The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}(10^3 + 1)^2 &= (x + 1)^2 \\ &= x^2 + 2x + 1 \\ &= (10^3)^2 + 2(10^3) + 1 \\ &= 1\,002\,001\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

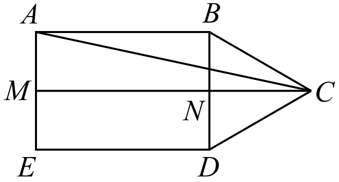
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b+2)$.
Step 3 moves $(a, -b+2)$ to $(-a, -b+2)$.
Thus, S_1 has coordinates $(-a, -b+2)$.
Step 1 moves $(-a, -b+2)$ to $(-a, b-2)$.
Step 2 moves $(-a, b-2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0

Now, $\frac{AD}{BD} = \frac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

<u>Case 1a: $b = 5$, $d = 3$, $a = 6$</u>

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1 \cdot 2 \cdot \cdots \cdot 199 \cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2} - b\right)^2 + 3\left(\frac{a}{2}\right)^2 = \frac{a^2}{4} - 2 \cdot \frac{a}{2} \cdot b + b^2 + \frac{3a^2}{4} = a^2 + b^2 - ab$$

Thus, if $K = \frac{a}{2} - b$ and $L = \frac{a}{2}$, we have $K^2 + 3L^2 = a^2 + b^2 - ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2} - a\right)^2 + 3\left(\frac{b}{2}\right)^2 = a^2 + b^2 - ab$$

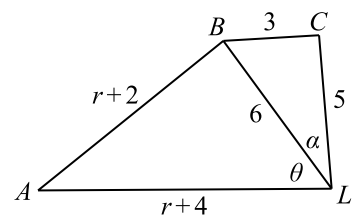
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \frac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

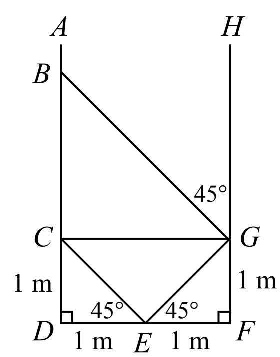
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1, 10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1 : 4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1 : 6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n + 1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \dfrac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

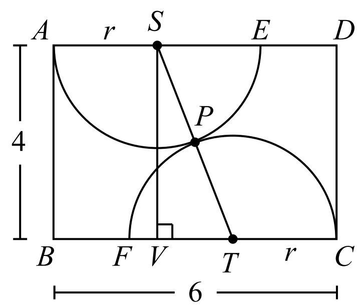
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2}\cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2\cdot EB\cdot BD\cdot\cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x)\cdot\tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.
Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.
This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.
Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.
Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.
We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.
We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} && \text{(knowing a formula for } g^{-1}) \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 && \text{(knowing a formula for } g^{-1}) \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.
Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.
Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\begin{aligned}\log_2(\sin x) + \log_2(\cos y) &= -\frac{3}{2}\\ \log_2(\sin x) - \log_2(\cos y) &= \frac{1}{2}\end{aligned}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\begin{aligned}\sin x \cos y &= 2^{-3/2} = \frac{1}{2\sqrt{2}}\\ \frac{\sin x}{\cos y} &= 2^{1/2} = \sqrt{2}\end{aligned}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}(10^3 + 1)^2 &= (x + 1)^2 \\ &= x^2 + 2x + 1 \\ &= (10^3)^2 + 2(10^3) + 1 \\ &= 1\,002\,001\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

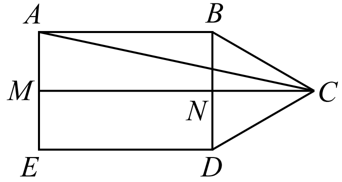
Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b + 2)$.
Step 3 moves $(a, -b + 2)$ to $(-a, -b + 2)$.
Thus, S_1 has coordinates $(-a, -b + 2)$.
Step 1 moves $(-a, -b + 2)$ to $(-a, b - 2)$.
Step 2 moves $(-a, b - 2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0

Now, $\frac{AD}{BD} = \frac{2}{1}$ and so $AD = 2BD = 4x$.

Suppose that M is the midpoint of AE and N is the midpoint of BD.

Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.

Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\cdots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

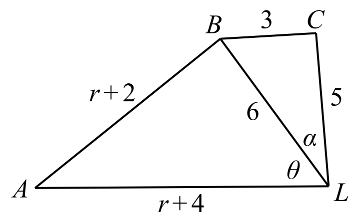
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

[illegible]

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

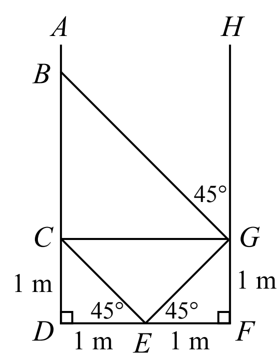
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1, 10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1 : 4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1 : 6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure $45°$ is isosceles.
This is because the measure of the third angle equals $180° - 90° - 45° = 45°$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90°$ and $\angle DCG = 90°$, then $\angle BGC = 180° - 90° - 45° = 45°$ and $\angle BCG = 90°$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\frac{a}{b} < \frac{4}{7}$ and $\frac{4}{7} < 1$, then $\frac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\frac{5}{9} < \frac{a}{a+15} < \frac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \frac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\frac{a}{b} = \frac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\frac{1}{2}$ are $10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\frac{5/16}{5/4}$ which equals $\frac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\frac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\frac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\frac{10 + 5d}{10 + 3d} = \frac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\frac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \le p \le 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \le p \le 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned}(1-3)(z+2) &= 0 \\ 1 + yz &= 9\end{aligned}$$

or

$$\begin{aligned}-2(z+2) &= 0 \\ yz &= 8\end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

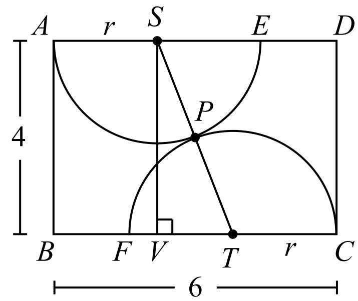
$$\begin{aligned}(x-3)(z+2) &= 0 \\ x + 2z &= 9\end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.
Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.
This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.
Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.
Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.
We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.
We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}) \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}) \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.
Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.
Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$$
$$\log_2(\sin x) - \log_2(\cos y) = \frac{1}{2}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\sin x \cos y = 2^{-3/2} = \frac{1}{2\sqrt{2}}$$
$$\frac{\sin x}{\cos y} = 2^{1/2} = \sqrt{2}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}(10^3 + 1)^2 &= (x + 1)^2 \\ &= x^2 + 2x + 1 \\ &= (10^3)^2 + 2(10^3) + 1 \\ &= 1\,002\,001\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

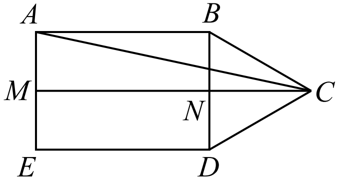
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b+2)$.
Step 3 moves $(a, -b+2)$ to $(-a, -b+2)$.
Thus, S_1 has coordinates $(-a, -b+2)$.
Step 1 moves $(-a, -b+2)$ to $(-a, b-2)$.
Step 2 moves $(-a, b-2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0

Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a=\frac{6+6\sqrt{2}}{3-3\sqrt{2}}=\frac{2+2\sqrt{2}}{1-\sqrt{2}}=\frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})}=\frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2}=-6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\dots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\dots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

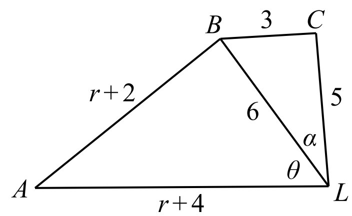
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

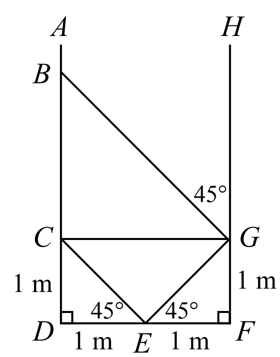
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1, 10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1 : 4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1 : 6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n + 1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\frac{a}{b} < \frac{4}{7}$ and $\frac{4}{7} < 1$, then $\frac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\frac{5}{9} < \frac{a}{a+15} < \frac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \frac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\frac{a}{b} = \frac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\frac{1}{2}$ are $10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\frac{5/16}{5/4}$ which equals $\frac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\frac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\frac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\frac{10 + 5d}{10 + 3d} = \frac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\frac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

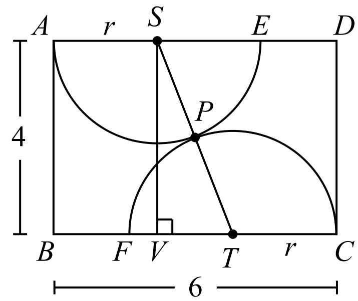
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.
Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.
This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.
Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.
Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.
We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.
We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}) \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}) \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.
Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.
Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$$
$$\log_2(\sin x) - \log_2(\cos y) = \frac{1}{2}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \le x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \le y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\sin x \cos y = 2^{-3/2} = \frac{1}{2\sqrt{2}}$$
$$\frac{\sin x}{\cos y} = 2^{1/2} = \sqrt{2}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \le x < 180^\circ$, it must be the case that $\sin x \ge 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \le x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \le y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}
(10^3 + 1)^2 &= (x + 1)^2 \\
&= x^2 + 2x + 1 \\
&= (10^3)^2 + 2(10^3) + 1 \\
&= 1\,002\,001
\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

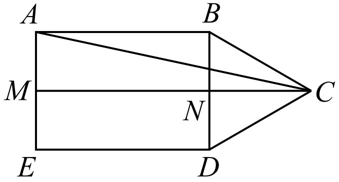
Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b + 2)$.
Step 3 moves $(a, -b + 2)$ to $(-a, -b + 2)$.
Thus, S_1 has coordinates $(-a, -b + 2)$.
Step 1 moves $(-a, -b + 2)$ to $(-a, b - 2)$.
Step 2 moves $(-a, b - 2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0

Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.

Suppose that M is the midpoint of AE and N is the midpoint of BD.

Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.

Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Case 1: $b = 5$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2 \cdots (397!)^2(399!)^2 \cdot 2^{200} \cdot (1 \cdot 2 \cdot \cdots \cdot 199 \cdot 200)}{200!}$$

Since $1 \cdot 2 \cdot \cdots \cdot 199 \cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2 \cdots (397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!) \cdots (397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

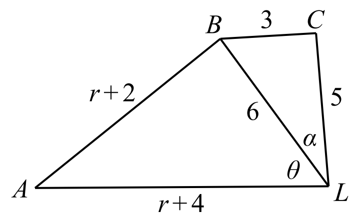
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.

Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.

The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.

Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.

Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.

The line with equation $y = x + 6$ has y-intercept 6.

(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.

Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.

When $x = 1$, we get $y = x + 9 = 10$.

Thus, these two lines intersect at $(1, 10)$.

Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.

Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.

From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).

The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.

Why is this the only possible value of m?

We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.

Thus, $b \geq 2$ and $c \geq 2$.

Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.

Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.

Thus, $c = 3$.

Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.

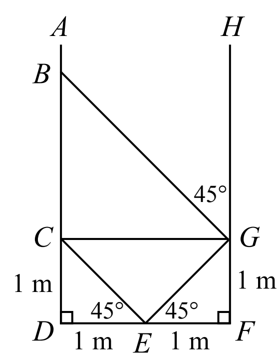
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.

Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1:4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1:6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \dfrac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

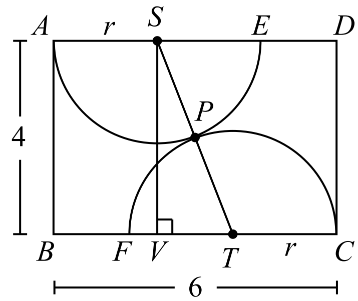
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.

Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.

This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.

Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.

Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.

We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.

We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} && \text{(knowing a formula for } g^{-1}\text{)} \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 && \text{(knowing a formula for } g^{-1}\text{)} \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.

Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.

Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\begin{aligned}\log_2(\sin x) + \log_2(\cos y) &= -\frac{3}{2} \\ \log_2(\sin x) - \log_2(\cos y) &= \frac{1}{2}\end{aligned}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\begin{aligned}\sin x \cos y &= 2^{-3/2} = \frac{1}{2\sqrt{2}} \\ \frac{\sin x}{\cos y} &= 2^{1/2} = \sqrt{2}\end{aligned}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}(10^3 + 1)^2 &= (x + 1)^2 \\ &= x^2 + 2x + 1 \\ &= (10^3)^2 + 2(10^3) + 1 \\ &= 1\,002\,001\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

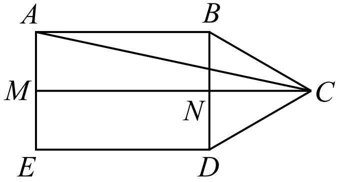
Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b+2)$.
Step 3 moves $(a, -b+2)$ to $(-a, -b+2)$.
Thus, S_1 has coordinates $(-a, -b+2)$.
Step 1 moves $(-a, -b+2)$ to $(-a, b-2)$.
Step 2 moves $(-a, b-2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0.

Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.

Suppose that M is the midpoint of AE and N is the midpoint of BD.

Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.

Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\cdots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

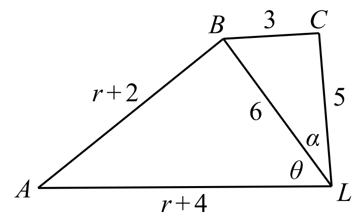
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

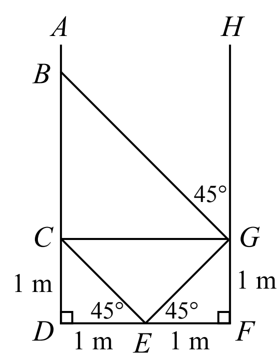
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1, 10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio 1 : 4, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is 1 : 6, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n + 1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2 \text{ m} + 1 \text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\frac{a}{b} < \frac{4}{7}$ and $\frac{4}{7} < 1$, then $\frac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\frac{5}{9} < \frac{a}{a+15} < \frac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \frac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\frac{a}{b} = \frac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\frac{1}{2}$ are $10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\frac{5/16}{5/4}$ which equals $\frac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\frac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\frac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\frac{10 + 5d}{10 + 3d} = \frac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\frac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

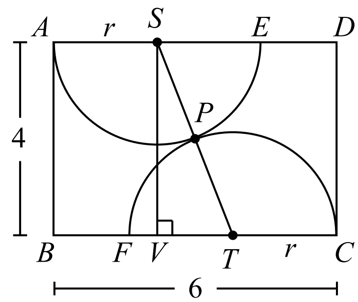
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.
Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.
This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.
Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.
Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.
We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.
We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}) \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}) \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.
Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.
Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\begin{aligned}\log_2(\sin x) + \log_2(\cos y) &= -\frac{3}{2} \\ \log_2(\sin x) - \log_2(\cos y) &= \frac{1}{2}\end{aligned}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\begin{aligned}\sin x \cos y &= 2^{-3/2} = \frac{1}{2\sqrt{2}} \\ \frac{\sin x}{\cos y} &= 2^{1/2} = \sqrt{2}\end{aligned}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}(10^3 + 1)^2 &= (x + 1)^2 \\ &= x^2 + 2x + 1 \\ &= (10^3)^2 + 2(10^3) + 1 \\ &= 1\,002\,001\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

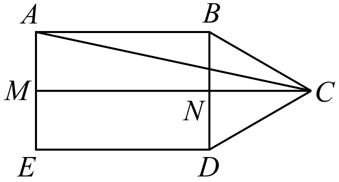
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

Suppose that S_0 has coordinates (a,b).
Step 1 moves (a,b) to $(a,-b)$.
Step 2 moves $(a,-b)$ to $(a,-b+2)$.
Step 3 moves $(a,-b+2)$ to $(-a,-b+2)$.
Thus, S_1 has coordinates $(-a,-b+2)$.
Step 1 moves $(-a,-b+2)$ to $(-a,b-2)$.
Step 2 moves $(-a,b-2)$ to $(-a,b)$.
Step 3 moves $(-a,b)$ to (a,b).
Thus, S_2 has coordinates (a,b), which are the same coordinates as S_0.

Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$

$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

<u>Case 1a: $b = 5$, $d = 3$, $a = 6$</u>

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\cdots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2} - b\right)^2 + 3\left(\frac{a}{2}\right)^2 = \frac{a^2}{4} - 2 \cdot \frac{a}{2} \cdot b + b^2 + \frac{3a^2}{4} = a^2 + b^2 - ab$$

Thus, if $K = \frac{a}{2} - b$ and $L = \frac{a}{2}$, we have $K^2 + 3L^2 = a^2 + b^2 - ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2} - a\right)^2 + 3\left(\frac{b}{2}\right)^2 = a^2 + b^2 - ab$$

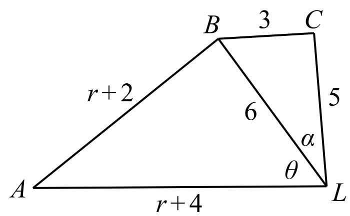
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

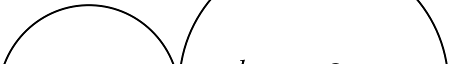
By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$



[illegible]

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x+6$.
The line with equation $y = x+6$ has y-intercept 6.

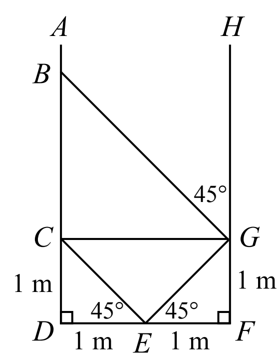
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x+7$ and $y = x+9$ intersect.
Equating values of y, we obtain $3x+7 = x+9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x+9 = 10$.
Thus, these two lines intersect at $(1,10)$.
Since all three lines pass through the same point, the line with equation $y = mx+17$ passes through $(1,10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1:4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1:6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \dfrac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

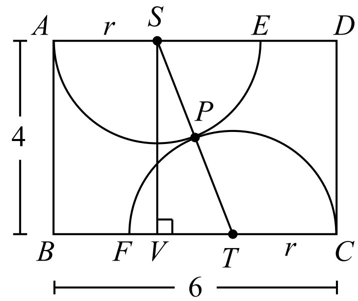
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45°$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30°$.
Since $\angle ABC = 135°$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135° - 45° - 30° = 60°$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60°$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60°) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.

Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.

This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.

Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.

Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.

We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.

We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} && \text{(knowing a formula for } g^{-1}\text{)} \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 && \text{(knowing a formula for } g^{-1}\text{)} \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.

Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.

Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$$
$$\log_2(\sin x) - \log_2(\cos y) = \frac{1}{2}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\sin x \cos y = 2^{-3/2} = \frac{1}{2\sqrt{2}}$$
$$\frac{\sin x}{\cos y} = 2^{1/2} = \sqrt{2}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$
\begin{aligned}
(10^3 + 1)^2 &= (x + 1)^2 \\
&= x^2 + 2x + 1 \\
&= (10^3)^2 + 2(10^3) + 1 \\
&= 1\,002\,001
\end{aligned}
$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2

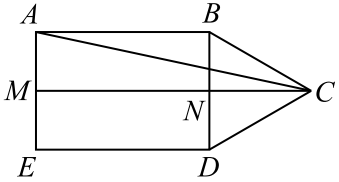
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b + 2)$.
Step 3 moves $(a, -b + 2)$ to $(-a, -b + 2)$.
Thus, S_1 has coordinates $(-a, -b + 2)$.
Step 1 moves $(-a, -b + 2)$ to $(-a, b - 2)$.
Step 2 moves $(-a, b - 2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0

Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n - 3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\cdots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

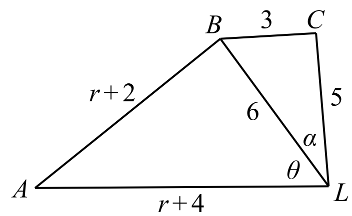
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned}
AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\
(r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\
12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\
\cos\theta &= \frac{4r+48}{12(r+4)} \\
\cos\theta &= \frac{r+12}{3r+12}
\end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned}
BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\
3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\
60\cos\alpha &= 36 + 25 - 9 \\
\cos\alpha &= \frac{52}{60} \\
\cos\alpha &= \frac{13}{15}
\end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned}
\cos 2\alpha &= 2\cos^2\alpha - 1 \\
&= 2 \cdot \frac{169}{225} - 1 \\
&= \frac{338}{225} - \frac{225}{225} \\
&= \frac{113}{225}
\end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

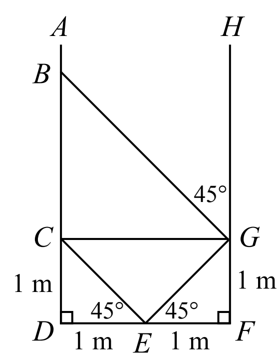
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1, 10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1 : 4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1 : 6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\frac{a}{b} < \frac{4}{7}$ and $\frac{4}{7} < 1$, then $\frac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\frac{5}{9} < \frac{a}{a+15} < \frac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \frac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\frac{a}{b} = \frac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\frac{1}{2}$ are $10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\frac{5/16}{5/4}$ which equals $\frac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\frac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\frac{10+5d}{10+3d}$.
Since these ratios are equal, then $\frac{10+5d}{10+3d} = \frac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\frac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

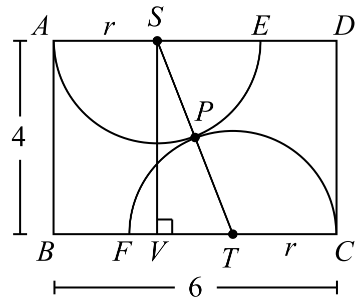
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.

Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.

This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.

Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.

Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.

We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.

We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}) \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}) \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.

Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.

Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\begin{aligned}\log_2(\sin x) + \log_2(\cos y) &= -\frac{3}{2}\\ \log_2(\sin x) - \log_2(\cos y) &= \frac{1}{2}\end{aligned}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\begin{aligned}\sin x \cos y &= 2^{-3/2} = \frac{1}{2\sqrt{2}}\\ \frac{\sin x}{\cos y} &= 2^{1/2} = \sqrt{2}\end{aligned}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}
(10^3 + 1)^2 &= (x + 1)^2 \\
&= x^2 + 2x + 1 \\
&= (10^3)^2 + 2(10^3) + 1 \\
&= 1\,002\,001
\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

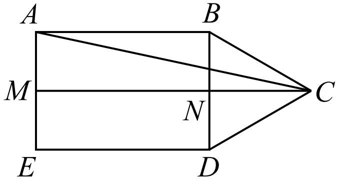
Solution 2
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b + 2)$.
Step 3 moves $(a, -b + 2)$ to $(-a, -b + 2)$.
Thus, S_1 has coordinates $(-a, -b + 2)$.
Step 1 moves $(-a, -b + 2)$ to $(-a, b - 2)$.
Step 2 moves $(-a, b - 2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0.

Now, $\frac{AD}{BD} = \frac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

$$a=\frac{6+6\sqrt{2}}{3-3\sqrt{2}}=\frac{2+2\sqrt{2}}{1-\sqrt{2}}=\frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})}=\frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2}=-6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\cdots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

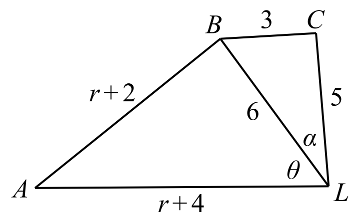
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

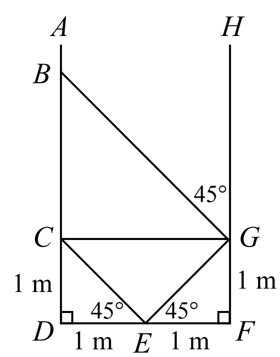
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1, 10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1, 10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1 : 4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1 : 6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n + 1 + \dfrac{15}{n}$ is an integer.
Since $n + 1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2 \text{ m} + 1 \text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\frac{a}{b} < \frac{4}{7}$ and $\frac{4}{7} < 1$, then $\frac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\frac{5}{9} < \frac{a}{a+15} < \frac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \frac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\frac{a}{b} = \frac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\frac{1}{2}$ are $10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\frac{5/16}{5/4}$ which equals $\frac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\frac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\frac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\frac{10 + 5d}{10 + 3d} = \frac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\frac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

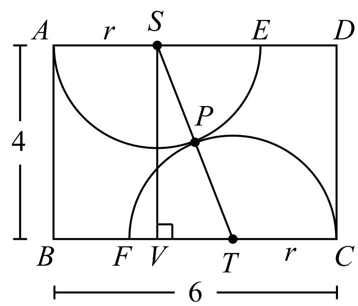
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2}\cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2\cdot EB\cdot BD\cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x)\cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.

Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.

This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.

Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.

Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.

We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.

We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}) \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}) \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.

Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.

Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$$
$$\log_2(\sin x) - \log_2(\cos y) = \frac{1}{2}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\sin x \cos y = 2^{-3/2} = \frac{1}{2\sqrt{2}}$$
$$\frac{\sin x}{\cos y} = 2^{1/2} = \sqrt{2}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

The sum of the digits of this integer is $1 + 2 + 1$ which equals 4.
To determine this integer without using a calculator, we can let $x = 10^3$.
Then

$$\begin{aligned}(10^3 + 1)^2 &= (x + 1)^2 \\ &= x^2 + 2x + 1 \\ &= (10^3)^2 + 2(10^3) + 1 \\ &= 1\,002\,001\end{aligned}$$

The slope of the line segment joining $A(0,8)$ and $C(8,2)$ is $\frac{8-2}{0-8}$ which equals $-\frac{3}{4}$.

Solution 2

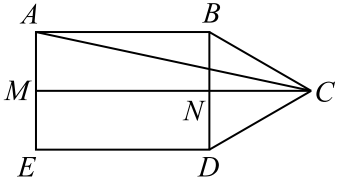
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC. Therefore, the slope of AC equals the slope of the line segment joining $M(3,9)$ to $N(7,6)$, which is $\frac{9-6}{3-7}$ or $-\frac{3}{4}$.

(c) Since $V(1,18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.

5. (a) *Solution 1*

Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b + 2)$.
Step 3 moves $(a, -b + 2)$ to $(-a, -b + 2)$.
Thus, S_1 has coordinates $(-a, -b + 2)$.
Step 1 moves $(-a, -b + 2)$ to $(-a, b - 2)$.
Step 2 moves $(-a, b - 2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0.

Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n - 3)d = 95$.

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

(b) Using the definition of f, the following equations are equivalent:

$$f(a) = 0$$
$$2a^2 - 3a + 1 = 0$$

Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2\cdot 2^{200}\cdot(1\cdot 2\cdot\cdots\cdot 199\cdot 200)}{200!}$$

Since $1\cdot 2\cdot\cdots\cdot 199\cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2}-b\right)^2+3\left(\frac{a}{2}\right)^2=\frac{a^2}{4}-2\cdot\frac{a}{2}\cdot b+b^2+\frac{3a^2}{4}=a^2+b^2-ab$$

Thus, if $K=\frac{a}{2}-b$ and $L=\frac{a}{2}$, we have $K^2+3L^2=a^2+b^2-ab$.

If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2}-a\right)^2+3\left(\frac{b}{2}\right)^2=a^2+b^2-ab$$

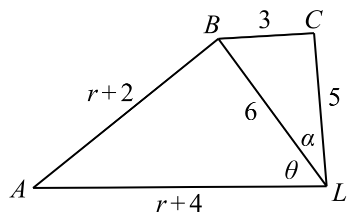
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

$$= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625}$$

[illegible]

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.

Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.

The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.

Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.

Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.

The line with equation $y = x + 6$ has y-intercept 6.

(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.

Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.

When $x = 1$, we get $y = x + 9 = 10$.

Thus, these two lines intersect at $(1,10)$.

Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1,10)$.

Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.

From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).

The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.

Why is this the only possible value of m?

We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.

Thus, $b \geq 2$ and $c \geq 2$.

Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.

Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.

Thus, $c = 3$.

Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.

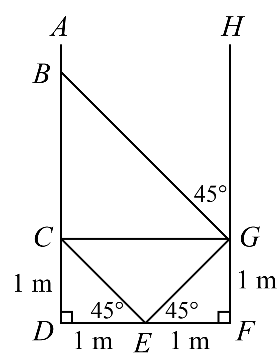
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.

Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1:4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5}\cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1:6$, which means that she has $6\cdot 20 = 120$ gold marbles.
Eleanor now has $20+120 = 140$ marbles, which means that she added $140-100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n}+\dfrac{n}{n}+\dfrac{15}{n} = n+1+\dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n+1+\dfrac{15}{n}$ is an integer.
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \dfrac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \le p \le 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \le p \le 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

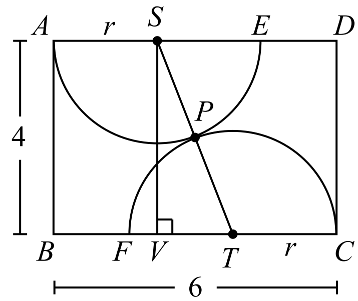
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45°$ and $BE = \sqrt{2}AB = \sqrt{2} \cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30°$.
Since $\angle ABC = 135°$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135° - 45° - 30° = 60°$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60°$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2 \cdot EB \cdot BD \cdot \cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60°) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x) \cdot \tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.
Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.
This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.
Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.
Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.
We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.
We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}\text{)} \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}\text{)} \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.
Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.
Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$$
$$\log_2(\sin x) - \log_2(\cos y) = \frac{1}{2}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\sin x \cos y = 2^{-3/2} = \frac{1}{2\sqrt{2}}$$
$$\frac{\sin x}{\cos y} = 2^{1/2} = \sqrt{2}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.