Dependent Types and Theorem Proving: Introduction to Dependent Types

Wojciech Kołowski

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Plan of lectures

- Lecture 1: Programming with dependent types.
- Lecture 2: Proving theorems with dependent types.
- Lecture 3: Differences between programming and proving.
- Lecture 4: Examples of bigger programs and longer proofs.
- Lecture 5: A deeper dive into F*.

Prerequisites

- To understand what we will be talking about, you should have a working knowledge of F# and the basic concepts of functional programming, namely:
- All about types: algebraic data types, sum types, product types, record types, pattern matching etc.
- All about functions: functions as first-class citizens, higher-order functions, recursive functions, currying etc.
- Even if you know these, you may be unfamiliar with the particular names – for example, “sum types” is a name used in academia and Haskell, but in F# they are better known as “discriminated unions”. I will make

Learning outcomes

- You will get basic familiarity with the ideas behind all dependently typed languages.
- You will learn about all the different kinds of dependent types and what they are good for.
- You won't be scared of all those obscure, scary and mysterious names and notations.
- You will begin to see logic and mathematics in a very different light, much closer to your day job (at least if you are a programmer working in F#).
- If you do the exercises, you will gain a basic proficiency in F*.

Introducing F*

- F* (pronounced "eff star") is a general-purpose purely functional programming language.
- Member of the ML family, syntactically most similar to F#.
- Aimed at program verification.
- Dependent types.
- Refinement types.
- Effect system.
- Not a .NET language.
- Neither compiled nor interpreted – it's a proof assistant, i.e. just a typechecker.
- To run a program, it has to be extracted to some other language, like F#, OCaml, C or WASM, and then compiled.

Useful links

- **Repo with all lecture materials**: `https://github.com/wkolowski/Dependent-Types-and-Theorem-Proving`
- **You can run F* inside your browser** (and have a nice tutorial guide you): `http://www.fstar-lang.org/tutorial/`
- GitHub: `https://github.com/FStarLang/FStar`
- Homepage: `http://www.fstar-lang.org/`
- Download: `http://www.fstar-lang.org/#download`
- Papers (not approachable for ordinary mortals): `http://www.fstar-lang.org/#papers`
- Talks/presentations (more approachable): `http://www.fstar-lang.org/#talks` (some of these are quite approachable if you're interested)

Code snippet no 1 - basics of F*

- We will now see some code that shows how these prerequisites look in F* (hint: basically the same as in F#).
- See the file `Lecture1/Prerequisites.fst`.

Why should we care about dependent types? 1/3

- Programs written in dynamically typed languages perform a lot of runtime checks.
- Beyond a certain size **dynamically typed software is hard to extend, refactor and maintain because errors manifest very late** in the development process, i.e. at runtime.
- Statically typed languages make the situation better, because they move typechecking to compile time, which means a lot of errors get caught much sooner.
- **Static typing is good**.

Why should we care about dependent types? 2/3

- But in simple functional languages like F# there's still plenty of runtime checks – division by zero, taking the head of empty list and a lot of user-defined checks which throw exceptions in case of failure.
- With dependent types, all runtime checks can be turned into static checks – **all errors are type errors**.
- This results in more extensible, refactorable and maintainable software (and also better performance – less stuff to do at runtime).
- We can not only get rid of runtime checks, dependent types can also replace most unit tests and property tests.
- **Dependent types bring static typing to its limits**.

Why should we care about dependent types? 3/3

- And when I say all errors are typing errors, I really mean it – with dependent types, we can express all properties, formulate all specifications and describe all mathematical objects.
- **Dependent types reveal a deep connection between functional programming and logic**.
- Despite their great power, dependent types are easy to understand and significantly simplify the language design.
- Have you ever heard about fancy Haskell stuff like multi-param typeclasses, GADTs, higher-rank types, higher-kinded types, existential types and so on?
- No? No problem – **with dependent types, we get all of that (and much more) for free**.

Matrix multiplication

- We can only multiply matrices whose dimensions match, i.e. we can multiply an $n \times m$ matrix by a $m \times k$ and get an $n \times k$ matrix as a result.
- How to model this in our favourite programming language without dependent types?
- The best we can do is to have **a type of matrices** `Matrix` and then matrix multiplication has type `matmult : Matrix -> Matrix -> Matrix`.
- What happens when we call it with matrices of the wrong dimensions?
- `matmult` $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **is well-typed, but will throw** an `IllegalArgumentException` or some other kind of runtime error, or maybe it will crash even less gracefully.

Matrix multiplication with dependent types

- In a language with dependent types we can create **a type of** $n \times m$ **matrices** `Matrix n m` and give multiplication the type `matmult : (n : ℕ) -> (m : ℕ) -> (k : ℕ) -> Matrix n m -> Matrix m k -> Matrix n k`
- Now `matmult` is a function which takes five arguments: the three matrix dimensions and the two matrices themselves.
- After giving it the dimensions of the first matrix from the previous slide, `matmult 2 2` has type `(k : ℕ) -> Matrix 2 2 -> Matrix 2 3 -> Matrix 2 3`.
- It is clear that `matmult 2 2 k` $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **is not well-typed** for any `k`, because the last argument is of type `Matrix 3 3`, but an argument of type `Matrix 2 k` was expected.

A nice paper

- Dependent types can also be used to keep track of units of measure.
- This is possible in F# too, but it's a built-in feature of the compiler, whereas the dependently typed solution is much more principled.
- It is also composable – we can keep track of both matrix dimensions and units.
- There's a nice paper about this: **Type systems for programs respecting dimensions** available at `https://fredriknf.com/papers/dimensions2021.pdf`

Array access

- When accessing the i-th element of an array, i must be smaller than the length of the array.
- How to model this in our favourite programming language without dependent types?
- We have `Array A`, **the type of arrays of** `A`s, and we can access its elements with a function `get : Array A -> int -> A`.
- What happens, when i is greater than the length of the array? Or, what happens when i is negative?
- `get [| 'a'; 'b'; 'c' ] 5` is well-typed, but will throw an `IndexOutOfBoundsException` or result in a segmentation fault.

Array access with dependent types 1/2

- In a language with dependent types we can have `Array A n`, **the type of arrays of `A`s whose length is `n`.**
- Then we have a few possibilities to model the type of get.
- `get : (n : ℕ) -> Array A n -> (i : ℕ) -> i < n -> A`.
- In this variant, the fourth argument of get is a proof that the index isn't out of bounds (we will cover proofs in the next lecture).
- We can't prove `5 < 3`, so we don't have any proof to feed into `get 3 [| 'a'; 'b'; 'c' |] 5 : 5 < 3 -> Char`.

Array access with dependent types 2/2

- `get : (n : ℕ) -> Array A n -> (i : ℕ{i < n }) -> A`.
- In this variant we use refinement types (which we will cover later today) to automatically guarantee that `i` isn't out of bounds.
- `get 3 [| 'a'; 'b'; 'c' |] 5` is not well-typed, because the typechecker can't prove `5 < 3`, and thus 5 is not of type `ℕ{5 < 3}`.

We're getting serious

- The above slides present nice fairy tales. . .
- . . . but how do dependent types actually work?
- And how to use them in F*?
- And what can ordinary programmers use them for besides number crunching with matrices and arrays?

Values and types

- To understand dependent types, first we have to understand **dependency**.
- And to understand dependency, we need to be aware of the distinction between **values** and **types**.
- By **values**, we mean the bread-and-butter of programming: numbers, strings, arrays, lists, functions, etc.
- It should be pretty obvious to you that in most languages, **types** are not of the same status as numbers or functions.

Dependencies

- Dependency is easy to understand. In fact, if you know basic F#, then you already know most of it, because in F#:
- **Values can depend on values**: we can think that the sum `n + m` is a number that depends on the numbers `n` and `m`. This dependency can be expressed as a function: `fun (n m : int) -> n + m`.
- **Values can depend on types**: for example, the identity function `fun (x : 'a) -> x` depends on the type `'a`.
- **Types can depend on types**: for example, the F# type `Set<'a>` depends on the type `'a`.

Naming the dependencies

- I bet you spotted the pattern in the previous slide, but it's a good idea to also have a name for the feature provided by each kind of dependency.
- Values can depend on values: (first-class) **functions**.
- Values can depend on types: **polymorphism** (i.e. "generics").
- Types can depend on types: **type operators**.

Dependent types

- There's yet another kind of dependency, which is not present in F#, but is present in F* and is the topic of this lecture.
- **Types can depend on values**: **dependent types**.
- But what are dependent types good for? You have been living your whole life without them, after all!

The running summary 1

- **Dependent types are types that can depend on values**.

Juggling dependencies

- Given a functional language like F#, how to enable types to depend on values?
- Of course we want to retain the other kinds of dependencies (values on values, values on types, types on types).
- It turns out it's best **throw away all kinds of dependencies besides the basic one** (values on values). . .
- . . . and then **turn types into values**!

Values and types

- So now values encompass both old, ordinary values (integers, tuples, functions, etc.) and new values (types).
- This way we get all four kinds of dependencies:
- Ordinary values can depend on ordinary values.
- Ordinary values can depend on type values.
- Type values can depend on type values.
- Type values can depend on ordinary values.

First-class types

- How do we turn types into values?
- In programming languages' parlance, this process is called **making types first-class citizens of the language**.
- But what does "first-class" mean, anyway?

What does "first-class" mean?

- In C, we can define functions that take an `int` and return an `int`.
- But we can't define a function that takes a function from `int`s to `int`s and returns an `int`.
- This means that functions from `int`s to `int`s are not treated the same as `int`s.
- As far as C is concerned, we can say that **integers are first-class, but functions are not first-class** (thus, they are "second-class").
- But C has function pointers, so you may be skeptical when I claim it doesn't have first-class functions.

Some heuristics

- **The concept of "first-class" is neither precisely defined nor exact**. Rather, it's more of a functional programming folklore that obeys the "I know it when I see it" principle.
- However there are some heuristics that can help you.
- **Something is first-class when it can be**:
- bound/assigned to variables.
- stored in data structures.
- passed to functions as an argument.
- returned from functions.
- constructed at runtime.
- nameless, i.e. it can exist without giving it any name.
- So, which of these is criteria is not fullfilled by C's function pointers?

A first-class quiz

- Let's have a little quiz to check if you get it.
- **Are the below language features first-class in F# or not?**
- Functions?
- Recursive functions?
- Arrays?
- Modules?
- Records?
- Types?

A type-based definition of "first-class"

- Heuristics from previous slides are nice...
- but I prefer to think about first-class-ness in a different way, which is better from the functional programming point of view.
- For a given programming, **a concept X is first-class if there is a type of all Xs**, loosely speaking.
- This means that a language has first-class functions if for any two types A and B there is a type $A \to B$ of all functions from A to B.

A first-class quiz (again)

- Let's try again. This time answer using the type-based definition of "first-class".
- **Are the below language features first-class in F# or not?**
- Functions?
- Recursive functions?
- Arrays?
- Modules?
- Records?
- Types?

A first-class quiz answers

- Functions? For any `'a` and `'b` there's a type of functions `'a -> 'b`.
- Recursive functions? There's no separate type of recursive functions, even though there's a syntactic distinction between `let` and `let rec`!
- Arrays? For any type `'a` there's a type of arrays, namely `array<'a>`.
- Modules? Modules don't have types, they have signatures. But signatures are not types, so modules are not first-class.
- Records? This one is mixed depending on how you understand it. On the one hand, for any kind of record you can imagine, there's a corresponding type. But on the other hand, there is no type of all record types.
- Types? There are types in F# and there are type variables `a'`, `b'`, `c'` etc., but we can't assign them any type!

Computing with first-class types

- Previously we learned that "X is first-class" means that there is a type of all Xs.
- For types, this means that we need to have a **type of types**.
- And that's it – we don't need anything else.
- Note: the phrase "types of types" sounds (and looks) bad, so we will call it **the universe of types**, or in short, just **the universe**.
- Because types are first-class in F*, we can assign them to variables, pass them to functions as arguments and return them from functions, and even compute types by recursion.

Code snippet no 2 - first-class types in F*

- It might a bit difficult to wrap your head around the idea of first-class types, so let's see how it plays out in F*.
- The code snippet can be found in `Lecture1/FirstClassTypes.fst`

The running summary 2

- Dependent types are types that can depend on values.
- **In dependently typed languages:**
- **There is a universe – a type whose elements are themselves types**.

So far so good

- So far so good, but we still don't know how dependent types work.
- We saw some in the examples, but they were left unexplained.
- We also saw some more in the last code snippet, but those were the crudest and most primitive dependent types in existence.
- **Now that we have learned about first-class types and the universe of types, we can learn dependent types proper**.

Dependent types by analogy

- We will introduce dependent types by analogy.
- **Each of the various kinds of dependent types out there is just a generalization of an ordinary non-dependent type that is well-known to functional programmers**:
- Dependent function types are a generalization of function types.
- Dependent pair types are a generalization of products.
- Dependent record types are a generalization of records.
- Inductive types are a generalization of algebraic data types.

Type families

- In the coming slides, we will often refer to **type families**.
- **A family of types indexed by type** `a` is just a function `a -> Type`.
- There can be many indices, like in `a -> b -> Type`.
- We have already seen examples in the last code snippet:
- `vec : Type -> nat -> Type` is a family of types whose members `vec a n` are lists of length `n` and elements of type `a`
- `matrix : Type -> nat -> nat -> Type` is a family of types whose members `matrix a n m` are $n \times m$ matrices with entries of type `a`.

Non-dependent functions

- Recall how ordinary function types work in F#.
- If `a : Type` is a type and `b : Type` is a type, then there is a type `a -> b : Type` of functions that take an element of `a` and return an element of `b`.
- We create functions of type `a -> b` by writing `fun (x : a) -> e` where `e` is an expression of type `b` in which `x` may occur.
- If we have a function `f : a -> b` and `x : a`, then we we can apply `f` to `x`, written `f x`, to get an element of type `b`.

Dependent functions

- Now, watch the analogy unfold. . .
- If `a : Type` is a type and `b : a -> Type` **is a family of types**, then there is a type `(x : a) -> b x` **of dependent functions** which take an element of `a` **named** `x` and return an element of `b x`.
- We create functions of type `(x : a) -> b x` by writing `fun (x : a) -> e` where `e` is an expression of type `b x` in which `x` may occur.
- If we have a function `f : (x : a) -> b x` and `x : a`, then we can apply `f` to `x`, written `f x`, to get an element of type `b x`.
- Hint: it's probably easiest to pronounce `(x : a) -> b x` as "for all `x` of type `a`, `b` of `x`". Thus is revealed the connection to logic, which we will see in the next lecture.

More dependent functions

- Of course, we can iterate the dependent function type to get a type of functions whose output type dependent on the value of many inputs.
- `(x : a) -> b x`
- `(x : a) -> ((y : b x) -> c x y)`
- Dependent function type associates to the right, just like ordinary function type, so we can drop the parentheses. We can also drop all but the last arrow.
- `(x : a) (y : b x) (z : c x y) -> d x y z`
- `(x : a) (y : b x) (z : c x y) (w : d x y z) -> e x y z w`
- etc.

Code snippet no 3 - dependent functions in F*

- Let's see how to use dependent functions in F*.
- See the code snippet `Lecture1/DependentFunctions.fst`

The running summary 3

- Dependent types are types that can depend on values.
- In dependently typed languages:
- There is a universe – a type whose elements are themselves types.
- **There is a type of dependent functions which are just like ordinary functions, but their output TYPE can depend on the VALUE of their input.**

Non-dependent pairs

- Recall how ordinary pairs work in F#.
- If `a : Type` is a type and `b : Type` is a type, then there is a type `a * b : Type` of pairs.
- To create a pair, we write `(x, y)` where `x` is of type `a` and `y` is of type `b`.
- To use a pair `p : a * b`, we use projections – we have `fst p : a` and `snd p : b`.
- We can also pattern match on pairs.

Dependent pairs

- Now, watch the analogy unfold...
- If `a : Type` is a type and `b : a -> Type` **is a family of types**, then there is **a type** `(x : a) & b x : Type` **of dependent pairs**.
- To create a dependent pair, we write `(| x, y |)` where `x` is of type `a` **and** `y` **is of type** `b x`.
- To use a pair `p : (x : a) & b x`, we use projections – we have `fst p : a` and `snd p : b (fst p)` (**note that the type of the second projection depends on the value of the first projection**).
- We can also pattern match on dependent pairs.

More dependent pairs

- **We can iterate the dependent pair type**, while dropping unneeded parentheses – analogously to what we did for dependent functions.
- `(x : a) & b x`
- `(x : a) & (y : b x) & c x y`
- `(x : a) & (y : b x) & (z : c x y) & d x y z`
- **But using iterated dependent pairs is very inconvenient!**
- To access component of a dependent quadruple `p` we would have to write `fst p`, `fst (snd p)`, `fst (snd (snd p))` and `snd (snd (snd p))`.

Dependent record types

- There's a better way than iterating dependent pair types: dependent record types.
- A record is basically a labeled tuple.
- **A dependent record is basically a labeled dependent tuple**.
- This means that the TYPES of later fields in a dependent record can depend on the VALUES of earlier fields.

Code snippet no 4 - dependent records in F*

- Let's see how dependent records work in F*.
- See the code snippet `Lecture1/DependentRecords.fst`

The running summary 4

- Dependent types are types that can depend on values.
- In dependently typed languages:
- There is a universe – a type whose elements are themselves types.
- There is a type of dependent functions which are just like ordinary functions, but their output TYPE can depend on the VALUE of their input.
- **Dependent record types are just like ordinary records, but the TYPES of later fields can depend on the VALUE of earlier fields.**

Inductive types refresher

- Recall how ordinary inductive types work in F# (where they are called discriminated unions; in Haskell, they are knwon as algebraic data types).
- To define an inductive type `I : Type`, we list its constructors.
- The constructors are ordinary functions which take some arguments (which may be of type `I`, i.e. the one that is being defined) and return an element of `I`.
- To create an element of `I`, we use one of the constructors and provide it with the arguments it requires.
- To use an element of `I`, we pattern match on it and for each case we provide an expression which will be computed if that case matches.

Inductive families 1/2

- Now watch the analogy unfold...
- To define an **inductive family** `I : a -> Type`, we list its constructors. Here `a` is some type that is already defined.
- The constructors are **dependent functions** which take some arguments (which **may be of type** `I y` for some `y : a`) and **return an element of the type** `I x`, for some `x : a`.
- To create an element of `I x`, we use one of the constructors and provide it with the arguments it requires.
- To use an element of `I x`, we pattern match on it and for each case we provide an expression which will be computed if that case matches.

Inductive families 2/2

- This time it's a bit harder to spot the analogy, so let's elaborate on it.
- Instead of a single type `I : Type`, we define a **family of types** `I : a -> Type` all at once.
- In this context, values of type `a` are called **indices** of the family `I`.
- We define a **separate type for each possible index**.
- To create a value that belongs to **some type** `I x` in the family, a constructor may require an argument that belongs to `I y`, **a different type** in the family.

Code snippet no 5 - inductive families in F*

- Let's see how inductive families work in F*.
- See the code snippet `Lecture1/InductiveFamilies.fst`

The running summary 5

- Dependent types are types that can depend on values.
- In dependently typed languages:
- There is a universe – a type whose elements are themselves types.
- There is a type of dependent functions which are just like ordinary functions, but their output TYPE can depend on the VALUE of their input.
- Dependent record types are just like ordinary records, but the TYPES of later fields can depend on the VALUE of earlier fields.
- Inductive families are just like ordinary inductive types, but the TYPES in the family can depend on the VALUE of the index.

Pi type and multiplication

- The dependent function type **is also known as the Pi type**.
- **This name comes from a notation**: `(x : a) -> b x` is sometimes written as $\prod_{x:\ a} b(x)$.
- **This notation comes from an analogy with multiplication**. In math $\prod_{k=0}^{n} a_k$ means $a_0 \cdot a_1 \cdot \ldots \cdot a_n$.
- **We can think about dependent function types in this way too**. For example, the type `(x : bool) -> p x` is equivalent to `p true * p false`.
- The result of multiplication is called a product, hence the dependent function type **is also known as the dependent product type**.
- As it turns out, the dependent function type **generalizes both the ordinary function type and the product type**, but in different ways.

Sigma type and addition

- The dependent pair type **is also known as the Sigma type**.
- **This name comes from a notation**: `(x : a) & b x` is sometimes written as $\sum_{x:\ a} b(x)$.
- **This notation comes from an analogy with addition**. In math $\sum_{k=0}^{n} a_k$ means $a_0 + a_1 + ... + a_n$.
- **We can think about dependent pair types in this way too**. For example, the type `(x : bool) & p x` is equivalent to `p true + p false` (where + just means a simple tagged union).
- The result of addition is called a sum, hence the dependent pair type **is also known as the dependent sum type**.
- As it turns out, the dependent pair type **generalizes both the product type and the sum type**, but in different ways.

Inductive types and polynomials 1/2

- An inductive type is **EITHER** constructor 1 applied to arguments `x1` **and** `x2` ... **and** `xN` **OR** constructor 2 applied to arguments ... **OR** constructor M applied to arguments ...
- In math, OR means **addition**, whereas AND means **multiplication**.
- So, an inductive type boils down to a **Sum of Products**.
- These products are made of two kinds of arguments: recursive arguments (whose type is the inductive type that is being defined) and non-recursive ones.
- If you think about it long enough, **inductive types correspond to polynomials**.

Inductive types and polynomials 2/2

- This could be hard to swallow, so let's see examples.
- Lists satisfy the equation $\mathtt{List}(A) = 1 + A \times \mathtt{List}(A)$.
- Here 1 corresponds to the `nil` constructor, whereas the A and $\mathtt{List}(A)$ on the right correspond to the arguments of the `cons` constructor.
- This corresponds to the polynomial $F(X) = 1 + A \times X$.
- $\mathtt{List}(A)$ is the least fixed point of this polynomial, i.e. the smallest type X that satisfies $F(X) = X$.
- Here "fixed point" corresponds to the fact that we create lists using constructors (`nil` and `cons`), whereas "least" corresponds to the fact that all lists are made of finitely many constructors.