Dependent Types and Theorem Proving: Introduction to Dependent Types

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Prerequisites

- To understand what we will be talking about, you should have a working knowledge of F# and the basic concepts of functional programming, namely:
- All about types: algebraic data types, sum types, product types, record types, pattern matching etc.
- All about functions: functions as first-class citizens, higher-order functions, recursive functions, currying etc.
- Even if you know these, you may be unfamiliar with the particular names – for example, “sum types” is a name used in academia and Haskell, but in F# they are better known as “discriminated unions”.

Learning outcomes

- You will get basic familiarity with the ideas behind all dependently typed languages.
- You will learn about all the different kinds of dependent types and what they are good for.
- You will be able to continue learning about dependent types on your own and won't be put off by all those obscure, scary and mysterious names and notations.

Introducing F*

- F* (pronounced "eff star") is a general-purpose purely functional programming language.
- Member of the ML family, syntactically most similar to F#.
- Aimed at program verification.
- Dependent types.
- Refinement types.
- Effect system.

Useful links

- **Repo with all lecture materials**: `https://github.com/wkolowski/Dependent-Types-and-Theorem-Proving`
- **You can run F* inside your browser** (and have a nice tutorial guide you): `http://www.fstar-lang.org/tutorial/`
- GitHub: `https://github.com/FStarLang/FStar`
- Homepage: `http://www.fstar-lang.org/`
- Download: `http://www.fstar-lang.org/#download`
- Papers (not approachable for ordinary mortals): `http://www.fstar-lang.org/#papers`
- Talks/presentations (more approachable): `http://www.fstar-lang.org/#talks` (some of these are quite approachable if you're interested)

Code snippet no. 1 - basics of F*

- We will now see some code that shows how these prerequisites look in F* (hint: basically the same as in F#).
- See the file Standalone/Code/Prerequisites.fst

Why should we care about dependent types? 1/3

- Programs written in dynamically typed languages perform a lot of runtime checks.
- Beyond a certain size **dynamically typed software is hard to extend, refactor and maintain because errors manifest very late** in the development process, i.e. at runtime.
- Statically typed languages make the situation better, because they move typechecking to compile time, which means a lot of errors get caught much sooner.
- **Static typing is good**.

Why should we care about dependent types? 2/3

- But in simple functional languages like F# there's still plenty of runtime checks – division by zero, taking the head of empty list and a lot of user-defined checks which throw exceptions in case of failure.
- With dependent types, all runtime checks can be turned into static checks – **all errors are type errors**.
- This results in more extensible, refactorable and maintainable software (and also better performance – less stuff to do at runtime).
- We can not only get rid of runtime checks, dependent types can also replace most unit tests and property tests.
- **Dependent types bring static typing to its limits**.

Why should we care about dependent types? 3/3

- And when I say all errors are typing errors, I really mean it – with dependent types, we can express all properties, formulate all specifications and describe all mathematical objects.
- **Dependent types reveal a deep connection between functional programming and logic**.
- Despite their great power, dependent types are easy to understand and significantly simplify the language design.
- Have you ever heard about fancy Haskell stuff like multi-param typeclasses, GADTs, higher-rank types, higher-kinded types, existential types and so on?
- No? No problem – **with dependent types, we get all of that (and much more) for free**.

Matrix multiplication

- We can only multiply matrices whose dimensions match, i.e. we can multiply an $n \times m$ matrix by a $m \times k$ and get an $n \times k$ matrix as a result.
- How to model this in our favourite programming language without dependent types?
- The best we can do is to have **a type of matrices** `Matrix` and then matrix multiplication has type `matmult : Matrix -> Matrix -> Matrix`.
- What happens when we call it with matrices of the wrong dimensions?
- `matmult` $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **is well-typed, but will throw** an `IllegalArgumentException` or some other kind of runtime error, or maybe it will crash even less gracefully.

Matrix multiplication with dependent types

- In a language with dependent types we can define `Matrix n m`, **the type of $n \times m$ matrices**, and give multiplication the type `matmult : (n : ℕ) -> (m : ℕ) -> (k : ℕ) -> Matrix n m -> Matrix m k -> Matrix n k`
- Now `matmult` is a function which takes five arguments: the three matrix dimensions and the two matrices themselves.
- After giving it the dimensions of the first matrix from the previous slide, `matmult 2 2` has type `(k : ℕ) -> Matrix 2 2 -> Matrix 2 3 -> Matrix 2 3`.
- It is clear that `matmult 2 2 k` $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **is not well-typed** for any `k`, because the last argument is of type `Matrix 3 3`, but an argument of type `Matrix 2 k` was expected.

Array access

- When accessing the *i*-th element of an array, *i* must be smaller than the length of the array.
- How to model this in our favourite programming language without dependent types?
- We can define `Array a`, **the type of arrays that hold elements of type** `a`, and we can access its elements with a function `get : Array a -> int -> a`.
- What happens, when *i* is greater than the length of the array? Or, what happens when *i* is negative?
- `get [| 'a'; 'b'; 'c' |] 5` is well-typed, but will throw an `IndexOutOfBoundsException` or result in a segmentation fault.

Array access with dependent types

- In a language with dependent types we can define `Array a n`, **the type of arrays of length *n* that hold elements of type** a, and we give array access the type `get : (n : ℕ) -> Array a n -> (i : int{0 <= i < n}) -> a`.
- We use refinement types (which we will cover later) to statically guarantee that `i` isn't out of bounds.
- `get 3 [| 'a'; 'b'; 'c' |] 5` is not well-typed, because the typechecker can't prove `0 <= 5 < 3`, and thus 5 is not of type `int{0 <= 5 < 3}`.

Values and types

- To understand dependent types, first we have to be aware of the distinction between **values** and **types** that is present in all the usual mainstream programming languages.
- By **values**, we mean the bread-and-butter of programming: numbers, strings, arrays, lists, functions, etc.
- It should be pretty obvious to you that in most languages, **types** are not of the same status as numbers or functions.

Dependencies

- Dependency is easy to understand. In fact, if you know basic F#, then you already know most of it, because in F#:
- **Values can depend on values**: we can think that the sum `n + m` is a number that depends on the numbers `n` and `m`. This dependency can be expressed as a **function**: `fun (n m : int) -> n + m`.
- **Values can depend on types**: for example, the identity function `fun (x : 'a) -> x` depends on the type `'a`. This kind of dependency is called **generics** (or, in academia, **polymorphism**).
- **Types can depend on types**: for example, the F# type `Set<'a>` depends on the type `'a`. This kind of dependency is called **type operators**.

Dependent types

- There's yet another kind of dependency, which is not present in F#, but is present in F* and is the topic of this lecture.
- **Types can depend on values**: **dependent types**.
- We have already seen two examples: the type `Matrix n m` depends on the numbers `n` and `m`, and the type `Array a n` depends on the type `a` and the number `n`.
- Given a functional language like F#, how to enable types to depend on values?
- Of course we want to retain the other kinds of dependencies (values on values, values on types, types on types).

Juggling dependencies

- It turns out it's best to **throw away all kinds of dependencies besides the basic one** (values on values, i.e. functions). . .
- . . . and then **turn types into values**!
- In other words: we want to make types first-class citizens of our language.
- Then we will be able to express all 4 kinds of dependencies using plain old functions.

What does “first-class” mean?

- **The concept of “first-class” is neither precisely defined nor exact**. Rather, it’s more of a functional programming folklore that obeys the “I know it when I see it” principle.
- However there are some heuristics that can help you.
- **Something is first-class when it can be**:
- bound/assigned to variables.
- stored in data structures.
- passed to functions as an argument.
- returned from functions.
- constructed at runtime.
- nameless, i.e. it can exist without giving it any name.

A type-based definition of “first-class”

- Heuristics from previous slides are nice. . .
- . . . but I prefer to think about first-class-ness in a different way, which is better from the functional programming point of view.
- For a given programming language, **a concept X is first-class if there is a type of all Xs**, loosely speaking.
- This means that, for example, a language has first-class functions if for any two types A and B there is a type $A \to B$ of all functions from A to B.

The Universe of Types

- For types, this means that we need to have a **type of types**.
- And that's it – we don't need anything else.
- Note: the phrase "types of types" sounds (and looks) bad, so we will call it **the universe of types**, or in short, just **the universe**.
- Because types are first-class in F*, we can assign them to variables, pass them to functions as arguments and return them from functions, and even compute types by recursion.

Type families

- In the coming slides, we will often refer to **type families**.
- **A family of types indexed by type** `a` is just a function `a -> Type`.
- There can be many indices, like in `a -> b -> Type`.
- We have already seen examples in the last code snippet:
- `Array : Type -> nat -> Type` is a family of types whose members `Array a n` are types of arrays of length `n` that hold elements of type `a`.
- `Matrix : nat -> nat -> Type` is a family of types whose members `Matrix n m` are $n \times m$ matrices.

Code snippet no. 2 - first-class types in F*

- It might a bit difficult to wrap your head around the idea of first-class types, so let's see how it plays out in F*.
- The code snippet can be found in Standalone/Code/Universe.fst

Dependent types by analogy

- We will introduce dependent types by analogy.
- **Each of the various kinds of dependent types out there is just a generalization of an ordinary non-dependent type that is well-known to functional programmers**:
- Dependent functions are a generalization of functions.
- Dependent pairs are a generalization of pairs.
- Dependent records are a generalization of records.
- Inductive types are a generalization of algebraic data types.

Non-dependent functions

- Recall how ordinary functions work in F#.
- If `a : Type` is a type and `b : Type` is a type, then there is a type `a -> b : Type` of functions that take an element of `a` and return an element of `b`.
- We create functions of type `a -> b` by writing `fun (x : a) -> e` where `e` is an expression of type `b` in which `x` may occur.
- If we have a function `f : a -> b` and `x : a`, then we we can apply `f` to `x`, written `f x`, to get an element of type `b`.

Dependent functions

- Now, watch the analogy unfold...
- If `a : Type` is a type and `b : a -> Type` **is a family of types**, then there is a type `(x : a) -> b x` **of dependent functions** which take an input of type `a` **named** `x` and return an output of type `b x`.
- We create functions of type `(x : a) -> b x` by writing `fun (x : a) -> e` where `e` is an expression of type `b x` in which `x` may occur.
- If we have a function `f : (x : a) -> b x` and an `x : a`, then we can apply `f` to `x`, written `f x`, to get an element of type `b x`.
- Hint: it's probably easiest to pronounce `(x : a) -> b x` as "for all `x` of type `a`, `b` of `x`". Thus is revealed the connection to logic, which we will revisit later.

More dependent functions

- Of course, we can iterate the dependent function type to get a type of functions whose output type depends on many input values.
- `(x : a) -> b x`
- `(x : a) -> ((y : b x) -> c x y)`
- Dependent function type associates to the right, just like ordinary function type, so we can drop the parentheses. We can also drop all but the last arrow.
- `(x : a) (y : b x) (z : c x y) -> d x y z`
- `(x : a) (y : b x) (z : c x y) (w : d x y z) -> e x y z w`
- etc.

Code snippet no. 3 - dependent functions in F*

- Let's see how to use dependent functions in F*.
- See the code snippet
  Standalone/Code/DependentFunctions.fst

Non-dependent pairs

- Recall how ordinary pairs work in F#.
- If `a : Type` is a type and `b : Type` is a type, then there is a type `a * b : Type` of pairs.
- To create a pair, we write `(x, y)` where `x` is of type `a` and `y` is of type `b`.
- To use a pair `p : a * b`, we use projections – we have `fst p : a` and `snd p : b`.
- We can also pattern match on pairs.

Dependent pairs

- Now, watch the analogy unfold. . .
- If `a : Type` is a type and `b : a -> Type` **is a family of types**, then there is **a type** `(x : a) & b x : Type` **of dependent pairs**.
- To create a dependent pair, we write `(| x, y |)` where `x` is of type `a` **and** `y` **is of type** `b x`.
- To use a pair `p : (x : a) & b x`, we use projections – we have `fst p : a` and `snd p : b (fst p)` (**note that the type of the second projection depends on the value of the first projection**).
- We can also pattern match on dependent pairs.

More dependent pairs

- **We can iterate the dependent pair type**, while dropping unneeded parentheses – analogously to what we did for dependent functions.
- `(x : a) & b x`
- `(x : a) & (y : b x) & c x y`
- `(x : a) & (y : b x) & (z : c x y) & d x y z`
- **But using iterated dependent pairs is very inconvenient!**
- To access components of a dependent quadruple `p` we need to write `fst p`, `fst (snd p)`, `fst (snd (snd p))` and `snd (snd (snd p))`.

Dependent record types

- There's a better way than iterating dependent pairs: dependent records.
- A record is basically a labeled tuple.
- **A dependent record is basically a labeled dependent tuple**.
- This means that the TYPES of later fields in a dependent record can depend on the VALUES of earlier fields.

Code snippet no. 4 - dependent records in F*

- Let's see how dependent records work in F*.
- See the code snippet Standalone/Code/DependentRecords.fst

Inductive types refresher

- Recall how ordinary inductive types work in F# (where they are called discriminated unions; in Haskell, they are knwon as algebraic data types).
- To define an inductive type `I : Type`, we list its constructors.
- The constructors are ordinary functions which take some arguments (which may be of type `I`, i.e. the one that is being defined) and return an element of `I`.
- To create an element of `I`, we use one of the constructors and provide it with the arguments it requires.
- To use an element of `I`, we pattern match on it and for each case we provide an expression which will be computed if that case matches.

Inductive families 1/2

- Now watch the analogy unfold...
- To define an **inductive family** `I : a -> Type`, we list its constructors. Here `a` is some type that is already defined.
- The constructors are **dependent functions** which take some arguments (which **may be of type** `I y` for some `y : a`) and **return an element of the type** `I x`, for some `x : a`.
- To create an element of `I x`, we use one of the constructors and provide it with the arguments it requires.
- To use an element of `I x`, we pattern match on it and for each case we provide an expression which will be computed if that case matches.

Inductive families 2/2

- This time it's a bit harder to spot the analogy, so let's elaborate on it.
- Instead of a single type `I : Type`, we define a **family of types** `I : a -> Type` all at once.
- In this context, values of type `a` are called **indices** of the family `I`.
- We define a **separate type for each possible index**.
- To create a value that belongs to **some type** `I x` in the family, a constructor may require an argument that belongs to `I y`, **a different type** in the family.

Code snippet no. 5 - inductive families in F*

- Let's see how inductive families work in F*.
- See the code snippet Standalone/Code/InductiveFamilies.fst

Summary of dependent types

- Dependent types are types that can depend on values.
- Dependently typed languages have:
- A universe – a type whose elements are themselves types.
- Dependent functions, which are just like ordinary functions, but their output TYPE can depend on the VALUE of their input.
- Dependent records, which are just like ordinary records, but the TYPES of later fields can depend on the VALUES of earlier fields.
- Inductive families, which are just like ordinary inductive types, but the TYPES in the family can depend on the VALUE of the index.

Some downsides of dependent types

- In dependently typed languages there is a lot of types.
- This is a blessing, because we can express all the complicated types and properties we need in order to guarantee correctness of our programs.
- But the richness of types also causes problems: it is often the case that there are many ways to define essentially the same type, which can give us a lot of headache.
- It also means we need to write a lot of boilerplate - for example, we need to define `map` separately for lists and vectors.

Refinement types to the rescue

- Beware: refinement types are not commonly considered to be dependent types!
- A refinement is just a different name for a function that returns a boolean.
- A refinement type is a type together with a refinement.
- In F* syntax: if `p : a -> bool`, then `x : a{p x}` is the type of values of type `a` for which `p` returns `true`.

Code snippet no. 6 - refinement types in F*

- Let’s see how refinement types work in F*.
- See the code snippet Standalone/Code/Refinements.fst

This is (almost) the end

- This is the end of the basic version of the slides.
- However, there are two bonus sections, which were initially supposed to be put into a continuation of this presentation (which was eventually cancelled).
- The next section describes the alternative names and notations used for dependent types in the literature.
- The last section is an (incomplete) introduction to logic. Towards the end, it explains the link between dependent types and logic.

Pi types and multiplication

- The dependent function type **is also known as the Pi type**.
- **This name comes from a notation**: `(x : a) -> b x` is sometimes written as $\prod_{x:\ a} b(x)$.
- **This notation comes from an analogy with multiplication**. In math $\prod_{k=0}^{n} a_k$ means $a_0 \cdot a_1 \cdot \ldots \cdot a_n$.
- **We can think about dependent function types in this way too**. For example, the type `(x : bool) -> p x` is equivalent to `p true * p false`.
- The result of multiplication is called a product, hence the dependent function type **is also known as the dependent product type**.
- As it turns out, the dependent function type **generalizes both the ordinary function type and the product type**, but in different ways.

Sigma types and addition

- The dependent pair type **is also known as the Sigma type**.
- **This name comes from a notation**: `(x : a) & b x` is sometimes written as $\sum_{x:\ a} b(x)$.
- **This notation comes from an analogy with addition**. In math $\sum_{k=0}^{n} a_k$ means $a_0 + a_1 + \ldots + a_n$.
- **We can think about dependent pair types in this way too**. For example, the type `(x : bool) & p x` is equivalent to `p true + p false` (where + just means a simple tagged union).
- The result of addition is called a sum, hence the dependent pair type **is also known as the dependent sum type**.
- As it turns out, the dependent pair type **generalizes both the product type and the sum type**, but in different ways.

Inductive types and polynomials 1/2

- An inductive type is **EITHER** constructor 1 applied to arguments `x1` **and** `x2` . . . **and** `xN` **OR** constructor 2 applied to arguments . . . **OR** constructor M applied to arguments . . .
- In math, OR means **addition**, whereas AND means **multiplication**.
- So, an inductive type boils down to a **Sum of Products**.
- These products are made of two kinds of arguments: recursive arguments (whose type is the inductive type that is being defined) and non-recursive ones.
- If you think about it long enough, **inductive types correspond to polynomials**.

Inductive types and polynomials 2/2

- This could be hard to swallow, so let's see examples.
- Lists satisfy the equation $\texttt{List}(A) = 1 + A \times \texttt{List}(A)$.
- Here 1 corresponds to the `nil` constructor, whereas the A and $\texttt{List}(A)$ on the right correspond to the arguments of the `cons` constructor.
- This corresponds to the polynomial $F(X) = 1 + A \times X$.
- $\texttt{List}(A)$ is the least fixed point of this polynomial, i.e. the smallest type X that satisfies $F(X) = X$.
- Here "fixed point" corresponds to the fact that we create lists using constructors (`nil` and `cons`), whereas "least" corresponds to the fact that all lists are made of finitely many constructors.

Boolean "logic"

- Being a programmer, you are good friends with the booleans, aren't you?
- There are two booleans, `true` and `false`.
- We can combine booleans `b` and `c` with the usual boolean functions:
- `not b` – "not b"
- `b && c` – "b and c"
- `b || c` – "b or c"

What is a logic

- Boolean logic is not an example of what logicians call a "logic", in the sense that it is not a "logical system", but merely a type with some unary and binary functions on it.
- A logic usually consists of:
- A definition of what **propositions** we're dealing with.
- A **semantics**, which tells us what these propositions mean.
- A **proof system**, which tells us which propositions can be proven and disproven.
- A **soundness theorem** which states that propositions proven true using the proof system are semantically true.
- Optionally, there may also be a **completeness theorem** which states that all semantically true propositions can be proven.

Propositions

- A proposition asserts that something is the case, irrespectively of whether this really is the case or not.
- Math example: "4 is a prime number."
- Software example: "For each input string x, if x is not malformed, my program produces as output an array of length at most 10."
- Hardware example: "This circuit implements addition of 16 bit integers."
- Real world example: "It's raining or I like trains."
- **Beware! Formal logic is not very good for reasoning about the real world!**

Propositional constants and connectives

- Propositions (usual letters: P, Q, R) are defined as follows:
- $\top$ – the true proposition.
- $\bot$ – the false proposition.
- $P, Q, R, \ldots$ – propositional variables.
- $\neg P$ – negation, read "not P".
- $P \vee Q$ – disjunction, read "P or Q".
- $P \wedge Q$ – conjunction, read "P and Q".
- $P \implies Q$ – implication, read "P implies Q" or "if P then Q".
- $P \iff Q$ – logical equivalence, read "P if and only if Q".

Classical logic

- Classical logic is the most widely known/taught/used logical system in the world.
- In classical logic, **we think of propositions as being either true or false.**
- Therefore, classical logic is the logic in which truth values are the booleans.
- The truth value of a propositional variable is determined by a **valuation** $v : \text{Var} \to \text{Bool}$.
- If $v(P) = \texttt{true}$, then P is considered to be true.
- Otherwise it's considered false.

Semantics of classical logic 1/2

- Given a valuation v : Var $\to$ Bool, the truth value of a proposition can be determined with a recursive function $[\![-]\!]$: Prop $\to$ Bool.
- $[\![\top]\!] = \texttt{true}$
- $[\![\bot]\!] = \texttt{false}$
- $[\![P]\!] = v(P)$, where P is a variable.
- $[\![\neg P]\!] = \texttt{not}\ [\![P]\!]$
- $[\![P \vee Q]\!] = [\![P]\!]\ \texttt{||}\ [\![Q]\!]$
- $[\![P \wedge Q]\!] = [\![P]\!]\ \texttt{\&\&}\ [\![Q]\!]$
- $[\![P \implies Q]\!] = (\texttt{not}\ [\![P]\!])\ \texttt{||}\ [\![Q]\!]$
- $[\![P \iff Q]\!] = [\![P]\!]\ \texttt{==}\ [\![Q]\!]$

Semantics of classical logic 2/2

- P is satisfiable when $[\![P]\!] = \texttt{true}$ for some valuation.
- P is falsifiable when $[\![P]\!] = \texttt{false}$ for some valuation.
- P is a tautology when $[\![P]\!] = \texttt{true}$ for all valuations.

Example

- Example: the proposition $P \implies Q$ is satisfiable (for $v(P) = \texttt{true}, v(Q) = \texttt{true}$).
- It is also falsifiable (for $v(P) = \texttt{true}, v(Q) = \texttt{false}$).
- Therefore, it is not a tautology.
- Example: the proposition $P \wedge Q \implies Q \wedge P$ is a tautology.
- We have $[\![P \wedge Q \implies Q \wedge P]\!] =$ $(\texttt{not}\ (v(P)\ \texttt{\&\&}\ v(Q)))\ \texttt{||}\ (v(P)\ \texttt{\&\&}\ v(Q))$.
- For any values of $v(P)$ and $v(Q)$ we always get $\texttt{true}$.

Classical logic is not that good

- We can check whether a proposition is a tautology by trying all possible valuations, but there are exponentially many of them.
- We can do better by defining a proof system with some axioms and inference rules, which would allow us to **prove** that a proposition is a tautology without trying all valuations.
- We won't do that because **classical logic is not the right logical system for proving programs correct**.
- We will use constructive logic instead.

Constructive logic 1/2

- In constructive logic, **propositions ARE NOT either true or false**.
- In constructive logic we usually think about propositions **in terms of their proofs.**
- In everyday language and also in mathematics as it is usually practiced, a "proof" means an argument by which one human demonstrates the truth of a statement to another human.
- In constructive logic, a proof is a formal object which **certifies that the given proposition has been proven**, in which case we say that the propositions holds.
- Meaning of propositions is determined by how we can prove them and how we can use their proofs to prove other propositions.

Constructive logic 2/2

- If we have a proof of P, we may think of it as “true” (although we shouldn’t think in terms of true and false).
- If we have a proof of $\neg P$, we may think that P is “false”.
- If we have neither proof, we don’t know anything about P.

Propositions are types, proofs are programs

- **We can represent propositions using types.**
- $\top$ corresponds to the `unit` type.
- $\bot$ corresponds to the `Empty` type (which can be defined as a disjoint union with zero constructors).
- $P \vee Q$ corresponds to a disjoint union `or` with constructors `inl : P -> or P Q` and `inr : Q -> or P Q`.
- $P \wedge Q$ corresponds to the product type `P * Q`.
- $P \implies Q$ corresponds to the function type `P -> Q`.
- $\neg P$ corresponds to the function type `P -> Empty`.
- $P \iff Q$ corresponds to the type `(P -> Q) * (Q -> P)`.
- **You already know how to prove these propositions – just write the appropriate program!**

Dependent types, predicates, relations and quantifiers

- What about dependent types?
- Type families `p : a -> Type` correspond to predicates, i.e. propositions that describe properties of objects of type `a`. Type families with many indices, like `r : a -> b -> Type`, correspond to relations, i.e. propositions that describe relationships between a thing of type `a` and a thing of type `b`.
- The dependent function type `(x : a) -> p x` corresponds to the universal quantifier $\forall x : a, p(x)$, which can be read as "each object `x` of type `a` satisfies the predicate `p`".
- The dependent pair type `(x : a) & p x` corresponds to the existential quantifier $\exists x : a, p(x)$, which can be read as "there exists an object `x` of type `a` which satisfies the predicate `p`".
- Inductive families are a very comfortable way of defining predicates and relations.