frac{1}{\vartheta_1^{r_1} \vartheta_2^{r_2}}\\
&\qquad\times \exp\left\{ -\frac{\gamma_{11}}{\vartheta_1} w_1 - \frac{\gamma_{21}}{\vartheta_2} w_2 + \left(\frac{\gamma_{11}}{\vartheta_1} + \frac{\gamma_{21}}{\vartheta_2}\right)\mu \right\}, \quad z \geq \mu,
\end{aligned}$$

and 0 otherwise. w_1, w_2 and z are realizations of W_1, W_2 and Z, respectively, based on the observed values $x_1, \ldots, x_{r_1}$ and $y_1, \ldots, y_{r_2}$. Some monotonicity considerations show that $\hat{\mu} = Z$ holds for arbitrary values of $\vartheta_j > 0$, $j = 1, 2$. Hence, we obtain for the likelihood function L the upper bound

$$\begin{aligned}
&L_1(\vartheta_1, \vartheta_2; x_i, 1 \leq i \leq r_1, y_j, 1 \leq j \leq r_2)\\
&\quad= \text{const} \cdot \frac{1}{\vartheta_1^{r_1} \vartheta_2^{r_2}} \exp\left\{ -\frac{\gamma_{11}}{\vartheta_1}(w_1 - z) - \frac{\gamma_{21}}{\vartheta_2}(w_2 - z)\right\},
\end{aligned} \tag{2.1}$$

with equality iff $\mu = z$. Considering the logarithm of (2.1) and applying the inequality $\log y \geq 1 - 1/y$, we obtain the upper bound

$$\begin{aligned}
&\log L_1(\vartheta_1, \vartheta_2; x_i, 1 \leq i \leq r_1, y_j, 1 \leq j \leq r_2)\\
&\quad\leq \text{const} - (r_1 + r_2) + r_1 \log\left(\frac{\gamma_{11}}{r_1}(w_1 - z)\right) + r_2 \log\left(\frac{\gamma_{21}}{r_2}(w_2 - z)\right)
\end{aligned}$$

with equality iff $\vartheta_j = \frac{\gamma_{j1}}{r_j}(w_j - z)$, $j = 1, 2$. This proves the assertion. □

Since MLEs of transformed parameters are obtained via a corresponding transformation of the MLEs (cf. (18), Theorem 5.28), we find the following result:

Theorem 2.2. *In the above situation, the MLE of $P(X < Y)$ is given by*

$$P^* = \frac{\frac{\gamma_{21}}{r_2}(W_2-\mu)}{\frac{\gamma_{11}}{r_1}(W_1-\mu)+\frac{\gamma_{21}}{r_2}(W_2-\mu)},$$

if the location parameter μ is known. It is given by

$$\hat{P} = \frac{\frac{\gamma_{21}}{r_2}(W_2-Z)}{\frac{\gamma_{11}}{r_1}(W_1-Z)+\frac{\gamma_{21}}{r_2}(W_2-Z)},$$

if μ is assumed to be unknown.

In contrast to the UMVUEs of P (cf. (12)) the resulting MLEs have a quite simple form. If $\alpha_1 = \ldots = \alpha_{n_1} = \beta_1 = \ldots = \beta_{n_2} = 1$ and $r_1 = r_2$, $n_1 = n_2$ the MLEs of P can be found in (3) and (4), p. 532. This situation corresponds to the case of order statistics with equal sample and censoring sizes.

Further results on the estimators are based on the knowledge of the distribution which we provide in the following theorem.

Lemma 2.3.

1. $\gamma_{j1}(W_j-\mu)$ *follows a gamma distribution with parameters r_j and ϑ,* i.e., $\gamma_{j1}(W_j-\mu) \sim \Gamma(r_j, \vartheta_j)$, $j = 1, 2$. *Moreover,* $\gamma_{j1}(W_j-\mu)$, $j = 1, 2$, *are independent random variables.*
2. Let $r_1, r_2 \geq 2$. $(W_1 - Z, W_2 - Z)$ *and Z are independent random variables, where* $Z \sim \mathrm{Exp}(\mu, (\frac{\gamma_{11}}{\vartheta_1}+\frac{\gamma_{21}}{\vartheta_2})^{-1})$. *The joint density function of* $(W_1 - Z, W_2 - Z)$ *is given by*

$$f^{W_1-Z,W_2-Z}(u_1,u_2) = \frac{\sigma_1^{r_1}\sigma_2^{r_2}}{(\sigma_1+\sigma_2)(r_1-1)!(r_2-1)!}$$
$$\times u_1^{r_1-2}u_2^{r_2-2}((r_1-1)u_2+(r_2-1)u_1)\exp(-\sigma_1 u_1-\sigma_2 u_2),$$

where $u_1, u_2 \geq 0$ and $\sigma_j = \gamma_{j1}/\vartheta_j$, $j = 1, 2$.

3. Let $r_1, r_2 \geq 2$. $V_j = \gamma_{j1}(W_j - Z)$, $j = 1, 2$, *are distributed according to a mixture of gamma distributions, i.e., their density functions are*

$$f^{V_1}(v_1) = \frac{\gamma_{11}\vartheta_2}{\gamma_{21}\vartheta_1+\gamma_{11}\vartheta_2}g_{r_1-1}(v_1)+\frac{\gamma_{21}\vartheta_1}{\gamma_{21}\vartheta_1+\gamma_{11}\vartheta_2}g_{r_1}(v_1) \quad and$$
$$f^{V_2}(v_2) = \frac{\gamma_{21}\vartheta_1}{\gamma_{21}\vartheta_1+\gamma_{11}\vartheta_2}g_{r_2-1}(v_2)+\frac{\gamma_{11}\vartheta_2}{\gamma_{21}\vartheta_1+\gamma_{11}\vartheta_2}g_{r_2}(v_2),$$

respectively, where $g_s(t) = \frac{t^{s-1}}{(s-1)!}\,\mathrm{e}^{-t}$, $t \geq 0$, $s \in \mathbf{N}$, *denotes the density function of a gamma distribution with parameters s and 1.*

Proof: The first assertion follows from the fact that normalized spacings of sequential order statistics from exponential distributions are i.i.d. random variables with a standard exponential distribution (cf. (13), p. 52). Hence, $\gamma_{j1}(W_j - \mu)$ is the sum of r_j i.i.d. standard exponentially distributed random variables. The independence is trivial.

The second result is derived from the joint distribution of (W_1, W_2, Z) given in (12). The last statement is deduced from the second one via integration. $\square$

The distributions f^{V_1} and f^{V_2} are particular binomial mixtures. For more information on binomial mixtures we refer to (19).

Remark 1.1. Lemma 2.3 includes results of (20) and (3), who were concerned with possibly censored samples from two exponential distributions $\operatorname{Exp}(\mu, \vartheta_1)$ and $\operatorname{Exp}(\mu, \vartheta_2)$. By analogy with their results it is possible to calculate the UMVUE $\tilde{\mu}$ of μ. The proof is based on the complete sufficiency of (Z, W_1, W_2), which is established similarly to (20) and to (3). (21) find the representation

$$\tilde{\mu} = Z - \frac{(W_1 - Z)(W_2 - Z)}{(r_1 - 1)(W_2 - Z) + (r_2 - 1)(W_1 - Z)}. \tag{2.2}$$

In the case of uncensored i.i.d. samples from exponential distributions, (2.2) yields the formulae of (20) and (3). The case $\vartheta_1 = \vartheta_2$ is considered in (17).

Applying Lemma 2.3 the distribution of the statistics P^* and $\hat{P}$ can be calculated. The distribution of P^* is known as a generalized three-parameter beta distribution. It is usually denoted by $\mathrm{G3B}(r_2, r_1; \lambda)$, where in our situation $\lambda = (r_2\vartheta_1)/(r_1\vartheta_2)$. Properties, extensions and further applications can be found in (22), (23), (24) and (25), p. 251. The density function of P^* is given by

$$\psi_{r_1, r_2; \lambda}(t) = \frac{\lambda^{r_2}}{B(r_1, r_2)} \frac{t^{r_2 - 1}(1 - t)^{r_1 - 1}}{[1 - (1 - \lambda)t]^{r_1 + r_2}}, \quad t \in [0, 1]. \tag{2.3}$$

A similar result holds in the case of an unknown location parameter. We study a more general form of the estimator $\hat{P}$, namely we consider the estimators

$$P_{a,b} = \frac{a(W_2 - Z)}{b(W_1 - Z) + a(W_2 - Z)}, \quad a, b > 0. \tag{2.4}$$

First, we derive the density function of $P_{a,b}$.

Lemma 2.5. *The density function $\varphi_{a,b}$ of $P_{a,b}$ is given by*

$$\varphi_{a,b}(t) = \binom{r_1 + r_2 - 2}{r_1 - 1} \frac{\sigma}{1 + \sigma} (\varepsilon\sigma)^{r_2 - 1} [(r_1 - 1)\varepsilon t + (r_2 - 1)(1 - t)] \times \frac{(1 - t)^{r_1 - 2} t^{r_2 - 2}}{[1 - (1 - \varepsilon\sigma)t]^{r_1 + r_2 - 1}}, \quad t \in [0, 1], \tag{2.5}$$

where $\sigma = (\gamma_{21}\vartheta_1)/(\gamma_{11}\vartheta_2)$ and $\varepsilon = b/a$.

Proof: We proceed by analogy with the proof of Theorem 3.1 in (12). The cumulative distribution function of $P_{a,b}$ possesses the integral representation

$$P(P_{a,b} \le t) = c \int_0^\infty u_2^{r_2-2}\, \mathrm{e}^{-\sigma_2 u_2} \times \int_{\delta(u_2,t)}^\infty [(r_1-1)u_2 + (r_2-1)u_1] u_1^{r_1-2}\, \mathrm{e}^{-\sigma_1 u_1}\, du_1\, du_2, \tag{2.6}$$

where $t \in [0,1]$, $\sigma_j = \gamma_{j1}/\vartheta_j$, $j = 1,2$, $c = (\sigma_1^{r_1}\sigma_2^{r_2})/[(\sigma_1+\sigma_2)(r_1-1)!(r_2-1)!]$ and $\delta(u_2,t) = a(1-t)u_2/(bt)$. Differentiating (2.6) w.r.t. t yields after some simplifications the integral expression

$$\varphi_{a,b}(t) = \frac{ac}{bt^2}\left[(r_1-1) + (r_2-1)\frac{a(1-t)}{bt}\right]\left(\frac{a(1-t)}{bt}\right)^{r_1-2} \times \int_0^\infty u_2^{r_1+r_2-2} \exp\left\{-\sigma_2 u_2 + \sigma_1 \frac{a(1-t)}{bt} u_2\right\} du_2.$$

Evaluation of this expression and some lengthy algebra lead to the representation (2.5). □

A closer look at the density function $\varphi_{a,b}$ yields the following remarkable result.

Corollary 2.6. The density function $\varphi_{a,b}$ is a mixture of generalized beta distributions, i.e.,

$$\varphi_{a,b}(t) = \frac{1}{1+\sigma}\psi_{r_1-1,r_2;\varepsilon\sigma}(t) + \frac{\sigma}{1+\sigma}\psi_{r_1,r_2-1;\varepsilon\sigma}(t), \tag{2.7}$$

where $\sigma = (\gamma_{21}\vartheta_1)/(\gamma_{11}\vartheta_2)$ and $\varepsilon = b/a$.

Proof: The density function $\varphi_{a,b}$ given in (2.5) can be written as

$$\varphi_{a,b}(t) = c_1 \frac{(1-t)^{r_1-2} t^{r_2-1}}{[1-(1-\varepsilon\sigma)t]^{r_1+r_2-1}} + c_2 \frac{(1-t)^{r_1-1} t^{r_2-2}}{[1-(1-\varepsilon\sigma)t]^{r_1+r_2-1}}$$

with appropriately chosen constants c_1 and c_2. Both ratios in the preceding formula are generalized beta distributions of the type (2.3) except for a normalizing factor. A straightforward calculation leads to formula (2.7).

In view of Lemma 2.3, Corollary 2.6 yields the following analogy: the estimator P^* is based on two (independent) gamma variates and its distribution is given by a generalized three-parameter beta distribution. The random variable $P_{a,b}$ is a ratio of (dependent) random variables distributed according to a mixture of gamma variates and its distribution is a mixture of generalized three-parameter beta distributions. The weights of the mixtures are the same in both set-ups.

Choosing $a = \gamma_{21}/r_2$ and $b = \gamma_{11}/r_1$ in (2.4), $P_{a,b}$ coincides with the MLE $\hat{P}$ of P. Hence, its density function is an immediate consequence of Lemma 2.5.

Theorem 2.7. *The density function of the MLE $\hat{P}$ is given by*

$$\varphi_{MLE}(t) = \binom{r_1 + r_2 - 2}{r_1 - 1} \frac{\sigma}{1+\sigma} \lambda^{r_2 - 1} \left[(r_1 - 1)\frac{r_2\gamma_{11}}{r_1\gamma_{21}} t + (r_2 - 1)(1-t)\right]$$
$$\times \frac{(1-t)^{r_1-2} t^{r_2-2}}{[1-(1-\lambda)t]^{r_1+r_2-1}}, \ t \in [0, 1],$$

where $\sigma = (\gamma_{21}\vartheta_1)/(\gamma_{11}\vartheta_2)$ and $\lambda = (r_2\vartheta_1)/(r_1\vartheta_2)$. Moreover, φ_{MLE} is a mixture of generalized beta distributions:

$$\varphi_{MLE}(t) = \frac{1}{1+\sigma}\psi_{r_1-1,r_2;\lambda}(t) + \frac{\sigma}{1+\sigma}\psi_{r_1,r_2-1;\lambda}(t).$$

Since the analogies between the case of a known and unknown location parameter are striking and the results can be deduced in a similar manner in both models, we consider subsequently the case of an unknown location parameter only.

3. MOMENTS OF $P_{a,b}$

From Corollary 2.6 the moments of the statistic $P_{a,b}$ can be calculated. Since the generalized three-parameter beta distribution is a particular Gauss hypergeometric distribution (cf. (26), (25), p. 253), its moments can be written in terms of hypergeometric functions $F(\alpha, \beta; \gamma; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j(\beta)_j}{(\gamma)_j j!} z^j$, where $(x)_j = x(x+1)\ldots(x+j-1)$, $j \geq 1$, denotes Pochhammer's symbol, $(x)_0 = 1$. Hence, we obtain

$$E\left(P_{a,b}^k\right) = \frac{1}{1+\sigma}(\varepsilon\sigma)^{r_2}\frac{B(r_2+k, r_1-1)}{B(r_2, r_1-1)}F(r_1+r_2-1, r_2+k;$$
$$\times r_1+r_2+k-1; 1-\varepsilon\sigma) + \frac{\sigma}{1+\sigma}(\varepsilon\sigma)^{r_2-1}\frac{B(r_2+k-1, r_1)}{B(r_2-1, r_1)}$$
$$\times F(r_1+r_2-1, r_2+k-1; r_1+r_2+k-1; 1-\varepsilon\sigma)$$

provided $0 < \varepsilon\sigma \leq 2$. If $\varepsilon\sigma > 2$ a similar representation can be obtained by interchanging some arguments of the hypergeometric functions, i.e., $1 - \varepsilon\sigma$ by $-(1-\varepsilon\sigma)/(2-\varepsilon\sigma)$ and r_1 by r_2. For details, we refer to (12). An alternative infinite series representation can be derived using a result of (22). In contrast to that infinite series expression, renewed applications of Gauss' recursions (cf. (27),

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p. 1071) lead to finite sum representations. We illustrate this method with the expectation of $P_{a,b}$. For convenience, suppose that $0 < \varepsilon\sigma \leq 2$. The expectation $E(P_{a,b})$ is given by

$$E(P_{a,b}) = \frac{(\varepsilon\sigma)^{r_2}}{1+\sigma}\left\{\frac{r_2}{r_1+r_2-1}F(r_1+r_2-1, r_2+1; r_1+r_2; 1-\varepsilon\sigma) + \frac{1}{\varepsilon}\frac{r_2-1}{r_2+r_1-1}F(r_1+r_2-1, r_2; r_1+r_2; 1-\varepsilon\sigma)\right\}.$$

Applying Lemma 5.3 of (12), we obtain

$$F(r_1+r_2-1, r_2; r_1+r_2; z) = \frac{r_1+r_2-1}{r_1-1}(1-z)^{-r_2} - \frac{r_2}{r_1-1}F(r_1+r_2-1, r_2+1; r_1+r_2; z)$$

and thus via Corollary 5.4 of (12)

$$E(P_{a,b}) = \frac{r_2-1}{(r_1-1)\varepsilon(1+\sigma)} + \frac{1}{1+\sigma}\left(1-\frac{r_2-1}{(r_1-1)\varepsilon}\right) \times\left\{\sum_{j=0}^{r_2-1}\frac{(r_1-1)_j}{(1-r_2)_j}(\varepsilon\sigma)^{j-r_2} + \frac{(-1)^{r_2+1}(\varepsilon\sigma)^{r_2}}{B(r_1-1, r_2)} \times\left((1-\varepsilon\sigma)^{1-r_1-r_2}\log(\varepsilon\sigma) + \sum_{j=0}^{r_1+r_2-1}\frac{(1-\varepsilon\sigma)^{1-j}}{r_1+r_2-j}\right)\right\}. \quad (3.1)$$

From this expression it is easily seen that for $a = r_1 - 1$, $b = r_2 - 1$ and $\gamma_{11} = \gamma_{21}$ the estimator $P_{a,b}$ is an unbiased estimator of P. This conclusion is not astonishing, because in this situation $P_{a,b}$ is the UMVUE of P as pointed out by (12).

4. AN INTERVAL ESTIMATE FOR *P*

Imitating an idea of (28), (7) propose an approximate confidence interval for the probability P. They observed that the distribution of the ratio $\lambda_1 T_1/(\lambda_1 T_1 + \lambda_2 T_2)$ (notation of (7)) is a binomial mixture of beta distributions where the binomial distribution has parameters 1 and $p = n_1\lambda_1/(n_1\lambda_1 + n_2\lambda_2)$. They propose to estimate p with methods applied in mixture models (e.g., see (29)). But, such an approach is not necessary in the given situation, because it is possible to estimate the weight p without drawing upon these methods. We illustrate this procedure by developing the approach of (7) in our more general situation. From Corollary 2.6 we conclude that the distribution of $P' = P_{a,b}$ with $b = \gamma_{11}\vartheta_2$ and $a = \gamma_{21}\vartheta_1$ is a binomial mixture of beta distributions, where the binomial distribution has parameters 1 and $p = 1/(1+\sigma) = \gamma_{11}\vartheta_2/(\gamma_{11}\vartheta_2 + \gamma_{21}\vartheta_1)$. Denoting by $I_x(r,s) = B(r,s)^{-1}\int_0^x t^{r-1}(1-t)^{s-1}\,dt$ the incomplete beta function ratio, the distribution function of P' is given by

$$\Phi(x) = \frac{1}{1+\sigma} I_x(r_2, r_1 - 1) + \left(1 - \frac{1}{1+\sigma}\right) I_x(r_2 - 1, r_1). \tag{4.1}$$

From (3.1) we conclude that $\tilde{P} = P_{a,b}$ with $a = r_1 - 1$ and $b = r_2 - 1$ is an unbiased estimator of $1/(1+\sigma)$. Moreover, since (W_1, W_2, Z) is complete sufficient, $\tilde{P}$ is even the UMVUE of $1/(1+\sigma)$. Another possible estimator is the MLE of $1/(1+\sigma)$, which is obtained by analogy with Theorem 2.2.

Continuing the approach of (7), we replace $1/(1+\sigma)$ by $\tilde{P}$ in (4.1) and compute the unique solution z_α of the equation

$$\tilde{\Phi}(x) = \tilde{P} I_x(r_2, r_1 - 1) + (1 - \tilde{P}) I_x(r_2 - 1, r_1) \overset{!}{=} \alpha, \quad \alpha \in (0,1).$$

The resulting approximate confidence interval for the probability P based on the estimated distribution function $\tilde{\Phi}$ is given by

$$\left[\frac{z_{\alpha/2}(r_1-1)(W_2-Z)}{(1-z_{\alpha/2})(r_1-1)(W_2-Z)+z_{\alpha/2}(r_2-1)(W_1-Z)},\right.$$
$$\left.\frac{z_{1-\alpha/2}(r_1-1)(W_2-Z)}{(1-z_{1-\alpha/2})(r_1-1)(W_2-Z)+z_{1-\alpha/2}(r_2-1)(W_1-Z)}\right].$$

This confidence interval reduces to the proposal of (7) in the case of two uncensored samples. But, it can be applied to various other sampling situations as well, e.g., type II censored data, estimation based on record values or progressive type II censored order statistics. Furthermore, mixtures of data types are allowed, e.g., one sample may be progressively censored and the other one may be consist of record values.

5. MORE SAMPLE CASE AND ASYMPTOTICS

In this section we consider the situation of s_1 and s_2 independent samples, i.e., $X_{*i}^{(1)}, \ldots, X_{*i}^{(n_{1i})} \sim \mathrm{WME}_{n_{1i}}(\mu, \vartheta_1, \tilde{\alpha}_i)$, $1 \le i \le s_1$ and $Y_{*i}^{(1)}, \ldots, Y_{*i}^{(n_{2i})} \sim \mathrm{WME}_{n_{2i}}(\mu, \vartheta_2, \tilde{\beta}_i)$, $1 \le i \le s_2$. Each sample is allowed to be type II censored with possibly different numbers r_{ji}, $1 \le i \le s_j$, $j = 1, 2$. Before focusing on the estimation procedure we introduce some notation. Let $\gamma_{1i}^{(k)} = (n_{1k} - i + 1)\alpha_{ik}$, $1 \le i \le n_{1k}$, $1 \le k \le s_1$, $\gamma_{2i}^{(k)} = (n_{2k} - i + 1)\beta_{ik}$, $1 \le i \le n_{2k}$, $1 \le k \le s_2$,

$Z = \min_{\substack{1\le i\le s_1\\1\le j\le s_2}} \{X_{*i}^{(1)}, Y_{*j}^{(1)}\}$ and define:

$$\bar{\gamma}_j = \sum_{i=1}^{s_j} \gamma_{j1}^{(i)}, \qquad R_j = \sum_{i=1}^{s_j} r_{ji}, \quad j = 1, 2,$$

$$W_{1i} = \frac{1}{\gamma_{11}^{(i)}} \sum_{k=1}^{r_{1i}} \gamma_{1k}^{(i)} \left(X_{*i}^{(k)} - X_{*i}^{(k-1)}\right), \quad W_{2i} = \frac{1}{\gamma_{21}^{(i)}} \sum_{k=1}^{r_{2i}} \gamma_{2k}^{(i)} \left(Y_{*i}^{(k)} - Y_{*i}^{(k-1)}\right),$$

$$X_{*i}^{(0)} = Y_{*i}^{(0)} = 0, \qquad V_j = \sum_{i=1}^{s_j} \gamma_{j1}^{(i)} W_{ji} - \bar{\gamma}_j Z, \quad j = 1, 2,$$

$$S_1 = \sum_{i=1}^{s_1} \gamma_{11}^{(i)} W_{1i} - \sum_{i=1}^{s_1} \gamma_{11}^{(i)} X_{*i}^{(1)}, \qquad S_2 = \sum_{i=1}^{s_2} \gamma_{21}^{(i)} W_{2i} - \sum_{i=1}^{s_2} \gamma_{21}^{(i)} Y_{*i}^{(1)},$$

$$T_1 = \sum_{i=1}^{s_1} \gamma_{11}^{(i)} \left(X_{*i}^{(1)} - Z\right), \qquad T_2 = \sum_{i=1}^{s_2} \gamma_{21}^{(i)} \left(Y_{*i}^{(1)} - Z\right).$$

With these notations we find for the likelihood function:

$$\begin{aligned} &L(\mu, \vartheta_1, \vartheta_2; x_{ij}, y_{ij}) \\ &\quad = \text{const} \cdot \frac{1}{\vartheta_1^{R_1} \vartheta_2^{R_2}} \exp\left\{ -\frac{\bar{\gamma}_1}{\vartheta_1} v_1 - \frac{\bar{\gamma}_2}{\vartheta_2} v_2 - \left(\frac{\bar{\gamma}_1}{\vartheta_1} + \frac{\bar{\gamma}_2}{\vartheta_2}\right)(z - \mu)\right\}, \quad z \ge \mu, \end{aligned}$$

where v_1, v_2 and z are realizations of V_1, V_2 and Z. This yields directly the MLEs of ϑ_1, ϑ_2, μ and P. In order to state the UMVUE of P we calculate the distribution of (V_1, V_2). Utilizing the above notations we find the decomposition

$$(V_1, V_2) = (S_1, S_2) + (T_1, T_2),$$

where (S_1, S_2) and (T_1, T_2) as well as S_1 and S_2 are pairwise independent random variables. First, we consider the cumulative distribution function of (T_1, T_2). Let $t_1, t_2 > 0$:

$$\begin{aligned} P(T_1 \le t_1, T_2 \le t_2) = \sum_{j=1}^{s_1} P\Bigg(&\sum_{i=1, i\ne j}^{s_1} \gamma_{11}^{(i)} \left(X_{*i}^{(1)} - X_{*j}^{(1)}\right) \le t_1, \\ &\sum_{i=1}^{s_2} \gamma_{21}^{(i)} \left(Y_{*i}^{(1)} - X_{*j}^{(1)}\right) \le t_2,\ X_{*j}^{(1)} \le Z \Bigg) \\ + \sum_{j=1}^{s_2} P\Bigg(&\sum_{i=1}^{s_1} \gamma_{11}^{(i)} \left(X_{*i}^{(1)} - Y_{*j}^{(1)}\right) \le t_1, \\ &\sum_{i=1, i\ne j}^{s_2} \gamma_{21}^{(i)} \left(Y_{*i}^{(1)} - Y_{*j}^{(1)}\right) \le t_2,\ Y_{*j}^{(1)} \le Z \Bigg) = \sum_{j=1}^{s_1} A_j + \sum_{j=1}^{s_2} B_j, \quad \text{say}. \end{aligned}$$

Now, we evaluate the summands A_j and B_j.

$$\begin{aligned} A_j &= \int_0^\infty P\Bigg(\sum_{i=1, i\neq j}^{s_1} \gamma_{11}^{(i)} X_{*i}^{(1)} \Big) \le t_1 + \big(\bar{\gamma}_1 - \gamma_{11}^{(j)}\big) z, z \le \min_{i \neq j} X_{*i}^{(1)}, \\ &\quad \times \sum_{i=1}^{s_2} \gamma_{21}^{(i)} Y_{*i}^{(1)} \le t_2 + \bar{\gamma}_2 z,\ z \le \min_i Y_{*i}^{(1)} \Bigg) dP^{X_{*j}^{(1)}}(z) \\ &\overset{(*)}{=} \int_0^\infty P\Bigg(\sum_{i=1, i\neq j}^{s_1} \gamma_{11}^{(i)} X_{*i}^{(1)} \Big) \le t_1 + \big(\bar{\gamma}_1 - \gamma_{11}^{(j)}\big) z, z \le X_{*i}^{(1)},\ i \neq j \Bigg) \\ &\quad \times P\Bigg(\sum_{i=1}^{s_2} \gamma_{21}^{(i)} Y_{*i}^{(1)} \le t_2 + \bar{\gamma}_2 z,\ z \le Y_{*i}^{(1)} \text{ for all } i \Bigg) dP^{X_{*j}^{(1)}}(z) \\ &\overset{(**)}{=} \left(\frac{\bar{\gamma}_1}{\vartheta_1} + \frac{\bar{\gamma}_2}{\vartheta_2} \right)^{-1} \frac{\gamma_{11}^{(j)}}{\vartheta_1} \Gamma_{t_1}(s_1 - 1, \vartheta_1) \Gamma_{t_2}(s_2, \vartheta_2), \end{aligned}$$

where $\Gamma_z(s, \vartheta)$ denotes the distribution function of a gamma variate with parameters s and ϑ. In $(*)$ we make use of the independence of the samples and in $(**)$ we apply formula (3.1) of (17). An analogous expression holds for B_j, such that we finally find

$$\begin{aligned} P(T_1 \le t_1, T_2 \le t_2) = \left(\frac{\bar{\gamma}_1}{\vartheta_1} + \frac{\bar{\gamma}_2}{\vartheta_2} \right)^{-1} \Bigg\{ & \frac{\bar{\gamma}_1}{\vartheta_1} \Gamma_{t_1}(s_1 - 1, \vartheta_1) \Gamma_{t_2}(s_2, \vartheta_2) \\ & + \frac{\bar{\gamma}_2}{\vartheta_2} \Gamma_{t_1}(s_1, \vartheta_1) \Gamma_{t_2}(s_2 - 1, \vartheta_2) \Bigg\}. \end{aligned}$$

Since $S_j \sim \Gamma(R_j - s_j + 1, \vartheta_j)$, $j = 1, 2$, we obtain

Theorem 5.1.

1. *(V_1, V_2) and Z are independent random variables.*
2. $Z \sim \mathrm{Exp}(\mu, (\frac{\bar{\gamma}_1}{\vartheta_1} + \frac{\bar{\gamma}_2}{\vartheta_2})^{-1})$.
3. *The joint distribution function of V_1, V_2 is given by*

$$\begin{aligned} P(V_1 \le v_1, V_2 \le v_2) = \left(\frac{\bar{\gamma}_1}{\vartheta_1} + \frac{\bar{\gamma}_2}{\vartheta_2} \right)^{-1} \Bigg\{ & \frac{\bar{\gamma}_1}{\vartheta_1} \Gamma_{v_1}(R_1 - 1, \vartheta_1) \Gamma_{v_2}(R_2, \vartheta_2) \\ & + \frac{\bar{\gamma}_2}{\vartheta_2} \Gamma_{v_1}(R_1, \vartheta_1) \Gamma_{v_2}(R_2 - 1, \vartheta_2) \Bigg\}. \end{aligned}$$

The distributional results presented in Sections 2 and 3 for the two-sample case, apply in this set-up as well. For brevity, we abstain from giving the details.

Similar to the two-sample case it is shown that

$$\hat{P}_R = \frac{R_1 V_2}{R_2 V_1 + R_1 V_2}$$

is the MLE of P based on the $s_1 + s_2$ samples, and that the statistic (V_1, V_2, Z) is complete sufficient. Moreover, we conclude from Theorem 5.1 and the two-sample case that the UMVUE $\tilde{P}_R$ of P is given by Theorem 2.1 in (12) with V_j, R_j and $\bar{\gamma}_j$ instead of $\gamma_{j1}(W_j - Z)$, r_j and γ_{j1}, respectively (cf. (5.2)). This result is deduced from the fact, that the estimator $\tilde{P}_R$ constructed this way is unbiased. This property is a consequence of the distribution given in Theorem 5.1. Moreover, all the distributional results of Section 2 hold (with obvious changes) as well.

In the given situation, it is near at hand to deal with asymptotics w.r.t. an increasing number of samples. First of all, we show that the sequence of MLEs $(\hat{P}_R)_R$ is strongly consistent. By analogy with Lemma 1 of (7) it is shown that $(V_j/R_j)_{R_j}$ is strongly consistent for $\vartheta_j > 0$, $j = 1, 2$.

Hence, the sequence of MLEs $(\hat{P}_R)_R$ is strongly consistent for arbitrary (R_1, R_2) with $\min\{R_1, R_2\} \to \infty$. Moreover, an application of the dominated convergence theorem yields that the MLE $\hat{P}_R$ is an asymptotically unbiased estimator of P (cf. (30), p. 11). Utilizing Theorem 5.1, i.e., the distribution of $\hat{P}_R$, we obtain the asymptotic normality of the MLE:

Theorem 5.2. *Let* $\lim_{R_1, R_2 \to \infty} R_1/(R_1 + R_2) = \delta \in (0, 1)$ *and* $\lim_{R_1, R_2 \to \infty} \bar{\gamma}_1/\bar{\gamma}_2 = \beta$.
Then the following assertions hold:

$$\sqrt{R}\left(\begin{pmatrix} V_1/R_1 \\ V_2/R_2 \end{pmatrix} - \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \begin{pmatrix} \vartheta_1/\delta & 0 \\ 0 & \vartheta_2/(1-\delta) \end{pmatrix}\right)$$

and

$$\sqrt{R}(\hat{P}_R - P) \xrightarrow{d} \mathcal{N}\left(0, \frac{P^2(1-P)^2}{\delta(1-\delta)}\right). \tag{5.1}$$

Proof: The first result follows directly from the joint cumulative distribution function of V_1 and V_2. The second result is a consequence of the continuous mapping theorem (cf. (30), p. 24).

Considering the asymptotic variance in (5.1) we conclude that it is preferable to choose the sequences (R_1) and (R_2) such that $R_1 \sim R_2$. This choice leads to the smallest asymptotic variance of $\hat{P}_R$, i.e., $4P^2(1-P)^2$.

By analogy with (28) we obtain a similar asymptotic result for the UMVUE of P. In the more sample situation the UMVUE is given by the piecewise defined estimator

$$\tilde{P}_R = \begin{cases} \dfrac{(R_1-1)\bar{\gamma}_1 V_2 + (R_2-1)\bar{\gamma}_2 V_1 (1-\frac{\bar{\gamma}_1}{\bar{\gamma}_2}) F(2-R_2,1;R_1;V_1/V_2)}{(R_1-1)\bar{\gamma}_1 V_2 + (R_2-1)\bar{\gamma}_2 V_1}, & V_1 \le V_2, \\ 1 - \dfrac{(R_2-1)\bar{\gamma}_2 V_1 + (R_1-1)\bar{\gamma}_1 V_2 (1-\frac{\bar{\gamma}_2}{\bar{\gamma}_1}) F(2-R_1,1;R_2;V_2/V_1)}{(R_1-1)\bar{\gamma}_1 V_2 + (R_2-1)\bar{\gamma}_2 V_1}, & V_1 > V_2. \end{cases} \tag{5.2}$$

Theorem 5.3. *Let* $\lim_{R_1,R_2\to\infty} R_1/(R_1+R_2) = \delta \in (0,1)$ *and* $\lim_{R_1,R_2\to\infty} \bar{\gamma}_1/\bar{\gamma}_2 = \beta$.
The UMVUE $\tilde{P}_R$ *and the MLE* $\hat{P}_R$ *are asymptotically equivalent in the sense that*

$$\sqrt{R}(\tilde{P}_R - \hat{P}_R) \to 0 \text{ a.e. w.r.t. } R \to \infty.$$

Moreover, the asymptotic distribution of $\sqrt{R}(\tilde{P}_R - P)$ *is given by* (5.1).

Proof: From Lemma 3.1 of (28) we deduce the expansion

$$F(2-R_2,1;R_1;V_1/V_2) = \frac{R_1 V_2}{(R_2-1)V_1 + R_1 V_2} + \mathcal{O}(1/R), \quad R = R_1 + R_2.$$

Considering the difference $\tilde{P}_R - \hat{P}_R$ given the assumption $V_1 \le V_2$, we obtain after some lengthy calculations

$$\tilde{P}_R - \hat{P}_R = \frac{1}{R}\frac{V_1 V_2}{(R_2-1)(R_1-1)}\left\{ -\frac{R}{R_1} + \frac{\bar{\gamma}_2}{\bar{\gamma}_1}\frac{R}{R_2} \right.$$
$$\left. -\left(1-\frac{\bar{\gamma}_2}{\bar{\gamma}_1}\right)\frac{R}{R_2}\frac{V_2}{R_2-1}\left(\frac{V_1}{R_1}+\frac{V_2}{R_2-1}\right)^{-1}\right\} + \mathcal{O}(1/R).$$

An analogous representation holds for $V_1 > V_2$. To sum up, we find $\tilde{P}_R - \hat{P}_R = \mathcal{O}(1/R)$. This yields the desired result. The asymptotic distribution of $\sqrt{R}(\tilde{P}_R - P)$ follows from Theorem 5.2. □

These asymptotic results can be utilized to derive approximate confidence bounds for the probability $P(X < Y)$. Moreover, the confidence interval given in Section 4 can be extended to the more sample case with obvious changes.

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