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Part 2: Estimating the Term-Structure
Part 3: Arbitrage Theory
Part 4: Short Rate Models
Part 5: Heath–Jarrow–Morton (HJM) Methodology
Part 6: Forward Measures
Part 7: Forwards and Futures
Part 8: Consistent Term-Structure Parametrizations
Part 9: Affine Processes
Part 10: Market Models

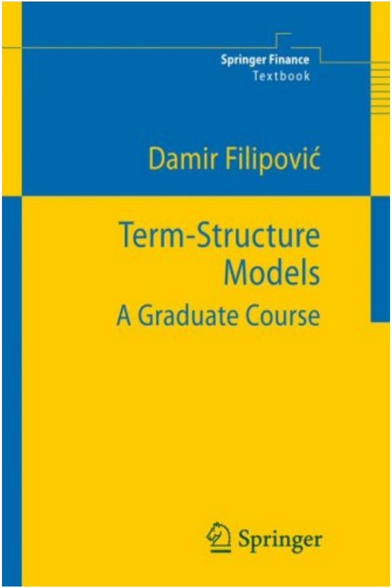
Term-Structure Models

A Graduate Course

Damir Filipović

Version 5 November 2009

Course Book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Zero-Coupon Bonds

Interest Rates

Money-Market Account and Short Rates

Coupon Bonds, Swaps and Yields

Fixed Coupon Bonds

Floating Rate Notes

Interest Rate Swaps

Yield and Duration

Market Conventions

Caps and Floors

Black's Formula

Swaptions

Black's Formula

Part I

Interest Rates and Related Contracts

Overview

- Bond = securitized form of a loan
- Bonds: primary financial instruments in the market where the time value of money is traded
- This chapter: basis concepts of interest rates and bond markets:
    - zero-coupon bonds
    - related interest rates
    - market conventions
    - market practice for pricing caps, floors and swaptions

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Zero-Coupon Bonds

- 1 euro today is worth more than 1 euro tomorrow
- zero-coupon bond pays 1 euro at maturity T
- time t value denoted by $P(t, T)$

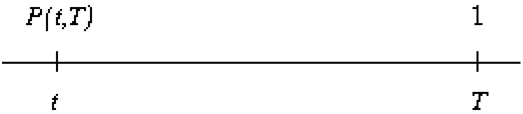

Figure: Cash flow of a T-bond.

Standing Assumptions

In theory we will assume that:

- there exists a frictionless market for all T-bonds
- $P(T,T) = 1$ for all T
- $P(t,T)$ is differentiable in T

In reality these assumptions are not always satisfied!

Q: why not assuming $P(t,T) \leq 1$?

Term-Structure

The term-structure of zero-coupon bond prices (or discount curve) $T \mapsto P(t, T)$ is smooth:

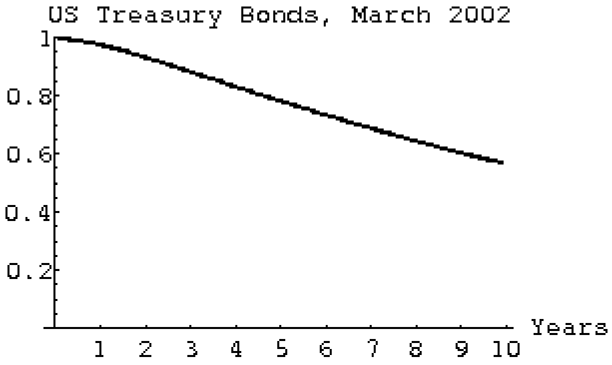

Figure: Term-structure $T \mapsto P(t, T)$.

Trajectories

Note: $t \mapsto P(t, T)$ is stochastic process:

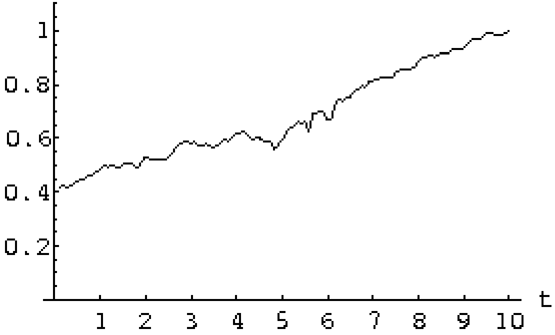

Figure: T-bond price process $t \mapsto P(t, T)$.

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Forward Rate Agreement (FRA)

FRA: current date t, expiry date $T > t$, maturity $S > T$:

- At t: sell one T-bond and buy $\frac{P(t,T)}{P(t,S)}$ S-bonds: zero net investment.
- At T: pay one euro.
- At S: receive $\frac{P(t,T)}{P(t,S)}$ euros.

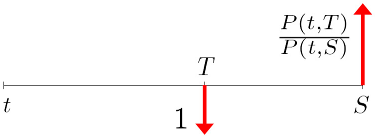

Figure: Net cash flow

Net effect: forward investment of one euro at time T yielding $\frac{P(t,T)}{P(t,S)}$ euros at S with certainty.

Simply Compounded Interest Rates

- simple forward rate for $[T, S]$ prevailing at t:

$$F(t; T, S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right),$$

which is equivalent to

$$1 + (S-T)F(t; T, S) = \frac{P(t,T)}{P(t,S)}.$$

- simple spot rate for $[t, T]$:

$$F(t, T) = F(t; t, T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right).$$

Continuously Compounded Interest Rates

- Continuously compounded forward rate for $[T, S]$ prevailing at t:

$$R(t; T, S) = -\frac{\log P(t, S) - \log P(t, T)}{S - T},$$

which is equivalent to

$$e^{R(t;T,S)(S-T)} = \frac{P(t, T)}{P(t, S)}.$$

- continuously compounded spot rate for $[t, T]$:

$$R(t, T) = R(t; t, T) = -\frac{\log P(t, T)}{T - t}.$$

Instantaneous Interest Rates

- let $S \downarrow T$:
- forward rate with maturity T prevailing at time t:

$$f(t,T) = \lim_{S \downarrow T} R(t;T,S) = -\frac{\partial \log P(t,T)}{\partial T}$$

which is equivalent to

$$P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\,du}.$$

$T \mapsto f(t,T)$ is called forward curve at time t.

- short rate at time t:

$$r(t) = f(t,t) = \lim_{T \downarrow t} R(t,T).$$

Market Example: LIBOR

- LIBOR (London Interbank Offered Rate): rate at which high-credit financial institutions can borrow in interbank market.
- maturities: from overnight to 12 months
- quoted on a simple compounding basis. E.g.: three-months forward LIBOR for period $[T, T + 1/4]$ at time t is

$$L(t, T) = F(t; T, T + 1/4).$$

- under normal conditions considered as risk-free, but ...
- ... LIBOR may reflect liquidity and credit risk (August 2007!)

Simple vs. Continuous Compounding

- annual rate R
- m-times compounded per year: $\left(1+\frac{R}{m}\right)^m$
- limit as $m \to \infty$:

$$\left(1+\frac{R}{m}\right)^m \to \mathrm{e}^R$$

continuous compounding

- Taylor: $\mathrm{e}^R = 1 + R + o(R)$
- Caution: $\mathrm{e}^{0.04} - 1.04 = 8.1 \times 10^{-4} = 8.1\,\mathrm{bp}$. Basis points (bp) matter!

Forward vs. Future Rates

- Can forward rates predict future spot rates?
- Thought experiment: deterministic world: all future rates are known today (t)
- Consequence: $P(t,S) = P(t,T)P(T,S)$ for all $t \leq T \leq S$
- This is equivalent to shifting forward curve:

$$f(t,S) = f(T,S) = r(S), \quad t \leq T \leq S$$

- In reality (non-deterministic): forecast of future short rate by forward rate have little predictive power

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Money-Market Account

- money-market account: instantaneous return $r(t)$:

$$dB(t) = r(t)B(t)dt$$

- with $B(0) = 1$ this is equivalent to: $B(t) = \mathrm{e}^{\int_0^t r(s)\,ds}$
- B is risk-free asset, r is risk-free rate of return
- B as numeraire: relate amounts of euro at different times

Proxies for the Short Rate

- $r(t)$ cannot be directly observed
- overnight interest rate not considered a good proxy (liquidity and microstructure effects)
- practiced solution: use longer rates as proxies, e.g. one- or three months LIBOR (liquid)

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Fixed Coupon Bonds

A (fixed) coupon bond is specified by

- coupon dates $T_1 < \cdots < T_n$ (T_n=maturity)
- fixed coupons $c_1, \ldots, c_n$
- a nominal value N

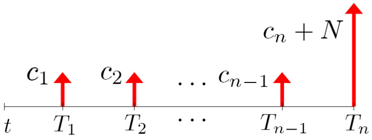

Figure: cash flow

The price at $t \le T_1$ is

$$p(t) = \sum_{i=1}^{n} P(t, T_i)c_i + P(t, T_n)N.$$

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Floating Rate Notes

A floating rate note is specified by

- reset/settlement dates $T_0 < \cdots < T_n$ (T_0=first reset date, T_n=maturity)
- a nominal value N
- floating coupon payments at $T_1, \ldots, T_n$

$$c_i = (T_i - T_{i-1})F(T_{i-1}, T_i)N$$

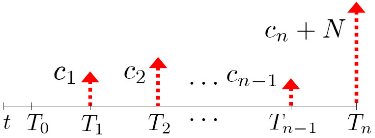

Figure: cash flow

Price at $t \leq T_0$ (replicate cash flow by buying N T_0-bonds):

$$p(t) = NP(t, T_0)$$

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Interest Rate Swaps

Exchange of fixed and floating coupon payments

A payer interest rate swap settled in arrears is specified by:

- reset/settlement dates $T_0 < T_1 < \cdots < T_n$ (T_0=first reset date, T_n=maturity)
- a fixed rate K
- a nominal value N
- for notational simplicity assume: $T_i - T_{i-1} \equiv \delta$

At T_i, $i \geq 1$, the holder of contract

- pays fixed $K\delta N$,
- receives floating $F(T_{i-1}, T_i)\delta N$.

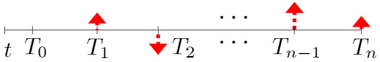

Figure: net cash flow

Swap Value

- value of payer interest rate swap at $t \leq T_0$:

$$\Pi_p(t) = N\left(P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^{n} P(t, T_i)\right)$$

$$= N\delta \sum_{i=1}^{n} P(t, T_i)\left(F(t; T_{i-1}, T_i) - K\right)$$

- value of receiver interest rate swap at $t \leq T_0$:
$\Pi_r(t) = -\Pi_p(t)$

Swap Rate

- forward (or par) swap rate makes $\Pi_r(t) = -\Pi_p(t) = 0$:

$$R_{swap}(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^n P(t,T_i)} = \sum_{i=1}^n w_i(t) F(t;T_{i-1},T_i)$$

with weights $w_i(t) = \frac{P(t,T_i)}{\sum_{j=1}^n P(t,T_j)}$

Market Quotes for Par Swap Rates

source: WestLB
URL: www.westlbmarkets.de/cms/sitecontent/ib/investmentbankinginternet/de/services/new_swapindikationen.standard.gid

Die Tabelle zeigt Preis-Indikationen für Forward-Swaps unterschiedlicher Vorlauf- und Swaplaufzeiten. Pro Laufzeiten-Kombination gibt es zwei quotierte Zinssätze. Den ersten Zinssatz empfangen Sie von der WestLB AG und Sie zahlen 6-M-Euribor, den zweiten Zinssatz zahlen Sie an die WestLB und empfangen 6-M-Euribor.

Letzte Aktualisierung: 14:59

Vorlauf	1M	3M	6M	1J	2J	3J	4J	5J
Swap								
1J	1.19-1.21	1.34-1.37	1.58-1.60	2.15-2.18	3.04-3.07	3.53-3.56	3.89-3.91	4.16-4.18
2J	1.70-1.73	1.86-1.89	2.09-2.11	2.59-2.61	3.28-3.31	3.71-3.73	4.02-4.04	4.25-4.27
3J	2.15-2.18	2.29-2.32	2.48-2.51	2.90-2.92	3.48-3.50	3.85-3.88	4.12-4.14	4.31-4.33
4J	2.49-2.52	2.61-2.64	2.78-2.81	3.13-3.15	3.64-3.66	3.96-3.99	4.20-4.22	4.36-4.39
5J	2.76-2.78	2.86-2.89	3.01-3.04	3.32-3.35	3.77-3.79	4.05-4.08	4.26-4.28	4.41-4.44
6J	2.97-3.00	3.07-3.09	3.20-3.22	3.47-3.50	3.87-3.89	4.12-4.15	4.31-4.34	4.47-4.49
7J	3.15-3.17	3.23-3.26	3.35-3.38	3.60-3.62	3.95-3.98	4.19-4.21	4.37-4.40	4.52-4.55
8J	3.29-3.31	3.37-3.39	3.47-3.50	3.70-3.72	4.02-4.05	4.25-4.28	4.43-4.46	4.57-4.60
9J	3.41-3.43	3.48-3.50	3.58-3.60	3.78-3.81	4.10-4.12	4.31-4.34	4.48-4.51	4.61-4.64
10J	3.51-3.53	3.58-3.60	3.67-3.70	3.87-3.89	4.16-4.19	4.37-4.39	4.52-4.55	4.64-4.67

Figure: forward swap rates from 25 Sep 09 (for illustration)

Swap Example

Swaps were developed because different companies could borrow at fixed or at floating rates in different markets. Example:

- company A is borrowing fixed at $5\frac{1}{2}\%$, but could borrow floating at LIBOR plus $\frac{1}{2}\%$;
- company B is borrowing floating at LIBOR plus 1%, but could borrow fixed at $6\frac{1}{2}\%$.

By agreeing to swap streams of cash flows both companies could be better off, and a mediating institution would also make money:

- company A pays LIBOR to the intermediary in exchange for fixed at $5\frac{3}{16}\%$ (receiver swap);
- company B pays the intermediary fixed at $5\frac{5}{16}\%$ in exchange for LIBOR (payer swap).

Swap Example cont'd

The net payments are as follows:

- company A is now paying LIBOR plus $\frac{5}{16}\%$ instead of LIBOR plus $\frac{1}{2}\%$;
- company B is paying fixed at $6\frac{5}{16}\%$ instead of $6\frac{1}{2}\%$;
- the intermediary receives fixed at $\frac{1}{8}\%$.

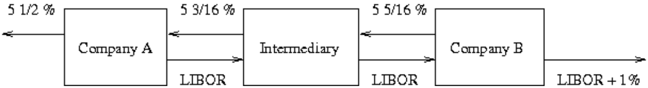

Figure: A swap with mediating institution.

Everyone seems to be better off (why?)

Interest Rate Swap Markets

- interest rate swap markets are over the counter
- but swap contracts exist in standardized form, e.g. by the ISDA (International Swaps and Derivatives Association, Inc.).
- swap markets are extremely liquid
- maturities from 1 to 30 years are standard, swap rate quotes available up to 60 years
- gives market participants, such as life insurers, opportunity to create synthetically long-dated investments

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Zero-Coupon Yield

- zero-coupon yield is the continuously compounded spot rate $R(t, T)$:

$$P(t, T) = \mathrm{e}^{-R(t,T)(T-t)}.$$

- $T \mapsto R(t, T)$ is called (zero-coupon) yield curve

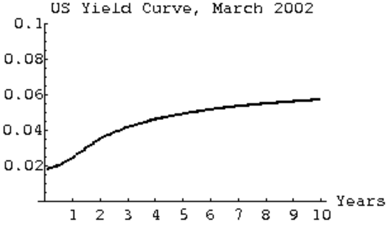

Figure: Yield curve $T \mapsto R(t, T)$.

- note: term "yield curve" is ambiguous

Yield-to-Maturity

- consider fixed coupon bond (short hand: c_n contains N)

$$p = \sum_{i=1}^{n} P(0, T_i) c_i$$

- bond's "internal rate of interest": (continuously compounded) yield-to-maturity y: unique solution to

$$p = \sum_{i=1}^{n} c_i \mathrm{e}^{-yT_i}.$$

- Schaefer [47]: yield-to-maturity is inadequate statistic for bond market:
  - coupon payments occurring at the same point in time are discounted by different discount factors, but
  - coupon payments at different points in time from the same bond are discounted by the same rate.

In reality, one would wish to do exactly the opposite !

Macaulay duration

- bond price change as function of y: Macaulay duration:

$$D_{Mac} = \frac{\sum_{i=1}^{n} T_i c_i \mathrm{e}^{-yT_i}}{p}$$

= weighted average of the coupon dates $T_1, \dots, T_n$ ("mean time to coupon payment")

= first-order sensitivity of bond price w.r.t. changes in the yield-to-maturity:

$$\frac{dp}{dy} = \frac{d}{dy}\left(\sum_{i=1}^{n} c_i \mathrm{e}^{-yT_i}\right) = -D_{Mac}\,p$$

(interest rate risk management!)

Duration

- write $y_i = R(0, T_i)$
- duration of the bond

$$D = \frac{\sum_{i=1}^{n} T_i c_i \mathrm{e}^{-y_i T_i}}{p} = \sum_{i=1}^{n} \frac{c_i P(0, T_i)}{p} T_i$$

= first-order sensitivity of bond price w.r.t. parallel shifts of yield curve:

$$\frac{d}{ds}\left(\sum_{i=1}^{n} c_i \mathrm{e}^{-(y_i+s)T_i}\right)\Big|_{s=0} = -Dp.$$

$\rightarrow$ duration is essentially for bonds (w.r.t. parallel shift of the yield curve) what delta is for stock options

Convexity

- bond equivalent of gamma is convexity:

$$C = \frac{d^2}{ds^2}\left(\sum_{i=1}^{n} c_i \mathrm{e}^{-(y_i+s)T_i}\right)\Big|_{s=0} = \sum_{i=1}^{n} c_i \mathrm{e}^{-y_i T_i}(T_i)^2$$

$\longrightarrow$ second-order approximation for bond price change Δp w.r.t. parallel shift Δy of yield curve:

$$\Delta p \approx -D p \Delta y + \frac{1}{2} C (\Delta y)^2$$

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Day-Count Conventions

- convention: measure time in units of years
- market evaluates year fraction between $t < T$ in different ways
- examples of day-count conventions $\delta(t, T)$:
  - actual/365: year has 365 days

$$\delta(t, T) = \frac{\text{actual number of days between } t \text{ and } T}{365}.$$

  - actual/360: as above but year counts 360 days
  - 30/360: months count 30 and years 360 days. Let $t = d_1/m_1/y_1$ and $T = d_2/m_2/y_2$

$$\delta(t, T) = \frac{\min(d_2, 30) + (30 - d_1)^+}{360} + \frac{(m_2 - m_1 - 1)^+}{12} + y_2 - y_1$$

Example: $t = 4$ January 2000 and $T = 4$ July 2002:

$$\delta(t, T) = \frac{4 + (30 - 4)}{360} + \frac{7 - 1 - 1}{12} + 2002 - 2000 = 2.5.$$

Coupon Bonds

Coupon bonds issued in the American (European) markets typically have semiannual (annual) coupon payments.
Debt securities issued by the US Treasury are divided into three classes:

- Bills: zero-coupon bonds with time to maturity less than one year.
- Notes: coupon bonds (semiannual) with time to maturity between 2 and 10 years.
- Bonds: coupon bonds (semiannual) with time to maturity between 10 and 30 years.[1]

STRIPS (separate trading of registered interest and principal of securities): synthetically created zero-coupon bonds, traded since August 1985

[1] 30-year Treasury bonds were not offered from 2002 to 2005.

Accrued Interest, Clean Price and Dirty Price

- recall coupon bond price formula

$$p(t) = \sum_{T_i \geq t} c_i P(t, T_i)$$

$\longrightarrow$ systematic discontinuities of price trajectory at $t = T_i$

- accrued interest at $t \in (T_{i-1}, T_i]$ is defined by

$$AI(i; t) = c_i \frac{t - T_{i-1}}{T_i - T_{i-1}}$$

- clean price (quoted) of coupon bond at $t \in (T_{i-1}, T_i]$ is

$$p_{clean}(t) = p(t) - AI(i; t)$$

$\longrightarrow$ dirty price (to pay) is

$$p(t) = p_{clean}(t) + AI(i; t)$$

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Yield-to-Maturity

see course book Section 2.5.4

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Caplets

- caplet with reset date T and settlement date $T + \delta$: pays the holder difference between simple market rate $F(T, T + \delta)$ (e.g. LIBOR) and strike rate κ
- cash flow at time $T + \delta$:

$$\delta(F(T, T + \delta) - \kappa)^+$$

Caps

- cap: strip of caplets, specified by
  - reset/settlement dates $T_0 < T_1 < \cdots < T_n$ (T_0=first reset date, T_n=maturity)
  - a cap rate κ
  - for notational simplicity assume: $T_i - T_{i-1} \equiv \delta$
- cap price at $t \leq T_0$ is

$$Cp(t) = \sum_{i=1}^{n} Cpl(t; T_{i-1}, T_i)$$

where $Cpl(t; T_{i-1}, T_i)$ is price of ith caplet

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Caps

At T_i, the holder of the cap receives

$$\delta(F(T_{i-1}, T_i) - \kappa)^+,$$

which is equivalent $(\dots)$ to cash flow

$$(1+\delta\kappa)\left(\frac{1}{1+\delta\kappa} - P(T_{i-1}, T_i)\right)^+$$

at T_{i-1} ($= (1+\delta\kappa)$ times put option on T_i-bond with strike price $1/(1+\delta\kappa)$ and maturity T_{i-1})

$\longrightarrow$ protects against rising interest rates

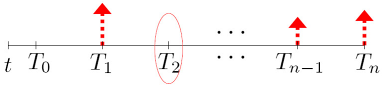

Figure: cash flow of cap

Floors

- floor: converse to a cap, protects against low rates
- strip of floorlets with cash flow at time T_i:

$$\delta(\kappa - F(T_{i-1}, T_i))^+$$

- ith floorlet price: $Fll(t; T_{i-1}, T_i)$
- floor price at $t \leq T_0$ is

$$Fl(t) = \sum_{i=1}^{n} Fll(t; T_{i-1}, T_i)$$

Caps, Floors and Swaps

- parity relation:

$$Cp(t) - Fl(t) = \Pi_p(t)$$

  value of a payer swap with rate κ, nominal one and same tenor structure as cap and floor
- cap/floor is ... at-the-money (ATM) if

$$\kappa = R_{swap} = \frac{P(0,T_0) - P(0,T_n)}{\delta \sum_{i=1}^{n} P(0,T_i)}$$

- ... in-the-money (ITM) if $\kappa < R_{swap}$
- ... out-of-the-money (OTM) if $\kappa > R_{swap}$

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Black's Formula

Black's formula for ith caplet value is

$$\begin{aligned} Cpl(t; T_{i-1}, T_i) = \delta P(t, T_i) \\ \times \left(F(t; T_{i-1}, T_i)\Phi(d_1) - \kappa\Phi(d_2)\right) \end{aligned}$$

where

$$d_{1,2} = \frac{\log\left(\frac{F(t;T_{i-1},T_i)}{\kappa}\right) \pm \frac{1}{2}\sigma(t)^2(T_{i-1} - t)}{\sigma(t)\sqrt{T_{i-1} - t}},$$

- Φ: standard Gaussian cumulative distribution function
- $\sigma(t)$: cap (implied) volatility (same for all caplets belonging to a cap)

Black's Formula cont'd

Black's formula assumes $F(T_{i-1}, T_i) = X(T_{i-1})$ where

$$dX = \sigma X\, dW, \quad X(t) = F(t; T_{i-1}, T_i)$$

and

$$Cpl(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}\left[(X(T_{i-1}) - \kappa)^+ \mid \mathcal{F}_t \right]$$

(to be justified later: in Market Models)

Black's Formula cont'd

Black's formula for ith floorlet is

$$Fl(t; T_{i-1}, T_i) = \delta P(t, T_i) \left(\kappa \Phi(-d_2) - F(t; T_{i-1}, T_i) \Phi(-d_1)\right)$$

- cap/floor prices are quoted in the market in terms of their implied volatilities
- typically: $t = 0$, $T_0 = \delta = T_i - T_{i-1}$ = three months (US market) or half a year (euro market)

Example of Cap Quotes

Table: US dollar ATM cap volatilities, 23 July 1999

Maturity (in years)	ATM vols (in %)
1	14.1
2	17.4
3	18.5
4	18.8
5	18.9
6	18.7
7	18.4
8	18.2
10	17.7
12	17.0
15	16.5
20	14.7
30	12.4

Example of Cap Quotes

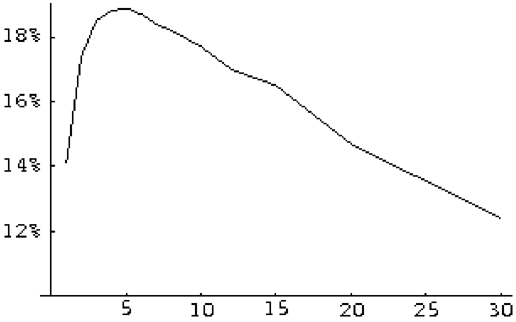

Figure: US dollar ATM cap volatilities, 23 July 1999.

It is a challenge for any market realistic interest rate model to match the given volatility curve.

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Swaptions

- payer (receiver) swaption with strike rate K: right to enter a payer (receiver) swap with fixed rate K at swaption maturity
- usually, swaption maturity = first reset date T_0 of underlying swap
- tenor of the swaption: underlying swap length $T_n - T_0$

Swaption Payoff

- swaption payoff at maturity

$$N\left(\sum_{i=1}^{n} P(T_0,T_i)\delta(F(T_0;T_{i-1},T_i)-K)\right)^+ = N\delta(R_{swap}(T_0)-K)^+\sum_{i=1}^{n} P(T_0,T_i)$$

cannot be decomposed into more elementary payoffs!

$\rightarrow$ dependence between different forward rates will enter valuation procedure

- payer (receiver) swaption is ATM, ITM, OTM if

$$K = R_{swap}(t), \quad K < (>) R_{swap}(t), \quad K > (<) R_{swap}(t)$$

- $x \times y$-swaption: maturity in x years, underlying swap y years long

Application: Callable Bond

Swaptions can be used to synthetically create callable bonds:

- company has issued 10-year bond with 4% coupon
- wants to add right to call bond (i.e. prepay bond) at par after 5 years
- cannot change original bond

Solution: buy a 5×5 receiver swaption with strike rate 4%:

- swaption cancels fixed coupon payments
- exchange of notionals between $t = 5$ and $T = 10$ is equivalent to paying floating

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Black's Formula

Black's price formula for payer and receiver swaption is

$$Swpt_p(t) = N\delta \left(R_{swap}(t)\Phi(d_1) - K\Phi(d_2) \right) \sum_{i=1}^{n} P(t, T_i),$$

$$Swpt_r(t) = N\delta \left(K\Phi(-d_2) - R_{swap}(t)\Phi(-d_1) \right) \sum_{i=1}^{n} P(t, T_i),$$

with

$$d_{1,2} = \frac{\log\left(\frac{R_{swap}(t)}{K}\right) \pm \frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}},$$

- Φ: standard Gaussian cumulative distribution function
- $\sigma(t)$: swaption implied volatility

Swaption Quotes

- swaption prices are quoted in terms of implied volatilities in matrix form
- note: accrual period $\delta = T_i - T_{i-1}$ for underlying swap can differ from prevailing δ for caps within the same market region!
- e.g. euro zone: caps are written on semiannual LIBOR ($\delta = 1/2$), while swaps pay annual coupons ($\delta = 1$)

Example of Swaption Quotes

Table: Black's implied volatilities (in %) of ATM swaptions on May 16, 2000. Maturities are 1,2,3,4,5,7,10 years, swaps lengths from 1 to 10 years

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4	15.8	14.6	13.8	13.3	12.9	12.6	12.3	12.0	11.7
2y	17.7	15.6	14.1	13.1	12.7	12.4	12.2	11.9	11.7	11.4
3y	17.6	15.5	13.9	12.7	12.3	12.1	11.9	11.7	11.5	11.3
4y	16.9	14.6	12.9	11.9	11.6	11.4	11.3	11.1	11.0	10.8
5y	15.8	13.9	12.4	11.5	11.1	10.9	10.8	10.7	10.5	10.4
7y	14.5	12.9	11.6	10.8	10.4	10.3	10.1	9.9	9.8	9.6
10y	13.5	11.5	10.4	9.8	9.4	9.3	9.1	8.8	8.6	8.4

Example of Swaption Quotes

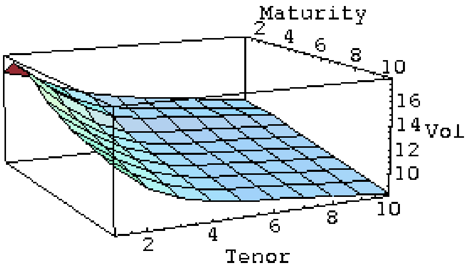

Figure: Black's implied volatilities (in %) of ATM swaptions on May 16, 2000.

An interest rate model for swaptions valuation must fit today's volatility surface.

A Bootstrapping Example

Non-parametric Estimation Methods

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Money Markets
Problems

Parametric Estimation Methods

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Smoothing Splines
Exponential–Polynomial Families

Principal Component Analysis

Principal Components of a Random Vector
Sample Principle Components
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Part II

Estimating the Term-Structure

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Principal Components of a Random Vector
Sample Principle Components
PCA of the

Overview

- in theory: assume given initial term-structure for all T
- reality: finitely many (possibly noisy) market quote observations
- pricing exotic derivatives: cash flow dates possibly do not match the predetermined finite time grid

$\longrightarrow$ interpolate the term-structure

- simples method: build up term structure from shorter maturities to longer maturities ("bootstrapping")

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- Problems

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- Sample Principle Components
- PCA of the Forward Curve
- Correlation

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Bootstrapping Example

Table: Yen data, 9 January 1996

LIBOR (%)		Futures		Swaps (%)	
o/n	0.49	20 Mar 96	99.34	2y	1.14
1w	0.50	19 Jun 96	99.25	3y	1.60
1m	0.53	18 Sep 96	99.10	4y	2.04
2m	0.55	18 Dec 96	98.90	5y	2.43
3m	0.56			7y	3.01
				10y	3.36

- spot date t_0: 11 January, 1996
- day-count convention: actual/360 (note: 1996 was leap year):

$$\delta(T,S) = \frac{\text{actual number of days between } T \text{ and } S}{360}$$

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Maturity Overlap

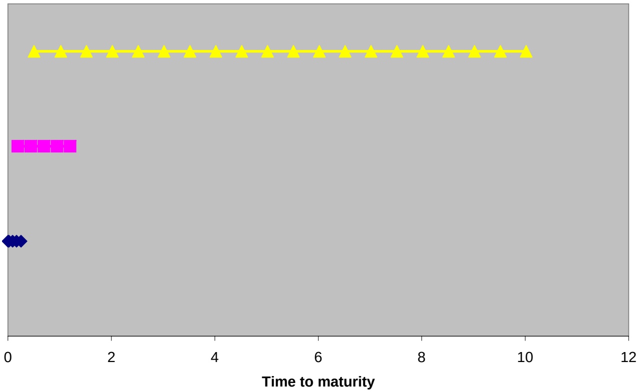

Figure: Overlapping maturity segments (from bottom up) of LIBOR, futures and swap markets.

First Column: LIBOR

- maturities $\{S_1, \ldots, S_5\} = \{12/1/96,\ 18/1/96,\ 13/2/96,\ 11/3/96,\ 11/4/96\}$
- zero-coupon bonds are

$$P(t_0, S_i) = \frac{1}{1 + \delta(t_0, S_i) F(t_0, S_i)}.$$

Second Column: Futures

- quoted as: futures price for settlement day $T_i = 100(1 - F_F(t_0; T_i, T_{i+1}))$
- $F_F(t_0; T_i, T_{i+1})$: futures rate for period $[T_i, T_{i+1}]$ prevailing at t_0
- reset/settlement dates

$$\{T_1, \dots, T_5\} = \{20/3/96, \dots 19/3/97\}$$

hence $\delta(T_i, T_{i+1}) \equiv 91/360$

- proxy: $F(t_0; T_i, T_{i+1}) = F_F(t_0; T_i, T_{i+1})$ (see later)

Second Column: Futures

- for $P(t_0, T_1)$: use geometric interpolation ($S_4 < T_1 < S_5$)

$$P(t_0, T_1) = P(t_0, S_4)^q \, P(t_0, S_5)^{1-q}$$

which is equivalent to linear interpolation of yields

$$R(t_0, T_1) = q \, R(t_0, S_4) + (1 - q) \, R(t_0, S_5)$$

where

$$q = \frac{\delta(T_1, S_5)}{\delta(S_4, S_5)} = \frac{22}{31} = 0.709677$$

- to derive $P(t_0, T_2), \ldots, P(t_0, T_5)$ use:

$$P(t_0, T_{i+1}) = \frac{P(t_0, T_i)}{1 + \delta(T_i, T_{i+1}) \, F(t_0; T_i, T_{i+1})}$$

Third Column: Swaps

- semiannual cash flows at dates

$$\{U_1, \dots, U_{20}\} = \left\{ \begin{array}{ll} 11/7/96, & 13/1/97, \\ 11/7/97, & 12/1/98, \\ 13/7/98, & 11/1/99, \\ 12/7/99, & 11/1/00, \\ 11/7/00, & 11/1/01, \\ 11/7/01, & 11/1/02, \\ 11/7/02, & 13/1/03, \\ 11/7/03, & 12/1/04, \\ 12/7/04, & 11/1, 05, \\ 11/7/05, & 11/1/06 \end{array} \right\}$$

- from data: $R_{swap}(t_0, U_i)$ for $i = 4, 6, 8, 10, 14, 20$

Third Column: Swaps

- recall

$$R_{swap}(t_0, U_n) = \frac{1 - P(t_0, U_n)}{\sum_{i=1}^{n} \delta(U_{i-1}, U_i)\, P(t_0, U_i)} \quad (\text{set } U_0 = t_0)$$

- overlap: $T_2 < U_1 < T_3$ and $T_4 < U_2 < T_5$
- linear interpolation of yields $\rightarrow$ $R(t_0, U_1)$, $R(t_0, U_2)$

$\rightarrow$ $P(t_0, U_1)$, $P(t_0, U_2)$ and hence $R_{swap}(t_0, U_1)$, $R_{swap}(t_0, U_2)$

- remaining swap rates by linear interpolation, e.g.

$$R_{swap}(t_0, U_3) = \frac{1}{2}(R_{swap}(t_0, U_2) + R_{swap}(t_0, U_4))$$

- inversion of above formula:

$$P(t_0, U_n) = \frac{1 - R_{swap}(t_0, U_n) \sum_{i=1}^{n-1} \delta(U_{i-1}, U_i)\, P(t_0, U_i)}{1 + R_{swap}(t_0, U_n)\delta(U_{n-1}, U_n)}$$

gives $P(t_0, U_n)$ for $n = 3, \ldots, 20$

Obtain Term Structure

- set $P(t_0, t_0) = 1$

$\longrightarrow$ have constructed term structure $P(t_0, t_i)$ for 30 points:

$$t_i = t_0, S_1, \ldots, S_4, T_1, S_5, T_2, U_1, T_3, T_4, U_2, T_5, U_3, \ldots, U_{20}$$

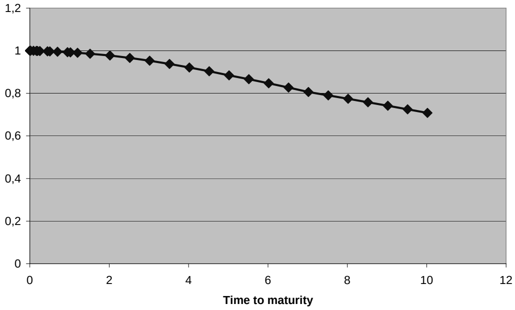

Figure: Zero-coupon bond curve.

Yield and Forward Rate Curves

- continuously compounded yield and forward rates: $R(t_0, t_i)$ and $R(t_0, t_i, t_{i+1})$
- "sawtooth": linear interpolation of swap rates inappropriate for implied forward rates

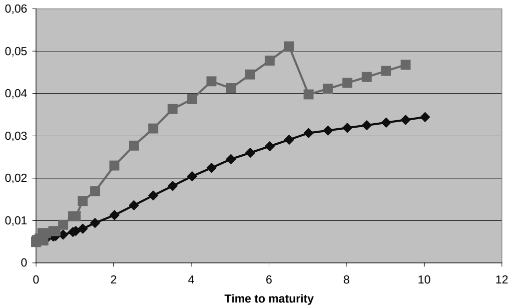

Figure: yields (lower curve), forward rates (upper curve)

Larger Time Scale

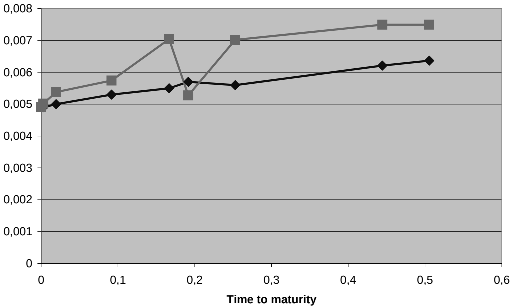

Figure: yields (lower curve), forward rates (upper curve)

- "sawtooth": systematic inconsistency of our use of LIBOR and futures rates data (we have treated futures rates as forward rates)

Futures vs. Forward Rates

- "sawtooth": systematic inconsistency of our use of LIBOR and futures rates data (we have treated futures rates as forward rates)
- in reality: futures rates often greater than forward rates
- difference is called convexity adjustment (model dependent)
- example: forward rate = futures rate $-\frac{1}{2}\sigma^2\tau^2$ where
  - τ = time to maturity of futures contract
  - σ = volatility parameter

(later: we will derive a more general formula)

Summary

- we constructed entire term-structure from relatively few instruments
- method exactly reconstructs market prices (desirable for interest rate option traders: marking to market), but ...
- ... forward curve (derivative $-\partial_T \log P(t_0, T)!$) irregular, sensitive to bond price variations/errors
- three curves resulting from LIBOR, futures and swaps: not coincident to common underlying curve
- bootstrapping: example of non-parametric estimation method

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Non-parametric Estimation Methods: General Problem

- finding today's (t_0) discount curve (term-structure) $x \mapsto D(x) = P(t_0, t_0 + x)$
- can be formulated as $p = C\,d + \epsilon$ where
  - p = column vector of n market prices
  - C = related cash flow matrix
  - $d = (D(x_1), \dots, D(x_N))^\top$
  - cash flow dates $t_0 < T_1 < \cdots < T_N$, $T_i - t_0 = x_i$
  - ϵ = vector of pricing errors, subject to being minimized
- including errors reasonable: prices never exact simultaneously quoted, bid ask spreads, allows for smoothing
- next: bring data from bond and money markets into above format

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Table: Market prices for UK gilts, 4/9/96

	Coupon (%)	Next coupon	Maturity date	Dirty price (p_i)
Bond 1	10	15/11/96	15/11/96	103.82
Bond 2	9.75	19/01/97	19/01/98	106.04
Bond 3	12.25	26/09/96	26/03/99	118.44
Bond 4	9	03/03/97	03/03/00	106.28
Bond 5	7	06/11/96	06/11/01	101.15
Bond 6	9.75	27/02/97	27/08/02	111.06
Bond 7	8.5	07/12/96	07/12/05	106.24
Bond 8	7.75	08/03/97	08/09/06	98.49
Bond 9	9	13/10/96	13/10/08	110.87

Formalization

- basic instruments: coupon bonds
- vector of quoted market bond prices $p = (p_1, \dots, p_n)^\top$,
- dates of all cash flows $t_0 < T_1 < \cdots < T_N$,
- bond $i = 1, \dots, n$ with cash flows $c_{i,j}$ at T_j (may be zero), form $n \times N$ cash flow matrix

$$C = (c_{i,j})_{\substack{1 \le i \le n \\ 1 \le j \le N}}$$

Example

- UK government bond (gilt) market on 4 September 1996, semiannual coupons, spot date 4/9/96, day-count convention actual/365

$\rightarrow$ $n = 9$, $N = 1 + 3 + 6 + 7 + 11 + 12 + 19 + 20 + 25 = 104$, $T_1 = 26/09/96$, $T_2 = 13/10/96$, $T_3 = 06/11/97$, ...

- note: no bonds have cash flows at same date $\Rightarrow$ N large

$$C = \begin{pmatrix}
0 & 0 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & 0 & \dots \\
6.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.125 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 & 0 & \dots \\
0 & 0 & 3.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 4.25 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.875 & 0 & \dots \\
0 & 4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{pmatrix}$$

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Data

Table: US money market, 6 October 1997

	Period	Rate	Maturity date
LIBOR	o/n	5.59375	9/10/97
	1m	5.625	10/11/97
	3m	5.71875	8/1/98
Futures	Oct 97	94.27	15/10/97
	Nov 97	94.26	19/11/97
	Dec 97	94.24	17/12/97
	Mar 98	94.23	18/3/98
	Jun 98	94.18	17/6/98
	Sep 98	94.12	16/9/98
	Dec 98	94	16/12/98
Swaps	2	6.01253	
	3	6.10823	
	4	6.16	
	5	6.22	
	7	6.32	
	10	6.42	
	15	6.56	
	20	6.56	
	30	6.56	

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Formalization

- LIBOR L, maturity T: $p = 1$ and $c = 1 + (T - t_0)L$ at T
- forward rate F for $[T, S]$: $p = 0$, $c_1 = -1$ at $T_1 = T$, $c_2 = 1 + (S - T)F$ at $T_2 = S$
- swap (receiver, swap rate K, tenor $t_0 \leq T_0 < \cdots < T_n$, $T_i - T_{i-1} \equiv \delta$): since

$$0 = -D(T_0 - t_0) + \delta K \sum_{j=1}^{n-1} D(T_j - t_0) + (1 + \delta K) D(T_n - t_0),$$

  we can choose
  - if $T_0 = t_0$: $p = 1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$,
  - if $T_0 > t_0$: $p = 0$, $c_0 = -1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$.

$\rightarrow$ at t_0: LIBOR and swaps have notional price 1, FRAs and forward swaps have notional price 0.

Example

- US money market on 6 October 1997, spot date 8/10/97, day-count convention actual/360
- LIBOR is for o/n (1/360), 1m (33/360), and 3m (92/360)
- futures are three-month rates ($\delta = 91/360$) taken as forward rates: quote $100(1 - F(t_0; T, T + \delta))$
- swaps are annual ($\delta = 1$) with first payment date 8/10/98

$\rightarrow$ $n = 3 + 7 + 9 = 19$, $N = 3 + (14 - 4) + 30 = 43$, $T_1 = 9/10/97$, $T_2 = 15/10/97$ (first future), $T_3 = 10/11/97$, ...

Cash Flow Matrix

first 14 columns of the 19×43 cash flow matrix C:

$$
\begin{array}{cccccccccccccc}
c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & c_{47} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & c_{58} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & c_{69} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{7,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{8,11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & c_{9,13} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{10,14} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{13,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{15,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{16,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{17,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{18,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{19,12} & 0 & 0
\end{array}
$$

with

$$c_{11} = 1.00016, \quad c_{23} = 1.00516, \quad c_{36} = 1.01461,$$

$$c_{47} = 1.01448, \quad c_{58} = 1.01451, \quad c_{69} = 1.01456, \quad c_{7,10} = 1.01459,$$

$$c_{8,11} = 1.01471, \quad c_{9,13} = 1.01486, \quad c_{10,14} = 1.01517$$

$$c_{11,12} = 0.060125, \quad c_{12,12} = 0.061082, \quad c_{13,12} = 0.0616,$$

$$c_{14,12} = 0.0622, \quad c_{15,12} = 0.0632, \quad c_{16,12} = 0.0642,$$

$$c_{17,12} = c_{18,12} = c_{19,12} = 0.0656.$$

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Problems

- typically $n \ll N$ and many entries of C are zero

$\Rightarrow$ quadratic optimization problem

$$\min_{d \in \mathbb{R}^N} \|p - C\,d\|^2$$

ill-posed: first-order condition $C^\top C \hat{d} = C^\top p$, and dimension of solution space $=$ $\dim \ker(C^\top C) = \dim \ker(C) \geq N - n$

- moreover: as many parameters as there are cash flow dates, and there is nothing to regularize the discount curve found from the regression $\Rightarrow$ discount factors of similar maturity can be very different $\Rightarrow$ ragged yield and forward curves
- alternative and better method: estimate a smooth yield curve parametrically from market rates ...

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Parametric Estimation Methods

- reduction of parameters and smooth term-structure of interest rates by using parameterized families of smooth curves
- class of linear families: fix a set of basis functions, preferably with compact support (why?)
- examples: B-splines (next), smoothing splines (later)
- non-linear families: Nelson–Siegel, Svensson (later)

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Cubic Splines

- piecewise cubic polynomial, everywhere twice differentiable
- interpolates values at $q+1$ knot points $\xi_0 < \cdots < \xi_q$
- general form is

$$\sigma(x) = \sum_{i=0}^{3} a_i x^i + \sum_{j=1}^{q-1} b_j (x - \xi_j)_+^3$$

$\rightarrow$ $q+3$ parameters $\{a_0, \ldots, a_3, b_1, \ldots, b_{q-1}\}$ (a kth-degree has $q+k$ parameters)

- uniquely characterized by σ' or σ'' at ξ_0 and ξ_q

Cubic B-splines

- introduce six extra knot points

$$\xi_{-3} < \xi_{-2} < \xi_{-1} < \xi_0 < \cdots < \xi_q < \xi_{q+1} < \xi_{q+2} < \xi_{q+3}$$

to obtain a basis for the cubic splines on $[\xi_0, \xi_q]$: $q+3$ B-splines

$$\psi_k(x) = \sum_{j=k}^{k+4} \left(\prod_{i=k, i\neq j}^{k+4} \frac{1}{\xi_i - \xi_j} \right) (x-\xi_j)_+^3, \quad k = -3, \dots, q-1$$

- B-spline ψ_k is zero outside $[\xi_k, \xi_{k+4}]$

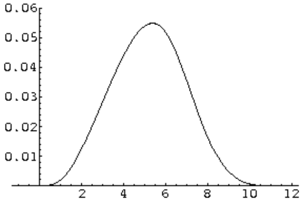

Figure: B-spline with knot points $\{0, 1, 6, 8, 11\}$

Estimating the Discount Function with Cubic B-splines

- Steeley [50]: use B-splines to estimate the discount curve:

$$D(x;z) = z_1\psi_1(x) + \cdots + z_m\psi_m(x)$$

- quadratic optimization problem: $\min_{z\in\mathbb{R}^m} \|p - C\Psi z\|^2$ with

$$d(z) = \begin{pmatrix} D(x_1;z) \\ \vdots \\ D(x_N;z) \end{pmatrix} = \begin{pmatrix} \psi_1(x_1) & \cdots & \psi_m(x_1) \\ \vdots & & \vdots \\ \psi_1(x_N) & \cdots & \psi_m(x_N) \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$$

- if the $n \times m$ matrix $A = C\Psi$ has full rank m, the unique unconstrained solution is $z^* = (A^\top A)^{-1}A^\top p$
- a reasonable constraint: $D(0;z) = \psi_1(0)z_1 + \cdots + \psi_m(0)z_m = 1$

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Example

- UK government bond market data from last section
- maximum time to maturity $x_{104} = 12.11$ [years]
- first bond is a zero-coupon bond: exact yield

$$y = -\frac{365}{72} \log \frac{103.822}{105} = -\frac{1}{0.197} \log 0.989 = 0.0572$$

- three different estimates with B-splines: 8, 7, 5 ...

Example

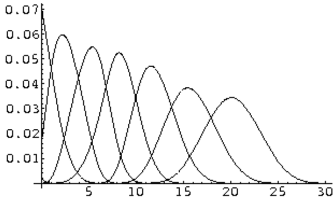

Figure: 8 (resp. first 7) B-splines with knots $\{-20, -5, -2, 0, 1, 6, 8, 11, 15, 20, 25, 30\}$.

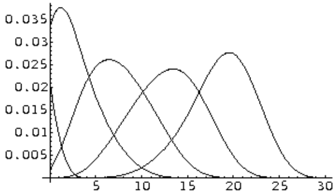

Figure: 5 B-splines with knot points $\{-10, -5, -2, 0, 4, 15, 20, 25, 30\}$.

Example

- 8 B-splines: $\min_{z\in\mathbb{R}^8} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.23$ with

$$z^* = \begin{pmatrix} 13.8641 \\ 11.4665 \\ 8.49629 \\ 7.69741 \\ 6.98066 \\ 6.23383 \\ -4.9717 \\ 855.074 \end{pmatrix}$$

- first 7 B-splines: $\min_{z\in\mathbb{R}^7} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.32$ with

$$z^* = \begin{pmatrix} 17.8019 \\ 11.3603 \\ 8.57992 \\ 7.56562 \\ 7.28853 \\ 5.38766 \\ 4.9919 \end{pmatrix}$$

- 5 B-splines: $\min_{z\in\mathbb{R}^5} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.39$ with

$$z^* = \begin{pmatrix} 15.652 \\ 19.4385 \\ 12.9886 \\ 7.40296 \\ 6.23152 \end{pmatrix}$$

Example: 8 B-splines

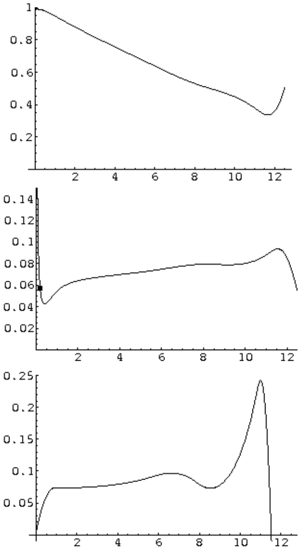

Figure: Discount curve, yield and forward curves for estimation with 8 B-splines. The dot is the exact yield of the first bond.

Example: 7 B-splines

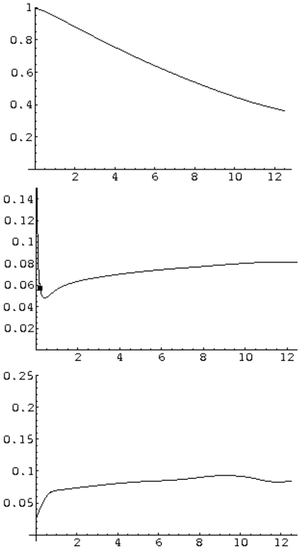

Figure: Discount curve, yield and forward curves for estimation with 7 B-splines. The dot is the exact yield of the first bond.

Example: 5 B-splines

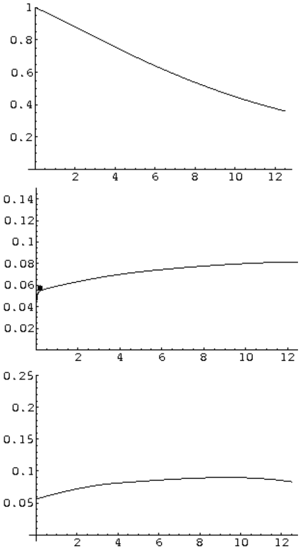

Figure: Discount curve, yield and forward curves for estimation with 5 B-splines. The dot is the exact yield of the first bond.

Summary: B-splines

- trade-off between quality (regularity) and the correctness of fit
- unstable and irregular yield and forward curves: short- and long-term maturities
- B-spline fits extremely sensitive to number and location of knot points

$\longrightarrow$ need criteria asserting smooth yield and forward curves that flatten towards long end

$\longrightarrow$ directly estimate the yield or forward curve

$\longrightarrow$ number and location of the knot points for splines adjusted to the data

$\longrightarrow$ smoothing splines . . .

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Problem

- data: N observed yields Y_i at T_i
- fitting requirement for forward curve $f(u)$:

$$\int_0^{T_i} f(u)\,du + \epsilon_i/\sqrt{\alpha} = T_i Y_i,$$

- smoothness criterion:

$$\int_0^T (f'(u))^2\,du$$

$\longrightarrow$ optimization problem: $\min_{f\in H} F(f)$ where

$$F(f) = \int_0^T (f'(u))^2\,du + \alpha \sum_{i=1}^N \left(Y_i T_i - \int_0^{T_i} f(u)\,du \right)^2$$

- α: tunes trade-off smoothness–correctness of fit
- H = Hilbert space of absolutely continuous functions with scalar product $\langle g,h\rangle_H = g(0)h(0) + \int_0^T g'(u)h'(u)\,du$

Lorimier's Theorem

Theorem (Lorimier [38])

The unique solution f is a second-order spline characterized by

$$f(u) = f(0) + \sum_{k=1}^{N} a_k h_k(u)$$

where $h_k \in C^1[0,T]$ is a second-order polynomial on $[0, T_k]$ with

$$h_k'(u) = (T_k - u)^+, \quad h_k(0) = T_k, \quad k = 1, \dots, N,$$

and $f(0)$ and a_k solve the linear system of equations

$$\sum_{k=1}^{N} a_k T_k = 0,$$

$$\alpha \left(Y_k T_k - f(0) T_k - \sum_{l=1}^{N} a_l \langle h_l, h_k \rangle_H \right) = a_k, \quad k = 1, \dots, N.$$

Lorimier's Theorem cont'd

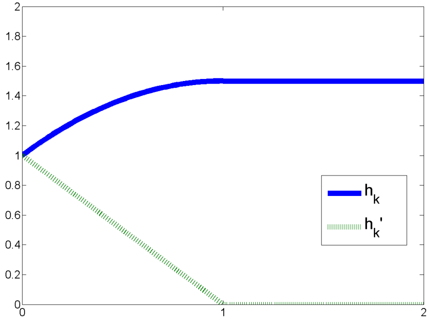

Figure: The function h_k and its derivative h_k' for $T_k = 1$.

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Exponential–Polynomial Families

- parametric curve families used by most central banks for term-structure estimation
- estimate forward curve $x \mapsto f(t_0, t_0 + x) = \phi(x) = \phi(x; z)$
- calibration to bond prices: nonlinear optimization problem

$$\min_{z \in \mathcal{Z}} \| p - C\, d(z) \| ,$$

  with

$$d_i(z) = \mathrm{e}^{-\int_0^{x_i} \phi(u;z)\, du}$$

  for payment tenor $0 < x_1 < \cdots < x_N$
- obvious modification for yield fitting
- examples: Nelson–Siegel and Svensson

Nelson–Siegel and Svensson Families

- Nelson–Siegel (NS): $\phi_{NS}(x;z) = z_1 + (z_2 + z_3 x)\mathrm{e}^{-z_4 x}$
- Svensson (S): $\phi_S(x;z) = z_1 + (z_2 + z_3 x)\mathrm{e}^{-z_5 x} + z_4 x \mathrm{e}^{-z_6 x}$
- both belong to general exponential–polynomial functions

$$p_1(x)\mathrm{e}^{-\alpha_1 x} + \cdots + p_n(x)\mathrm{e}^{-\alpha_n x}$$

where p_i denote polynomials of degree n_i

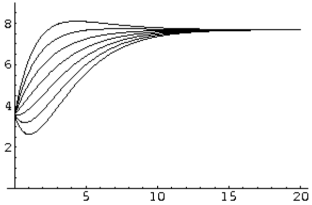

Figure: NS curves for $z_1 = 7.69$, $z_2 = -4.13$, $z_4 = 0.5$ and 7 different values for $z_3 = 1.76, 0.77, -0.22, -1.21, -2.2, -3.19, -4.18$.

Nelson–Siegel and Svensson Families

Table: Overview of estimation procedures by several central banks. Bank for International Settlements (BIS) 1999 [5]. NS is for Nelson–Siegel, S for Svensson, wp for weighted prices

Central bank	Method	Minimized error
Belgium	S or NS	wp
Canada	S	wp
Finland	NS	wp
France	S or NS	wp
Germany	S	yields
Italy	NS	wp
Japan	smoothing splines	prices
Norway	S	yields
Spain	S	wp
Sweden	S	yields
UK	S	yields
USA	smoothing splines	bills: wp bonds: prices

Desirable Features for Curve Families

- Flexible: curves shall fit wide range of term structures
- Parsimonious: number of factors not too large (curse of dimensionality).
- Regular: prefer smooth yield/forward curves that flatten out towards the long end
- Consistent: curve families compatible with dynamic interest rate models (explained and exploited in more detail below)

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Principal Component Analysis

- major problem in term-structure estimation: high dimensionality
- aim: find basis shapes of the yield curve (increments)
- principal component analysis (PCA): dimension reduction technique in multivariate analysis
- key mathematical principle: spectral decomposition for symmetric $n \times n$ matrix Q: $Q = ALA^\top$ where
    - $L = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix of eigenvalues of Q with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$;
    - A is an orthogonal matrix (that is, $A^{-1} = A^\top$) whose columns $a_1, \dots, a_n$ are the normalized eigenvectors of Q ($Qa_i = \lambda_i a_i$), which form orthonormal basis of $\mathbb{R}^n$.

$A^\top$ denotes the transpose of A

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PCA of Random Vector (1)

- X: n-dimensional random vector with mean $\mu = \mathbb{E}[X]$, covariance matrix $Q = \mathrm{cov}[X]$
- Q symmetric and positive semi-definite: $Q = ALA^\top$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and eigenvectors $a_1, \dots, a_n$
- principal components transform of X: $Y = A^\top(X - \mu)$ (recentering and rotation of X)
- $Y_i = a_i^\top(X - \mu) = i$th principal component of X
- $a_i = i$th vector of loadings of X
- note: $Y_i =$ projection of $X - \mu$ onto a_i

$\longrightarrow$ obtain decomposition

$$X = \mu + AY = \mu + \sum_{i=1}^{n} Y_i a_i$$

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PCA of Random Vector (2)

- can show: $\mathbb{E}[Y] = 0$ and $\mathrm{Cov}[Y] = A^\top Q A = A^\top A L A^\top A = L$

$\Rightarrow$ principal components of X are uncorrelated, have variances $\mathrm{Var}[Y_i] = \lambda_i$

- can show: $\mathrm{Var}[a_1^\top X] = \max\left\{\mathrm{Var}[b^\top X] \mid b^\top b = 1\right\}$: Y_1 has maximal variance among all standardized linear combinations of X. For $i = 2, \dots, n$: Y_i has maximal variance among all such linear combinations orthogonal to first $i-1$ linear combinations

PCA of Random Vector (3)

- observe that

$$\sum_{i=1}^{n} \mathrm{Var}[X_i] = \mathrm{trace}(Q) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mathrm{Var}[Y_i]$$

- hence $\frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{n} \lambda_i}$ = amount of variability in X explained by the first k principal components $Y_1, \dots, Y_k$
- application: X = high-dimensional stationary model for (daily changes of) the forward curve. If first k principal components $Y_1, \dots, Y_k$ explain significant amount (e.g. 99%) of variability in X then approximate

$$X \approx \mu + \sum_{i=1}^{k} Y_i a_i .$$

$\Rightarrow$ loadings $a_1, \dots, a_k$ are main components of stochastic forward curve movements

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Sample PCs

- multivariate data observations $x = [x(1), \dots, x(N)]$
- each column $x(t) = (x_1(t), \dots, x_n(t))^\top$ is sample realization of random vector $X(t)$
- $X(t)$ identically distributed as X: mean $\mu = \mathbb{E}[X]$, covariance matrix $Q = \mathrm{cov}[X]$
- empirical mean $\hat{\mu} = \frac{1}{N}\sum_{t=1}^{N} x(t)$
- empirical $n \times n$ covariance matrix (positive semi-definite)

$$\hat{Q}_{ij} = \mathrm{Cov}[x_i, x_j] = \frac{1}{N}\sum_{t=1}^{N}(x_i(t) - \hat{\mu}_i)(x_j(t) - \hat{\mu}_j)$$

- PCA decomposition: $x = \hat{\mu} + \sum_{i=1}^{n} y_i \hat{a}_i$ with
  - $\hat{Q} = \hat{A}\hat{L}\hat{A}^\top$, loadings $\hat{A} = (\hat{a}_1 \mid \cdots \mid \hat{a}_n)$
  - empirical principal components $y = \hat{A}^\top(x - \hat{\mu})$
  - $\mathrm{Cov}[y_i, y_j] = \frac{1}{N}\sum_{t=1}^{N} y_i(t) y_j(t) = \begin{cases} \hat{\lambda}_i, & \text{if } i = j, \\ 0, & \text{else} \end{cases}$

Sample PCs cont'd

- empirical mean $\hat{\mu}$ and covariance matrix $\hat{Q}$ standard estimators for true parameters μ and Q, if observations $X(t)$ are either independent or at least serially uncorrelated (i.e. $\mathrm{Cov}[X(t), X(t+h)] = 0$ for all $h \neq 0$)
- if this kind of stationarity of time series $X(t)$ is in doubt, the standard practice is to differentiate and to consider the increments $\Delta X(t) = X(t) - X(t-1)$
- as illustrated in the following example ...

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PCA of Forward Curve

- increments of forward curve:

$$x_i(t) = R(t+\Delta t; t+\Delta t+\tau_{i-1}, t+\Delta t+\tau_i) - R(t; t+\tau_{i-1}, t+\tau_i)$$

  for times *to* maturity $0 = \tau_0 < \cdots < \tau_n$

- example (Rebonato [44, Section 3.1]): UK market data from 1989–1992, original maturity spectrum divided into eight distinct buckets, i.e. $n = 8$

PCA of Forward Curve (2)

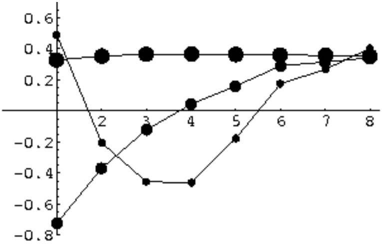

Figure: First three forward curve loadings

Stylized facts:

- first loading roughly flat: parallel shifts of the forward curve (affects the average rate);
- second loading upward sloping: tilting of the forward curve (affects the slope)
- third loading hump-shaped: flex (affecting the curvature)

PCA of Forward Curve (3)

Table: Explained variance of the principal components

PC	Explained variance (%)
1	92.17
2	6.93
3	0.61
4	0.24
5	0.03
6–8	0.01

- First 3 PCs explain more than 99% of variance of x: forward curves can be approximated by linear combination of first three loadings, with small relative error
- these features are very typical (stylized facts), to be expected in most PCA of forward curve (increments). See also Carmona and Tehranchi [14, Section 1.7]. PCA of forward curve goes back to Litterman and Scheinkman [37].

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Correlation

- typical example (Brown and Schaefer [13]): US Treasury term structure 1987–1994
- correlation for changes of forward rates of maturities 0, 0.5, 1, 1.5, 2, 3 years:

$$\begin{pmatrix} 1 & 0.87 & 0.74 & 0.69 & 0.64 & 0.6 \\ & 1 & 0.96 & 0.93 & 0.9 & 0.85 \\ & & 1 & 0.99 & 0.95 & 0.92 \\ & & & 1 & 0.97 & 0.93 \\ & & & & 1 & 0.95 \\ & & & & & 1 \end{pmatrix}$$

Correlation (2)

- first row of this correlation matrix:

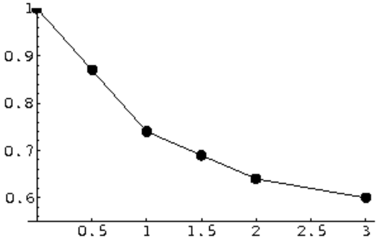

Figure: Correlation between the short rate and instantaneous forward rates for the US Treasury curve 1987–1994.

- in stylized fashion: de-correlation occurs quickly $\Rightarrow$ exponentially decaying correlation structure is plausible

Part III

Arbitrage Theory

Overview

- Recall fundamental arbitrage principles in a Brownian-motion-driven financial market
- Basics of stochastic calculus (without proofs)
- Standard terminology: used without further explanation. Consult text books on stochastic calculus.
- Main pillars for financial applications:
  - Itô's formula
  - Girsanov's change of measure theorem
  - Martingale representation theorem

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Usual Stochastic Setup

- Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$
- Usual conditions:
  - completeness: $\mathcal{F}_0$ contains all of the null sets
  - right-continuity: $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for $t \geq 0$
- d-dimensional $(\mathcal{F}_t)$-Brownian motion $W = (W_1, \dots, W_d)^\top$
- Infinite time horizon (w.l.o.g.): $\mathcal{F} = \mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$
- Sometimes also : $\mathcal{F}_t = \mathcal{F}_t^W$ (for hedging/completeness)

Stochastic Processes

- Convention: "$X = Y$" means "$X = Y$ a.s." ($\mathbb{P}[X = Y] = 1$)
- Borel σ-algebras: $\mathcal{B}[0, t]$, $\mathcal{B}(\mathbb{R}_+)$, or simply $\mathcal{B}$, etc.
- Stochastic process $X = X(\omega, t)$ is called:
  - adapted if $\Omega \ni \omega \mapsto X(\omega, t)$ is $\mathcal{F}_t$-measurable $\forall t \geq 0$,
  - progressively measurable if $\Omega \times [0, t] \ni (\omega, s) \mapsto X(\omega, s)$ is $\mathcal{F}_t \otimes \mathcal{B}[0, t]$-measurable $\forall t \geq 0$.
- Progressive implies adapted
- Progressive $\Rightarrow \int_0^t X(s)\, ds$, $X(t \wedge \tau)$ (stopping time τ), etc. are adapted
- Notation: Prog = progressive σ-algebra, generated by all progressive processes, on $\Omega \times \mathbb{R}_+$. Fact: progressive $\Leftrightarrow$ Prog-measurable (Proposition 1.41 in [42])

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Stochastic Integrands

- $\mathcal{L}^2 :=$ set of $\mathbb{R}^d$-valued progressive processes $h = (h_1, \dots, h_d)$ with

$$\mathbb{E}\left[\int_0^\infty \|h(s)\|^2\, ds\right] < \infty$$

- $\mathcal{L} :=$ set of $\mathbb{R}^d$-valued progressive processes $h = (h_1, \dots, h_d)$ with

$$\int_0^t \|h(s)\|^2\, ds < \infty \quad \text{for all } t > 0$$

- Obvious: $\mathcal{L}^2 \subset \mathcal{L}$

Stochastic Integral

Theorem (Stochastic Integral)

For every $h \in \mathcal{L}$ one can define the stochastic integral

$$(h \bullet W)_t = \int_0^t h(s)\, dW(s) = \sum_{j=1}^{d} \int_0^t h_j(s)\, dW_j(s).$$

with the following properties:

1. *The process $h \bullet W$ is a continuous local martingale.*
2. *Linearity: $(\lambda g + h) \bullet W = \lambda (g \bullet W) + h \bullet W$, for $g, h \in \mathcal{L}$ and $\lambda \in \mathbb{R}$.*
3. *For any stopping time τ, the stopped integral equals*

$$\int_0^{t\wedge\tau} h(s)\, dW(s) = \int_0^t 1_{\{s\leq\tau\}} h(s)\, dW(s) \quad \text{for all } t > 0.$$

Stochastic Integral cont'd

Theorem (Stochastic Integral cont'd)

4. *If $h \in \mathcal{L}^2$ then $h \bullet W$ is a martingale and the Itô isometry holds:*

$$\mathbb{E}\left[\left(\int_0^\infty h(s)\,dW(s)\right)^2\right] = \mathbb{E}\left[\int_0^\infty \|h(s)\|^2\,ds\right].$$

5. *Dominated convergence: if $(h_n) \subset \mathcal{L}$ is a sequence with $\lim_n h_n = 0$ pointwise and such that $|h_n| \leq k$ for some finite constant k then $\lim_n \sup_{s \leq t} |(h_n \bullet W)_s| = 0$ in probability for all $t > 0$.*

Proof.

See [45, Section 2, Chapter IV]. □

Itô processes

- Convention: stochastic integrands = row vectors, Brownian motion = column vector
- Itô process := drift + continuous local martingale

$$X(t) = X(0) + \int_0^t a(s)\,ds + \int_0^t \rho(s)\,dW(s)$$

where $\rho \in \mathcal{L}$ and a is progressive process with $\int_0^t |a(s)|\,ds < \infty \;\forall t > 0$

Lemma

The above decomposition of X is unique in the sense that

$$X(t) = X(0) + \int_0^t a'(s)\,ds + \int_0^t \rho'(s)\,ds$$

implies $a' = a$ and $\rho' = \rho$ $d\mathbb{P} \otimes dt$-a.s.

Proof.

This follows from Proposition (1.2) in [45, Chapter IV]. □

Itô processes cont'd

- Notation:
$$dX(t) = a(t)\,dt + \rho(t)\,dW(t)$$
or, shorter,
$$dX = a\,dt + \rho\,dW$$
- $\mathcal{L}^2(X) :=$ set of progressive $h = (h_1, \ldots, h_d)$ with $\mathbb{E}\left[\int_0^\infty |h(s)a(s)|^2\,ds\right] < \infty$ and $h\rho \in \mathcal{L}^2$
- $\mathcal{L}(X) :=$ set of progressive $h = (h_1, \ldots, h_d)$ with $\int_0^t |h(s)a(s)|\,ds < \infty$ for all $t > 0$ and $h\rho \in \mathcal{L}$
- For $h \in \mathcal{L}(X)$: define stochastic integral w.r.t. X as

$$\int_0^t h(s)\,dX(s) = \int_0^t h(s)a(s)\,ds + \int_0^t h(s)\rho(s)\,dW(s)$$

Quadratic Variation and Covariation

- let $Y(t) = Y(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s)$ be another Itô process
- Define covariation process of X and Y as

$$\langle X, Y\rangle_t = \int_0^t \rho(s)\sigma(s)^\top ds$$

- $\langle X, X\rangle$ called quadratic variation process of X
- Fact (Theorem (1.8) and Definition (1.20) in [45, Chapter IV]):

$$\langle X, Y\rangle_t = \lim \sum_{i=0}^{m} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \quad \text{in probability,}$$

for any sequence of partitions $0 = t_0 < t_1 < \cdots < t_m = t$ with $\max_i |t_{i+1} - t_i| \to 0$

Lévy's Characterization Theorem

- Fact ([49, Section 6.4] or Exercise (1.27) in [45, Chapter IV]): $\langle W_i, W_j \rangle_t = \delta_{ij} t$

Theorem (Lévy's Characterization Theorem)

An $\mathbb{R}^d$-valued continuous local martingale X vanishing at $t = 0$ is a Brownian motion if and only if $\langle X_i, X_j \rangle_t = \delta_{ij} t$ for every $1 \leq i, j \leq d$.

Proof.

See Theorem (3.6) in [45, Chapter IV]. □

Multivariate Itô Processes

- Definition: $X = (X_1, \dots, X_n)^\top$ n-dimensional Itô process if every X_i is an Itô process
- Definition: $\mathcal{L}^2(X)$ ($\mathcal{L}(X)$) $:=$ set of progressive processes $h = (h_1, \dots, h_n)$ such that h_i is in $\mathcal{L}^2(X_i)$ ($\mathcal{L}(X_i)$) for all i
- In this sense: $\mathcal{L}^2 = \mathcal{L}^2(W)$ and $\mathcal{L} = \mathcal{L}(W)$
- Stochastic integral of $h \in \mathcal{L}(X)$ w.r.t. to X is defined coordinate-wise as

$$(h \bullet X)_t = \int_0^t h(s)\, dX(s) = \sum_{i=1}^n \int_0^t h_i(s)\, dX_i(s)$$

Itô's Formula

Core formula of stochastic calculus:

Theorem (Itô's Formula)

Let $f \in C^2(\mathbb{R}^n)$. Then $f(X)$ is an Itô process and

$$f(X(t)) = f(X(0)) + \sum_{i=1}^{n} \int_0^t \frac{\partial f(X(s))}{\partial x_i}\, dX_i(s) + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f(X(s))}{\partial x_i \partial x_j}\, d\langle X_i, X_j\rangle_s .$$

Proof.

See Theorem (3.3) in [45, Chapter IV]. □

Integration by Parts Formula

Corollary (for $f(x,y) = xy$): integration by parts formula

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s)\,dY(s) + \int_0^t Y(s)\,dX(s) + \langle X, Y\rangle_t.$$

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Definitions

Ingredients:

- $b : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ $\mathrm{Prog} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable
- $\sigma : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ $\mathrm{Prog} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable
- ξ: some $\mathcal{F}_0$-measurable initial value

Definitions

Definition

A process X is solution of the stochastic differential equation

$$\begin{aligned} dX(t) &= b(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) \\ X(0) &= \xi \end{aligned} \tag{1}$$

if X is an Itô process satisfying

$$X(t) = \xi + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s).$$

We say that X is unique if any other solution X' of (1) is indistinguishable from X, that is, $X(t) = X'(t)$ for all $t \geq 0$ a.s. If $b(\omega, t, x) = b(t, x)$ and $\sigma(\omega, t, x) = \sigma(t, x)$: solution X of (1) is also called a (time-inhomogeneous) diffusion with diffusion matrix $a(t, x) = \sigma(t, x)\sigma(t, x)^\top$ and drift $b(t, x)$.

Existence and Uniqueness

Theorem

Suppose $b(t,x)$ and $\sigma(t,x)$ satisfy the Lipschitz and linear growth conditions

$$\|b(t,x)-b(t,y)\|+\|\sigma(t,x)-\sigma(t,y)\|\leq K\|x-y\|,$$
$$\|b(t,x)\|^2+\|\sigma(t,x)\|^2\leq K^2(1+\|x\|^2),$$

for all $t\geq 0$ and $x,y\in\mathbb{R}^n$, where K is some finite constant. Then, for every time–space initial point $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}^n$, there exists a unique solution $X=X^{(t_0,x_0)}$ of the stochastic differential equation

$$\begin{aligned} dX(t) &= b(t_0+t,X(t))\,dt+\sigma(t_0+t,X(t))\,dW(t) \\ X(0) &= x_0. \end{aligned} \tag{2}$$

Proof.

See [35, Theorem 5.2.9]. □

Existence and Uniqueness

Note: existence and uniqueness hold sometimes without Lipschitz condition on $\sigma(t, x)$, see "Affine Processes" below

Markov Property

Theorem

Suppose $b(t,x)$ and $a(t,x) = \sigma(t,x)\sigma(t,x)^\top$ are continuous in (t,x), and assume that for every time–space initial point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a unique solution $X = X^{(t_0,x_0)}$ of the stochastic differential equation (2). Then X has the Markov property. That is, for every bounded measurable function f on $\mathbb{R}^n$, there exists a measurable function F on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ such that

$$\mathbb{E}[f(X(T)) \mid \mathcal{F}_t] = F(t, T, X(t)), \quad t \leq T.$$

In words, the $\mathcal{F}_t$-conditional distribution of $X(T)$ is a function of t, T and $X(t)$ only.

Proof.

Follows from [35, Theorem 4.20]. □

Remarks

- Continuity assumption on the diffusion matrix $a(t,x)$ rather than on $\sigma(t,x)$, since $a(t,x)$ determines law of X. Ambiguity with $\sigma(t,x)$: $\sigma\, DD^\top \sigma^\top = \sigma\sigma^\top$ for any orthogonal $d \times d$-matrix D.
- For most practical purposes: assume $\sigma(t,x)$ itself is continuous in (t,x)
- Time-inhomogeneous diffusion X in (2) can be regarded as $\mathbb{R}_+ \times \mathbb{R}^n$-valued homogeneous diffusion $(X_0', \dots, X_n')(t) = (t_0 + t, X(t))$. Calendar time at inception $(t=0)$ is then $X_0'(0) = t_0$. Accordingly, t measures relative time with respect to t_0.

Stochastic Exponential

Define: stochastic exponential of an Itô process X by

$$\mathcal{E}_t(X) = \mathrm{e}^{X(t) - \frac{1}{2}\langle X, X\rangle_t}$$

Lemma

Let X and Y be Itô processes.

1. *$U = \mathcal{E}(X)$ is a positive Itô process and the unique solution of the stochastic differential equation*

$$dU = U\,dX, \quad U(0) = \mathrm{e}^{X(0)}.$$

2. *$\mathcal{E}(X)$ is continuous local martingale if X is local martingale.*
3. *$\mathcal{E}(0) = 1$.*
4. *$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y)\,\mathrm{e}^{\langle X,Y\rangle}$.*
5. *$\mathcal{E}(X)^{-1} = \mathcal{E}(-X)\,\mathrm{e}^{\langle X,X\rangle}$.*

Proof.

Stochastic calculus (exercise). □

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Financial Market

Financial market $S = (S_0, \dots, S_n)^\top$ with

- a risk-free asset (money-market account)

$$dS_0 = S_0 \, r \, dt, \quad S_0(0) = 1,$$

- n risky assets ($i = 1, \dots, n$)

$$dS_i = S_i \left(\mu_i \, dt + \sigma_i \, dW \right), \quad S_i(0) > 0$$

Term-Structure Models

Damir Filipović

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Financial Market

Standing assumption: short rates r, appreciation rates μ_i, volatility $\sigma_i = (\sigma_{i1}, \dots, \sigma_{id})$ progressive and s.t.

$$X_0(t) = \int_0^t r(s)\, ds$$

$$X_i(t) = \int_0^t \mu_i(s)\, ds + \int_0^t \sigma_i(s)\, dW(s)$$

are well-defined Itô processes ($i = 1, \dots, n$)

Corollary

For all i: $S_i(t) = S_i(0)\mathcal{E}_t(X_i)$ are positive Itô processes.

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Self-Financing Portfolios

- Definition: a portfolio/trading strategy is an $\mathbb{R}^{n+1}$-valued progressive process $\phi = (\phi_0, \dots, \phi_n)$
- Corresponding value process: $V = \phi\, S = \sum_{i=0}^{n} \phi_i\, S_i$
- Portfolio ϕ is self-financing for S if $\phi \in \mathcal{L}(S)$ and

$$dV = \phi\, dS = \sum_{i=0}^{n} \phi_i\, dS_i$$

In words: there is no in- or outflow of capital

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Numeraires

- All prices are in terms of a numeraire (e.g. euros)
- Any traded instrument qualifies as a numeraire: S_0 (often), or S_p for some $p \leq n$
- Notation: calligraphic letters for discounted values

$$\mathcal{S} = \frac{S}{S_p}$$

$$\mathcal{V} = \frac{V}{S_p} = \sum_{i=0}^{n} \phi_i \, \mathcal{S}_i$$

Invariance of Self-Financing Property

Lemma

Let $\phi \in \mathcal{L}(S) \cap \mathcal{L}(\mathcal{S})$. Then ϕ is self-financing for S if and only if it is self-financing for $\mathcal{S}$, in particular

$$d\mathcal{V} = \phi \, d\mathcal{S} = \sum_{\substack{i=0 \\ i \neq p}}^{n} \phi_i \, d\mathcal{S}_i. \tag{3}$$

Proof.

Exercise. $\square$

Note: Since $d\mathcal{S}_p \equiv 0$, the number of summands in (3) reduces to n.

Construct Self-Financing Portfolio

Given

- initial wealth $V(0)$
- any $(\phi_0, \dots, \phi_{p-1}, \phi_{p+1}, \dots, \phi_n) \in \mathcal{L}(\mathcal{S}_0, \dots, \mathcal{S}_{p-1}, \mathcal{S}_{p+1}, \dots, \mathcal{S}_n)$

Aim: construct ϕ_p s.t. $\phi = (\phi_0, \dots, \phi_n)$ is self-financing

- From above lemma: discounted value process is

$$\mathcal{V}(t) = V(0) + \sum_{\substack{i=0 \\ i \neq p}}^{n} \int_0^t \phi_i(s)\, d\mathcal{S}_i(s)$$

- Hence define $\phi_p(t) = \mathcal{V}(t) - \sum_{\substack{i=0 \\ i \neq p}}^{n} \phi_i(t)\mathcal{S}_i(t)$
- Since $\mathcal{S}_p \equiv 1 \Rightarrow \phi = (\phi_0, \dots, \phi_n) \in \mathcal{L}(\mathcal{S})$: ϕ is self-financing for $\mathcal{S}$
- To be checked from case to case: $\phi \in \mathcal{L}(S)$

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Arbitrage Portfolios

- Definition: arbitrage portfolio := self-financing portfolio ϕ with value process satisfying

$$V(0) = 0 \quad \text{and} \quad V(T) \geq 0 \quad \text{and} \quad \mathbb{P}[V(T) > 0] > 0$$

for some $T > 0$.

- If no arbitrage portfolios exist for any $T > 0$ we say the model is arbitrage-free.

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Example of Arbitrage

Lemma

Suppose there exists a self-financing portfolio with value process

$$dU = U\,k\,dt,$$

for some progressive process k. If the market is arbitrage-free then necessarily

$$r = k, \quad d\mathbb{P} \otimes dt\text{-a.s.}$$

Proof I

After discounting with S_0 we obtain

$$\mathcal{U}(t) = \frac{U(t)}{S_0(t)} = U(0)\mathrm{e}^{\int_0^t (k(s)-r(s))\,ds}.$$

Then

$$\psi(t) = 1_{\{k(t)>r(t)\}}$$

yields a self-financing strategy with discounted value process

$$\mathcal{V}(t) = \int_0^t \psi(s)\,d\mathcal{U}(s)$$
$$= \int_0^t \left(1_{\{k(s)>r(s)\}}(k(s)-r(s))\mathcal{U}(s)\right)\,ds \geq 0.$$

Proof II

Hence absence of arbitrage requires

$$
\begin{aligned}
0 &= \mathbb{E}[\mathcal{V}(T)] \\
&= \int_{\mathcal{N}} \underbrace{\left(1_{\{k(\omega,t)>r(\omega,t)\}}(k(\omega,t)-r(\omega,t))\mathcal{U}(\omega,t)\right)}_{>0 \text{ on } \mathcal{N}} d\mathbb{P}\otimes dt
\end{aligned}
$$

where

$$
\mathcal{N} = \{(\omega,t) \mid k(\omega,t) > r(\omega,t)\}
$$

is a measurable subset of $\Omega \times [0,T]$. But this can only hold if $\mathcal{N}$ is a $d\mathbb{P}\otimes dt$-nullset. Using the same arguments with changed signs proves the lemma ($\rightarrow$ exercise).

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Equivalent Martingale Measures

- When is a model arbitrage-free?
- For simplicity of notation: fix S_0 as a numeraire

Definition

An equivalent (local) martingale measure (E(L)MM) $\mathbb{Q} \sim \mathbb{P}$ has the property that the discounted price processes $\mathcal{S}_i$ are $\mathbb{Q}$-(local) martingales for all i.

- Need to understand how W transforms under equivalent change of measure ...

Girsanov's Theorem

Theorem (Girsanov's Change of Measure Theorem)

Let $\gamma \in \mathcal{L}$ be such that the stochastic exponential $\mathcal{E}(\gamma \bullet W)$ is a uniformly integrable martingale with $\mathcal{E}_\infty(\gamma \bullet W) > 0$. Then

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_\infty(\gamma \bullet W)$$

defines an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$, and the process

$$W^*(t) = W(t) - \int_0^t \gamma(s)^\top ds$$

is a $\mathbb{Q}$-Brownian motion.

Proof.

See Theorem (1.12) in [45, Chapter VIII]. □

Note: $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}_t(\gamma \bullet W)$

Novikov's Condition

Theorem (Novikov's Condition)

If

$$\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\int_0^\infty \|\gamma(s)\|^2\,ds}\right] < \infty$$

then $\mathcal{E}(\gamma \bullet W)$ *is a uniformly integrable martingale with* $\mathcal{E}_\infty(\gamma \bullet W) > 0$.

Proof.

See Proposition (1.15) in [45, Chapter VIII] for uniform integrability of $\mathcal{E}(\gamma \bullet W)$, and Proposition (1.26) in [45, Chapter IV] for finiteness of $(\gamma \bullet W)_\infty$ which is equivalent to $\mathcal{E}_\infty(\gamma \bullet W) > 0$. $\square$

Note: Novikov's condition is only sufficient but not necessary, see exercise in "Affine Processes".

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Market Price of Risk

Assume:

- $\mathbb{Q}$ ELMM of the form $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_\infty(\gamma \bullet W)$
- Girsanov transformed Brownian motion $W^*(t) = W(t) - \int_0^t \gamma(s)^\top ds$

Integration by parts $\Rightarrow$ $\mathcal{S}$-dynamics ($i = 1, \dots, n$):

$$\begin{aligned} d\mathcal{S}_i &= \mathcal{S}_i\,(\mu_i - r)\,dt + \mathcal{S}_i\,\sigma_i\,dW \\ &= \mathcal{S}_i\left(\mu_i - r + \sigma_i\gamma^\top\right)dt + \mathcal{S}_i\sigma_i\,dW^* \end{aligned}$$

$\mathcal{S}$ is a $\mathbb{Q}$-local martingale $\Rightarrow$ $d\mathbb{Q}\otimes dt$-a.s.

$$-\sigma_i\,\gamma^\top = \mu_i - r \quad \text{for all } i = 1, \dots, n.$$

Economic interpretation:

- right: excess of return over the risk free rate r for asset i
- left: linear combination of volatilities σ_{ij} (risk factor W_j) with factor loadings $-\gamma_j$

Market Price of Risk cont'd

$\Rightarrow$ $-\gamma :=$ the market price of risk vector

- Main point: $-\gamma$ the same for all risky assets $i = 1, \dots, n$
- Conversely: if $\gamma \in \mathcal{L}$ satisfies $-\sigma_i \gamma^\top = \mu_i - r$ and Novikov's condition, then $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_\infty(\gamma \bullet W)$ defines an ELMM $\mathbb{Q}$
- Final note: $\mathcal{S}_i = S_i(0)\mathcal{E}(\sigma_i \bullet W^*) \Rightarrow$ if σ_i satisfies Novikov condition $\forall i = 1, \dots, n$ then $\mathbb{Q}$ an EMM

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An Arbitrage Strategy

- Local martingales $\Rightarrow$ be alert to pitfalls!
- Example: $\exists \phi \in \mathcal{L}$ s.t. local martingale $M(t) = \int_0^t \phi(s)\, dW(s)$ satisfies $M(1) = 1$ (Dudley's Representation Theorem 12.1 in [49])
- Looks like discounted value process of a self-financing strategy (in the Bachelier model [3] $\mathcal{S} = W$)

$\Rightarrow$ arbitrage!

- Solution: M is unbounded from below (a true local martingale). In reality, no lender would provide us with an infinite credit line.

Admissible Strategies

Definition

A self-financing strategy ϕ is admissible if its discounted value process $\mathcal{V}$ is a $\mathbb{Q}$-martingale for some ELMM $\mathbb{Q}$.

- Caution: admissibility is sensitive with respect to the choice of numeraire (see Delbaen and Schachermayer [22])

Useful local martingale property result (generalizing "Stochastic Integral" Theorem):

Lemma

The discounted value process $\mathcal{V}$ of an admissible strategy is a $\mathbb{Q}$-local martingale under every ELMM $\mathbb{Q}$.

Proof

Proof.

By assumption, $d\mathcal{V} = \phi\, d\mathcal{S}$ is the stochastic integral with respect to the continuous $\mathbb{Q}$-local martingale $\mathcal{S}$. The statement now follows from Proposition (2.7) in [45, Chapter IV] and Proposition (1.5) in [45, Chapter VIII]. $\square$

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First Fundamental Theorem of Asset Pricing

Lemma

Suppose there exists an ELMM $\mathbb{Q}$. Then the model is arbitrage-free, in the sense that there exists no admissible arbitrage strategy.

Proof.

Indeed, let $\mathcal{V}$ be the discounted value process of an admissible strategy, with $\mathcal{V}(0) = 0$ and $\mathcal{V}(T) \geq 0$. Since $\mathcal{V}$ is a $\mathbb{Q}$-martingale for some ELMM $\mathbb{Q}$, we have

$$0 \leq \mathbb{E}_{\mathbb{Q}}[\mathcal{V}(T)] = \mathcal{V}(0) = 0,$$

whence $\mathcal{V}(T) = 0$. $\square$

First Fundamental Theorem of Asset Pricing

As for converse statement:

- Absence of arbitrage among admissible strategies not sufficient for the existence of an ELMM
- Delbaen and Schachermayer [21]: "no free lunch with vanishing risk" (some form of asymptotic arbitrage) is equivalent to the existence of an ELMM
- Technical details far from trivial, beyond this course
- Custom in financial engineering: consider existence of an ELMM as "essentially equivalent" to the absence of arbitrage

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Contingent Claims

Definition

A contingent claim due at T (or T-claim) is an $\mathcal{F}_T$-measurable random variable X.

- Examples: cap, floor, swaption

Main problems:

- How can one hedge against the financial risk involved in trading contingent claims?
- What is a fair price for a contingent claim X?

Attainable Claims

Definition

A contingent claim X due at T is attainable if $\exists$ admissible strategy ϕ which replicates, or hedges, X. That is, its value process V satisfies $V(T) = X$

- Simple example: S_1 = price process of T-bond. Contingent claim $X \equiv 1$ due at T: attainable by buy-and-hold strategy $\phi_0 \equiv 0$, $\phi_1 = 1_{[0,T]}$, with value process $V = S_1$.

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Representation Theorem

We know: stochastic integral w.r.t. W is a local martingale. The converse holds true if the filtration is not too large:

Theorem (Representation Theorem)

Assume that the filtration

$$(\mathcal{F}_t) \text{ is generated by the Brownian motion } W. \tag{4}$$

Then every $\mathbb{P}$-local martingale M has a continuous modification and there exists $\psi \in \mathcal{L}$ such that

$$M(t) = M(0) + \int_0^t \psi(s)\, dW(s).$$

Consequently, every equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ can be represented in the form (15.3) for some $\gamma \in \mathcal{L}$.

Proof.

See Theorem (3.5) in [45, Chapter V]. The last statement follows since $M(t) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$ is a positive martingale. $\square$

Complete Markets

Definition

The market model is complete if, on any finite time horizon $T > 0$, every T-claim X with bounded discounted payoff $X/S_0(T)$ is attainable.

- Note: completeness $\not\Leftrightarrow$ absence of arbitrage
- However . . .

Second Fundamental Theorem of Asset Pricing

Theorem (Second Fundamental Theorem of Asset Pricing)

Assume $\mathcal{F}_t = \mathcal{F}_t^W$ and there exists an ELMM $\mathbb{Q}$. Then the following are equivalent:

1. *The model is complete.*
2. *The ELMM $\mathbb{Q}$ is unique.*
3. *The $n \times d$-volatility matrix $\sigma = (\sigma_{ij})$ is $d\mathbb{P} \otimes dt$-a.s. injective.*
4. *The market price of risk $-\gamma$ is $d\mathbb{P} \otimes dt$-a.s. unique.*

Under any of these conditions, every T-claim X with $\mathbb{E}_{\mathbb{Q}}\left[\frac{|X|}{S_0(T)}\right] < \infty$ is attainable.

Second Fundamental Theorem of Asset Pricing

Note: Property 3 $\Rightarrow$ number of risk factors $d \leq n$ number of risky assets: randomness generated by the d noise factors dW can be fully absorbed by the n discounted price increments $d\mathcal{S}_1, \dots, d\mathcal{S}_n$

Proof I

$1 \Rightarrow 2$: Let $A \in \mathcal{F}_T$. By definition there exists an admissible strategy ϕ with discounted value process $\mathcal{V}$ satisfying $\mathcal{V}(t) = \mathbb{E}_{\mathbb{Q}}[1_A \mid \mathcal{F}_t]$ for some ELMM $\mathbb{Q}$. This implies that $|\mathcal{V}| \leq 1$. In view of Lemma 15.6, $\mathcal{V}$ is thus a martingale under any ELMM. Now let $\mathbb{Q}'$ be any ELMM. Then $\mathbb{Q}'[A] = \mathcal{V}(0) = \mathbb{Q}[A]$, and hence $\mathbb{Q} = \mathbb{Q}'$.

$2 \Rightarrow 3$: See Proposition 8.2.1 in [41].

$3 \Rightarrow 4 \Rightarrow 2$: This follows from the linear market price of risk equation (2) and the last statement of the representation theorem.

$3 \Rightarrow 1$: Let X be a claim due at some $T > 0$ satisfying $\mathbb{E}_{\mathbb{Q}}\left[\frac{|X|}{S_0(T)}\right] < \infty$ for some ELMM $\mathbb{Q}$ (this holds in particular if $X/S_0(T)$ is bounded). We define the $\mathbb{Q}$-martingale

$$Y(t) = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)} \mid \mathcal{F}_t\right], \quad t \leq T.$$

Proof II

By Bayes' rule we obtain

$$Y(t)D(t) = D(t)\mathbb{E}_{\mathbb{Q}}[Y(T) \mid \mathcal{F}_t] = \mathbb{E}[Y(T)D(T) \mid \mathcal{F}_t],$$

with the density process $D(t) = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = \mathcal{E}_t(\gamma \bullet W)$. Hence YD is a $\mathbb{P}$-martingale and by the representation theorem there exists some $\psi \in \mathcal{L}$ such that

$$Y(t)D(t) = Y(0) + \int_0^t \psi(s)\, dW(s).$$

Applying Itô's formula yields

$$d\left(\frac{1}{D}\right) = -\frac{1}{D}\gamma\, dW + \frac{1}{D}\|\gamma\|^2\, dt,$$

Proof III

and

$$
\begin{aligned}
dY = d\left((YD)\frac{1}{D}\right) &= YD\, d\left(\frac{1}{D}\right) + \frac{1}{D}\, d(YD) + d\left\langle YD, \frac{1}{D}\right\rangle \\
&= \left(\frac{1}{D}\psi - Y\gamma\right) dW - \left(\frac{1}{D}\psi - Y\gamma\right)\gamma^\top dt \\
&= \left(\frac{1}{D}\psi - Y\gamma\right) dW^*
\end{aligned}
$$

where $dW^* = dW - \gamma\, dt$ denotes the Girsanov transformed $\mathbb{Q}$-Brownian motion. Note that we just have shown that the martingale representation property also holds for W^* under $\mathbb{Q}$. Since σ is injective, there exists some $d \times n$-matrix-valued progressive process σ^{-1} such that $\sigma^{-1}\sigma$ equals the $d \times d$-identity matrix. If we define $\phi = (\phi_1, \dots, \phi_n)$ via

$$
\phi_i = \frac{\left(\left(\frac{1}{D}\psi - Y\gamma\right)\sigma^{-1}\right)_i}{\mathcal{S}_i},
$$

Proof IV

it follows that

$$dY = \left(\frac{1}{D}\psi - Y\gamma\right)\sigma^{-1}\sigma\, dW^* = \sum_{i=1}^{n} \phi_i \mathcal{S}_i \sigma_i\, dW^* = \sum_{i=1}^{n} \phi_i\, d\mathcal{S}_i.$$

Hence ϕ yields an admissible strategy with discounted value process satisfying

$$\mathcal{V}(t) = Y(t) = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)}\right] + \sum_{i=1}^{n}\int_0^t \phi_i(s)\, d\mathcal{S}_i(s),$$

and in particular $\mathcal{V}(T) = Y(T) = X/S_0(T)$. Notice that ϕ is admissible since $\mathcal{V}$ is by construction a true $\mathbb{Q}$-martingale. This also proves the last statement of the theorem.

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Arbitrage Price

Complete model: unique arbitrage price

$$\Pi(t) = S_0(t)\mathcal{V}(t) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)} \mid \mathcal{F}_t\right].$$

Any other price would yield arbitrage!
Illustration for $t = 0$ (see Proposition 2.6.1 in [41]): suppose price $p > \Pi(0)$. Then

- at $t = 0$: sell short the claim and receive p. Invest $p - \Pi(0)$ in the money-market account, replicate claim with remaining initial capital $\Pi(0)$
- at T: clear short position in the claim, we are left with $p - \Pi(0) > 0$ units of $S_0(T) \Rightarrow$ arbitrage (similar for $p < \Pi(0)$)

Incomplete Markets

- Our bond markets are often complete: infinitely many traded assets
- But: real markets are generically incomplete (e.g. price jumps)
- Vast literature on pricing and hedging in incomplete markets

Our approach (custom):

- exogenously specify a particular ELMM $\mathbb{Q}$ (the market price of risk $-\gamma$)
- price a T-claim X by $\mathbb{Q}$-expectation: $\Pi(t) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)} \mid \mathcal{F}_t\right]$

$\Rightarrow$ Consistent arbitrage price: $S_0, \ldots, S_n, \Pi$ is arbitrage-free

State-Price Density

Define

$$\pi(t) = \frac{1}{S_0(t)} \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}.$$

Bayes' rule implies

$$\Pi(t) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)} \mid \mathcal{F}_t\right] = S_0(t)\frac{\mathbb{E}\left[\frac{X}{S_0(T)}\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T} \mid \mathcal{F}_t\right]}{\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}}$$

$$= \frac{\mathbb{E}\left[X\pi(T) \mid \mathcal{F}_t\right]}{\pi(t)}$$

In particular $\Pi(0) = \mathbb{E}[X\pi(T)]$

Definition

π is called the state-price density process

State-Price Density

- Example: T-bond price

$$P(t,T) = \mathbb{E}\left[\frac{\pi(T)}{\pi(t)} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_0(t)}{S_0(T)} \mid \mathcal{F}_t\right]$$

- Show: If $\mathbb{Q}$ is an EMM then $S_i\pi$ are $\mathbb{P}$-martingales.

Generalities

Diffusion Short-Rate Models

Examples

Inverting the Forward Curve

Affine Term-Structures

Some Standard Models

Vasiček Model

CIR Model

Dothan Model

Ho–Lee Model

Hull–White Model

Part IV

Short Rate Models

Overview

- Earliest stochastic interest rate models: models of the short rates
- Introduction to diffusion short-rate models in general
- Survey of some standard models
- Particular focus is on affine term-structures

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General Assumptions

- Usual stochastic setup $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$
- $\mathbb{P}$ objective probability measure
- short rates follow Itô process

$$dr(t) = b(t)\,dt + \sigma(t)\,dW(t)$$

- money-market account $B(t) = \mathrm{e}^{\int_0^t r(s)\,ds}$
- no arbitrage: $\exists$ EMM $\mathbb{Q}$ of the form $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_\infty(\gamma \bullet W)$, s.t. discounted bond price process $P(t,T)/B(t)$ is a $\mathbb{Q}$-martingale and $P(T,T) = 1$ for all $T > 0$
- Girsanov: $W^*(t) = W(t) - \int_0^t \gamma(s)^\top ds$ is $\mathbb{Q}$-Brownian motion
- for strategies involving a continuum of bonds see Björk et al. [6] and Carmona and Tehranchi [14]

Consequences

- $\mathbb{Q}$ is true EMM: $P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\,ds} \mid \mathcal{F}_t \right]$
- under $\mathbb{Q}$: $dr(t) = \left(b(t) + \sigma(t)\,\gamma(t)^\top \right)\, dt + \sigma(t)\, dW^*(t)$
- If $\mathcal{F}_t = \mathcal{F}_t^W$ then for any $T > 0$ $\exists$ progressive $\mathbb{R}^d$-valued process $v(\cdot, T)$ s.t. $(\dots)$

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + v(t,T)\,dW^*(t).$$

Hence under objective probability $\mathbb{P}$:

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) - v(t,T)\,\gamma(t)^\top \right)\, dt + v(t,T)\,dW(t)$$

$\rightarrow$ market price of risk: $-\gamma$ = excess of instantaneous return over $r(t)$ in units of volatility $v(t,T)$

Specification of Market Price of Risk

- How to specify market price of risk (mpr)?
- general equilibrium: market price of risk endogenous, see Cox, Ingersoll and Ross [19]
- arbitrage theory: unable to identify market price of risk: money-market account B alone cannot be used to replicate bond payoffs: model is incomplete
- in line with non-uniqueness of EMM: can be any equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$

Summary

- A short-rate model is not fully determined without the exogenous specification of the market price of risk.
- custom: postulate $\mathbb{Q}$-dynamics of r

$\longrightarrow$ implies $\mathbb{Q}$-dynamics of all bonds

$\longrightarrow$ contingent claims priced by $\mathbb{Q}$-expectation of discounted payoffs

- market price of risk (and hence $\mathbb{P}$): inferred by statistical methods from historical observations of price movements

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Diffusion Short-Rate Models

- stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ with martingale measure $\mathbb{Q}$
- W^* one-dimensional $\mathbb{Q}$-Brownian motion $(d = 1)$
- $\mathcal{Z} \subset \mathbb{R}$ closed interval with non-empty interior
- b and σ continuous functions on $\mathbb{R}_+ \times \mathcal{Z}$
- Assumption: for all $(t_0, r_0) \in \mathcal{Z}$ there exists a unique $\mathcal{Z}$-valued solution $r = r^{(t_0,r_0)}$ of SDE

$$dr(t) = b(t_0+t, r(t))\,dt + \sigma(t_0+t, r(t))\,dW^*(t), \quad r(0) = r_0$$

- sufficient conditions for existence and uniqueness available
- recall: Markov property of r

Term-Structure Equation

Lemma

Let $T > 0$, and Φ continuous function on $\mathcal{Z}$. Assume $F = F(t,r)$ in $C^{1,2}([0,T] \times \mathcal{Z})$ is a solution to boundary value problem on $[0,T] \times \mathcal{Z}$ (term-structure equation for Φ):

$$\partial_t F(t,r) + b(t,r)\partial_r F(t,r) + \frac{1}{2}\sigma^2(t,r)\partial_r^2 F(t,r) - rF(t,r) = 0$$
$$F(T,r) = \Phi(r).$$

Then $M(t) = F(t,r(t))\mathrm{e}^{-\int_0^t r(u)\,du}$, $t \leq T$, is a local martingale.

If in addition either:

1. $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left|\partial_r F(t,r(t))\mathrm{e}^{-\int_0^t r(u)\,du}\sigma(t,r(t))\right|^2 dt\right] < \infty$, *or*
2. *M is uniformly bounded,*

then M is a true martingale, and

$$F(t,r(t)) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(u)\,du}\Phi(r(T)) \mid \mathcal{F}_t\right], \quad t \leq T.$$

Term-Structure Equation: Proof

Proof.

We can apply Itô's formula to M and obtain

$$
\begin{aligned}
dM(t) = \Big(&\partial_t F(t,r(t)) + b(t,r(t))\partial_r F(t,r(t)) \\
&+ \frac{1}{2}\sigma^2(t,r)\partial_r^2 F(t,r(t)) - r(t)F(t,r(t))\Big) \mathrm{e}^{-\int_0^t r(u)\,du}\,dt \\
&+ \partial_r F(t,r(t))\mathrm{e}^{-\int_0^t r(u)\,du}\sigma(t,r(t))\,dW^*(t) \\
= {} & \partial_r F(t,r(t))\mathrm{e}^{-\int_0^t r(u)\,du}\sigma(t,r(t))\,dW^*(t).
\end{aligned}
$$

Hence M is a local martingale.
Either Condition 1 or 2 implies M is true martingale. Since

$$
M(T) = \Phi(r(T))\mathrm{e}^{-\int_0^T r(u)\,du}
$$

we obtain

$$
F(t,r(t))\mathrm{e}^{-\int_0^t r(u)\,du} = M(t) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r(u)\,du}\Phi(r(T)) \mid \mathcal{F}_t\right].
$$

Multiplying both sides by $\mathrm{e}^{\int_0^t r(u)\,du}$ yields the claim. $\square$

Term-Structure Equation: Pricing Efficiency

- term-structure equation for Φ: solution $F(t, r(t))$ = price of T-claim $\Phi(r(T))$
- for $\Phi \equiv 1$: $P(t, T) = F(t, r(t); T)$
- pricing algorithm computationally efficient?
- ok: solving PDEs in less than three space dimensions numerically feasible, but ...
- ... nuisance: have to solve a PDE for every $T > 0$, just to get bond prices
- problem: term-structure calibration

$\Rightarrow$ short-rate models admitting closed-form solutions for bond prices favorable

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Examples

1. Vasiček [52]: $\mathcal{Z} = \mathbb{R}$,

$$dr(t) = (b + \beta r(t))\, dt + \sigma\, dW^*(t),$$

2. Cox–Ingersoll–Ross (henceforth CIR) [19]: $\mathcal{Z} = \mathbb{R}_+$, $b \geq 0$,

$$dr(t) = (b + \beta r(t))\, dt + \sigma \sqrt{r(t)}\, dW^*(t),$$

3. Dothan [23]: $\mathcal{Z} = \mathbb{R}_+$,

$$dr(t) = \beta r(t)\, dt + \sigma r(t)\, dW^*(t),$$

4. Black–Derman–Toy [7]: $\mathcal{Z} = \mathbb{R}_+$,

$$dr(t) = \beta(t) r(t)\, dt + \sigma(t) r(t)\, dW^*(t),$$

Examples continued

5. Black–Karasinski [8]: $\mathcal{Z} = \mathbb{R}_+$, $\ell(t) = \log r(t)$,

$$d\ell(t) = (b(t) + \beta(t)\ell(t))\,dt + \sigma(t)\,dW^*(t),$$

6. Ho–Lee [29]: $\mathcal{Z} = \mathbb{R}$,

$$dr(t) = b(t)\,dt + \sigma\,dW^*(t),$$

7. Hull–White extended Vasiček [30]: $\mathcal{Z} = \mathbb{R}$,

$$dr(t) = (b(t) + \beta(t)r(t))\,dt + \sigma(t)\,dW^*(t),$$

8. Hull–White extended CIR [30]: $\mathcal{Z} = \mathbb{R}_+$, $b(t) \geq 0$,

$$dr(t) = (b(t) + \beta(t)r(t))\,dt + \sigma(t)\sqrt{r(t)}\,dW^*(t).$$

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Inverse Problem

- specification of short-rate model parameters fully specifies initial term-structure $T \mapsto P(0,T) = F(0, r(0); T)$ and hence forward curve
- conversely: invert term-structure equation to match given initial forward curve
- example Vasiček model: $P(0,T) = F(0, r(0); T, b, \beta, \sigma)$ parameterized curve family with three degrees of freedom b, β, σ (for given $(r(0))$
- often too restrictive: poor fit of the current data

$\Rightarrow$ time-inhomogeneous short-rate models, such as the Hull–White extensions: time-dependent parameters = infinite degree of freedom $\Rightarrow$ perfect fit of any given curve

- usually functions $b(t)$ etc. are fully determined by the initial term-structure
- explicit examples in the following ...

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Affine Term-Structure Models

- recall: diffusion short rate model

$$dr(t) = b(t_0 + t, r(t))\, dt + \sigma(t_0 + t, r(t))\, dW^*(t) \quad (5)$$

- recall: short-rate models admitting closed-form bond prices $F(t, r; T)$ favorable
- Definition: r affine term-structure (ATS) model if $F(t, r; T) = \exp(-A(t, T) - B(t, T)r)$
- note: $F(T, r; T) = 1$ implies $A(T, T) = B(T, T) = 0$
- nice: short-rate ATS models can be completely characterized ...

Characterization

Proposition

The short-rate model (5) provides ATS if and only if its diffusion and drift terms are of the form

$$\sigma^2(t,r) = a(t) + \alpha(t) r \quad \text{and} \quad b(t,r) = b(t) + \beta(t) r, \tag{6}$$

for some continuous functions a, α, b, β, and the functions A and B satisfy the system of ordinary differential equations, for all $t \leq T$,

$$\partial_t A(t,T) = \frac{1}{2} a(t) B^2(t,T) - b(t) B(t,T), \quad A(T,T) = 0, \tag{7}$$

$$\partial_t B(t,T) = \frac{1}{2} \alpha(t) B^2(t,T) - \beta(t) B(t,T) - 1, \quad B(T,T) = 0. \tag{8}$$

Proof

Proof.

Insert $F(t, r; T) = \exp(-A(t, T) - B(t, T)r)$ in term-structure equation: short-rate model (5) provides an ATS if and only if

$$\frac{1}{2}\sigma^2(t, r)B^2(t, T) - b(t, r)B(t, T) = \partial_t A(t, T) + (\partial_t B(t, T) + 1)r \tag{9}$$

holds for all $t \leq T$ and $r \in \mathcal{Z}$.

By inspection: specification (6)–(8) satisfies (9). This proves sufficiency.

Necessity of (6)–(8): ...

$\square$

Further Specification of Affine Parameters

- $\mathcal{Z} = \mathbb{R}$: necessarily $\alpha(t) = 0$ and $a(t) \geq 0$, and b, β arbitrary: Hull–White extension of the Vasiček model
- $\mathcal{Z} = \mathbb{R}_+$: necessarily $a(t) = 0$, $\alpha(t) \geq 0$ and $b(t) \geq 0$ (otherwise the process would cross zero), and β arbitrary: Hull–White extension of the CIR model
- general affine processes are discussed later
- recall list: all short-rate models except the Dothan, Black–Derman–Toy and Black–Karasinski models have an ATS

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Vasiček: $dr = (b + \beta r)\, dt + \sigma\, dW^*$

- explicit solution:

$$r(t) = r(0)\mathrm{e}^{\beta t} + \frac{b}{\beta}\left(\mathrm{e}^{\beta t} - 1\right) + \sigma \mathrm{e}^{\beta t} \int_0^t \mathrm{e}^{-\beta s}\, dW^*(s)$$

$\Rightarrow$ $r(t)$ Gaussian process with mean

$$\mathbb{E}_{\mathbb{Q}}\left[r(t)\right] = r(0)\mathrm{e}^{\beta t} + \frac{b}{\beta}\left(\mathrm{e}^{\beta t} - 1\right)$$

and variance

$$\mathrm{Var}_{\mathbb{Q}}[r(t)] = \sigma^2 \mathrm{e}^{2\beta t} \int_0^t \mathrm{e}^{-2\beta s}\, ds = \frac{\sigma^2}{2\beta}\left(\mathrm{e}^{2\beta t} - 1\right)$$

- $\mathbb{Q}[r(t) < 0] > 0$: not satisfactory (but probability usually small)
- Vasiček assumed constant market price of risk (on finite time horizon) $\Rightarrow$ objective $\mathbb{P}$-dynamics of $r(t)$ also of the above form

Vasiček: $dr = (b + \beta r)\, dt + \sigma\, dW^*$

If $\beta < 0$:

- $\Rightarrow$ $r(t)$ mean-reverting with mean reversion level $b/|\beta|$
- $\Rightarrow$ $r(t)$ converges to a Gaussian r.v. with mean $b/|\beta|$ and variance $\sigma^2/(2|\beta|)$, for $t \to \infty$

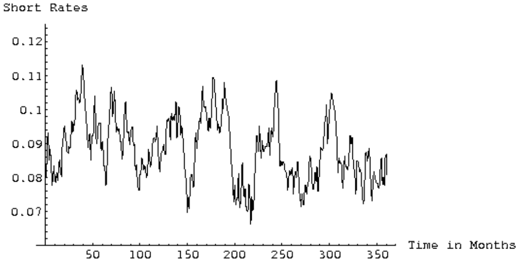

Figure: Vasiček short-rate process for $\beta = -0.86$, $b/|\beta| = 0.09$ (mean reversion level), $\sigma = 0.0148$ and $r(0) = 0.08$.

Vašiček: $dr = (b + \beta r)\,dt + \sigma\,dW^*$

- ATS equations (7)–(8) become

$$\partial_t A(t,T) = \frac{\sigma^2}{2} B^2(t,T) - bB(t,T), \quad A(T,T) = 0,$$
$$\partial_t B(t,T) = -\beta B(t,T) - 1, \quad B(T,T) = 0.$$

- explicit solution:

$$B(t,T) = \frac{1}{\beta}\left(\mathrm{e}^{\beta(T-t)} - 1\right)$$

$$\begin{aligned} A(t,T) &= A(T,T) - \int_t^T \partial_s A(s,T)\,ds \\ &= -\frac{\sigma^2}{2}\int_t^T B^2(s,T)\,ds + b\int_t^T B(s,T)\,ds \\ &= \frac{\sigma^2\left(4\mathrm{e}^{\beta(T-t)} - \mathrm{e}^{2\beta(T-t)} - 2\beta(T-t) - 3\right)}{4\beta^3} \\ &\quad + b\frac{\mathrm{e}^{\beta(T-t)} - 1 - \beta(T-t)}{\beta^2} \end{aligned}$$

Vasiček: $dr = (b + \beta r)\, dt + \sigma\, dW^*$

- recall: closed-form bond prices
  $P(t, T) = \exp(-A(t, T) - B(t, T) r(t))$
- will see below: also closed-form bond option prices

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CIR: $dr = (b + \beta r)\, dt + \sigma \sqrt{r}\, dW^*$

- CIR SDE has unique nonnegative solution for $r(0) \geq 0$
- if $b \geq \sigma^2/2$ then $r > 0$ whenever $r(0) > 0$ (e.g. Lamberton and Lapeyre [36, Proposition 6.2.4])
- ATS equation becomes non linear (Riccati equation):

$$\partial_t B(t,T) = \frac{\sigma^2}{2} B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0$$

- explicit solution:

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma - \beta)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}$$

where $\gamma = \sqrt{\beta^2 + 2\sigma^2}$

CIR: $dr = (b + \beta r)\, dt + \sigma \sqrt{r}\, dW^*$

- integration yields

$$A(t,T) = -\frac{2b}{\sigma^2} \log\left(\frac{2\gamma e^{(\gamma-\beta)(T-t)/2}}{(\gamma-\beta)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right)$$

$\Rightarrow$ closed-form bond prices
$P(t,T) = \exp\left(-A(t,T) - B(t,T)r(t)\right)$

- will see below: also closed-form bond option prices

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Dothan: $dr = \beta r\, dt + \sigma r\, dW^*$

- Dothan [23] starts from drift-less geometric Brownian motion under objective probability measure $\mathbb{P}$:

$$dr(t) = \sigma r(t)\, dW(t)$$

- constant market price of risk gives above $\mathbb{Q}$-dynamics
- easily integrated: for $s \leq t$:

$$r(t) = r(s) \exp\left(\left(\beta - \sigma^2/2\right)(t-s) + \sigma(W^*(t) - W^*(s))\right)$$

$\Rightarrow$ $\mathcal{F}_s$-conditional distribution of $r(t)$ is lognormal with mean and variance

$$\mathbb{E}_{\mathbb{Q}}[r(t) \mid \mathcal{F}_s] = r(s) \mathrm{e}^{\beta(t-s)}$$

$$\mathrm{Var}_{\mathbb{Q}}[r(t) \mid \mathcal{F}_s] = r^2(s) \mathrm{e}^{2\beta(t-s)} \left(\mathrm{e}^{\sigma^2(t-s)} - 1\right)$$

Lognormal Short-Rate Models

+ lognormal short-rate models (Dothan, Black–Derman–Toy, Black–Karasinski): positive interest rates
– but no closed-form bond (option) prices
– major drawback: explosion of the money-market account:

$$\mathbb{E}_\mathbb{Q}[B(\Delta t)] = \mathbb{E}_\mathbb{Q}\left[\mathrm{e}^{\int_0^{\Delta t} r(s)\,ds}\right] \approx \mathbb{E}_\mathbb{Q}\left[\mathrm{e}^{\frac{r(0)+r(\Delta t)}{2}\Delta t}\right]$$

- fact: $\mathbb{E}_\mathbb{Q}\left[\mathrm{e}^{\mathrm{e}^{Y}}\right] = \infty$ for Gaussian Y

$\Rightarrow$ $\mathbb{E}_\mathbb{Q}[B(\Delta t)] = \infty$ for arbitrarily small Δt

- similarly: Eurodollar future price $= \infty$ (see later)
- idea of lognormal rates taken up in mid-90s by Sandmann and Sondermann [46] and others $\rightarrow$ market models with lognormal LIBOR or swap rates (studied below)

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Ho–Lee: $dr = b(t)\,dt + \sigma\,dW^*$

- ATS equations (7)–(8) become

$$\partial_t A(t,T) = \frac{\sigma^2}{2}B^2(t,T) - b(t)B(t,T), \quad A(T,T) = 0,$$
$$\partial_t B(t,T) = -1, \quad B(T,T) = 0$$

- explicit solution

$$B(t,T) = T - t,$$
$$A(t,T) = -\frac{\sigma^2}{6}(T-t)^3 + \int_t^T b(s)(T-s)\,ds$$

Ho–Lee: $dr = b(t)\,dt + \sigma\,dW^*$

$\Rightarrow$ forward curve

$$f(t,T) = \partial_T A(t,T) + \partial_T B(t,T) r(t)$$
$$= -\frac{\sigma^2}{2}(T-t)^2 + \int_t^T b(s)\,ds + r(t)$$

$\Rightarrow$ $b(s) = \partial_s f_0(s) + \sigma^2 s$ gives a perfect fit of observed initial forward curve $f_0(T)$

- plugging back into ATS:

$$f(t,T) = f_0(T) - f_0(t) + \sigma^2 t(T-t) + r(t)$$

- integrate:

$$P(t,T) = \mathrm{e}^{-\int_t^T f_0(s)\,ds + f_0(t)(T-t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)}$$

Ho–Lee: $dr = b(t)\,dt + \sigma\,dW^*$

- interesting:

$$r(t) = r(0) + \int_0^t b(s)\,ds + \sigma W^*(t) = f_0(t) + \frac{\sigma^2 t^2}{2} + \sigma W^*(t)$$

$\Rightarrow$ $r(t)$ fluctuates along the modified initial forward curve

$\Rightarrow$ forward vs. future rate: $f_0(t) = \mathbb{E}_{\mathbb{Q}}[r(t)] - \frac{\sigma^2 t^2}{2}$

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Hull–White: $dr = (b(t) + \beta r)\, dt + \sigma\, dW^*$

- ATS equation for $B(t, T)$ as in Vasiček model:

$$\partial_t B(t, T) = -\beta B(t, T) - 1, \quad B(T, T) = 0$$

- explicit solution

$$B(t, T) = \frac{1}{\beta}\left(\mathrm{e}^{\beta(T-t)} - 1\right).$$

- ATS equation for $A(t, T)$:

$$A(t, T) = -\frac{\sigma^2}{2}\int_t^T B^2(s, T)\, ds + \int_t^T b(s) B(s, T)\, ds$$

Hull–White: $dr = (b(t) + \beta r)\, dt + \sigma\, dW^*$

- notice $\partial_T B(s,T) = -\partial_s B(s,T)$:

$$
\begin{aligned}
f_0(T) &= \partial_T A(0,T) + \partial_T B(0,T) r(0) \\
&= \frac{\sigma^2}{2}\int_0^T \partial_s B^2(s,T)\, ds + \int_0^T b(s)\partial_T B(s,T)\, ds \\
&\quad + \partial_T B(0,T) r(0) \\
&= \underbrace{-\frac{\sigma^2}{2\beta^2}\left(\mathrm{e}^{\beta T} - 1\right)^2}_{=:g(T)} + \underbrace{\int_0^T b(s)\mathrm{e}^{\beta(T-s)}\, ds + \mathrm{e}^{\beta T} r(0)}_{=:\phi(T)}.
\end{aligned}
$$

- ϕ satisfies

$$
\partial_T \phi(T) = \beta\phi(T) + b(T), \quad \phi(0) = r(0).
$$

Hull–White: $dr = (b(t) + \beta r)\, dt + \sigma\, dW^*$

- since $\phi = f_0 + g$: conclude

$$\begin{aligned} b(T) &= \partial_T \phi(T) - \beta\phi(T) \\ &= \partial_T (f_0(T) + g(T)) - \beta(f_0(T) + g(T)). \end{aligned}$$

- plugging in $(\dots)$:

$$\begin{aligned} f(t,T) = f_0(T) &- \mathrm{e}^{\beta(T-t)} f_0(t) \\ &- \frac{\sigma^2}{2\beta^2}\left(\mathrm{e}^{\beta(T-t)} - 1\right)\left(\mathrm{e}^{\beta(T-t)} - \mathrm{e}^{\beta(T+t)}\right) \\ &+ \mathrm{e}^{\beta(T-t)} r(t). \end{aligned}$$

Part V

Heath–Jarrow–Morton (HJM) Methodology

Overview

- Have seen above: short-rate models not always flexible enough to calibrating them to the observed initial term-structure
- In late 1980s: Heath, Jarrow and Morton (henceforth HJM) [27] proposed a new framework for modeling the entire forward curve directly
- This chapter: provides essentials of HJM framework

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Recap: Usual Stochastic Setup

- filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$
- usual conditions:
  - completeness: $\mathcal{F}_0$ contains all of the null sets
  - right-continuity: $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for $t \geq 0$
- d-dimensional $(\mathcal{F}_t)$-Brownian motion $W = (W_1, \dots, W_d)^\top$
- infinite time horizon (w.l.o.g.): $\mathcal{F} = \mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$

Assumptions

- given $\mathbb{R}$- and $\mathbb{R}^d$-valued stochastic process $\alpha = \alpha(\omega, t, T)$ and $\sigma = (\sigma_1(\omega, t, T), \dots, \sigma_d(\omega, t, T))$ s.t.
  - **(HJM.1)** α and σ are $\mathrm{Prog} \otimes \mathcal{B}$-measurable
  - **(HJM.2)** $\int_0^T \int_0^T |\alpha(s,t)|\, ds\, dt < \infty$ for all T
  - **(HJM.3)** $\sup_{s,t \le T} \|\sigma(s,t)\| < \infty$ for all T (note: this is a ω-wise boundedness assumption)
- given integrable initial forward curve $T \mapsto f(0,T)$
- for every T: forward rate follows Itô dynamics ($t \le T$)

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\, ds + \int_0^t \sigma(s,T)\, dW(s) \quad (10)$$

Assumptions

$\Rightarrow$ very general setup: only substantive economic restriction is continuous sample paths assumption (and the finite number of random drivers $W_1, \dots, W_d$)

- integrals in (10) well defined by **(HJM.1)–(HJM.3)**
- from Fubini corollary below: implied short-rate process

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s)$$

has progressive modification and satisfies $\int_0^t |r(s)|\,ds < \infty$ a.s. for all t

- hence money-market account $B(t) = \mathrm{e}^{\int_0^t r(s)ds}$ well defined

Properties

Lemma

For every maturity T, the zero-coupon bond price $P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\,du}$ follows Itô process ($t \le T$)

$$P(t,T) = P(0,T) + \int_0^t P(s,T)\,(r(s) + b(s,T))\,ds + \int_0^t P(s,T) v(s,T)\,dW(s),$$

where

$$v(s,T) = -\int_s^T \sigma(s,u)\,du,$$

is the T-bond volatility and

$$b(s,T) = -\int_s^T \alpha(s,u)\,du + \frac{1}{2}\|v(s,T)\|^2.$$

Proof

Using the classical and stochastic Fubini Theorem below twice, we calculate

$$
\begin{aligned}
\log P(t,T) &= -\int_t^T f(t,u)\,du \\
&= -\int_t^T f(0,u)\,du - \int_t^T \int_0^t \alpha(s,u)\,ds\,du - \int_t^T \int_0^t \sigma(s,u)\,dW(s)\,du \\
&= -\int_t^T f(0,u)\,du - \int_0^t \int_t^T \alpha(s,u)\,du\,ds - \int_0^t \int_t^T \sigma(s,u)\,du\,dW(s) \\
&= -\int_0^T f(0,u)\,du - \int_0^t \int_s^T \alpha(s,u)\,du\,ds - \int_0^t \int_s^T \sigma(s,u)\,du\,dW(s) \\
&\quad + \int_0^t f(0,u)\,du + \int_0^t \int_s^t \alpha(s,u)\,du\,ds + \int_0^t \int_s^t \sigma(s,u)\,du\,dW(s) \\
&= -\int_0^T f(0,u)\,du + \int_0^t \left(b(s,T) - \frac{1}{2}\|v(s,T)\|^2 \right) ds + \int_0^t v(s,T)\,dW(s) \\
&\quad + \int_0^t \underbrace{\left(f(0,u) + \int_0^u \alpha(s,u)\,ds + \int_0^u \sigma(s,u)\,dW(s) \right)}_{=r(u)} du \\
&= \log P(0,T) + \int_0^t \left(r(s) + b(s,T) - \frac{1}{2}\|v(s,T)\|^2 \right) ds + \int_0^t v(s,T)\,dW(s).
\end{aligned}
$$

Itô's formula now implies the assertion.

Discounted Price

Corollary

For every maturity T, the discounted bond price process satisfies

$$\frac{P(t,T)}{B(t)} = P(0,T) + \int_0^t \frac{P(s,T)}{B(s)} b(s,T)\, ds + \int_0^t \frac{P(s,T)}{B(s)} v(s,T)\, dW(s).$$

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Absence of Arbitrage

- investigate restrictions on the dynamics (10) under no arbitrage
- assume given $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_\infty(\gamma \bullet W)$ for some $\gamma \in \mathcal{L}$
- Girsanov transformed $\mathbb{Q}$-Brownian motion: $dW^* = dW - \gamma^\top dt$
- $\mathbb{Q}$ an ELMM for the bond market if $\frac{P(t,T)}{B(t)}$ is a $\mathbb{Q}$-local martingale for every T

HJM Drift Condition

Theorem (HJM Drift Condition)

$\mathbb{Q}$ is an ELMM if and only if

$$b(t,T) = -v(t,T)\,\gamma(t)^\top \quad \text{for all } T,\ d\mathbb{P}\otimes dt\text{-a.s.}$$

In this case, the $\mathbb{Q}$-dynamics of the forward rates $f(t,T)$ are of the form

$$f(t,T) = f(0,T) + \int_0^t \underbrace{\left(\sigma(s,T)\int_s^T \sigma(s,u)^\top du\right)}_{\text{HJM drift}} ds + \int_0^t \sigma(s,T)$$

and the discounted T-bond price satisfies

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}_t(v(\cdot,T)\bullet W^*)$$

for $t \le T$.

Proof I

In view of "Discounted Price" corollary we find that

$$d\frac{P(t,T)}{B(t)} = \frac{P(t,T)}{B(t)}\left(b(t,T) + v(t,T)\,\gamma(t)^\top\right)\,dt + \frac{P(t,T)}{B(t)}v(t,T)\,dW^*(t).$$

Hence $\frac{P(t,T)}{B(t)}$, $t \leq T$, is a $\mathbb{Q}$-local martingale if and only if $b(t,T) = -v(t,T)\,\gamma(t)^\top$ $d\mathbb{P}\otimes dt$-a.s. Since $v(t,T)$ and $b(t,T)$ are both continuous in T, we deduce that $\mathbb{Q}$ is an ELMM if and only if $b(t,T) = -v(t,T)\,\gamma(t)^\top$ for all T, $d\mathbb{P}\otimes dt$-a.s.

Differentiating both sides in T yields

$$-\alpha(t,T) + \sigma(t,T)\int_t^T \sigma(t,u)^\top du = \sigma(t,T)\,\gamma(t)^\top$$

Proof II

for all T, $d\mathbb{P} \otimes dt$-a.s. Insert this in (10). The expression for $P(t,T)/B(t)$ now follows from Stochastic Exponential Lemma.

Market Price of Risk

- follows from Theorem above:

$$dP(t,T) = P(t,T)\left(r(t) - v(t,T)\,\gamma(t)^\top\right)dt + P(t,T)v(t,T)\,dW(t)$$

$\Rightarrow$ $-\gamma$ = market price of risk for the bond market

- striking feature of HJM framework: $\mathbb{Q}$-law of $f(t,T)$ only depends on volatility $\sigma(t,T)$, not on $\mathbb{P}$-drift $\alpha(t,T)$

$\Rightarrow$ option pricing only depends on σ, similar to Black–Scholes stock price model

When Is $\frac{P(t,T)}{B(t)}$ $\mathbb{Q}$-martingale?

Corollary

Suppose HJM drift condition holds. Then $\mathbb{Q}$ is EMM if

1. *Novikov condition* $\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\frac{1}{2}\int_0^T \|v(t,T)\|^2\,dt}\right] < \infty$ *for all* T*; or*
2. *forward rates are nonnegative:* $f(t,T) \geq 0$ *for all* $t \leq T$*.*

Proof.

Novikov condition is sufficient for
$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}_t(v(\cdot,T)\bullet W^*)$ to be a $\mathbb{Q}$-martingale.
If $f(t,T) \geq 0$, then $0 \leq \frac{P(t,T)}{B(t)} \leq 1$ is a uniformly bounded local martingale, and hence a true martingale. □

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Simple Example

- interplay between short-rate models and HJM framework?
- example (simplest HJM model): constant $\sigma(t, T) \equiv \sigma > 0$:

$$f(t, T) = f(0, T) + \sigma^2 t \left(T - \frac{t}{2} \right) + \sigma W^*(t)$$

$\Rightarrow$ short rates follow Ho–Lee model:

$$r(t) = f(t, t) = f(0, t) + \frac{\sigma^2 t^2}{2} + \sigma W^*(t)$$

General Result

Proposition

Suppose $f(0,T)$, $\alpha(t,T)$ and $\sigma(t,T)$ are differentiable in T with $\int_0^T |\partial_u f(0,u)|\,du < \infty$ and such that **(HJM.1)–(HJM.3)** *are satisfied for $\alpha(t,T)$ and $\sigma(t,T)$ replaced by $\partial_T \alpha(t,T)$ and $\partial_T \sigma(t,T)$.*

Then the short-rate process is an Itô process of the form

$$r(t) = r(0) + \int_0^t \zeta(u)\,du + \int_0^t \sigma(u,u)\,dW(u)$$

where

$$\zeta(u) = \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u)\,ds + \int_0^u \partial_u \sigma(s,u)\,dW(s).$$

Proof

Recall first that

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s).$$

Applying the Fubini Theorem 25.1 below to the stochastic integral gives

$$\begin{aligned}\int_0^t \sigma(s,t)\,dW(s) &= \int_0^t \sigma(s,s)\,dW(s) + \int_0^t (\sigma(s,t) - \sigma(s,s))\,dW(s)\\ &= \int_0^t \sigma(s,s)\,dW(s) + \int_0^t \int_s^t \partial_u \sigma(s,u)\,du\,dW(s)\\ &= \int_0^t \sigma(s,s)\,dW(s) + \int_0^t \int_0^u \partial_u \sigma(s,u)\,dW(s)\,du.\end{aligned}$$

Moreover, from the classical Fubini Theorem we deduce in a similar way that

$$\int_0^t \alpha(s,t)\,ds = \int_0^t \alpha(s,s)\,ds + \int_0^t \int_0^u \partial_u \alpha(s,u)\,ds\,du,$$

and finally

$$f(0,t) = r(0) + \int_0^t \partial_u f(0,u)\,du.$$

Combining these formulas, we obtain the desired result.

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HJM Models

- HJM model: $\sigma(\omega, t, T) = \sigma(t, T, f(\omega, t, T))$ for appropriate function σ
- simplest choice: deterministic $\sigma(t, T)$ not depending on ω

$\Rightarrow$ Gaussian distributed forward rates $f(t, T) \Rightarrow$ simple bond option price formulas (see later)

- particular case: Ho-Lee model $\sigma(t, T) \equiv \sigma$ (seen above)

HJM Models: Existence and Uniqueness

- shown in HJM [27] and Morton [40]:
- assume: $\sigma(t, T, f)$ uniformly bounded, jointly continuous, and Lipschitz continuous in f
- assume: continuous initial forward curve $f(0, T)$
- then $\exists$ unique jointly continuous solution $f(t, T)$ of

$$df(t,T) = \left(\sigma(t,T,f(t,T)) \int_t^T \sigma(t,u,f(t,u))\,du\right)dt \\ + \sigma(t,T,f(t,T))\,dW(t)$$

- remarkable: boundedness condition on σ cannot be substantially weakened as following example shows . . .

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Proportional Volatility

- single Brownian motion ($d = 1$)
- $\sigma(t, T, f(t, T)) = \sigma f(t, T)$ for some constant $\sigma > 0$: positive and Lipschitz continuous but not bounded
- solution of HJM equation must satisfy

$$f(t, T) = f(0, T) \mathrm{e}^{\sigma^2 \int_0^t \int_s^T f(s,u)\, du\, ds} \mathrm{e}^{\sigma W(t) - \frac{\sigma^2}{2} t} \tag{11}$$

- claim: there is no finite-valued solution to this expression (following Avellaneda and Laurence [2, Section 13.6])

Proportional Volatility

- assume for simplicity: $f(0,T) \equiv 1$ and $\sigma = 1$
- differentiating both sides of (11) in T:

$$\partial_T f(t,T) = f(t,T) \int_0^t f(s,T)\,ds = \frac{1}{2}\partial_t \left(\int_0^t f(s,T)\,ds \right)^2$$

- integrating from $t=0$ to $t=1$ and interchanging order of differentiation and integration:

$$\partial_T \int_0^1 f(s,T)\,ds = \frac{1}{2} \left(\int_0^1 f(s,T)\,ds \right)^2$$

- solving this differential equation path-wise for $X(T) = \int_0^1 f(s,T)\,ds$, $T \geq 1$, we obtain as unique solution

$$X(T) = \frac{X(1)}{1 - \frac{X(1)}{2}(T-1)}$$

Proportional Volatility

- since $X(1) > 0$: $X(T) \uparrow \infty$ for $T \uparrow \tau$ where $\tau = 1 + \frac{2}{X(1)}$ is a finite random time
- conclude: $f(\omega, t, \tau(\omega))$ must become $+\infty$ for some $t \leq 1$, for almost all ω.
- nonexistence of HJM models with proportional volatility encouraged development of LIBOR market models (see later)

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Fubini's Theorem

Theorem (Fubini's theorem for Stochastic Integrals)

Consider the $\mathbb{R}^d$-valued stochastic process $\phi = \phi(\omega, t, s)$ with two indices, $0 \le t, s \le T$, satisfying the following properties:

1. ϕ *is* $\mathrm{Prog}_T \otimes \mathcal{B}[0, T]$*-measurable;*
2. $\sup_{t,s} \|\phi(t, s)\| < \infty$.[2]

Then $\lambda(t) = \int_0^T \phi(t, s)\, ds \in \mathcal{L}$*, and there exists a* $\mathcal{F}_T \otimes \mathcal{B}[0, T]$*-measurable modification* $\psi(s)$ *of* $\int_0^T \phi(t, s)\, dW(t)$ *with* $\int_0^T \psi^2(s)\, ds < \infty$ *a.s. Moreover,* $\int_0^T \psi(s)\, ds = \int_0^T \lambda(t)\, dW(t)$*, that is,*

$$\int_0^T \left(\int_0^T \phi(t, s)\, dW(t) \right) ds = \int_0^T \left(\int_0^T \phi(t, s)\, ds \right) dW(t).$$

[2] Note that this is a ω-wise boundedness assumption.

Fubini's Theorem: Proof

see course book Section 6.5

Fubini's Theorem: Corollary

Corollary

Let ϕ be as in Theorem 25.1. Then the process

$$\int_0^s \phi(t,s)\,dW(t), \quad s \in [0,T],$$

has a progressive modification $\pi(s)$ with $\int_0^T \pi^2(s)\,ds < \infty$ a.s.

Proof.

For $\phi(\omega,t,s) = K(\omega,t)f(s)$, with bounded progressive process K and bounded measurable function f, the process

$$\int_0^s \phi(t,s)\,dW(t) = f(s)\int_0^s K(t)\,dW(t)$$

is clearly progressive and path-wise square integrable. Now use a similar monotone class and localization argument as in the proof of Theorem 25.1 ($\rightarrow$ exercise). $\square$

Monotone Class Theorem

Theorem (Monotone Class Theorem)

Suppose the set $\mathcal{H}$ consists of real-valued bounded functions defined on a set Ω with the following properties:

1. *$\mathcal{H}$ is a vector space;*
2. *$\mathcal{H}$ contains the constant function 1_Ω;*
3. *if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ monotone, for some bounded function f on Ω, then $f \in \mathcal{H}$.*

If $\mathcal{H}$ contains a collection $\mathcal{M}$ of real-valued functions, which is closed under multiplication (that is, $f, g \in \mathcal{M}$ implies $fg \in \mathcal{M}$). Then $\mathcal{H}$ contains all real-valued bounded functions that are measurable with respect to the σ-algebra which is generated by $\mathcal{M}$ (that is, $\sigma\{f^{-1}(A) \mid A \in \mathcal{B},\ f \in \mathcal{M}\}$).

Proof.

see e.g. Steele [49, Section 12.6]. □

T-Bond as Numeraire

Bond Option Pricing

Example: Vasiček Short-Rate Model

Black–Scholes Model with Gaussian Interest Rates

Example: Black–Scholes–Vasiček Model

Part VI

Forward Measures

Overview

- we replace risk-free numeraire by another traded asset, such as the T-bond
- change of numeraire technique proves most useful for option pricing and provides the basis for the market models studied below
- we derive explicit option price formulas for Gaussian HJM models
- this includes the Vasiček short-rate model and some extension of the Black–Scholes model with stochastic interest rates

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T-Forward Measure

- HJM setup from above: $\exists$ EMM $\mathbb{Q}$ for all T-bonds, W^* the respective $\mathbb{Q}$-Brownian motion
- recall: T-bond volatility $v(t,T) = -\int_t^T \sigma(t,u)\,du$
- fix $T>0$: $\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{P(0,T)B(T)}$ defines probability measure $\mathbb{Q}^T \sim \mathbb{Q}$ on $\mathcal{F}_T$ (why?) and for $t \le T$:

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \mid \mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)B(t)}$$

- $\mathbb{Q}^T$ is called the T-forward measure
- from above: $\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \mathcal{E}_t\left(v(\cdot,T)\bullet W^*\right)$

$\Rightarrow$ Girsanov's Theorem: $W^T(t) = W^*(t) - \int_0^t v(s,T)^\top ds$, $t \le T$, is a $\mathbb{Q}^T$-Brownian motion

Fundamental Property

Lemma

For any $S > 0$, the T-bond discounted S-bond price process

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)} \mathcal{E}_t\left(\sigma_{S,T} \bullet W^T\right), \quad t \leq S \wedge T$$

is a $\mathbb{Q}^T$-martingale, where we define

$$\sigma_{S,T}(t) = -\sigma_{T,S}(t) = v(t,S) - v(t,T) = \int_S^T \sigma(t,u)\,du. \tag{12}$$

Moreover, the T- and S-forward measures are related by

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^T}\Big|_{\mathcal{F}_t} = \frac{P(t,S)\,P(0,T)}{P(t,T)\,P(0,S)} = \mathcal{E}_t\left(\sigma_{S,T} \bullet W^T\right), \quad t \leq S \wedge T.$$

Proof

Let $u \le t \le S \wedge T$. Bayes' rule gives

$$\mathbb{E}_{\mathbb{Q}^T}\left[\frac{P(t,S)}{P(t,T)} \mid \mathcal{F}_u\right] = \frac{\mathbb{E}_{\mathbb{Q}}\left[\frac{P(t,T)}{P(0,T)B(t)}\frac{P(t,S)}{P(t,T)} \mid \mathcal{F}_u\right]}{\frac{P(u,T)}{P(0,T)B(u)}} = \frac{\frac{P(u,S)}{B(u)}}{\frac{P(u,T)}{B(u)}}$$

$$= \frac{P(u,S)}{P(u,T)},$$

which proves that $P(t,S)/P(t,T)$ is a martingale. The stochastic exponential representation follows from Stochastic Exponential Lemma and the representation of $P(t,T)/B(t)$ ($\rightarrow$ exercise). The second claim follows from the identity

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^T}\mid_{\mathcal{F}_t} = \frac{d\mathbb{Q}^S}{d\mathbb{Q}}\mid_{\mathcal{F}_t}\ \frac{d\mathbb{Q}}{d\mathbb{Q}^T}\mid_{\mathcal{F}_t}.$$

Pricing with Stochastic Interest Rates

$\Rightarrow$ collection of EMMs: each $\mathbb{Q}^T$ corresponds to different numeraire: T-bond

- $\mathbb{Q}$ is called the risk-neutral (or spot) measure
- simpler pricing formulas: let X be a T-claim s.t. $\mathbb{E}_{\mathbb{Q}}\left[\frac{|X|}{B(T)}\right] < \infty$
- arbitrage price at $t \leq T$: $\pi(t) = B(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{B(T)} \mid \mathcal{F}_t\right]$
- computation: have to know joint distribution of $1/B(T)$ and $X \Rightarrow$ double integral (rather hard work)

Pricing with Stochastic Interest Rates

- if $1/B(T)$ and X were independent under $\mathbb{Q}$ conditional on $\mathcal{F}_t$: $\pi(t) = P(t,T)\mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t]$
- a much simpler formula, since:
  - only have to compute single integral $\mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t]$;
  - $P(t,T)$ observable at t, and does not have to be computed within the model
- but: independence of $1/B(T)$ and X unrealistic assumption for interest rate sensitive claims X !
- good news: above simple formula holds—under $\mathbb{Q}^T$ ...

Forward Measure Pricing

Proposition

Under above assumptions: $\mathbb{E}_{\mathbb{Q}^T}[|X|] < \infty$ *and*

$$\pi(t) = P(t,T)\mathbb{E}_{\mathbb{Q}^T}[X \mid \mathcal{F}_t].$$

Proof

Proof.
Bayes' rule yields

$$\mathbb{E}_{\mathbb{Q}^T}[|X|] = \mathbb{E}_{\mathbb{Q}}\left[\frac{|X|}{P(0,T)B(T)}\right] < \infty \quad \text{(by assumption).}$$

Moreover, again by Bayes' rule,

$$\begin{aligned}\pi(t) &= P(0,T)B(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{P(0,T)B(T)} \mid \mathcal{F}_t\right] \\ &= P(0,T)B(t)\frac{P(t,T)}{P(0,T)B(t)}E_{\mathbb{Q}^T}[X \mid \mathcal{F}_t] \\ &= P(t,T)\mathbb{E}_{\mathbb{Q}^T}[X \mid \mathcal{F}_t],\end{aligned}$$

as desired. $\square$

Application: Expectation Hypothesis

Lemma (Expectation Hypothesis)

If $\sigma(\cdot, T) \in \mathcal{L}^2$, the expectation hypothesis holds under the T-forward measure:

$$f(t,T) = \mathbb{E}_{\mathbb{Q}^T}\left[r(T) \mid \mathcal{F}_t\right] \quad \textit{for } t \leq T.$$

Proof.

Under $\mathbb{Q}^T$ we have

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\, dW^T(s). \tag{13}$$

Hence, if $\sigma(\cdot, T) \in \mathcal{L}^2$ then $f(t,T)$, $t \leq T$, is a $\mathbb{Q}^T$-martingale. □

A Word of Warning

In view of equation (13) it is tempting to "specify" a forward rate model by postulating the dynamics of $f(\cdot, T)$ under $\mathbb{Q}^T$ for each maturity T separately without reference to some underlying $\mathbb{Q}$. However, it is far from clear whether a common risk-neutral measure $\mathbb{Q}$, tying all $\mathbb{Q}^T$'s, exists in this case. On the other hand, we note that this is exactly the approach in the LIBOR market model developed below. The important difference being that there one considers finitely many maturities only.

Application: Dybvig–Ingersoll–Ross Theorem

- Dybvig–Ingersoll–Ross [24]: long rates can never fall
- recall: zero-coupon yield $R(t,T) = \frac{1}{T-t}\int_t^T f(t,s)\,ds$
- define: asymptotic long rate $R_\infty(t) = \lim_{T\to\infty} R(t,T)$

Lemma (Dybvig–Ingersoll–Ross Theorem)

For all $s < t$ the long rates satisfy $R_\infty(s) \le R_\infty(t)$ if they exist.

Proof

Let $s < t$ be such that $R_\infty(s)$ and $R_\infty(t)$ exist. Then $p(u) = \lim_{T\to\infty} P(t,T)^{\frac{1}{T}} = \mathrm{e}^{-R_\infty(u)}$ exist for $u \in \{s,t\}$, and it remains to prove that $p(s) \geq p(t)$.

Under the t-forward measure $\mathbb{Q}^t$, we have

$$\frac{P(s,T)}{P(s,t)} = \mathbb{E}_{\mathbb{Q}^t}[P(t,T) \mid \mathcal{F}_s],$$

and thus

$$p(s) = \lim_{T\to\infty} \mathbb{E}_{\mathbb{Q}^t}[P(t,T) \mid \mathcal{F}_s]^{\frac{1}{T}}.$$

Now let $X \geq 0$ be any bounded random variable with $\mathbb{E}_{\mathbb{Q}^t}[X] = 1$. Using the $\mathcal{F}_s$-conditional versions of Fatou's lemma, Hölder's inequality and dominated convergence, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^t}\left[X\,p(t)\right] &= \mathbb{E}_{\mathbb{Q}^t}\left[\lim\inf_{T\to\infty} X\,P(t,T)^{\frac{1}{T}}\right] \leq \mathbb{E}_{\mathbb{Q}^t}\left[\lim\inf_{T\to\infty} \mathbb{E}_{\mathbb{Q}^t}\left[X\,P(t,T)^{\frac{1}{T}} \mid \mathcal{F}_s\right]\right] \\
&\leq \mathbb{E}_{\mathbb{Q}^t}\left[\lim\inf_{T\to\infty} \mathbb{E}_{\mathbb{Q}^t}\left[X^{\frac{T}{T-1}} \mid \mathcal{F}_s\right]^{\frac{T-1}{T}} \mathbb{E}_{\mathbb{Q}^t}\left[\,P(t,T) \mid \mathcal{F}_s\right]^{\frac{1}{T}}\right] \\
&= \mathbb{E}_{\mathbb{Q}^t}\left[X\,p(s)\right].
\end{aligned}$$

Since X was arbitrary with the stated properties, we conclude that $p(t) \leq p(s)$, and the lemma is proved.

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Bond Option Pricing: General

- consider European call option on S-bond, expiry date $T < S$, strike price K
- arbitrage price at $t = 0$ (for simplicity):

$$\pi = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r(s)\,ds}\,(P(T,S)-K)^+\right]$$

- decompose

$$\begin{aligned}\pi &= \mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}P(T,S)\,1_{\{P(T,S)\ge K\}}\right]\\ &\quad - K\mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}\,1_{\{P(T,S)\ge K\}}\right]\\ &= P(0,S)\mathbb{Q}^S\left[P(T,S)\ge K\right]\\ &\quad - KP(0,T)\mathbb{Q}^T\left[P(T,S)\ge K\right]\end{aligned}$$

Bond Option Pricing: General

- observe that

$$\mathbb{Q}^S[P(T,S) \geq K] = \mathbb{Q}^S\left[\frac{P(T,T)}{P(T,S)} \leq \frac{1}{K}\right]$$
$$\mathbb{Q}^T[P(T,S) \geq K] = \mathbb{Q}^T\left[\frac{P(T,S)}{P(T,T)} \geq K\right].$$

$\Rightarrow$ look for lognormal $\frac{P(T,T)}{P(T,S)}$, $\frac{P(T,S)}{P(T,T)}$ under $\mathbb{Q}^S$, $\mathbb{Q}^T$ ...

Bond Option Pricing: Gaussian Models

- assume: $\sigma(t,T) = (\sigma_1(t,T),\dots,\sigma_d(t,T))$ deterministic

$\Rightarrow$ $f(t,T)$ Gaussian distributed

$\Rightarrow$ $\frac{P(T,T)}{P(T,S)}$, $\frac{P(T,S)}{P(T,T)}$ lognormal under $\mathbb{Q}^S$, $\mathbb{Q}^T$

$\Rightarrow$ closed-form option price formula ...

Bond Option Pricing: Gaussian Models

Proposition

Under the above Gaussian assumption, the bond option price is

$$\pi = P(0,S)\Phi[d_1] - KP(0,T)\Phi[d_2],$$

where Φ is the standard Gaussian cumulative distribution function,

$$d_{1,2} = \frac{\log\left[\frac{P(0,S)}{KP(0,T)}\right] \pm \frac{1}{2}\int_0^T \|\sigma_{T,S}(s)\|^2\, ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2\, ds}},$$

and $\sigma_{T,S}(s)$ is given in (12).

- similar closed-form expression for put option (...)

Proof

Proof.

It is enough to observe that

$$\frac{\log \frac{P(T,T)}{P(T,S)} - \log \frac{P(0,T)}{P(0,S)} + \frac{1}{2}\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}}$$

and

$$\frac{\log \frac{P(T,S)}{P(T,T)} - \log \frac{P(0,S)}{P(0,T)} + \frac{1}{2}\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}}$$

are standard Gaussian distributed under $\mathbb{Q}^S$ and $\mathbb{Q}^T$, respectively. □

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26 Bond Option Pricing

Example: Vasiček Short-Rate Model

27 Black–Scholes Model with Gaussian Interest Rates

Example: Black–Scholes–Vasiček Model

Vasiček Model

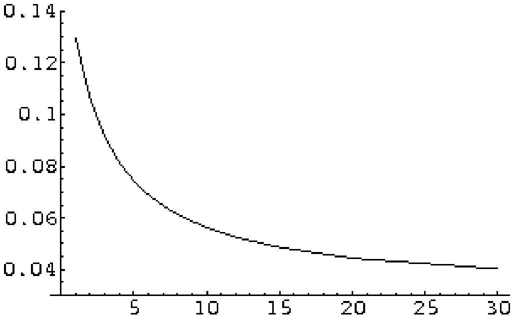
- Vasiček short-rate model ($d = 1$):
  $dr = (b + \beta r)\, dt + \sigma\, dW^*$

$\Rightarrow$ $(\ldots)$ $df(t, T) = \alpha(t, T)\, dt + \sigma e^{\beta(T-t)}\, dW^*$

$\Rightarrow$ $\sigma(t, T) = \sigma e^{\beta(T-t)}$ deterministic

Vasiček Model

- example: $\beta = -0.86$, $b/|\beta| = 0.09$ (mean reversion level), $\sigma = 0.0148$ and $r(0) = 0.08$
- ATM cap prices and Black volatilities for: $t_0 = 0$ (today), $T_0 = 1/4$ (first reset date), and $T_i - T_{i-1} \equiv 1/4$, $i = 1, \dots, 119$ (maturity of the last cap is $T_{119} = 30$)
- contrast to market curve: Vasiček model cannot produce humped volatility curves



Vasiček Model

Table: Vasiček ATM cap prices and Black volatilities

Maturity	ATM prices	ATM vols
1	0.00215686	0.129734
2	0.00567477	0.106348
3	0.00907115	0.0915455
4	0.0121906	0.0815358
5	0.01503	0.0743607
6	0.017613	0.0689651
7	0.0199647	0.0647515
8	0.0221081	0.0613624
10	0.025847	0.0562337
12	0.028963	0.0525296
15	0.0326962	0.0485755
20	0.0370565	0.0443967
30	0.0416089	0.0402203

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Generalized Black–Scholes

- Black–Scholes model [9]: stock S, money-market account B following $\mathbb{Q}$-dynamics

$$dB = Br\,dt, \quad B(0) = 1,$$
$$dS = Sr\,dt + S\Sigma\,dW^*, \quad S(0) > 0,$$

- constant volatility $\Sigma = (\Sigma_1, \ldots, \Sigma_d) \in \mathbb{R}^d$
- new: r stochastic within above Gaussian HJM setup

Generalized Black–Scholes

- consider European call option on S, maturity T, strike price K
- arbitrage price at $t = 0$ (for simplicity):

$$\pi = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(T)}(S(T)-K)^+\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\frac{S(T)}{B(T)}1_{\{S(T)\geq K\}}\right] - K\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(T)}1_{\{S(T)\geq K\}}\right]$$

- recall T-forward measure $\mathbb{Q}^T$
- similarly: choose S as numeraire and define EMM $\mathbb{Q}^{(S)} \sim \mathbb{Q}$ on $\mathcal{F}_T$ via

$$\frac{d\mathbb{Q}^{(S)}}{d\mathbb{Q}} = \frac{S(T)}{S(0)\,B(T)} = \mathcal{E}_T(\Sigma \bullet W^*)$$

- Girsanov: $W^{(S)}(t) = W^*(t) - \Sigma^\top t$, $t \leq T$, $\mathbb{Q}^{(S)}$-Brownian motion

Generalized Black–Scholes

- change of measure (Bayes):
  $\pi = S(0)\mathbb{Q}^{(S)}[S(T) \geq K] - KP(0,T)\mathbb{Q}^{T}[S(T) \geq K]$
- remains: compute probabilities

$$\mathbb{Q}^{(S)}[S(T) \geq K] = \mathbb{Q}^{(S)}\left[\frac{P(T,T)}{S(T)} \leq \frac{1}{K}\right],$$

$$\mathbb{Q}^{T}[S(T) \geq K] = \mathbb{Q}^{T}\left[\frac{S(T)}{P(T,T)} \geq K\right]$$

- observe: $\frac{P(t,T)}{S(t)}$ is $\mathbb{Q}^{(S)}$-martingale and $\frac{S(t)}{P(t,T)}$ is $\mathbb{Q}^{T}$ martingale for $t \leq T$

Generalized Black–Scholes

- Itô's formula (to practice stochastic calculus):

$$
\begin{aligned}
d\frac{S(t)}{P(t)} &= \frac{1}{P(t)}\,dS(t) - \frac{S(t)}{P(t)^2}\,dP(t) \\
&\quad - \frac{1}{P(t)^2}\,d\langle S, P\rangle_t + \frac{S(t)}{P(t)^3}\,d\langle P, P\rangle_t \\
&= (\cdots)\,dt + \frac{S(t)}{P(t)}\,(\Sigma - v(t,T))\,dW^*(t)
\end{aligned}
$$

  (omitted parameter T in $P(t,T)$)
- recall: T-bond volatility $v(t,T) = -\int_t^T \sigma(t,u)\,du$
- no need to compute drift term: volatility unaffected by change of measure $\Rightarrow \mathbb{Q}^T$-dynamics:

$$
d\frac{S(t)}{P(t,T)} = \frac{S(t)}{P(t,T)}\,(\Sigma - v(t,T))\,dW^T(t)
$$

Generalized Black–Scholes

$\Rightarrow$ stochastic exponential equation:

$$\frac{S(T)}{P(T,T)} = \frac{S(0)}{P(0,T)}\mathcal{E}_T\left((\Sigma - v(\cdot,T)) \bullet W^T\right)$$

is lognormally distributed under $\mathbb{Q}^T$

- along similar calculations (. . .):

$$\frac{P(T,T)}{S(T)} = \frac{P(0,T)}{S(0)}\mathcal{E}_T\left(-(\Sigma - v(\cdot,T)) \bullet W^{(S)}\right)$$

is lognormally distributed under $\mathbb{Q}^{(S)}$

Generalized Black–Scholes Option Price Formula

Proposition

In the above generalized Black–Scholes model, the option price is

$$\pi = S(0)\Phi[d_1] - KP(0,T)\Phi[d_2],$$

where Φ is the standard Gaussian cumulative distribution function and

$$d_{1,2} = \frac{\log\left[\frac{S(0)}{KP(0,T)}\right] \pm \frac{1}{2}\int_0^T \|\Sigma - v(t,T)\|^2\, dt}{\sqrt{\int_0^T \|\Sigma - v(t,T)\|^2\, dt}}. \tag{14}$$

Note that $v(t,T) = 0$ yields the classical Black–Scholes option price formula for constant short rate.

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Black–Scholes–Vasiček

- special case of generalized Black–Scholes formula: Vasiček short-rate model
- $\dim W^* = d = 2$
- Vasiček short-rate dynamics:

$$dr = (b + \beta r)\, dt + \sigma\, dW^*,$$

where $\sigma = (\sigma_1, \sigma_2)$

- note: this corresponds to standard representation $dr = (b + \beta r)\, dt + \|\sigma\|\, d\mathcal{W}^*$ for the one-dimensional $\mathbb{Q}$-Brownian motion $\mathcal{W}^* = \frac{\sigma_1 W_1^* + \sigma_2 W_2^*}{\|\sigma\|}$

$\Rightarrow$ $\mathbb{R}^2$-valued T-bond volatility

$$v(t, T) = -\sigma \int_t^T \mathrm{e}^{\beta(T-s)}\, ds = \frac{\sigma}{\beta}\left(1 - \mathrm{e}^{\beta(T-t)}\right)$$

Black–Scholes–Vasiček

- tedious but elementary computation yields

$$\int_0^T \|\Sigma - v(t,T)\|^2 \, dt$$
$$= \|\Sigma\|^2 \, T + 2\Sigma\,\sigma^\top \frac{e^{\beta T} - 1 - \beta T}{\beta^2} + \|\sigma\|^2 \frac{e^{2\beta T} - 4e^{\beta T} + 2\beta T + 3}{2\beta^3}$$

  for aggregate volatility in (14)

- relation between option price π and instantaneous covariation $d\langle S, r\rangle/dt = \Sigma\,\sigma^\top$ of S and r: π monotone increasing in $\int_0^T \|\Sigma - v(t,T)\|^2 \, dt$, which again is increasing in $\Sigma\,\sigma^\top$ since $e^{\beta T} - 1 - \beta T > 0$

Black–Scholes–Vasiček

$\Rightarrow$ π increases with increasing covariation between S and r

- for negative covariation, π may be smaller than classical Black–Scholes option price with constant short rates ($\sigma = 0$)

Part VII

Forwards and Futures

Overview

- we discuss two common types of term contracts:
- forwards: mainly traded over the counter (OTC)
- futures: actively traded on many exchanges
- underlying in both cases: a T-claim $\mathcal{Y}$, e.g. exchange rate, interest rate, commodity such as copper, any traded or non-traded asset, an index, etc.
- special discussion: interest rate futures, futures rates and forward rates in the Gaussian HJM model

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Interest Rate Futures

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Forward Contracts

- assume HJM setup, and let $\mathcal{Y}$ denote a T-claim
- forward contract on $\mathcal{Y}$, contracted at t, with time of delivery $T > t$ and forward price $f(t; T, \mathcal{Y})$ defined by following payment scheme:
    - at T: holder (long position) pays $f(t; T, \mathcal{Y})$ and receives $\mathcal{Y}$ from underwriter (short position)
    - at t: forward price chosen such that present value of forward contract is zero:

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\left(\mathcal{Y} - f(t; T, \mathcal{Y})\right) \mid \mathcal{F}_t\right] = 0$$

- this is equivalent to

$$f(t; T, \mathcal{Y}) = \frac{1}{P(t,T)}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\mathcal{Y} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}^T}\left[\mathcal{Y} \mid \mathcal{F}_t\right]$$

Forward Contracts

- examples: the forward price at t of:
  1. a dollar delivered at T is 1
  2. an S-bond delivered at $T \leq S$ is $\frac{P(t,S)}{P(t,T)}$
  3. any traded asset S delivered at T is $\frac{S(t)}{P(t,T)}$
- note: forward price $f(s;T,\mathcal{Y})$ has to be distinguished from (spot) price at time s of the forward contract entered at time $t \leq s$, which is

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_s^T r(u)\,du}\,(\mathcal{Y} - f(t;T,\mathcal{Y}))\mid \mathcal{F}_s\right] \\ = P(s,T)\,(f(s;T,\mathcal{Y}) - f(t;T,\mathcal{Y}))$$

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Futures Contracts

- futures contract on $\mathcal{Y}$ with time of delivery T defined as:
  - at every $t \leq T$: there is a market quoted futures price $F(t; T, \mathcal{Y})$, which makes the futures contract on $\mathcal{Y}$, if entered at t, equal to zero
  - at T: holder (long position) pays $F(T; T, \mathcal{Y})$ and receives $\mathcal{Y}$ from underwriter (short position)
  - marking to market or resettlement: during any infinitesimal time interval $(t, t + \Delta t]$: holder receives (or pays, if negative) $F(t; T, \mathcal{Y}) - F(t + \Delta t; T, \mathcal{Y})$

$\Rightarrow$ continuous cash flow between the two parties of a futures contract, they are required to keep certain amount of money as safety margin

Futures Contracts

- trading volumes in futures are huge
- one of reasons: often difficult to trade/hedge directly in underlying object (e.g. an index including illiquid instruments, or a commodity such as copper, gas or electricity)
- holding short position in futures: no need to physically deliver the underlying object if you exit contract before delivery date
- selling short makes it possible to hedge against the underlying

Futures Price Process

- Suppose $\mathbb{E}_{\mathbb{Q}}[|\mathcal{Y}|] < \infty$, then futures price process $=$ $\mathbb{Q}$-martingale:

$$F(t; T, \mathcal{Y}) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{Y} \mid \mathcal{F}_t\right]$$

- consequence: forward $=$ futures price if interest rates deterministic

Heuristic Argument

- w.l.o.g $F(t) = F(t; T, \mathcal{Y})$ is an Itô process
- cumulative discounted cash flow of futures contract in $(t, T]$: $V = \lim_N V_N$ with

$$V_N = \sum_{i=1}^{N} \frac{1}{B(t_i)} \left(F(t_i) - F(t_{i-1}) \right)$$

  limit over sequence of partitions $t = t_0 < \cdots < t_N = T$ with $\max_i |t_i - t_{i-1}| \to 0$ for $N \to \infty$
- can rewrite

$$V_N = \sum_{i=1}^{N} \frac{1}{B(t_{i-1})} \left(F(t_i) - F(t_{i-1}) \right) + \sum_{i=1}^{N} \left(\frac{1}{B(t_i)} - \frac{1}{B(t_{i-1})} \right) \left(F(t_i) - F(t_{i-1}) \right)$$

Heuristic Argument

- B continuous $\Rightarrow 1/B \in \mathcal{L}(F)$
- elementary stochastic calculus: $V_N \to V$ in probability with

$$V = \int_t^T \frac{1}{B(s)}\, dF(s) + \int_t^T d\left\langle \frac{1}{B}, F \right\rangle_s = \int_t^T \frac{1}{B(s)}\, dF(s)$$

Heuristic Argument

- present value = zero:

$$\mathbb{E}_{\mathbb{Q}}\left[\int_t^T \frac{1}{B(s)}\, dF(s) \mid \mathcal{F}_t\right] = 0$$

- consequence:

$$M(t) = \int_0^t \frac{1}{B(s)}\, dF(s) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{1}{B(s)}\, dF(s) \mid \mathcal{F}_t\right]$$

is $\mathbb{Q}$-martingale, $t \leq T$

- assume: $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T B(s)^2\, d\langle M, M\rangle_s\right] = \mathbb{E}_{\mathbb{Q}}\left[\langle F, F\rangle_T\right] < \infty$

$\Rightarrow$ $B \in \mathcal{L}^2(M)$ and

$$F(t) = \int_0^t B(s)\, dM(s)$$

is $\mathbb{Q}$-martingale for $t \leq T$, as desired

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Eurodollar Futures

- Interest rate futures: can be divided into futures on short-term instruments and futures on coupon bonds
- here: we only consider an example from the first group
- Eurodollars = deposits of US dollars in institutions outside of the US
- LIBOR is the interbank rate of interest for Eurodollar loans
- Eurodollar futures contract is tied to LIBOR
  - introduced by the International Money Market (IMM) of the Chicago Mercantile Exchange (CME) in 1981
  - designed to protect its owner from fluctuations in the 3-month (=1/4 year) LIBOR
  - maturity (delivery) months are March, June, September and December

Formal Definition

- fix maturity T
- $L(T)$ = 3-month spot LIBOR for period $[T, T+1/4]$
- market quote of Eurodollar futures contract on $L(T)$ at $t \le T$ is

$$1 - L_F(t,T) \quad [100 \text{ per cent}]$$

  where $L_F(t,T)$ = corresponding futures rate (compare with bootstrapping example above)
- futures price, used for the marking to market, defined by

$$F(t; T, L(T)) = 1 - \frac{1}{4} L_F(t,T) \quad [\text{million dollars}]$$

$\Rightarrow$ change of 1 basis point (0.01%) in futures rate $L_F(t,T)$ leads to cash flow of $10^6 \times 10^{-4} \times \frac{1}{4} = 25$ [dollars]

Formal Definition

- definition: $L_F(T, T) = L(T)$

$\Rightarrow$ final price $F(T; T, L(T)) = 1 - \frac{1}{4}L(T) = \mathcal{Y}$: underlying $\mathcal{Y}$ is a synthetic value, no physical delivery at maturity, settlement is made in cash

- since $F(t; T, L(T)) = \mathbb{E}_{\mathbb{Q}}[F(T; T, L(T)) \mid \mathcal{F}_t]$ we obtain explicit formula for the futures rate

$$L_F(t, T) = \mathbb{E}_{\mathbb{Q}}[L(T) \mid \mathcal{F}_t]$$

A Mathematicians Intuition

- underlying $\mathcal{Y} = P(T, T + 1/4)$
- futures price $= \mathbb{E}_{\mathbb{Q}} \left[P(T, T + 1/4) \mid \mathcal{F}_t \right]$
- exact: $P(T, T + 1/4) = 1 - \frac{1}{4} L(T) P(T, T + 1/4)$
- approximate: $P(T, T + 1/4) \approx 1 - \frac{1}{4} L(T)$

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Forward vs. Futures

- let S be price process of a traded asset with $\mathbb{Q}$-dynamics

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \rho(t)\,dW^*(t)$$

- fix delivery date T
- forward price of S for delivery at T:

$$f(t; T, S(T)) = \frac{S(t)}{P(t,T)}$$

- futures price of S for delivery at T

$$F(t; T, S(T)) = \mathbb{E}_{\mathbb{Q}}[S(T) \mid \mathcal{F}_t]$$

- aim: establish relationship between the two prices under Gaussian assumption

Forward vs. Futures

Proposition

Suppose $\rho(t)$ and T-bond volatility $v(t,T)$ are deterministic. Then

$$F(t;T,S(T)) = f(t;T,S(T))\,\mathrm{e}^{\int_t^T (v(s,T)-\rho(s))\,v(s,T)^\top ds}$$

for $t \le T$.

Hence, if the instantaneous covariation of $S(t)$ and $P(t,T)$ is negative,

$$\frac{d\langle S, P(\cdot,T)\rangle_t}{dt} = S(t)P(t,T)\,\rho(t)\,v(t,T)^\top \le 0,$$

then the futures price dominates the forward price.

Proof

Proof.

Write $\mu(s) = v(s,T) - \rho(s)$. It is clear that

$$f(t;T,S(T)) = \frac{S(0)}{P(0,T)} \mathcal{E}_t(\mu \bullet W^*) \exp\left(\int_0^t \mu(s)\, v(s,T)^\top ds\right).$$

Since $\mathcal{E}(\mu \bullet W^*)$ is a $\mathbb{Q}$-martingale and $\rho(s)$ and $v(s,T)$ are deterministic, we obtain

$$\begin{aligned} F(t;T,S(T)) &= \mathbb{E}_{\mathbb{Q}}[f(T;T,S(T)) \mid \mathcal{F}_t] \\ &= f(t;T,S(T))\, \mathrm{e}^{\int_t^T \mu(s)\, v(s,T)^\top ds}, \end{aligned}$$

as desired. $\square$

Forward vs. Futures Rates

Lemma (Convexity Adjustments)

Assume Gaussian HJM framework. Then relation between instantaneous forward and futures rates:

$$f(t,T) = \mathbb{E}_{\mathbb{Q}}[r(T) \mid \mathcal{F}_t] - \int_t^T \left(\sigma(s,T) \int_s^T \sigma(s,u)^\top du \right) ds$$

and simple forward and futures rates:

$$\begin{aligned} F(t;T,S) &= \mathbb{E}_{\mathbb{Q}}[F(T,S) \mid \mathcal{F}_t] \\ &\quad - \frac{P(t,T)}{(S-T)P(t,S)} \left(\mathrm{e}^{\int_t^T \left(\int_T^S \sigma(s,v)\, dv \int_s^S \sigma(s,u)^\top du \right) ds} - 1 \right) \end{aligned}$$

Forward vs. Futures Rates

Hence, if

$$\sigma(s,v)\,\sigma(s,u)^\top \geq 0 \quad \text{for all } s \leq u \wedge v$$

then futures rates are always greater than the corresponding forward rates.

Proof.

Exercise $\square$

Part VIII

Consistent Term-Structure Parametrizations

Overview

- practitioners and academics have vital interest in parameterized term-structure models
- in this chapter: take up a point left open at the end of estimation chapter: exploit whether parameterized curve families $\phi(\cdot, z)$, used for estimating the forward curve, go well with arbitrage-free interest rate models
- recall BIS document [5]: rich source of cross-sectional data (daily estimations of the parameter z) for the Nelson–Siegel and Svensson families
- suggests that calibrating a diffusion process Z for the parameter z would lead to an accurate factor model for the forward curve
- conditions for absence of arbitrage can be formulated in terms of the drift and diffusion of Z and derivatives of ϕ
- these conditions turn out to be surprisingly restrictive in some cases

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32 Consistency Condition

33 Affine Term-Structures

34 Polynomial Term-Structures
- Special Case: $m = 1$
- General Case: $m \geq 1$

35 Exponential–Polynomial Families
- Nelson–Siegel Family
- Svensson Family

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- Svensson Family

Short Rate Models Revisited

- seen earlier: every time-homogeneous diffusion short-rate model $r(t)$ induces forward rates of the form

$$f(t, T) = \phi(T - t, r(t))$$

  for some deterministic function ϕ

+ computational merits (e.g. Jamshidian decomposition)

– unrealistic implications: e.g. family of attainable forward curves $\{\phi(\cdot, r) \mid r \in \mathbb{R}\}$ is only one-dimensional

→ term-structure movements explained by single state variable $r(t)$: conflicts with above principal component analysis (2-3 factors needed for statistically accurate description of forward curve movements)

– maturity-specific risk: if $d = 1$ e.g. a bond option with maturity five years could be perfectly hedged by the money-market account and a bond of maturity 30 years

Multi-factor Models

- to gain more flexibility: multiple factors $m \geq 1$
- m-factor model := interest rate model of the form

$$f(t, T) = \phi(T - t, Z(t))$$

- deterministic ϕ
- m-dim state space process Z

Assumptions

- state space $\mathcal{Z} \subset \mathbb{R}^m$ closed with non-empty interior
- $\phi \in C^{1,2}(\mathbb{R}_+ \times \mathcal{Z})$
- $b : \mathcal{Z} \to \mathbb{R}^m$ continuous
- $\rho : \mathcal{Z} \to \mathbb{R}^{m \times d}$ measurable, s.t. diffusion matrix $a(z) = \rho(z)\rho(z)^\top$ continuous in $z \in \mathcal{Z}$
- W^*: d-dim Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$
- $\forall z \in \mathcal{Z}$ $\exists$ unique $\mathcal{Z}$-valued solution $Z = Z^z$ of

$$\begin{aligned} dZ(t) &= b(Z(t))\,dt + \rho(Z(t))\,dW^*(t) \\ Z(0) &= z \end{aligned}$$

- NA: $\mathbb{Q}$ is risk-neutral measure for bond prices

$$P(t, T) = \exp\left(-\int_0^{T-t} \phi(x, Z^z(t))\,dx\right)$$

for all $z \in \mathcal{Z}$

Remark

Time-inhomogeneous models are included by identifying one component, say Z_1, with calendar time. We therefore set $dZ_1 = dt$, which is equivalent to $b_1 \equiv 1$ and $\rho_{1j} \equiv 0$ for $j = 1, \ldots, d$. Calendar time at inception is now $Z_1(0) = z_1$, and t, T, etc. accordingly denote relative time with respect to z_1. The NA assumption for all $z \in \mathcal{Z}$ now means, in particular, that absence of arbitrage holds relative to any initial calendar time z_1.

First Consequences

- short rates given by $r(t) = \phi(0, Z(t))$

$\Rightarrow$ NA holds if and only if

$$\frac{\exp\left(-\int_0^{T-t}\phi(x, Z^z(t))\,dx\right)}{\exp\left(\int_0^t \phi(0, Z^z(s))\,ds\right)}, \quad t \leq T,$$

is a $\mathbb{Q}$-local martingale, for all $z \in \mathcal{Z}$

- aim: find consistency condition for a, b and ϕ such that NA holds
- possible way: apply Itô's formula and set drift $= 0$
- our way: first embed in HJM framework and use HJM drift condition ...

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Consistency Condition: Derivation

apply Itô's formula to $f(t, T) = \phi(T - t, Z(t))$:

$$df(t, T) = \Big(- \partial_x \phi(T - t, Z(t)) + \sum_{i=1}^{m} b_i(Z(t)) \partial_{z_i} \phi(T - t, Z(t)) + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(Z(t)) \partial_{z_i} \partial_{z_j} \phi(T - t, Z(t)) \Big) \, dt + \sum_{i=1}^{m} \sum_{j=1}^{d} \partial_{z_i} \phi(T - t, Z(t)) \rho_{ij}(Z(t)) \, dW_j^*(t).$$

Consistency Condition: Derivation

$\Rightarrow$ induced forward rate model is of the HJM type with

$$\alpha(t,T) = -\partial_x \phi(T-t, Z(t)) + \sum_{i=1}^{m} b_i(Z(t)) \partial_{z_i} \phi(T-t, Z(t)) + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(Z(t)) \partial_{z_i} \partial_{z_j} \phi(T-t, Z(t))$$

$$\sigma_j(t,T) = \sum_{i=1}^{m} \partial_{z_i} \phi(T-t, Z(t)) \rho_{ij}(Z(t)), \quad j = 1, \dots, d$$

satisfying **(HJM.1)–(HJM.3)**

Consistency Condition: Derivation

NA $\Leftrightarrow$ HJM drift condition:

$$
\begin{aligned}
&-\phi(T-t,Z(t))+\phi(0,Z(t))+\sum_{i=1}^m b_i(Z(t))\partial_{z_i}\Phi(T-t,Z(t))\\
&\qquad+\frac{1}{2}\sum_{i,j=1}^m a_{ij}(Z(t))\partial_{z_i}\partial_{z_j}\Phi(T-t,Z(t))\\
&=\frac{1}{2}\sum_{j=1}^d\left(\sum_{i=1}^m \partial_{z_i}\Phi(T-t,Z(t))\rho_{ij}(Z(t))\right)^2\\
&=\frac{1}{2}\sum_{k,l=1}^m a_{kl}(Z(t))\partial_{z_k}\Phi(T-t,Z(t))\partial_{z_l}\Phi(T-t,Z(t))
\end{aligned}
$$

where we define $\Phi(x,z)=\int_0^x \phi(u,z)\,du$

- to hold a.s. for all $t \leq T$ and $z = Z(0)$, now let $t \to 0$ ...

Consistency Condition

Proposition (Consistency Condition)

NA holds if and only if

$$\partial_x \Phi(x,z) = \phi(0,z) + \sum_{i=1}^m b_i(z)\partial_{z_i}\Phi(x,z) + \frac{1}{2}\sum_{i,j=1}^m a_{ij}(z)\left(\partial_{z_i}\partial_{z_j}\Phi(x,z) - \partial_{z_i}\Phi(x,z)\partial_{z_j}\Phi(x,z)\right) \tag{15}$$

for all $(x,z) \in \mathbb{R}_+ \times \mathcal{Z}$.

Terminology

Definition

The pair of characteristics $\{a, b\}$ and the forward curve parametrization ϕ are consistent if NA, or equivalently the above consistency condition (15), holds.

Interpretations of (15)

- pricing: take $\phi(0,z)$, a, b as given and solve PDE (15) with initial condition $\Phi(0,z) = 0$
- inverse problem: given parametric estimation method ϕ given, find a and b such that (15) is satisfied for all (x,z)
- it turns out that the latter approach is quite restrictive on possible choices of a and b ...

First Consequence

Proposition

Suppose that the functions

$$\partial_{z_i}\Phi(\cdot, z) \quad \textit{and} \quad \frac{1}{2}\left(\partial_{z_i}\partial_{z_j}\Phi(\cdot, z) - \partial_{z_i}\Phi(\cdot, z)\partial_{z_j}\Phi(\cdot, z)\right),$$

for $1 \leq i \leq j \leq m$, are linearly independent for all z in some dense subset $\mathcal{D} \subset \mathcal{Z}$. Then there exists one and only one consistent pair $\{a, b\}$.

Proof

Proof.

Set $M = m + m(m+1)/2$, the number of unknown functions b_k and $a_{kl} = a_{lk}$. Let $z \in \mathcal{D}$. Then there exists a sequence $0 \leq x_1 < \cdots < x_M$ such that the $M \times M$-matrix with kth row vector built by

$$\partial_{z_i} \Phi(x_k, z) \quad \text{and} \quad \frac{1}{2}\left(\partial_{z_i}\partial_{z_j}\Phi(x_k, z) - \partial_{z_i}\Phi(x_k, z)\partial_{z_j}\Phi(x_k, z)\right),$$

for $1 \leq i \leq j \leq m$, is invertible. Thus, $b(z)$ and $a(z)$ are uniquely determined by (15). This holds for each $z \in \mathcal{D}$. By continuity of b and a hence for all $z \in \mathcal{Z}$. $\square$

Practical Implications

- suppose parameterized curve family $\{\phi(\cdot, z) \mid z \in \mathcal{Z}\}$ used for daily forward curve estimation in terms of state variable z.
- above proposition: any consistent $\mathbb{Q}$-diffusion model Z for z is fully determined by ϕ
- moreover: diffusion matrix $a(z)$ of Z not affected by equivalent measure transformation

$\Rightarrow$ statistical calibration only possible for drift of the model (or equivalently, for the market price of risk), since observations of z are made under objective measure $\mathbb{P} \sim \mathbb{Q}$ and $d\mathbb{Q}/d\mathbb{P}$ is left unspecified by our consistency considerations

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Affine Term-Structure (ATS)

- simplest case: time-homogeneous affine term-structure (ATS)

$$\phi(x, z) = g_0(x) + g_1(x) z_1 + \cdots g_m(x) z_m$$

- second-order z-derivatives vanish, (15) simplifies:

$$\begin{aligned} g_0(x) - g_0(0) + \sum_{i=1}^{m} z_i (g_i(x) - g_i(0)) \\ = \sum_{i=1}^{m} b_i(z) G_i(x) - \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(z) G_i(x) G_j(x), \quad (16) \end{aligned}$$

where we define

$$G_i(x) = \int_0^x g_i(u)\, du$$

Affine Term-Structure (ATS)

- assume: $m + m(m+1)/2$ functions $G_1, \dots, G_m,\ G_1 G_1,\ G_1 G_2, \dots, G_m G_m$ linearly independent

$\Rightarrow$ can invert and solve the linear equation (16) for a and b (as in proof of above proposition)

- left-hand side of (16) affine in $z \Rightarrow a$, b affine:

$$a_{ij}(z) = a_{ij} + \sum_{k=1}^{m} \alpha_{k;ij} z_k,$$

$$b_i(z) = b_i + \sum_{j=1}^{m} \beta_{ij} z_j,$$

for parameters a_{ij}, $\alpha_{k;ij}$, b_i and β_{ij}

Affine Term-Structure (ATS)

plugging back into (16) and matching terms ⇒ Riccati equations

$$\partial_x G_0(x) = g_0(0) + \sum_{i=1}^{m} b_i G_i(x) - \frac{1}{2} \sum_{i,j=1}^{m} a_{ij} G_i(x) G_j(x) \tag{17}$$

$$\partial_x G_k(x) = g_k(0) + \sum_{i=1}^{m} \beta_{ki} G_i(x) - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_{k;ij} G_i(x) G_j(x) \tag{18}$$

Summary

Proposition

If $\{a, b\}$ is consistent with above ATS, then a and b are affine, and G_i solve system of Riccati equations (17)–(18) with initial conditions $G_i(0) = 0$.

Conversely, suppose a and b are affine, and let $g_i(0)$ be some given constants. If the functions G_i solve the system of Riccati equations (17)–(18) with initial conditions $G_i(0) = 0$, then the above ATS is consistent with $\{a, b\}$.

Discussion

- this proposition extends earlier result on time-homogeneous affine short-rate models with

$$A(t,T) = G_0(T-t) \quad \text{and} \quad B(t,T) = G_1(T-t)$$

- note 1: we did not have to assume linear independence of B and B^2: this assumption becomes necessary as soon as $m \geq 2$ ($\rightarrow$ exercise)
- note 2: we have freedom to choose constants $g_i(0)$ related to short rates by

$$r(t) = f(t,t) = g_0(0) + g_1(0)Z_1(t) + \cdots + g_m(0)Z_m(t)$$

- typical choice: $g_1(0) = 1$ and all other $g_i(0) = 0 \Rightarrow Z_1(t)$ is (non-Markovian) short-rate process

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Polynomial Term-Structure (PTS)

- polynomial term-structure (PTS) $\phi(x,z) = \sum_{|\mathbf{i}|=0}^{n} g_{\mathbf{i}}(x)\, z^{\mathbf{i}}$
- multi-index notation: $\mathbf{i} = (i_1, \dots, i_m)$, $|\mathbf{i}| = i_1 + \cdots + i_m$ and $z^{\mathbf{i}} = z_1^{i_1} \cdots z_m^{i_m}$
- n = degree of the PTS: maximal k with $g_{\mathbf{i}} \neq 0$ for some $|\mathbf{i}| = k$

Maximal Degree Problem

- $n = 1$: ATS
- $n = 2$: quadratic term-structure (QTS), intensively studied in the literature (e.g. Ahn et al. [1])
- question: do we gain something by looking at $n = 3$ and higher-degree PTS models?
- surprising answer: no
- we show that there is no consistent PTS for $n > 2$ ...
- for simplicity $m = 1$ only (for general case see course book Section 9.4.2)

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Maximal Degree Problem I

- special case $m = 1$: PTS now reads

$$\phi(x, z) = \sum_{i=0}^{n} g_i(x)\, z^i$$

- define $G_i(x) = \int_0^x g_i(u)\, du$

Theorem (Maximal Degree Problem I)

Suppose that G_i and $G_i G_j$ are linearly independent functions, for $1 \leq i \leq j \leq n$, and that $\rho \not\equiv 0$.
Then consistency implies $n \in \{1, 2\}$. Moreover, $b(z)$ and $a(z)$ are polynomials in z with $\deg b(z) \leq 1$ in any case (QTS and ATS), and $\deg a(z) = 0$ if $n = 2$ (QTS) and $\deg a(z) \leq 1$ if $n = 1$ (ATS).

Proof

Equation (15) can be rewritten

$$\sum_{i=0}^{n} (g_i(x) - g_i(0))\, z^i = \sum_{i=0}^{n} G_i(x) B_i(z) - \sum_{i,j=0}^{n} G_i(x) G_j(x) A_{ij}(z)$$

where we define

$$B_i(z) = b(z) i z^{i-1} + \frac{1}{2} a(z) i(i-1) z^{i-2},$$
$$A_{ij}(z) = \frac{1}{2} a(z) ij z^{i-1} z^{j-1}.$$

By assumption we can solve above linear equation for B and A, and thus $B_i(z)$ and $A_{ij}(z)$ are polynomials in z of order less than or equal n.

Proof cont'd

In particular, this holds for

$$B_1(z) = b(z) \quad \text{and} \quad 2A_{11}(z) = a(z).$$

But then, since $a \not\equiv 0$ by assumption, $2A_{nn}(z) = a(z)n^2 z^{2n-2}$ cannot be a polynomial of order less than or equal n unless $2n - 2 \leq n$, which implies $n \leq 2$. The theorem is thus proved for $n = 1$. For $n = 2$, we obtain $\deg a(z) = 0$ and thus $B_2(z) = 2b(z)z + a(z)$. Hence also in this case $\deg b(z) \leq 1$, and the theorem is proved for $m = 1$.

Maximal Degree Problem II

Relax linear independence hypothesis on G_i, $G_i G_j$:

Theorem (Maximal Degree Problem II)

Suppose that:

1. $\sup \mathcal{Z} = \infty$;
2. *b and ρ satisfy a linear growth condition*

$$|b(z)| + |\rho(z)| \leq C(1 + |z|), \quad z \in \mathcal{Z},$$

for some finite constant C;

3. *$a(z)$ is asymptotically bounded away from zero:*

$$\liminf_{z \to \infty} a(z) > 0.$$

Then consistency implies $n \in \{1, 2\}$.

Note: linear growth condition 2 is standard assumption for non-explosion of Z

Proof

Again, we consider equation (15), which reads

$$\sum_{i=0}^{n} (g_i(x) - g_i(0))\, z^i = b(z) \sum_{i=0}^{n} G_i(x) i z^{i-1} + \frac{1}{2} a(z) \left(\sum_{i,j=0}^{n} G_i(x) i(i-1) z^{i-2} - \left(\sum_{i=0}^{n} G_i(x) i z^{i-1} \right)^2 \right). \tag{19}$$

We argue by contradiction and assume that $n > 2$, which implies $2n - 2 > n$. Dividing (19) by z^{2n-2}, for $z \neq 0$, yields

. . .

Proof cont'd

$$\frac{1}{2}a(z)\frac{\left(\sum_{i=0}^{n} G_i(x) i z^{i-1}\right)^2}{z^{2n-2}} = \frac{b(z)}{z}\frac{\sum_{i=0}^{n} G_i(x) i z^{i-1}}{z^{2n-3}}$$
$$+ \frac{a(z)}{2z^2}\frac{\sum_{i,j=0}^{n} G_i(x) i(i-1) z^{i-2}}{z^{2n-4}} - \frac{\sum_{i=0}^{n} \left(g_i(x) - g_i(0)\right) z^i}{z^{2n-2}}.$$

By assumption 1 this holds for all z large enough. The right-hand side converges to zero, for $z \to \infty$, by assumption 2. Taking the $\liminf$ of the left-hand side yields by 3, that

$$\frac{1}{2}\liminf_{z\to\infty} a(z) G_n^2(x) n^2 > 0,$$

a contradiction. Thus $n \leq 2$.

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Maximal Degree Problem I

Theorem (Maximal Degree Problem I)

Suppose that $G_{\mathbf{i}_\mu}$ and $G_{\mathbf{i}_\mu}G_{\mathbf{i}_\nu}$ are linearly independent functions, $1 \leq \mu \leq \nu \leq N$, and that $\rho \not\equiv 0$.
Then consistency implies $n \in \{1, 2\}$. Moreover, $b(z)$ and $a(z)$ are polynomials in z with $\deg b(z) \leq 1$ in any case (QTS and ATS), and $\deg a(z) = 0$ if $n = 2$ (QTS) and $\deg a(z) \leq 1$ if $n = 1$ (ATS).

Maximal Degree Problem II

Theorem (Maximal Degree Problem II)

Suppose that:

1. $\mathcal{Z}$ *is a cone;*
2. b *and* ρ *satisfy a linear growth condition*

$$\|b(z)\| + \|\rho(z)\| \leq C(1 + \|z\|), \quad z \in \mathcal{Z},$$

   for some finite constant C*;*

3. $a(z)$ *becomes uniformly elliptic for* $\|z\|$ *large enough:*

$$\langle a(z)v, v\rangle \geq k(z)\|v\|^2, \quad v \in \mathbb{R}^m,$$

   for some function $k : \mathcal{Z} \to \mathbb{R}_+$ *with*

$$\liminf_{z\in\mathcal{Z},\|z\|\to\infty} k(z) > 0.$$

Then consistency implies $n \in \{1, 2\}$*.*

Note: linear growth condition 2 is standard assumption for non-explosion of Z

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Nelson–Siegel Family

Recall Nelson–Siegel: $\phi_{NS}(x,z) = z_1 + (z_2 + z_3 x)\mathrm{e}^{-z_4 x}$

Proposition

The unique solution to (15) *for* ϕ_{NS} *is*

$$a(z) = 0,\ b_1(z) = b_4(z) = 0,\ b_2(z) = z_3 - z_2 z_4,\ b_3(z) = -z_3 z_4.$$

The corresponding state process is

$$\begin{aligned}
Z_1(t) &\equiv z_1,\\
Z_2(t) &= (z_2 + z_3 t)\,\mathrm{e}^{-z_4 t},\\
Z_3(t) &= z_3 \mathrm{e}^{-z_4 t},\\
Z_4(t) &\equiv z_4,
\end{aligned}$$

where $Z(0) = (z_1, \ldots, z_4)$ *denotes the initial point. Hence there is no non-trivial consistent diffusion process* Z *for the Nelson–Siegel family.*

Proof.

$\rightarrow$ Exercise

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Svensson Family

Recall Svensson: $\phi_S(x, z) = z_1 + (z_2 + z_3 x)\mathrm{e}^{-z_5 x} + z_4 x \mathrm{e}^{-z_6 x}$

Proposition

The only non-trivial consistent HJM model for the Svensson family is the Hull–White extended Vasiček short-rate model

$$dr(t) = \left(z_1 z_5 + z_3 \mathrm{e}^{-z_5 t} + z_4 \mathrm{e}^{-2 z_5 t} - z_5 r(t)\right) dt + \sqrt{z_4 z_5}\, \mathrm{e}^{-z_5 t} d\mathcal{W}^*($$

where $(z_1, \ldots, z_5)$ are given by the initial forward curve

$$f(0, x) = z_1 + (z_2 + z_3 x)\mathrm{e}^{-z_5 x} + z_4 x \mathrm{e}^{-2 z_5 x}$$

and $\mathcal{W}^*$ is some $\mathbb{Q}$-standard Brownian motion. The form of the corresponding state process Z is given in the proof below.

Discussion

- Svensson family admits no consistently varying exponents: $Z_6(t) \equiv z_6 = 2z_5 \equiv 2Z_5(t)$

$\Rightarrow$ exponents, z_5 and $z_6 = 2z_5$, be rather considered as model parameters than factors

- hypothesis empirically tested on US bond price data in Sharef [48]:
  - constant exponents hypothesis could not be rejected, while the relation $z_6 = 2z_5$ does
  - statistical properties of z very sensitive on the numerical term-structure estimation procedure
- Becker and Bouwman [4]: inter-temporal smoothing device can substantially improve parameter stability and smoothness
- open problem: explore consistent inter-temporal smoothing devices

Proof

The consistency equation (15) becomes, after differentiating both sides in x,

$$q_1(x) + q_2(x)\mathrm{e}^{-z_5 x} + q_3(x)\mathrm{e}^{-z_6 x} + q_4(x)\mathrm{e}^{-2z_5 x} + q_5(x)\mathrm{e}^{-(z_5+z_6)x} + q_6(x)\mathrm{e}^{-2z_6 x} = 0, \quad (20)$$

for some polynomials $q_1, \ldots, q_6$. Indeed, we assume for the moment that

$$z_5 \neq z_6, \quad z_5 + z_6 \neq 0 \quad \text{and } z_i \neq 0 \text{ for all } i = 1, \ldots, 6. \quad (21)$$

Proof cont'd

Then the terms involved in (20) are

$$\partial_x \phi_S(x,z) = (-z_2 z_5 + z_3 - z_3 z_5 x)\mathrm{e}^{-z_5 x} + (z_4 - z_4 z_6 x)\mathrm{e}^{-z_6 x},$$

$$\nabla_z \phi_S(x,z) = \begin{pmatrix} 1 \\ \mathrm{e}^{-z_5 x} \\ x\mathrm{e}^{-z_5 x} \\ x\mathrm{e}^{-z_6 x} \\ (-z_2 x - z_3 x^2)\mathrm{e}^{-z_5 x} \\ -z_4 x^2 \mathrm{e}^{-z_6 x} \end{pmatrix},$$

$$\partial_{z_i}\partial_{z_j}\phi_S(x,z) = 0 \quad \text{for } 1 \leq i,j \leq 4,$$

Proof cont'd

$$\nabla_z \partial_{z_5} \phi_S(x,z) = \begin{pmatrix} 0 \\ -x\mathrm{e}^{-z_5 x} \\ -x^2\mathrm{e}^{-z_5 x} \\ 0 \\ (z_2 x^2 + z_3 x^3)\mathrm{e}^{-z_5 x} \\ 0 \end{pmatrix},$$

$$\nabla_z \partial_{z_6} \phi_S(x,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x^2\mathrm{e}^{-z_6 x} \\ 0 \\ z_4 x^3\mathrm{e}^{-z_6 x} \end{pmatrix},$$

Proof cont'd

$$\int_0^x \nabla_z \phi_S(u, z)\, du$$

$$= \begin{pmatrix} x \\ -\frac{1}{z_5} e^{-z_5 x} + \frac{1}{z_5} \\ \left(-\frac{x}{z_5} - \frac{1}{z_5^2}\right) e^{-z_5 x} + \frac{1}{z_5^2} \\ \left(-\frac{x}{z_6} - \frac{1}{z_6^2}\right) e^{-z_6 x} + \frac{1}{z_6^2} \\ \left(\frac{z_3}{z_5} x^2 + \left(\frac{z_2}{z_5} + \frac{2z_3}{z_5^2}\right) x + \frac{z_2}{z_5^2} + \frac{2z_3}{z_5^3}\right) e^{-z_5 x} - \frac{z_2}{z_5^2} - \frac{z_3}{z_5^3} \\ \left(\frac{z_4}{z_6} x^2 + \frac{2z_4}{z_6^2} x + \frac{2z_4}{z_6^3}\right) e^{-z_6 x} - \frac{z_4}{z_6^3} \end{pmatrix}.$$

Proof cont'd

Straightforward calculations lead to

$$q_1(x) = -a_{11}(z)x + \cdots,$$
$$q_4(x) = a_{55}(z)\frac{z_3^2}{z_5}x^4 + \cdots,$$
$$q_6(x) = a_{66}(z)\frac{z_4^2}{z_6}x^4 + \cdots,$$
$$\deg q_2, \deg q_3, \deg q_5 \leq 3,$$

where $\cdots$ stands for lower-order terms in x. Because of (21) we conclude that

$$a_{11}(z) = a_{55}(z) = a_{66}(z) = 0.$$

But a is a positive semi-definite symmetric matrix. Hence

$$a_{1j}(z) = a_{j1}(z) = a_{5j}(z) = a_{j5}(z) = a_{6j}(z) = a_{j6}(z) = 0 \quad j \leq 6.$$

Proof cont'd

Taking this into account, expression (20) simplifies considerably. We are left with

$$\begin{aligned} q_1(x) &= b_1(z), \\ \deg q_2, \deg q_3 &\leq 1, \\ q_4(x) &= a_{33}(z)\frac{1}{z_5}x^2 + \cdots, \\ q_5(x) &= a_{34}(z)\left(\frac{1}{z_5} + \frac{1}{z_6}\right)x^2 + \cdots, \\ q_6(x) &= a_{44}(z)\frac{1}{z_6}x^2 + \cdots. \end{aligned}$$

Because of (21) we know that the exponents $-2z_5$, $-(z_5 + z_6)$ and $-2z_6$ are mutually different. Hence

$$b_1(z) = a_{3j}(z) = a_{j3}(z) = a_{4j}(z) = a_{j4}(z) = 0 \quad j \leq 6.$$

Only $a_{22}(z)$ is left as positive candidate among the components of $a(z)$. The remaining terms are

$$\begin{aligned} q_2(x) &= (b_3(z) + z_3z_5)x + b_2(z) - z_3 - \frac{a_{22}(z)}{z_5} + z_2z_5, \\ q_3(x) &= (b_4(z) + z_4z_6)x - z_4, \\ q_4(x) &= a_{22}(z)\frac{1}{z_5}, \end{aligned}$$

while $q_1 = q_5 = q_6 = 0$.

Proof cont'd

If $2z_5 \neq z_6$ then also $a_{22}(z) = 0$. If $2z_5 = z_6$ then the condition $q_3 + q_4 = q_2 = 0$ leads to

$$\begin{aligned} a_{22}(z) &= z_4 z_5, \\ b_2(z) &= z_3 + z_4 - 2_5 z_2, \\ b_3(z) &= -z_5 z_3, \\ b_4(z) &= -2 z_5 z_4. \end{aligned}$$

We derived the above results under the assumption (21). But the set of z where (21) holds is dense $\mathcal{Z}$. By continuity of $a(z)$ and $b(z)$ in z, the above results thus extend for all $z \in \mathcal{Z}$. In particular, all Z_i's but Z_2 are deterministic; Z_1, Z_5 and Z_6 are even constant.

Proof cont'd

Thus, since

$$a(z) = 0 \quad \text{if } 2z_5 \neq z_6,$$

we only have a non-trivial process Z if

$$Z_6(t) \equiv 2Z_5(t) \equiv 2Z_5(0). \tag{22}$$

In that case we have, writing for short $z_i = Z_i(0)$,

$$\begin{aligned} Z_1(t) &\equiv z_1, \\ Z_3(t) &= z_3 \mathrm{e}^{-z_5 t}, \\ Z_4(t) &= z_4 \mathrm{e}^{-2z_5 t} \end{aligned} \tag{23}$$

and

$$dZ_2(t) = \left(z_3 \mathrm{e}^{-z_5 t} + z_4 \mathrm{e}^{-2z_5 t} - z_5 Z_2(t)\right) dt + \sum_{j=1}^{d} \rho_{2j}(t)\, dW_j^*(t), \tag{24}$$

where $\rho_{2j}(t)$ (not necessarily deterministic) are such that

$$\sum_{j=1}^{d} \rho_{2j}^2(t) = a_{22}(Z(t)) = z_4 z_5 \mathrm{e}^{-2z_5 t}.$$

Proof cont'd

By Lévy's characterization theorem we have that

$$\mathcal{W}^*(t) = \sum_{j=1}^{d} \int_0^t \frac{\rho_{2j}(s)}{\sqrt{z_4 z_5 \mathrm{e}^{-z_5 s}}} \, dW_j^*(s)$$

is a real-valued standard Brownian motion. Hence the corresponding short-rate process

$$r(t) = \phi_S(0, Z(t)) = z_1 + Z_2(t)$$

satisfies

$$dr(t) = \left(z_1 z_5 + z_3 \mathrm{e}^{-z_5 t} + z_4 \mathrm{e}^{-2 z_5 t} - z_5 r(t)\right) dt + \sqrt{z_4 z_5} \mathrm{e}^{-z_5 t} \, d\mathcal{W}^*$$

Hence the proposition is proved.

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Overview

- We have seen above: affine diffusion induces affine term-structure
- In this chapter:
  - Study affine processes in detail
  - Elaborate on their nice analytical properties
  - Applications in option pricing by Fourier transform

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Standing Assumptions

- fix dimension $d \geq 1$
- closed state space $\mathcal{X} \subset \mathbb{R}^d$ with non-empty interior
- $b : \mathcal{X} \to \mathbb{R}^d$ continuous
- $\rho : \mathcal{X} \to \mathbb{R}^{d \times d}$ measurable and s.t. diffusion matrix $a(x) = \rho(x)\rho(x)^\top$ continuous in $x \in \mathcal{X}$
- W: d-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$
- for every $x \in \mathcal{X}$ there exists a unique solution $X = X^x$ of

$$
\begin{aligned}
dX(t) &= b(X(t))\,dt + \rho(X(t))\,dW(t) \\
X(0) &= x.
\end{aligned}
$$

Definition

Definition

We call X affine if the $\mathcal{F}_t$-conditional characteristic function of $X(T)$ is exponential affine in $X(t)$: there exist $\mathbb{C}$- and $\mathbb{C}^d$-valued functions $\phi(t,u)$ and $\psi(t,u)$, respectively, with jointly continuous t-derivatives such that $X = X^x$ satisfies

$$\mathbb{E}\left[\mathrm{e}^{u^\top X(T)} \mid \mathcal{F}_t\right] = \mathrm{e}^{\phi(T-t,u)+\psi(T-t,u)^\top X(t)} \tag{25}$$

for all $u \in \mathrm{i}\mathbb{R}^d$, $t \le T$ and $x \in \mathcal{X}$.

- $\mathrm{Re}\left(\phi(T-t,u)+\psi(T-t,u)^\top X(t)\right) \le 0$
- ϕ, ψ uniquely determined by (25) with $\phi(0,u)=0$, $\psi(0,u)=u$

Necessary Conditions

Theorem

Suppose X is affine. Then the diffusion matrix $a(x)$ and drift $b(x)$ are affine in x. That is,

$$a(x) = a + \sum_{i=1}^{d} x_i \alpha_i$$

$$b(x) = b + \sum_{i=1}^{d} x_i \beta_i = b + \mathcal{B}x$$

for some $d \times d$-matrices a and α_i, and d-vectors b and β_i, where we denote by

$$\mathcal{B} = (\beta_1, \ldots, \beta_d)$$

the $d \times d$-matrix with ith column vector β_i, $1 \le i \le d$.

Necessary Conditions cont'd

Theorem (cont'd)

Moreover, ϕ and $\psi = (\psi_1, \ldots, \psi_d)^\top$ solve the system of Riccati equations

$$
\begin{aligned}
\partial_t \phi(t,u) &= \frac{1}{2}\psi(t,u)^\top a\,\psi(t,u) + b^\top \psi(t,u) \\
\phi(0,u) &= 0 \\
\partial_t \psi_i(t,u) &= \frac{1}{2}\psi(t,u)^\top \alpha_i\,\psi(t,u) + \beta_i^\top \psi(t,u), \quad 1 \le i \le d, \\
\psi(0,u) &= u.
\end{aligned}
\tag{26}
$$

In particular, ϕ is determined by ψ via simple integration:

$$
\phi(t,u) = \int_0^t \left(\frac{1}{2}\psi(s,u)^\top a\,\psi(s,u) + b^\top \psi(s,u) \right) ds.
$$

Sufficient Conditions

Theorem

Conversely, suppose the diffusion matrix $a(x)$ and drift $b(x)$ are affine and suppose there exists a solution (ϕ, ψ) of the Riccati equations (26) such that $\mathrm{Re}\,(\phi(t,u) + \psi(t,u)^\top x) \leq 0$ for all $t \geq 0$, $u \in \mathrm{i}\mathbb{R}^d$ and $x \in \mathcal{X}$. Then X is affine with conditional characteristic function (25).

Proof

Suppose X is affine. For $T > 0$ and $u \in \mathrm{i}\mathbb{R}^d$ define the complex-valued Itô process

$$M(t) = \mathrm{e}^{\phi(T-t,u)+\psi(T-t,u)^\top X(t)}.$$

We can apply Itô's formula, separately to the real and imaginary parts of M, and obtain

$$dM(t) = I(t)\,dt + \psi(T-t,u)^\top \rho(X(t))\,dW(t), \quad t \le T,$$

with

$$I(t) = -\partial_T \phi(T-t,u) - \partial_T \psi(T-t,u)^\top X(t)$$
$$+\psi(T-t,u)^\top b(X(t)) + \frac{1}{2}\psi(T-t,u)^\top a(X(t))\,\psi(T-t,u).$$

Since M is a martingale, we have $I(t) = 0$ for all $t \le T$ a.s.

Proof cont'd

Letting $t \to 0$, by continuity of the parameters, we thus obtain

$$\begin{aligned}\partial_T \phi(T,u) &+ \partial_T \psi(T,u)^\top x \\ &= \psi(T,u)^\top b(x) + \frac{1}{2}\psi(T,u)^\top a(x)\,\psi(T,u)\end{aligned}$$

for all $x \in \mathcal{X}$, $T \geq 0$, $u \in \mathrm{i}\mathbb{R}^d$. Since $\psi(0,u) = u$, this implies that a and b are affine of the stated form. Plugging this back into the above equation and separating first-order terms in x yields the Riccati equations (26).

Proof cont'd

Conversely, suppose a and b are affine. Let (ϕ, ψ) be a solution of the Riccati equations (26) such that $\phi(t,u) + \psi(t,u)^\top x$ has a nonpositive real part for all $t \geq 0$, $u \in \mathrm{i}\mathbb{R}^d$ and $x \in \mathcal{X}$. Then M, defined as above, is a uniformly bounded (substantial!) local martingale, and hence a martingale, with $M(T) = \mathrm{e}^{u^\top X(T)}$. Therefore $\mathbb{E}[M(T) \mid \mathcal{F}_t] = M(t)$, for all $t \leq T$, which is (25), and the theorem is proved.

ODE Facts

K = placeholder for either $\mathbb{R}$ or $\mathbb{C}$

Lemma

Consider the system of ordinary differential equations

$$\begin{aligned} \partial_t f(t,u) &= R(f(t,u)) \\ f(0,u) &= u, \end{aligned} \tag{27}$$

where $R : K^d \to K^d$ is a locally Lipschitz continuous function. Then the following holds:

1. *For every $u \in K^d$, there exists a life time $t_+(u) \in (0,\infty]$ such that there exists a unique solution $f(\cdot,u) : [0,t_+(u)) \to K \times K^d$ of (27).*

ODE Facts cont'd

Lemma (cont'd)

2. *The domain $\mathcal{D}_K = \{(t,u) \in \mathbb{R}_+ \times K^d \mid t < t_+(u)\}$ is open in $\mathbb{R}_+ \times K^d$ and maximal in the sense that either $t_+(u) = \infty$ or $\lim_{t \uparrow t_+(u)} \|f(t,u)\| = \infty$ for all $u \in K^d$.*
3. *The t-section $\mathcal{D}_K(t) = \{u \in K^d \mid (t,u) \in \mathcal{D}_K\}$ is open in K^d, and non-expanding in t:*

$$K^d = \mathcal{D}_K(0) \supseteq \mathcal{D}_K(t_1) \supseteq \mathcal{D}_K(t_2) \quad 0 \le t_1 \le t_2.$$

   In fact, we have $f(s, \mathcal{D}_K(t_2)) \subseteq \mathcal{D}_K(t_1)$ for all $s \le t_2 - t_1$.
4. *If R is analytic on K^d then f is an analytic function on $\mathcal{D}_K$.*

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Necessary Parameter Conditions

There is an implicit trade-off between the parameters a, α_i, b, β_i and the state space $\mathcal{X}$:

- a, α_i, b, β_i must be such that X does not leave the set $\mathcal{X}$;
- a, α_i must be such that $a + \sum_{i=1}^{d} x_i \alpha_i$ is symmetric and positive semi-definite for all $x \in \mathcal{X}$.

Canonical State Space

- standing assumption: canonical state space

$$\mathcal{X} = \mathbb{R}_+^m \times \mathbb{R}^n$$

  for some integers $m, n \geq 0$ with $m + n = d$
- covers essentially all applications appearing in the finance literature (except semi-definite matrix-valued)
- notation: $I = \{1, \dots, m\}$ and $J = \{m+1, \dots, m+n\}$
- notation: $\mu_M = (\mu_i)_{i \in M}$ and $\nu_{MN} = (\nu_{ij})_{i \in M, j \in N}$

Characterization

Theorem

The process X on the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$ is affine if and only if $a(x)$ and $b(x)$ are affine for admissible parameters a, α_i, b, β_i in the following sense:

$$\begin{aligned} a,\ \alpha_i \quad & \textit{are symmetric positive semi-definite,} \\ a_{II} = 0 \quad & (\textit{and thus } a_{IJ} = a_{JI}^\top = 0), \\ \alpha_j = 0 \quad & \textit{for all} \quad j \in J \\ \alpha_{i,kl} = \alpha_{i,lk} = 0 \quad & \textit{for } k \in I \setminus \{i\}, \textit{ for all } 1 \le i, l \le d, \\ b \in \mathbb{R}_+^m \times \mathbb{R}^n, & \\ \mathcal{B}_{IJ} = 0, & \\ \mathcal{B}_{II} \quad & \textit{has nonnegative off-diagonal elements.} \end{aligned} \tag{28}$$

Characterization cont'd

Theorem (cont'd)

In this case, the corresponding system of Riccati equations (26) *simplifies to*

$$
\begin{aligned}
\partial_t \phi(t,u) &= \frac{1}{2}\psi_J(t,u)^\top a_{JJ}\,\psi_J(t,u) + b^\top \psi(t,u)\\
\phi(0,u) &= 0\\
\partial_t \psi_i(t,u) &= \frac{1}{2}\psi(t,u)^\top \alpha_i\,\psi(t,u) + \beta_i^\top \psi(t,u), \quad i \in I,\\
\partial_t \psi_J(t,u) &= \mathcal{B}_{JJ}^\top \psi_J(t,u),\\
\psi(0,u) &= u,
\end{aligned}
\tag{29}
$$

and there exists a unique global solution $(\phi(\cdot,u),\psi(\cdot,u)) : \mathbb{R}_+ \to \mathbb{C}_- \times \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$ *for all initial values* $u \in \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$. *In particular, the equation for* ψ_J *forms an autonomous linear system with unique global solution* $\psi_J(t,u) = \mathrm{e}^{\mathcal{B}_{JJ}^\top t}\, u_J$ *for all* $u_J \in \mathbb{C}^n$.

Illustration $d = 3$, $m \leq 1$

- $m = 0$: $\alpha(x) \equiv a$ for some positive semi-definite symmetric 3×3-matrix a
- $m = 1$:

$$a = \begin{pmatrix} 0 & 0 & 0 \\ & + & * \\ & & + \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} + & * & * \\ & + & * \\ & & + \end{pmatrix}$$

Illustration $d = 3$, $m = 2$

$$a = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & + \end{pmatrix}$$

$$\alpha_1 = \begin{pmatrix} + & 0 & * \\ & 0 & 0 \\ & & + \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ & + & * \\ & & + \end{pmatrix}$$

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Illustration $d = 3$, $m = 3$

$$a = 0\,, \quad \alpha_1 = \begin{pmatrix} + & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ & + & 0 \\ & & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & + \end{pmatrix}$$

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Sketch of Proof: Necessity

- $a(x)$ symmetric positive semi-definite for all $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$ $\Leftrightarrow \alpha_j = 0$ for all $j \in J$, and a and α_i symmetric positive semi-definite for all $i \in I$.
- stochastic invariance: $\mathbb{R}^m_+ \times \mathbb{R}^n = \bigcap_{i \in I} \{x \mid \mathbb{R}^d \mid e_i^\top x \geq 0\}$

Lemma

$X^x(t) \in \mathbb{R}^m_+ \times \mathbb{R}^n$ for all $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$ only if for all boundary points x with $x_k = 0$ for some $k \in I$:

1. *diffusion parallel to boundary:* $e_k^\top a(x)\, e_k = 0$
2. *drift inward pointing:* $e_k^\top b(x) \geq 0$

$\Rightarrow$ implies admissibility conditions

Sketch of Proof: Sufficiency

- solution $\psi(t,u)$ of Riccati equations
  - is $\mathbb{C}^m_- \times \mathrm{i}\mathbb{R}^n$-valued
  - has life time $t_+(u) = \infty$

  for all $u \in \mathbb{C}^m_- \times \mathrm{i}\mathbb{R}^n$
- $\phi(t,u)$ is $\mathbb{C}_-$-valued

$\Rightarrow$ $\mathrm{Re}\,(\phi(t,u) + \psi(t,u)^\top x) \le 0$ for all $t \ge 0$, $u \in \mathrm{i}\mathbb{R}^d$ and $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$

Extension of Affine Transform Formula

notation: $\mathcal{S}(U) = \{z \in \mathbb{C}^k \mid \mathrm{Re}\, z \in U\}$

Theorem

Suppose X is affine. Let $\mathcal{D}_K$ ($K = \mathbb{R}$ or $\mathbb{C}$) denote the maximal domain for the system of Riccati equations, and let $\tau > 0$. Then:

1. $\mathcal{S}(\mathcal{D}_\mathbb{R}(\tau)) \subset \mathcal{D}_\mathbb{C}(\tau)$.
2. $\mathcal{D}_\mathbb{R}(\tau) = M(\tau)$ *where*

$$M(\tau) = \left\{ u \in \mathbb{R}^d \mid \mathbb{E}\left[\mathrm{e}^{u^\top X^x(\tau)} \right] < \infty \text{ for all } x \in \mathbb{R}^m_+ \times \mathbb{R}^n \right\}.$$

3. $\mathcal{D}_\mathbb{R}(\tau)$ *and* $\mathcal{D}_\mathbb{R}$ *are convex sets.*

Extension of Affine Transform Formula cont'd

Theorem (cont'd)

Moreover, for all $0 \le t \le T$ *and* $x \in \mathbb{R}_+^m \times \mathbb{R}^n$*:*

4. (25) *holds for all* $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T-t))$.
5. (25) *holds for all* $u \in \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$.
6. $M(t) \supseteq M(T)$.

Part 6 states: $\mathbb{E}\left[\mathrm{e}^{u^\top X^x(T)}\right] < \infty$ for all $x \in \mathbb{R}_+^m \times \mathbb{R}^n$, for some given T and $u \in \mathbb{R}^d$, implies $\mathbb{E}\left[\mathrm{e}^{u^\top X^x(t)}\right]$ for all $x \in \mathbb{R}_+^m \times \mathbb{R}^n$ and $t \le T$.

Proof.

Technical. See course book Section 10.7.3. □

Key Message

Corollary

Suppose that either side of (25) *is well defined for some* $t \leq T$ *and* $u \in \mathbb{R}^d$. *Then* (25) *holds, implying that both sides are well defined in particular, for* u *replaced by* $u + \mathrm{i}v$ *for any* $v \in \mathbb{R}^d$.

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Assumptions

- X affine on canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$
- pricing: interpret $\mathbb{P} = \mathbb{Q}$ as risk-neutral measure
- $W = W^*$ as $\mathbb{Q}$-Brownian motion
- note: affine property of X not preserved under equivalent change of measure in general, see e.g. Cheridito, Filipović and Yor [17].
- affine short-rate model: $r(t) = c + \gamma^\top X(t)$, for $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$
- special cases for $d = 1$: Vasiček and CIR short-rate models
- recall: affine term-structure model induces an affine short-rate model

General Problem

- consider T-claim with payoff $f(X(T))$ s.t. $\mathbb{E}\left[\mathrm{e}^{-\int_0^T r(s)\,ds}\,|f(X(T))|\right] < \infty$
- arbitrage price at $t \le T$: $\pi(t) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\,f(X(T)) \mid \mathcal{F}_t\right]$
- particular example: T-bond with $f \equiv 1$
- aim: derive analytic/numerically tractable pricing formula!
- first step: derive formula for $\mathcal{F}_t$-conditional characteristic function of $X(T)$ under the T-forward measure = (up to normalization with $P(t,T)$)

$$\mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\,\mathrm{e}^{u^\top X(T)} \mid \mathcal{F}_t\right]$$

for $u \in \mathrm{i}\mathbb{R}^d$

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T-forward Characteristic Function

Theorem

Let $\tau > 0$. The following statements are equivalent:

1. $\mathbb{E}\left[\mathrm{e}^{-\int_0^\tau r(s)\,ds}\right] < \infty$ *for all $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$ (e.g. $\gamma \in \mathbb{R}^m_+ \times \{0\}$)*
2. *There exists a unique solution $(\Phi(\cdot, u), \Psi(\cdot, u)) : [0, \tau] \to \mathbb{C} \times \mathbb{C}^d$ of*

$$
\begin{aligned}
\partial_t \Phi(t,u) &= \frac{1}{2}\Psi_J(t,u)^\top a_{JJ}\,\Psi_J(t,u) + b^\top \Psi(t,u) - c,\\
\Phi(0,u) &= 0,\\
\partial_t \Psi_i(t,u) &= \frac{1}{2}\Psi(t,u)^\top \alpha_i\,\Psi(t,u) + \beta_i^\top \Psi(t,u) - \gamma_i, \quad i \in I,\\
\partial_t \psi_J(t,u) &= \mathcal{B}_{JJ}^\top \Psi_J(t,u) - \gamma_J,\\
\Psi(0,u) &= u
\end{aligned}
\tag{30}
$$

for $u = 0$.

T-forward Characteristic Function cont'd

Theorem (cont'd)

Moreover, let $\mathcal{D}_K$ ($K = \mathbb{R}$ or $\mathbb{C}$) denote the maximal domain for the system of Riccati equations. If either of the above conditions holds then $\mathcal{D}_{\mathbb{R}}(S)$ is a convex open neighborhood of 0 in $\mathbb{R}^d$, and $\mathcal{S}(\mathcal{D}_{\mathbb{R}}(S)) \subset \mathcal{D}_{\mathbb{C}}(S)$, for all $S \le \tau$. Further:

$$\mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\,\mathrm{e}^{u^\top X(T)} \mid \mathcal{F}_t\right] = \mathrm{e}^{\Phi(T-t,u)+\Psi(T-t,u)^\top X(t)} \tag{31}$$

for all $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(S))$, $t \le T \le t+S$ and $x \in \mathbb{R}_+^m \times \mathbb{R}^n$.

Proof

Enlarge the state space: define

$$Y(t) = y + \int_0^t \left(c + \gamma^\top X(s) \right) ds, \quad y \in \mathbb{R}.$$

Then $X' = (X, Y)$ is an $\mathbb{R}_+^m \times \mathbb{R}^{n+1}$-valued diffusion process with diffusion matrix $a' + \sum_{i \in I} x_i \alpha_i'$ and drift $b' + \mathcal{B}' x'$ where

$$a' = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha_i' = \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix},$$

$$b' = \begin{pmatrix} b \\ c \end{pmatrix}, \quad \mathcal{B}' = \begin{pmatrix} \mathcal{B} & 0 \\ \gamma^\top & 0 \end{pmatrix}$$

form admissible parameters. We claim that X' is an affine process.

Proof cont'd

Indeed, the candidate Riccati equations read, for $i \in I$:

$$\begin{aligned}
\partial_t \phi'(t,u,v) &= \frac{1}{2}\psi'_J(t,u,v)^\top a_{JJ}\,\psi'_J(t,u,v) \\
&\quad + b^\top \psi'_{\{1,\dots,d\}}(t,u,v) + \boxed{cv}, \\
\phi'(0,u,v) &= 0, \\
\partial_t \psi'_i(t,u,v) &= \frac{1}{2}\psi'(t,u,v)^\top \alpha_i\,\psi'(t,u,v) \\
&\quad + \beta_i^\top \psi'_{\{1,\dots,d\}}(t,u,v) + \boxed{\gamma_i v}, \\
\partial_t \psi'_J(t,u,v) &= \mathcal{B}_{JJ}^\top \psi'_J(t,u,v) + \boxed{\gamma_J v}, \\
\partial_t \psi'_{d+1}(t,u,v) &= 0, \\
\psi'(0,u,v) &= \begin{pmatrix} u \\ v \end{pmatrix}.
\end{aligned} \tag{32}$$

Here we replaced the constant solution $\psi'_{d+1}(\cdot,u,v) \equiv v$ by v in the boxes.

Proof cont'd

Theorem 38.1 carries over: there exists a unique global $\mathbb{C}_- \times \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^{n+1}$-valued solution $(\phi'(\cdot, u, v), \psi'(\cdot, u, v))$ of (32) for all $(u, v) \in \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n \times \mathrm{i}\mathbb{R}$. The second part of Theorem 37.2 thus asserts that X' is affine with conditional characteristic function

$$\mathbb{E}\left[\mathrm{e}^{u^\top X(T)+vY(T)} \mid \mathcal{F}_t\right] \\ = \mathrm{e}^{\phi'(T-t,u,v)+\psi'_{\{1,\dots,d\}}(T-t,u,v)^\top X(t)+vY(t)}$$

for all $(u, v) \in \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n \times \mathrm{i}\mathbb{R}$ and $t \leq T$.

Proof cont'd

The theorem now follows from Theorem 38.3 once we set $\Phi(t,u) = \phi'(t,u,-1)$ and $\Psi(t,u) = \psi'_{\{1,\dots,d\}}(t,u,-1)$. Indeed, it is clear by inspection that $\mathcal{D}_K(S) = \{u \in K^d \mid (u,-1) \in \mathcal{D}'_K(S)\}$ where $\mathcal{D}'_K$ denotes the maximal domain for the system of Riccati equations (32).

Bond Price Formula

Corollary

For any maturity $T \leq \tau$, the T-bond price at $t \leq T$ is given as

$$P(t,T) = \mathrm{e}^{-A(T-t)-B(T-t)^\top X(t)}$$

where we define $A(t) = -\Phi(t,0)$, $B(t) = -\Psi(t,0)$. Moreover, for $t \leq T \leq S \leq \tau$, the $\mathcal{F}_t$-conditional characteristic function of $X(T)$ under the S-forward measure $\mathbb{Q}^S$ is given by

$$\mathbb{E}_{\mathbb{Q}^S}\left[\mathrm{e}^{u^\top X(T)} \mid \mathcal{F}_t\right] = \frac{\mathrm{e}^{-A(S-T)+\Phi(T-t,u-B(S-T))+\Psi(T-t,u-B(S-T))^\top X(t)}}{P(t,S)} \tag{33}$$

for all $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T) + B(S-T))$, which contains $\mathrm{i}\mathbb{R}^d$.

Proof

The bond price formula follows from the theorem with $u = 0$. Now let $t \leq T \leq S \leq \tau$. In view of the flow property $\Psi(T, -B(S-T)) = -B(S)$, we know that $-B(S-T) \in \mathcal{D}_{\mathbb{R}}(T)$, and thus $\mathcal{S}(\mathcal{D}_{\mathbb{R}}(T) + B(S-T))$ contains $\mathrm{i}\mathbb{R}^d$. Moreover, for $u \in \mathcal{S}(\mathcal{D}_{\mathbb{R}}(T) + B(S-T))$, we obtain from (31) by nested conditional expectation

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-\int_t^S r(s)\,ds}\mathrm{e}^{u^\top X(T)} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\,\mathbb{E}\left[\mathrm{e}^{-\int_T^S r(s)\,ds} \mid \mathcal{F}_T\right]\mathrm{e}^{u^\top X(T)} \mid \mathcal{F}_t\right] \\
&= \mathrm{e}^{-A(S-T)}\mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\mathrm{e}^{(u-B(S-T))^\top X(T)} \mid \mathcal{F}_t\right] \\
&= \mathrm{e}^{-A(S-T)+\Phi(T-t,u-B(S-T))+\Psi(T-t,u-B(S-T))^\top X(t)}.
\end{aligned}
$$

Normalizing by $P(t,S)$ yields the S-forward characteristic function.

Affine Pricing

- recall: $\pi(t) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds} f(X(T)) \mid \mathcal{F}_t\right]$
- either: we recognize $\mathcal{F}_t$-conditional distribution $Q(t, T, dx)$ of $X(T)$ under T-forward measure $\mathbb{Q}^T$ from its characteristic function above, then compute by (numerical) integration

$$\pi(t) = P(t, T) \int_{\mathbb{R}^d} f(x)\, Q(t, T, dx).$$

- or: employ Fourier transform ...

Affine Pricing Formula I

Theorem

Suppose either condition 1 or 2 of Theorem 39.1 is met for some $\tau \geq T$, and let $\mathcal{D}_{\mathbb{R}}$ denote the maximal domain for the system of Riccati equations (30). *Assume that f satisfies*

$$f(x) = \int_{\mathbb{R}^q} \mathrm{e}^{(v+\mathrm{i}L\lambda)^\top x} \widetilde{f}(\lambda)\, d\lambda, \quad dx\text{-a.s.}$$

for some $v \in \mathcal{D}_{\mathbb{R}}(T)$ and $d \times q$-matrix L, and some integrable function $\widetilde{f} : \mathbb{R}^q \to \mathbb{C}$, for a positive integer $q \leq d$. Then the price $\pi(t)$ is well defined and given by the formula

$$\pi(t) = \int_{\mathbb{R}^q} \mathrm{e}^{\Phi(T-t, v+\mathrm{i}L\lambda)+\Psi(T-t, v+\mathrm{i}L\lambda)^\top X(t)} \widetilde{f}(\lambda)\, d\lambda. \quad (34)$$

Proof

By assumption, we have

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^T r(s)\,ds}\,|f(X(T))|\right] \le \mathbb{E}\left[\int_{\mathbb{R}^q} \mathrm{e}^{-\int_0^T r(s)\,ds}\mathrm{e}^{v^\top X(T)}\,|\widetilde{f}(\lambda)|\,d\lambda\right] < \infty.$$

Hence we may apply Fubini's theorem to change the order of integration, which gives

$$\begin{aligned}\pi(t) &= \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\int_{\mathbb{R}^q} \mathrm{e}^{(v+\mathrm{i}L\lambda)^\top X(T)}\widetilde{f}(\lambda)\,d\lambda \mid \mathcal{F}_t\right] \\ &= \int_{\mathbb{R}^q} \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\,\mathrm{e}^{(v+\mathrm{i}L\lambda)^\top X(T)} \mid \mathcal{F}_t\right]\widetilde{f}(\lambda)\,d\lambda \\ &= \int_{\mathbb{R}^q} \mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v+\mathrm{i}L\lambda)^\top X(t)}\,\widetilde{f}(\lambda)\,d\lambda,\end{aligned}$$

as desired.

Affine Pricing Formula II

Theorem

Suppose either condition 1 or 2 of Theorem 39.1 is met for some $\tau \geq T$, and let $\mathcal{D}_{\mathbb{R}}$ denote the maximal domain for the system of Riccati equations (30). Assume that f is of the form

$$f(x) = \mathrm{e}^{v^\top x}\, h(L^\top x)$$

for some $v \in \mathcal{D}_{\mathbb{R}}(T)$ and $d \times q$-matrix L, and some integrable function $h : \mathbb{R}^q \to \mathbb{R}$, for a positive integer $q \leq d$. Define the bounded function

$$\widetilde{f}(\lambda) = \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}\lambda^\top y}\, h(y)\, dy, \quad \lambda \in \mathbb{R}^q.$$

1. *If $\widetilde{f}(\lambda)$ is an integrable function in $\lambda \in \mathbb{R}^q$ then the assumptions of Theorem 39.3 are met.*

Affine Pricing Formula II cont'd

Theorem (cont'd)

2. *If $v = Lw$, for some $w \in \mathbb{R}^q$, and $\mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v+\mathrm{i}L\lambda)^\top X(t)}$ is an integrable function in $\lambda \in \mathbb{R}^q$ then the $\mathcal{F}_t$-conditional distribution of the $\mathbb{R}^q$-valued random variable $Y = L^\top X(T)$ under the T-forward measure $\mathbb{Q}^T$ admits the continuous density function*

$$q(t,T,y) = \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q} \mathrm{e}^{-(w+\mathrm{i}\lambda)^\top y} \times \frac{\mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v+\mathrm{i}L\lambda)^\top X(t)}}{P(t,T)}\, d\lambda.$$

In either case, the integral in (34) is well defined and the price formula (34) holds.

Fourier Transform

For the proof, we recall the fundamental inversion formula from Fourier analysis ([51, Chapter I, Corollary 1.21]):

Lemma

Let $g : \mathbb{R}^q \to \mathbb{C}$ be an integrable function with integrable Fourier transform

$$\hat{g}(\lambda) = \int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}\lambda^\top y} g(y)\, dy.$$

Then the inversion formula

$$g(y) = \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}\lambda^\top y} \hat{g}(\lambda)\, d\lambda$$

holds for dy-almost all $y \in \mathbb{R}^q$.

Proof of 1

Under the assumption of 1, the Fourier inversion formula applied to $h(y)$ yields the representation (39.3). Hence Theorem 39.3 applies.

Proof of 2

We denote by $q(t, T, dy)$ the $\mathcal{F}_t$-conditional distribution of $Y = L^\top X(T)$ under the T-forward measure $\mathbb{Q}^T$. From (33) we infer the characteristic function of the bounded (why?) measure $\mathrm{e}^{w^\top y}\, q(t, T, dy)$:

$$\int_{\mathbb{R}^q} \mathrm{e}^{(w+\mathrm{i}\lambda)^\top y}\, q(t, T, dy) = \mathbb{E}\left[\mathrm{e}^{(w+\mathrm{i}\lambda)^\top L^\top X(T)} \mid \mathcal{F}_t\right]$$

$$= \frac{\mathrm{e}^{\Phi(T-t, v+\mathrm{i}L\lambda)+\Psi(T-t, v+\mathrm{i}L\lambda)^\top X(t)}}{P(t, T)}, \quad \lambda \in \mathbb{R}^q.$$

By assumption, this is an integrable function in λ on $\mathbb{R}^q$. The Fourier inversion formula thus applies and the injectivity of the characteristic function (see e.g. [53, Section 16.6]) yields that $q(t, T, dy)$ admits the continuous density function as stated.

Proof of 2 cont'd

Moreover, we then obtain

$$
\begin{aligned}
&P(t,T)\int_{\mathbb{R}^q} |\mathrm{e}^{w^\top y}\, h(y)|q(t,T,y)\,dy \\
&\le \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q}\int_{\mathbb{R}^q} |h(y)|\left|\mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v+\mathrm{i}L\lambda)^\top X(t)}\right| \,d\lambda\,dy <
\end{aligned}
$$

Hence again we can apply Fubini's theorem to change the order of integration, which gives

$$
\begin{aligned}
\pi(t) &= P(t,T)\int_{\mathbb{R}^q} \mathrm{e}^{w^\top y}\, h(y)\, q(t,T,y)\,dy \\
&= \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q}\int_{\mathbb{R}^q} \mathrm{e}^{w^\top y}\, h(y)\,\mathrm{e}^{-(w+\mathrm{i}\lambda)^\top y}\,\mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v} \\
&= \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q}\left(\int_{\mathbb{R}^q} h(y)\,\mathrm{e}^{-\mathrm{i}\lambda^\top y}\,dy\right)\,\mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v+\mathrm{i}L}
\end{aligned}
$$

which is (34).

Discussion

- integral in pricing formula

$$\pi(t) = \int_{\mathbb{R}^q} \mathrm{e}^{\Phi(T-t, v+\mathrm{i}L\lambda)+\Psi(T-t, v+\mathrm{i}L\lambda)^\top X(t)} \widetilde{f}(\lambda)\, d\lambda$$

  has to be computed numerically in general
- important: integral is over $\mathbb{R}^q$, where possibly $q \ll d$
- example bond options: $\widetilde{f}$ is in closed form and $q = 1$ (will see this below ...)

Discussion cont'd

reflection on affine pricing formula:

- payoff $f(X(T)) = \int_{\mathbb{R}^q} \mathrm{e}^{(v+\mathrm{i}L\lambda)^\top X(T)}\,\widetilde{f}(\lambda)\,d\lambda$ is decomposed into linear combination of (a continuum) of complex-valued basis "payoffs" $\mathrm{e}^{(v+\mathrm{i}L\lambda)^\top X(T)}$ with weights $\widetilde{f}(\lambda)$
- by nature of affine process X: these basis claims admit closed-form complex-valued "prices"

$$\pi_{v+\mathrm{i}L\lambda}(t) = \mathrm{e}^{\Phi(T-t,v+\mathrm{i}L\lambda)+\Psi(T-t,v+\mathrm{i}L\lambda)^\top X(t)}$$

- linearity of pricing: price of $f(X(T))$ is given as linear combination of $\pi_{v+\mathrm{i}L\lambda}(t)$ with same weights $\widetilde{f}(\lambda)$

Discussion cont'd

$\longrightarrow$ power of affine diffusion processes!

$\longrightarrow$ suggests to explore other types of diffusion processes that admit closed-form prices for some well specified basis of payoff functions (e.g. [15, 10]): open area of research

- examples of non-trivial Fourier decompositions? Yes! ...

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Call Option

Lemma

Let $K > 0$. For any $y \in \mathbb{R}$ the following identities hold:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{(w+\mathrm{i}\lambda)y}\,\frac{K^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)}\,d\lambda$$

$$= \begin{cases} (K-\mathrm{e}^{y})^{+} & \text{if } w < 0, \\ (\mathrm{e}^{y}-K)^{+} - \mathrm{e}^{y} & \text{if } 0 < w < 1, \\ (\mathrm{e}^{y}-K)^{+} & \text{if } w > 1. \end{cases}$$

The middle case ($0 < w < 1$) obviously also equals $(K-\mathrm{e}^{y})^{+} - K$.

Proof

Proof.

Let $w < 0$. Then the function $h(y) = \mathrm{e}^{-wy}(K - \mathrm{e}^{y})^{+}$ is integrable on $\mathbb{R}$. An easy calculation shows that its Fourier transform

$$\hat{h}(\lambda) = \int_{\mathbb{R}} \mathrm{e}^{-(w+\mathrm{i}\lambda)y}(K - \mathrm{e}^{y})^{+}\, dy = \frac{K^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)}$$

is also integrable on $\mathbb{R}$. Hence the Fourier inversion formula applies, and we conclude that the claimed identity holds for $w < 0$. The other cases follow by similar arguments ($\rightarrow$ exercise). $\square$

Exchange Option

choose $K = \mathrm{e}^z$ in above lemma:

Corollary

For any $y, z \in \mathbb{R}$ the following identities hold:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{(w+\mathrm{i}\lambda)y-(w-1+\mathrm{i}\lambda)z}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)}\,d\lambda = \begin{cases} (\mathrm{e}^y-\mathrm{e}^z)^+ & \text{if } w>1, \\ (\mathrm{e}^y-\mathrm{e}^z)^+-\mathrm{e}^y & \text{if } 0<w<1. \end{cases}$$

Exchange Option cont'd

- two asset prices $S_i = \mathrm{e}^{X_{m+i}}$, $i = 1, 2$
- exchange option (Magrabe option): option to exchange c_2 units of asset S_2 against c_1 units of asset S_1 at T
- payoff at T: $f(X(T)) = \left(c_1 \mathrm{e}^{X_{m+1}(T)} - c_2 \mathrm{e}^{X_{m+2}(T)}\right)^+$
- meets assumptions of affine pricing formula with $q = 1$, $v = w e_{m+1} + (1 - w) e_{m+2}$, $L = e_{m+1} - e_{m+2}$, and

$$\widetilde{f}(\lambda) = \frac{c_1^{w+\mathrm{i}\lambda} c_2^{-(w-1+\mathrm{i}\lambda)}}{2\pi(w + \mathrm{i}\lambda)(w - 1 + \mathrm{i}\lambda)},$$

for some $w > 1$

- remains to be checked from case to case: whether $v \in \mathcal{D}_{\mathbb{R}}(T)$

Spread Option

Hurd and Zhou [31]: double Fourier integral representation:

Lemma

Let $w = (w_1, w_2)^\top \in \mathbb{R}^2$ be such that $w_2 < 0$ and $w_1 + w_2 > 1$. Then for any $y = (y_1, y_2)^\top \in \mathbb{R}^2$ the following identity holds:

$$(\mathrm{e}^{y_1} - \mathrm{e}^{y_2} - 1)^+ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{e}^{(w+\mathrm{i}\lambda)^\top y} \times \frac{\Gamma(w_1 + w_2 - 1 + \mathrm{i}(\lambda_1 + \lambda_2))\,\Gamma(-w_2 - \mathrm{i}\lambda_2)}{\Gamma(w_1 + 1 + \mathrm{i}\lambda_1)}\, d\lambda_1\, d\lambda_2,$$

where the Gamma function $\Gamma(z) = \int_0^\infty t^{-1+z}\,\mathrm{e}^{-t}\,dt$ is defined for all complex z with $\mathrm{Re}\,(z) > 0$.

Proof.

Technical. See course book Section 10.3. $\square$

Spread Option

- two asset prices $S_i = \mathrm{e}^{X_{m+i}}$, $i = 1, 2$
- spread option, strike price $K > 0$, payoff at T:

$$f(X(T)) = \left(\mathrm{e}^{X_{m+1}(T)} - \mathrm{e}^{X_{m+2}(T)} - K\right)^+$$

- meets assumptions of affine pricing formula with $q = 2$, $v = w_1 e_{m+1} + w_2 e_{m+2}$, $L = (e_{m+1}, e_{m+2})$, and

$$\widetilde{f}(\lambda) = \frac{\Gamma(w_1 + w_2 - 1 + \mathrm{i}(\lambda_1 + \lambda_2))\,\Gamma(-w_2 - \mathrm{i}\lambda_2)}{(2\pi)^2 K^{w_1+w_2+\mathrm{i}(\lambda_1+\lambda_2)}\Gamma(w_1 + 1 + \mathrm{i}\lambda_1)},$$

for some $w_2 < 0$ and $w_1 > 1 - w_2$

- remains to be checked from case to case: whether $v \in \mathcal{D}_{\mathbb{R}}(T)$

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Assumptions

- assume: either condition 1 or 2 of Theorem 39.1 is met
- European call option on S-bond, expiry date $T < S \leq \tau$, strike price K
- call option lemma $\Rightarrow$ payoff function:

$$\left(\mathrm{e}^{-A(S-T)-B(S-t)^\top x} - K\right)^+ = \int_{\mathbb{R}} \mathrm{e}^{-(w+\mathrm{i}\lambda)B(S-t)^\top x}\, \widetilde{f}(w,\lambda)\, d\lambda$$

where we define, for any real $w > 1$,

$$\widetilde{f}(w,\lambda) = \frac{1}{2\pi}\, \mathrm{e}^{-(w+\mathrm{i}\lambda)A(S-T)}\, \frac{K^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)},$$

- similar formula for put options (...)

Affine Bond Option Pricing Formula

Corollary

There exists some $w_- < 0$ and $w_+ > 1$ such that $-B(S-T)w \in \mathcal{D}_{\mathbb{R}}(T)$ for all $w \in (w_-, w_+)$, and the line integral

$$\Pi(w,t) = \int_{\mathbb{R}} \mathrm{e}^{\Phi(T-t,-(w+\mathrm{i}\lambda)B(S-T))} \times \mathrm{e}^{\Psi(T-t,-(w+\mathrm{i}\lambda)B(S-T))^\top X(t)} \widetilde{f}(w,\lambda)\, d\lambda$$

is well defined for all $w \in (w_-, w_+) \setminus \{0,1\}$ and $t \leq T$. Moreover, the time t prices of the European call and put option on the S-bond with expiry date T and strike price K are given by any of the following identities: ...

Affine Bond Option Pricing Formula cont'd

Corollary (cont'd)

$$
\begin{aligned}
\pi_{call}(t) &= \begin{cases} \Pi(w,t), & \text{if } w \in (1, w_+), \\ \Pi(w,t) + P(t,S), & \text{if } w \in (0,1), \end{cases} \\
&= P(t,S)q(t,S,\mathcal{I}) - KP(t,T)q(t,T,\mathcal{I}) \\
\pi_{put}(t) &= \begin{cases} \Pi(w,t) + KP(t,T), & \text{if } w \in (0,1), \\ \Pi(w,t), & \text{if } w \in (w_-, 0), \end{cases} \\
&= KP(t,T)q(t,T,\mathbb{R}\setminus\mathcal{I}) - P(t,S)q(t,S,\mathbb{R}\setminus\mathcal{I})
\end{aligned}
$$

where $\mathcal{I} = (A(S-T) + \log K, \infty)$, *and* $q(t,S,dy)$ *and* $q(t,T,dy)$ *denote the* $\mathcal{F}_t$*-conditional distributions of the real-valued random variable* $Y = -B(S-T)^\top X(T)$ *under the* S*- and* T*-forward measure, respectively.*

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Proof

Proof.

See course book Section 10.3.2 □

Consequences

- The pricing of European call and put bond options in the present d-dimensional affine factor model boils down to the computation of a line integral $\Pi(w, t)$, which is a simple numerical task!
- Moreover, in case the distributions $q(t, S, dy)$ and $q(t, T, dy)$ are explicitly known, the pricing is reduced to the computation of the respective probabilities of the exercise events $\mathcal{I}$ and $\mathbb{R} \setminus \mathcal{I}$
- In the following two subsections, we illustrate this approach for the Vasiček and CIR short-rate models.

Example: Vasiček Model

- recall: $dr = (b + \beta r)\, dt + \sigma\, dW$
- state space: $\mathbb{R}$, and we set $r = X$

$\Rightarrow$ Riccati equations:

$$\Phi(t,u) = \frac{1}{2}\sigma^2 \int_0^t \Psi^2(s,u)\, ds + b \int_0^t \Psi(s,u)\, ds$$

$$\partial_t \Psi(t,u) = \beta \Psi(t,u) - 1,$$

$$\Psi(0,u) = u$$

- for all $u \in \mathbb{C}$ $\exists$ unique global solution:

$$\Phi(t,u) = \frac{1}{2}\sigma^2 \left(\frac{u^2}{2\beta}(\mathrm{e}^{2\beta t} - 1) + \frac{1}{2\beta^3}(\mathrm{e}^{2\beta t} - 4\mathrm{e}^{\beta t} + 2\beta t + 3) - \frac{u}{\beta^2}(\mathrm{e}^{2\beta t} - 2\mathrm{e}^{\beta t} + 2\beta) \right) + b \left(\frac{\mathrm{e}^{\beta t} - 1}{\beta} u - \frac{\mathrm{e}^{\beta t} - 1 - \beta t}{\beta^2} \right)$$

$$\Psi(t,u) = \mathrm{e}^{\beta t} u - \frac{\mathrm{e}^{\beta t} - 1}{\beta}$$

Example: Vasiček Model cont'd

$\Rightarrow$ affine S-forward characteristic function formula holds for all $u \in \mathbb{C}$ and $t \leq T$:

$$\mathbb{E}_{\mathbb{Q}^S}\left[\mathrm{e}^{ur(T)} \mid \mathcal{F}_t\right] = \exp\left(\frac{1}{2}\sigma^2 \frac{\mathrm{e}^{2\beta(T-t)} - 1}{2\beta} u^2 + \text{low order } u\text{-terms}\right)$$

$\Rightarrow$ under S-forward measure, $r(T)$ is $\mathcal{F}_t$-conditionally Gaussian distributed with variance $\sigma^2 \frac{\mathrm{e}^{2\beta(T-t)}-1}{2\beta}$

- in line with earlier findings!
- straightforward calculation: $\mathcal{F}_t$-conditional $\mathbb{Q}^S$-mean of $r(T)$ $(\dots)$
- re-derive bond option price formula for the Vasiček model from Gaussian HJM formula above ($\rightarrow$ exercise)

Example: CIR Model

- recall: $dr = (b + \beta r)\,dt + \sigma\sqrt{r}\,dW$
- state space: $\mathbb{R}_+$, and we set $r = X$

$\Rightarrow$ Riccati equations:

$$\Phi(t,u) = b\int_0^t \Psi(s,u)\,ds,$$

$$\partial_t \Psi(t,u) = \frac{1}{2}\sigma^2\Psi^2(t,u) + \beta\Psi(t,u) - 1,$$

$$\Psi(0,u) = u.$$

Example: CIR Model cont'd

- for all $u \in \mathbb{C}_-$ $\exists$ unique global $\mathbb{C}_-$-valued solution:

$$\Phi(t,u) = \frac{2b}{\sigma^2} \log\left(\frac{2\theta \mathrm{e}^{\frac{(\theta-\beta)t}{2}}}{L_3(t) - L_4(t)u}\right)$$

$$\Psi(t,u) = -\frac{L_1(t) - L_2(t)u}{L_3(t) - L_4(t)u}$$

where $\theta = \sqrt{\beta^2 + 2\sigma^2}$ and

$$L_1(t) = 2\left(\mathrm{e}^{\theta t} - 1\right)$$

$$L_2(t) = \theta\left(\mathrm{e}^{\theta t} + 1\right) + \beta\left(\mathrm{e}^{\theta t} - 1\right)$$

$$L_3(t) = \theta\left(\mathrm{e}^{\theta t} + 1\right) - \beta\left(\mathrm{e}^{\theta t} - 1\right)$$

$$L_4(t) = \sigma^2\left(\mathrm{e}^{\theta t} - 1\right).$$

Example: CIR Model cont'd

- elementary algebraic manipulations $(\dots) \Rightarrow S$-forward characteristic function of $r(T)$:

$$\mathbb{E}_{\mathbb{Q}^S}\left[\mathrm{e}^{ur(T)} \mid \mathcal{F}_t\right] = \frac{\mathrm{e}^{\frac{C_2(t,T,S)r(t)C_1(t,T,S)u}{1-C_1(t,T,S)u}}}{(1 - C_1(t,T,S)u)^{\frac{2b}{\sigma^2}}}$$

where

$$C_1(t,T,S) = \frac{L_3(S-T)L_4(T-t)}{2\theta L_3(S-t)}$$

$$C_2(t,T,S) = \frac{L_2(T-t)}{L_4(T-t)} - \frac{L_1(S-t)}{L_3(S-t)}$$

$\Rightarrow$ $\mathcal{F}_t$-conditional distribution of $Z(t,T) = \frac{2r(T)}{C_1(t,T,S)}$ under $\mathbb{Q}^S$ is noncentral χ^2 with $\frac{4b}{\sigma^2}$ degrees of freedom and parameter of noncentrality $2C_2(t,T,S)r(t)$

- see next lemma ...

Noncentral χ^2-Distribution

Lemma (Noncentral χ^2-Distribution)

The noncentral χ^2-distribution with $\delta > 0$ degrees of freedom and noncentrality parameter $\zeta > 0$ has density function

$$f_{\chi^2(\delta,\zeta)}(x) = \frac{1}{2} \mathrm{e}^{-\frac{x+\zeta}{2}} \left(\frac{x}{\zeta}\right)^{\frac{\delta}{4}-\frac{1}{2}} I_{\frac{\delta}{2}-1}(\sqrt{\zeta x}), \quad x \geq 0$$

and characteristic function

$$\int_{\mathbb{R}_+} \mathrm{e}^{ux} f_{\chi^2(\delta,\zeta)}(x)\, dx = \frac{\mathrm{e}^{\frac{\zeta u}{1-2u}}}{(1-2u)^{\frac{\delta}{2}}}, \quad u \in \mathbb{C}_-.$$

Here $I_\nu(x) = \sum_{j\geq 0} \frac{1}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j+\nu}$ denotes the modified Bessel function of the first kind of order $\nu > -1$.

Proof.

See e.g. [33, Chapter 29]. □

Noncentral χ^2-Distribution cont'd

Noncentral χ^2 = generalization of distribution of the sum of the squares of independent normal distributed random variables:

- fix $\delta \in \mathbb{N}$, reals $\nu_1, \dots, \nu_\delta \in \mathbb{R}$, and define $\zeta = \sum_{i=1}^{\delta} \nu_i^2$
- let $N_1, \dots, N_\delta$ be independent standard normal distributed rvs
- define $Z = \sum_{i=1}^{\delta} (N_i + \nu_i)^2$
- direct integration shows: characteristic function of Z equals

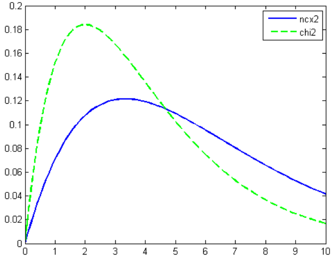
$$\mathbb{E}[\mathrm{e}^{uZ}] = \frac{\mathrm{e}^{\frac{\zeta u}{1-2u}}}{(1-2u)^{\frac{\delta}{2}}}, \quad u \in \mathbb{C}_-$$

- above lemma $\Rightarrow$ Z noncentral χ^2-distributed with δ degrees of freedom and noncentrality parameter ζ
- good to know: noncentral χ^2-distribution hard coded in most statistical software packages

$\Rightarrow$ explicit bond option price formulas for the CIR model!

Noncentral χ^2-Distribution cont'd

```
x = (0:0.1:10)';
ncx2 = ncx2pdf(x,4,2);
chi2 = chi2pdf(x,4);
plot(x,ncx2,'b-','LineWidth',2)
hold on
plot(x,chi2,'g--','LineWidth',2)
legend('ncx2','chi2')
```



Example: CIR Model cont'd

- numerical example: $\sigma^2 = 0.033$, $b = 0.08$, $\beta = -0.9$, $r_0 = 0.08$
- set $t = 0$, $T = 1$ and $S = 2$
- numerical integration shows: line integral $\Pi(w, 0)$ in above bond option pricing corollary behaves numerically stable for w ranging between $(-1, 2) \setminus \{0, 1\}$ (see figure).
- on the other hand, we know that $\Pi(w, 0)$ diverges for $w \to -\infty$ (why?)

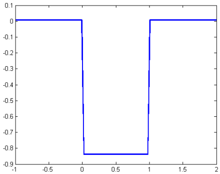


Figure: Line integral $\Pi(w, 0)$ as a function of w.

Example: CIR Model cont'd

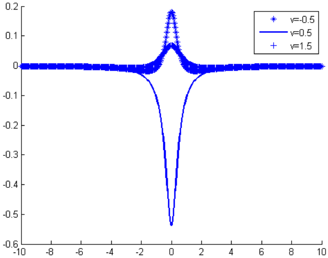


Figure: Real part of the integrand of $\Pi(w, 0)$, for $w = -0.5$, 0.5, 1.5, as a function of λ.

Example: CIR Model cont'd

- the resulting ATM call and put option strike price is $K = 0.9180$
- the call and put option price $\pi_{call}(0) = \pi_{put}(0) = 0.0078$ can now be computed by any of the formulas in the bond option pricing corollary ($\rightarrow$ exercise)

Example: CIR Model cont'd

- application: compute ATM cap prices and implied Black volatilities ($\rightarrow$ exercise)
- tenor: $t = 0$ (today), $T_0 = 1/4$ (first reset date), and $T_i - T_{i-1} \equiv 1/4$, $i = 1, \dots, 119$ (the maturity of the last cap is $T_{119} = 30$)
- the following table and figure show the ATM cap prices and implied Black volatilities for a range of maturities
- like the Vasiček model, the CIR model seems incapable of producing humped volatility curves

Example: CIR Model cont'd

Table: CIR ATM cap prices and Black volatilities

Maturity	ATM prices	ATM vols
1	0.0073	0.4506
2	0.0190	0.3720
3	0.0302	0.3226
4	0.0406	0.2890
5	0.0501	0.2647
6	0.0588	0.2462
7	0.0668	0.2316
8	0.0742	0.2198
10	0.0871	0.2017
12	0.0979	0.1886
15	0.1110	0.1744
20	0.1265	0.1594
30	0.1430	0.1442

Example: CIR Model cont'd

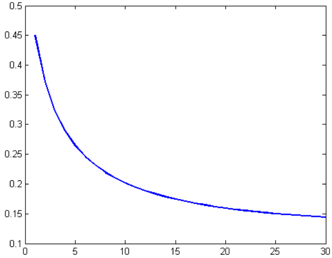


Figure: CIR ATM cap Black volatilities.

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Heston Model

- Heston [28]: generalizes Black–Scholes model by assuming stochastic volatility
- interest rates: constant $r(t) \equiv r \geq 0$
- one risky asset (stock) $S = \mathrm{e}^{X_2}$
- $X = (X_1, X_2)$: affine process on $\mathbb{R}_+ \times \mathbb{R}$:

$$dX_1 = (k + \kappa X_1)\, dt + \sigma \sqrt{2X_1}\, dW_1$$
$$dX_2 = (r - X_1)\, dt + \sqrt{2X_1} \left(\rho\, dW_1 + \sqrt{1-\rho^2} dW_2 \right)$$

for $k, \sigma \geq 0$, $\kappa \in \mathbb{R}$, $\rho \in [-1, 1]$

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Heston Model cont'd

- implied risk-neutral stock dynamics:

$$dS = Sr\,dt + S\sqrt{2X_1}\,d\mathcal{W}$$

for Brownian motion $\mathcal{W} = \rho W_1 + \sqrt{1-\rho^2}\,W_2$

$\Rightarrow$ $\sqrt{2X_1}$ = stochastic volatility of the price process S

- possibly non-zero covariation $d\langle S, X_1\rangle = 2\rho\sigma S X_1\,dt$

Heston Model cont'd

- system of Riccati equations for X equivalent to

$$
\begin{aligned}
\phi(t,u) &= k\int_0^t \psi_1(s,u)\,ds + ru_2 t\\
\partial_t\psi_1(t,u) &= \sigma^2\psi_1^2(t,u) + (2\rho\sigma u_2+\kappa)\psi_1(t,u) + u_2^2 - u_2\\
\psi_1(0,u) &= u_1\\
\psi_2(t,u) &= u_2,
\end{aligned}
$$

- Riccati lemma below: $\exists$ explicit global solution ϕ,ψ if $u_1\in\mathbb{C}_-$ and $0\le \mathrm{Re}\,u_2\le 1$
- in particular $u=(0,1)$: $\phi(t,0,1)=rt$, $\psi(t,0,1)=(0,1)^\top$
- Theorem 38.3 $\Rightarrow$ $\mathbb{E}[S(T)]<\infty$ and (EMM):

$$
\begin{aligned}
\mathbb{E}[\mathrm{e}^{-rT}S(T)\mid\mathcal{F}_t] &= \mathrm{e}^{-rT}\mathbb{E}[\mathrm{e}^{X_2(T)}\mid\mathcal{F}_t]\\
&= \mathrm{e}^{-rT}\mathrm{e}^{r(T-t)+X_2(t)} = \mathrm{e}^{-rt}S(t)
\end{aligned}
$$

Term-Structure Models

Damir Filipović

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Riccati Lemma

Lemma

Consider the Riccati differential equation

$$\partial_t G = AG^2 + BG - C, \quad G(0,u) = u, \tag{35}$$

where $A, B, C \in \mathbb{C}$ and $u \in \mathbb{C}$, with $A \neq 0$ and $B^2 + 4AC \in \mathbb{C} \setminus \mathbb{R}_-$. Let $\sqrt{\cdot}$ denote the analytic extension of the real square root to $\mathbb{C} \setminus \mathbb{R}_-$, and define $\theta = \sqrt{B^2 + 4AC}$.

1. *The function*

$$G(t,u) = -\frac{2C\left(e^{\theta t} - 1\right) - \left(\theta\left(e^{\theta t} + 1\right) + B\left(e^{\theta t} - 1\right)\right)u}{\theta\left(e^{\theta t} + 1\right) - B\left(e^{\theta t} - 1\right) - 2A\left(e^{\theta t} - 1\right)u}$$

   is the unique solution of equation (35) on its maximal interval of existence $[0, t_+(u))$. Moreover,

$$\int_0^t G(s,u)ds = \frac{1}{A}\log\left(\frac{2\theta e^{\frac{\theta - B}{2}t}}{\theta(e^{\theta t} + 1) - B(e^{\theta t} - 1) - 2A(e^{\theta t} - 1)u}\right).$$

2. *If, moreover, $A > 0$, $B \in \mathbb{R}$, $\mathrm{Re}(C) \geq 0$ and $u \in \mathbb{C}_-$ then $t_+(u) = \infty$ and $G(t,u)$ is $\mathbb{C}_-$-valued.*

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Heston Model cont'd

- European call option, maturity T, strike price K, price:

$$\pi(t) = \mathrm{e}^{-r(T-t)}\mathbb{E}\left[(S(T)-K)^+ \mid \mathcal{F}_t\right]$$

- fix $w > 1$ small enough with $(0, w) \in \mathcal{D}_{\mathbb{R}}(T)$ ($\rightarrow$ exercise):

$$\pi(t) = \mathrm{e}^{-r(T-t)} \int_{\mathbb{R}} \mathrm{e}^{\phi(T-t,0,w+\mathrm{i}\lambda)+\psi_1(T-t,0,w+\mathrm{i}\lambda)X_1(t)} \times \mathrm{e}^{(w+\mathrm{i}\lambda)X_2(t)}\,\widetilde{f}(\lambda)\,d\lambda$$

with $\widetilde{f}(\lambda) = \frac{1}{2\pi}\frac{K^{-(w-1+\mathrm{i}\lambda)}}{(w+\mathrm{i}\lambda)(w-1+\mathrm{i}\lambda)}$

- alternatively: fix $0 < w < 1$ and then

$$\pi(t) = S(t) + \mathrm{e}^{-r(T-t)} \int_{\mathbb{R}} \mathrm{e}^{\phi(T-t,0,w+\mathrm{i}\lambda)+\psi_1(T-t,0,w+\mathrm{i}\lambda)X_1(t)} \times \mathrm{e}^{(w+\mathrm{i}\lambda)X_2(t)}\,\widetilde{f}(\lambda)\,d\lambda$$

Heston Model cont'd

numerical example: $X_1(0) = 0.02$, $X_2(0) = 0$, $\sigma = 0.1$, $\kappa = -2.0$, $k = 0.02$, $r = 0.01$, $\rho = 0.5$

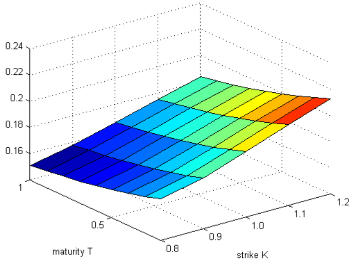

Figure: Implied volatility surface for the Heston model.

Heston Model cont'd

Table: Call option prices in the Heston model

T–K	0.8	0.9	1.0	1.1	1.2
0.2	0.2016	0.1049	0.0348	0.0074	0.0012
0.4	0.2037	0.1120	0.0478	0.0168	0.0053
0.6	0.2061	0.1183	0.0571	0.0245	0.0100
0.8	0.2088	0.1239	0.0646	0.0310	0.0144
1.0	0.2115	0.1291	0.0711	0.0368	0.0186

Heston Model cont'd

Table: Black–Scholes implied volatilities for the call option prices in the Heston model

T–K	0.8	0.9	1.0	1.1	1.2
0.2	0.1715	0.1786	0.1899	0.2017	0.2126
0.4	0.1641	0.1712	0.1818	0.1930	0.2033
0.6	0.1585	0.1656	0.1755	0.1858	0.1954
0.8	0.1544	0.1612	0.1704	0.1799	0.1889
1.0	0.1513	0.1579	0.1664	0.1751	0.1835

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Assumption

- let X be affine on the canonical state space $\mathbb{R}^m_+ \times \mathbb{R}^n$ with admissible parameters a, α_i, b, β_i

$\Rightarrow$ for any $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$ the process $X = X^x$ satisfies

$$dX = (b + \mathcal{B}X)\,dt + \rho(X)\,dW$$
$$X(0) = x,$$

with $\rho(x)\rho(x)^\top = a + \sum_{i \in I} x_i \alpha_i$

Affine Transformations

- let Λ be an invertible $d \times d$-matrix

$\Rightarrow$ the linear transform $Y = \Lambda X$ satisfies

$$dY = \left(\Lambda b + \Lambda \mathcal{B} \Lambda^{-1} Y\right) dt + \Lambda \rho \left(\Lambda^{-1} Y\right) dW$$
$$Y(0) = \Lambda x$$

- hence Y has affine drift $\Lambda b + \Lambda \mathcal{B} \Lambda^{-1} y$ and diffusion matrix $\Lambda \alpha(\Lambda^{-1} y)\Lambda^{\top}$

Affine Transformations cont'd

- affine short-rate model $r(t) = c + \gamma^\top X(t)$ can be expressed in terms of $Y(t)$:

$$r(t) = c + \gamma^\top \Lambda^{-1} Y(t)$$

$\Rightarrow$ Y specifies an affine short-rate model producing the same short rates (and bond prices etc.) as X

- identification problem: $\exists$? unique canonical representation for affine short-rate models with the same observable implications?
- literature: [20, 18, 34, 16] and others
- will show: diffusion matrix $\alpha(x)$ can always be brought into block-diagonal form . . .

Affine Transformations cont'd

Lemma

There exists some invertible $d \times d$-matrix Λ with $\Lambda(\mathbb{R}_+^m \times \mathbb{R}^n) = \mathbb{R}_+^m \times \mathbb{R}^n$ such that $\Lambda\alpha(\Lambda^{-1}y)\Lambda^\top$ is block-diagonal of the form

$$\Lambda\alpha(\Lambda^{-1}y)\Lambda^\top = \begin{pmatrix} \mathrm{diag}(y_1, \dots, y_q, 0, \dots, 0) & 0 \\ 0 & p + \sum_{i \in I} y_i \pi_i \end{pmatrix}$$

for some integer $0 \le q \le m$ and symmetric positive semi-definite $n \times n$ matrices $p, \pi_1, \dots, \pi_m$. Moreover, Λb and $\Lambda \mathcal{B} \Lambda^{-1}$ meet the admissibility conditions in lieu of b and $\mathcal{B}$.

Proof

$\Lambda\alpha(x)\Lambda^\top$ is block-diagonal for all $x = \Lambda^{-1}y$ if and only if $\Lambda a \Lambda^\top$ and $\Lambda\alpha_i\Lambda^\top$ are block-diagonal for all $i \in I$.

Define the invertible $d \times d$-matrix

$$\Lambda = \begin{pmatrix} I_m & 0 \\ D & I_n \end{pmatrix} \tag{36}$$

where the $n \times m$-matrix $D = (\delta_1, \dots, \delta_m)$ has ith column vector

$$\delta_i = \begin{cases} -\frac{\alpha_{i,iJ}}{\alpha_{i,ii}}, & \text{if } \alpha_{i,ii} > 0 \\ 0, & \text{else.} \end{cases}$$

Then $\Lambda(\mathbb{R}^m_+ \times \mathbb{R}^n) = \mathbb{R}^m_+ \times \mathbb{R}^n$ and

$$D\alpha_{i,II} = -\alpha_{i,JI}, \quad i \in I.$$

Proof cont'd

From here we easily verify that

$$\Lambda\alpha_i = \begin{pmatrix} \alpha_{i,II} & \alpha_{i,IJ} \\ 0 & D\alpha_{i,IJ} + \alpha_{i,JJ} \end{pmatrix},$$

and thus

$$\Lambda\alpha_i\Lambda^\top = \begin{pmatrix} \alpha_{i,II} & 0 \\ 0 & D\alpha_{i,IJ} + \alpha_{i,JJ} \end{pmatrix}.$$

Since $\Lambda a \Lambda^\top = a$, the first assertion is proved (modulo permutation of first m indices).
The admissibility conditions for Λb and $\Lambda\mathcal{B}\Lambda^{-1}$ can easily be checked as well.

Canonical Representation

Combining the preceding findings:

Theorem (Canonical Representation)

Any affine short-rate model $r(t) = c + \gamma^\top X(t)$, after some modification of γ if necessary, admits an $\mathbb{R}_+^m \times \mathbb{R}^n$-valued affine state process X with block-diagonal diffusion matrix of the form

$$\alpha(x) = \begin{pmatrix} \mathrm{diag}(x_1, \dots, x_q, 0, \dots, 0) & 0 \\ 0 & a + \sum_{i \in I} x_i \alpha_{i,JJ} \end{pmatrix}$$

for some integer $0 \le q \le m$.

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Existence Problem

- Standing assumption always was existence and uniqueness for
$$dX = (b + \mathcal{B}X)\,dt + \rho(X)\,dW$$
$$X(0) = x$$
- Problem: $\rho(x)\rho(x)^\top = a + \sum_{i\in I} x_i\alpha_i$. Hence $\rho(x)$ is not Lipschitz in general.
- Question: exists a solution at all?
- Recall affine characterization theorem 37.2: admissibility conditions on a, α_i, but not on $\rho(x)$
- Indeed: can replace $\rho(x)$ by $\rho(x)D$ since $\rho(x)DD^\top\rho(x)^\top = \rho(x)\rho(x)^\top$ for any orthogonal $d\times d$-matrix D

Law of X

- Theorem 38.1: ϕ and ψ in affine transform formula uniquely determined by admissible parameters a, α_i, b, β_i as solution of the Riccati equations
- ϕ and ψ uniquely determine the law of the process X by iteration:

$$\mathbb{E}\left[\mathrm{e}^{u_1^\top X(t_1)+u_2^\top X(t_2)}\right] = \mathbb{E}\left[\mathrm{e}^{u_1^\top X(t_1)}\mathbb{E}\left[\mathrm{e}^{u_2^\top X(t_2)} \mid \mathcal{F}_{t_1}\right]\right]$$

$$= \mathbb{E}\left[\mathrm{e}^{u_1^\top X(t_1)}\mathrm{e}^{\phi(t_2-t_1,u_2)+\psi(t_2-t_1,u_2)^\top X(t_1)}\right]$$

$$= \mathrm{e}^{\phi(t_2-t_1,u_2)+\phi(t_1,u_1+\psi(t_2-t_1,u_2))+\psi(t_1,u_1+\psi(t_2-t_1,u_2))^\top x}$$

$\Rightarrow$ law of X: uniquely determined by a, α_i, b, β_i but can be realized by infinitely many variants of the above SDE

- Need canonical choice of $\rho(x)$ ...

Canonical Affine SDE

- Recall canonical representation: block-diagonal diffusion after linear transformation $X \mapsto \Lambda X$:

$$\rho(x)\rho(x)^\top = \begin{pmatrix} \mathrm{diag}(x_1,\dots,x_q,0,\dots,0) & 0 \\ 0 & a+\sum_{i\in I} x_i\alpha_{i,JJ} \end{pmatrix}$$

- Obvious: $\rho(x) = \rho(x_I)$
- Set $\rho_{IJ}(x) \equiv 0$, $\rho_{JI}(x) \equiv 0$, and

$$\rho_{II}(x_I) = \mathrm{diag}(\sqrt{x_1},\dots,\sqrt{x_q},0,\dots,0)$$

- Determine $\rho_{JJ}(x_I)$ via Cholesky factorization of $a+\sum_{i\in I} x_i\alpha_{i,JJ}$ = unique lower triangular matrix with positive diagonal elements and

$$\rho_{JJ}(x_I)\rho_{JJ}(x_I)^\top = a + \sum_{i\in I} x_i\alpha_{i,JJ}$$

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Canonical Affine SDE cont'd

- Affine stochastic differential equation now reads

$$\begin{aligned} dX_I &= (b_I + \mathcal{B}_{II} X_I)\,dt + \rho_{II}(X_I)\,dW_I \\ dX_J &= (b_J + \mathcal{B}_{JI} X_I + \mathcal{B}_{JJ} X_J)\,dt + \rho_{JJ}(X_I)\,dW_J \\ X(0) &= x \end{aligned} \tag{37}$$

- Yamada–Watanabe (1971): $\exists$ unique $\mathbb{R}^m_+ \times \mathbb{R}^n$-valued solution $X = X^x$, for any $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$

Summary

We thus have shown:

Theorem

Let a, α_i, b, β_i be admissible parameters. Then there exists a measurable function $\rho : \mathbb{R}_+^m \times \mathbb{R}^n \to \mathbb{R}^{d\times d}$ with $\rho(x)\rho(x)^\top = a + \sum_{i\in I} x_i\alpha_i$, and such that, for any $x \in \mathbb{R}_+^m \times \mathbb{R}^n$, there exists a unique $\mathbb{R}_+^m \times \mathbb{R}^n$-valued solution $X = X^x$ of the above affine SDE.

Moreover, the law of X is uniquely determined by a, α_i, b, β_i, and does not depend on the particular choice of ρ.

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Heuristic Derivation From HJM

LIBOR Market Model

LIBOR Dynamics Under Different Measures

Implied Bond Market

Implied Money-Market Account

Swaption Pricing

Forward Swap Measure

Analytic Approximations

Monte Carlo Simulation of the LIBOR Market Model

Volatility Structure and Calibration

Part X

Market Models

Overview

- Instantaneous forward rates: not easy to estimate
- Directly model observables such as LIBOR
- Breakthroughs 1997:
  - Miltersen et al. [39], Brace et al. [11]: HJM-type lognormal LIBOR rate model
  - Jamshidian [32]: framework for arbitrage-free LIBOR and swap rate models not based on HJM
- Principal idea: choose other numeraire than risk-free account
- Both approaches lead to Black's formula for either caps (LIBOR models) or swaptions (swap rate models)
- Because of this they are usually referred to as "market models"

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- LIBOR Dynamics Under Different Measures

45 Implied Bond Market

46 Implied Money-Market Account

47 Swaption Pricing
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Heuristic Derivation

- Point of departure: HJM setup from above
- Recall: forward δ-period LIBOR

$$L(t,T) = F(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)$$

- Seen in Chapter "Forward Measures": $P(t,T)/P(t,T+\delta)$ is a $\mathbb{Q}^{T+\delta}$-martingale:

$$d\left(\frac{P(t,T)}{P(t,T+\delta)}\right) = \frac{P(t,T)}{P(t,T+\delta)}\sigma_{T,T+\delta}(t)\,dW^{T+\delta}(t)$$

where $\sigma_{T,T+\delta}(t) = \int_T^{T+\delta}\sigma(t,u)\,du$

- Hence

$$\begin{aligned} dL(t,T) &= \frac{1}{\delta}\,d\left(\frac{P(t,T)}{P(t,T+\delta)}\right) \\ &= \frac{1}{\delta}(\delta L(t,T)+1)\sigma_{T,T+\delta}(t)\,dW^{T+\delta}(t) \end{aligned}$$

Heuristic Derivation cont'd

- Suppose: $\exists\ \mathbb{R}^d$-valued deterministic function $\lambda(t,T)$ s.t.

$$\sigma_{T,T+\delta}(t) = \frac{\delta L(t,T)}{\delta L(t,T)+1}\lambda(t,T) \tag{38}$$

- Plugging in: $dL(t,T) = L(t,T)\lambda(t,T)\,dW^{T+\delta}(t)$
- This is equivalent to, for $s \le t \le T$,

$$L(t,T) = L(s,T)\mathrm{e}^{\int_s^t \lambda(u,T)\,dW^{T+\delta}(u) - \frac{1}{2}\int_s^t \|\lambda(u,T)\|^2\,du}$$

$\Rightarrow$ $\mathcal{F}_t$-conditional $\mathbb{Q}^{T+\delta}$-distribution of $\log L(T,T)$ is Gaussian with mean $= \log L(t,T) - \frac{1}{2}\int_t^T \|\lambda(s,T)\|^2\,ds$ and variance $= \int_t^T \|\lambda(s,T)\|^2\,ds$

Heuristic Derivation cont'd

⇒ Caplet (reset date T, settlement date $T+\delta$, strike rate κ) has time t price

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{T+\delta} r(s)\,ds}\delta(L(T,T)-\kappa)^+ \mid \mathcal{F}_t\right] \\
&= P(t,T+\delta)\mathbb{E}_{\mathbb{Q}^{T+\delta}}\left[\delta(L(T,T)-\kappa)^+ \mid \mathcal{F}_t\right] \\
&= \delta P(t,T+\delta)\left(L(t,T)\Phi(d_1(t,T))-\kappa\Phi(d_2(t,T))\right)
\end{aligned}$$

where Φ = standard Gaussian cdf, and

$$d_{1,2}(t,T)=\frac{\log\left(\frac{L(t,T)}{\kappa}\right)\pm\frac{1}{2}\int_t^T\|\lambda(s,T)\|^2\,ds}{\left(\int_t^T\|\lambda(s,T)\|^2\,ds\right)^{\frac{1}{2}}}$$

- This is just Black's formula with $\sigma(t)^2=\frac{1}{T-t}\int_t^T\|\lambda(s,T)\|^2\,ds$

Heuristic Derivation cont'd

- Question: ∃? HJM model satisfying (38)?
- Answer: yes, but the construction and proof are not easy
- Idea: rewrite (38)

$$\int_T^{T+\delta} \sigma(t,u)\,du = \left(1 - e^{-\int_T^{T+\delta} f(t,u)\,du}\right)\lambda(t,T)$$

- Differentiating in T gives

$$\begin{aligned}\sigma(t,T+\delta) &= \sigma(t,T)\\ &\quad + (f(t,T+\delta) - f(t,T))e^{-\int_T^{T+\delta} f(t,u)\,du}\lambda(t,T)\\ &\quad + \left(1 - e^{-\int_T^{T+\delta} f(t,u)\,du}\right)\partial_T\lambda(t,T)\end{aligned}$$

Heuristic Derivation cont'd

- This recurrence relation can be solved by forward induction, once $\sigma(t,\cdot)$ is determined on $[0,\delta)$ (typically, $\sigma(t,T)=0$ for $T\in[0,\delta)$)

$\Rightarrow$ Complicated dependence of σ on the forward curve

$\Rightarrow$ Existence and uniqueness for HJM equations: carried out by [11], see also [25, Section 5.6]

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Direct Approach

- Direct approach to LIBOR models: outside of HJM framework
- Instead of risk-neutral martingale measure: work under forward measures with numeraires = bonds
- Fix finite time horizon $T_M = M\delta$, $M \in \mathbb{N}$
- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T_M]}, \mathbb{Q}^{T_M})$ carrying a d-dimensional Brownian motion $W^{T_M}(t)$, $t \in [0, T_M]$
- Notation suggests that $\mathbb{Q}^{T_M}$ will play the role of the T_M-forward measure
- Write $T_m = m\delta$, $m = 0, \ldots, M$
- Aim: construct a model for the M forward LIBOR rates with maturities $T_0, \ldots, T_{M-1}$

Direct Approach cont'd

Take as given:

- $\forall m \le M-1$: an $\mathbb{R}^d$-valued deterministic bounded measurable function $\lambda(t, T_m)$ = volatility of $L(t, T_m)$
- an initial positive and nonincreasing discrete term-structure $P(0, T_m)$, $m \le M$, and hence nonnegative initial forward LIBOR rates

$$L(0, T_m) = \frac{1}{\delta}\left(\frac{P(0, T_m)}{P(0, T_{m+1})} - 1\right), \quad m = 0, \dots, M-1$$

Direct Approach cont'd

- Backward induction: postulate first that

$$dL(t, T_{M-1}) = L(t, T_{M-1})\lambda(t, T_{M-1})\, dW^{T_M}(t),$$

  $t \in [0, T_{M-1}]$, which is equivalent to

$$L(t, T_{M-1}) = L(0, T_{M-1})\mathcal{E}_t\left(\lambda(\cdot, T_{M-1}) \bullet W^{T_M}\right)$$

- Motivated by (38): define $\mathbb{R}^d$-valued bounded progressive process, $t \in [0, T_{M-1}]$,

$$\sigma_{T_{M-1},T_M}(t) = \frac{\delta L(t, T_{M-1})}{\delta L(t, T_{M-1}) + 1}\lambda(t, T_{M-1})$$

- Induces equivalent probability measure $\mathbb{Q}^{T_{M-1}}$ on $\mathcal{F}_{T_{M-1}}$:

$$\frac{d\mathbb{Q}^{T_{M-1}}}{d\mathbb{Q}^{T_M}} = \mathcal{E}_{T_{M-1}}\left(\sigma_{T_{M-1},T_M} \bullet W^{T_M}\right) \propto \delta L(T_{M-1}, T_{M-1})+1$$

- Girsanov: $W^{T_{M-1}}(t) = W^{T_M}(t) - \int_0^t \sigma_{T_{M-1},T_M}(s)^\top\, ds$ is a $\mathbb{Q}^{T_{M-1}}$-Brownian motion, $t \in [0, T_{M-1}]$

Intermezzo: Sanity Check

- Recall from part "Forward Measures": in HJM framework

$$\frac{d\mathbb{Q}^{T_{M-1}}}{d\mathbb{Q}^{T_M}}\,|_{\mathcal{F}_t} = \frac{P(t,T_{M-1})\,P(0,T_M)}{P(t,T_M)\,P(0,T_{M-1})} = \mathcal{E}_t\left(\sigma_{T_{M-1},T_M} \bullet W^{T_M}\right)$$

- Moreover

$$\frac{P(t,T_{M-1})}{P(t,T_M)} = \delta L(t,T_{M-1}) + 1$$

Direct Approach cont'd

- Hence we can postulate

$$dL(t, T_{M-2}) = L(t, T_{M-2})\lambda(t, T_{M-2})\, dW^{T_{M-1}}(t),$$

$t \in [0, T_{M-2}]$, which is equivalent to

$$L(t, T_{M-2}) = L(0, T_{M-2})\mathcal{E}_t\left(\lambda(\cdot, T_{M-2}) \bullet W^{T_{M-1}}\right)$$

- Define $\mathbb{R}^d$-valued bounded progressive process, $t \in [0, T_{M-2}]$,

$$\sigma_{T_{M-2},T_{M-1}}(t) = \frac{\delta L(t, T_{M-2})}{\delta L(t, T_{M-2}) + 1}\lambda(t, T_{M-2})$$

- Induces probability measure $\mathbb{Q}^{T_{M-2}} \sim \mathbb{Q}^{T_{M-1}}$ on $\mathcal{F}_{T_{M-2}}$:

$$\frac{d\mathbb{Q}^{T_{M-2}}}{d\mathbb{Q}^{T_{M-1}}} = \mathcal{E}_{T_{M-2}}\left(\sigma_{T_{M-2},T_{M-1}} \bullet W^{T_{M-1}}\right)$$

- Girsanov: $W^{T_{M-2}}(t) = W^{T_{M-1}}(t) - \int_0^t \sigma_{T_{M-2},T_{M-1}}(s)^\top\, ds$ is a $\mathbb{Q}^{T_{M-2}}$-Brownian motion, $t \in [0, T_{M-2}]$

Term-Structure Models

Damir Filipović

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Direct Approach cont'd

- Repeating this procedure leads to a family of lognormal martingales $(L(t, T_m))_{t\in[0,T_m]}$ under their respective measures $\mathbb{Q}^{T_{m+1}}$.

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Find dynamics of $L(t, T_m)$ under any $\mathbb{Q}^{T_{n+1}}$:

Lemma

Let $0 \le m, n \le M-1$. Then the dynamics of $L(t, T_m)$ under $\mathbb{Q}^{T_{n+1}}$ is given according to the three cases, for $t \in [0, T_m \wedge T_{n+1}]$,

$$\frac{dL(t, T_m)}{L(t, T_m)} = \lambda(t, T_m)$$

$$\cdot \begin{cases} -\left(\sum_{l=m+1}^{n} \sigma_{T_l, T_{l+1}}(t)^\top dt + dW^{T_{n+1}}(t)\right), & m < n, \\ dW^{T_{n+1}}(t), & m = n, \\ \left(\sum_{l=n+1}^{m} \sigma_{T_l, T_{l+1}}(t)^\top dt + dW^{T_{n+1}}(t)\right), & m > n. \end{cases}$$

Proof

Proof.
This follows from the obvious equality

$$W^{T_{i+1}}(t) = W^{T_{j+1}}(t) - \sum_{l=i+1}^{j} \int_0^t \sigma_{T_l, T_{l+1}}(s)^\top \, ds, \quad t \in [0, T_{i+1}],$$

(→ exercise). □

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Forward Prices

- What can be said about bond prices?
- Define forward price process ($1 \le m \le M$):

$$\frac{P(t, T_{m-1})}{P(t, T_m)} = \delta L(t, T_{m-1}) + 1, \quad t \in [0, T_{m-1}]$$

- Clearly:

$$d\left(\frac{P(t, T_{m-1})}{P(t, T_m)}\right) = \delta \, dL(t, T_{m-1})$$
$$= \frac{P(t, T_{m-1})}{P(t, T_m)} \sigma_{T_{m-1}, T_m}(t) \, dW^{T_m}(t)$$

- Hence

$$\frac{P(t, T_{m-1})}{P(t, T_m)} = \frac{P(0, T_{m-1})}{P(0, T_m)} \mathcal{E}_t\left(\sigma_{T_{m-1}, T_m} \bullet W^{T_m}\right)$$

is a $\mathbb{Q}^{T_m}$-martingale, $t \in [0, T_{m-1}]$

Forward Prices cont'd

- Extension: define T_m-forward price for all T_k-bonds:

$$\frac{P(t,T_k)}{P(t,T_m)} = \begin{cases} \frac{P(t,T_k)}{P(t,T_{k+1})} \cdots \frac{P(t,T_{m-1})}{P(t,T_m)}, & k < m \\ \left(\frac{P(t,T_m)}{P(t,T_{m+1})}\right)^{-1} \cdots \left(\frac{P(t,T_{k-1})}{P(t,T_k)}\right)^{-1}, & k > m, \end{cases}$$

for $t \in [0, T_k \wedge T_m]$

- Following lemma is in accordance with part "Forward Measures" ...

Forward Prices cont'd

Lemma

For every $1 \le k \neq m \le M$, the forward price process satisfies

$$\frac{P(t,T_k)}{P(t,T_m)} = \frac{P(0,T_k)}{P(0,T_m)} \mathcal{E}_t \left(\sigma_{T_k,T_m} \bullet W^{T_m} \right), \quad t \in [0, T_k \wedge T_m],$$

for the $\mathbb{R}^d$-valued bounded progressive process

$$\sigma_{T_k,T_m} = \begin{cases} \sum_{l=k}^{m-1} \sigma_{T_l,T_{l+1}}, & k < m \\ -\sum_{l=m}^{k-1} \sigma_{T_l,T_{l+1}}, & k > m. \end{cases} \tag{39}$$

Hence $\frac{P(t,T_k)}{P(t,T_m)}$, $t \in [0, T_k \wedge T_m]$, is a positive $\mathbb{Q}^{T_m}$-martingale.

Proof

Suppose first $k < m$. Then (5) and (3) imply

$$\frac{P(t,T_k)}{P(t,T_m)} = \prod_{l=k}^{m-1} \frac{P(t,T_l)}{P(t,T_{l+1})} = \frac{P(0,T_k)}{P(0,T_m)} \prod_{l=k}^{m-1} \mathcal{E}_t\left(\sigma_{T_l,T_{l+1}} \bullet W^{T_{l+1}}\right)$$

$$= \frac{P(0,T_k)}{P(0,T_m)} \exp\left[\int_0^t \sum_{l=k}^{m-1} \sigma_{T_l,T_{l+1}}(s)\left(dW^{T_m}(s) - \sum_{i=l+1}^{m-1} \sigma_{T_i,T_{i+1}}(s)^\top ds\right)\right.$$

$$\left. -\frac{1}{2}\int_0^t \sum_{l=k}^{m-1} \|\sigma_{T_l,T_{l+1}}(s)\|^2 ds\right]$$

$$= \frac{P(0,T_k)}{P(0,T_m)} \exp\left[\int_0^t \sum_{l=k}^{m-1} \sigma_{T_l,T_{l+1}}(s)\, dW^{T_m}(s) - \frac{1}{2}\int_0^t \left\|\sum_{l=k}^{m-1} \sigma_{T_l,T_{l+1}}(s)\right\|^2 ds\right].$$

Hence

$$\frac{P(t,T_k)}{P(t,T_m)} = \frac{P(0,T_k)}{P(0,T_m)} \mathcal{E}_t\left(\left(\sum_{l=k}^{m-1} \sigma_{T_l,T_{l+1}}\right) \bullet W^{T_m}\right),$$

as desired. The case $k > m$ follows by similar argumentation (exercise).

Nominal Bond Prices

- Can derive nominal T_n-bond prices

$$P(T_m, T_n) = \prod_{k=m+1}^{n} \frac{P(T_m, T_k)}{P(T_m, T_{k-1})}$$
$$= \prod_{k=m+1}^{n} \frac{1}{\delta L(T_m, T_{k-1}) + 1}$$

only at dates $t = T_m$, for $0 \leq m < n \leq M$

- However: continuous time bond price dynamics $P(t, T_n)$ not determined
- Knowledge of forward LIBOR rates $L(t, T)$ for all maturities $T \in [0, T_{M-1}]$ would be necessary: see section "Continuous-Tenor Case" below

Forward Pricing

- Notwithstanding: can now consistently compute T_m-forward price for any T_m-contingent claim X with $\mathbb{E}_{\mathbb{Q}^{T_m}}[|X|] < \infty$:

$$\frac{\pi(t)}{P(t,T_m)} = \mathbb{E}_{\mathbb{Q}^{T_m}}\left[X \mid \mathcal{F}_t\right]$$

- Change of numeraire:

Lemma

The T_m-forward price price satisfies

$$\frac{\pi(t)}{P(t,T_m)} = \frac{P(t,T_n)}{P(t,T_m)} \mathbb{E}_{\mathbb{Q}^{T_n}}\left[\frac{X}{P(T_m,T_n)} \mid \mathcal{F}_t\right],$$

for all $m < n \leq M$.

Proof

By last lemma: for $t \in [0, T_k]$

$$\frac{d\mathbb{Q}^{T_k}}{d\mathbb{Q}^{T_{k+1}}}|_{\mathcal{F}_t} = \mathcal{E}_t\left(\sigma_{T_k,T_{k+1}} \bullet W^{T_{k+1}}\right) = \frac{P(0,T_{k+1})}{P(0,T_k)} \frac{P(t,T_k)}{P(t,T_{k+1})}.$$

Hence

$$\begin{aligned}\frac{d\mathbb{Q}^{T_m}}{d\mathbb{Q}^{T_n}}|_{\mathcal{F}_t} &= \prod_{k=m}^{n-1} \frac{d\mathbb{Q}^{T_k}}{d\mathbb{Q}^{T_{k+1}}}|_{\mathcal{F}_t} = \prod_{k=m}^{n-1} \frac{P(0,T_{k+1})}{P(0,T_k)} \frac{P(t,T_k)}{P(t,T_{k+1})} \\ &= \frac{P(0,T_n)}{P(0,T_m)} \frac{P(t,T_m)}{P(t,T_n)}.\end{aligned}$$

Bayes' rule now yields the assertion.

Caplet Pricing

From section "Heuristic Derivation" we now obtain:

Corollary

Let $m+1 < n \leq M$. The time T_m price of the nth caplet with reset date T_{n-1}, settlement date T_n and strike rate κ is

$$
\begin{aligned}
Cpl(T_m; T_{n-1}, T_n) &= \delta P(T_m, T_n) \\
&\quad \times \left(L(T_m, T_{n-1}) \Phi(d_1(n; T_m)) - \kappa \Phi(d_2(n; T_m)) \right)
\end{aligned}
$$

where Φ = standard Gaussian cdf, and

$$
d_{1,2}(n; T_m) = \frac{\log\left(\frac{L(T_m, T_{n-1})}{\kappa}\right) \pm \frac{1}{2} \int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 \, ds}{\left(\int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 \, ds\right)^{\frac{1}{2}}}.
$$

Caplet Pricing cont'd

This is exactly Black's formula for the caplet price with

$$\sigma(T_m)^2 = \frac{1}{T_{n-1} - T_m} \int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 \, ds.$$

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Definition

- Define discrete-time implied money-market account process

$$B^*(0) = 1,$$
$$B^*(T_m) = (1 + \delta L(T_{m-1}, T_{m-1}))B^*(T_{m-1}), \quad 1 \leq m \leq M$$

- Equivalently:

$$B^*(T_n) = B^*(T_m) \prod_{k=m}^{n-1} \frac{1}{P(T_k, T_{k+1})}, \quad m < n \leq M$$

- Interpretation: $B^*(T_m)$ = cash amount accumulated up to time T_m by rolling over a series of zero-coupon bonds with the shortest maturities available
- By construction: B^* nondecreasing and $B^*(T_m)$ is $\mathcal{F}_{T_{m-1}}$-measurable for all $m = 1, \dots, M$

First Properties

Define function $\eta : [0, T_{M-1}] \to \mathbb{N}$ by $T_{\eta(t)-1} \leq t < T_{\eta(t)}$

Lemma

For all $t \in [0, T_{M-1}]$ we have

$$\mathbb{E}_{\mathbb{Q}^{T_M}}\left[B^*(T_M)P(0,T_M) \mid \mathcal{F}_t\right] = \mathcal{E}_t\left(\sigma_{T_0,T_{M-1}} \bullet W^{T_M}\right),$$

where we define (in accordance with σ_{T_k,T_m} for $k \geq 1$) the $\mathbb{R}^d$-valued bounded progressive process

$$\sigma_{T_0,T_{M-1}}(t) = \sum_{k=\eta(t)}^{M-1} \sigma_{T_k,T_{k+1}}(t).$$

In particular, for all $0 \leq m \leq M$ we have

$$\mathbb{E}_{\mathbb{Q}^{T_M}}\left[B^*(T_M) \mid \mathcal{F}_{T_m}\right] = \frac{B^*(T_m)}{P(T_m,T_M)}.$$

Proof

Proof.
Follows from above lemma for $P(t, T_k)/P(t, T_m)$ ($\rightarrow$ exercise). $\square$

Risk-Neutral Measure

- In view of preceding lemma: can define $\mathbb{Q}^* \sim \mathbb{Q}^{T_M}$ on $\mathcal{F}_{T_M}$ by

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_M}} = B^*(T_M)P(0,T_M)$$

- We have

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} = \begin{cases} \mathcal{E}_t\left(\sigma_{T_0,T_{M-1}} \bullet W^{T_M}\right), & t \in [0, T_{M-1}], \\ B^*(T_m)\frac{P(0,T_M)}{P(T_m,T_M)}, & \text{if } t = T_m,\ m \le M-1 \end{cases}$$

- Interpretation: $\mathbb{Q}^*$ = risk-neutral martingale measure
- Also called "spot LIBOR measure" (Jamshidian [32])

Risk-Neutral Pricing

Bayes' rule implies:

Lemma

The time T_k price of the T_m-contingent claim X from "Forward Pricing" lemma above satisfies

$$\pi(T_k) = B^*(T_k)\mathbb{E}_{\mathbb{Q}^*}\left[\frac{X}{B^*(T_m)} \mid \mathcal{F}_{T_k}\right],$$

for all $k \le m$.

- Consequence ($\pi(T_k) = P(T_k, T_m)$ for $X = 1$): discrete-time process

$$\left(\frac{P(T_k, T_m)}{B^*(T_k)}\right)_{k=0,\dots,m}$$

is a $\mathbb{Q}^*$-martingale w.r.t. $(\mathcal{F}_{T_k})$

$\Rightarrow$ We constructed a full discrete-time interest rate model

Risk-Neutral LIBOR Dynamics

Girsanov: $W^*(t) = W^{T_M}(t) - \int_0^t \sigma_{T_0,T_{M-1}}(s)\,ds$ is a $\mathbb{Q}^*$-Brownian motion, $t \in [0, T_{M-1}]$

Lemma

Let $0 \le m \le M-1$. Then the dynamics of $L(t, T_m)$ under $\mathbb{Q}^*$ is given according to, $t \in [0, T_m]$

$$\frac{dL(t,T_m)}{L(t,T_m)} = \lambda(t,T_m) \sum_{k=\eta(t)}^{m} \sigma_{T_k,T_{k+1}}(t)^\top dt + \lambda(t,T_m)\,dW^*(t).$$

Proof.

Follows from "LIBOR dynamics" lemma above for $n = M-1$ and the definition of W^*. □

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Problem Formulation

- Consider a payer swaption with nominal 1, strike rate K, maturity T_μ, underlying tenor $T_\mu, T_{\mu+1}, \ldots, T_\nu$ (T_μ is the first reset date and T_ν the maturity of the underlying swap), for some $\mu < \nu \le M$
- Recall: payoff at maturity T_μ is

$$\Pi = \delta \left(\sum_{m=\mu+1}^{\nu} P(T_\mu, T_m)(L(T_\mu, T_{m-1}) - K) \right)^+$$

- By above pricing lemmas: swaption price at $t = 0$

$$\pi = P(0, T_\mu)\mathbb{E}_{\mathbb{Q}^{T_\mu}}[\Pi] = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{\Pi}{B^*(T_\mu)}\right]$$

$\Rightarrow$ Need joint distribution (under $\mathbb{Q}^{T_\mu}$ or $\mathbb{Q}^*$) of

$$L(T_\mu, T_\mu), L(T_\mu, T_{\mu+1}), \ldots, L(T_\mu, T_{\nu-1})$$

- No analytic formula in LIBOR market model ...

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Definition

- Corresponding forward swap rate at $t \leq T_\mu$ is

$$R_{swap}(t) = \frac{P(t, T_\mu) - P(t, T_\nu)}{\delta \sum_{k=\mu+1}^{\nu} P(t, T_k)} = \frac{1 - \frac{P(t,T_\nu)}{P(t,T_\mu)}}{\delta \sum_{k=\mu+1}^{\nu} \frac{P(t,T_k)}{P(t,T_\mu)}}.$$

$\Rightarrow$ $R_{swap}(t)$ available in our LIBOR market model

- Define positive $\mathbb{Q}^{T_\mu}$-martingale $D(t) = \sum_{k=\mu+1}^{\nu} \frac{P(t,T_k)}{P(t,T_\mu)}$, $t \in [0, T_\mu]$
- Induces forward swap measure $\mathbb{Q}^{swap} \sim \mathbb{Q}^{T_\mu}$ on $\mathcal{F}_{T_\mu}$ by

$$\frac{d\mathbb{Q}^{swap}}{d\mathbb{Q}^{T_\mu}} = \frac{D(T_\mu)}{D(0)}.$$

Change of Numeraire

Lemma

The forward swap rate process $R_{swap}(t)$, $t \in [0, T_\mu]$, is a positive $\mathbb{Q}^{swap}$-martingale.
Moreover, there exists some d-dimensional $\mathbb{Q}^{swap}$-Brownian motion W^{swap} and an $\mathbb{R}^d$-valued progressive swap volatility process ρ^{swap} such that

$$dR_{swap}(t) = R_{swap}(t)\rho^{swap}(t)\, dW^{swap}(t), \quad t \in [0, T_\mu].$$

- Note: $\rho^{swap}(t)$ explicitly available in principle

Proof

Let $0 \leq m \leq M$ and $0 \leq s \leq t \leq T_m \wedge T_\mu$. Then

$$\mathbb{E}_{\mathbb{Q}^{swap}}\left[\frac{P(t,T_m)}{P(t,T_\mu)D(t)} \mid \mathcal{F}_s\right]$$
$$= \frac{1}{D(s)}\mathbb{E}_{\mathbb{Q}^{T_\mu}}\left[\frac{P(t,T_m)}{P(t,T_\mu)D(t)}D(t) \mid \mathcal{F}_s\right] = \frac{1}{D(s)}\frac{P(s,T_m)}{P(s,T_\mu)}.$$

On the other hand, from above formula for $R_{swap}(t)$:

$$R_{swap}(t) = \frac{1}{\delta D(t)} - \frac{P(t,T_\nu)}{\delta P(t,T_\mu)D(t)}.$$

Hence $R_{swap}(t)$ is a positive $\mathbb{Q}^{swap}$-martingale.
The representation of $R_{swap}(t)$ in terms of W^{swap} and ρ^{swap} follows from above lemma for $P(t,T_k)/P(t,T_m)$ and Girsanov's theorem.

Swap Measure Swaption Pricing

- Recall: swaption payoff at maturity can be written as

$$\delta D(T_\mu)\,(R_{swap}(T_\mu)-K)^+$$

- Hence the price equals

$$\begin{aligned}\pi &= \delta P(0,T_\mu)\mathbb{E}_{\mathbb{Q}^{T_\mu}}\left[D(T_\mu)\,(R_{swap}(T_\mu)-K)^+\right]\\ &= \delta P(0,T_\mu)D(0)\mathbb{E}_{\mathbb{Q}^{swap}}\left[(R_{swap}(T_\mu)-K)^+\right]\\ &= \delta\sum_{k=\mu+1}^{\nu} P(0,T_k)\mathbb{E}_{\mathbb{Q}^{swap}}\left[(R_{swap}(T_\mu)-K)^+\right]\end{aligned}$$

Lognormal Hypothesis

- Hypothesis **(H)**: $\rho^{swap}(t)$ is deterministic
- Consequence: $\log R_{swap}(T_\mu)$ Gaussian distributed under $\mathbb{Q}^{swap}$ with mean $= \log R_{swap}(0) - \frac{1}{2}\int_0^{T_\mu} \|\rho^{swap}(t)\|^2\,dt$ and variance $= \int_0^{T_\mu} \|\rho^{swap}(t)\|^2\,dt$

$\Rightarrow$ Swaption price would be

$$\pi = \delta \sum_{k=\mu+1}^{\nu} P(0,T_k)\left(R_{swap}(0)\Phi(d_1) - K\Phi(d_2)\right)$$

with

$$d_{1,2} = \frac{\log\left(\frac{R_{swap}(0)}{K}\right) \pm \frac{1}{2}\int_0^{T_\mu} \|\rho^{swap}(t)\|^2\,dt}{\left(\int_0^{T_\mu} \|\rho^{swap}(t)\|^2\,dt\right)^{\frac{1}{2}}}$$

- This is Black's formula with volatility $\sigma^2 = \frac{1}{T_\mu}\int_0^{T_\mu} \|\rho^{swap}(t)\|^2\,dt$

Lognormal Hypothesis Valid?

- Fact (without proof): ρ^{swap} cannot be deterministic in our lognormal LIBOR setup
- Alternative for swaption pricing: model forward swap rates directly and postulate that they are lognormal under the forward swap measures (the so-called swap market model)
- Carried out by Jamshidian [32], and computationally improved by Pelsser [43]
- But (without proof): then forward LIBOR rate volatility cannot be deterministic

$\Rightarrow$ Either one gets Black's formula for caps or for swaptions, but not simultaneously for both

$\Rightarrow$ In lognormal forward LIBOR model swaption prices have to be approximated

- via Monte Carlo methods
- via analytic approximation ...

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Freezing the Weights

- Recall: forward swap rate can be written as weighted sum of forward LIBOR rates

$$R_{swap}(t) = \sum_{m=\mu+1}^{\nu} w_m(t) L(t, T_{m-1})$$

with weights

$$w_m(t) = \frac{P(t, T_m)}{D(t)P(t, T_\mu)} = \frac{\frac{1}{1+\delta L(t,T_\mu)} \cdots \frac{1}{1+\delta L(t,T_{m-1})}}{\sum_{k=\mu+1}^{\nu} \frac{1}{1+\delta L(t,T_\mu)} \cdots \frac{1}{1+\delta L(t,T_{k-1})}}$$

- Empirical studies show: variability of w_m small compared to variability of $L(t, T_{m-1})$

$\rightarrow$ Approximate $R_{swap}(t) \approx \sum_{m=\mu+1}^{\nu} w_m(0) L(t, T_{m-1})$

Freezing the Weights cont'd

$\Rightarrow$ $\mathbb{Q}^{T_\mu}$-dynamics, $t \in [0, T_\mu]$

$$dR_{swap}(t) \approx (\cdots)\, dt + \sum_{m=\mu+1}^{\nu} w_m(0) L(t, T_{m-1}) \lambda(t, T_{m-1})\, dW^{T_\mu}$$

$\Rightarrow$ Forward swap volatility satisfies

$$\|\rho^{swap}(t)\|^2 = \frac{d\,\langle \log R_{swap}, \log R_{swap}\rangle_t}{dt} \approx \sum_{k,l=\mu+1}^{\nu} \frac{w_k(0) L(t, T_{k-1}) \lambda(t, T_{k-1})}{R_{swap}(t)} \times \frac{w_l(0) L(t, T_{l-1})\, \lambda(t, T_{l-1})^\top}{R_{swap}(t)}$$

Further Approximation

- Further approximation: $\|\rho^{swap}(t)\|^2 \approx \tilde{\rho}^{swap}(t)^2$ with

$$\tilde{\rho}^{swap}(t)^2 = \sum_{k,l=\mu+1}^{\nu} \frac{w_k(0)L(0,T_{k-1})\lambda(t,T_{k-1})}{R_{swap}(0)} \times \frac{w_l(0)L(0,T_{l-1})\lambda(t,T_{l-1})^\top}{R_{swap}(0)}$$

- Lévy's characterization theorem: $\mathcal{W}(t) = \int_0^t \frac{\rho^{swap}(s)\,dW^{swap}(s)}{\|\rho^{swap}(s)\|}$ is a $\mathbb{Q}^{swap}$-Brownian motion, $t \in [0, T_\mu]$

Summary

$\Rightarrow$ $\tilde{\rho}^{swap}(t)$ deterministic and

$$
\begin{aligned}
dR_{swap}(t) &= R_{swap}(t)\|\rho^{swap}(t)\|\, d\mathcal{W}(t) \\
&\approx R_{swap}(t)\tilde{\rho}^{swap}(t)\, d\mathcal{W}(t)
\end{aligned}
$$

$\Rightarrow$ Approximate swaption price by Black's formula with

$$
\sigma^2 = \frac{1}{T_\mu}\int_0^{T_\mu} \tilde{\rho}^{swap}(t)^2\, dt
$$

- "Rebonato's formula" (Rebonato [44])
- Goodness of this approximation numerically tested by several authors ([12, Chapter 8]): "the approximation is satisfactory in general"

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Towards an Euler Scheme

- Seen in swaption case above: option pricing typically requires Monte Carlo simulation
- Many ways to simulate forward LIBOR rates: e.g. Glasserman [26]
- Here: sketch particular Euler scheme
- Focus on risk-neutral measure $\mathbb{Q}^*$ (albeit the following can be carried out under any forward measure)
- Aim: simulate entire M-vector $(L(t, T_0), \dots, L(t, T_{M-1}))^\top$

Towards an Euler Scheme

- Transform: $H_m(t) = \log L(t, T_m)$ satisfies (Itô's formula)

$$dH_m(t) = \alpha_m(t)\, dt + \lambda(t, T_m)\, dW^*(t), \quad t \le T_m$$

with drift term

$$\alpha_m(t) = \lambda(t, T_m) \sum_{k=\eta(t)}^{m} \frac{\delta \mathrm{e}^{H_k(t)}}{1 + \delta \mathrm{e}^{H_k(t)}} \lambda(t, T_k)^\top - \frac{1}{2} \|\lambda(t, T_m)\|^2$$

- Advantage of simulating H_m:
  - keeps $L(t, T_m) = \exp(H_m(t))$ positive
  - H_m has Gaussian increments $\Rightarrow$ improves convergence of Euler scheme

General pricing problem

- General pricing problem: T_n-claim with payoff $f\left(H_n(T_n), \dots, H_{M-1}(T_n)\right)$
- Price at $t = 0$: $\pi = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{f(H_n(T_n),\dots,H_{M-1}(T_n))}{B^*(T_n)}\right]$

Euler Scheme

- Fix time grid $t_i = i\Delta t$, $i = 0, \dots, N$, with $\Delta t = T_n/N$ for N large enough
- $Z(1), \dots, Z(N)$: sequence of independent standard normal random vectors in $\mathbb{R}^d$
- Euler approximation for H_m: $1 \le i \le N$

$$H_m(t_i) = H_m(t_{i-1}) + \alpha_m(t_{i-1})\,\Delta t + \lambda(t_{i-1}, T_m)\,Z(i)\,\sqrt{\Delta t} \tag{40}$$

- Monte Carlo principle:
  1. simulate via Euler scheme K independent copies $\Pi^{(1)}, \dots, \Pi^{(K)}$ of $\Pi = \frac{f(H_n(T_n),\dots,H_{M-1}(T_n))}{B^*(T_n)}$
  2. estimate π via averaging: $\overline{\Pi} = \frac{1}{K}\sum_{j=1}^{K} \Pi^{(j)}$

Efficiency

Three considerations are important for the efficiency of this simulation estimator: bias, variance, and computing time

Bias: introduced via Euler approximation (40): $\mathbb{E}_{\mathbb{Q}^*}[\overline{\Pi}]$ differs from target value $\pi = \mathbb{E}_{\mathbb{Q}^*}[\Pi]$. Bias is reduced by increasing number of time discretization steps N. (In our example: bias is already negligible for $\Delta t = 1/12$, can assume that $\mathbb{E}_{\mathbb{Q}^*}[\overline{\Pi}] \approx \pi$.)

Efficiency cont'd

Variance: central limit theorem: as number of replications K increases, the simulation estimation error $\overline{\Pi} - \pi$ approximately normal distributed with mean zero and approximate standard deviation of

$$s_\pi = \sqrt{\frac{\sum_{j=1}^{K}(\Pi^{(j)} - \overline{\Pi})}{K\,(K-1)}}.$$

s_π is called standard error of the Monte Carlo simulation: $\overline{\Pi} \pm s_\pi$ is an asymptotically (as $K \to \infty$) valid 68% confidence interval for the true value π.

Computing time: obvious trade-off between bias and variance for a given computing capacity, which has to be carefully balanced in general.

Thorough treatment of Monte Carlo: e.g. Glasserman [26]

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Volatility Estimation

- So far: volatility factors $\lambda(t, T_m)$ = exogenous deterministic functions
- In practice: chosen to calibrate to
  - market prices of liquidly traded derivatives, such as caps and swaptions
  - historical time series
- Note: model is automatically calibrated to initial bond prices
- Volatility estimation: huge topic, see e.g. Brigo and Mercurio [12]
- Here: brief discussion of two approaches:
  - historical volatility estimation via principal component analysis (PCA)
  - volatility calibration to market quotes of caps and swaptions . . .

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Sketch of PCA

- Assume: $\lambda(t, T_m) = \lambda(T_m - t)$, for all m
- Given: N observations $x(1), \dots, x(N)$ of random vectors $X(i) = (X_1(i), \dots, X_M(i))^\top$ where $(1 \le m \le M)$

  $$X_m(i) = \log L(i\delta, (i+m-1)\delta) - \log L((i-1)\delta, (i+m-1)\delta)$$

  (sliding LIBOR curve increments)
- Euler approximation (40) & neglect drift term (see next page) $\Rightarrow$

  $$X_m(i) \approx \lambda(m\delta)\, Z(i) \sqrt{\delta} \tag{41}$$

$\Rightarrow$ $X(i)$ approximately i.i.d. with zero mean

Sketch of PCA: Drift Term

- Recall $\mathbb{Q}^*$-drift term of $\log L(i\delta, (i+m-1)\delta)$:

$$\alpha_m((i-1)\delta) = -\frac{1}{2}\|\lambda(m\delta)\|^2 + \lambda(m\delta)\sum_{k=1}^{m}\frac{\delta L((i-1)\delta, (i+k-1)\delta)}{1+\delta L((i-1)\delta, (i+k-1)\delta)}\lambda(k\delta)^\top$$

- For $\mathbb{P}$: to be corrected by

$$\text{market price of risk} \times \delta\,\|\lambda(m\delta)\|$$

$\Rightarrow$ Drift term of order

$$\delta\,\|\lambda(m\delta)\| \times \max\{\|\lambda(m\delta)\|, \text{market price of risk}\}$$

$\Rightarrow$ neglected

Sketch of PCA cont'd

- empirical mean $\hat{\mu} = \frac{1}{N}\sum_{t=1}^{N} x(t)$
- empirical covariance matrix

$$\hat{Q}_{ij} = \mathrm{Cov}[x_i, x_j] = \frac{1}{N}\sum_{t=1}^{N}(x_i(t) - \hat{\mu}_i)(x_j(t) - \hat{\mu}_j)$$

- PCA decomposition:

$$x(i) = \hat{\mu} + \sum_{j=1}^{M} \hat{a}_j\, y_j(i) \approx \sum_{j=1}^{M} \hat{a}_j\, y_j(i)$$

with
  - $\hat{Q} = \hat{A}\hat{L}\hat{A}^\top$, loadings $\hat{A} = (\hat{a}_1 \mid \cdots \mid \hat{a}_n)$
  - empirical principal components $y = \hat{A}^\top(x - \hat{\mu})$: y_j uncorrelated, nonincreasing order $\mathrm{Var}[y_1] \geq \mathrm{Var}[y_2] \geq \cdots$
- Compare with (41) $\Rightarrow$ estimate $(1 \leq m \leq M)$

$$\lambda_j(m\delta) = \sqrt{\frac{\mathrm{Var}[y_j]}{\delta}}\, \hat{a}_{jm}$$

PCA Stylized Facts

- First 2–3 principal components y_j enough to explain most of the variance of x
- First three loadings $\hat{a}_j$ (i.e. volatility curves $s \mapsto \lambda_j(s)$) are typically of the form as seen in Section "PCA of the Forward Curve": flat, upward (or downward) sloping, and hump-shaped

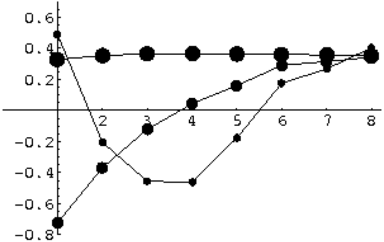

Figure: First three forward curve loadings

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Volatility vs. Correlation

- Recall Black's caplet price formula: function of $\|\lambda(t, T_m)\|$ only

$\Rightarrow$ No gain in flexibility in matching caplet implied volatilities by taking number d of Brownian motions greater than one

- Potential value of a multi-factor model lies in capturing correlations between forward LIBOR rates of different maturities.
- Euler approximation (40) $\Rightarrow$ instantaneous correlation between increments of $\log L(t, T_m)$ and $\log L(t, T_n)$ approximately

$$\rho_{mn}(t) = \frac{\lambda(t, T_m)\,\lambda(t, T_n)^\top}{\|\lambda(t, T_m)\|\,\|\lambda(t, T_n)\|}$$

- $\rho_{mn}(t)$ chosen to match market quotes of swaptions, or historical correlations as indicated in above PCA

Volatility vs. Correlation cont'd

- Formalize dual aspect volatility vs. correlation:

$$\lambda(t, T_m) = \sigma_m(t)\,\ell_m(t)$$

  where
  - $\sigma_m(t) = \|\lambda(t, T_m)\| =$ volatility of $L(t, T_m)$: calibrated to caplet prices
  - $\ell_m(t) = \lambda(t, T_m)/\|\lambda(t, T_m)\|$ captures correlation between different rates: $\rho_{mn}(t) = \ell_m(t)\,\ell_n(t)^\top$

Volatility Specifications

- Assume as given: market quotes of all caplets $Cpl(T_{n-1}, T_n) = Cpl(0; T_{n-1}, T_n)$ by their implied volatilities $\sigma_{Cpl(T_{n-1},T_n)}$ $(1 \le n \le M-1)$:

$$\int_0^{T_n} \sigma_n(t)^2 \, dt = \sigma^2_{Cpl(T_n,T_{n+1})} \, T_n$$

- Specification 1: $\sigma_m(t) = \sigma(T_m - t)$ = function of time to maturity

$\Rightarrow$ $\int_{T_{n-1}}^{T_n} \sigma(t)^2 \, dt =$

$$= \begin{cases} \sigma^2_{Cpl(T_1,T_2)} \, T_1, & n = 1 \\ \sigma^2_{Cpl(T_n,T_{n+1})} \, T_n - \sigma^2_{Cpl(T_{n-1},T_n)} \, T_{n-1}, & 2 \le n \le M-1 \end{cases}$$

- Drawback: requires $\sigma^2_{Cpl(T_n,T_{n+1})} \, T_n \ge \sigma^2_{Cpl(T_{n-1},T_n)} \, T_{n-1}$, which is not satisfied by caplet data in general !

Volatility Specifications cont'd

- Specification 2: $\sigma_m(t) \equiv \sigma_m$ independent of t, consistent

$\Rightarrow$ Calibration easy: $\sigma_n = \sigma_{Cpl(T_n,T_{n+1})}$, $1 \leq n \leq M-1$

- Drawback: stipulates that volatility does not change over time: not plausible for long-matured forward LIBOR rates
- Specification 3: parametric form $\sigma_m(t) = v_m \, \mathrm{e}^{-\beta(T_m - t)}$, common exponent β, individual factors v_m
- Note: specification under-determined: the system

$$v_n^2 \frac{1-\mathrm{e}^{-2\beta T_n}}{2\beta} = \sigma^2_{Cpl(T_n,T_{n+1})} \, T_n, \quad 1 \leq n \leq M-1,$$

leaves one degree of freedom (to be fixed by some additional data point)

- Note: for $\beta = 0$ we obtain back Specification 2
- More systematic classification of admissible volatility specifications: see Brigo and Mercurio [12, Section 6.3]

Correlation Specifications

- Common assumption: $\ell_m(t) \equiv \ell_m$ (e.g. in [12, Section 6.3])

$\Rightarrow$ Analytic approximation formula ("Rebonato's") for implied swaption volatility:

$$\sigma_{swp}^2 \approx \frac{1}{T_\mu} \sum_{k,l=\mu}^{\nu-1} \rho_{k,l} \times \frac{w_{k+1}(0) w_{l+1}(0) L(0,T_k) L(0,T_l) \int_0^{T_\mu} \sigma_k(t)\sigma_l(t)\,dt}{R_{swap}^2(0)}$$

- Estimator for ρ, given initial term-structure $L(0, T_k)$ and σ_m calibrated to caplet quotes
- Caution (euro zone): $\delta = 1/2$ for caplets, $\delta = 1$ for swaps !
- Alternative to analytic approximation formula: iterative Monte Carlo simulations

Correlation Specifications cont'd

- Specification of ℓ_m goes together with choice of d
- For illustration: consider two extreme cases $d = 1$ and $d = M - 1$
- Remark: recall stylized facts on "Correlation": ρ_{mn} exponentially decaying $\rho_{mn} = \mathrm{e}^{-\gamma |T_m - T_n|}$. Specifications I and II then correspond to $\gamma = 0$ and $\gamma = \infty$

Correlation Specification I

- Specification I: $d = 1$ and $\ell_m = 1$, for all m
- instantaneous correlation between increments of $\log L(t, T_m)$ and $\log L(t, T_n)$ is one

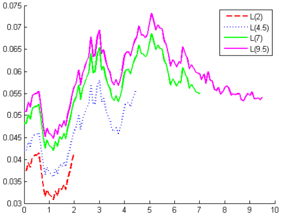

Figure: Perfectly correlated case: trajectories of $L(\cdot, T)$ for $T = 2, 4.5, 7, 9.5$.

Correlation Specification II

- Specification II: $d = M - 1$ and $\ell_m = e_m^\top$, ρ=unit matrix

$\Rightarrow$ SDEs decoupled ($t \le T_m$, $1 \le m \le M-1$):

$$dH_m(t) = \left(\frac{\delta \mathrm{e}^{H_m(t)}}{1 + \delta \mathrm{e}^{H_m(t)}} \sigma_m(t)^2 - \frac{1}{2} \sigma_m(t)^2 \right) dt + \sigma_m(t)\, dW_m(t)$$

$\Rightarrow$ independent forward LIBOR rates

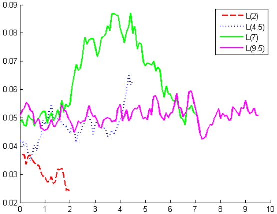

Figure: Independent case: trajectories of $L(\cdot, T)$ for $T = 2, 4.5, 7, 9.5$.

Caplet Calibration

- Usual market quotes: prices in bp on caps and floors, see tables (or to be inferred from implied volatilities)

$\Rightarrow$ Have to strip caplet and floorlet volatilities

- Illustration: bootstrapping method analogous to part "Estimating the Term-Structure"
- Tenor: $T_i = i/2$, $i = 0, \ldots, 20$, where $T_1 = 1/2 =$ first caplet reset date, $T_{20} = 10 =$ maturity of last cap
- Prevailing initial forward LIBOR curve given in following table

Forward LIBOR Curve

Table: Forward LIBOR curve (in %) on 18 November 2008

T_i	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
$L(0, T_i)$	4.228	2.791	3.067	3.067	3.728	3.728	4.051	4.051	4.199	4.199

T_i	5	5.5	6	6.5	7	7.5	8	8.5	9	9.5
$L(0, T_i)$	4.450	4.450	4.626	4.626	4.816	4.816	4.960	4.960	5.088	5.088

Cap Prices

Table: Euro cap prices (in basis points) on 18 November 2008

T–K	3.5%	4%	4.5%	5%	5.5%	6%	6.5%	7%	7.5%
2	25.0	11.0	5.0	2.5	1.5	1.0	0.5	0.0	0.0
3	77.0	40.5	21.5	12.0	7.0	4.0	2.5	1.5	1.5
4	148.5	86.0	48.5	27.0	16.0	10.0	6.5	4.5	4.0
5	230.5	140.5	82.0	47.5	28.5	17.5	11.5	8.0	7.5
6	325.5	206.0	125.5	74.5	45.5	29.0	19.0	13.5	12.5
7	431.5	283.5	178.0	109.0	68.0	44.5	29.5	21.0	20.5
8	545.5	368.5	238.0	149.5	95.0	62.5	42.5	30.0	29.0
9	664.0	459.0	304.5	196.5	127.0	85.0	58.5	42.0	40.0
10	786.0	554.5	376.5	248.5	164.0	111.0	77.0	56.0	53.0

Floor Prices

Table: Euro floor prices (in basis points) on 18 November 2008

T–K	3%	2.75%	2.5%	2.25%	2%	1.75%	1.5%	1.25%	1%
2	69.5	50.0	34.0	23.0	14.5	9.0	5.5	3.5	1.5
3	92.0	66.5	47.0	32.0	21.5	14.0	9.0	5.0	2.5
4	110.0	80.5	58.0	40.5	28.5	19.5	13.0	8.0	4.0
5	127.0	94.0	68.5	49.0	35.0	25.0	17.0	11.0	5.5
6	142.0	107.0	78.5	58.0	42.5	31.0	21.5	13.5	7.5
7	157.5	119.5	89.5	67.0	50.0	37.0	26.5	16.5	9.0
8	172.5	132.0	101.0	76.5	58.5	43.5	31.0	20.0	10.5
9	187.5	145.0	112.0	86.5	66.5	50.0	36.0	23.5	13.0
10	201.5	157.5	122.5	95.5	74.0	56.5	41.0	27.5	15.5

Boostrapping Caplet Volatilities

- Consider: caps at strike rate $K = 3.5\%$
- First cap, maturity $T_4 = 2$ years:

$$Cp(T_4) = Cpl(T_1, T_2) + Cpl(T_2, T_3) + Cpl(T_3, T_4)$$

- Infer implied caplet volatility by inverting Black's formula:

$$\sigma_{Cpl(T_1,T_2)} = \sigma_{Cpl(T_2,T_3)} = \sigma_{Cpl(T_3,T_4)} = 29.3\%$$

- Next cap matures in $T_6 = 3$ years:

$$Cp(T_6) - Cp(T_4) = Cpl(T_4, T_5) + Cpl(T_5, T_6)$$

- Infer implied caplet volatility $\sigma_{Cpl(T_4,T_5)} = \sigma_{Cpl(T_5,T_6)} = 20.8\%$, without altering the previous ones, etc.

Boostrapping Caplet Volatilities cont'd

Table: Implied volatilities (in %) for caplets $Cpl(T_{i-1}, T_i)$ at strike rate 3.5%

i	1	2	3	4	5	6	7	8	9	10
$\sigma_{Cpl(T_{i-1},T_i)}$	n/a	29.3	29.3	29.3	20.8	20.8	18.3	18.3	17.8	17.8

i	11	12	13	14	15	16	17	18	19	20
$\sigma_{Cpl(T_{i-1},T_i)}$	16.3	16.3	16.7	16.7	16.1	16.1	15.7	15.7	15.7	15.7

Calibrate LIBOR Market Model

- Parametric specification: $\sigma_m(t) = v_m \, \mathrm{e}^{-\beta(T_m - t)}$
- Recall: $v_n^2 \, \frac{1-\mathrm{e}^{-2\beta T_n}}{2\beta} = \sigma^2_{Cpl(T_n, T_{n+1})} \, T_n$

$\Rightarrow$ $v_1, \dots, v_{19}$ as functions on β (= degree of freedom)

- Sanity check: MC algorithm reproduces 3.5% market cap prices independently the correlation specification ...

Price a Swaption

- Price the at-the-money 4×6-swaption: maturity in 4 years, underlying swap 6 years long
- Tenor: first reset date $T_8 = 4$, and annual(!) coupon payments at $T_{10} = 5, \ldots, T_{20} = 10$
- Swaption price depends on β and on correlation specifications I ($d = 1$, $\ell_m = 1$) or II ($d = M - 1$, $\ell_m = e_m^\top$).
- Compute by MC and analytic approximation formula

Results

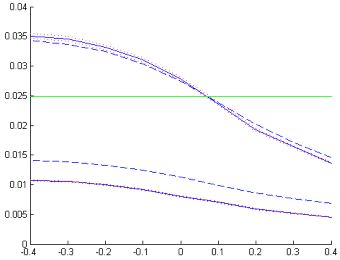

Figure: The swaption price as function of β. The straight horizontal line indicates the real market quote of 248 bp. The upper curves are for the correlation specification I, the lower curves are for specification II. The solid lines show the Monte Carlo simulation based prices with standard errors indicated by the dotted lines. The dashed lines show the respective prices based on the analytic approximation formula.

Discussion

- Actual market quote for this swaption was 248 bp
- For correlation specification I: obtain an estimate $\beta \approx 0.07$
- For correlation specification II: diversification effect between underlying independent LIBOR rates

$\Rightarrow$ Lower aggregate volatility and thus lower swaption price

- Cannot calibrate to actual market price: indicates that specification II systematically underprices swaptions
- Analytic approximation error: 7–10 pb (specification I), 20–35 bp (specification II)
- More systematic tests for quality of analytical approximation in [12, Chapter 8]

Outlook: Volatility Smile

- Snag: bootstrapped caplet implied volatilities depend on strike rate in general (volatility smile/skew)

$\Rightarrow$ Lognormal LIBOR market model can only match term-structure of caplet volatility for one strike rate at a time (no matter how many driving Brownian motions d)

- Situation similar to Black–Scholes stock market model: incapable of fitting market option prices across all strikes (note: Heston stochastic volatility model can produce volatility skews)
- Possible extensions:
  - $\lambda(t, T_m) = \lambda(\omega, t, T_m)$ progressive process (analogous to Heston model)
  - replace driving Brownian motion by Lévy process
- Much research over last decade to achieve a good fitting of market option data, see e.g. Brigo and Mercurio [12, Part IV] for a detailed overview

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Additional Assumptions

- Aim: specify all forward LIBOR rates $L(t, T)$, $T \in [0, T_{M-1}]$: fill the gaps between the T_js
- Each forward LIBOR $L(t, T)$ will follow a lognormal process under $T + \delta$-forward measure

Additional assumptions:

- stochastic basis: $\mathcal{F}_t = \mathcal{F}_t^{W^{T_M}}$ (for representation theorem)
- $\forall T \in [0, T_{M-1}]$: an $\mathbb{R}^d$-valued deterministic bounded measurable function $\lambda(t, T_m)$ = volatility of $L(t, T_m)$
- an initial positive and nonincreasing term-structure $P(0, T)$, $T \in [0, T_M]$, and hence nonnegative initial forward LIBOR rates

$$L(0, T) = \frac{1}{\delta}\left(\frac{P(0, T)}{P(0, T+\delta)} - 1\right), \quad T \in [0, T_{M-1}]$$

Construction 1st Step

- Construct discrete-tenor model for $L(t, T_m)$, $m = 0, \dots, M - 1$, as in Section "LIBOR Market Model"

2nd Step: Interpolate for $T \in [T_{M-1}, T_M]$

- No forward LIBOR for $T \in [T_{M-1}, T_M]$ (not defined)
- Recall

$$P(0, T_m) = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{1}{B^*(T_m)}\right], \quad m \le M$$

- By monotonicity of $P(0, T)$: $\exists$ unique nondecreasing $\alpha : [T_{M-1}, T_M] \to [0, 1]$ with
  1. $\alpha(T_{M-1}) = 0$ and $\alpha(T_M) = 1$
  2. $\log B^*(T) := (1 - \alpha(T)) \log B^*(T_{M-1}) + \alpha(T) \log B^*(T_M)$ satisfies

$$P(0, T) = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{1}{B^*(T)}\right], \quad T \in [T_{M-1}, T_M]$$

2nd Step cont'd

- For $T \in [T_{M-1}, T_M]$ ($B^*(T)$ is $\mathcal{F}_T$-measurable and positive): define T-forward measure $\mathbb{Q}^T \sim \mathbb{Q}^*$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^*} = \frac{1}{B^*(T)P(0,T)}$$

- Note: $\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_M}} = \frac{d\mathbb{Q}^T}{d\mathbb{Q}^*}\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_M}} = \frac{B^*(T_M)P(0,T_M)}{B^*(T)P(0,T)}$
- Representation Theorem: $\exists$ unique $\sigma_{T,T_M} \in \mathcal{L}$ such that

$$\begin{aligned}\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} &= \mathbb{E}_{\mathbb{Q}^{T_M}}\left[\frac{B^*(T_M)P(0,T_M)}{B^*(T)P(0,T)} \mid \mathcal{F}_t\right] \\ &= \mathcal{E}_t\left(\sigma_{T,T_M} \bullet W^{T_M}\right), \quad t \in [0,T]\end{aligned}$$

- Girsanov: $W^T(t) = W^{T_M}(t) - \int_0^t \sigma_{T,T_M}(s)^\top\, ds$ is a $\mathbb{Q}^T$-Brownian motion, $t \in [0,T]$

3rd Step: Backward Induction

- Fix $T \in [T_{M-2}, T_{M-1}]$
- Can now define forward LIBOR $L(t, T)$:

$$dL(t,T) = L(t,T)\lambda(t,T)\,dW^{T+\delta}(t),$$
$$L(0,T) = \frac{1}{\delta}\left(\frac{P(0,T)}{P(0,T+\delta)} - 1\right)$$

- This defines bounded progressive $\sigma_{T,T+\delta}(t) = \frac{\delta L(t,T)}{\delta L(t,T)+1}\lambda(t,T)$
- T-Forward measure defined by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T+\delta}} = \mathcal{E}_T\left(\sigma_{T,T+\delta} \bullet W^{T+\delta}\right)$$

3rd Step cont'd

Note: $\sigma_{T,T_M} = \sigma_{T,T+\delta} + \sigma_{T+\delta,T_M}$ satisfies $(t \in [0, T])$

$$
\begin{aligned}
\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_M}}\Big|_{\mathcal{F}_t} &= \frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T+\delta}}\Big|_{\mathcal{F}_t} \frac{d\mathbb{Q}^{T+\delta}}{d\mathbb{Q}^{T_M}}\Big|_{\mathcal{F}_t} \\
&= \mathcal{E}_t\left(\sigma_{T,T+\delta} \bullet W^{T+\delta}\right) \mathcal{E}_t\left(\sigma_{T+\delta,T_M} \bullet W^{T_M}\right) \\
&= \mathcal{E}_t\left(\sigma_{T,T_M} \bullet W^{T_M}\right)
\end{aligned}
$$

- Proceeding by backward induction $\Rightarrow$ forward measure $\mathbb{Q}^T$, $\mathbb{Q}^T$-Brownian motion W^T for all $T \in [0, T_M]$

$\Rightarrow$ forward LIBOR $L(t, T)$ for all $T \in [0, T_{M-1}]$

Zero-Coupon Bonds

- Finally: obtain zero-coupon bond prices for all maturities
- Fix $0 \leq T \leq S \leq T_M$
- Note: $\sigma_{T,S} = \sigma_{T,T_M} - \sigma_{S,T_M}$ satisfies ($t \in [0, T]$)

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^T}|_{\mathcal{F}_t} = \frac{d\mathbb{Q}^S}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} \frac{d\mathbb{Q}^{T_M}}{d\mathbb{Q}^T}|_{\mathcal{F}_t} = \mathcal{E}_t\left(-\sigma_{T,S} \bullet W^T\right)$$

- By part "Forward Measures": consistently define forward price process

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)} \frac{d\mathbb{Q}^S}{d\mathbb{Q}^T}|_{\mathcal{F}_t} = \frac{P(0,S)}{P(0,T)} \mathcal{E}_t\left(-\sigma_{T,S} \bullet W^T\right)$$

Term-Structure Models

Damir Filipović

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- In particular for $t = T$:

$$P(T,S) = \frac{P(0,S)}{P(0,T)} \mathcal{E}_T \left(-\sigma_{T,S} \bullet W^T \right)$$

- Exercise: $\frac{P(t,T)}{B^*(t)}$ is $\mathbb{Q}^*$-martingale
- Note: $P(T,S)$ may be greater than 1, unless $S - T = m\delta$ for some $m \in \mathbb{N}$

$\Rightarrow$ Even though all δ-period forward LIBOR $L(t,T)$ are nonnegative, there may be negative interest rates for other than δ periods.

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