ADVANCES IN MATHEMATICS 75, 1–118 (1989)

Formal Power Series of Logarithmic Type

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TO THE MEMORY OF JEANETTE OPPENHEIMER

CONTENTS

1. INTRODUCTION

Since its beginning with Barrow and Newton, the calculus of finite differences has been viewed—whether admittedly or not—as a formal method of computation with special functions. Until the nineteenth century, when the rigors of convergence were to be gradually imposed, mathematicians handling difference equations and series involving polynomials and

* Supported under a National Science Foundation Graduate Fellowship.
† Partially supported by the National Science Foundation under Grant MCS-8104855.

0001-8708/89 $7.50

logarithms were not beset by doubts as to the correctness of their manipulations, and in fact, their results have seldom turned out to be incorrect, even by the standards of our day.

There were, however, some embarrassing exceptions—which mathematicians in this century have chosen largely to ignore. Perhaps the best known of these is the Euler–MacLaurin summation formula (cf. Section 5.1). When applied to any "function" other than a polynomial, this formula gives a divergent series, which nonetheless can be used to obtain astonishingly good numerical approximations, and which can be used without fear in formal manipulations. It is of little help to justify such manipulations by appealing to Poincaré's definition of an asymptotic expansion. Most Euler–MacLaurin series contain logarithmic terms and other functions growing slower than any polynomial, whose asymptotic expansion according to Poincaré would equal zero. The suggestion first made by Dubois-Reymond, and later taken up by G. H. Hardy—that the notion of an asymptotic expansion be reinforced by logarithmic scales—has not been developed, nor is it clear how the difficulties of its implementation are to be surmounted, or even whether such difficulties are worth surmounting.

Browbeaten by demands of rigor that would eventually be seen as spurious, the early difference-equationists of this century—such stalwarts as Milne-Thompson, Nörlund, Pincherle, and Steffensen—went to great lengths to devise acceptable definitions of the "natural" solutions of difference equations. With due respect to our predecessors, we submit that their proposals can nowadays be classified as pointless. What is needed instead in order to justify formal manipulations—we can state nowadays with the confidence that comes after fifty years of local algebra—is an extension of the notion of formal power series that will include, besides powers of x, powers series in other special functions of classical analysis, most notably exponentials and logarithms. It is the purpose of this work to carry out such an extension, one that includes the formal theory of infinite series in polynomials and logarithms.

The difficulty of this program, and perhaps the reason why it has not been previously carried out, is the algebraic unwieldiness of the functions $x^n(\log x)^t$, where n is an integer and t a nonnegative integer. The coefficients in the Taylor expansions in powers of a of the functions $(x+a)^n\,(\log(x+a))^t$ do not seem to follow much rhyme or reason, and unless these coefficients are somehow made easy to handle, no simple operational calculus can be obtained. We resolve this difficulty by introducing another basis for the vector space spanned by the functions $x^n(\log x)^t$, a basis whose members we call the *harmonic logarithms*. We denote by $\lambda_n^{(t)}(x)$ the harmonic logarithm of order t and degree n. For positive integers t and integers n, or for $t=0$ and nonnegative integers n, the harmonic logarithms of order t and n span the subspace $L^{(t)}$ of the logarithmic algebra spanned by $x^n(\log x)^t$ over the same values of n and t. In particular, $L^{(0)}$ is the subspace of ordinary polynomials in x. When t is a positive integer, however, the variable n is allowed to vary over all integers, positive or negative. One has, for example, for $t=1$,

$$\lambda_n^{(1)}(x)=\begin{cases} x^n(\log x - 1 - \frac{1}{2} - \cdots - 1/n) & \text{for } n \geqslant 0, \text{ and} \\ x^n & \text{for } n<0. \end{cases} \tag{1}$$

The harmonic logarithms turn out to satisfy an identity which "logarithmically" generalizes the binomial theorem for polynomials, to wit:

$$\lambda_n^{(t)}(x+a)=\sum_{k\geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k \lambda_{n-k}^{(t)}(x).$$

The coefficients $\lfloor {n \atop k} \rceil$ are generalizations of the binomial coefficients (and, in fact, coincide with the binomial coefficients in all cases in which the binomial coefficients are defined). We propose to call them the *Roman coefficients.* They satisfy identities similar to those satisfied by the binomial coefficients, and they are defined for all integers n and k.

The ring of formal series of logarithmic type is now defined in two steps in terms of the harmonic logarithms. First, one completes the subspace $L^{(t)}$ into a vector space $\mathscr{L}^{(t)}$ in such a way as to obtain formal series of the form

$$\sum_{n \leqslant d} b_n \lambda_n^{(t)}(x), \tag{2}$$

and second, one takes the algebraic direct sum $\mathscr{L}$ of the vector spaces $\mathscr{L}^{(t)}$, thereby obtaining the *logarithmic algebra.* Thus, a formal series of logarithmic type is a *finite* sum of (infinite) formal series of the form (2) for different values of t. Note that for $t=0$, one obtains nothing more than polynomials in x.

The ring of formal differential operators with constant coefficients acts on the logarithmic algebra $\mathscr{L}$. More pleasingly, the ring of all formal *Laurent* series in the derivative $\mathbf{D}$ acts on the subspace $\mathscr{L}^{(+)}$ which is the direct sum of all subspaces $\mathscr{L}^{(t)}$ for t positive. The subspace $\mathscr{L}^{(+)}$ turns out to be the "largest" subspace of the logarithmic algebra on which the derivative operator is invertible. This fact (together with certain commutation relations explained in the text) leads to a definition of natural solutions of difference equations which is more general than the ones previously given by Milne-Thompson and Nörlund, and which coincides with them in all cases where both are defined.

The remainder of this work develops logarithmic analogs of various notions that were previously only known for polynomials, notably the logarithmic analog of Appell polynomials and the logarithmic analog of the theory of sequences of binomial type, which we call Roman graded sequences. Several special cases are worked out, notably the logarithmic analogs of Bernoulli and Hermite polynomials, of Gould and Laguerre polynomials, and of the factorial powers of the calculus of finite differences. In the last example, one finds that the Gauss ψ-function (the logarithmic derivative of the gamma function) is one term in the logarithmic extension of the lower factorial function. Thus, classical identities satisfied by the ψ-function are seen to be trivial consequences of general logarithmic identities. We stress the fact that the examples given here are only a sampling of the special functions that can be brought under the logarithmic umbrella.

We are profoundly indebted to the work of S. Roman, whose theory of formal series provided the guiding thread for the present theory, as well as to Roberto Matarazzo, who first suggested (while a student at a summer course in Cortona) that Roman's theory could be made more concrete by introducing the functions given by Eq. (1). We are also indebted to the many contributors to the theory of polynomial sequences of binomial type and to the umbral calculus, to Barnabei, Brini, Nicoletti, Joni, Garsia, Kahaner, Odlyzko, Bender, Goldman, Ueno, and Watanbe to name only a few. Finally, we would like to especially thank Askey, Chen, Knuth, Niederhausen, and Roman for their insightful comments. In particular, we would like to credit: Chen for providing the second proof of Proposition 3.3.5 and Theorem 4.3.8, and for giving an alternate definition of the harmonic logarithm (Theorem 4.2.2); Chen and Knuth for having independently simplified Proposition 3.2.5; Knuth for independently arriving at Proposition 3.1.2, for discovering Propositions 3.2.7 and 5.3.2, and for devising the third proof of Theorem 4.3.8; and Roman for arriving at the correct statements of Proposition 5.3.1 and Theorems 4.3.7 and 5.6.2.

We suggest the reader begin by looking at the synopsis (Section 2), and then proceed directly to the section of examples (Section 8). After lightly scanning Section 3 on Roman coefficients and harmonic numbers, the reader may proceed to the main body of the text, namely, to Sections 4–7.

2. Synopsis

Given the algebra $K[x]$ of polynomials in the variable x over a field K of characteristic zero, it is often convenient to extend this algebra into the algebra of formal power series $K[[x]]$. This is done by completing the algebra $K[x]$ in the topology for which the sequence of polynomials $(x^n)_{n \geqslant 0}$ tends to zero. The algebra $K[[x]]$ is an integral domain whose quotient field is the field of formal Laurent series.

Our first objective will be to devise an analogous process for another algebra, namely, the algebra L spanned by the logarithmic powers

$$x^n(\log x)^t, \tag{3}$$

where n is an arbitrary integer, and t is a nonnegative integer. In order to do this, it is necessary to introduce another basis of the algebra L, whose elements we denote by

$$\lambda_n^{(t)}(x),$$

where n is an integer, and t is a nonnegative integer; they are called the *harmonic logarithms*, and they will be described shortly. Each harmonic logarithm will be a finite combination of logarithmic powers as in Eq. (3).

To define the harmonic logarithms (Definition 4.2.1), we require a graded sequence of rational numbers which will be called the *harmonic numbers* (Definition 3.3.3). They are defined as

$$c_n^{(k)} = \frac{(-1)^k}{k!} \lfloor n \rceil ! \, [\mathbf{D}^k (x)_n]_{x=0},$$

where the *Roman factorial* $\lfloor n \rceil !$ is defined as

$$\lfloor n \rceil ! = \begin{cases} (-1)^{n+1}/(-n-1)! & \text{for} \quad n<0, \text{ and} \\ n! & \text{for} \quad n \geqslant 0, \end{cases}$$

and the classical lower factorial of degree k is defined as

$$(x)_k = \begin{cases} \prod_{i=0}^{k-1} (x-i) & \text{for} \quad k \geqslant 0, \text{ and} \\ \prod_{i=k}^{-1} (x-i)^{-1} & \text{for} \quad k<0. \end{cases}$$

For example, for $n \geqslant 0$, one computes the following sample values of the harmonic numbers (Eqs. (11) and (12)):

$$c_n^{(1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$c_n^{(2)} = 1 + \frac{1}{2}\left(1+\frac{1}{2}\right) + \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right).$$

The harmonic numbers turn out to have interesting combinatorial connections—some of which we discuss in the text.

In terms of the harmonic numbers, we define the *harmonic logarithm* of order t and degree n by the formula

$$\lambda_n^{(t)}(x) = x^n \sum_{k=0}^{t} (-1)^k (t)_k \, c_n^{(k)} (\log x)^{t-k}.$$

(A more elegant expression will be given shortly.) See Tables 4.1 through 4.3 for examples.

If we set

$$\lfloor n \rceil = \begin{cases} n & \text{for} \quad n \neq 0, \text{ and} \\ 1 & \text{for} \quad n = 0, \end{cases}$$

then one easily verifies the differential equation

$$\mathbf{D}\lambda_n^{(t)}(x) = \lfloor n \rceil \lambda_{n-1}^{(t)}(x).$$

(See Theorem 4.3.8.)

Thus, the subspaces $L^{(t)}$ spanned by the harmonic logarithms of order t are invariant under the action of the operator $\mathbf{D}$.

By specifying a suitable topology (Definition 4.3.4), we obtain, as the completion of the algebra L, the algebra $\mathscr{L}$ of *formal power series of logarithmic type*. An element $p(x) \in \mathscr{L}$ is informally defined as a finite sum

$$p(x) = \sum_{t=0}^{j} p^{(t)}(x),$$

where each $p^{(t)}(x)$ is an infinite series

$$p^{(t)}(x) = \sum_{n \leqslant d_t} b_n^{(t)} \lambda_n^{(t)}(x).$$

The elements $p^{(t)}(x)$ are said to be *homogeneous* of *order* t. For example, a series $p^{(0)}(x)$, homogeneous of order 0, is an ordinary polynomial, and a homogeneous series of order 1 is an infinite sum of the form

$$p^{(1)}(x) = \sum_{n=0}^{d} b_n^{(1)} x^n \left(\log(x) - 1 - \frac{1}{2} - \cdots - \frac{1}{n} \right) + \sum_{n<0} b_n^{(1)} x^n.$$

Each of the subspaces $\mathscr{L}^{(t)}$ spanned by the homogeneous elements of order t is invariant under the action of $\mathbf{D}$.

If

$$f(\mathbf{D}) = \sum_{k \geqslant 0} a_k \mathbf{D}^k \tag{4}$$

is a formal differential operator, then the action $f(\mathbf{D})$ on any formal power series of logarithmic type $p(x)$ is well defined. Moreover, it turns out than when restricted to the subspace $\mathscr{L}^{(+)}$ spanned by all homogeneous elements of positive order, every operator as in Eq. (4) is invertible. (See Proposition 4.3.9.) In other words, on the subspace $\mathscr{L}^{(+)}$ we have an action of all formal *Laurent* operators in **D**; that is, of all formal differential operators of the form

$$f(\mathbf{D}) = \sum_{k \geqslant d} a_k \mathbf{D}^k,$$

whe d is an *arbitrary* integer, positive or negative. The invertibility of the derivative operator in the subspace $\mathscr{L}^{(+)}$ generated by the harmonic logarithms of strictly positive order turns out to be very useful. It allows us to redefine the harmonic logarithm (Corollary 4.3.10) as

$$\lambda_n^{(t)}(x) = \lfloor n \rceil ! \, \mathbf{D}^{-n} (\log x)^t$$

for n an integer, and t a positive integer.

The harmonic logarithms $\lambda_n^{(t)}(x)$ can be viewed as generalizations of the ordinary powers x^n. In fact, they satisfy an analog of the binomial theorem. To derive this analog, we are led to generalize the notion of a binomial coefficient.

One defines the *Roman coefficients* as

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{\lfloor n \rceil !}{\lfloor k \rceil ! \, \lfloor n-k \rceil !}$$

where both n and k are *arbitrary* integers. When $n \geqslant k \geqslant 0$, the Roman coefficients equal the ordinary binomial coefficients $\binom{n}{k}$. The Roman coefficients give an extension of the binomial coefficients which seem to be the "right" one. In fact, over the field of complex numbers one establishes the Taylor expansion

$$\lambda_n^{(t)}(x+a) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k \lambda_{n-k}^{(t)}(x). \tag{5}$$

The simplest example is the classical,

$$\begin{aligned} \lambda_0^{(1)}(x+a) &= \log(x+a) \\ &= \sum_{k \geqslant 0} \left\lfloor \begin{matrix} 0 \\ -k \end{matrix} \right\rceil a^k \lambda_{-k}^{(1)}(x) \\ &= \log x + \sum_{k > 0} \frac{(-1)^{k+1} a^k}{k x^k}, \end{aligned}$$

which is the Taylor expansion of $\log(x+a)$. By a special device, the Taylor expansion in Eq. (5) can be made to make sense over any field of characteristic zero, and leads to the definition of the shift operator

$$E^a \lambda_n^{(t)}(x) = \lambda_n^{(t)}(x+a).$$

One easily verifies (see Proposition 5.2.1) that $E^a = e^{a\mathbf{D}}$. The Binomial Theorem is a trivial special case of Eq. (5).

The purpose of this work is to derive "logarithmic analogs" of facts relating polynomials and their derivatives. Such analogs often throw a new light on the "classical" version for polynomials. By way of example, recall the symmetric inner product

$$\langle p(x) \mid q(x) \rangle = [p(\mathbf{D})\, q(x)]_{x=0} \tag{6}$$

defined on polynomials $p(x)$ and $q(x)$. The Hilbert space defined by this inner product is of frequent occurrence in Quantum Field Theory (in the *Boson calculus*) and in other circumstances. Remarkably, that inner product (Eq. (6)) can be extended to a symmetric inner product defined on the subspace spanned by the harmonic logarithms.

For $p^{(t)}(x) \in \mathscr{L}^{(t)}$, let

$$p^{(t)}(x) = \sum_{n \leqslant d} b_n \lambda_n^{(t)}(x), \tag{7}$$

and define (Definition 5.5.1)

$$\langle p^{(s)}(x) \rangle_t = b_0\, \delta_{st},$$

and if

$$p(x) = p^{(0)}(x) + \cdots + p^{(j)}(x),$$

set

$$\langle p(x) \rangle = \langle p^{(0)}(x) \rangle_0 + \cdots + \langle p^{(j)}(x) \rangle_j.$$

The linear functional $\langle\ \rangle$ is called the *augmentation*. It is the logarithmic analog of evaluation at zero. One obtains the following extension of Taylor's formula (Theorem 5.5.5) valid for all formal power series of logarithmic type:

$$p(x) = \sum_{t \geqslant 0} \sum_{n \in \mathbf{Z}} \frac{\langle \mathbf{D}^n p(x) \rangle_t}{\lfloor n \rceil!} \lambda_n^{(t)}(x).$$

In Section 7.2, we define a logarithmic analog of the usual inner product on polynomials. Again, this inner product is symmetric (though it is not definite), and the harmonic logarithms form an orthonormal basis.

We next describe the structure of shift-invariant operators on the logarithmic algebra $\mathscr{L}$. It is well known (see, for example, "Finite Operator Calculus," by Rota, Kahaner, and Odlyzko) that every continuous linear operator T on the algebra $\mathscr{L}^{(0)}$ of polynomials which is *shift-invariant* (that is, such that $TE^a = E^aT$ for all scalars a) can be written as

$$T = \sum_{n \geq 0} b_n \mathbf{D}^n.$$

Such an operator will be called a *differential operator*. On the subspace $\mathscr{L}^{(+)}$ spanned by formal logarithmic series of *positive* order one finds a greater variety of shift-invariant operators, namely:

1. *Laurent operators* of the form

$$T = \sum_{n \geq d} b_n \mathbf{D}^n.$$

2. *Elementary shift-invariant operators*, defined as

$$E_{st} \lambda_n^{(u)}(x) = \begin{cases} \lambda_n^{(s)}(x) & \text{if } t = u, \text{ and} \\ 0 & \text{if } t \neq u. \end{cases}$$

3. Combinations thereof. (See Proposition 5.4.3.)

After these preliminaries, we can state the main purpose of the present work. A great many sequences of polynomials that occur in formal mathematical analysis are defined by functional equations involving the derivative operator and the operator of multiplication by x (which we denote by $\mathbf{x}$). We have seen that the logarithmic extension of the derivative operator $\mathbf{D}$ is again the derivative. On the other hand, the logarithmic extension of the operator of multiplication by x is an operator σ, which we call the *standard Roman shift*, defined (Definition 7.2.1) as

$$\sigma \lambda_n^{(t)}(x) = \begin{cases} \lambda_{n+1}^{(t)}(x) & \text{if } n \neq -1, \text{ and} \\ 0 & \text{if } n = -1. \end{cases}$$

We have $\sigma x^n = x^{n+1}$ for $n \neq -1$; in particular, the restriction of σ to the subspace $\mathscr{L}^{(0)}$ of ordinary polynomials in x is indeed the operator $\mathbf{x}$ of multiplication by x.

The operators $\mathbf{D}$ and σ satisfy several remarkable identities. They obey the commutation relation

$$\mathbf{D}\sigma - \sigma\mathbf{D} = \mathbf{I}.$$

Moreover, the identity (Theorem 8.1.10)

$$e^{-a\sigma} \mathbf{D} e^{a\sigma} p(x) = (\mathbf{D} - \mathbf{I})\, p(x)$$

holds for all $p(x)$ in the span of the harmonic logarithms. Relative to the inner product of Section 7.2, the operators $\mathbf{D}$ and σ turn out to be adjoint.

Thus, "any" fact about polynomials that can be stated using the operators $\mathbf{x}$ and $\mathbf{D}$ has, in principle, a logarithmic extension involving the operators σ and $\mathbf{D}$, where the harmonic logarithms $\lambda_n^{(t)}(x)$ play the role of the powers x^n. In this way, one generates new special functions, whose study throws light on the given polynomial sequence. In several instances, extending a polynomial sequence logarithmically results in sequences of well-known special functions, and one obtains a pleasing unification.

We shall consider the logarithmic extension of Appell sequences, and of sequences of polynomials of binomial type.

We define (Definition 5.6.1) a *graded sequence of formal power series of logarithmic type* (or simply a *logarithmic graded sequence*) to be a doubly indexed sequence $p_n^{(t)}(x)$ of logarithmic formal power series (that is, elements of $\mathscr{L}$) such that

1. For n negative, $p_n^{(0)}(x)=0$. In other words, the left end of the first row of Table 2.1 is identically zero.

2. For n nonnegative, $p_n^{(0)}(x)$ is a logarithmic series of degree n and order 0. In other words, the right side of the first row of Table 2.1 consists of polynomials of increasing degree; $p_n^{(0)}(x)$ is a polynomial of degree n.

3. For t positive, $p_n^{(t)}(x)$ is a logarithmic series of degree n and order t. Thus, row t of Table 2.1 consists of logarithmic series of order t. They are arranged in order of degree with there being exactly one series of each degree.

4. (Regularity) For all integers n, nonnegative integers s, and positive integers t, $E_{st}\,p_n^{(t)}(x)=p_n^{(s)}(x)$. In other words, each row of Table 2.1 (other than the first) is mapped onto another row by an elementary shift operator.

We deal throughout with logarithmic graded sequences.

TABLE 2.1

A Graded Sequence of Formal Power Series of Logarithmic Type

$\cdots$	0	0	0	$p_0^{(0)}(x)$	$p_1^{(0)}(x)$	$p_2^{(0)}(x)$	$p_3^{(0)}(x)$	$\cdots$
$\cdots$	$p_{-3}^{(1)}(x)$	$p_{-2}^{(1)}(x)$	$\boxed{p_{-1}^{(1)}(x)}$	$p_0^{(1)}(x)$	$p_1^{(1)}(x)$	$p_2^{(1)}(x)$	$p_3^{(1)}(x)$	$\cdots$
$\cdots$	$p_{-3}^{(2)}(x)$	$p_{-2}^{(2)}(x)$	$p_{-1}^{(2)}(x)$	$p_0^{(2)}(x)$	$p_1^{(2)}(x)$	$p_2^{(2)}(x)$	$p_3^{(2)}(x)$	$\cdots$
$\cdots$	$p_{-3}^{(3)}(x)$	$p_{-2}^{(3)}(x)$	$p_{-1}^{(3)}(x)$	$p_0^{(3)}(x)$	$p_1^{(3)}(x)$	$p_2^{(3)}(x)$	$p_3^{(3)}(x)$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

An *Appell* logarithmic graded sequence $p_n^{(t)}(x)$ is a sequence defined as

$$p_n^{(t)}(x) = T^{-1}\lambda_n^{(t)}(x)$$

for some differential operator

$$T = \sum_{k \geqslant 0} b_k \mathbf{D}^k,$$

where $b_0 \neq 0$. (See Proposition 5.7.2) This definition is analogous to the classical definition, and in fact, the homogeneous component of order zero $(p_n^{(0)}(x))_{n \geqslant 0}$ of an Appell graded sequence turns out, not unexpectedly, to be an ordinary Appell polynomial sequence. The classical characterization of sequences of Appell polynomials turns out to have logarithmic analogs, in which binomial coefficients are replaced by Roman coefficients; for example, one finds

$$p_n^{(t)}(x+a) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k p_{n-k}^{(t)}(x),$$

and (Corollary 5.7.3)

$$p_n^{(t)}(x+a) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_k^{(0)}(a)\, \lambda_{n-k}^{(t)}(x).$$

Most importantly, the *summation formula* (Eq. (28))

$$E^a = \sum_{k \geqslant 0} \frac{p_k^{(0)}(a)}{k!} T\mathbf{D}^k$$

gives an expansion of the shift-operator E^a which remains valid in all of the logarithmic algebra.

By way of example, we consider the logarithmic extension of the Bernoulli and Hermite polynomials.

The Bernoulli polynomials $B_n(x)$ are defined as

$$B_n(x) = J^{-1}x^n,$$

where the *Bernoulli Operator*

$$J = \frac{e^{\mathbf{D}} - I}{\mathbf{D}}.$$

The logarithmic Bernoulli graded sequence (Definition 5.7.5) is defined similarly:

$$B_n^{(t)}(x) = J^{-1}\lambda_n^{(t)}(x).$$

Classically, one uses the identity

$$\Delta B_n(x) = nx^{n-1}$$

(where $\Delta f(x) = f(x+1) - f(x)$) to obtain closed form expressions of sums of powers of integers

$$\sum_{i=1}^{x} i^n = \frac{1}{n+1} B_{n+1}(x).$$

Using logarithmic Bernoulli graded sequences one can obtain closed form expressions for any summation

$$\sum_{i=j}^{k} i^l (\log i)^n.$$

More importantly, the Euler–MacLaurin summation formula, namely,

$$I = B_0 J + B_1 \Delta + \frac{B_2}{2!} \Delta \mathbf{D} + \frac{B_3}{3!} \Delta \mathbf{D}^2 + \cdots,$$

where $B_k = B_k^{(0)}(0)$ are the Bernoulli numbers, gives identities like (Eq. (31))

$$\begin{aligned} &\frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n} \\ &\quad = B_0[\log(x+n+1) - \log(x)] + B_1[(x+n+1)^{-1} - x^{-1}] \\ &\qquad + \frac{B_2}{2!}[-(x+n+1)^{-2} + x^{-2}] + \frac{B_3}{3!}[2(x+n+1)^{-3} - 2x^{-3}] + \cdots. \end{aligned}$$

Classically, the right side was interpreted as an "asymptotic expansion"; in the present theory, the right side is a *convergent* series in the topology of the ring of formal power series of logarithmic type, and one has an *identity*. This is the first of several examples of Theorem 5.6.2 which shows how the topology of the ring of formal power series of logarithmic type turns asymptotic series into convergent series. Another example is Stirling's formula, which is obtained by appliying the Euler–MacLaurin formula to $\log x$, thereby obtaining (Eq. (32))

$$\begin{aligned} &\log(x(x+1)\cdots(x+n)) \\ &\quad = B_0((x+n+1)\log(x+n+1) - x\log x - n - 1) \\ &\qquad + B_1(\log(x+n+1) - \log x) + \frac{B_2}{2!}\left[\frac{1}{x+n+1} - \frac{1}{x}\right] + \cdots. \end{aligned}$$

Again, the right side is convergent, and the identity holds over any field of characteristic zero.

We turn next to the logarithmic extension of the Hermite polynomials. Classically, the Hermite polynomials $H_n(x)$ are defined as

$$H_n(x) = e^{-\mathbf{D}^2/2} x^n,$$

that is (over $\mathbf{R}$ or $\mathbf{C}$),

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} H_n(x+t)\, dt = x^n.$$

We define the logarithmic Hermite graded sequence (Definition 5.7.6) as

$$H_n^{(t)}(x) = e^{-\mathbf{D}^2/2} \lambda_n^{(t)}(x).$$

Evidently, we have $H_n^{(0)}(x) = H_n(x)$. Furthermore, for n negative we have (Eq. (35))

$$H_n^{(1)}(x) = \sum_{k \geqslant 0} \left(-\frac{1}{2}\right)^k \frac{\lfloor n \rceil!}{\lfloor k \rceil! \lfloor n-2k \rceil!} x^{n-2k}.$$

Classically, the right side is the asymptotic expansion of $H_n(x)$; in the present theory, the right side is a convergent series.

One has (Eq. (34)), as in the classical case,

$$H_n^{(t)}(x) = \sum_{k \geqslant 0} \left(-\frac{1}{2}\right)^k \frac{\lfloor n \rceil!}{\lfloor k \rceil! \lfloor n-2k \rceil!} \lambda_{n-2k}^{(t)}(x).$$

Most of the properties of Hermite polynomials have logarithmic generalizations.

We next describe the logarithmic extension of the theory of polynomial sequences of binomial type. Recall (Definition 6.5.1) that a sequence of polynomials of binomial type is defined by the identity

$$p_n(x+a) = \sum_{k \geqslant 0} \binom{n}{k} p_k(a)\, p_{n-k}(x).$$

Examples of such sequences include:

1. The ordinary powers of x, x^n;
2. The lower factorials, $(x)_n$;
3. The Abel polynomials, $x(x-na)^{n-1}$; and
4. The Laguerre polynomials, $L_n(x)$.

We say that a graded sequence of formal power series of logarithmic type is a *Roman graded sequence* when it satisfies any of the following four conditions; each of which turns out (Theorem 6.5.4) to be equivalent to any of the others.

1. **Binomial Type Condition**: $p_n^{(t)}(x+a)=\sum_{k \geqslant 0} \left\lfloor {n \atop k} \right\rceil p_k^{(0)}(a)\, p_{n-k}^{(t)}(x)$.
2. **Basic Condition**: The following three conditions hold:
   2.1. $\langle p_n^{(t)}(x)\rangle = 0$ for $n \neq 0$,
   2.2. $\langle p_0^{(t)}(x)\rangle = 1$, and
   2.3. There exists a delta operator, that is, a differential operator of the form $f(\mathbf{D})=\sum_{k \geqslant 1} a_k \mathbf{D}^k$ (where $a_1 \neq 0$) such that $f(\mathbf{D})\, p_n^{(t)}(x) = \lfloor n \rceil\, p_{n-1}^{(t)}(x)$. In other words, the relationship between a Roman graded sequence $p_n^{(t)}(x)$ and its delta operator $f(\mathbf{D})$ is analogous to the relationship between the harmonic logarithms $\lambda_n^{(t)}(x)$ and the derivative $\mathbf{D}$.
3. **Roman Condition**: The identity

$$\langle f(\mathbf{D})\, g(\mathbf{D})\, p_n^{(t)}(x)\rangle_t = \sum_{k=c}^{n-d} \left\lfloor {n \atop k} \right\rceil \langle f(\mathbf{D})\, p_k^{(t)}(x)\rangle_t \times \langle g(\mathbf{D})\, p_{n-k}^{(t)}(x)\rangle_t \tag{8}$$

is satisfied for all Laurent operators $f(\mathbf{D})$ and $g(\mathbf{D})$ (of degrees c and d, respectively) when t is positive, and the identity is satisfied by all differential operators when t is zero.

4. **Coefficient Condition**: If the coefficients of $p_n^{(t)}(x)$ are given by $p_n^{(t)}(x)=\sum_{k \leqslant n} b_{nk}^{(t)} \lambda_k^{(t)}(x)$, then

$$\left\lfloor {i+j \atop i} \right\rceil b_{n,i+j}^{(t)} = \sum_{k=j}^{n-i} \left\lfloor {n \atop k} \right\rceil b_{kj}^{(t)} b_{n-k,i}^{(t)}. \tag{9}$$

The theory of Roman graded sequences extends the theory of sequences of polynomials of binomial type. Although such an extension is straightforward, the identities for special functions that result upon applying them to special Roman graded sequences are sometimes far from trivial.

We have first of all the Expansion Theorem (Theorem 6.2.4)

$$g(\mathbf{D}) = \sum_k \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x)\rangle_t}{\lfloor k \rceil !} f(\mathbf{D})^k$$

for any *Laurent* operator $g(\mathbf{D})$ and positive integer t.

Second, we have a version of Taylor's formula (Theorem 6.2.5)

$$p(x)=\sum_{t\geqslant 0}\sum_{n}\frac{\langle f(\mathbf{D})^{n}\,p(x)\rangle_{t}}{\lfloor n\rceil!}\,p_n^{(t)}(x)$$

for any logarithmic series $p(x)\in\mathscr{L}$.

Our most important results concerning Roman graded sequences were obtained by seeking closed form expressions for Roman graded sequences. In this case, one obtains formulae of striking simplicity, which are even simpler than the corresponding formulae for sequences of polynomials. The classical formulae for polynomials are obtained by restricting to $\mathscr{L}^{(0)}$.

There are two main results for Roman graded sequences. The first has to do with explicit formulae for their computation. The basic result is the following: Let, as above, $f(\mathbf{D})$ be the delta operator belonging to the Roman graded sequence $p_n^{(t)}(x)$. Then (by Theorem 7.2.8) for $t=1$ (for simplicity), we have

$$p_{-1}^{(1)}(x)=f'(\mathbf{D})x^{-1}.$$

The series $p_{-1}^{(1)}(x)$ is called the *residual series*. It is indicated by a box in Table 2.1. In general, we have the *transfer formula*

$$p_n^{(t)}(x)=f'(\mathbf{D})\,g(\mathbf{D})^{-n-1}\,\lambda_n^{(t)}(x)$$

for all n, where $g(\mathbf{D})=f(\mathbf{D})/\mathbf{D}$.

Similarly, one derives (Corollary 7.2.11) when $n\neq -1$, 0:

$$p_n^{(t)}(x)=\sigma g(\mathbf{D})^{-n}\,\lambda_{n-1}^{(t)}(x).$$

One also derives the Recurrence Formula (Theorem 7.2.6) when $n\neq -1$:

$$p_n^{(t)}(x)=\sigma f'(\mathbf{D})^{-1}\,p_{n-1}^{(t)}(x),$$

for all nonnegative t.

We consider some examples of Roman graded sequences. Consider the *forward difference operator* $\Delta=E-\mathbf{I}$ (so that $\Delta'=E$) along with its Roman graded sequence $(x)_n^{(t)}$. We have (Eq. (44)) as an application of the transfer formula the residual series

$$(x)_{-1}^{(1)}=\Delta' x^{-1}=\frac{1}{x+1},$$

and the recursion formula gives (Proposition 8.1.2)

$$(x)_n^{(1)}=\frac{1}{(x+1)\cdots(x-n)}=(x)_n$$

for n negative, and (Proposition 8.1.3)

$$(x)_n^{(0)} = x(x-1)\cdots(x-n+1) = (x)_n$$

for n nonnegative. For $n=0$, we proceed as follows. From

$$(x)_0^{(1)} = \Delta' \frac{\mathbf{D}}{\Delta} \log x$$

and the Euler–MacLaurin formula, we have (Eq. (46))

$$\begin{aligned}(x)_0^{(1)} &= \sum_{k \geqslant 0} \frac{B_k}{k!} \mathbf{D}^k \log(x+1) \\ &= \log(x+1) + \frac{B_1}{1+x} - \frac{B_2}{2(1+x)^2} + \frac{B_3}{3(1+x)^3} - \cdots.\end{aligned}$$

Thus, we find that $(x)_0^{(1)} = \psi(x+1)$ coincides with the classical ψ-function—the logarithmic derivative of the gamma function—introduced by Gauss. Similarly, one finds that $(x)_1^{(1)}$ and $(x)_2^{(1)}$ coincide with the digamma and trigamma functions of Gauss.

The logarithmic binomial identity (Eq. (47))

$$(x+a)_0^{(1)} = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} 0 \\ k \end{matrix} \right\rceil (a)_k \, (x)_{-k}^{(1)}$$

gives trivially a classical identity (Eq. (48)) satisfied by the ψ-function, that is:

$$\psi(x+a+1) = \psi(x+1) + \sum_{k \geqslant 0} \frac{(-1)^{k+1} a(a-1)\cdots(a-k+1)}{k(x+1)(x+2)\cdots(x+k)}.$$

Similar identities can be obtained for the digamma and trigamma functions.

The expansion theorem gives an extension of Newton's formula of the calculus of finite differences that is convergent for all formal logarithmic series (Proposition 8.1.4):

$$p(x) = \sum_{t \geqslant 0} \sum_{n} \frac{a_n^{(t)}}{\lfloor n \rceil!} (x)_n^{(t)},$$

where

$$a_n^{(t)} = \langle \Delta^n p(x) \rangle_t.$$

For example, we have the *identity* (Eq. (49))

$$\frac{1}{x}=\sum_{k\geqslant 0}\frac{1}{k!\,(x+1)(x+2)\cdots(x+k+1)}.$$

Thus, the classical theory of factorial series is subsumed into the present theory and rendered purely formal, that is, independent of any convergence questions in the complex field.

We note that the solution of a difference equation

$$\Delta p(x)=q(x),$$

where $q(x)$ is a given formal logarithmic series in $\mathscr{L}^{(+)}$, is uniquely defined, since the operator $\mathbf{D}$ is invertible in $\mathscr{L}^{(+)}$. We thus succeed in giving a simple definition of the natural solution of a difference equation which avoids the *ad hoc* techniques used by Milne-Thompson, Nörlund, and others.

We next consider the logarithmic extension of the Abel polynomials. It is the Roman graded sequence corresponding to the delta operator $f(\mathbf{D})=E^a\mathbf{D}$. Here we have immediately (Eq. (51))

$$A_n^{(1)}(x)=x(x-an)^{n-1}$$

for all n negative, and (Eq. (50))

$$A_n^{(0)}(x)=x(x-an)^{n-1}$$

for all n positive; these are the classical Abel polynomials. The Abel series of degree zero and order one has the unexpectedly simple form (Eq. (52))

$$A_0^{(1)}(x)=(E^a\mathbf{D})'\,E^{-a}\lambda_1^{(1)}(x)=(a\mathbf{D}+\mathbf{I})\log x=\frac{a}{x}+\log x.$$

More generally (Eq. (54)):

$$A_n^{(t)}(x)=\lambda_n^{(t)}(x-na)+a\lfloor n\rceil\,\lambda_{n-1}^{(t)}(x-na).$$

Since

$$A_0^{(1)}(x+b)=\sum_{k\geqslant 0}\begin{bmatrix}0\\k\end{bmatrix}A_k^{(0)}(b)\,A_{n-k}^{(1)}(x),$$

we have the (previously unknown) identity (Eq. (53))

$$\frac{a}{x+b}+\log(x+b)=\frac{a}{x}+\log x+\sum_{k\geqslant 1}\frac{(-1)^{k+1}\,b(b-ak)^{k-1}\,x}{k(x+ak)^{k+1}}.$$

Again, one obtains an expansion theorem of any formal logarithmic series as a convergent series in the $A_n^{(t)}(x)$;

$$p(x) = \sum_{t \geqslant 0} \sum_{n} \frac{a_n^{(t)}}{\lfloor n \rceil!} A_n^{(t)}(x)$$

with

$$a_n^{(t)} = \langle E^{na} \mathbf{D}^n p(x) \rangle_t.$$

For example (Eq. (55)),

$$\log x = \sum_{k \leqslant 0} \begin{bmatrix} 0 \\ k \end{bmatrix} (ka)^{-k} A_k^{(1)}(x),$$

which is again an identity that we have not seen in the literature.

The second main result we prove about Roman graded sequences is the solution of the problem of connection constants. Given two Roman graded sequences $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$ associated with delta operators $f(\mathbf{D})$ and $g(\mathbf{D})$, respectively, we wish to compute the coefficients in the expansion

$$p_n^{(t)}(x) = \sum_k c_{nk}^{(t)} q_k^{(t)}(x).$$

By regularity, $c_{nk}^{(t)} = c_{nk}^{(s)}$. Furthermore, setting

$$r_n^{(t)}(x) = \sum_k c_{nk}^{(t)} \lambda_k^{(t)}(x),$$

one proves (Proposition 7.1.7) that the graded sequence $r_n^{(t)}(x)$ is the Roman graded sequence whose delta operator is $g(f^{(-1)})$.

Thus, the coefficients $c_{nk}^{(t)}$ can be computed by the transfer formula or the recurrence formula. For example (Eq. (56)),

$$\frac{1}{(x-1)\cdots(x-n)} = \sum_{k \geqslant 0} (-1)^{n+k} \binom{-n-1}{k} \times \frac{(n+k-1)!}{(n-1)!\,(x+1)\cdots(x+n+k)}$$

for n a positive integer.

3. Roman Coefficients and Harmonic Numbers

3.1. *The Roman Factorial*

We begin by giving the definition of a generalization of the factorial $n!$ which makes sense for negative values of n as well. It will be seen that the definition below is the correct one.

DEFINITION 3.1.1 (Roman Factorial). For every integer n, define n *Roman factorial* to be

$$\lfloor n \rceil ! = \begin{cases} (-1)^{n+1}/(-n-1)! & \text{for } n < 0, \text{ and} \\ n! & \text{for } n \geqslant 0, \end{cases}$$

where $n! = 1 \times 2 \times 3 \times \cdots \times n$.

Equivalently, $\lfloor n \rceil !$ can be defined recursively by the requirements

$$\lfloor 0 \rceil ! = 1$$
$$\lfloor n \rceil ! = \lfloor n \rceil \lfloor n-1 \rceil !,$$

where "*Roman n*" $\lfloor n \rceil$ is defined to be

$$\lfloor n \rceil = \begin{cases} n & \text{for } n \neq 0, \text{ and} \\ 1 & \text{for } n = 0. \end{cases}$$

If we adopt Iverson's notation for the moment, and write logical expressions in parentheses to mean 1 if true and 0 if false, then we have the following proposition:

PROPOSITION 3.1.2 (Knuth). *For any integer n,*

$$\lfloor n \rceil ! \, \lfloor -n-1 \rceil ! = (-1)^{n+(n<0)}.$$

This extension of the notion of factorial leads to a generalization of the definition of Binomial Coefficients where both arguments may be negative.

TABLE 3.1

Roman Factorials, $\lfloor n \rceil !$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\lfloor n \rceil !$	$-\frac{1}{120}$	$\frac{1}{24}$	$-\frac{1}{6}$	$\frac{1}{2}$	-1	1	1	1	2	6	24	120	720

3.2. *The Roman Coefficients*

3.2.1. *Definition*

DEFINITION 3.2.1 (Roman Coefficients). For every pair of integers n, k, define the *Roman Coefficient* (read: "Roman n choose k") to be

$$\left\lfloor {n \atop k} \right\rceil = \frac{\lfloor n \rceil!}{\lfloor k \rceil! \, \lfloor n-k \rceil!}.$$

The relationship between Roman Coefficients and binomial coefficients is the following:

PROPOSITION 3.2.2. *Let n and k be integers. Depending on what region of the Cartesian plane the point (n, k) is in, the following formulae apply:*

Region 1—If $n \geqslant k \geqslant 0$, then $\lfloor {n \atop k} \rceil = \binom{n}{k}$.
Region 2—If $k \geqslant 0 > n$, then $\lfloor {n \atop k} \rceil = (-1)^k \binom{-n+k-1}{k}$.
Region 3—If $0 > n \geqslant k$, then $\lfloor {n \atop k} \rceil = (-1)^{n+k} \binom{-k-1}{n-k}$.

TABLE 3.2

Roman Coefficients, $\lfloor {n \atop k} \rceil$

n \ k	-4	-3	-2	-1	0	1	2	3	4	5	6
6	$-1/840$	$1/252$	$-1/56$	$1/7$	1	6	15	20	15	6	1
5	$-1/504$	$1/168$	$-1/42$	$1/6$	1	5	10	10	5	1	$1/6$
4	$-1/280$	$1/105$	$-1/30$	$1/5$	1	4	6	4	1	$1/5$	$-1/30$
3	$-1/140$	$1/60$	$-1/20$	$1/4$	1	3	3	1	$1/4$	$-1/20$	$1/60$
2	$-1/80$	$1/30$	$-1/12$	$1/3$	1	2	1	$1/3$	$-1/12$	$1/30$	$-1/60$
1	$-1/20$	$1/12$	$-1/6$	$1/2$	1	1	$1/2$	$-1/6$	$1/12$	$-1/20$	$1/30$
0	$-1/4$	$1/3$	$-1/2$	1	1	1	$-1/2$	$1/3$	$-1/4$	$1/5$	$-1/6$
-1	-1	1	-1	1	1	-1	1	-1	1	-1	1
-2	3	-2	1	-1	1	-2	3	-4	5	-6	7
-3	-3	1	$-1/2$	$-1/2$	1	-3	6	-10	15	-21	28
-4	1	$-1/3$	$-1/6$	$-1/3$	1	-4	10	-20	35	-56	84
-5	$-1/4$	$-1/12$	$-1/12$	$-1/4$	1	-5	15	-35	70	-126	210

TABLE 3.3

Region 1

n \ k	0	1	2	3	4	5	6	7
7	1	7	21	35	35	21	7	1
6	1	6	15	20	15	6	1	
5	1	5	10	10	5	1		
4	1	4	6	4	1			
3	1	3	3	1				
2	1	2	1					
1	1	1						
0	1							

TABLE 3.4

Region 2

n \ k	0	1	2	3	4	5	6
-1	1	-1	1	-1	1	-1	1
-2	1	-2	3	-4	5	-6	7
-3	1	-3	6	-10	15	-21	28
-4	1	-4	10	-20	35	-56	84
-5	1	-5	15	-35	70	-126	210

TABLE 3.5

Region 3

n \ k	-6	-5	-4	-3	-3	-1
-1	-1	1	-1	1	-1	1
-2	5	-4	3	-2	1	
-3	-10	6	-3	1		
-4	10	-4	1			
-5	-5	1				
-6						

Region 4—If $k > n \geqslant 0$, then

$$\left[\begin{matrix} n \\ k \end{matrix}\right] = (-1)^{n+k} \frac{1}{n-k} \binom{k}{n}^{-1}$$

$$= (-1)^{n+k+1} \frac{1}{n+1} \binom{k}{n+1}^{-1}$$

$$= (-1)^{n+k+1} \frac{1}{k} \binom{k-1}{n}^{-1}.$$

Region 5—If $n \geqslant 0 > k$, then

$$\left[\begin{matrix} n \\ k \end{matrix}\right] = (-1)^{k} \frac{1}{k} \binom{n-k}{n}^{-1}$$

$$= (-1)^{k} \frac{1}{k-n} \binom{n-k-1}{n}^{-1}$$

$$= (-1)^{k+1} \frac{1}{n+1} \binom{n-k-1}{n+1}^{-1}$$

$$= \left[\begin{matrix} n \\ n-k \end{matrix}\right],$$

where the pair $(n, n-k)$ lies in region 4 (defined above).

Region 6—If $0 > k > n$, then

$$\left[\begin{matrix} n \\ k \end{matrix}\right] = \frac{1}{n-k} \binom{-n-1}{-k-1}^{-1}$$

$$= \frac{1}{k} \binom{-n-1}{-k}^{-1}$$

$$= \frac{1}{n+1} \binom{-n-2}{-k-1}^{-1}$$

$$= (-1)^{k+1} \left[\begin{matrix} k-n-1 \\ -n-1 \end{matrix}\right]$$

$$= (-1)^{k+1} \left[\begin{matrix} k-n-1 \\ k \end{matrix}\right],$$

where the pair $(k-n-1, -n-1)$ lies in region 4 (defined above), and the pair $(k-n-1, k)$ lies in region 5 (defined above).

TABLE 3.6

Region 4

n \ k	1	2	3	4	5	6	7
6							1/7
5						1/6	−1/42
4					1/5	−1/30	1/105
3				1/4	−1/20	1/60	−1/140
2			1/3	−1/12	1/30	−1/60	1/105
1		1/2	−1/6	1/12	−1/20	1/30	−1/42
0	1	−1/2	1/3	−1/4	1/5	−1/6	1/7

TABLE 3.7

Region 5

n \ k	−4	−3	−1	−1
6	−1/840	1/252	−1/56	1/7
5	−1/504	1/168	−1/42	1/6
4	−1/280	1/105	−1/30	1/5
3	−1/140	1/60	−1/20	1/4
2	−1/80	1/30	−1/12	1/3
1	−1/20	1/12	−1/6	1/2
0	−1/4	1/3	−1/2	1

TABLE 3.8

Region 6

n \ k	−6	−5	−4	−3	−2	−1
−2						−1
−3					−1/2	−1/2
−4				−1/3	−1/6	−1/3
−5			−1/4	−1/12	−1/12	−1/4
−6		−1/5	−1/20	−1/30	−1/20	−1/5
−7	−1/6	−1/30	−1/60	−1/60	−1/30	−1/6

Note that in regions 1, 2, and 3, the Roman coefficients equal binomial coefficients up to a permutation and a change of sign. In regions 4, 5, and 6, the Roman coefficients are expressed simply in terms of the *reciprocals* of the binomial coefficients. Furthermore, regions 4, 5, and 6 are identical up to permutation and change of sign. Thus, all of the Roman coefficients are related in a simple way to those in the first quadrant (regions 1 and 4). In particular, the Roman coefficients always equal integers or the reciprocals of integers.

3.2.2. *Properties*

Several binomial coefficient identities extend to Roman coefficients. We give a few examples here. Others will be given later in the text.

PROPOSITION 3.2.3 (Complementation). *For any pair of integers n, k,*

$$\left\lfloor {n \atop k} \right\rceil = \left\lfloor {n \atop n-k} \right\rceil.$$

PROPOSITION 3.2.4 (Iteration). *For all integers n, k, and r,*

$$\left\lfloor {n \atop k} \right\rceil \left\lfloor {k \atop r} \right\rceil = \left\lfloor {n \atop r} \right\rceil \left\lfloor {n-r \atop k-r} \right\rceil.$$

PROPOSITION 3.2.5 (Pascal's Triangle). *Suppose one of the following equivalent conditions holds true*:

1. *Of the six regions mentioned in Proposition* 3.2.2, *the points* (n, k), $(n-1, k)$, *and* $(n-1, k-1)$ *all lie in the same region.*
2. *The following conditions hold*:

$$\operatorname{sign}(n) = \operatorname{sign}(n-1),$$
$$\operatorname{sign}(k) = \operatorname{sign}(k-1),$$
$$\operatorname{sign}(n-k) = \operatorname{sign}(n-k-1),$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ -1 & \text{if } x < 0. \end{cases}$$

3. *The following conditions hold*:

$$n \neq 0$$
$$k \neq 0$$
$$n \neq k.$$

Then we have

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil + \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil.$$

Note that the signum function given in condition 2 is not the usual definition of the sign function.

Proof of Proposition 3.2.5. Since under these conditions $\lfloor n \rceil = n$, $\lfloor k \rceil = k$, and $\lfloor n-k \rceil = n-k$,

$$\begin{aligned}
\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil + \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil &= \frac{\lfloor n-1 \rceil !}{\lfloor n-k-1 \rceil ! \, \lfloor k \rceil !} + \frac{\lfloor n-1 \rceil !}{\lfloor n-k \rceil ! \, \lfloor k-1 \rceil !} \\
&= \lfloor n-k \rceil \left(\frac{\lfloor n-1 \rceil !}{\lfloor n-k \rceil ! \, \lfloor k \rceil !} \right) + \lfloor k \rceil \left(\frac{\lfloor n-1 \rceil !}{\lfloor n-k \rceil ! \, \lfloor k \rceil !} \right) \\
&= \lfloor n \rceil \left(\frac{\lfloor n-1 \rceil !}{\lfloor n-k \rceil ! \, \lfloor k \rceil !} \right) \\
&= \frac{\lfloor n \rceil !}{\lfloor n-k \rceil ! \, \lfloor k \rceil !} \\
&= \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil. \quad \blacksquare
\end{aligned}$$

COROLLARY 3.2.6. *For r nonnegative and (n, k), $(n+r, k)$, $(n, k+1)$, and $(n+r+1, k+1)$ in the same region (as defined in Theorem 3.2.2), we have*

$$\sum_{m=n}^{n+r} \left\lfloor \begin{matrix} m \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n+r+1 \\ k+1 \end{matrix} \right\rceil - \left\lfloor \begin{matrix} n \\ k+1 \end{matrix} \right\rceil.$$

Proof. Induction on r. $\blacksquare$

Contrast this corollary with this classical result involving binomical coefficients in which for $n \geqslant k \geqslant 0$,

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

If we again adopt Iverson's notation (Proposition 3.1.2), we have the following beautiful identity:

PROPOSITION 3.2.7 (Knuth's Rotation/Reflection Law). *For any integers n and k,*

$$(-1)^{k+(k>0)} \left\lfloor \begin{matrix} -n \\ k-1 \end{matrix} \right\rceil = (-1)^{n+(n>0)} \left\lfloor \begin{matrix} -k \\ n-1 \end{matrix} \right\rceil.$$

Proof. By Proposition 3.1.2, we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n+k+(n<0)+(k<0)} \begin{bmatrix} -k-1 \\ -n-1 \end{bmatrix}. \quad ∎$$

3.3. *The Harmonic Numbers*

We proceed to define the Harmonic numbers and derive some remarkable identities satisfied by them. Many of these identities will be used in the text. Recall:

DEFINITION 3.3.1 (Lower Factorial). For k an integer, let $(x)_k$ denote the *lower factorial* of degree k in the variable x. It is defined directly as follows:

$$(x)_k = \begin{cases} \prod_{i=0}^{k-1} (x-i) = x(x-1)\cdots(x-k+1) & \text{for } k \geq 0, \text{ and} \\ \prod_{i=k}^{-1} (x-i)^{-1} = 1/(x+1)(x+2)\cdots(x-k) & \text{for } k < 0. \end{cases}$$

Equivalently, $(x)_k$ can be defined recursively by the requirements

$$(x)_0 = 1$$
$$(x)_k = (x-k+1)(x)_{k-1} \qquad \text{for all } k.$$

We make a brief digression on notation in order to prevent any possible confusion. We use the symbol $(x)_n$ to denote the "falling powers" $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$; however, many researchers—for example, Askey and Henrici—reserve this notation for the "rising powers" $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$. Actually, as Knuth has pointed out, Pochhammer, who devised this notation, did not intend either of these definitions; he used $(x)_n$ to denote $x(x-1)\cdots(x-n+1)/n!$

Note that we now have the following alternate definition of the Roman factorial in terms of the lower factorial:

$$\lfloor n \rceil ! = (n)_{n-\operatorname{sign}(n)},$$

where sign(n) is as defined classically, or as defined in Proposition 3.2.5.

Now, we generalize the notion of Stirling numbers of the first kind:

DEFINITION 3.3.2 (Stirling Numbers of the First Kind). For all integers n, we define the *Stirling numbers* $s(n, k)$ to be the coefficients of the expansion of $(y)_n$ in terms of formal power series. That is,

$$(y)_n = \sum_{k \geq 0} s(n, k)\, y^k.$$

Note that for n positive this corresponds to the usual definition of Stirling numbers of the first kind.

DEFINITION 3.3.3 (Harmonic Numbers). Let n be an integer, and let k be a nonnegative integer; define the *harmonic number*, $c_n^{(k)}$, of order k and degree n to be

$$c_n^{(k)} = (-1)^k \frac{\lfloor n \rceil !}{k!} [\mathbf{D}^k (x)_{-n}]_{x=0}$$
$$= (-1)^k \lfloor n \rceil ! \, s(-n, k).$$

THEOREM 3.3.4. *For all integers n and all positive integers k,*

$$nc_n^{(k)} - c_n^{(k-1)} = \lfloor n \rceil c_{n-1}^{(k)},$$

and

$$nc_n^{(0)} = \lfloor n \rceil c_{n-1}^{(0)}.$$

Proof 1. From the recursion

$$(y)_n = (y - n + 1)(y)_{n-1},$$

TABLE 3.9

Harmonic Numbers, $c_n^{(k)}$

$k \backslash n$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	0	0	0	0	0	0	1	1	1	1	1	1
1	-1	-1	-1	-1	-1	-1	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$
2	$-\frac{137}{60}$	$-\frac{25}{12}$	$-\frac{11}{6}$	$-\frac{3}{2}$	-1	0	0	1	$\frac{7}{4}$	$\frac{85}{36}$	$\frac{415}{144}$	$\frac{12,019}{3,600}$
3	$-\frac{15}{8}$	$-\frac{35}{24}$	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{15}{8}$	$\frac{576}{216}$	$\frac{5845}{1728}$	$\frac{874,853}{216,000}$
4	$-\frac{17}{24}$	$-\frac{5}{12}$	$-\frac{1}{6}$	0	0	0	0	1	$\frac{31}{16}$	$\frac{3661}{1296}$	$\frac{76,111}{20,736}$	$\frac{58,067,611}{12,960,000}$
5	$-\frac{3}{24}$	$-\frac{1}{24}$	0	0	0	0	0	1	$\frac{63}{32}$	$\frac{22,631}{7,776}$	$\frac{952,525}{248,832}$	$\frac{3,673,451,957}{777,600,000}$
6	$-\frac{1}{120}$	0	0	0	0	0	0	1	$\frac{127}{64}$	$\frac{137,845}{46,656}$	$\frac{11,679,655}{2,985,984}$	$\frac{226,576,031,859}{46,656,000,000}$

we have

$$s(n+1, k) = s(n, k-1) - ns(n, k).$$

Thus,

$$\begin{aligned} nc_n^{(k)} - c_n^{(k-1)} &= (-1)^k \lfloor n \rceil! \,(ns(-n, k) + s(-n, k-1)) \\ &= (-1)^k \lfloor n \rceil! \, s(-n+1, k) \\ &= \lfloor n \rceil c_{n-1}^{(k)}. \quad \blacksquare \end{aligned}$$

Proof 2. For $n = 0$, Theorem is immediate. Now, for $n \neq 0$, consider the identity

$$(x+1)_{-n} = (x+1)(x)_{-n-1}$$

and differentiate k times:

$$\begin{aligned} \mathbf{D}^k(x+1)_{-n} &= \mathbf{D}^k(x+1)(x)_{-n-1} \\ \mathbf{D}^k((x)_{-n} - n(x)_{-n-1}) &= \sum_{i=0}^{k} \binom{k}{i} (\mathbf{D}^i(x+1))(\mathbf{D}^{k-i}(x)_{-n-1}) \\ \mathbf{D}^k(x)_{-n} - n\mathbf{D}^k(x)_{-n-1} &= (x+1)\,\mathbf{D}^k(x)_{-n-1} + k\mathbf{D}^{k-1}(x)_{-n-1} \\ \mathbf{D}^k(x)_{-n} &= (x+n+1)\,\mathbf{D}^k(x)_{-n-1} + k\mathbf{D}^{k-1}(x)_{-n-1} \end{aligned}$$

since $(x+1)_n - (x)_n = n(x)_{n-1}$. Multiply by $(-1)^k n!/k!$, and set $x = 0$ to obtain

$$c_{n+1}^{(k)} = c_n^{(k)} + \frac{1}{n+1} c_{n+1}^{(k-1)},$$

from which the conclusion follows. $\blacksquare$

3.3.1. *Harmonic Numbers of Nonnegative Degree*

In this section, we derive several identities which hold for harmonic numbers of nonnegative degree.

Recall that a *linear partition* ρ is a nonincreasing infinite sequence, $(\rho_i)_{i \geqslant 1}$, of nonnegative integers which is eventually zero. For example, $\rho = (17, 2, 2, 1, 0, 0, ...)$ is a linear partition. Each nonzero ρ_i is called a *part* of ρ. In the above example, the multiset of parts of ρ is $\{1, 2, 2, 17\}$. The number of parts of ρ is denoted $l(\rho)$. A linear partition ρ is said to be a *partition of* n if the sum of its parts is n, and we write $\rho \vdash n$. The product of the parts of ρ is denoted by $\pi(\rho)$.

The set of all linear partitions is denoted by $\mathscr{P}$. A linear partition is said to have *distinct parts* if its multiset of parts is, in fact, a set. The set of all linear partitions with distinct parts is denoted by $\mathscr{P}^*$.

The zero sequence is a linear partition. It has no parts. Thus, it is a partition of zero with distinct parts, and the product of its parts is one.

Proposition 3.3.5 and its corollaries give enumerative interpretations of the harmonic numbers of nonnegative degree.

PROPOSITION 3.3.5 (Harmonic Relation). *For any nonnegative integers n and k, the harmonic numbers satisfy the identity*

$$c_n^{(k)} = \sum_{\substack{\rho \in \mathscr{P} \\ l(\rho) = k \\ \rho_1 \leqslant n}} \pi(\rho)^{-1};$$

that is, the harmonic number of order k and degree n is equal to the sum over all linear partitions ρ with k parts and with no part greater than n of the reciprocal of the product of the parts of the partition ρ.

We offer two proofs:

Proof 1. Let n and k be as above, and define

$$d_n^{(k)} = \sum_{\substack{\rho \in \mathscr{P} \\ l(\rho) = k \\ \rho_1 \leqslant n}} \pi(\rho)^{-1}.$$

By Theorem 3.3.4 it will suffice to verify the recursion

$$nd_n^{(k)} - d_n^{(k-1)} = nd_{n-1}^{(k)} \tag{10}$$

for n and k positive, since the boundary conditions are easy to verify.

Consider the following series of equalities:

$$\begin{aligned} nd_n^{(k)} - d_n^{(k-1)} &= \Bigg(n \sum_{\substack{\mu \in \mathscr{P} \\ l(\mu) = k \\ \mu_1 \leqslant n}} \pi(\mu)^{-1}\Bigg) - \Bigg(\sum_{\substack{\mu \in \mathscr{P} \\ l(\mu) = k-1 \\ \mu_1 \leqslant n}} \pi(\mu)^{-1}\Bigg) \\ &= \Bigg(n \sum_{\substack{\mu \in \mathscr{P} \\ l(\mu) = k \\ \mu_1 \leqslant n}} \pi(\mu)^{-1}\Bigg) - \Bigg(n \sum_{\substack{\mu \in \mathscr{P} \\ l(\mu) = k \\ \mu_1 = n}} \pi(\mu)^{-1}\Bigg) \\ &= n \sum_{\substack{\mu \in \mathscr{P} \\ l(\mu) = k \\ \mu_1 < n}} \pi(\mu)^{-1} \\ &= nd_{n-1}^{(k)}. \end{aligned}$$

Thus, Eq. (10) holds. ∎

Proof 2. For n positive,

$$\begin{aligned} c_n^{(k)} &= (-1)^k \frac{\lfloor n \rceil!}{k!} [\mathbf{D}^k (y)_{-n}]_{y=0} \\ &= \frac{(-1)^k \lfloor n \rceil!}{n!\, k!} \left[\mathbf{D}^k \frac{1}{(1+y)(1+y/2)\cdots(1+y/n)} \right]_{y=0} \\ &= \frac{(-1)^k}{k!} \left[\mathbf{D}^k \frac{1}{(1+y)(1+y/2)\cdots(1+y/n)} \right]_{y=0}, \end{aligned}$$

which equals $(-1)^k$ times the coefficient of y^k in

$$\frac{1}{(1+y)(1+y/2)\cdots(1+y/n)},$$

and this is merely the coefficient of y^k in

$$\left(\sum_{\rho_1 \geqslant 0} y^{\rho_1}\right)\left(\sum_{\rho_2 \geqslant 0} (y/2)^{\rho_2}\right)\cdots\left(\sum_{\rho_n \geqslant 0} (y/n)^{\rho_n}\right).$$

That is,

$$c_n^{(k)} = \sum_{\substack{\rho \in \mathscr{P} \\ l(\rho) = k \\ \rho_1 \leqslant n}} \pi(\rho)^{-1}. \quad \blacksquare$$

Thus, we see from Proposition 3.3.5 why the $c_n^{(k)}$ have been called harmonic numbers; they are generalizations of the partial sums of the harmonic series. For $k = 1$, Proposition 3.3.5 yields a sum over partitions of length one with no part greater than n. There are n such partitions; they are in the integers from 1 to n. Thus, the sum is the sum of the reciprocals of the first n integers, so that

$$c_n^{(1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{11}$$

For the harmonic numbers $c_n^{(2)}$, we obtain similarly

$$c_n^{(2)} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right). \tag{12}$$

For the harmonic numbers $c_n^{(3)}$, we obtain

$$\begin{aligned} c_n^{(3)} = 1 & \\ & + \frac{1}{2}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right)\right] \\ & + \frac{1}{3}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\right] \\ & \vdots \\ & + \frac{1}{n}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right]. \end{aligned}$$

Now, we can derive a host of other identities satisfied by the harmonic numbers. In particular, Proposition 3.3.6 turns out to be very useful in the calculation of harmonic numbers.

PROPOSITION 3.3.6 (Knuth). *Let n and k be nonnegative integers (not both zero). Then $c_n^{(k)}$ is given by the finite sum*

$$c_n^{(k)} = \sum_{m \geq 1} \binom{n}{m} (-1)^{m-1} m^{-k}.$$

Proof. By recursion using Theorem 3.3.4. ▮

COROLLARY 3.3.7. *For all nonnegative integers n and k,*

$$c_n^{(k)} = n \sum_{\substack{\rho \in \mathscr{P} \\ l(\rho) = k+1 \\ \rho_1 = n}} \pi(\rho)^{-1},$$

where the sum is over all linear partitions ρ with $k+1$ parts and whose largest part is n.

Proof. Add the part n to each partition ρ being summed over in Corollary 3.3.5. ▮

COROLLARY 3.3.8. *Let n and k be nonnegative integers. Then*

$$\begin{aligned} c_n^{(k)} &= \sum_{\substack{M \subseteq \{1,2,\ldots,n\} \\ |M| = k}} \left(\prod_{m \in M} m^{-1}\right) \\ &= n \sum_{\substack{M \subseteq \{1,2,\ldots,n\} \\ |M| = k+1 \\ n \in M}} \left(\prod_{m \in M} m^{-1}\right), \end{aligned}$$

where all of the sums are over multisets M, *and all of the products are computed with the proper multiplicities.*

Proof. Every linear partition is associated with a unique multiset of positive numbers called its parts. In the identities from Proposition 3.3.5 and Corollary 3.3.7, sum over these multisets instead of the partitions themselves. ▮

COROLLARY 3.3.9. *Let n and k be nonnegative integers. Then*

$$c_n^{(k)} = \sum \left(\prod_{i=1}^{n} i^{-m_i} \right),$$

where the sum ranges over all sequences, $(m_i)_{i=1}^n$, of n nonnegative integers which sum to k.

Proof. Every linear partition is determined by the number m_i of times each integer i occurs as a part. Hence, we can sum over sequences of non-negative integers m_i. ▮

We note that

$$\lim_{k \to \infty} c_n^{(k)} = n$$

for all nonnegative n. Whereas for n negative, we shall see instead that

$$\sum_{k \geqslant 0} c_n^{(k)} = n.$$

3.3.2. *Harmonic Numbers of Negative Degree*

The harmonic numbers of negative degree have similar expansions.

PROPOSITION 3.3.10. *Let k be a nonnegative integer, and let n be a positive integer. Then*

$$c_{-n}^{(k)} = - \sum_{\substack{\mu \in \mathscr{P}^* \\ l(\mu) = k-1 \\ \mu_1 < n}} \pi(\mu)^{-1},$$

that is, the harmonic number of order k and degree $-n$ is equal to the sum over all linear partitions, μ, with $k-1$ parts all of which are distinct and less than n, of the reciprocal of the product of the parts of μ.

Recall that the trivial partition has no parts, and therefore the product of its parts is one; however, there are no partitions with -1 parts.

Proof of Proposition 3.3.10. Note that

$$
\begin{aligned}
c^{(k)}_{-n} &= (-1)^k \frac{\lfloor -n \rceil !}{k!} \left[\mathbf{D}^k (x)_n\right]_{x=0} \\
&= \frac{(-1)^{n+k+1}}{k!\,(n-1)!} \left[\mathbf{D}^k \prod_{i=0}^{n-1} (x-i)\right]_{x=0} \\
&= \frac{(-1)^{n+k+1}}{k!\,(n-1)!} \left[\mathbf{D}^k \sum_{\substack{\mu \in \mathscr{P}^* \\ \mu_1 < n}} (-1)^{l(\mu)} x^{n-l(\mu)} \pi(\mu)\right]_{x=0} \\
&= \frac{(-1)^{n+k+1}}{k!\,(n-1)!} \left[\sum_{\substack{\mu \in \mathscr{P}^* \\ \mu_1 < n}} \frac{(n-l(\mu))!}{(n-l(\mu)-k)!} (-1)^{l(\mu)} x^{n-l(\mu)-k}\, \pi(\mu)\right]_{x=0} \\
&= \frac{(-1)^{n+k+1}}{k!\,(n-1)!} \sum_{\substack{\mu \in \mathscr{P}^* \\ \mu_1 < n \\ l(\mu) = n-k}} k!\,(-1)^{n-k}\, \pi(\mu) \\
&= -\frac{1}{(n-1)!} \sum_{\substack{\nu \in \mathscr{P}^* \\ \nu_1 < n \\ l(\nu) = k-1}} (n-1)!\, \pi(\nu)^{-1} \\
&= -\sum_{\substack{\nu \in \mathscr{P}^* \\ \nu_1 < n \\ l(\nu) = k-1}} \pi(\nu)^{-1}. \quad \blacksquare
\end{aligned}
$$

Note also that $c_n^{(k)} = 0$ if $k > -n > 0$ or if $k = 0$ and $n < 0$, since there is no partition with k distinct parts all less than k, or with -1 parts.

By way of example, let us consider the extreme cases. If $k = -n > 0$, then we must have $k-1$ distinct parts less than k. There is only one way to do this; we must use the partition consisting of the integers from 1 through $k-1$. Thus, $c^{(k)}_{-k} = -1/(k-1)!$.

Conversely, for $k = 1$ and $n < 0$, we sum over partitions with no parts. The trivial partition is the only such partition, so $c_n^{(1)} = -1$.

For $k = 2$, the harmonic numbers of negative degree are, except for sign, the partial sums of the harmonic series

$$
c^{(2)}_{-n} = -1 - \frac{1}{2} - \cdots - \frac{1}{n-1}. \tag{13}
$$

Again, as happened for harmonic numbers of nonnegative degree, larger values of k give generalizations of the harmonic series. For example,

$$c_{-n}^{(3)} = -\frac{1}{2} - \frac{1}{3}\left(1 + \frac{1}{2}\right) - \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$$
$$- \cdots - \frac{1}{n-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-2}\right). \tag{14}$$

Some useful equivalent formulations of Proposition 3.3.10 follow:

COROLLARY 3.3.11. *Let k be a nonnegative integer, and let n be a positive integer. Then*

$$c_{-n}^{(k)} = -n \sum_{\substack{\mu \in \mathscr{P}^* \\ l(\mu) = k \\ \mu_1 = n}} \pi(\mu)^{-1}.$$

Proof. Add the part n to every partition μ being summed over in Proposition 3.3.10. ∎

COROLLARY 3.3.12. *Let k be a nonnegative integer, and let n be a positive integer. Then*

$$c_{-n}^{(k)} = - \sum_{\substack{S \subseteq \{1, 2, \ldots, n-1\} \\ |S| = k-1}} \left(\prod_{s \in S} s^{-1}\right)$$
$$= -n \sum_{\substack{S \subseteq \{1, 2, \ldots, n\} \\ |S| = k \\ n \in S}} \left(\prod_{s \in S} s^{-1}\right),$$

where the sums range over sets S.

Proof. These identities can be obtained from Proposition 3.3.10 and Corollary 3.3.11 by summing over the set of parts of μ instead of μ itself. ∎

COROLLARY 3.3.13. *For all nonnegative integers k, and all positive integers n,*

$$c_{-n}^{(k)} = -n \sum_{\substack{\mu \vdash n \\ l(\mu) = k}} (\pi(\mu)^{-1}) \left(\prod_{j=1}^{n} m_i(\mu)!\right)^{-1},$$

where the sum ranges over partitions ρ of the number n into exactly k parts, and $m_i(\rho)$ denote the number of times i occurs as part of ρ.

Proof. $|s(n,k)|$ is the number of permutations of n letters with k cycles. The number of permutations of cycle type ρ is $n!\,z_\rho^{-1}$, where $z_\rho = \prod_{i\geqslant 1} i^{m_i} m_i!$ and m_i is the number of parts of ρ equal to i. ▮

Recall that for n nonnegative, $\lim_{k\to\infty} c_n^{(k)} = n$. We now have, by contrast,

PROPOSITION 3.3.14. *For n negative,*

$$\sum_{k=1}^{-n} c_n^{(k)} = n. \tag{15}$$

Furthermore,

$$\sum_{k\geqslant 0} c_n^{(k)} = n. \tag{16}$$

Proof 1. It will suffice to prove Eq. (15), since $c_n^{(k)} = 0$ if $k=0$ or if $k > -n$. Let us expand the left-hand side of Eq. (15) as

$$\begin{aligned}
\sum_{k=1}^{-n} c_n^{(k)} &= \sum_{k=1}^{-n} (-1)^k\, s(-n,k) \lfloor n \rceil ! \\
&= -\frac{1}{(-n-1)!} \sum_{k=1}^{-n} |s(-n,k)| \\
&= -\frac{1}{(-n-1)} \sum_{k=1}^{-n} |\{\pi\in S_{-n} : \pi \text{ has } k \text{ cycles}\}| \\
&= -\frac{(-n)!}{(-n-1)!} \\
&= n. \quad ▮
\end{aligned}$$

Proof 2. It will suffice to verify Eq. (16). Expand $(y)_n$ as

$$(y)_n = \sum_{k\geqslant 0} \frac{(-1)^k\, c_{-n}^{(k)} x^k}{\lfloor -n \rceil !},$$

and evaluate at $y=-1$:

$$(-1)(-1-1)\cdots(-1-n+1) = \sum_{k\geqslant 0} \frac{(-1)^k\, c_{-n}^{(k)}}{\lfloor -n \rceil !}(-1)^k$$

$$(-1)^n\, n! = \sum_{k\geqslant 0} (-1)^{n-1}\,(n-1)!\; c_{-n}^{(k)}$$

$$-n = \sum_{k\geqslant 0} c_{-n}^{(k)}. \quad ▮$$

4. Logarithmic Power Series

4.1. *The formal Logarithm*

Our objective is to define a generalization of the ring of formal power series which includes a series of powers of the logarithm function $\log x$.

Definition 4.1.1. Let K be a field of characteristic zero, and let $K[x, x^{-1}]$ denote the field of Laurent series with finitely many terms over the field K. In other words,

$$K[x, x^{-1}] = \{x^n p(x): p(x) \in K[x] \text{ and } n \in \mathbf{Z}\}$$
$$= \left\{\sum_{i=j}^{k} a_i x^i: j, k \in \mathbf{Z},\ j \leqslant k, \text{ and for } j \leqslant i \leqslant k,\ a_i \in K\right\}.$$

Note that $K[x, x^{-1}]$ is a K-algebra.

If $p(x) = \sum_{i=j}^{k} a_i x^i \neq 0$, then the degree of $p(x)$ is defined to be $\deg(p(x)) = \max\{i: a_i \neq 0\}$. We set $\deg(0) = -\infty$.

The *derivative* $\mathbf{D}$ is the operator on $K[x, x^{-1}]$ acting in the usual way;

$$\mathbf{D} \sum_{i=j}^{k} a_i x^i = \sum_{i=j}^{k} i a_i x^{i-1}.$$

The operator $\mathbf{D}$ is a derivation.

We adjoin a logarithm to the algebra $K[x, x^{-1}]$, which will behave as expected under differentiation. To this end, adjoin the formal element $\log x$ to the algebra $K[x, x^{-1}]$. In other words, let $K[x, x^{-1}, \log x]$ be the algebra of polynomials in the "variable" $\log x$, whose coefficients are Laurent series from the field $K[x, x^{-1}]$.

The derivative operator $\mathbf{D}$ is extended to $K[x, x^{-1}, \log x]$ by setting $\mathbf{D} \log x = x^{-1}$. There is a unique such derivation, which we again denote by $\mathbf{D}$. The derivative is the linear operator defined on $K[x, x^{-1}, \log x]$ by

$$\mathbf{D}(x^n (\log x)^t) = n x^{n-1} (\log x)^t + t x^{n-1} (\log x)^{t-1}. \tag{17}$$

Our objective is to complete the algebra $K[x, x^{-1}, \log x]$ in a natural "local" topology. To this end, we will begin by defining another basis of this algebra.

4.2. *The Harmonic Logarithms*

As demonstrated by Eq. (17), the action of the derivative on the basis $x^n (\log x)^t$ of the K-algebra $K[x, x^{-1}, \log x]$ is quite unwieldy. Remarkably, another basis of this algebra can be found which behaves much like the powers x^n do in the algebra of polynomials, at least as far as differentiation goes. To motivate the definition that follows, we anticipate the fact (Corollary 4.3.10) that the harmonic logarithms will turn out to be equal to $\lfloor n \rceil !\, \mathbf{D}^{-n}(\log x)^t$.

DEFINITION 4.2.1 (Harmonic Logarithm). If t is a nonnegative integer and n is an integer, we define the *harmonic logarithm* $\lambda_n^{(t)}(x)$ of degree n and order t to be the element of $K[x, x^{-1}, \log x]$ defined by

$$\lambda_n^{(t)}(x) = x^n \sum_{k=0}^{t} (-1)^k \, (t)_k \, c_n^{(k)} (\log x)^{t-k}.$$

Note that for n negative, $\lambda_n^{(0)}(x) = 0$.

THEOREM 4.2.2 (Alternate Definition of Harmonic Logarithm). *For all integers n and nonnegative integers t,*

$$\lambda_n^{(t)}(x) = x^n \lfloor n \rceil ! \, (x\mathbf{D})_{-n} \, (\log x)^t.$$

Proof. Note that the operator $x\mathbf{D}$ acts on powers of $\log x$ just as the derivative $\mathbf{D}$ acts on powers of x:

$$x\mathbf{D}(\log x)^t = t(\log x)^{t-1}.$$

TABLE 4.1

Harmonic Logarithms of Order Zero

$\vdots \quad \vdots$
$\lambda_n^{(0)}(x) = x^n$
$\vdots \quad \vdots$
$\lambda_2^{(0)}(x) = x^2$
$\lambda_1^{(0)}(x) = x$
$\lambda_0^{(0)}(x) = 1$
$\lambda_{-1}^{(0)}(x) = 0$
$\lambda_{-2}^{(0)}(x) = 0$
$\vdots \quad \vdots$
$\lambda_{-n}^{(0)}(x) = 0$
$\vdots \quad \vdots$

TABLE 4.2

Harmonic Logarithms of Order One

$$\vdots \quad \vdots$$
$$\lambda_n^{(1)}(x) = x^n\left(\log x - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right)$$
$$\vdots \quad \vdots$$
$$\lambda_2^{(1)}(x) = x^2\left(\log x - 1 - \frac{1}{2}\right)$$
$$\lambda_1^{(1)}(x) = x(\log x - 1)$$
$$\lambda_0^{(1)}(x) = \log x$$
$$\lambda_{-1}^{(1)}(x) = x^{-1}$$
$$\lambda_{-2}^{(1)}(x) = x^{-2}$$
$$\vdots \quad \vdots$$
$$\lambda_{-n}^{(1)}(x) = x^{-n}$$
$$\vdots \quad \vdots$$

TABLE 4.3

Harmonic Logarithms of Order Two

$$\vdots \quad \vdots$$
$$\lambda_n^{(2)}(x) = x^n\left[(\log x)^2 - \left(2 + \frac{2}{2} + \cdots + \frac{2}{n}\right)\log x + 2 + \frac{2}{2}\left(1 + \frac{1}{2}\right) + \cdots + \frac{2}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right]$$
$$\vdots \quad \vdots$$
$$\lambda_2^{(2)}(x) = x^2\left[(\log x)^2 - \left(2 + \frac{2}{2}\right)\log x + 2 + \frac{2}{2}\left(1 + \frac{1}{2}\right)\right]$$
$$\lambda_1^{(2)}(x) = x[(\log x)^2 - 2\log x + 2]$$
$$\lambda_0^{(2)}(x) = (\log x)^2$$
$$\lambda_{-1}^{(2)}(x) = 2x^{-1}\log x$$
$$\lambda_{-2}^{(2)}(x) = 2x^{-2}[\log x - 1]$$
$$\lambda_{-3}^{(2)}(x) = 2x^{-3}\left[\log(x) - 1 - \frac{1}{2}\right]$$
$$\vdots \quad \vdots$$
$$\lambda_{-n}^{(2)}(x) = 2x^{-n}\left[\log(x) - 1 - \frac{1}{2} - \cdots - \frac{1}{n-1}\right]$$
$$\vdots \quad \vdots$$

Thus,

$$\begin{aligned}
\lambda_n^{(t)}(x) &= x^n \sum_{k \geqslant 0} (-1)^k (t)_k c_n^{(k)} (\log x)^{t-k} \\
&= x^n \llcorner n \urcorner! \sum_{k \geqslant 0} s(-n, k)(t)_k (\log x)^{t-k} \\
&= x^n \llcorner n \urcorner! \sum_{k \geqslant 0} s(-n, k)(x\mathbf{D})^k (\log x)^t \\
&= x^n \llcorner n \urcorner! (x\mathbf{D})_{-n} (\log x)^t. \quad \blacksquare
\end{aligned}$$

PROPOSITION 4.2.3. *The set of nonzero harmonic logarithms $\lambda_n^{(t)}(x)$ is a basis for $K[x, x^{-1}, \log x]$.*

Proof. When n is an integer, and t is a nonnegative integer, set

$$L_n^{(t)}(x) = \begin{cases} \lambda_n^{(t)}(x) & \text{for} \quad n \geqslant 0, \text{ and} \\ (1/(t+1))\, \lambda_n^{(t+1)}(x) & \text{for} \quad n < 0. \end{cases}$$

It will suffice to prove that the linear transformation that maps $x^n(\log x)^t$ to $L_n^{(t)}(x)$ is unitriangular in the basis of monomials $x^n(\log x)^t$. In other words, we must show that the coefficient of $x^n(\log x)^s$ in $L_n^t(x)$ is zero for $s > t$, and one for $s = t$, since $L_n^{(t)}(x)$ is homogeneous in x of degree n.

1. Assume $n \geqslant 0$. Clearly, for $s > t$, the coefficient is zero. For $s = t$, coefficient is $(-1)^0\, 0!\, \binom{t}{0} c_n^{(0)} = 1$.

2. Assume $n < 0$. As above, we are done for $s > t + 1$. For $s = t + 1$, the desired coefficient is $-(1/(t+1))(t+1)_0\, c_n^{(0)}$. However, $c_n^{(0)} = 0$, so the coefficient is zero as desired. Finally, for $s = t$, we have $(t+1)_1\, c_n^{(1)}/(t+1)$, which equals unity. $\blacksquare$

COROLLARY 4.2.4. *Every $p(x) \in K[x, x^{-1}, \log x]$ can be uniquely written as a finite sum of the form*

$$p(x) = \sum_{n=0}^{j} b_n^{(0)} \lambda_n^{(0)}(x) + \sum_{t=1}^{k} \sum_{n=i}^{j} b_n^{(t)} \lambda_n^{(t)}(x). \tag{18}$$

Let $p(x) \in K[x, x^{-1}, \log x]$, and let the coefficients $b_n^{(t)}$ be as in Eq. (18). If $b_n^{(t)} \neq 0$ implies that $t = t_0$, then $p(x)$ is said to be *homogeneous* of order t_0.

Let $L^{(t)}$ denote the subspace of $K[x, x^{-1}, \log x]$ consisting of those elements which are homogeneous of order t. Thus, $K[x, x^{-1}, \log x]$ is the direct sum of the subspaces $L^{(t)}$.

For any nonzero logarithmic series $p(x) \in K[x, x^{-1}, \log(x)]$, we define the *degree* of $p(x)$ to be $\deg(p(x)) = \max\{n\colon b_n^{(t)} \neq 0 \text{ for some } t\}$, where the constants $b_n^{(t)}$ are given by Eq. (18). By convention, $\deg(0) = -\infty$.

4.3. *The Logarithmic Algebra*

We define a topology on $L^{(t)}$ as follows: A sequence $(p_n(x))_{n \geqslant 0}$ in $L^{(t)}$ is Cauchy if for every integer d, there is an integer N, such that for any $n, m \geqslant N$, the difference $p_N(x) - p_n(x)$ is of degree at most d.

In other words,

PROPOSITION 4.3.1. *A sequence $(p_n(x))_{n \geqslant 0}$ in $L^{(t)}$ is Cauchy if and only if*

1. *The degree of $p_n(x)$ is bounded above, and*

2. *The coefficients b_{nk} of $p_n(x) = \sum_{k \leqslant n} b_{nk} \lambda_k^{(t)}(x)$ are eventually constant for all fixed k as n goes to infinity.*

Proof. (If) Suppose $\deg(p_n(x)) \leqslant d$ for all n, and that for $k \leqslant d$, b_{nk} is constant for $n \geqslant N_k$. Now, for any k, let $N = 0$ if $k > d$ and otherwise let $N = \max\{N_j\colon k \leqslant j \leqslant d\}$. Now, any difference $p_N(x) - p_n(x)$ for $n \geqslant N$ will be of degree at most k, so $(p_n^{(t)})_{n \geqslant 0}$ is Cauchy.

(Only If) If the degree of $p_n(x)$ is not bounded above, then the difference $p_{n+1}(x) - p_n(x)$ will have arbitrarily *high* degree, so the sequence will not be Cauchy. If b_{nk} is not eventually constant, then it varies frequently, and thus the difference $p_{n+1}(x) - p_n(x)$ will frequently have degree at *least* k. ∎

DEFINITION 4.3.2. $\mathscr{L}^{(t)}$ is defined to be the completion of $L^{(t)}$ in this topology. Its elements are said to be *homogeneous logarithmic series* of order t.

PROPOSITION 4.3.3. *An element $p^{(t)}(x)$ of $\mathscr{L}^{(t)}$ can be uniquely expressed as the following convergent infinite sum*

$$p^{(t)}(x) = \sum_{n \leqslant d} b_n \lambda_n^{(t)}(x).$$

Proof. $p^{(t)}(x)$ is the limit of some Cauchy sequence $(p_n^{(t)}(x))_{n \geqslant 0}$. Let d be an upper bound on the degree of $p_n^{(t)}(x)$. Without loss of generality, we may assume that $\deg(p_{n+1}^{(t)}(x) - p_n^{(t)}(x)) \leqslant d - n$. Thus, b_n is the coefficient of $\lambda_n^{(t)}(x)$ in the expression $p_{d-n}^{(t)}(x)$. ∎

DEFINITION 4.3.4 (Logarithmic Algebra). The *logarithmic algebra* $\mathscr{L}$ is the (algebraic) direct sum

$$\mathscr{L} = \bigoplus_{t \geq 0} \mathscr{L}^{(t)}.$$

Members of $\mathscr{L}$ will be called *formal power series of logarithmic type*, or simply *logarithmic series*; they are *finite* sums of elements of $\mathscr{L}^{(t)}$ as t ranges over the nonnegative integers.

In other words, every formal series of logarithmic type $p(x)$ is a convergent sum—in the topology just defined—of the form

$$p(x) = p^{(0)}(x) + p^{(1)}(x) + p^{(2)}(x) + \cdots + p^{(s)}(x),$$

where

$$p^{(0)}(x) = \sum_{n=0}^{d} b_n^{(0)} x^n,$$

and for t positive,

$$p^{(t)}(x) = \sum_{n \leq d} b_n^{(t)} \lambda_n^{(t)}(x).$$

Thus, a formal power series of logarithmic type is a *finite* sum of homogeneous series $p^{(t)}(x)$.

Logarithmic series of order zero are ordinary polynomials, that is, $\mathscr{L}^{(0)} = K[x]$. Although $\mathscr{L}^{(0)}$ is a subalgebra of $\mathscr{L}$ none of the other subspaces $\mathscr{L}^{(t)}$ are closed under multiplication.

The *positive logarithmic subspace* $\mathscr{L}^{(+)}$ is the direct sum

$$\mathscr{L}^{(+)} = \bigoplus_{t > 0} \mathscr{L}^{(t)}.$$

Note that $\mathscr{L}^{(+)}$ is not closed under multiplication, since

$$\begin{aligned} \lambda_1^{(1)}(x)\, \lambda_{-1}^{(1)}(x) &= \log x - 1 \\ &= \lambda_0^{(1)}(x) - \lambda_0^{(0)}(x) \\ &\notin \mathscr{L}^{(+)}. \end{aligned}$$

The algebra $\mathscr{L}$ of formal power series of logarithmic type is a generalization of the algebra of polynomials, and it is our objective to extend to this algebra various properties of the algebra of formal power series. Such extensions will be informally called "*logarithmic extensions.*" We begin by establishing a fundamental property:

THEOREM 4.3.5. *The logarithmic algebra, $\mathscr{L}$, is a topological algebra.*

Proof. It will suffice to show that multiplication is continuous. Suppose $(p_n(x))_{n\geqslant 0}$ converges to $p(x)$ and $(q_n(x))_{n\geqslant 0}$ converges to $q(x)$. Without loss of generality, we may assume that $p_n(x)-p(x)$ and $q_n(x)-q(x)$ are of degree $-n$. Let a be the maximum degree attained by $q_n(x)$. The integer a is well defined, since $(q_n(x))_{n\geqslant 0}$ converges. Let b be the degree of $p(x)$. Thus, the product $(p_n(x)-p(x))\,q_n(x)$ is of degree at most $a-n$, and $p(x)(q_n(x)-q(x))$ is of degree $b-n$. Hence, $p_n(x)\,q_n(x)-p(x)\,q(x)$ is of degree at most $a+b-n$ which is eventually arbitrary small. Thus, $(p_n(x)\,q_n(x))_{n\geqslant 0}$ converges to $p(x)\,q(x)$. ∎

We note for further use the following:

PROPOSITION 4.3.6. *For each nonnegative integer t,* $(\log x)^t=\lambda_0^{(t)}(x)$.

Proof. Immediate from Definition 4.2.1. ∎

The following theorem is used to justify applying the results of this paper to problems involving the complex or real numbers.

THEOREM 4.3.7. *Let S be an open subset of the complex plane for which $\{|z|\colon z\in S\}$ is unbounded. Suppose $(p_n(x))_{n\geqslant 0}$ is a sequence of formal logarithmic series which converges to a formal series $p(x)$ which represents a function from S to the complex numbers. Then the sequence $(p_n(x))_{n\geqslant 0}$ is asymptotic to $p(x)$ as $|x|$ goes to infinity* within S.

Proof. Now, for $n\geqslant 0$,

$$\frac{\lambda_n^{(t)}(x)}{\lambda_{n+1}^{(t)}(x)}=\frac{\mathcal{O}(x^n(\log x)^t)}{\mathcal{O}(x^{n+1}(\log x)^t)}$$

as $|x|\to\infty$, so

$$\lim_{|x|\to\infty}\frac{\lambda_n^{(t)}(x)}{\lambda_{n+1}^{(t)}(x)}=0.$$

Similarly, for $n<0$,

$$\lim_{|x|\to\infty}\frac{\lambda_n^{(t)}(x)}{\lambda_{n+1}^{(t)}(x)}=0. \quad ∎$$

The following result is fundamental. It shows that the harmonic logarithms behave under differentiation like ordinary powers, provided the ordinary factorial $n!$ is replaced by the Roman factorial:

THEOREM 4.3.8. *Let n be an integer, and t be a nonnegative integer. Then*

$$\mathbf{D}\lambda_{n+1}^{(t)}(x) = \lfloor n+1 \rceil \lambda_n^{(t)}(x).$$

More generally, for any nonnegative integer i,

$$\mathbf{D}^i \lambda_{n+i}^{(t)}(x) = \frac{\lfloor n+i \rceil!}{\lfloor n \rceil!} \lambda_n^{(t)}(x). \tag{19}$$

We have three proofs.

Proof 1. We have

$$\begin{aligned}
\mathbf{D}\lambda_{n+1}^{(t)}(x) &= \sum_{k=0}^{t} (-1)^k (t)_k \, c_{n+1}^{(k)} \, \mathbf{D}x^{n+1}(\log x)^{t-k} \\
&= \sum_{k=0}^{t} (-1)^k (t)_k \, c_{n+1}^{(k)} [(n+1)\, x^n (\log x)^{t-k} \\
&\quad + (t-k)\, x^n (\log x)^{t-k-1}] \\
&= \sum_{k=0}^{t} (-1)^k (t)_k \left((n+1) c_{n+1}^{(k)} - c_{n+1}^{(k-1)}\right) x^{n+1} (\log x)^{t-k}.
\end{aligned}$$

By Theorem 3.3.4, $(n+1)c_{n+1}^{(k)} - c_{n+1}^{(k-1)} = \lfloor n+1 \rceil c_n^{(k)}$, hence the conclusion. ∎

Proof 2. We have

$$\begin{aligned}
\mathbf{D}\lambda_{n+1}^{(t)}(x) &= \mathbf{D}x^{n+1} \lfloor n+1 \rceil! \, (x\mathbf{D})_{-n-1} (\log x)^t \\
&= \lfloor n+1 \rceil! \, (n+1)\, x^n (x\mathbf{D})_{-n-1} \\
&\quad + \lfloor n+1 \rceil! \, x^{n+1} \mathbf{D}(x\mathbf{D})_{-n-1} (\log x)^t \\
&= \lfloor n+1 \rceil! \, x^n (n+1+x\mathbf{D})(x\mathbf{D})_{-n-1} (\log x)^t \\
&= \lfloor n+1 \rceil \lfloor n \rceil! \, x^n (x\mathbf{D})_{-n} (\log x)^t \\
&= \lfloor n+1 \rceil \lambda_n^{(t)}(x). \quad ∎
\end{aligned}$$

Recall that $\lambda_n^{(0)}(x) = 0$ for n negative.

The third proof will be deferred until after Proposition 5.3.2.

PROPOSITION 4.3.9. *The restriction of the derivative operator* $\mathbf{D}$ *to* $\mathscr{L}^{(+)}$ *is a bijection.*

Proof. For $t>0$, define a continuous linear operator by

$$\lambda_n^{(t)}(x) \mapsto \lfloor n+1 \rceil^{-1} \lambda_{n+1}^{(t)}(x)$$

for all integers n. It is clear this is the inverse of $\mathbf{D}$. ∎

On $\mathscr{L}^{(+)}$, we write $\mathbf{D}^{-1}$ to denote the inverse of $\mathbf{D}$. Equation (19) is therefore valid for all integers i and n when t is a positive integer. In particular, we have the following simple formula for the harmonic logarithms:

COROLLARY 4.3.10. *Let t be a positive integer, and n be an arbitrary integer. Then*

$$\lambda_n^{(t)}(x) = \lfloor n \rceil! \, \mathbf{D}^{-n} (\log x)^t.$$

Finally, here is a result which has been previously alluded to.

COROLLARY 4.3.11. *For all positive integers t, the subspace $\mathscr{L}^{(t)}$ is the minimal closed invariant subspace of $\mathscr{L}^{(+)}$ under the action of* $\mathbf{D}$ *and* $\mathbf{D}^{-1}$ *which contains* $(\log x)^t$.

Proof. Proposition 4.3.6 and Theorem 4.3.8. ∎

5. SHIFT-INVARIANT OPERATORS

5.1. *The Operator Topology*

It is a classical result that the algebra of formal differential operators $\sum_{n \geq 0} a_n \mathbf{D}^n$ acts on the vector space $\mathscr{L}^{(0)}$ of polynomials. In view of the fact that the derivative operator is invertible in $\mathscr{L}^{(t)}$ for t positive (Proposition 4.3.9), we can define on $\mathscr{L}^{(+)}$ the action of a more general class of differential operators, namely, Laurent series in the derivative. To this end, we begin by defining a topology on the ring of continuous linear operators acting on formal power series of logarithmic type.

Let $\mathscr{M}$ be a closed subspace of $\mathscr{L}$. We say that a sequence $(\theta_n)_{n \geq 0}$ of continuous linear operators of $\mathscr{M}$ into itself converges in the operator topology on $\mathscr{M}$ when for every $p(x) \in \mathscr{M}$ the sequence $(\theta_n p(x))_{n \geq 0}$ converges.

PROPOSITION 5.1.1. *Let* $\mathscr{M} = \mathscr{L}$, $\mathscr{L}^{(+)}$, *or* $\mathscr{L}^{(t)}$. *Then the set of con-*

tinuous linear operators in the operator topology of $\mathcal{M}$ is a complete topological K-algebra whose operations are given by

$$(\theta\phi)\,p(x) = \theta(\phi p(x))$$
$$(a\theta)\,p(x) = a(\theta p(x))$$
$$(\theta + \phi)\,p(x) = (\theta p(x)) + (\phi p(x)).$$

Proof. Let $(\theta_n)_{n \geqslant 0}$ and $(\phi_n)_{n \geqslant 0}$ be convergent sequences of continuous linear operators on $\mathcal{M}$. Thus, for any $p(x) \in \mathcal{M}$, the sequences

$$(\theta_n\, p(x))_{n \geqslant 0} \qquad \text{and} \qquad (\phi_n\, p(x))_{n \geqslant 0}$$

are Cauchy. In particular,

$$(\theta_n \phi_k\, p(x))_{n \geqslant 0}$$

is Cauchy for any $k \geqslant 0$, and since θ_k is continuous,

$$(\theta_k \phi_n\, p(x))_{n \geqslant 0}$$

is also Cauchy for any $k \geqslant 0$. Hence,

$$(\theta_n \phi_n\, p(x))_{n \geqslant 0}$$

is Cauchy, and

$$(\theta_n \phi_n)_{n \geqslant 0}$$

converges.

$((\theta_n + \phi_n)\, p(x))_{n \geqslant 0}$ converges since $\mathcal{M}$ is a topological space. Hence, $(\theta_n + \phi_n)_{n \geqslant 0}$ converges.

The ring is complete since $\theta(p(x)) = \lim_{n \to \infty} \theta_n\, p(x)$ is the limit of a Cauchy sequence $(\theta_n)_{n \geqslant 0}$. ∎

We list below some notable operators:

1. (Derivative) The derivative operator $\mathbf{D}$ is defined on all of $\mathcal{L}$.

2. (Antiderivative) The inverse of the derivative operator—denoted by $\mathbf{D}^{-1}$—is defined on $\mathcal{L}^{(+)}$, but not on $\mathcal{L}^{(0)}$.

3. (Shift Operator) For any field element $a \in K$, the *shift operator*, $E^a\colon \mathcal{L} \to \mathcal{L}$, is the unique continuous algebra isomorphism such that

$$E^a x = x + a,$$

and

$$E^a \log x = \log x + \sum_{j > 0} (-1)^{j+1} \frac{a^j}{j x^j}.$$

Denote E^1 by E.

4. (Elementary Shift-Invariant Operator) For any pair of nonnegative integers s and t, the *elementary shift-invariant operator* from t to s—denoted E_{st}—is the (continuous) K-linear map on the logarithmic algebra defined by

$$E_{st}\lambda_n^{(u)}(x) = \begin{cases} \lambda_n^{(s)}(x) & \text{if } t = u, \text{ and} \\ 0 & \text{if } t \neq u, \end{cases}$$

where s, t, and u are nonnegative integers, and n is an integer (nonnegative if $u = 0$).

5. (Projection Map) E_{tt}-$\mathbf{proj}_t$ is the projection map $\mathscr{L} \to \mathscr{L}^{(t)}$. In other words, for nonnegative integers s and t, and integers n,

$$\mathbf{proj}_s(\lambda_n^{(t)}(x)) = \begin{cases} \lambda_n^{(t)}(x) & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases}$$

These projections commute with $\mathbf{D}$. Note, however, that not all continuous, linear projections which commute with $\mathbf{D}$ are expressible in terms of these projections.

A subspace $\mathscr{M}$ of $\mathscr{L}$ will be said to be *shift-invariant* if it is invariant under the shift operator E^a for all $a \in K$. An operator θ on a shift-invariant subspace $\mathscr{M}$ will be said to be *shift-invariant* when $E^a\theta = \theta E^a$ for all $a \in K$. For example, all the operators listed above will be shown to be shift-invariant.

A continuous linear operator θ on $\mathscr{L}$ or $\mathscr{L}^{(+)}$ is said to be a *regular* operator if it commutes with every elementary shift-invariant operator E_{st} (except possibly when $t = 0$), that is, such that

$$\theta E_{st} = E_{st}\theta$$

for $t \neq 0$. For example, $\mathbf{D}$, $\mathbf{D}^{-1}$, and E^a are regular operators.

A regular shift-invariant operator on $\mathscr{L}^{(+)}$ will be called a *Laurent operator*. The set of Laurent operators will be denoted by Γ, and the set of differential operators by $\Gamma^{(+)}$.

Clearly every differential operator restricts to a Laurent operator.

This notation is not as counterintuitive as it may seem; we will show that $\Gamma^{(+)}$ is generated by the nonnegative powers of $\mathbf{D}$ whereas Γ is generated by all powers of $\mathbf{D}$. That is the meaning of the $(+)$ in the symbol $\Gamma^{(+)}$.

PROPOSITION 5.1.2. 1. *The set of Laurent operators Γ is a complete topological ring in the operator topology of $\mathscr{L}^{(+)}$.*

2. *The set of differential operators $\Gamma^{(+)}$ is a complete topological ring in the operator topology of $\mathscr{L}$.*

Proof. Γ and $\Gamma^{(+)}$ clearly are K-algebras, so it will suffice to show that the limit of any Cauchy sequence of Laurent (resp. differential) operators is again a Laurent (resp. differential) operator.

Let $(\theta_n)_{n \geqslant 0}$ be such a Cauchy sequence, and let θ be its limit. Now,

$$\begin{aligned}\theta E^a p(x) &= \lim_{n \to \infty} \theta_n E^a p(x) \\ &= \lim_{n \to \infty} E^a \theta_n p(x) \\ &= E^a \lim_{n \to \infty} \theta_n p(x)\end{aligned}$$

since E^a is a continuous operator. This in turn equals $E^a \theta p(x)$, so θ is shift-invariant. *Mutatis mutandis*, we have $E_{st}\theta = \theta E_{st}$, so θ is regular. ∎

Our objective is to obtain structural characterizations of Laurent and differential operators.

We shall write infinite series

$$\sum_{k \geqslant d} \theta_k$$

of operators, which are understood to denote the limits of their partial sums.

5.2. *Taylor's Formula*

We shall derive analogs of Taylor's formula in the Logarithmic algebra $\mathscr{L}$. We begin by giving the following alternate definition of the shift operator:

PROPOSITION 5.2.1. *For each field element $a \in K$, we have*

$$E^a = \sum_{k \geqslant 0} \frac{a^k}{k!} \mathbf{D}^k = e^{a\mathbf{D}}. \tag{20}$$

Proof. For each field element $a \in K$, define

$$T^a = \sum_{k \geqslant 0} \frac{a^k \mathbf{D}^k}{k!}.$$

We wish to show that $T^a = E^a$. Since $\mathscr{L}$ is the completion of $K[x, x^{-1}, \log x]$, it will suffice to show that T^a is an algebra homomorphism, and agrees with E^a on x, and $\log(x)$. Now,

$$\begin{aligned} T^a x &= \sum_{k\geqslant 0} \frac{a^k}{k!} \mathbf{D}^k x \\ &= \begin{bmatrix} n \\ 0 \end{bmatrix} a^0 \lambda_1^{(0)}(x) + \begin{bmatrix} n \\ 1 \end{bmatrix} a^1 \lambda_0^{(0)}(x) \\ &= x + a \\ &= E^a x, \end{aligned}$$

and

$$\begin{aligned} T^a \log x &= \sum_{k\geqslant 0} \frac{a^k}{k!} \mathbf{D}^k \log x \\ &= \sum_{k\geqslant 0} \begin{bmatrix} 0 \\ k \end{bmatrix} a^k \lambda_{-k}^{(1)}(x) \\ &= \log x + \sum_{k>0} \frac{(-1)^{k+1} a^k}{k x^k}, \end{aligned}$$

so by definition,

$$T^a \log x = E^a \log x.$$

Hence, E^a and T^a agree on x and $\log x$.

To show T^a is an algebra homomorphism, choose a pair of formal power series of logarithmic type $p(x)$, $q(x) \in \mathscr{L}$. Then

$$\begin{aligned} T^a(p(x)\, q(x)) &= \sum_{k\geqslant 0} \frac{a^k}{k!} \mathbf{D}^k(p(x)\, q(x)) \\ &= \sum_{k\geqslant 0} \frac{a^k}{k!} \sum_{i+j=k} \binom{k}{i} (\mathbf{D}^i p(x))(\mathbf{D}^j q(x)) \\ &= \sum_{i,j\geqslant 0} \frac{a^{i+j}}{i!\, j!} (\mathbf{D}^i p(x))(\mathbf{D}^j q(x)) \\ &= (T^a p(x))(T^a q(x)). \quad \blacksquare \end{aligned}$$

Thus, $E^a E^b = E^{a+b}$.

The Binomial Theorem is a trivial consequence of Proposition 5.2.1 relative to the harmonic logarithms of order zero—the powers of x. Whereas, by considering the harmonic logarithms of order one and nonnegative degree, $\lambda_n^{(1)}(x)$ with $n \geqslant 0$, we obtain the following identity for harmonic logarithms:

$$
\begin{aligned}
&(1+a)^n\left(\log(1+a)-1-\frac{1}{2}-\cdots-\frac{1}{n}\right) \\
&\quad=\left[(x+a)^n\left(\log(x+a)-1-\frac{1}{2}-\cdots-\frac{1}{n}\right)\right]_{x=1} \\
&\quad=\left[\sum_{i=0}^{n}\left\lfloor \begin{matrix} n \\ i \end{matrix}\right\rceil a^{n-i}x^i\left(\log x-1-\frac{1}{2}-\cdots-\frac{1}{i}\right)+\sum_{i>n}\left\lfloor \begin{matrix} n \\ i \end{matrix}\right\rceil a^i x^{n-i}\right]_{x=1} \\
&\quad=-\sum_{i=0}^{n}\left\lfloor \begin{matrix} n \\ i \end{matrix}\right\rceil a^{n-i}\left(1+\frac{1}{2}+\cdots+\frac{1}{i}\right)+\sum_{i>n}\left\lfloor \begin{matrix} n \\ i \end{matrix}\right\rceil a^i,
\end{aligned}
$$

and therefore:

$$
\begin{aligned}
&(1+a)^n\log(1+a) \\
&\quad=((1+a)^n-1)c_n^{(1)}-\sum_{i=0}^{n-1}\binom{n}{i}a^{n-i}c_i^{(1)}+\sum_{i>n}\left\lfloor \begin{matrix} n \\ i \end{matrix}\right\rceil a^i \\
&\quad=((1+a)^n-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-na\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) \\
&\qquad-\binom{n}{2}a^2\left(1+\frac{1}{2}+\cdots+\frac{1}{n-2}\right)-\cdots-\frac{3}{2}\binom{n}{2}a^{n-2}-na^{n-1} \\
&\qquad+a^{n+1}\left[\begin{matrix} n \\ -1 \end{matrix}\right]+a^{n+2}\left[\begin{matrix} n \\ -2 \end{matrix}\right]+\cdots.
\end{aligned}
$$

5.3. *Complex or Real Analysis*

Proposition 5.2.1 can be summarized as stating that

$$E^a f(x)=e^{a\mathbf{D}}f(x)=f(x+a)$$

for all logarithmic series $f(x)\in\mathscr{L}$. Note that this identity is tautological, since we cannot "evaluate" the variable x. However, in the case of complex numbers, we can, and we have:

PROPOSITION 5.3.1. *Suppose the following*:

- *The base field K is the field of complex numbers.*
- *$f(x)$ is a formal power series of logarithmic type*:

$$f(x)=\sum_{n,t}a_n^{(t)}\lambda_n^{(t)}(x). \tag{21}$$

- *When $\lambda_n^{(t)}(x)$ is considered as a function as opposed to as a formal expression, we have the following convergence within an open disk in the complex plane (centered at say a and with radius b): The summation in Eq.* (21) *when summed in some order converges to a* function $\tilde{f}(x)$.
- $\tilde{f}(x)$ *is analytic in the open disk centered at a with radius b.*
- ε *is a complex number with* $|\varepsilon| < b$.

Then the formal series $g(x) = E^{\varepsilon} f(x)$ *converges (within a neighborhood of a) to the function* $\tilde{g}(x) = \tilde{f}(x+\varepsilon)$ *regardless of the order of summation.*

Proof. We observe

$$\begin{aligned} E^{\varepsilon} f(x)|_{x=z} &= e^{\varepsilon \mathbf{D}} \tilde{f}(x) \\ &= \sum_{k \geqslant 0} \frac{\varepsilon^k}{k!} \mathbf{D}^k \tilde{f}(x)|_{x=z} \\ &= \sum_{k \geqslant 0} \frac{x^k}{k!} [\mathbf{D}^k f(x)]_{x=z}|_{x=\varepsilon}, \end{aligned}$$

which converges by assumption. ∎

For example, $x^n (\log x)^t$ and $\lambda_n^{(t)}(x)$ and finite linear combinations thereof satisfy the conditions of the above proposition since they are analytic in the punctured plane.

Now, we have the following identity in the complex numbers where $\Gamma(z)$ denotes the Gamma function:

PROPOSITION 5.3.2 (Knuth). 1. *For any integer n and nonnegative integer k,* $(-1)^k c_n^{(k)}$ *is the coefficient of* z^k *in the expansion of*

$$\frac{\lfloor n \rceil! \, \Gamma(z+1)}{\Gamma(z+n+1)}.$$

2. *For any integer n,*

$$\frac{\lfloor n \rceil! \, x^{n+z} \Gamma(z+1)}{\Gamma(z+n+1)} = \sum_{t \geqslant 0} \lambda_n^{(t)}(x) \frac{z^t}{t!}.$$

Proof. (1) The coefficient of z^k is by definition

$$\frac{\lfloor n \rceil!}{k!} [\mathbf{D}^k (x)_{-n}]_{x=0}$$

which in turn is by Definition 3.3.3 equal to $(-1)^k c_n^{(k)}$.

(2) Apply part 1 to Definition 4.2.1. ∎

This gives us our third proof of Theorem 4.3.8.

Proof 3 *of Theorem* 4.3.8. We have

$$\begin{aligned}\sum_{t\geqslant 0} \mathbf{D}\lambda_n^{(t)}(x)\frac{z^t}{t!} &= \mathbf{D}\frac{\lfloor n\rceil!\, x^{n+z}\Gamma(z+1)}{\Gamma(z+n+1)}\\ &= \frac{(n+z)\lfloor n\rceil!\, x^{n+z-1}\Gamma(z+1)}{\Gamma(z+n+1)}\\ &= \lfloor n\rceil\frac{\lfloor n-1\rceil!\, x^{(n-1)+z}\,\Gamma(z+1)}{\Gamma(z+(n-1)+1)}\\ &= \sum_{t\geqslant 0}\lfloor n\rceil\,\lambda_{n-1}^{(t)}(x)\frac{z^t}{t!},\end{aligned}$$

so $\mathbf{D}\lambda_n^{(t)}(x)=\lfloor n\rceil\lambda_{n-1}^{(t)}(x)$. ∎

5.4. *Characterization of Various Classes of Operators*

5.4.1. *Laurent Operators*

Every Laurent operator θ maps $\mathscr{L}^{(t)}$ into itself for every positive integer t. We denote by θ its restriction to $\mathscr{L}^{(t)}$, by an abuse of notation, and we say that θ is a Laurent operator of $\mathscr{L}^{(t)}$ into itself.

THEOREM 5.4.1. *θ is a Laurent operator on the positive logarithmic algebra $\mathscr{L}^{(+)}$ if and only if there exists a convergent series*

$$\theta=\sum_{k\geqslant d} a_k\,\mathbf{D}^k,$$

where $a_k\in K$, and d is an integer.

If $a_d\neq 0$, we say that θ is a Laurent operator of *degree d*.

Proof. Let G be the set of operators expressed by such a convergent series.

$(G\subseteq\Gamma)$ $E^a\in G$ by Proposition 5.2.1. Also, G is clearly commutative, so its members are shift-invariant. Finally, $\mathbf{D}$ is regular. Hence, all members of G are regular.

$(\Gamma\subseteq G)$ Let θ be a Laurent operator. By regularity, θ is determined by its action on the harmonic logarithms of order t for any particular positive t:

$$\theta\lambda_n^{(t)}(x)=\sum_{m\in Z} e_{nm}\lambda_m^{(t)}(x).$$

Notice that since $\deg(\theta\lambda_n^{(t)}(x))$ is finite, we have, for sufficiently small m, $e_{nm}=0$. Next,

$$\begin{aligned}
\theta E^a\lambda_n^{(t)}(x) &= \sum_{m\geq 0}\left\lfloor \begin{matrix} n \\ m\end{matrix}\right\rceil a^m\theta\lambda_{n-m}^{(t)}(x)\\
&= \sum_{m\geq 0}\sum_{k\in\mathbf{Z}}\left\lfloor \begin{matrix} n \\ m\end{matrix}\right\rceil a^m e_{n-m,k}\lambda_k^{(t)}(x)\\
E^a\theta\lambda_n^{(t)}(x) &= \sum_{i\in\mathbf{Z}} e_{ni}E^a\lambda_i^{(t)}(x)\\
&= \sum_{i\in\mathbf{Z}} e_{ni}\sum_{j\geq 0}\left\lfloor \begin{matrix} i \\ j\end{matrix}\right\rceil a^j\lambda_{i-j}^{(t)}(x)\\
&= \sum_{i\in\mathbf{Z}}\sum_{k\leq i}\left\lfloor \begin{matrix} i \\ k\end{matrix}\right\rceil a^{i-k}e_{ni}\lambda_k^{(t)}(x)\\
&= \sum_{m\in\mathbf{Z}}\sum_{k\geq 0}\left\lfloor \begin{matrix} m+k \\ k\end{matrix}\right\rceil a^m e_{n,m+k}\lambda_k^{(t)}(x).
\end{aligned}$$

Equating coefficients of $a^m\lambda_k^{(t)}(x)$ we obtain

$$\left\lfloor \begin{matrix} n \\ m\end{matrix}\right\rceil e_{n-m,k} = \left\lfloor \begin{matrix} m+k \\ k\end{matrix}\right\rceil e_{n,m+k} \qquad \text{for all} \quad n, m, \text{and } k.$$

In particular, setting $m=n$, we find that

$$e_{0,k} = \left\lfloor \begin{matrix} n+k \\ k\end{matrix}\right\rceil e_{n+k,k}.$$

Hence, θ is determined by the sequence $(e_{0,k})_{k\in\mathbf{Z}}$, and therefore equals the operator

$$\sum_{n\in\mathbf{Z}}\frac{e_{n,0}}{\lfloor n\rceil!}\mathbf{D}^n.$$

Note that $e_{n,0}$ is zero for sufficiently small n. ▌

In view of the preceding theorem, we see that the K-algebra of Laurent operators is isomorphic to the ring of formal Laurent series in the "variable" $\mathbf{D}$. This isomorphism is easily seen to be an isomorphism of topological K-algebras. In particular, every Laurent operator on $\mathscr{L}^{(+)}$ is *invertible*. Hence, every differential equation of the form

$$f(\mathbf{D})\, p(x) = q(x),$$

where $f(\mathbf{D}) \in \Gamma$ is a Laurent operator and $q(x) \in \mathscr{L}^{(+)}$, has a unique solution $p(x)$ in $\mathscr{L}^{(+)}$.

5.4.2. *Differential Operators*

In analogy with the preceding result for Laurent operators, we obtain the following structure theorem for differential operators:

COROLLARY 5.4.2. *θ is a differential operator on the logarithmic algebra $\mathscr{L}$ if and only if there exists a convergent series such that*

$$\theta = \sum_{k \geqslant 0} a_k \mathbf{D}^k,$$

where $a_k \in K$.

Proof. Let $G^{(+)}$ be the set of such convergent series.

The set of shift-invariant operators on the algebra of polynomials $\mathscr{L}^{(0)}$ is well known to be $G^{(+)}$, so $G^{(+)} \subseteq \Gamma^{(+)}$. Conversely, by the above theorem $G^{(+)}$ contains all those elements of Γ which are well defined on $\mathscr{L}^{(0)}$. ∎

Thus, $\Gamma^{(+)}$ is naturally isomorphic to the ring of formal power series over K. As opposed to Laurent operators, a differential operator is invertible in $\Gamma^{(+)}$ if and only if it is of degree 0.

Furthermore, $\Gamma^{(+)}$ can be characterized as the set of Laurent operators of nonnegative degree.

5.4.3. *Shift-Invariant Operators*

The rings of shift-invariant operators on $\mathscr{L}$ and $\mathscr{L}^{(+)}$ are structured as follows:

PROPOSITION 5.4.3. 1. *The ring of shift-invariant continuous linear operators on the positive logarithmic subspace $\mathscr{L}^{(+)}$ is the closure, in the operator topology, of the span of the operators $\mathbf{D}^n E_{st}$, where n is an integer and s and t are positive integers.*

2. *The ring of shift-invariant continuous linear operators on the logarithmic algebra $\mathscr{L}$ is the closure, in the operator topology, of the span of the operators $\mathbf{D}^n E_{st}$, where either n is an integer and s and t are positive integers, or n and s are nonnegative integers and $t = 0$.*

Proof. Let $\mathscr{D}$ and $\mathscr{D}_0$ be the two closures defined above, and let $\mathscr{R}$ and $\mathscr{R}_0$ be the two rings defined above.

($\mathscr{D} \subseteq \mathscr{R}$ and $\mathscr{D}_0 \subseteq \mathscr{R}_0$) Elementary shift-invariant operators commute with the derivative. Thus, they commute with all Laurent and differential operators. Hence, they are in fact shift-invariant.

Observe that $E_{st} D^n$ is continuous, linear, and shift-invariant when n is an integer and s, t are positive integers, and when n and s are nonnegative and

t is zero. We conclude that every operator in $\mathscr{D}$ and $\mathscr{D}_0$ is continuous, linear, and shift-invariant.

($\mathscr{R} \subseteq \mathscr{D}$) Let θ be a continuous, linear, shift-invariant operator. For each pair of positive integers, s, t, define $\theta_{st} = \mathbf{proj}_s \theta \mathbf{proj}_t$. Obviously, $\theta = \sum_{s,t>0} \theta_{st}$. It will suffice to show that for all positive integers s and t, there is a Laurent operator $f_{st}(\mathbf{D}) \in \Gamma$ such that $\theta_{st} = f(\mathbf{D}) E_{ts}$.

However, $E_{ts}\theta_{st}$ is a continuous, linear, shift-invariant operator on $\mathscr{L}^{(t)}$. Thus, $E_{ts}\theta_{st} = f(\mathbf{D}) E_{tt}$ for some Laurent operator $f(\mathbf{D})$. Hence, $\theta_{ts} = E_{ts} f(\mathbf{D}) E_{tt} = f(\mathbf{D}) E_{ts}$ as desired.

($\mathscr{R}_0 \subseteq \mathscr{D}_0$) Similarly, it will suffice to show that θ_{s0} (as defined above) is equal to zero for s positive. Assume not towards contradiction. By the reasoning above, $\theta_{s0} = f(\mathbf{D}) E_{s0}$ for some nonzero differential operator $f(\mathbf{D})$. Let $g(\mathbf{D})$ be the inverse of $f(\mathbf{D})$. Since $g(\mathbf{D})$ is a shift-invariant operator, the product $g(\mathbf{D})\theta$ is also shift-invariant. Hence, without loss of generality, $\theta_{s0} = E_{s0}$. We calculate that $E_{s0} E1 = E_{s0} 1 = (\log x)^s$. However, we also know that $E E_{s0} 1 = E(\log x)^s = (E(\log x))^s$, but $\log x \neq E \log x$. Contradiction. ∎

5.5. *The Augmentation*

DEFINITION 5.5.1 (Augmentation). For s a nonnegative integer, we define the *augmentation of order* s to be the continuous linear functional $\langle\ \rangle_s \colon \mathscr{L} \to K$ such that $\langle \lambda_n^{(t)}(x) \rangle_s = \delta_{s,t}\, \delta_{n,0}$. We define the *total augmentation* to be the continuous linear functional $\langle\ \rangle \colon \mathscr{L} \to K$ such that $\langle \lambda_n^{(t)}(x) \rangle = \delta_{n,0}$.

Thus, the total augmentation is the sum of the augmentations of the various orders.

When $\langle\ \rangle_0$ is restricted to $\mathscr{L}^{(0)}$, the augmentation reduces the evaluation of a polynomial at $x = 0$. That is, $\langle p(x) \rangle_0 = p(0)$ for $p(x) \in \mathscr{L}^{(0)}$.

An augmentation of positive order can be viewed as a generalization of evaluation at $x = 0$; it is closely related to the residue of complex variable theory. (Recall that the residue of a Laurent series is its coefficient of x^{-1}.) For instance, for $p(x) \in \mathscr{L}^{(1)}$,

$$\langle p(x) \rangle_1 = \operatorname{Res}(\mathbf{D} p(x)).$$

In general, for t positive and $p(x) \in \mathscr{L}^{(t)}$,

$$\langle p(x) \rangle_t = \frac{1}{t!} \operatorname{Res}(\mathbf{D}(x\mathbf{D})^{t-1} p(x)). \tag{22}$$

Note that Eq. (22) also holds for $p(x) \in \bigoplus_{s=0}^{t} \mathscr{L}^{(s)}$.

We derive formulae relating the augmentation to the derivative. The basic identity is:

PROPOSITION 5.5.2. 1. *Given integers m and n, and positive integers s and t,*

$$\langle \mathbf{D}^m \lambda_n^{(t)}(x) \rangle_s = \lfloor n \rceil! \, \delta_{mn} \, \delta_{st},$$

and

$$\langle \mathbf{D}^m \lambda_n^{(t)}(x) \rangle = \lfloor n \rceil! \, \delta_{mn}.$$

2. *Given nonnegative integers m and n,*

$$\langle \mathbf{D}^m \lambda_n^{(0)}(x) \rangle_0 = \lfloor n \rceil! \, \delta_{mn}.$$

More generally, we have:

PROPOSITION 5.5.3. 1. *If $f(\mathbf{D})$ is a Laurent operator given by the convergent sum*

$$f(\mathbf{D}) = \sum_k a_k \, \mathbf{D}^k,$$

and $p(x) \in \mathscr{L}^{(+)}$ is given by the convergent sum

$$p(x) = \sum_{t>0} \sum_n b_n^{(t)} \lambda_n^{(t)}(x),$$

then the augmentation of order t $\langle f(\mathbf{D}) \, p(x) \rangle_t$ is given by the finite sum

$$\langle f(\mathbf{D}) \, p(x) \rangle_t = \sum_k \lfloor k \rceil! \, a_k b_k^{(t)}.$$

2. *Similarly,*

PROPOSITION 5.5.4. *If $f(\mathbf{D})$ is a differential operator given by the convergent sum*

$$f(\mathbf{D}) = \sum_{k \geqslant 0} a_k \, \mathbf{D}^k,$$

and the formal power series of logarithmic type $p(x)$ is given by the convergent sum

$$p(x) = \sum_{t \geqslant 0} \sum_n b_n^{(t)} \lambda_n^{(t)}(x),$$

then the augmentation of order t $\langle f(\mathbf{D})\, p(x)\rangle_t$ is given by the finite sum

$$\langle f(\mathbf{D})\, p(x)\rangle_t = \sum_{k \geqslant 0} \lfloor k \rceil !\, a_k b_k^{(t)}.$$

Some special cases are of interest. The augmentation of the derivative of a formal power series of logarithmic type can be described by $\langle \mathbf{D}^k p(x)\rangle_t = \lfloor k \rceil !\, b_k^{(t)}$, where $p(x)$ is as in Proposition 5.5.3 or 5.5.4. Similarly, the augmentation of the action of a differential operator on a harmonic logarithm is $\langle f(\mathbf{D})\, \lambda_n^{(t)}(x)\rangle_t = \lfloor n \rceil !\, a_n$, where t is a positive integer, n is an integer, and $f(\mathbf{D}) = \sum_{k \geqslant c} a_k \mathbf{D}^k$ as in Proposition 5.5.3 or 5.5.4.

The augmentation leads us to a version of Taylor's formula for formal power series of logarithmic type:

THEOREM 5.5.5 (Logarithmic Taylor's Formula). *Let $p(x) \in \mathscr{L}^{(t)}$ for some nonnegative integer t. Let $d = \deg(p(x))$. Then we have the following expansion in $\mathscr{L}^{(t)}$:*

$$p(x) = \sum_n \frac{\langle \mathbf{D}^n p(x)\rangle_t}{\lfloor n \rceil !} \lambda_n^{(t)}(x).$$

The series on the right is convergent since $\langle \mathbf{D}^n p(x)\rangle_t = 0$ for all $n > d$.

Proof. By linearity and continuity, it suffices to consider the case $p(x) = \lambda_n^{(t)}(x)$. This special case was treated above (Proposition 5.5.2). ∎

Thus, we can see that a formal power series of logarithmic type is determined by the augmentations of its derivatives.

Conversely, a Laurent operator can be recovered from the augmentations of its action on the harmonic logarithms of a particular order $t > 0$, as the following theorem shows:

THEOREM 5.5.6 (Expansion Theorem). 1. *Let $f(\mathbf{D})$ be a Laurent operator, and let t be a positive integer, then we have the convergent series*

$$f(\mathbf{D}) = \sum_n \frac{\langle f(\mathbf{D})\, \lambda_n^{(t)}(x)\rangle_t}{\lfloor n \rceil !} \mathbf{D}^n.$$

2. *Similarly, if $f(\mathbf{D})$ is a differential operator, and t is a nonnegative integer, then we have the convergent series*

$$f(\mathbf{D}) = \sum_{n \geqslant 0} \frac{\langle f(\mathbf{D})\, \lambda_n^{(t)}(x)\rangle_t}{\lfloor n \rceil !} \mathbf{D}^n.$$

The following argument will be used repeatedly in the next two sections—often implicitly.

PROPOSITION 5.5.7 (Spanning Argument). 1. *Let* $p(x) \in \mathscr{L}^{(+)}$. *If* $\langle f(\mathbf{D})\, p(x) \rangle_t = 0$ *for all positive integers* t *and all Laurent operators* $f(\mathbf{D}) \in \Gamma$, *then* $p(x) = 0$.

2. *Let* t *be a positive integer and* $f(\mathbf{D}) \in \Gamma$ *be a Laurent operator. If* $\langle f(\mathbf{D})\, p(x) \rangle_t = 0$ *for all* $p(x) \in \mathscr{L}^{(t)}$, *then* $f(\mathbf{D}) = 0$.

3. *Similarly, for* $\mathscr{L}^{(0)}$, *let* $p(x) \in \mathscr{L}^{(0)}$. *If* $\langle f(\mathbf{D})\, p(x) \rangle_0 = 0$ *for all differential operators* $f(\mathbf{D}) \in \Gamma^{(+)}$, *then* $p(x) = 0$.

4. *Let* t *be a nonnegative integer, and* $f(\mathbf{D}) \in \Gamma^{(+)}$ *be a differential operator. If for all* $p(x) \in \mathscr{L}^{(t)}$, $\langle f(\mathbf{D})\, p(x) \rangle_t = 0$, *then* $f(\mathbf{D}) = 0$.

5.6. *Graded Sequences*

DEFINITION 5.6.1 (Graded Sequences of Formal Power Series of Logarithmic Type). The sequence $p_n^{(t)}(x)$ (for t a nonnegative integer and n an arbitrary integer) is called a *graded sequence of formal power series of logarithmic type* if the following conditions hold:

1. For all integers n, and for all nonnegative integers t, $p_n^{(t)}(x)$ is a homogeneous formal power series of logarithmic type of order t.

2. For n negative, $p_n^{(0)}(x) = 0$.

3. For n nonnegative, $p_n^{(0)}(x)$ is a logarithmic series of degree n. In other words, $p_n^{(0)}(x)$ is a polynomial of degree n.

4. For t positive, $p_n^{(t)}(x)$ is of degree n.

5. (Regularity) For all integers n, positive integers t, and nonnegative integers s, $E_{st}\, p_n^{(t)}(x) = p_n^{(s)}(x)$.

The logarithmic series $p_{-1}^{(1)}(x)$ is called the *residual series* of the graded sequence $p_n^{(t)}(x)$ (indicated by a box in Table 5.1).

The *principal subsequence* of $p_n^{(t)}(x)$ is the subsequence $(\tilde{p}_n(x))_{n \in \mathbf{Z}}$ defined by

$$\tilde{p}_n(x) = \begin{cases} p_n^{(0)}(x) & \text{for } \; n \geqslant 0, \text{ and} \\ p_n^{(1)}(x) & \text{for } \; n < 0. \end{cases}$$

TABLE 5.1

A Graded Sequence of Formal Power Series of Logarithmic Type

$\cdots$	0	0	0	$\tilde{p}_0(x)$	$\tilde{p}_1(x)$	$\tilde{p}_2(x)$	$\tilde{p}_3(x)$	$\cdots$
$\cdots$	$\tilde{p}_{-3}(x)$	$\tilde{p}_{-2}(x)$	$\boxed{\tilde{p}_{-1}(x)}$	$p_0^{(1)}(x)$	$p_1^{(1)}(x)$	$p_2^{(1)}(x)$	$p_3^{(1)}(x)$	$\cdots$
$\cdots$	$p_{-3}^{(2)}(x)$	$p_{-2}^{(2)}(x)$	$p_{-1}^{(2)}(x)$	$p_0^{(2)}(x)$	$p_1^{(2)}(x)$	$p_2^{(2)}(x)$	$p_3^{(2)}(x)$	$\cdots$
$\cdots$	$p_{-3}^{(3)}(x)$	$p_{-2}^{(3)}(x)$	$p_{-1}^{(3)}(x)$	$p_0^{(3)}(x)$	$p_1^{(3)}(x)$	$p_2^{(3)}(x)$	$p_3^{(3)}(x)$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

For example, $\lambda_n^{(t)}(x)$ is a graded sequence of formal power series of logarithmic type. Its residual series is $1/x$, and its principal subsequence is the sequence of powers of x, $(x^n)_{n \in \mathbf{Z}}$.

The nonzero elements of any graded sequence form a pseudobasis for $\mathscr{L}$. Thus, for any pair of graded sequences $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$, there is a unique continuous linear operator θ such that $\theta p_n^{(t)}(x) = q_n^{(t)}(x)$.

The restriction of a graded sequence to any particular t forms a pseudobasis for $\mathscr{L}^{(t)}$.

THEOREM 5.6.2. *Let $p_n^{(t)}(x)$ be a graded sequence. Then*

1. *Every series $g(x) \in \mathscr{L}$ can be uniquely written as a finite sum*

$$g(x) = \sum_t g^{(t)}(x),$$

where $g^{(t)}(x) \in \mathscr{L}^{(t)}$.

2. *Each $g^{(t)}(x)$ above can be uniquely represented in the topology of the complex numbers as an asymptotic expansion*

$$g^{(t)}(x) \sim \sum_n a_n^{(t)} p_n^{(t)}(x)$$

as $x \to \infty$.

Proof. (1) This was observed in the remarks immediately following Definition 4.3.4.

(2) As mentioned before $\{p_n^{(t)}(x) \neq 0\}$ is a pseudobasis for $\mathscr{L}$ so

$$g^{(t)}(x) = \sum_n a_n^{(t)} p_n^{(t)}(x)$$

for a unique choice of constants $a_n^{(t)}$.

So by Theorem 4.3.7, we are done. ∎

5.7. *Appell Graded Sequences*

By way of introduction to the spirit of the present work, as well as an application of the preceding theory, we sketch the logarithmic generalization of the notion of an Appell sequence of polynomials, that is, a sequence $(p_n(x))_{n \geqslant 0}$ of polynomials satisfying the identity

$$p_n(x+a) = \sum_{k \geqslant 0} \binom{n}{k} a^{n-k} p_k(x).$$

By Corollary 4.3.11, the action of the derivative operator $\mathbf{D}$ on the space $\mathscr{L}$ of formal power series of logarithmic type is naturally decomposable as a direct sum of the minimal invariant subspaces $\mathscr{L}^{(t)}$. For each positive integer t, the subspace $\mathscr{L}^{(t)}$ is the minimal invariant subspace of $\mathscr{L}^{(+)}$ under the action of the operators $\mathbf{D}$ and $\mathbf{D}^{-1}$ which contains $(\log x)^t$. It is not yet clear, however, that the basis of the spaces $\mathscr{L}^{(t)}$ provided by the harmonic logarithms, $\lambda_n^{(t)}(x)$, is determined by intrinsic algebraic properties. In order to derive the properties that single out the harmonic logarithms as the natural basis for $\mathscr{L}^{(t)}$, we are led to a generalization to formal power series of logarithmic type of the classical theory of Appell polynomials.

DEFINITION 5.7.1 (Appell Graded Sequence). A graded sequence $p_n^{(t)}(x)$ is called an *Appell graded sequence* if for all integers n, for all nonnegative integers t, and for all field elements a, the following identity is satisfied:

$$E^a p_n^{(t)}(x) = p_n^{(t)}(x+a) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k p_{n-k}^{(t)}(x). \tag{23}$$

Note that the sum on the right is a convergent infinite sum when t and a are nonzero.

PROPOSITION 5.7.2. *For a graded sequence $p_n^{(t)}(x)$, the following statements are equivalent:*

1. *$p_n^{(t)}(x)$ is an Appell graded sequence.*
2. *For all integers n and nonnegative integers t,*

$$\mathbf{D} p_n^{(t)}(x) = \lfloor n \rceil\, p_{n-1}^{(t)}(x). \tag{24}$$

3. *There is a differential operator T of degree 0 such that for all integers n and nonnegative integers t,*

$$p_n^{(t)}(x) = T^{-1} \lambda_n^{(t)}(x).$$

Proof. (1 implies 3) Let T be the continuous, linear operator in $\mathscr{L}$ mapping $p_n^{(t)}(x)$ to $\lambda_n^{(t)}(x)$. We have

$$\begin{aligned}
TE^a p_n^{(t)}(x) &= \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k T p_{n-k}^{(t)}(x) \\
&= \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k \lambda_{n-k}^{(t)}(x) \\
&= E^a \lambda_n^{(t)}(x) \\
&= E^a T p_n^{(t)}(x).
\end{aligned}$$

By Corollary 5.4.2, T is a differential operator of degree 0. Thus, we have

$$p_n^{(t)}(x) = T^{-1} \lambda_n^{(t)}(x). \tag{25}$$

(3 implies 2) We have the following series of identities:

$$\begin{aligned}
\mathbf{D} p_n^{(t)}(x) &= \mathbf{D}\, T^{-1} \lambda_n^{(t)}(x) \\
&= T^{-1} \mathbf{D}\, \lambda_n^{(t)}(x) \\
&= \lfloor n \rceil\, T^{-1} \lambda_{n-1}^{(t)}(x) \\
&= \lfloor n \rceil\, p_{n-1}^{(t)}(x).
\end{aligned}$$

(2 implies 1) We immediately have

$$\begin{aligned}
E^a p_n^{(t)}(x) &= \sum_{k \geqslant 0} \frac{a^k}{k!} \mathbf{D}^k p_n^{(t)}(x) \\
&= \sum_{k \geqslant 0} a^k \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_{n-k}^{(t)}(x). \quad \blacksquare
\end{aligned}$$

Note that given an Appell graded sequence $p_n^{(t)}(x)$, its component of order zero $(p_n^{(0)}(x))_{n \geqslant 0}$ is an ordinary Appell sequence of polynomials.

We have the following alternative characterization of Appell graded sequences:

COROLLARY 5.7.3. *A graded sequence* $p_n^{(t)}(x)$ *is an Appell graded sequence only if*

$$p_n^{(t)}(x+a) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_k^{(0)}(a) \lambda_{n-k}^{(t)}(x). \tag{26}$$

Proof. By the symmetry of variables in Eq. (23), we have

$$p_n^{(0)}(x+a)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix} p_k^{(0)}(a)x^{n-k}.$$

Let the operator T be as in Proposition 5.7.2, and apply it to both sides of the equality; we infer that for n nonnegative,

$$E^a x^n=(x+a)^n=\sum_{k=0}^{n}\frac{p_k^{(0)}(a)}{k!}T\mathbf{D}^k x^n. \tag{27}$$

We are allowed to evaluate $p_k^{(0)}(x)$ at $x=a$ since $p_k^{(0)}(x)$ is a *polynomial.* By the spanning argument, we have the *summation formula*:

$$E^a=\sum_{k\geq 0}\frac{p_k^{(0)}(a)}{k!}T\mathbf{D}^k. \tag{28}$$

By the characterization of differential operators (Corollary 5.4.2), this identity holds in the entire logarithmic algebra $\mathscr{L}$. We have therefore

$$E^a p_n^{(t)}(x)=\sum_{k\geq 0}\frac{p_k^{(0)}(a)}{k!}T\mathbf{D}^k p_n^{(t)}(x),$$

and simplifying we obtain

$$p_n^{(t)}(x+a)=\sum_{k\geq 0}\begin{bmatrix}n\\k\end{bmatrix} p_k^{(0)}(a)\,\lambda_{n-k}^{(t)}(x). \quad ∎$$

Note that in contrast to Eq. (27), the sum on the right in Eq. (26) is in general infinite.

We now deduce an explicit expression for an Appell graded sequence as a linear combination of harmonic logarithms.

PROPOSITION 5.7.4. *Let $p_n^{(t)}(x)$ be an Appell graded sequence, then for all integers n and nonnegative integers t,*

$$p_n^{(t)}(x)=\sum_{k\geq 0}\begin{bmatrix}n\\k\end{bmatrix} p_k^{(0)}(0)\,\lambda_{n-k}^{(t)}(x).$$

Proof. Set $a=0$ in Corollary 5.7.3. ∎

Setting $a=0$ in Eq. (28), we obtain

$$\mathbf{I}=\left(\sum_{k\geq 0}\frac{p_k^{(0)}(0)}{k!}\mathbf{D}^k\right)T. \tag{29}$$

We shall now give some examples of Appell graded sequences.

5.7.1. *Harmonic Graded Sequence*

First, we remark that the graded sequence of harmonic logarithms is characterized among all Appell graded sequences by the fact that

$$\langle \lambda_n^{(t)}(x) \rangle_t = \lfloor n \rceil !\, \delta_{n,0}.$$

This is the characterization of the harmonic graded sequence we had previously announced.

5.7.2. *Bernoulli Graded Sequence*

Next, we consider the logarithmic extension of Bernoulli polynomials.

DEFINITION 5.7.5 (Logarithmic Bernoulli Sequence). Define the *Bernoulli operator* J by

$$Jp(x) = \int_x^{x+1} p(t)\, dt$$

for $p(x) \in \mathscr{L}$, that is,

$$J = \frac{e^{\mathbf{D}} - I}{\mathbf{D}}.$$

The Appell graded sequence $B_n^{(t)}(x)$ defined as

$$B_n^{(t)}(x) = J^{-1} \lambda_n^{(t)}(x)$$

will be called the *logarithmic Bernoulli graded sequence*. For $t = 0$, we obtain the ordinary *Bernoulli polynomials*, and the sequence

$$B_n^{(0)}(0) = B_n$$

is the sequence of *Bernoulli numbers*.

The logarithmic Bernoulli graded sequence can be computed by Proposition 5.7.4:

$$B_n^{(t)}(x) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil B_k \lambda_{n-k}^{(t)}(x). \tag{30}$$

See Table 5.2.

In particular, the residual series is given by

$$B_{-1}^{(1)}(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \cdots.$$

For the logarithmic Bernoulli graded sequence, Eq. (29) gives the *Euler–MacLaurin summation formula*. In the present context, it has impor-

TABLE 5.2

Logarithmic Bernoulli Graded Sequence, $B_n^{(t)}(x)$

$B_{-2}^{(t)} = \lambda_{-2}^{(t)}(x) + \lambda_{-3}^{(t)}(x) + \frac{1}{2}\lambda_{-4}^{(t)}(x) - \frac{1}{6}\lambda_{-6}^{(t)}(x) + \frac{1}{6}\lambda_{-8}^{(t)}(x) - \cdots$	
$B_{-1}^{(t)} = \lambda_{-1}^{(t)}(x) + \frac{1}{2}\lambda_{-2}^{(t)}(x) + \frac{1}{6}\lambda_{-3}^{(t)}(x) - \frac{1}{30}\lambda_{-5}^{(t)}(x) + \frac{1}{42}\lambda_{-7}^{(t)}(x) - \cdots$	residual Bernoulli series
$B_0^{(t)} = \lambda_0^{(t)}(x) - \frac{1}{2}\lambda_{-1}^{(t)}(x) - \frac{1}{12}\lambda_{-2}^{(t)}(x) + \frac{1}{120}\lambda_{-4}^{(t)}(x) - \frac{1}{252}\lambda_{-6}^{(t)}(x) + \cdots$	
$B_1^{(t)} = \lambda_1^{(t)}(x) - \frac{1}{2}\lambda_0^{(t)}(x) + \frac{1}{12}\lambda_{-1}^{(t)}(x) - \frac{1}{360}\lambda_{-3}^{(t)}(x) + \frac{1}{1260}\lambda_{-5}^{(t)}(x) - \cdots$	
$B_2^{(t)} = \lambda_2^{(t)}(x) - \lambda_1^{(t)}(x) + \frac{1}{6}\lambda_0^{(t)}(x) + \frac{1}{360}\lambda_{-2}^{(t)}(x) - \frac{1}{2520}\lambda_{-4}^{(t)}(x) + \cdots$	

tant applications. Since $\mathbf{D}J = e^{\mathbf{D}} - I = \Delta$, where Δ is the classical forward difference operator $\Delta p(x) = p(x+1) - p(x)$, the Euler–MacLaurin formula can be written as

$$I = B_0 J + B_1\,\Delta + \frac{B_2}{2!}\Delta\mathbf{D} + \frac{B_3}{3!}\Delta\mathbf{D}^2 + \cdots.$$

Applying it to an arbitrary formal power series of logarithmic type $p(x)$ we obtain

$$\begin{aligned} p(x) + p(x+1) + \cdots + p(x+n) &\\ = B_0\left[\int_x^{x+n+1} p(t)\,dt\right] &+ B_1[p(x+n+1) - p(x)] \\ + \frac{B_2}{2!}[p'(x+n+1) - p'(x)] &+ \frac{B_3}{3!}[p''(x+n+1) - p''(x)] + \cdots. \end{aligned}$$

We stress the fact that this formula is an *identity* in the logarithmic algebra, and not just an asymptotic formula. For example, for $p(x) = 1/x$, we obtain

$$\begin{aligned} \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n} &\\ = B_0[\log(x+n+1) - \log(x)] &+ B_1[(x+n+1)^{-1} - x^{-1}] \\ + \frac{B_2}{2!}[-(x+n+1)^{-2} + x^{-2}] &+ \frac{B_3}{3!}[2(x+n+1)^{-3} - 2x^{-3}] + \cdots. \end{aligned} \tag{31}$$

Again, this is an *identity*, and not just an asymptotic expansion.

For another example, let $p(x) = \log x$. We then obtain

$$\begin{aligned}
&\log(x(x+1)\cdots(x+n)) \\
&\quad = B_0((x+n+1)\log(x+n+1) - x\log x - n - 1) \\
&\qquad + B_1(\log(x+n+1) - \log x) + \frac{B_2}{2!}\left[\frac{1}{x+n+1} - \frac{1}{x}\right] + \cdots. \qquad (32)
\end{aligned}$$

This formula gives a version of Stirling's formula for the factorial which is valid over any field of characteristic zero. It can be specialized to give the classical formula over the complex numbers, or a generalization of it valid over p-adic fields.

We note that $J = \Delta\mathbf{D}^{-1}$, and thus for t positive,

$$\begin{aligned}
\Delta B_n^{(t)}(x) &= \Delta J^{-1}\lambda_n^{(t)}(x) \\
&= \mathbf{D}\lambda_n^{(t)}(x) \\
&= \lfloor n \rceil\, \lambda_{n-1}^{(t)}(x).
\end{aligned}$$

Summing, we obtain

$$\begin{aligned}
&\lambda_n^{(t)}(x) + \lambda_n^{(t)}(x+1) + \cdots + \lambda_n^{(t)}(x+k) \\
&\quad = \lfloor n \rceil^{-1}\, [B_{n+1}^{(t)}(x+k+1) - B_{n+1}^{(t)}(x)].
\end{aligned}$$

For example,

$$\log(x(x+1)\cdots(x+k)) = B_1^{(1)}(x+k+1) - B_1^{(1)}(x).$$

Clearly, any similar sum of $x^k(\log x)^t$ can be evaluated in closed form by the logarithmic Bernoulli graded sequence.

5.7.3. *Hermite Graded Sequence*

Our next example of an Appell graded sequence is the logarithmic Hermite graded sequence.

DEFINITION 5.7.6 (Logarithmic Hermite Graded Sequence). Let the *Weierstrass operator* be $W = e^{\mathbf{D}^2/2}$. The *logarithmic Hermite graded sequence* $H_n^{(t)}(x)$ is defined as

$$H_n^{(t)}(x) = W^{-1}\lambda_n^{(t)}(x).$$

For n nonnegative, $H_n^{(0)}(x) = H_n(x)$ is the usual Hermite *polynomial.*

The logarithmic Hermite graded sequence can be computed via Proposition 5.7.4:

$$H_n^{(t)}(x) = \sum_{k \geq 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil H_n^{(0)}(0)\, \lambda_{n-k}^{(t)}(x). \tag{33}$$

(See Table 5.3.) Recall that

$$E_{0t} H_n^{(t)}(x) = H_n(x),$$

in other words, the "positive" terms of Eq. (33) equal the ordinary Hermite polynomials, except that $\lambda_n^{(t)}(x)$ replaces x^n. Thus, we have a logarithmic generalization of the Hermite polynomials. All classical properties of Hermite polynomials may be extended to the logarithmic Hermite graded sequence. For example, we have the explicit expression

$$\begin{aligned} H_n^{(t)}(x) &= e^{-\mathbf{D}^2/2} \lambda_n^{(t)}(x) \\ &= \sum_{k \geq 0} \left(-\frac{1}{2}\right)^k \frac{\lfloor n \rceil!}{k!\, \lfloor n-2k \rceil!} \lambda_{n-2k}^{(t)}(x), \end{aligned} \tag{34}$$

so that for n negative

$$H_n^{(1)}(x) = \sum_{k \geq 0} \left(-\frac{1}{2}\right)^k \frac{(2k-n-1)!}{k!\,(-n-1)!} x^{n-2k}. \tag{35}$$

The right side is the classical asymptotic expansion of the Hermite series $H_n^{(1)}(x)$; in the present context, it is a convergent series, and one term of the Hermite graded sequence. We trivially have

$$\mathbf{D} H_n^{(t)}(x) = \lfloor n \rceil\, H_{n-1}^{(t)}(x),$$

and

$$E^a H_n^{(t)}(x) = \sum_{k \geq 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil a^k H_{n-k}^{(t)}(x).$$

TABLE 5.3

Logarithmic Hermite Graded Sequence $H_n^{(t)}(x)$

$H_{-2}^{(t)}(x) = \lambda_{-2}^{(t)}(x) - 6\lambda_{-4}^{(t)}(x) + 60\lambda_{-6}^{(t)}(x) - 840\lambda_{-8}^{(t)}(x) - \cdots$	
$H_{-1}^{(t)}(x) = \lambda_{-1}^{(t)}(x) - 2\lambda_{-3}^{(t)}(x) + 12\lambda_{-5}^{(t)}(x) - 120\lambda_{-7}^{(t)}(x) - \cdots$	residual Hermite series
$H_0^{(t)}(x) = \lambda_0^{(t)}(x) + \lambda_{-2}^{(t)}(x) - 3\lambda_{-4}^{(t)}(x) + 20\lambda_{-6}^{(t)}(x) + \cdots$	
$H_1^{(t)}(x) = \lambda_1^{(t)}(x) - \lambda_{-1}^{(t)}(x) + \lambda_{-3}^{(t)}(x) - 4\lambda_{-5}^{(t)}(x) + \cdots$	
$H_2^{(t)}(x) = \lambda_2^{(t)}(x) - 2\lambda_0^{(t)}(x) - \lambda_{-2}^{(t)}(x) + 2\lambda_{-4}^{(t)}(x) + \cdots$	

Finally, every logarithmic series $p(x)$ has a unique convergent expansion in terms of the logarithic Hermite graded sequence:

$$p(x)=\sum_{n,t}\frac{\langle W\mathbf{D}^n p(x)\rangle_t}{\lfloor n\rceil!}H_n^{(t)}(x).$$

6. Roman Graded Sequences

In this section, we introduce the central concept of this work. It is known that the operational calculus of formal differential operators is intimately associated with sequences of polynomials of binomial type, that is, with sequences of polynomials $p_n(x)$ satisfying the binomial identity

$$p_n(x+a)=\sum_{k=0}^{n}\binom{n}{k}p_k(x)\,p_{n-k}(a).$$

A good many sequences of polynomials occurring in combinatorics and in the theory of special functions turn out to be of binomial type. We give here the logarithmic generalization of this notion; such graded sequences of formal power series of logarithmic type are called Roman graded sequences. We will derive five equivalent characterizations of such graded sequences.

We anticipate the fact (Theorem 6.5.4) that the five notions introduced below, namely,

1. Roman graded sequence (Definition 6.1.3),
2. associated graded sequence (Definition 6.2.3),
3. conjugate graded sequence (Definition 6.4.1),
4. basic graded sequence (Definition 6.3.1), and
5. graded sequence of logarithmic binomial type (Definition 6.5.2),

will be shown to coincide.

6.1. *Roman Graded Sequences*

We begin by deriving a formula for the action of a product of two Laurent operators on the harmonic logarithm.

Proposition 6.1.1. *Let $f(\mathbf{D})$ and $g(\mathbf{D})$ be Laurent operators. Then we have the following finite sum*

$$\langle f(\mathbf{D})\,g(\mathbf{D})\,\lambda_n^{(t)}(x)\rangle_t=\sum_k\left\lfloor \begin{matrix} n\\ k\end{matrix}\right\rceil\langle f(\mathbf{D})\,\lambda_k^{(t)}(x)\rangle_t \times\langle g(\mathbf{D})\,\lambda_{n-k}^{(t)}(x)\rangle_t$$

for t positive.

Proof. Let $f(\mathbf{D})=\sum_{k\geqslant c} a_k \mathbf{D}^k$, and $g(\mathbf{D})=\sum_{k\geqslant d} b_k \mathbf{D}^k$. Their product is

$$f(\mathbf{D})\,g(\mathbf{D})=\sum_{l\geqslant c+d}\left(\sum_{k=c}^{l-d} a_k b_{l-k}\right)\mathbf{D}^l.$$

Hence, by Theorem 5.5.3,

$$\begin{aligned}\langle f(\mathbf{D})\,g(\mathbf{D})\,\lambda_n^{(t)}(x)\rangle_t &= \lfloor n\rceil! \sum_{k=c}^{n-d} a_k b_{n-k}\\ &= \sum_{k=c}^{n-d} \frac{\lfloor n\rceil!}{\lfloor k\rceil!\,\lfloor n-k\rceil!}(\lfloor k\rceil!\,a_k)(\lfloor n-k\rceil!\,b_{n-k})\\ &= \sum_{k=c}^{n-d} \left\lfloor \begin{matrix} n\\ k\end{matrix}\right\rceil \langle f(\mathbf{D})\,\lambda_k^{(t)}(x)\rangle_t \,\langle g(\mathbf{D})\,\lambda_{n-k}^{(t)}(x)\rangle_t. \quad\blacksquare\end{aligned}$$

Similarly, for differential operators, we have more generally:

PROPOSITION 6.1.2. *Let $f(\mathbf{D})$ and $g(\mathbf{D})$ be differential operators. Then we have the following finite sum*

$$\begin{aligned}\langle f(\mathbf{D})\,g(\mathbf{D})\,\lambda_n^{(t)}(x)\rangle_t = \sum_k \left\lfloor \begin{matrix} n\\ k\end{matrix}\right\rceil &\langle f(\mathbf{D})\,\lambda_k^{(t)}(x)\rangle_t\\ &\times \langle g(\mathbf{D})\,\lambda_{n-k}^{(t)}(x)\rangle_t\end{aligned}$$

for all nonnegative integers t.

The extension to products of more than two operators follows easily by induction.

We introduce Roman graded sequences by the following definition. It will shortly be seen that simpler alternate definitions can be given.

DEFINITION 6.1.3 (Roman Graded Sequence). Let $p_n^{(t)}(x)$ be a graded sequence of formal power series of logarithmic type. The graded sequence is a *Roman graded sequence* if for all integers n and nonnegative integers t, we have the *following* finite sum

$$\begin{aligned}\langle f(\mathbf{D})\,g(\mathbf{D})\,p_n^{(t)}(x)\rangle_t = \sum_k \left\lfloor \begin{matrix} n\\ k\end{matrix}\right\rceil &\langle f(\mathbf{D})\,p_k^{(t)}(x)\rangle_t\\ &\times \langle g(\mathbf{D})\,p_{n-k}^{(t)}(x)\rangle_t,\end{aligned} \tag{36}$$

where $f(\mathbf{D})$ and $g(\mathbf{D})$ are Laurent operators if t is positive and are differential operators if t is zero.

For example, $\lambda_n^{(t)}(x)$ is a Roman graded sequence.

We next define the evaluation operator, which plays the role of the exponential function in the present logarithmic context.

DEFINITION 6.1.4 (Evaluation Operator). For every $a \in K$, and for every integer m, the *evaluation operator* $\xi_{a,m}$ is the Laurent operator

$$\xi_{a,m} = \sum_{k \geq m} \frac{a^k}{\lfloor k \rceil!} \mathbf{D}^k.$$

Thus, if s and t are positive integers, and n is an integer, then

$$\langle \xi_{a,m} \lambda_n^{(t)}(x) \rangle_s = \begin{cases} a^n & \text{if both } n \geq m \text{ and } s = t \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\xi_{a,m} \in \Gamma^{(+)}$ if and only if m is nonnegative.

We wish to show that a graded sequence is a Roman graded sequence if and only if Eq. (36) holds when $f(\mathbf{D})$ and $g(\mathbf{D})$ are evaluation operators. The proof depends on the following preliminary

LEMMA 6.1.5. *If $p_n^{(t)}(x)$ is a graded sequence of formal power series of logarithmic type with coefficients given by*

$$p_n^{(t)}(x) = \sum_{k \leq n} b_{nk} \lambda_k^{(t)}(x),$$

then Eq. (36) *holds for all evaluation operators if and only if for all integers n, i, and j,*

$$\left\lfloor \begin{matrix} i+j \\ i \end{matrix} \right\rceil b_{n,i+j} = \sum_{k=j}^{n-i} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil b_{kj} b_{n-k,i}. \tag{37}$$

Proof. Let l, m, and n be integers. Let a and b be field elements, and let t be a positive integer.

On the one hand,

$$\begin{aligned} \langle \xi_{a,m} \xi_{b,l} p_n^{(t)}(x) \rangle_t &= \sum_{k \leq n} b_{nk} \langle \xi_{a,m} \xi_{b,l} \lambda_k^{(t)}(x) \rangle_t \\ &= \sum_{k \leq n} \sum_{j=m}^{k-l} \left\lfloor \begin{matrix} k \\ j \end{matrix} \right\rceil \\ &\qquad \times b_{nk} \langle \xi_{a,m} \lambda_j^{(t)}(x) \rangle_t \langle \xi_{b,l} \lambda_{k-j}^{(t)}(x) \rangle_t \end{aligned}$$

since $\lambda_n^{(t)}(x)$ is a Roman graded sequence. This in turn equals

$$\begin{aligned}\langle \xi_{a,m}\xi_{b,l}p_n^{(t)}(x)\rangle_t &= \sum_{j=m}^{n-l}\sum_{k=j+l}^{n} \left\lfloor \begin{matrix} k \\ j \end{matrix} \right\rceil b_{nk} a^j b^{k-j} \\ &= \sum_{j=m}^{n-l}\sum_{i=l}^{n-j} \left(\left\lfloor \begin{matrix} i+j \\ j \end{matrix} \right\rceil b_{n,i+j} \right) a^j b^i,\end{aligned}$$

where $i=k-j$. On the other hand,

$$\begin{aligned}&\sum_{k=m}^{n-l} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle \xi_{a,m} p_k^{(t)}(x)\rangle_t \langle \xi_{b,l} p_{n-k}^{(t)}(x)\rangle_t \\ &\quad = \sum_{k=m}^{n-l} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left(\sum_{j=m}^{k} b_{kj} \langle \xi_{a,m}\lambda_j^{(t)}(x)\rangle_t \right) \\ &\quad\quad \times \left(\sum_{i=l}^{n-k} b_{n-k,i} \langle \xi_{b,l}\lambda_i^{(t)}(x)\rangle_t \right) \\ &\quad = \sum_{j=m}^{n-l}\sum_{i=l}^{n-j} \left(\sum_{k=j}^{n-i} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil b_{kj} b_{n-k,i} \right) a^j b^i.\end{aligned}$$

Equating coefficients of $a^j b^i$ concludes the proof. ∎

As an immediate consequence, we obtain

Proposition 6.1.6. *A graded sequence $p_n^{(t)}(x)$ is a Roman graded sequence if and only if Eq. (36) holds whenever $f(\mathbf{D})$ and $g(\mathbf{D})$ are evaluation operators.*

Proof. By Lemma 6.1.5, it will suffice to show that Eq. (37) implies that the graded sequence $p_n^{(t)}(x)$ is a Roman graded sequence.

(t positive) By continuity, it will suffice to consider the case $f(\mathbf{D})=\mathbf{D}^c$, and $g(\mathbf{D})=\mathbf{D}^d$ for integers c, d. Let $p_n^{(t)}(x)=\sum_{k\leqslant n} b_{nk}\lambda_k^{(t)}(x)$ as before. Now,

$$\begin{aligned}\langle \mathbf{D}^c\mathbf{D}^d p_n^{(t)}(x)\rangle_t &= \lfloor c+d \rceil !\, b_{n,c+d} \\ &= \lfloor c \rceil !\, \lfloor d \rceil !\, \sum_{k=c}^{n-d} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil b_{k,c} b_{n-k,d}\end{aligned}$$

by Eq. (37). This in turn equals

$$\sum_{k=c}^{n-d} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle \mathbf{D}^c p_k^{(t)}(x)\rangle_t \langle \mathbf{D}^d p_{n-k}^{(t)}(x)\rangle_t,$$

and the conclusion follows.

($t=0$) We need only consider the case in which c and d are nonnegative. Hence, only the evaluation operators of nonnegative order are relevant, and the same argument applies. ∎

We note the following consequences of the preceeding arguments:

PORISM 6.1.7. *A graded sequence is a Roman graded sequence if and only if Eq.* (37) *holds for its coefficients.*

6.2. *Associated Graded Sequences*

DEFINITION 6.2.1 (Delta Operator). A *delta operator* is a differential operator of degree one.

We proceed to derive an altogether different characteristic property of Roman graded sequences, which uses delta operators. Such differential operators may be viewed as playing the role of the derivative—much like the forward difference operator Δ (Definition 8.1.1) acts on the polynomial sequence of lower factorials $(x)_n$ (Definition 3.3.1).

Observe the important fact that if $f(\mathbf{D})$ is a delta operator, then the series of powers of $(f(\mathbf{D})^n)_{n\in\mathbf{Z}}$, is a pseudobasis for the K-vector space of Laurent operators Γ. That is, for every Laurent operator $g(\mathbf{D})$, there is an integer c and a sequence $(a_k)_{k\geqslant c}$ such that $g(\mathbf{D})=\sum_{k\geqslant c} a_k f(\mathbf{D})^k$. Similarly, the series of nonnegative powers of $f(\mathbf{D})$, $(f(\mathbf{D})^n)_{n\geqslant 0}$ is a pseudobasis for $\Gamma^{(+)}$.

PROPOSITION 6.2.2. *Let $f(\mathbf{D})$ be a delta operator in Γ. Then there is a unique graded sequence of formal power series of logarithmic type, $p_n^{(t)}(x)$, such that*

$$\langle f(\mathbf{D})^k\, p_n^{(t)}(x)\rangle_t = \lfloor n\rceil!\, \delta_{nk}, \tag{38}$$

where n and k are integers, and t is a nonnegative integer.

Proof. Write

$$p_n^{(t)}(x)=\sum_{j\leqslant d} b_{nj}^{(t)}\lambda_j^{(t)}(x),$$

and let $f(\mathbf{D})=\sum_{i\geqslant 1} a_i\mathbf{D}^i$. Note that $a_1\neq 0$. Let $f(\mathbf{D})^k=\sum_{i\geqslant k} a_i^{(k)}\mathbf{D}^i$ so that $a_k^{(k)}=(a_1)^k\neq 0$. In this notation, Eq. (38) is equivalent to the equation

$$\sum_{j=k}^{d} \lfloor j\rceil!\, a_j^{(k)} b_{nj}^{(t)} = \lfloor n\rceil!\, \delta_{nk}.$$

Hence, the coefficients $b_{nj}^{(t)}$ can be computed recursively as

$$b_{nj}^{(t)} = \frac{1}{\lfloor j \rceil !\, a_j^{(j)}} \left(\lfloor n \rceil !\, \delta_{nj} - \sum_{i>j} \lfloor i \rceil !\, a_i^{(j)} b_{ni}^{(t)} \right).$$

The recursion is well defined since $b_{nj}^{(t)} = 0$ for $j > n$. Finally, notice that $p_n^{(t)}(x)$ is regular since $b_{nj}^{(t)} = b_{nj}^{(s)}$, and that $\deg(p_n^{(t)}(x)) = n$ since $b_{nn}^{(t)} = \lfloor n \rceil !/\lfloor n \rceil !\, a_n^{(n)} \neq 0$. ∎

DEFINITION 6.2.3 (Associated Graded Sequence). Let $f(\mathbf{D})$ be a delta operator. The unique graded sequence mentioned in Proposition 6.2.2 is called the *associated graded sequence* of the delta operator $f(\mathbf{D})$.

For example, $\lambda_n^{(t)}(x)$ is the graded sequence associated with the delta operator $\mathbf{D}$ since

$$\langle \mathbf{D}^k \lambda_n^{(t)}(x) \rangle_t = \lfloor n \rceil !\, \delta_{nk}.$$

We can now generalize Theorem 5.5.6 to explicitly determine the coefficients in the expansion of an arbitrary Laurent operator in terms of the powers of a delta operator:

THEOREM 6.2.4 (Expansion Theorem). *Let the graded sequence $p_n^{(t)}(x)$ be associated with the delta operator $f(\mathbf{D})$. Then for all Laurent operators $g(\mathbf{D})$ and positive integers t, we have the convergent sum*

$$g(\mathbf{D}) = \sum_k \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x) \rangle_t}{\lfloor k \rceil !} f(\mathbf{D})^k. \tag{39}$$

When t is zero, Eq. (39) *holds for all differential operators $g(\mathbf{D})$.*

Proof. By Definition 5.5.1 we have

$$\begin{aligned} &\left\langle \sum_{k \geq d} \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x) \rangle_t}{\lfloor k \rceil !} f(\mathbf{D})^k\, p_n^{(t)}(x) \right\rangle_t \\ &\quad = \sum_{k \geq d} \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x) \rangle_t}{\lfloor k \rceil !} \langle f(\mathbf{D})^k\, p_n^{(t)}(x) \rangle_t \\ &\quad = \langle g(\mathbf{D})\, p_n^{(t)}(x) \rangle_t, \end{aligned}$$

where $d = \deg(g(\mathbf{D}))$. The conclusion follows by the spanning argument. ∎

Dually, we obtain the explicit form of the expansion of an arbitrary formal series of logarithmic type as a linear combination of elements of a Roman graded sequence. This gives a useful generalization of Theorem 5.5.5.

THEOREM 6.2.5 (Logarithmic Taylor's Theorem). *Let $p_n^{(t)}(x)$ be the graded sequence associated with the delta operator $f(\mathbf{D})$. Then for every formal power series of logarithmic type $p(x)$ we have the convergent sum*

$$p(x) = \sum_{t \geqslant 0} \sum_{n} \frac{\langle f(\mathbf{D})^n \, p(x) \rangle_t}{\lfloor n \rceil !} \, p_n^{(t)}(x).$$

Proof. For all $p(x)$, $q(x) \in \mathscr{L}$, it is clear that

$$p(x) = q(x)$$

if and only if for all field elements $a \in K$, integers m, and positive integers t,

$$\langle \xi_{a,m} \, p(x) \rangle_t = \langle \xi_{a,m} q(x) \rangle_t.$$

When we apply the Expansion Theorem where $g(\mathbf{D})$ is set equal to $\xi_{a,m}^{(t)}$, we find that

$$\xi_{a,m} = \sum_{k \geqslant m} \frac{\langle \xi_{a,m} \, p_k^{(t)}(x) \rangle_t}{\lfloor k \rceil !} \, f(\mathbf{D})^k.$$

Let $d = \deg(p(x))$. Thus,

$$\begin{aligned} \langle \xi_{a,m} \, p(x) \rangle_t &= \sum_{k=m}^{d} \frac{\langle \xi_{a,m} \, p_k^{(t)}(x) \rangle_t}{\lfloor k \rceil !} \langle f(\mathbf{D})^k \, p(x) \rangle_t \\ &= \left\langle \xi_{a,m} \sum_{k=m}^{d} p_k^{(t)}(x) \frac{\langle f(\mathbf{D})^k \, p(x) \rangle_t}{\lfloor k \rceil !} \right\rangle_t . \end{aligned}$$

Therefore, by Part 1 of the spanning argument and the above remarks, we conclude that

$$p(x) = \sum_{\substack{t \geqslant 0 \\ n \leqslant d}} \frac{\langle f(\mathbf{D})^k \, p(x) \rangle_t}{\lfloor k \rceil !} \, p_n^{(t)}(x). \quad \blacksquare$$

6.3. *Basic Graded Sequences*

DEFINITION 6.3.1 (Basic Graded Sequence). Let $f(\mathbf{D})$ be a delta operator. A graded sequence of formal power series of logarithmic type $p_n^{(t)}(x)$ is called the *basic graded sequence* for $f(\mathbf{D})$ if

1. For all nonnegative t, $\langle p_0^{(t)}(x)\rangle_t = 1$;

2. For all $n > 0$ (and thus all $n \neq 0$), and all nonnegative t, $\langle p_n^{(t)}(x)\rangle_t = 0$;

3. For all integers n, and nonnegative integers t, $f(\mathbf{D})\, p_n^{(t)}(x) = \lfloor n \rceil\, p_{n-1}^{(t)}(x)$.

THEOREM 6.3.2. 1. *Let $p_n^{(t)}(x)$ be a logarithmic graded sequence. Such a graded sequence is the basic graded sequence for the delta operator $f(\mathbf{D})$ if and only if it is the associated graded sequence for the delta operator $f(\mathbf{D})$.*

2. *Every delta operator has a unique basic sequence.*

3. *Every basic sequence is basic for a unique delta operator.*

Proof. (1: If) Properties 1 and 2 of Definition 6.3.1 follow from Definition 6.2.3. Property 3 follows from the following series of equalities, and the spanning argument:

$$\begin{aligned}\langle f(\mathbf{D})^k\, (f(\mathbf{D})\, p_n^{(t)}(x))\rangle_t &= \langle f(\mathbf{D})^{k+1}\, p_n^{(t)}(x)\rangle_t \\ &= \lfloor n \rceil!\, \delta_{n,k+1} \\ &= \lfloor n \rceil(\lfloor n-1 \rceil!\, \delta_{n-1,k}) \\ &= \lfloor n \rceil \langle f(\mathbf{D})^k\, p_{n-1}^{(t)}(x)\rangle_t.\end{aligned}$$

(1: Only if) By induction we have $f(\mathbf{D})^k\, p_n^{(t)}(x) = (\lfloor n \rceil!/\lfloor n-k \rceil!) \times p_{n-k}^{(t)}(x)$. Hence,

$$\begin{aligned}\langle f(\mathbf{D})^k\, p_n^{(t)}(x)\rangle_t &= \left\langle \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!}\, p_{n-k}^{(t)}(x)\right\rangle_t \\ &= \lfloor n \rceil!\, \delta_{n-k,0} \\ &= \lfloor n \rceil!\, \delta_{n,k}.\end{aligned}$$

Hence, $p_n^{(t)}(x)$ is the associated graded sequence of $f(\mathbf{D})$ by definition.

(2 and 3) Immediate from 1. ∎

The simplest example of a basic graded sequence is the graded sequence of harmonic logarithms $\lambda_n^{(t)}(x)$; it is basic graded sequence for the delta operator $\mathbf{D}$. It can be viewed as the natural logarithmic extension of the sequence of powers of x. More generally, every sequence of binomial type (and every factor sequence) has a natural logarithmic extension into a basic graded sequence related to the same delta operator.

Note that $p_0^{(0)}(x)$ is a nonzero constant, so that $f(\mathbf{D})\, p_0^{(0)}(x) = 0$.

For t positive, $\mathbf{D}$ is invertible on $\mathscr{L}^{(t)}$. Hence, we have an unexpected bounty worth pointing out explicitly:

PROPOSITION 6.3.3. *Let $p_n^{(t)}(x)$ be the associated graded sequence of formal power series of logarithmic type of the delta operator $f(\mathbf{D})$. If n and k are integers, and t is positive, then*

$$p_{n+k}^{(t)}(x) = \frac{\lfloor n+k \rceil!}{\lfloor n \rceil!} f(\mathbf{D})^{-k} p_n^{(t)}(x).$$

Less surprisingly, if n, k, and t are nonnegative, then

$$p_{n-k}^{(t)}(x) = \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!} f(\mathbf{D})^{k} p_n^{(t)}(x).$$

We may now connect the notion of an associated graded sequence with that of a Roman graded sequence.

PROPOSITION 6.3.4. *Let $p_n^{(t)}(x)$ be an associated graded sequence, and let $f(\mathbf{D}) \in \Gamma$ be a Laurent operator. Then we have the convergent sum*

$$f(\mathbf{D})\, p_n^{(t)}(x) = \sum_k \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle f(\mathbf{D})\, p_k^{(t)}(x) \rangle_t\, p_{n-k}^{(t)}(x)$$

for all positive integers t, and also for $t=0$ if $f(\mathbf{D})$ is in fact a differential operator.

Proof. Suppose $p_n^{(t)}(x)$ is associated with the delta operator $g(\mathbf{D})$. Then for all integers j,

$$\begin{aligned} g(\mathbf{D})^j\, p_n^{(t)}(x) &= \frac{\lfloor n \rceil!}{\lfloor n-j \rceil!} p_{n-j}^{(t)}(x) \\ &= \sum_k \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle g(\mathbf{D})^j\, p_k^{(t)}(x) \rangle_t\, p_{n-k}^{(t)}(x). \end{aligned}$$

Since $\{g(\mathbf{D})^j : j \in \mathbf{Z}\}$ is a pseudobasis, we can use continuity and linearity to replace $g(\mathbf{D})^j$ by $f(\mathbf{D})$. ∎

As a consequence we obtain:

PROPOSITION 6.3.5. *A graded sequence of formal power series of logarithmic type is an associated graded sequence if and only if it is a Roman graded sequence.*

Thus, every Roman graded sequence is associated with a unique delta operator.

Proof of Proposition 6.3.5. (Only if) Suppose $p_n^{(t)}(x)$ is the associated graded sequence for the delta operator $f(\mathbf{D})$. Then for all nonnegative integers t, and for all integers m and l, and for all integers i and j such that $m \leqslant i \leqslant n-l$ and $l \leqslant j \leqslant n-m$ we have

$$\begin{aligned}\langle f(\mathbf{D})^i f(\mathbf{D})^j p_n^{(t)}(x)\rangle_t &= \lfloor n\rceil!\, \delta_{i+j,n}\\ &= \sum_{k=m}^{n-l} \begin{bmatrix} n\\ k\end{bmatrix} \lfloor k\rceil!\, \delta_{i,k} \lfloor n-k\rceil!\, \delta_{j,n-k}\\ &= \sum_{k=m}^{n-l} \begin{bmatrix} n\\ k\end{bmatrix} \langle f(\mathbf{D})^i p_k^{(t)}(x)\rangle_t\\ &\quad \times \langle f(\mathbf{D})^j p_{n-k}^{(t)}(x)\rangle_t.\end{aligned}$$

Let a and b be elements of the field K. If we write $\xi_{a,m} = \sum_{i \geqslant m} a_i f(\mathbf{D})^i$ and $\xi_{b,l} = \sum_{j \geqslant l} b_j f(\mathbf{D})^j$ then

$$\begin{aligned}\langle \xi_{a,m} \xi_{b,l} p_n^{(t)}(x)\rangle_t &= \sum_{i \geqslant m} \sum_{j \geqslant l} a_i b_j \langle f(\mathbf{D})^i f(\mathbf{D})^j p_n^{(t)}(x)\rangle_t\\ &= \sum_{i \geqslant m} \sum_{j \geqslant l} \sum_{k=m}^{n-l} a_i b_j \begin{bmatrix} n\\ k\end{bmatrix}\\ &\quad \times \langle f(\mathbf{D})^i p_k^{(t)}(x)\rangle_t \langle f(\mathbf{D})^j p_{n-k}^{(t)}(x)\rangle_t\\ &= \sum_{k=m}^{n-l} \begin{bmatrix} n\\ k\end{bmatrix} \langle \xi_{a,m} p_k^{(t)}(x)\rangle_t \langle \xi_{b,l} p_{n-k}^{t}(x)\rangle_t.\end{aligned}$$

Hence, Eq. (36) is satisfied for all evaluation operators. Thus, $p_n^{(t)}(x)$ is a Roman graded sequence.

(If) Conversely, let $p_n^{(t)}(x)$ be a Roman graded sequence. We define a sequence of Laurent operators $(f_k(\mathbf{D}))_{k \in \mathbf{Z}}$ by the relation

$$\langle f_k(\mathbf{D})\, p_n^{(1)}(x)\rangle_1 = \lfloor n\rceil!\, \delta_{nk}.$$

By the spanning argument, $f_k(\mathbf{D})$ is well defined. Now,

$$\langle f_k(\mathbf{D})\, \lambda_n^{(1)}(x)\rangle_1 = 0$$

for $n < k$, and

$$\langle f_k(\mathbf{D})\, \lambda_k^{(1)}(x)\rangle_1 \neq 0,$$

hence $\deg(f_k(\mathbf{D}))=k$. In particular, $f_1(\mathbf{D})$ is a delta operator. Since $p_n^{(t)}(x)$ is a Roman graded sequence, we infer that

$$\begin{aligned}\langle f_i(\mathbf{D})\, f_j(\mathbf{D})\, p_n^{(1)}(x)\rangle_1 &= \sum_{k=i}^{n-j} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle f_i(\mathbf{D})\, p_k^{(1)}(x)\rangle_1 \\ &\quad \times \langle f_j(\mathbf{D})\, p_{n-k}^{(1)}(x)\rangle_1 \\ &= \lfloor n \rceil!\, \delta_{n,i+j} \\ &= \langle f_{i+j}(\mathbf{D})\, p_n^{(1)}(x)\rangle_1 .\end{aligned}$$

The spanning argument shows that $f_i(\mathbf{D})\, f_j(\mathbf{D}) = f_{i+j}(\mathbf{D})$. Hence, by induction, $f_i(\mathbf{D}) = f_1(\mathbf{D})^i$. Therefore, $p_n^{(t)}(x)$ is the associated graded sequence of $f_1(\mathbf{D})$. ∎

6.4. *Conjugate Graded Sequences*

We give yet another characterization of Roman graded sequences based on delta operators.

DEFINITION 6.4.1 (Conjugate Graded Sequence). Let $f(\mathbf{D}) \in \Gamma$ be a delta operator. Its *conjugate graded sequence* $q_n^{(t)}(x)$ is defined as

$$q_n^{(t)}(x) = \sum_{k \leqslant n} \frac{\langle f(\mathbf{D})^k\, \lambda_n^{(t)}(x)\rangle_t}{\lfloor k \rceil!}\, \lambda_k^{(t)}(x),$$

where n is an integer, and t is a nonnegative integer.

Since $f(\mathbf{D})$ is a delta operator, $q_n^{(t)}(x)$ is indeed a graded sequence of formal power series of logarithmic type.

The canonical example of a conjugate graded sequence is the graded sequence of harmonic logarithms; it is the conjugate graded sequence of the delta operator $\mathbf{D}$. Indeed, we have

$$\lambda_n^{(t)}(x) = \sum_{k \leqslant n} \langle \mathbf{D}^k \lambda_n^{(t)}(x)\rangle_t \frac{\lambda_k^{(t)}(x)}{\lfloor k \rceil!}.$$

In the above example, the conjugate graded sequence of a delta operator is a Roman graded sequence. This fact is true in general:

PROPOSITION 6.4.2. *A graded sequence of formal power series of logarithmic type $q_n^{(t)}(x)$ is a Roman graded sequence if and only if it is the conjugate graded sequence of a delta operator. Moreover, in this case, such a delta operator is unique.*

Proof. (If) Let $q_n^{(t)}(x)$ be the conjugate graded sequence of the delta operator $f(\mathbf{D})$. Now, $q_n^{(t)}(x) = \sum_{k \leqslant n} b_{nk} \lambda_k^{(t)}(x)$ where

$$b_{nk} = \frac{\langle f(\mathbf{D})^k \lambda_n^{(t)}(x) \rangle_t}{\lfloor k \rceil!}.$$

It will suffice to show that the b_{nk} satisfy Eq. (37). For integers i and j, we have

$$\sum_{k=j}^{n-i} \begin{bmatrix} n \\ k \end{bmatrix} b_{kj} b_{n-k,i} = \sum_{k=j}^{n-i} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\langle f(\mathbf{D})^j \lambda_k^{(t)}(x) \rangle_t}{\lfloor j \rceil!} \frac{\langle f(\mathbf{D})^i \lambda_{n-k}^{(t)}(x) \rangle_t}{\lfloor i \rceil!}$$
$$= \frac{\langle f(\mathbf{D})^{i+j} \lambda_n^{(t)}(x) \rangle_t}{\lfloor i \rceil! \lfloor j \rceil!}$$

since $\lambda_n^{(t)}(x)$ is a Roman graded sequence. The last expression equals $\lfloor {}^{i+j}_{\ i} \rceil b_{n,i+j}$. Hence, Eq. (37) holds.

(Only if) Conversely, suppose that $q_n^{(t)}(x)$ is a Roman graded sequence. Let

$$q_n^{(t)}(x) = \sum_{k \leqslant n} b_{nk} \lambda_k^{(t)}(x).$$

For k an integer, define the Laurent operator $f_k(\mathbf{D})$ by $\langle f_k(\mathbf{D}) \lambda_n^{(t)}(x) \rangle_t = \lfloor k \rceil! \, b_{nk}$. By the spanning argument, this condition defines $f_k(\mathbf{D})$. In fact, we have

$$f_1(\mathbf{D}) = \sum_{n \in \mathbf{Z}} \frac{b_{n,1}}{\lfloor n \rceil!} \mathbf{D}^n.$$

Since $b_{n,1} = 0$ for all $n \leqslant 0$, but $b_{1,1} \neq 0$, $f_1(\mathbf{D})$ is a delta operator. Now, $\langle f_{i+j}(\mathbf{D}) \lambda_n^{(t)}(x) \rangle_t = \lfloor i+j \rceil! \, b_{n,i+j}$ by definition, and that expression equals $\sum_{k=i}^{n-j} \lfloor {}^n_k \rceil \langle f_i(\mathbf{D}) \lambda_k^{(t)}(x) \rangle_i \langle f_j(\mathbf{D}) \lambda_{n-k}^{(t)}(x) \rangle_t$ by Eq. (37). This in turn equals $\langle f_i(\mathbf{D}) f_j(\mathbf{D}) \lambda_n^{(t)}(x) \rangle_t$ since $\lambda_n^{(t)}(x)$ is a Roman graded sequence:

$$\langle f_{i+j}(\mathbf{D}) \lambda_n^{(t)}(x) \rangle_t = \langle f_i(\mathbf{D}) f_j(\mathbf{D}) \lambda_n^{(t)}(x) \rangle_t.$$

We infer $f_{i+j}(\mathbf{D}) = f_i(\mathbf{D}) f_j(\mathbf{D})$, again by the spanning argument. Thus, by induction, $f_1(\mathbf{D})^i = f_i(\mathbf{D})$. Therefore,

$$q_n^{(t)}(x) = \sum_{k \leqslant n} \frac{\langle f_1(\mathbf{D})^k \lambda_n^{(t)}(x) \rangle_t}{\lfloor k \rceil!} \lambda_k^{(t)}(x).$$

In other words, $q_n^{(t)}(x)$ is the conjugate graded sequence of the delta operator $f_1(\mathbf{D})$ (and no other). ∎

6.5. *Graded Sequences of Logarithmic Binomial Type*

In view of Propositions 6.4.2 and 6.3.5, we see that a delta operator is related to two Roman graded sequences—its associated (or basic) graded sequence and its conjugate graded sequence. On the other hand, a Roman graded sequence is related to two delta operators—one for which it is the associated graded sequence, and one for which it is the conjugate graded sequence. To further clarify these relationships, we briefly recall some facts about sequences of polynomials of binomial type.

DEFINITION 6.5.1 (Polynomial Sequence of Binomial Type). A sequence of polynomials

$$(p_n(x))_{n \geqslant 0}$$

is of *binomial type* if each polynomial $p_n(x)$ is of degree n, and for all field elements a, and all nonnegative integers n,

$$p_n(x+a) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)\, p_{n-k}(x).$$

It is a basic result of the Umbral calculus of Roman and Rota that every sequence of binomial type is associated with a unique delta operator and *vice versa*. $(p_n(x))_{n \geqslant 0}$ is the associated sequence of the delta operator $f(\mathbf{D})$ if and only if $f(\mathbf{D})\, p_n(x) = np_{n-1}(x)$ for positive n, and $p_n(0) = \delta_{n,0}$ for nonnegative n.

In the present theory, the logarithmic analogs of sequences of binomial type are the Roman graded sequences defined above. Again, we begin with a new definition equivalent to that of a Roman graded sequence.

DEFINITION 6.5.2 (Graded Sequence of Logarithmic Binomial Type). Let $p_n^{(t)}(x)$ be a graded sequence of formal power series of logarithmic type. The graded sequence is of *logarithmic binomial type* if for all integers n, nonnegative integers t, and constants $a \in K$, the following identity is satisfied:

$$p_n^{(t)}(x+a) = \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_k^{(0)}(a)\, p_{n-k}^{(t)}(x). \tag{40}$$

For $t=0$, we obtain simply the definition of a sequence of polynomials of binomial type. For $t=1$ and n negative, we obtain the factor sequences of *The Umbral Calculus*. Thus, the notion of a Roman graded sequence subsumes both the notion of a polynomial sequence of binomial type and the notion of a factor sequence. In view of the present theory, the older notion of factor sequence can be seen as obsolete.

We next prove that every graded sequence of logarithmic binomial type is a Roman graded sequence, and conversely.

THEOREM 6.5.3. *A logarithmic graded sequence $p_n^{(t)}(x)$ is a basic graded sequence for some delta operator $f(\mathbf{D})$ if and only if it is a graded sequence of logarithmic binomial type.*

Proof. (Only if) By iteration of Property 3 of basic polynomials (Definition 6.3.1), we see that

$$f(\mathbf{D})^k\, p_n^{(t)}(x) = \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!}\, p_{n-k}^{(t)}(x).$$

Hence, for $k = n$,

$$\langle f(\mathbf{D})^n\, p_n^{(t)}(x) \rangle = \lfloor n \rceil!$$

whereas for $k < n$,

$$\langle f(\mathbf{D})^k\, p_n^{(t)}(x) \rangle = 0.$$

Thus, we may trivially express $p_n^{(t)}(x)$ in the form

$$p_n^{(t)}(x) = \sum_{k \leqslant n} \frac{p_k^{(t)}(x)}{\lfloor k \rceil!} \langle f(\mathbf{D})^k\, p_n^{(t)}(x) \rangle_t.$$

Since any logarithmic graded sequence is a pseudobasis of $\mathscr{L}$, this holds—by linearity and continuity—for all formal power series of logarithmic type $p(x)$; that is, we have

$$p(x) = \sum_{k,t} \frac{p_k^{(t)}(x)}{\lfloor k \rceil!} \langle f(\mathbf{D})^k\, p(x) \rangle_t.$$

In particular, let $p(x) = E^a p_n^{(t)}(x)$. Then

$$\begin{aligned}
E^a p_n^{(t)}(x) &= \sum_{k \leqslant n} \frac{p_k^{(t)}(x)}{\lfloor k \rceil!} \langle E^a f(\mathbf{D})^k\, p_n^{(t)}(x) \rangle_t \\
&= \sum_{k \leqslant n} \frac{p_k^{(t)}(x)}{\lfloor k \rceil!} \left\langle E^a \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!}\, p_{n-k}^{(t)}(x) \right\rangle_t \\
&= \sum_{k \leqslant n} \frac{p_k^{(t)}(x)}{\lfloor k \rceil!} \left\langle E^a \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!}\, p_{n-k}^{(0)}(x) \right\rangle_0 \\
&= \sum_{k \leqslant n} \frac{\lfloor n \rceil!\, p_k^{(t)}(x)}{\lfloor n-k \rceil!\, \lfloor k \rceil!}\, p_{n-k}^{(0)}(a) \\
&= \sum_{k \geqslant 0} \begin{bmatrix} n \\ k \end{bmatrix} p_k^{(0)}(a)\, p_{n-k}^{(t)}(x).
\end{aligned}$$

Thus, $p_n^{(t)}(x)$ is a graded sequence of logarithmic binomial type.

(If) By the logarithmic binomial identity,

$$p_0^{(0)}(x) = p_0^{(0)}(0)\, p_0^{(0)}(x),$$

but $p_0^{(0)}(x) \in K$ is merely a constant, so we have

$$p_0^{(0)}(x) = 1.$$

For n positive, we have

$$0 = p_n^{(0)}(x) - p_n^{(0)}(x) = \sum_{k>0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_k^{(0)}(0)\, p_{n-k}^{(0)}(x),$$

and $p_n^{(0)}(x)$ is a basis for $\mathscr{L}^{(0)}$, so $p_k^{(0)}(0) = 0$ for $k > 0$. Thus, by regularity,

$$\langle p_n^{(t)}(x) \rangle_t = \delta_{n,0}$$

for all integers n, and nonnegative integers t.

Define a continuous, linear operator Q by the identity

$$Qp_n^{(t)}(x) = \lfloor n \rceil\, p_{n-1}^{(t)}(x).$$

We must show that Q is a delta operator. Obviously, Q is regular, and it lowers the degree of any logarithmic series by one, so it will suffice to show that Q is shift-invariant. This we verify by the following string of identities

$$\begin{aligned} QE^a p_n^{(t)}(x) &= Q \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil p_k^{(0)}(a)\, p_{n-k}^{(t)}(x) \\ &= \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lfloor n-k \rceil\, p_k^{(0)}(a)\, p_{n-k-1}^{(t)}(x) \\ &= \sum_{k \geqslant 0} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil \lfloor n \rceil\, p_k^{(0)}(a)\, p_{n-k-1}^{(t)}(x) \\ &= \lfloor n \rceil\, E^a p_{n-1}^{(t)}(x) \\ &= E^a Q p_n^{(t)}(x). \quad \blacksquare \end{aligned}$$

We summarize the results of the preceding sections in the following theorem:

THEOREM 6.5.4. *For any graded sequence of formal power series of logarithmic type $p_n^{(t)}(x)$, the following statements are equivalent*:

1. *$p_n^{(t)}(x)$ is a Roman graded sequence (Definition* 6.1.3).
2. *$p_n^{(t)}(x)$ is associated with some delta operator (Definition* 6.2.3).

3. $p_n^{(t)}(x)$ *is the conjugate graded sequence of some delta operator* (*Definition* 6.4.1).

4. $p_n^{(t)}(x)$ *is the basic graded sequence of some delta operator* (*Definition* 6.3.1).

5. $p_n^{(t)}(x)$ *is a graded sequence of logarithmic binomial type* (*Definition* 6.5.2).

6. $p_n^{(t)}(x)$ *is a graded sequence of formal power series of logarithmic type whose coefficients*

$$p_n^{(t)}(x) = \sum_{k \leqslant n} b_{nk} \lambda_k^{(t)}(x)$$

satisfy Eq. (37).

Furthermore, in this case $p_n^{(t)}(x)$ *is associated with and basic for the same delta operator.*

Proof. Condition 1 is equivalent to Condition 6 by Porism 6.1.7, it is equivalent to Condition 3 by Proposition 6.4.2, and it is equivalent to Condition 2 by Proposition 6.3.5. Next, Condition 2 is equivalent to Condition 4 by Theorem 6.3.2. Finally, Condition 4 is equivalent to Condition 5 by Theorem 6.5.3. ∎

We note the following useful identities for Roman graded sequences:

PROPOSITION 6.5.5. *Let* $p_n^{(t)}(x)$ *be a Roman graded sequence associated with the delta operator* $f(\mathbf{D})$, *let* t *be a positive integer, and let* n *be an integer. Then*

$$p_{n+1}^{(t)}(x) = \lfloor n+1 \rceil f(\mathbf{D})^{-1} p_n^{(t)}(x).$$

PORISM 6.5.6. *Let* $p_n^{(t)}(x)$ *be a Roman graded sequence with coefficients given by*

$$p_n^{(t)}(x) = \sum_{k \leqslant n} b_{nk} \lambda_k^{(t)}(x).$$

Then for any integer n, *and for* r *and* t *nonnegative integers, we have*

$$\mathbf{D}^r p_n^{(t)}(x) = \sum_{k \geqslant r} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil b_{kr} r!\, p_{n-k}^{(t)}(x).$$

Proof. Left to reader. ∎

7. Relations Among Roman Graded Sequences

7.1. *Transfer Operators*

We note a consequence of the results of the preceding section that is worth pointing out. Let $p_n^{(t)}(x)$ be the associated graded sequence for the delta operator $f(\mathbf{D})$. In general, the conjugate graded sequence for the operator $f(\mathbf{D})$ will be some other Roman graded sequence, say $q_n^{(t)}(x)$. This graded sequence will be in turn the associated graded sequence for another delta operator, say $g(\mathbf{D})$. What is the relationship between $f(\mathbf{D})$ and $g(\mathbf{D})$? And between $p_n^{(t)}(x)$ and $g(\mathbf{D})$?

We shall prove the remarkable fact that the formal power series $f(\mathbf{D})$ and $g(\mathbf{D})$ are inverses to each other in the sense of functional composition, and $p_n^{(t)}(x)$ is the conjugate graded sequence for $g(\mathbf{D})$. Actually, the results we shall obtain will be more sweeping, and lead to a powerful technique for establishing identities among formal power series of logarithmic type.

Definition 7.1.1 (Composition of Delta Operators). Let $f(\mathbf{D})$ and $g(\mathbf{D})$ be delta operators with $g(\mathbf{D}) = \sum_{k \geqslant 1} a_k \mathbf{D}^k$. We define the *composition* $g(f)$ to be the delta operator expressed by the following convergent summation

$$g(f) = \sum_{k \geqslant 1} a_k f(\mathbf{D})^k.$$

If $g(f) = \mathbf{D} = f(g)$, then $g(\mathbf{D})$ and $f(\mathbf{D})$ are called *compositional inverses*, and we write $g(\mathbf{D}) = f^{(-1)}(\mathbf{D})$.

All delta operators have unique compositional inverses.

We shall occasionally use the boldface $\mathbf{p}$ to denote a logarithmic graded sequence $p_n^{(t)}(x)$. Thus,

Definition 7.1.2 (Umbral Composition of Logarithmic Graded Sequences). Let $\mathbf{p}$ and $\mathbf{q}$ be logarithmic graded sequences, and let $q_n^{(t)}(x) = \sum_{k \leqslant n} b_{nk} \lambda_k^{(t)}(x)$. We define the *umbral composition* of the $\mathbf{q}$ graded sequence with the $\mathbf{p}$ graded sequence as the graded sequence defined by the convergent summation

$$q_n^{(t)}(\mathbf{p}) = \sum_{k \leqslant n} b_{nk} p_n^{(t)}(x).$$

Notice that $\mathbf{D}$ is a two-sided identity for the group of delta operators under composition, and that the graded sequence of harmonic logarithms is a two-sided identity for the semigroup formed by the operation of composition of Roman graded sequences. We shall show that this semigroup is actually a group, and that the two groups are naturally isomorphic. The isomorphism is given by the function which associates each delta operator with its associated graded sequence. The crucial role in obtaining these results is played by the notion of a transfer operator which we proceed to define:

DEFINITION 7.1.3 (Transfer Operator). Let $p_n^{(t)}(x)$ be a Roman graded sequence. The *transfer operator* associated with the graded sequence is the continuous linear operator $\tau_{\mathbf{p}}\colon \mathscr{L} \to \mathscr{L}$ defined as

$$\tau_{\mathbf{p}} \lambda_n^{(t)}(x) = p_n^{(t)}(x)$$

for n an integer, and t a nonnegative integer.

If $p_n^{(t)}(x)$ is the associated graded sequence for the delta operator $f(\mathbf{D})$, we will frequently write τ_f for the transfer operator associated with $p_n^{(t)}(x)$.

Notice that transfer operators are regular operators.

DEFINITION 7.1.4 (Adjoint). If θ is a continuous linear operator on $\mathscr{L}^{(t)}$ for some positive integer t, then the *adjoint* of θ is defined to be the linear operator $\mathbf{adj}(\theta)\colon \Gamma \to \Gamma$ such that

$$\langle [\mathbf{adj}(\theta)\, f(\mathbf{D})]\, p(x) \rangle_t = \langle f(\mathbf{D})\, \theta p(x) \rangle_t$$

for all formal power series of logarithmic type $p(x) \in \mathscr{L}^{(t)}$, and for all Laurent operators $f(\mathbf{D}) \in \Gamma$.

If θ is a continuous, linear operator on $\mathscr{L}^{(0)}$, then the *adjoint* of θ is defined to be the linear operator $\mathbf{adj}(\theta)\colon \Gamma^{(+)} \to \Gamma^{(+)}$ such that

$$\langle [\mathbf{adj}(\theta)\, f(\mathbf{D})]\, p(x) \rangle_0 = \langle f(\mathbf{D})\, \theta p(x) \rangle_0 \tag{41}$$

for all formal power series of logarithmic type $p(x) \in \mathscr{L}^{(0)}$, and for all differential operators $f(\mathbf{D}) \in \Gamma^{(+)}$.

By the spanning argument, the adjoint is well defined.

Let θ be a continuous linear operator on $\mathscr{L}$ or $\mathscr{L}^{(+)}$ which maps $\mathscr{L}^{(t)}$ to itself. The adjoint of the restriction of θ to $\mathscr{L}^{(t)}$ will be denoted by $\mathbf{adj}(\theta)^{(t)}$. If θ is a regular, continuous, linear operator on $\mathscr{L}^{(+)}$ or all of $\mathscr{L}$, the adjoint of θ is the unique operator which coincides with $\mathbf{adj}(\theta)^{(t)}$ for t positive.

The adjoint of nonregular operators on $\mathscr{L}$ or $\mathscr{L}^{(+)}$ can be similarly defined; however, we omit the discussion, since it would be an unnecessary digression.

EXAMPLES. 1. Let $f(\mathbf{D})$ be a Laurent operator. Then

$$[\mathbf{adj}(f(\mathbf{D}))](g(\mathbf{D})) = f(\mathbf{D})\, g(\mathbf{D}).$$

In other words, the adjoint of a Laurent operator $f(\mathbf{D})$ is the operator of multiplication by $f(\mathbf{D})$.

2. We define a regular, continuous, linear operator σ as follows (for all $t \geqslant 0$ and all $n \in \mathbf{Z}$):

$$\sigma\lambda_n^{(t)}(x) = \begin{cases} \lambda_{n+1}^{(t)}(x) & \text{if } n \neq -1, \text{ and} \\ 0 & \text{if } n = -1. \end{cases}$$

We call the operator σ the *standard Roman shift*. Note that it is not a shift-invariant operator. We have in particular $\sigma x^n = x^{n+1}$ for $n \neq -1$. Thus, the standard Roman shift is a logarithmic generalization of the operator $\mathbf{x}$ of multiplication by x. The standard Roman shift has as its adjoint the operator $\mathbf{adj}(\sigma)(\mathbf{D}^n) = n\mathbf{D}^{n-1}$ for all integers n, since

$$\langle k\mathbf{D}^{k-1}\lambda_n^{(t)}(x)\rangle_t = \lfloor k \rceil!\,(1-\delta_{0,k})\delta_{n+1,k} = \langle \mathbf{D}^k\sigma\lambda_n^{(t)}(x)\rangle_t.$$

The operator $\mathbf{adj}(\sigma)$ is called the *Pincherle derivative*.

PROPOSITION 7.1.5. *If τ is a transfer operator, then its adjoint $\mathbf{adj}(\tau)$ is an automorphism of Γ, and $\mathbf{adj}(\tau)^{(0)}$ is an automorphism of $\Gamma^{(+)}$.*

Proof. We will show that $\mathbf{adj}(\tau)$ is an monomorphism and that it acts on delta operators. This will imply that $\mathbf{adj}(\tau)$ preserves degree, and thus, that $\mathbf{adj}(\tau)$ is an automorphism.

Let τ be associated with the Roman graded sequence $p_n^{(t)}(x)$.

(Injectivity) Assume that $\mathbf{adj}(\tau)\, f(\mathbf{D}) = \mathbf{adj}(\tau)\, g(\mathbf{D})$ for some pair of Laurent operators $f(\mathbf{D})$ and $g(\mathbf{D})$. Thus, for all $p(x) \in \mathscr{L}^{(1)}$,

$$\langle f(\mathbf{D})\, \tau p(x)\rangle_1 = \langle g(\mathbf{D})\, \tau p(x)\rangle_1.$$

In particular, for all n,

$$\langle f(\mathbf{D})\, p_n^{(1)}(x)\rangle_1 = \langle g(\mathbf{D})\, p_n^{(1)}(x)\rangle_1.$$

By the spanning argument, we infer $f(\mathbf{D}) = g(\mathbf{D})$.

(Morphism) By definition, $\mathbf{adj}(\tau)$ is continuous and linear, so we need only confirm that $\mathbf{adj}(\tau)$ preserves multiplication. Let n be an integer, and let $f(\mathbf{D})$ and $g(\mathbf{D})$ be Laurent operators $c = \deg(f(\mathbf{D}))$, and $d = \deg(g(\mathbf{D}))$. Then we have

$$\begin{aligned}
&\langle \mathbf{adj}(\tau)(f(\mathbf{D})\, g(\mathbf{D}))\, \lambda_n^{(1)}(x) \rangle_1 \\
&\quad = \langle f(\mathbf{D})\, g(\mathbf{D})\, \tau \lambda_n^{(1)}(x) \rangle_1 \\
&\quad = \langle f(\mathbf{D})\, g(\mathbf{D})\, p_n^{(1)}(x) \rangle_1 \\
&\quad = \sum_{k=c}^{n-d} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle f(\mathbf{D})\, p_k^{(1)}(x) \rangle_1 \langle g(\mathbf{D})\, p_{n-k}^{(1)}(x) \rangle_1 \\
&\quad = \sum_{k=c}^{n-d} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle f(\mathbf{D})\, \tau \lambda_k^{(1)}(x) \rangle_1 \langle g(\mathbf{D})\, \tau \lambda_{n-k}^{(1)}(x) \rangle_1 \\
&\quad = \sum_{k=c}^{n-d} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \langle [\mathbf{adj}(\tau)\, f(\mathbf{D})]\, \lambda_k^{(1)}(x) \rangle_1 \langle [\mathbf{adj}(\tau)\, g(\mathbf{D})]\, \lambda_{n-k}^{(1)}(x) \rangle_1 \\
&\quad = \langle [\mathbf{adj}(\tau)\, f(\mathbf{D})][\mathbf{adj}(\tau)\, g(\mathbf{D})]\, \lambda_n^{(1)}(x) \rangle_1.
\end{aligned}$$

Thus, $\mathbf{adj}(\tau)[f(\mathbf{D})\, g(\mathbf{D})] = (\mathbf{adj}(\tau)\, f(\mathbf{D}))(\mathbf{adj}(\tau)\, g(\mathbf{D}))$ by the spanning argument.

(Action on delta operators) Finally, suppose $p_n^{(t)}(x)$ is the associated graded sequence for the delta operator $f(\mathbf{D})$, then for all integers n,

$$\begin{aligned}
\langle \mathbf{adj}(\tau)\, f(\mathbf{D})\, \lambda_n^{(1)}(x) \rangle_1 &= \langle f(\mathbf{D})\, p_n^{(1)}(x) \rangle_1 \\
&= \lfloor n \rceil!\, \delta_{n,1} \\
&= \langle \mathbf{D} \lambda_n^{(1)}(x) \rangle_1
\end{aligned}$$

and by the spanning argument, $\mathbf{adj}(\tau)\, f(\mathbf{D}) = \mathbf{D}$. The conclusion now follows by the Expansion Theorem. ∎

The most important properties of transfer operators are stated in the following proposition.

PROPOSITION 7.1.6. 1. *A transfer operator maps Roman graded sequences to Roman graded sequences.*

2. *If* $\tau\colon p_n^{(1)}(x) \mapsto q_n^{(t)}(x)$ *is a continuous linear operator, where the* $p_n^{(t)}(x)$ *and* $q_n^{(t)}(x)$ *are the associated graded sequences for the delta operators* $f(\mathbf{D})$ *and* $g(\mathbf{D})$, *respectively, then* $\mathbf{adj}(\tau)\, g(\mathbf{D}) = f(\mathbf{D})$.

3. *The operator* τ *above is a transfer operator.*

Proof. (1) Let $\tau\colon \lambda_n^{(t)}(x) \mapsto p_n^{(t)}(x)$ be a transfer operator. By

Proposition 7.1.5, $\mathbf{adj}(\tau)$ is an isomorphism of Γ. Let $q_n^{(t)}(x)$ be the associated graded sequence for the delta operator $g(\mathbf{D})$. Therefore,

$$\langle \mathbf{adj}(\tau)^{-1}\, g(\mathbf{D})^k\, \tau q_n^{(t)}(x)\rangle_t = \langle g(\mathbf{D})^k\, q_n^{(t)}(x)\rangle_t = \llcorner n \urcorner!\, \delta_{nk}.$$

Hence, $\tau q_n^{(t)}(x)$ is the associated graded sequence for the delta operator $\mathbf{adj}(\tau)^{-1}\, g(\mathbf{D})$.

(2) We have the following sequence of equalities

$$\begin{aligned}\langle \mathbf{adj}(\tau)\, g(\mathbf{D})\, p_n^{(1)}(x)\rangle_1 &= \langle g(\mathbf{D})\, \tau p_n^{(1)}(x)\rangle_1\\ &= \langle g(\mathbf{D})\, q_n^{(1)}(x)\rangle_1\\ &= \llcorner n \urcorner!\, \delta_{n,1}\\ &= \langle f(\mathbf{D})\, p_n^{(1)}(x)\rangle_1.\end{aligned}$$

Hence, by the spanning argument, $\mathbf{adj}(\tau)\, g(\mathbf{D}) = f(\mathbf{D})$.

(3) More generally, for an arbitrary Laurent operator $\sum_{j \leqslant k} a_j\, g(\mathbf{D})^j$, we have by Proposition 7.1.5, $\mathbf{adj}(\tau) \sum_{j \leqslant k} a_j\, g(\mathbf{D})^j = \sum a_j f(\mathbf{D})^j$. Specialize to the case of $\sum_{j \geqslant k} a_j\, \mathbf{D}^j = f^{(-1)}(\mathbf{D})^k$ to get

$$\mathbf{adj}(\tau)(f^{(-1)}(g))^k = \mathbf{D}^k,$$

and hence

$$\langle (f^{(-1)}(g))^k\, \tau\lambda_n^{(t)}(x)\rangle_t = \langle \mathbf{D}^k \lambda_n^{(t)}(x)\rangle_t.$$

In other words, $\tau\lambda_n^{(t)}(x)$ is associated with the delta operator $f^{(-1)}(g)$. ∎

The following results illustrate the applications of the preceding proposition.

PROPOSITION 7.1.7. *Let $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$ be the associated graded sequences of the delta operators $g(\mathbf{D})$ and $f(\mathbf{D})$, respectively. Then the composition $g(f)$ is the delta operator with associated graded sequence $q_n^{(t)}(\mathbf{p})$.*

Proof. If τ: $\lambda_n^{(t)}(x) \mapsto p_n^{(t)}(x)$ is a transfer operator, then as in the proof of part 1 of Proposition 7.1.6, $\tau q_n^{(t)}(x)$ is the associated graded sequence for $(\mathbf{adj}(\tau))^{-1}\, g(\mathbf{D})$. However, $\tau q_n^{(t)}(x) = q_n^{(t)}(\mathbf{p})$ by Definition 7.1.1. Moreover, part 2 of the same proposition asserts that $\mathbf{adj}(\tau)\, f(\mathbf{D}) = \mathbf{D}$, so $\mathbf{adj}(\tau)^{-1}\, \mathbf{D} = f(\mathbf{D})$, and, thus, $\mathbf{adj}(\tau)^{-1}\, g(\mathbf{D}) = g(f)$. The conclusion follows. ∎

COROLLARY 7.1.8. *The set of Roman graded sequences is closed under umbral composition.*

COROLLARY 7.1.9 (Inverses). *Let $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$ be the Roman graded sequences associated with the delta operators $f(\mathbf{D})$ and $g(\mathbf{D})$, respectively. Then the following statements are equivalent:*

1. $f(g) = \mathbf{D}$.
2. $g(f) = \mathbf{D}$.
3. $q_n^{(t)}(\mathbf{p}) = \lambda_n^{(t)}(x)$.
4. $p_n^{(t)}(\mathbf{q}) = \lambda_n^{(t)}(x)$.

COROLLARY 7.1.10. *The associated Roman graded sequence for the delta operator $f(\mathbf{D})$ is the conjugate Roman graded sequence for the delta operator $f^{(-1)}(\mathbf{D})$.*

Proof. Theorem 6.2.5 and Corollary 7.1.9. ∎

7.2. *Explicit Formulae for Roman Graded Sequences*

The following definition extends to all Roman graded sequences the notion of the standard Roman shift (Example 2 following Definition 7.1.4):

DEFINITION 7.2.1 (Roman Shift). If $p_n^{(t)}(x)$ is a Roman graded sequence, the *Roman shift* relative to $p_n^{(t)}(x)$ is the continuous linear operator $\sigma_{\mathbf{p}}\colon \mathscr{L} \to \mathscr{L}$ defined by

$$\sigma_{\mathbf{p}}\, p_n^{(t)}(x) = \begin{cases} p_{n+1}^{(t)}(x) & \text{if } n \neq -1, \text{ and} \\ 0 & \text{if } n = -1, \end{cases} \tag{42}$$

for all integers n and nonnegative integers t (except $n = -1$ and $t = 0$).

If $p_n^{(t)}(x)$ is associated with the delta operator $f(\mathbf{D})$, we also write σ_f instead of $\sigma_{\mathbf{p}}$. When no graded sequence of delta operator has been specified, we assume $\sigma = \sigma_{\mathbf{D}} = \sigma_{\lambda}$, that is, that σ is the standard Roman shift as previously defined.

We *digress* to indicate how one can define a symmetric nondegenerate bilinear form associated with any Roman graded sequence $p_n^{(t)}(x)$. Consider the principal subsequence $\tilde{p}_n(x)$ (Definition 5.6.1). Define an inner product

$$\langle \tilde{p}_n(x) \mid \tilde{p}_m(x) \rangle = \delta_{mn} \lfloor n \rceil !.$$

One verifies that relative to this bilinear form, the operator $f(\mathbf{D})$ is adjoint to the Roman shift σ_f. For example, for the harmonic logarithm one has $\tilde{\lambda}_n(x) = x^n$ for all integers n, and

$$\langle \mathbf{D} x^n \mid x^m \rangle = \langle x^n \mid \sigma x^m \rangle,$$

where σ is the standard Roman shift, which restricts to the operator $\mathbf{x}$ of multiplication by x. The principal sequence $\tilde{p}_n(x)$ is the set of eigenfunctions of the operator $\sigma_f f(\mathbf{D})$ with eigenvalue n. For example, the sequence x^n for n an integer is the set of eigenvalues of the operator $\mathbf{xD}$.

If K is the field of real numbers, the bilinear form is not in general definite, since

$$\langle 1+x^{-2} \mid 1+x^{-2} \rangle = 0.$$

However, if K is the field of complex numbers, then we can define a Hermitian form

$$\langle p(x) \mid q(x) \rangle_c = -\langle \overline{p(x)} \mid q(x) \rangle,$$

so we have

$$\begin{aligned} \langle (ix)^n \mid (ix)^n \rangle_c &= \langle \overline{(ix)^n} \mid (ix)^n \rangle \\ &= \begin{cases} -\langle x^n \mid x^n \rangle & \text{for } n \text{ even, and} \\ \langle x^n \mid x^n \rangle & \text{for } n \text{ odd,} \end{cases} \end{aligned}$$

and thus for n negative

$$\langle (ix)^n \mid (ix)^n \rangle_c = \frac{1}{(-n-1)!} > 0.$$

Extending by linearity, we obtain a Hermitian inner product which is positive definite, where the sequence $h_n(x)$ defined as

$$h_n(x) = \begin{cases} x^n & \text{for } n \geqslant 0, \text{ and} \\ (ix)^n & \text{for } n < 0 \end{cases}$$

is a complete orthogonal sequence in the Hilbert space obtained by completion relative to this Hermitian inner product. One can therefore develop a spectral theory of the operator $\mathbf{xD}$ in this Hilbert space (rather than with the smaller one including only positive powers of x, as is done classically). *End of digression.*

PROPOSITION 7.2.2. 1. *A regular, continuous, linear operator θ defined on $\mathscr{L}^{(+)}$ is a Roman shift if and only if* $\mathbf{adj}(\theta)$ *is a continuous, everywhere defined derivation of Γ, the algebra of Laurent operators, for which* $\mathbf{adj}(\theta f(\mathbf{D})) = 1$ *for some delta operator* $f(\mathbf{D})$.

2. *A regular, continuous, linear operator θ defined on the logarithmic algebra $\mathscr{L}$ is a Roman shift if and only if* $\mathbf{adj}(\theta)$ *is a continuous, everywhere defined, derivation of $\Gamma^{(+)}$, the algebra of differential operators, for which* $\mathbf{adj}(\theta f(\mathbf{D})) = 1$ *for some delta operator* $f(\mathbf{D})$.

Proof. (Only if) Let $p_n^{(t)}(x)$ be a Roman graded sequence of formal power series of logarithmic type. By Proposition 6.4.2, $p_n^{(t)}(x)$ is associated with a delta operator $f(\mathbf{D})$. Now, we have

$$\begin{aligned}\langle \mathbf{adj}(\sigma_f)\, f(\mathbf{D})^k\, p_n^{(1)}(x)\rangle_1 &= \langle f(\mathbf{D})^k\, \sigma_f\, p_n^{(1)}(x)\rangle_1\\ &= \delta_{n,-1}\langle f(\mathbf{D})^k\, p_{n+1}^{(1)}(x)\rangle_1\\ &= (n+1)\lfloor n\rceil!\,\delta_{n+1,k}\\ &= k\lfloor n\rceil!\,\delta_{n,k-1}\\ &= \langle kf(\mathbf{D})^{k-1}\, p_n^{(1)}(x)\rangle_1.\end{aligned}$$

Hence, by the spanning argument $\mathbf{adj}(\sigma_f)\, f(\mathbf{D})^k = kf(\mathbf{D})^{k-1}$. Also, $\mathbf{adj}(\sigma_f)\, f(\mathbf{D}) = 1$. By the continuity of $\mathbf{adj}(\sigma_f)$ and the spanning argument, the result follows.

(If) Conversely, suppose $\mathbf{adj}(\theta)$ is a continuous, everywhere defined derivation of Γ such that

$$\mathbf{adj}(\theta)\, f(\mathbf{D}) = 1.$$

Let $\sigma_{\mathbf{p}}$ be the shift associated with $p_n^{(t)}(x)$, the associated graded sequence for $f(\mathbf{D})$. We shall show that $\theta = \sigma_{\mathbf{p}}$.

$$\begin{aligned}\langle f(\mathbf{D})^k\, \theta p_n^{(1)}(x)\rangle_1 &= \langle \mathbf{adj}(\theta)\, f(\mathbf{D})^k\, p_n^{(1)}(x)\rangle_1\\ &= \langle kf(\mathbf{D})^{k-1}\, p_n^{(1)}(x)\rangle_1\\ &= k\lfloor n\rceil!\,\delta_{n,k-1}\\ &= \langle f(\mathbf{D})^k\, \sigma_p\, p_n^{(1)}(x)\rangle_1.\end{aligned}$$

Thus by the spanning argument, $\theta = \sigma_{\mathbf{p}}$. ∎

Next, we derive the chain rule for Roman shifts.

PROPOSITION 7.2.3 (Chain Rule). *Suppose σ_f and σ_g are Roman shift operators. Then*

$$\mathbf{adj}(\sigma_f) = (\mathbf{adj}(\sigma_f)\, g(\mathbf{D}))\, \mathbf{adj}(\sigma_g).$$

Proof. For any Laurent operator $h(\mathbf{D}) = \sum_{k\geqslant n} a_k\, g(\mathbf{D})^k$,

$$\begin{aligned}\mathbf{adj}(\sigma_f)\, h(\mathbf{D}) &= \sum_{k\geqslant n} ka_k\, g(\mathbf{D})^{k-1}\, \mathbf{adj}(\sigma_f)\, g(\mathbf{D})\\ &= [\mathbf{adj}(\sigma_f)\, g(\mathbf{D})][\mathbf{adj}(\sigma_g)\, h(\mathbf{D})],\end{aligned}$$

so $\mathbf{adj}(\sigma_f) = (\mathbf{adj}(\sigma_f)\, g(\mathbf{D}))\, \mathbf{adj}(\sigma_g)$. ∎

The following proposition allows us to relate two Roman shift operators.

PROPOSITION 7.2.4. *If σ_f and σ_g are regular shift operators, then*

$$\sigma_f = \sigma_g(\mathbf{adj}(\sigma_f)\, g(\mathbf{D}))^{-1}.$$

Proof. For any $h(\mathbf{D}) \in \Gamma$ and $p(x) \in \mathscr{L}$, we have

$$\begin{aligned} h(\mathbf{D})\, \sigma_f\, p(x) &= \mathbf{adj}(\sigma_f)\, h(\mathbf{D})\, p(x) \\ &= (\mathbf{adj}(\sigma_g)\, h(\mathbf{D}))(\mathbf{adj}(\sigma_f)\, g(\mathbf{D}))\, p(x) \end{aligned}$$

by the chain rule. This in turn equals $h(\mathbf{D})\, \sigma_g(\mathbf{adj}(\sigma_f)\, g(\mathbf{D}))\, p(x)$. ∎

DEFINITION 7.2.5 (Pincherle Derivative). Let $f(\mathbf{D})$ be a Laurent operator. We can define its *Pincherle derivative* $f'(\mathbf{D})$ by any of the following equivalent formulations:

1. The Pincherle derivative is the continuous, linear map

$$\begin{aligned} {}' : \Gamma &\to \Gamma \\ \mathbf{D}^n &\mapsto n\mathbf{D}^{n-1}. \end{aligned}$$

2. $f'(\mathbf{D}) = f(\mathbf{D})\sigma - \sigma f(\mathbf{D})$. (Recall Definition 7.2.1.)

3. The Pincherle derivative is the adjoint of the standard Roman shift σ. In other words $f'(\mathbf{D}) = \mathbf{adj}(\sigma)\, f(\mathbf{D})$.

We are now ready to derive the following recurrence formula for Roman graded sequences:

THEOREM 7.2.6 (Recurrence Formula). *If $p_n^{(t)}(x)$ is the associated graded sequence of delta operator $f(\mathbf{D})$ then for all integers $n \neq -1$ and for all non-negative integers t,*

$$p_{n+1}^{(t)}(x) = \sigma(f'(\mathbf{D}))^{-1}\, p_n^{(t)}(x),$$

where σ is the standard Roman shift.

Proof. Let n be an integer other than -1, then by Proposition 7.2.4,

$$\begin{aligned} p_{n+1}^{(t)}(x) &= \sigma_f\, p_n^{(t)}(x) \\ &= \sigma(\mathbf{adj}(\sigma)\, f(\mathbf{D}))^{-1}\, p_n^{(t)}(x) \\ &= \sigma f'(\mathbf{D})^{-1}\, p_n^{(t)}(x). \quad ∎ \end{aligned}$$

Next, we give an explicit formula for the associated graded sequence of a delta operator in terms of the residual series of the graded sequence of harmonic logarithms.

PROPOSITION 7.2.7. *If $p_n^{(t)}(x)$ is the associated graded sequence of formal power series of logarithmic type for the delta operator $f(\mathbf{D})$, then when n is an integer, and t is a positive integer,*

$$p_n^{(t)}(x) = \lfloor n \rceil!\, f'(\mathbf{D})\, f(\mathbf{D})^{-1-n}\, \lambda_{-1}^{(t)}(x).$$

Proof. Let $q_n^{(t)}(x)$ be the graded sequence defined by

$$q_n^{(t)}(x) = \lfloor n \rceil!\, f'(\mathbf{D})\, f(\mathbf{D})^{-1-n}\, \lambda_{-1}^{(t)}(x).$$

It will suffice to verify that $q_n^{(t)}(x)$ is the basic graded sequence for $f(\mathbf{D})$. $q_n^{(t)}(x)$ is indeed a *graded sequence* since $\deg(f'(\mathbf{D})\, f(\mathbf{D})^{-1-n}) = -1-n$. Then, if $n \neq 0$, we have

$$\begin{aligned} \langle q_n^{(t)}(x) \rangle_t &= \langle \lfloor n \rceil!\, f'(\mathbf{D})\, f(\mathbf{D})^{-1-n}\, \lambda_{-1}^{(t)}(x) \rangle_t \\ &= \frac{\lfloor n \rceil!}{-n} \langle (f(\mathbf{D})^{-n})'\, \lambda_{-1}^{(t)}(x) \rangle_t. \end{aligned}$$

Now, $\langle (f(\mathbf{D})^{-n})'\, \lambda_{-1}^{(t)}(x) \rangle_t$ equals the coefficient of $\mathbf{D}^{-1}$ in the Laurent operator $(f(\mathbf{D})^{-n})'$. But in the expansion of the Pincherle derivative, $g'(\mathbf{D})$, of any Laurent operator $g(\mathbf{D}) \in \Gamma$ the coefficient of $\mathbf{D}^{-1}$ equals zero. Hence, $\langle q_n^{(t)}(x) \rangle_t = 0$. Thus, Property 2 of Definition 6.3.1 is satisfied.

Now, consider the case $n = 0$, and t positive. Then we have

$$\langle q_0^{(t)}(x) \rangle_t = \langle f'(\mathbf{D})\, f(\mathbf{D})^{-1}\, \lambda_{-1}^{(t)}(x) \rangle_t.$$

Say $f(\mathbf{D}) = \sum_{k \geq 1} a_k \mathbf{D}^k$ with $a_1 \neq 0$. The coefficient of $\mathbf{D}^{-1}$ in $f(\mathbf{D})^{-1}$ is a_1^{-1}, and the coefficient of 1 in $f'(\mathbf{D})$ is a_1. Neither operator has any terms of lower degree. Hence, the coefficient of $\mathbf{D}^{-1}$ in $f'(\mathbf{D})\, f(\mathbf{D})^{-1}$ is 1, and there are no terms of lower degree. Hence, $\langle q_0^{(t)}(x) \rangle_t = 1$. Thus, Property 1 of Definition 6.3.1 is satisfied.

Finally,

$$\begin{aligned} f(\mathbf{D})\, q_n^{(t)}(x) &= \lfloor n \rceil!\, f'(\mathbf{D})\, f(\mathbf{D})^{-n}\, \lambda_{-1}^{(t)}(x) \\ &= \lfloor n \rceil\, q_{n-1}^{(t)}(x) \end{aligned}$$

so Property 3 is also satisfied. Therefore, $p_n^{(t)}(x) = q_n^{(t)}(x)$. ∎

Proposition 7.2.7 is unusual in that it does not readily generalize to $\mathscr{L}^{(0)}$ because for $t = 0$, $\lambda_{-1}^{(t)}(x)$ equals 0. Moreover, for $n \geq 0$, $f(\mathbf{D})^{-1-n}$ has negative degree, and thus is not a member of $\Gamma^{(+)}$.

Nonetheless, its consequence—the transfer formula—still holds for $\mathscr{L}^{(0)}$:

THEOREM 7.2.8 (The Transfer Formula). *If $p_n^{(t)}(x)$ is the associated graded sequence for the delta operator $f(\mathbf{D})$, then for all integers n and nonnegative integers t,*

$$p_n^{(t)}(x) = f'(\mathbf{D}) \left(\frac{\mathbf{D}}{f(\mathbf{D})} \right)^{n+1} \lambda_n^{(t)}(x).$$

In particular, the residual series is given by

$$p_{-1}^{(1)}(x) = f'(\mathbf{D}) x^{-1}.$$

Proof. For t positive, the conclusion is immediate from the preceding proposition. To derive the general result from t positive, we need the following definition. The *lowering operator* L: $\mathscr{L}^{(+)} \to \mathscr{L}$ is the continuous K-linear surjection $L = \sum_{n>0} E_{n-1,n}$. More specifically, we have for t positive

$$L\lambda_n^{(t)}(x) = \lambda_n^{(t-1)}(x).$$

Thus, L is a continuous, linear, shift-invariant operator. In terms of the lowering operator L, we have for all nonnegative integers t, and all integers n,

$$\begin{aligned} p_n^{(t)}(x) &= Lp_n^{(t+1)}(x) \\ &= Lf'(\mathbf{D}) \left(\frac{\mathbf{D}}{f(\mathbf{D})} \right)^{n+1} \lambda_n^{(t+1)}(x) \\ &= f'(\mathbf{D}) \left(\frac{\mathbf{D}}{f(\mathbf{D})} \right)^{n+1} L\lambda_n^{(t+1)}(x) \\ &= f'(\mathbf{D}) \left(\frac{\mathbf{D}}{f(\mathbf{D})} \right)^{n+1} \lambda_n^{(t)}(x). \quad \blacksquare \end{aligned}$$

The following generalization of the transfer formula is often useful:

COROLLARY 7.2.9. *Let $f(\mathbf{D})$ and $g(\mathbf{D})$ be the delta operators for the Roman graded sequences $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$, respectively, then for all integers n and nonnegative integers t,*

$$p_n^{(t)}(x) = f'(\mathbf{D})\, g'(\mathbf{D})^{-1} \left(\frac{g(\mathbf{D})}{f(\mathbf{D})} \right)^{n+1} q_n^{(t)}(x).$$

The following variant of the transfer formula is also useful:

PROPOSITION 7.2.10. *In the notation of Proposition* 7.2.7,

$$p_n^{(t)}(x) = g(\mathbf{D})^{-n}\, \lambda_n^{(t)}(x) - (g(\mathbf{D})^{-n})'\, \lambda_{n-1}^{(t)}(x)$$

for $n \neq 0$, *where* $g(\mathbf{D}) = \mathbf{D}^{-1} f(\mathbf{D})$.

Proof. By Proposition 7.2.7, $p_n^{(t)}(x) = f'(\mathbf{D})\, g(\mathbf{D})^{-1-n}\, \lambda_n^{(t)}(x)$. However,

$$\begin{aligned} f'(\mathbf{D})\, g(\mathbf{D})^{-1-n} &= (\mathbf{D}\, g(\mathbf{D}))'\, g(\mathbf{D})^{-n-1} \\ &= (\mathbf{D}' g(\mathbf{D}) + \mathbf{D}\, g'(\mathbf{D}))\, g(\mathbf{D})^{-n-1} \\ &= g(\mathbf{D})^{-n} + \mathbf{D}\, g'(\mathbf{D})\, g(\mathbf{D})^{-n-1} \\ &= g(\mathbf{D})^{-n} + (1/n)(g(\mathbf{D})^{-n})'\, \mathbf{D} \end{aligned}$$

so that

$$f'(\mathbf{D})\, g(\mathbf{D})^{-1-n}\, \lambda_n^{(t)}(x) = g(\mathbf{D})^{-n}\, \lambda_n^{(t)}(x) + (g(\mathbf{D})^{-n})'\, \lambda_{n-1}^{(t)}(x). \quad \blacksquare$$

Note that Proposition 7.2.10 does not in general hold for $n = 0$.

COROLLARY 7.2.11. *In the notation of Proposition* 7.2.10,

$$p_n^{(t)}(x) = \sigma g(\mathbf{D})^{-n}\, \lambda_{n-1}^{(t)}(x)$$

for $n \neq 0$, 1, *where* σ *is the standard Roman shift.*

Proof. By Proposition 7.2.10,

$$\begin{aligned} p_n^{(t)}(x) &= g(\mathbf{D})^{-n}\, \lambda_n^{(t)}(x) - (g(\mathbf{D})^{-n})'\, \lambda_{n-1}^{(t)}(x) \\ &= g(\mathbf{D})^{-n}\, \lambda_n^{(t)}(x) - g(\mathbf{D})^{-n}\, \sigma\lambda_{n-1}^{(t)}(x) \\ &\quad + \sigma g(\mathbf{D})^{-n}\, \lambda_{n-1}^{(t)}(x). \quad \blacksquare \end{aligned}$$

We conclude with an important remark about graded sequences of formal power series of logarithmic type. Let $p_n^{(t)}(x)$ be a Roman graded sequence and let

$$p_n^{(t)}(x) = \sum_k c_{nk}^{(t)} \lambda_k^{(t)}(x).$$

Then by regularity

1. For t and s positive, and for any integers n and k, $c_{nk}^{(t)} = c_{nk}^{(s)}$, and
2. For t, s, n, and k nonnegative integers, $c_{nk}^{(t)} = c_{nk}^{(s)}$.

In view of this, we see that in computations with Roman graded sequences it suffices for most purposes to compute in the subspace $\mathscr{L}^{(1)}$. In other words, even in computations with polynomials it is preferable to deal with logarithms first!

7.3. *Composition of Formal Series*

Let $f(\mathbf{D})$ be a delta operator. All of the formulae for composition and inversions of series (in particular, the various versions of the Lagrange inversion formula) are consequences of the following theorem.

PROPOSITION 7.3.1. 1. *If $f(\mathbf{D})$ is the delta operator with associated graded sequence $p_n^{(t)}(x)$, then for every Laurent operator $g(\mathbf{D})$, we have the following convergent series*:

$$g(f^{(-1)})=\sum_k \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x)\rangle_t}{\lfloor k\rceil!}\mathbf{D}^k,$$

where t is a positive integer.

2. *This identity holds for $t=0$ as well whenever $g(\mathbf{D})$ is a differential operator.*

Proof. By the expansion theorem,

$$g(\mathbf{D})=\sum_{k\geqslant d} \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x)\rangle_t}{\lfloor k\rceil!} f(\mathbf{D})^k,$$

where $d=\deg(g(\mathbf{D}))$. Hence,

$$\begin{aligned} g(f^{(-1)}) &= \sum_{k\geqslant d} \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x)\rangle_t}{\lfloor k\rceil!} f(f^{(-1)})^k \\ &= \sum_{k\geqslant d} \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x)\rangle_t}{\lfloor k\rceil!}\mathbf{D}^k. \quad \blacksquare \end{aligned}$$

Theorem 7.3.1 has many versions and many corollaries. We first deduce from the Expansion Theorem the convergent expansion

$$\begin{aligned} \sum_k \frac{\langle g(\mathbf{D})\, p_k^{(t)}(x)\rangle_t}{\lfloor k\rceil!}\mathbf{D}^k &= g(f^{(-1)}) \\ &= \sum_k \frac{\langle g(f^{(-1)})\, \lambda_k^{(t)}(x)\rangle_t}{\lfloor k\rceil!}\mathbf{D}^k. \end{aligned}$$

Hence, $\langle g(\mathbf{D})\, p_n^{(t)}(x)\rangle_t = \langle g(f^{(-1)})\, \lambda_n^{(t)}(x)\rangle_t$, a result which was also established in Section 6.2. An application of the Transfer Formula gives

$$\begin{aligned}\lfloor n\rceil!\, \langle g(\mathbf{D})\, f'(\mathbf{D})\, f(\mathbf{D})^{-1-n}\, \lambda_n^{(1)}(x)\rangle_1 &= \langle g(\mathbf{D})\, p_n^{(1)}(x)\rangle_1 \\ &= \langle g(f^{(-1)})\, \lambda_n^{(1)}(x)\rangle_1.\end{aligned}$$

By the spanning argument, we have the following corollary.

COROLLARY 7.3.2. *Let $f(\mathbf{D})$ be a delta operator, and let $g(\mathbf{D})$ be any Laurent operator. Then we have the convergent sum*

$$g(f^{(-1)}) = \sum_k \langle g(\mathbf{D})\, f'(\mathbf{D})\, f(\mathbf{D})^{-1-k}\, \lambda_{-1}^{(1)}(x)\rangle_1\, \mathbf{D}^k.$$

Let d be an integer. By taking $g(\mathbf{D}) = \mathbf{D}^d$, we obtain powers of $f^{(-1)}(\mathbf{D})$.

COROLLARY 7.3.3. *If $f(\mathbf{D})$ is a delta operator with associated graded sequence $p_n^{(t)}(x)$, then we have the convergent expansions*

$$\begin{aligned}f^{(-1)}(\mathbf{D})^d &= \sum_k \frac{\langle \mathbf{D}^d p_k^{(1)}(x)\rangle_1}{\lfloor k\rceil!}\, \mathbf{D}^k \\ &= \sum_k \frac{\langle f'(\mathbf{D})\, f(\mathbf{D})^{-1-k}\, \lambda_{-1-d}^{(1)}(x)\rangle_1}{\lfloor -1-d\rceil!}\, \mathbf{D}^k.\end{aligned}$$

8. EXAMPLES

8.1. *Roman Graded Sequences*

We prefix some general considerations about the computation of the Roman graded sequences $p_n^{(t)}(x)$ associated with a delta operator $f(\mathbf{D})$. The crucial step is the computation of the residual series $p_{-1}^{(1)}(x)$. This is given by the simple formula

$$p_{-1}^{(1)}(x) = f'(\mathbf{D})\,\frac{1}{x}.$$

Once the residual series is known, any of the series $p_n^{(1)}(x)$ can be obtained from the residual series by applying a suitable power of $f(\mathbf{D})$, that is, by the Transfer Formula:

$$p_n^{(1)}(x) = f'(\mathbf{D})\left(\frac{f(\mathbf{D})}{\mathbf{D}}\right)^{-1-n} x^n$$

for $n<0$, and

$$p_n^{(1)}(x)=f'(\mathbf{D})\left(\frac{f(\mathbf{D})}{\mathbf{D}}\right)^{-1-n} x^n\left(\log x-1-\frac{1}{2}-\cdots-\frac{1}{n}\right)$$

for $n \geqslant 0$. If

$$p_n^{(1)}(x)=\sum_k c_{nk}\lambda_k^{(1)}(x),$$

then it follows from regularity that

$$p_n^{(t)}(x)=\sum_k c_{nk}\lambda_k^{(t)}(x)$$

for all nonnegative t. Note that for $t=0$ we obtain

$$p_n^{(0)}(x)=\sum_{k=0}^{n} c_{nk}x^k,$$

which is the sequence of polynomials of binomial type associated with the delta operator $f(\mathbf{D})$; thus we see that even in the case of polynomials, it may be speedier to compute via the logarithmic graded sequence.

We next make some general remarks about solutions of differential equations of infinite order (and thus, in particular, of difference equations). The differential equation

$$f(\mathbf{D})\, p(x)=q(x), \tag{43}$$

where $q(x)\in\mathscr{L}^{(+)}$ is given, and $f(\mathbf{D})$ is any Laurent operator, has a unique solution $p(x)\in\mathscr{L}^{(+)}$. For example, $q(x)$ may be any rational function of x whose numerator is of smaller degree than the denominator.

TABLE 8.1

Examples of Roman Graded Sequences

Delta operator	Associated graded sequence	Inverse operator	Conjugate graded sequence
$\mathbf{D}$	$\lambda_n^{(t)}(x)$	$\mathbf{D}$	$\lambda_n^{(t)}(x)$
$\Delta=E-\mathbf{I}$	$(x)_n^{(t)}$	$\log(\mathbf{I}+\mathbf{D})$	$\phi_n^{(t)}(x)$
$\nabla=\mathbf{I}-E^{-1}$	$\langle x\rangle_n^{(t)}$	$-\log(\mathbf{I}-\mathbf{D})$	$q_n^{(t)}(x)$
$A_a=\mathbf{D}E^a$	$A_n^{(t)}(x)$		$\mu_n^{(t)}(x)$
$E^a(E-1)$	$G_n^{(t)}(x)$		$g_n^{(t)}(x)$
$K=\dfrac{\mathbf{D}}{\mathbf{D}-\mathbf{I}}$	$L_n^{(t)}(x)$	$K=\dfrac{\mathbf{D}}{\mathbf{D}-\mathbf{I}}$	$L_n^{(t)}(x)$

If $q(x) \in \mathscr{L}$ (for example, if $q(x)$ is an arbitrary rational function) the solution may not be unique. We can nevertheless define the *natural solution* of Eq. (43) as follows. Define the operators

$$R = E_{10} + E_{21} + E_{32} + \cdots$$
$$L = E_{01} + E_{12} + E_{23} + \cdots.$$

L is called the *lowering operator*, and R is called the *raising operator*. Now, let $s(x) = Rq(x)$, and consider the differential equation

$$f(\mathbf{D})\, r(x) = s(x).$$

Since $s(x) \in \mathscr{L}^{(+)}$, this differential equation has a unique solution $r(x) \in \mathscr{L}^{(+)}$. Now, set $p(x) = Lr(x)$ to obtain a solution of Eq. (43). The present definition agrees with (and is simpler than) all other definitions given of a natural solution over the complex numbers, yet it is valid over any field of characteristic zero.

A notable example is the difference equation

$$\Delta p(x) = 1/x.$$

By the above remarks, it has a unique solution in $\mathscr{L}^{(+)}$, which turns out to be the ψ-function $\psi(x)$, the logarithmic derivative of the gamma function. Thus, the theory of the ψ-function can be developed purely formally.

8.1.1. *Lower Factorial*

Other than the Harmonic logarithms—which we have already discussed at great length—our first example of a Roman graded sequence is the logarithmic lower factorial.

DEFINITION 8.1.1 (Forward Difference Operator). Define the *forward difference operator* $\Delta = E - \mathbf{I} = e^{\mathbf{D}} - \mathbf{I}$. Let $(x)_n^{(t)}$ denote its associated graded sequence; it will be called the *logarithmic lower factorial graded sequence*. Let $\phi_n^{(t)}(x)$ denote its conjugate graded sequence; it will be called the *logarithmic exponential graded sequence.*

Now, $\Delta' = E$, so by Theorem 7.2.8, we calculate the residual series

$$(x)_{-1}^{(1)} = Ex^{-1} = \frac{1}{x+1}. \tag{44}$$

In general,

TABLE 8.2

Logarithmic Lower Factorials, $(x)_n^{(l)}$

$(x)_2^{(0)} = x(x-1)$	$(x)_2^{(1)} = \lambda_2^{(1)}(x) - \lambda_1^{(1)}(x) + \frac{B_3^{(3)}}{3} x^{-1} - \frac{B_4^{(3)}}{12} x^{-2} + \cdots$
$(x)_1^{(0)} = x$	$(x)_1^{(1)} = x \log(x) - x + \frac{B_2^{(2)}}{2} x^{-1} - \frac{B_3^{(2)}}{6} x^{-2} + \cdots$
$(x)_0^{(0)} = 1$	$(x)_0^{(1)} = \log(x+1) + \frac{B_1}{1+x} - \frac{B_2}{2(1+x)^2} + \frac{B_3}{3(1+x)^3} - \cdots$
	$(x)_{-1}^{(1)} = \frac{1}{x+1}$
	$(x)_{-2}^{(1)} = \frac{1}{(x+1)(x+2)}$

PROPOSITION 8.1.2. *For n positive,*

$$(x)_{-n}^{(1)} = \frac{1}{(x+1)\cdots(x+n)} = (x)_{-n}.$$

(*Recall Definition* 3.3.1.)

Proof. We will proceed by induction. The case $n=1$ amounts to Eq. (44), which we verified above. Now, suppose proposition hold for n then

$$\begin{aligned}(x)_{-n-1}^{(1)} &= \lfloor -n \rceil^{-1} \Delta (x)_{-n}^{(1)} \\ &= \frac{1}{n}\left(\frac{1}{(x+1)\cdots(x+n)} - \frac{1}{(x+2)\cdots(x+n+1)}\right) \\ &= \frac{1}{(x+1)\cdots(x+n+1)}. \quad \blacksquare\end{aligned}$$

Similarly, we have the classical result (page 133 of "The Umbral Calculus")

PROPOSITION 8.1.3. *For n nonnegative,* $(x)_n^{(0)} = x(x-1)\cdots(x-n+1) = (x)_n$.

Now, we will determine the series $(x)_0^{(1)}$. Again by Theorem 7.2.8, we have

$$(x)_0^{(1)} = E\frac{\mathbf{D}}{\Delta} \log x$$

$$\int_x^{x+1} (t)_0^{(1)}\, dt = \log(x+1).$$

Now,

$$\frac{\mathbf{D}}{\Delta} = \sum_{k \geqslant 0} \frac{B_k}{k!} \mathbf{D}^k \tag{45}$$

by the Euler–MacLaurin formula). Thus,

$$\begin{aligned}(x)_0^{(1)} &= \sum_{k \geqslant 0} \frac{B_k}{k!} \mathbf{D}^k \log(x+1) \\ &= \log(x+1) + \frac{B_1}{1+x} - \frac{B_2}{2(1+x)^2} + \frac{B_3}{3(1+x)^3} - \cdots.\end{aligned} \tag{46}$$

Thus, we find that $(x)_0^{(1)} = \psi(x+1)$ coincides with the classical ψ-function (the logarithmic derivative of the gamma function) introduced by Gauss. Similarly, one finds that $(x)_1^{(1)}$ and $(x)_2^{(1)}$ coincide with the digamma and trigamma functions.

The classical expansion

$$\left(\frac{D}{\Delta}\right)^u = \sum_{k \geqslant 0} \frac{B_k^{(u)}}{k!} \mathbf{D}^k$$

defines the graded sequence $B_k^{(u)}$ of *Bernoulli numbers* of *order k*. In terms of these higher-order Bernoulli numbers, we obtain for $n \geqslant 0$:

$$\begin{aligned}(x)_n^{(t)} &= \left(\frac{\mathbf{D}}{\Delta}\right)^{1+n} E\lambda_n^{(t)}(x) \\ &= \sum_{k \geqslant 0} \frac{B_k^{(n+1)}}{k!} \mathbf{D}^k \lambda_n^{(t)}(x+1) \\ &= \sum_{k \geqslant 0} B_k^{(n+1)} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(x+1).\end{aligned}$$

Let us calculate $(x)_n^{(2)}$ for $n < 0$. To begin with, by Theorem 7.2.8, we can calculate the residual series of order 2:

$$\begin{aligned}(x)_{-1}^{(2)} &= E(2x^{-1} \log x) \\ &= \frac{2\log(x+1)}{x+1}.\end{aligned}$$

Continuing in this way,

$$\begin{aligned}(x)_{-2}^{(2)} &= -\Delta(x)_{-1}^{(2)}\\ &= (x)_{-1}^{(2)} - (x+1)_{-1}^{(2)}\\ &= 2\left(\frac{\log(x+1)}{x+1} - \frac{\log(x+2)}{x+2}\right)\\ &= 2\left(\frac{\log(x+1) - (x+1)\log((x+2)/(x+1))}{(x+1)(x+2)}\right),\end{aligned}$$

and

$$\begin{aligned}(x)_{-3}^{(2)} &= -\frac{1}{2}\Delta(x)_{-2}^{(2)}\\ &= \frac{\log(x+1) - (x+1)\log((x+2)/(x+1))}{(x+1)(x+2)}\\ &\quad - \frac{\log(x+2) - (x+2)\log((x+3)/(x+2))}{(x+2)(x+3)}\\ &= \frac{2\log(x+1)}{(x+1)(x+2)(x+3)} + \frac{\log((x+3)/(x+1))}{(x+3)}.\end{aligned}$$

In general for n positive,

$$(x)_{-n}^{(2)} = \frac{2\log(x+1)}{(x+1)\cdots(x+n)} + \frac{2\log((x+n)/(x+1)}{(n-1)!\,(x+n)}.$$

The identity

$$(x+a)_0^{(1)} = \sum_{k\geqslant 0} \left\lfloor \begin{matrix} 0 \\ k \end{matrix} \right\rceil (a)_k\,(x)_{-k}^{(1)} \tag{47}$$

gives a classical identity satisfied by the ψ-function, that is,

$$\psi(x+a+1) = \psi(x+1) + \sum_{k\geqslant 0} \frac{(-1)^{k+1}\,a(a-1)\cdots(a-k+1)}{k(x+1)(x+2)\cdots(x+k)}. \tag{48}$$

Similar identities can be obtained for the digamma and trigamma functions.

Theorem 6.2.5 gives the following generalization of Newton's expansion:

PROPOSITION 8.1.4. *Every formal power series of logarithmic type $p(x)$ can be uniquely expanded as a convergent series*

$$p(x) = \sum_{n,t} \frac{a_n^{(t)}}{\lfloor n \rceil!}\,(x)_n^{(t)},$$

where the coefficients $a_n^{(t)}$ *are given by*

$$a_n^{(t)} = \langle \Delta^n p(x) \rangle_t.$$

For example,

$$\frac{1}{x} = \sum_{k \geq 0} \frac{1}{k!} \frac{1}{(x+1)\cdots(x+k+1)}. \tag{49}$$

We digress to indicate the meaning of such equalities. Formally, we merely mean that when both sides of the equality are expanded in terms of harmonic logarithms $\lambda_n^{(t)}(x)$ the resulting coefficients will be identical. However, because of such results as Theorem 5.6.2, we are allowed to make computations in the real or complex numbers, and thus obtain "asymptotic" expansions. In the example above, if $x = 100$, then the right side and left side are both approximately 0.01. In fact, the error is about 0.0000015 when you compute 20 or more terms of the summation. If $x = 69$, and you compute the first 14 terms of the summation, you find the left side is 0.14488 and the right side is 0.14493.

8.1.2. *Upper Factorial*

DEFINITION 8.1.5 (Backward Difference Operator). Define the *backward difference operator* $\nabla = \mathbf{I} - E^{-1} = \mathbf{I} - e^{-\mathbf{D}}$. Let $\langle x \rangle_n^{(t)}$ denote its associated graded sequence; it will be called the *logarithmic upper factorial graded sequence.*

As before, the residual series is given by

$$\langle x \rangle_{-1}^{(1)} = E^{-1} x^{-1} = \frac{1}{x-1},$$

TABLE 8.3

Logarithmic Upper Factorials, $\langle x \rangle_n^{(t)}$

$\langle x \rangle_2^{(0)} = x(x+1)$	$\langle x \rangle_2^{(1)} = \lambda_2^{(1)}(x) + \lambda_1^{(1)}(x) - \frac{B_3^{(3)}}{3} x^{-1} - \frac{B_4^{(3)}}{12} x^{-2} - \cdots$
$\langle x \rangle_1^{(0)} = x$	$\langle x \rangle_1^{(1)} = -x \log(x) + x - \frac{B_2^{(2)}}{2} x^{-1} - \frac{B_3^{(2)}}{6} x^{-2} - \cdots$
$\langle x \rangle_0^{(0)} = 1$	$\langle x \rangle_0^{(1)} = \log(x-1) - \frac{B_1}{x-1} + \frac{B_2}{2(x-1)^2} - \frac{B_3}{3(x-1)^3} + \cdots$
	$\langle x \rangle_{-1}^{(1)} = \frac{1}{x-1}$
	$\langle x \rangle_{-2}^{(1)} = \frac{1}{(x-1)(x-2)}$

and in general for n positive

$$\langle x\rangle_{-n}^{(1)}=\frac{1}{(x-1)\cdots(x-n)}.$$

Similarly, for n nonnegative, we have the classical result (page 134 of "The Umbral Calculus")

$$\langle x\rangle_{n}^{(0)}=x(x+1)\cdots(x+n-1).$$

For $n=0$ we have:

PROPOSITION 8.1.6.

$$\langle x\rangle_{0}^{(1)}=\log(x-1)-\frac{B_1}{x-1}+\frac{B_2}{2(x-1)^2}-\frac{B_3}{3(x-1)^3}+\cdots.$$

Proof. By Theorem 7.2.8, we have

$$\langle x\rangle_{0}^{(1)}=E^{-1}\frac{\mathbf{D}}{\nabla}\log x.$$

From the Euler–MacLaurin formula, namely from

$$\begin{aligned}\frac{\mathbf{D}}{\nabla}&=\frac{\mathbf{D}}{\mathbf{I}-e^{-\mathbf{D}}}\\&=\frac{-\mathbf{D}}{e^{-\mathbf{D}}-\mathbf{I}}\\&=\sum_{k\geqslant 0}\frac{B_k}{k!}(-\mathbf{D})^k\\&=\sum_{k\geqslant 0}(-1)^k\frac{B_k}{k!}\mathbf{D}^k,\end{aligned}$$

we infer

$$\begin{aligned}\langle x\rangle_{0}^{(1)}&=\sum_{k\geqslant 0}(-1)^k\frac{B_k}{k!}\mathbf{D}^k\log(x+1)\\&=\log(x-1)-\frac{B_1}{x-1}+\frac{B_2}{2(x-1)^2}-\frac{B_3}{3(x-1)^3}+\cdots.\end{aligned}$$

We have, in terms of the Bernoulli numbers of higher order,

$$\left(\frac{\mathbf{D}}{\nabla}\right)^u=\sum_{k\geqslant 0}(-1)^k\frac{B_k^{(u)}}{k!}\mathbf{D}^k.$$

Hence, for $n \geqslant 0$,

$$\begin{aligned}\langle x\rangle_n^{(t)} &= \left(\frac{\mathbf{D}}{\nabla}\right)^{1+n} E^{-1}\lambda_n^{(t)}(x)\\ &= \sum_{k\geqslant 0} (-1)^k \frac{B_k^{(n+1)}}{k!} \mathbf{D}^k \lambda_n^{(t)}(x-1)\\ &= \sum_{k\geqslant 0} (-1)^k B_k^{(n+1)} \left\lceil \begin{matrix} n\\ k\end{matrix} \right\rfloor \lambda_{n-k}^{(t)}(x-1).\end{aligned}$$

As with the upper factorial graded sequence, we compute $\langle x\rangle_n^{(2)}$ as follows starting with the residual series of order 2.

$$\begin{aligned}\langle x\rangle_{-1}^{(2)} &= E^{-1}(2x^{-1}\log x)\\ &= \frac{2\log(x-1)}{x-1}\\ \langle x\rangle_{-2}^{(2)} &= -\nabla\langle x\rangle_{-1}^{(2)}\\ &= 2\left(\frac{\log(x-2)}{x-2} - \frac{\log(x-1)}{x-1}\right)\\ &= 2\,\frac{\log(x-2)-(x-2)\log((x-1)/(x-2))}{(x-1)(x-2)},\end{aligned}$$

and in general

$$\langle x\rangle_{-n}^{(2)} = \frac{2\log(x-n)}{(x-1)\cdots(x-n)} + \frac{2\log((x-1)/(x-n))}{(n-1)!\,(x-n)}.$$

8.1.3. *Abel*

The logarithmic extension of the Abel polynomials turns out to be surprisingly pleasing.

DEFINITION 8.1.7 (Abel Operator). Define the *Abel operator* $A_a = \mathbf{D}E^a$. Its associated graded sequence $A_n^{(t)}(x)$ will be called the *logarithmic Abel graded sequence*. Its conjugate graded sequence $\mu_n^{(t)}(x)$ will be called the *logarithmic inverse Abel graded sequence.*

By Corollary 7.2.11, for $n \neq 0$, 1, we obtain

$$\begin{aligned}A_n^{(t)}(x) &= \sigma E^{-na}\lambda_{n-1}^{(t)}(x)\\ &= \sigma \sum_{k\geqslant 0} \left\lceil \begin{matrix} n-1\\ k\end{matrix} \right\rfloor (-na)^k \lambda_{n-k-1}^{(t)}(x)\\ &= \sum_{\substack{k\leqslant n\\ k\neq 0}} \left\lceil \begin{matrix} n-1\\ k\end{matrix} \right\rfloor (-na)^k \lambda_k^{(t)}(x).\end{aligned}$$

TABLE 8.4

Logarithmic Abel Graded Sequence, $A_n^{(t)}(x)$

$A_2^{(0)}(x)=x(x-2a)$	$A_2^{(1)}(x)=\sigma\lambda_1^{(1)}(x-2a)$
$A_1^{(0)}(x)=x$	$A_1^{(1)}(x)=\sigma\log(x-a)$
$A_0^{(0)}(x)=1$	$A_0^{(1)}(x)=\log(x)+\frac{a}{x}$
	$A_{-1}^{(1)}(x)=x(x+a)^{-2}$
	$A_{-2}^{(1)}(x)=x(x+2a)^{-3}$

In particular, for n positive, we obtain the classical Abel polynomials:

$$A_n^{(0)}(x)=x(x-na)^{n-1}, \tag{50}$$

and for n negative, we still have

$$A_n^{(1)}(x)=x(x-na)^{n-1}. \tag{51}$$

Thus, the residual series is

$$A_{-1}^{(1)}=\frac{x}{(x+a)^2}.$$

The series of degree zero is computed via Theorem 7.2.8. It turns out to be very simple:

$$\begin{aligned}A_0^{(1)}(x)&=A'_aE^{-a}\log x\\&=(\mathbf{I}+a\mathbf{D})\log x\\&=\log x+a/x.\end{aligned} \tag{52}$$

From the logarithmic binomial identity

$$A_0^{(1)}(x+b)=\sum_{k\geqslant 0}\left\lceil\begin{matrix}0\\k\end{matrix}\right\rfloor A_k^{(0)}(b)\,A_{-k}^{(1)}(x),$$

we infer the remarkable identity (which we believe to be new)

$$\frac{a}{x+b}+\log(x+b)=\frac{a}{x}+\log x+\sum_{k\geqslant 1}\frac{(-1)^{k+1}\,b(b-ak)^{k-1}\,x}{k(x+ak)^{k+1}}. \tag{53}$$

For example, in the real numbers we can substitute here the values $a=1$, $b=2$, and $x=5$. If we compute the first 12 terms of the series, the left-hand and right-hand sides are both approximately 2.0887673.

In general, by Theorem 7.2.8,

$$\begin{aligned} A_n^{(t)}(x) &= E^{-na}(\mathbf{I} + a\mathbf{D})\,\lambda_n^{(t)}(x) \\ &= \lambda_n^{(t)}(x - na) + a\lfloor n \rceil\, \lambda_{n-1}^{(t)}(x - na). \end{aligned} \tag{54}$$

For example,

$$A_1^{(1)}(x) = x \log(x - a) + a - x.$$

Again, by Theorem 6.2.5 every formal power series of logarithmic type can be expanded in terms of Abel series:

$$p(x) = \sum_{n,t} \frac{a_n^{(t)}}{\lfloor n \rceil !} A_n^{(t)}(x),$$

where

$$a_n^{(t)} = \langle E^{na}\, \mathbf{D}^n p(x) \rangle_t.$$

For example,

$$\begin{aligned} \log x &= \sum_{k \leqslant 0} (ka)^{-k} \begin{bmatrix} 0 \\ k \end{bmatrix} A_k^{(1)}(x) \\ &= A_0^{(1)}(x) + aA_{-1}^{(1)}(x) - 2a^2 A_{-2}^{(1)}(x) + 9a^3 A_{-3}^{(1)}(x) - \cdots \\ &= \log x + \frac{a}{x} - \sum_{k > 0} \left(\frac{-k}{x + ka} \right)^{k+1} a^k x. \end{aligned} \tag{55}$$

That is,

$$x^{-2} = \sum_{k > 0} \left(\frac{-ka}{x + ka} \right)^{k+1}.$$

8.1.4. *Gould*

DEFINITION 8.1.8 (Logarithmic Gould Graded Sequence). The *logarithmic Gould graded sequence* $G_n^{(t)}(x)$ is the Roman graded sequence associated with the delta operator $E^a \Delta = E^{a+1} - E^a$.

The Pincherle derivative of $E^a \Delta$ is $(a+1)E^{a+1} - aE^a$, so the residual series is given by

$$\begin{aligned} G_{-1}^{(1)}(x) &= ((a+1)E^{a+1} - aE^a)\frac{1}{x} \\ &= \frac{a+1}{x + a + 1} - \frac{a}{x + a} \\ &= \frac{x}{(x + a)(x + a + 1)}. \end{aligned}$$

TABLE 8.5

Logarithmic Gould Graded Sequence, $G_n^{(t)}(x)$

$$G_2^{(0)}(x) = x(x-2a-1)$$

$$G_1^{(0)}(x) = x$$

$$G_0^{(0)}(x) = 1$$

$$G_2^{(1)}(x) = \langle x\rangle_2^{(1)} - a\langle x\rangle_1^{(1)} + \sum_{k\geq 2} (-1)^{n+k} \times \binom{na}{k} \frac{\lfloor n-k-1\rceil!}{\lfloor n-1\rceil!\,(x-1)\cdots(x-n+k)}$$

$$G_1^{(1)}(x) = x\log(x) - x + \sum_{k\geq 2} B_k^{(2)} \left\lfloor \begin{matrix} 2 \\ k \end{matrix} \right\rceil x^{1-k} + \sum_{k\geq 2} \left(\binom{a}{k} + a(k+1)\binom{a-1}{k-1}\right) \frac{(-1)^k}{(k-1)!\,(x-1)\cdots(x-k+1)}$$

$$G_0^{(1)}(x) = \log(x+1) + \frac{B_1}{x+1} - \frac{2!\,B_2}{(x+1)^2} + \cdots + \frac{a}{x+1} - \frac{a}{(x+1)(x+2)} + \frac{a}{2!\,(x+1)(x+2)(x+3)} + \cdots$$

$$G_{-1}^{(1)}(x) = \frac{x}{(x+a)(x+a+1)}$$

$$G_{-2}^{(1)}(x) = \frac{x}{(x+2a)(x+2a+1)(x+2a+2)}$$

Since Roman graded sequences are basic, we have

$$\begin{aligned} G_{-2}^{(1)}(x) &= -E^a\,\Delta G_{-1}^{(1)}(x) \\ &= E^a\left(\frac{x}{(x+a)(x+a+1)} - \frac{x+1}{(x+a+1)(x+a+2)}\right) \\ &= \frac{x}{(x+2a)(x+2a+1)(x+2a+2)} \\ &= \frac{x}{x+2a}\,(x+2a)_{-2}. \end{aligned}$$

Similarly, by induction we have for n positive

$$G_{-n}^{(1)}(x) = x(x+na-1)_{-n-1},$$

and by induction for n nonnegative

$$G_n^{(0)}(x) = x(x - na - 1)_{n-1}.$$

See Section 8.2.6 for an explicit computation of $G_n^{(1)}(x)$ for $n \geq 0$.

8.1.5. *Lageurre*

Our final example of a Roman graded sequence is the logarithmic Laguerre graded sequence.

Definition 8.1.9 (Laguerre Operator). Define the *Laguerre operator* $K = \mathbf{D}/(\mathbf{D} - \mathbf{I})$. Define the *logarithmic Laguerre graded sequence* to be its associated graded sequence denoted $L_n^{(t)}(x)$.

Now, the Pincherle derivative of the Laguerre operator is given by $K' = -(\mathbf{D} - \mathbf{I})^{-2}$, so by Theorem 7.2.8,

$$\begin{aligned} L_n^{(t)}(x) &= -(\mathbf{D} - \mathbf{I})^{-2}(\mathbf{D} - \mathbf{I})^{n+1}\lambda_n^{(t)}(x) \\ &= -(\mathbf{D} - \mathbf{I})^{n-1}\lambda_n^{(t)}(x). \end{aligned}$$

Hence, for n positive,

$$\begin{aligned} L_n^{(t)}(x) &= -\left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \mathbf{D}^{n-k-1}\right)\lambda_n^{(t)}(x) \\ &= \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \mathbf{D}^{n-k}\lambda_n^{(t)}(x) \\ &= \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{n!}{k!}\lambda_k^{(t)}(x). \end{aligned}$$

TABLE 8.6

Logarithmic Laguerre Graded Sequence, $L_n^{(t)}(x)$

$L_2^{(0)}(x) = x^2 - 2x$	$L_2^{(1)}(x) = \lambda_2^{(1)}(x) - 2\lambda_1^{(1)}(x)$
$L_1^{(0)}(x) = -x$	$L_1^{(1)}(x) = -\lambda_1^{(1)}(x)$
$L_0^{(0)}(x) = 1$	$L_0^{(1)}(x) = \log(x) - \sum_{k \geq 2} (-1)^k (k-2)!\, x^{1-k}$
	$L_{-1}^{(1)}(x) = \sum_{k \geq 1} (-1)^k k!\, x^{-k}$
	$L_{-2}^{(1)}(x) = -\sum_{k \geq 2} (-1)^k (k-1)k!/2x^k$

Observe the remarkable fact that $L_n^{(t)}(x)$ does not contain any terms of negative degree. In particular,

$$L_1^{(t)}(x) = -\lambda_1^{(t)}(x)$$

from which we can compute

$$\begin{aligned} L_0^{(t)}(x) &= KL_1^{(t)}(x) \\ &= (\mathbf{D} + \mathbf{D}^2 + \mathbf{D}^3 + \cdots)\, \lambda_1^{(t)}(x) \\ &= \sum_{k \geqslant 1} \lfloor 1-k \rceil!^{-1}\, \lambda_{1-k}^{(t)}(x) \\ &= \log x + \sum_{k<0} (-1)^{k-1} (-k-1)!\, \lambda_k^{(t)}(x). \end{aligned}$$

In particular, we obtain the expression

$$L_0^{(1)}(x) = \log(x) + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots.$$

To calculate the negative terms, we proceed as follows. Let n be positive, then

$$\begin{aligned} L_{-n}^{(t)}(x) &= \lfloor -n \rceil!^{-1}\, K^{n+1} L_1^{(t)}(x) \\ &= -\lfloor -n \rceil!\, (-1)^{n+1} (\mathbf{D} + \mathbf{D}^2 + \cdots)^{n+1}\, \lambda_1^{(t)}(x) \\ &= \lfloor -n \rceil!\, (-1)^n \left(\sum_{k \geqslant 0} \binom{-n-1}{k} \mathbf{D}^{k+n+1} \right) \lambda_1^{(t)}(x) \\ &= \sum_{k \geqslant 0} (-1)^n \binom{-n-1}{k} \frac{\lfloor -n \rceil!}{\lfloor -n-k \rceil!}\, \lambda_{-n-k}^{(t)}(x) \\ &= \sum_{k \geqslant 0} (-1)^{n+k} \binom{-n-1}{k} \frac{(n+k-1)!}{(n-1)!}\, \lambda_{-n-k}^{(t)}(x). \end{aligned}$$

Finally, note the following amazing fact.

THEOREM 8.1.10. *For any finite linear combination of harmonic logarithms $p(x)$, and any $a \in K$,*

$$(e^{-a\sigma}\, \mathbf{D} e^{a\sigma})\, p(x) = (\mathbf{D} - \alpha \mathbf{I})\, p(x).$$

By $e^{-a\sigma}\, \mathbf{D} e^{a\sigma} p(x)$ we merely mean to indicate the expression

$$\sum_{i \geqslant 0} \sum_{j \geqslant 0} \frac{(-1)^i a^{i+j}}{i!\, j!}\, \sigma^i\, \mathbf{D} \sigma^j p(x),$$

which we assert does in fact converge to the indicated value. The "operator" $e^{a\sigma}$ cannot be extended to all of $\mathscr{L}$.

The corresponding classical identity

$$e^{-x}\,\mathbf{D}e^{x} = \mathbf{D} + \mathbf{I}$$

is associated with the classical identity

$$\mathbf{Dx} - \mathbf{xD} = \mathbf{I},$$

which corresponds to the logarithmic identity

$$\mathbf{D}\sigma - \sigma\mathbf{D} = \mathbf{D}' = \mathbf{I}.$$

Proof of Theorem 8.1.10. By linearity, it will suffice to consider the case $p(x) = \lambda_n^{(t)}(x)$.

(n positive) Classical proof can be applied *mutatis mutandis.*

($n = 0$) We calculate

$$\begin{aligned}
e^{-a\sigma}\,\mathbf{D}e^{a\sigma}\lambda_0^{(t)}(x) &= \sum_{i\geqslant 0}\sum_{j\geqslant 0}\frac{(-1)^i a^{i+j}}{i!\,j!}\sigma^i\,\mathbf{D}\sigma^j\lambda_0^{(t)}(x)\\
&= \lambda_{-1}^{(t)}(x) - a\sum_{i\geqslant 0}\sum_{j\geqslant 0}\frac{(-1)^i a^{i+j}}{i!\,(j+1)!}\sigma^i\,\mathbf{D}\sigma^j\lambda_1^{(t)}(x)\\
&= \lambda_{-1}^{(t)}(x) - a\sum_{i\geqslant 0}\sum_{j\geqslant 0}\frac{(-1)^i a^{i+j}}{i!\,j!}\sigma^i\lambda_j^{(t)}(x)\\
&= \lambda_{-1}^{(t)}(x) - a\sum_{k\geqslant 0}\frac{a^k}{k!}\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\right)\lambda_k^{(t)}(x)\\
&= \lambda_{-1}^{(t)}(x) - a\lambda_0^{(t)}(x)\\
&= (\mathbf{D} - a\mathbf{I})\,\lambda_0^{(t)}(x).
\end{aligned}$$

(n negative) Similarly, we compute

$$\begin{aligned}
e^{-a\sigma}\,\mathbf{D}e^{a\sigma}\lambda_n^{(t)}(x) &= \sum_{i\geqslant 0}\sum_{j\geqslant 0}\frac{(-1)^i a^{i+j}}{i!\,j!}\sigma^i\,\mathbf{D}\sigma^j\lambda_n^{(t)}(x)\\
&= \sum_{k\geqslant 0}\sum_{i=0}^{k}\frac{(-1)^i a^k}{i!\,(k-i)!}\sigma^i\,\mathbf{D}\sigma^{k-i}\lambda_n^{(t)}(x)\\
&= \sum_{k=0}^{-n-1}\sum_{i=0}^{k}\frac{(-1)^i a^k(n+k-i)}{i!\,(k-i)!}\lambda_{n+k-1}^{(t)}(x)\\
&= n\lambda_{n-1}^{(t)}(x) + a\lambda_n^{(t)}(x)\\
&= (\mathbf{D} - a\mathbf{I})\,\lambda_0^{(t)}(x). \quad \blacksquare
\end{aligned}$$

From Theorem 8.1.10, we derive a logarithmic extension of the classical Rodrigues' formula for Laguerre polynomials:

$$L_n^{(t)}(x) = -e^{\sigma}\,\mathbf{D}^{n-1} e^{-\sigma} \lambda_n^{(t)}(x).$$

8.2. *Connection Constants*

Given two graded sequences $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$, we would like to express one in terms of the other:

$$p_n^{(t)}(x) = \sum_{k \leqslant n} a_{nk} q_k^{(t)}(x).$$

The constants a_{nk} are called the *connection constants* from $q_n^{(t)}(x)$ to $p_n^{(t)}(x)$.

If $p_n^{(t)}(x)$ and $q_n^{(t)}(x)$ are the Roman graded sequences associated with the delta operators $f(\mathbf{D})$ and $g(\mathbf{D})$, respectively, then by Proposition 7.1.7,

$$r_n^{(t)}(x) = \sum_{k \leqslant n} a_{nk} \lambda_n^{(t)}(x)$$

is the Roman graded sequence associated with $g(f^{(-1)})$, where $f^{(-1)}(\mathbf{D})$ is the compositional inverse of $f(\mathbf{D})$. Thus, to determine the connection constants it will suffice to calculate $r_n^{(t)}(x)$. This easy device for the computation of connection constants is the most effective application of the present theory.

8.2.1. *Upper Factorial to Lower Factorial*

To express $(x)_n^{(t)}$ in terms of $\langle x \rangle_n^{(t)}$ we first calculate

$$\begin{aligned}
\Delta(\nabla^{(-1)}) &= e^{-\log(\mathbf{I}-\mathbf{D})} - \mathbf{I} \\
&= \frac{\mathbf{I}}{\mathbf{I}-\mathbf{D}} - \mathbf{I} \\
&= \frac{\mathbf{D}}{\mathbf{I}-\mathbf{D}} \\
&= -K(\mathbf{D}),
\end{aligned}$$

where $K(\mathbf{D})$ is the Laguerre operator (Definition 8.1.9). Thus, $r_n^{(t)}(x)$ is very simply related to the logarithmic Laguerre graded sequence,

$$r_n^{(t)}(x) = ML_n^{(t)}(x),$$

where $M\lambda_n^{(t)}(x) = (-1)^n \lambda_n^{(t)}(x)$. Note that $Mx^n = (-x)^n$, so $r_n^{(0)}(x) = L_n^{(0)}(-x)$, and for $n < 0$, $r_n^{(1)}(x) = L_n^{(1)}(-x)$.

We now apply the results from Section 8.1.5.

Thus, for series of positive degree

$$(x)_n^{(t)} = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} \langle x \rangle_k^{(t)},$$

TABLE 8.7

Lower Factorial in Terms of Upper Factorials

$(x)_2^{(0)} = \langle x\rangle_2^{(0)} + 2\langle x\rangle_1^{(0)}$	$(x)_2^{(1)} = \langle x\rangle_2^{(1)}(x) + 2\langle x\rangle_1^{(1)}$
$(x)_1^{(0)} = \langle x\rangle_1^{(0)}$	$(x)_1^{(1)} = \langle x\rangle_1^{(1)}$
$(x)_0^{(0)} = \langle x\rangle_0^{(0)}$	$(x)_0^{(1)} = \langle x\rangle_0^{(1)} - \sum_{k\geq 2} (k-2)!\,\langle x\rangle_{1-k}^{(1)}$
	$(x)_{-1}^{(1)} = -\sum_{k\geq 1} k!\,\langle x\rangle_{-k}^{(1)}$
	$(x)_{-2}^{(1)} = \sum_{k\geq 2} (k-1)k!\,\langle x\rangle_{-k}^{(1)}/2$

for $n=0$,

$$(x)_0^{(t)} = \langle x\rangle_0^{(t)} - \sum_{k\geq 2} (k-2)!\,\langle x\rangle_{1-k}^{(t)},$$

and for series of negative degree

$$(x)_n^{(t)} = \sum_{k\leq 0} \binom{-k}{n} \frac{(-k-1)!}{(-n-1)!} \langle x\rangle_{-k}^{(t)}.$$

Similarly, from Theorem 8.1.10, we have identities like

$$(x)_n^{(t)} = -e^{\sigma\nabla} \nabla^{n-1} e^{-\sigma\nabla} \langle x\rangle_n^{(t)}.$$

8.2.2. *Lower Factorial to Upper Factorial*

Conversely, to compute $\langle x\rangle_n^{(t)}$ in terms of $(x)_n^{(t)}$, we need the delta operator

$$\nabla(\Delta^{(-1)}) = \frac{\mathbf{D}}{\mathbf{D}+\mathbf{I}} = K(-\mathbf{D}).$$

Hence, the connection constants from $(x)_n^{(t)}$ to $\langle x\rangle_n^{(t)}$ are given by the coefficients of $(-1)^n L_n^{(t)}(x)$. Hence, for $n>0$,

$$\langle x\rangle_n^{(t)} = \sum_{k=1}^{n} (-1)^{n+k} \binom{n-1}{k-1} \frac{n!}{k!} (x)_k^{(t)}$$

$$\langle x\rangle_0^{(t)} = (x)_0^{(t)} + \sum_{k\geq 2} (-1)^k (k-2)!\,(x)_{1-k}^{(t)}$$

$$\langle x\rangle_{-n}^{(t)} = \sum_{k\geq 0} (-1)^{n+k} \binom{-n-1}{k} \frac{(n+k-1)!}{(n-1)!} (x)_{-n-k}^{(t)}. \tag{56}$$

TABLE 8.8

Upper Factorial in Terms of Lower Factorial

$\langle x\rangle_2^{(0)}=(x)_2^{(0)}-2(x)_1^{(0)}$	$\langle x\rangle_2^{(1)}=(x)_2^{(1)}\langle x\rangle-2(x)_1^{(1)}$
$\langle x\rangle_1^{(0)}=(x)_1^{(0)}$	$\langle x\rangle_1^{(1)}=(x)_1^{(1)}$
$\langle x\rangle_0^{(0)}=(x)_0^{(0)}$	$\langle x\rangle_0^{(1)}=(x)_0^{(1)}-\sum_{k\geqslant 2}(-1)^k(k-2)!\,(x)_{1-k}^{(1)}$
	$\langle x\rangle_{-1}^{(1)}=-\sum_{k\geqslant 1}(k)!\,(x)_{-k}^{(1)}$
	$\langle x\rangle_{-2}^{(1)}=\sum_{k\geqslant 2}(-1)^k\frac{k!\,k(k-1)}{2}(x)_{-k}^{(1)}$

8.2.3. *Laguerre to Harmonic*

If $p_n^{(t)}(x)$ is the Roman graded sequence associated with the delta operator $f(\mathbf{D})$, then finding the connection constants from $p_n^{(t)}(x)$ to $\lambda_n^{(t)}(x)$ is tantamount to finding the graded sequence associated with $f^{(-1)}(\mathbf{D})$, that is, the conjugate graded sequence for $f(\mathbf{D})$.

Note that since $x/(x-1)$ is the compositional inverse of itself, the logarithmic Laguerre graded sequence is self-conjugate, so we have

$$\lambda_n^{(t)}(x)=\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}L_k^{(t)}(x)$$

$$\lambda_0^{(t)}(x)=L_0^{(t)}(x)+\sum_{k<0}(-1)^{k+1}(-k-1)!\,L_k^{(t)}(x)$$

$$\lambda_{-n}^{(t)}(x)=\sum_{k\geqslant 0}(-1)^{n+k}\binom{-n-1}{k}\frac{(n+k-1)!}{(n-1)!}L_{-n-k}^{(t)}(x),$$

and from Theorem 8.1.10,

$$\lambda_n^{(t)}(x)=L_n^{(t)}(\mathbf{L})=-e^{\sigma\mathbf{L}}K^{n-1}e^{-\sigma\mathbf{L}}L_n^{(t)}(x).$$

8.2.4. *Lower Factorial to Harmonic*

Similarly, the connection constants from $(x)_n^{(t)}$ to $\lambda_n^{(t)}(x)$ are given by $\phi_n^{(t)}(x)$—the logarithmic exponential graded sequence—which we will now compute.

The relevant delta operator is

$$\log(\mathbf{I}+\mathbf{D})=\sum_{k>0}(-1)^{k+1}\frac{\mathbf{D}^k}{k}.$$

By Theorem 7.2.8, we can calculate the residual series:

$$\phi_{-1}^{(t)}(x) = \frac{\mathbf{I}}{\mathbf{I}+\mathbf{D}} \lambda_{-1}^{(t)}(x)$$

$$= \sum_{k \geqslant 0} (-1)^k \mathbf{D}^k \lambda_{-1}^{(t)}(x)$$

$$= \sum_{k \geqslant 0} (-1)^k \lfloor -k-1 \rceil !^{-1} \lambda_{-1-k}^{(t)}(x)$$

$$= \sum_{k \geqslant 0} k! \, \lambda_{-1-k}^{(t)}(x).$$

By Theorems 7.2.6 and 8.1.10, for $n \neq -1$, we obtain the recursion formula

$$\phi_{n+1}^{(t)}(x) = \sigma(\mathbf{I}+\mathbf{D}) \phi_n^{(t)}(x) = \sigma e^{-\sigma} \mathbf{D} e^{\sigma} \phi_n^{(t)}(x).$$

8.2.5. *Abel to Harmonic*

The inverse of the Abel operator $A_a^{(-1)}$ is not easily calculated. Nevertheless, we may calculate the logarithmic inverse Abel graded sequence, using conjugate graded sequences.

Thus, by Definition 6.4.1,

$$\mu_n^{(t)}(x) = \sum_{k \leqslant n} \frac{\langle \mathbf{D}^k E^{ka} \lambda_n^{(t)}(x) \rangle_t}{\lfloor k \rceil !} \lambda_k^{(t)}(x).$$

Now,

$$\mathbf{D}^k E^{ka} \lambda_n^{(t)}(x) = \mathbf{D}^k \sum_{j \geqslant 0} \begin{bmatrix} n \\ j \end{bmatrix} (ka)^j \lambda_{n-j}^{(t)}(x)$$

$$= \sum_{j \geqslant 0} (ka)^j \frac{\lfloor n \rceil !}{j! \lfloor n-j-k \rceil !} \lambda_{n-j-k}^{(t)}(x).$$

Thus,

$$\mu_n^{(t)}(x) = \sum_{k \leqslant n} \frac{(ka)^{n-k} (\lfloor n \rceil ! / (n-k)! \, 0!)}{\lfloor k \rceil !} \lambda_k^{(t)}(x)$$

$$= \sum_{k \leqslant n} (ka)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \lambda_k^{(t)}(x).$$

Hence, the remarkable identity

$$\lambda_n^{(t)}(x) = \sum_{k \leqslant n} (ka)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} A_k^{(t)}(x).$$

8.2.6. *Upper Factorial to Gould*

The relevant operator is $f(\mathbf{D}) = -\mathbf{D}(\mathbf{I}-\mathbf{D})^{-a}$, whose associated graded sequence is

$$\begin{aligned} r_n^{(t)}(x) &= f'(\mathbf{D}) \left(\frac{f(\mathbf{D})}{\mathbf{D}}\right)^{-n-1} \lambda_n^{(t)}(x) \\ &= (-1)^n \left((a-1)\mathbf{D}+\mathbf{I}\right)(\mathbf{I}-\mathbf{D})^{an-1} \lambda_n^{(t)}(x). \end{aligned} \tag{57}$$

As expected, for $a=1$ we get a variant on the Laguerre graded sequence, and for $a=0$ we get a variant on the harmonic graded sequence.

For $n \neq 0$, 1, instead of Eq. (57) we may use Corollary 7.2.11:

$$\begin{aligned} r_n^{(t)}(x) &= (-1)^n \sigma(\mathbf{I}-\mathbf{D})^{na} \lambda_{n-1}^{(t)}(x) \\ &= (-1)^n \sigma \left(\sum_{k \geqslant 0} (-1)^k \binom{na}{k} \mathbf{D}^k \right) \lambda_{n-1}^{(t)}(x) \\ &= \sum_{\substack{k \geqslant 0 \\ k \neq n}} (-1)^{n+k} \binom{na}{k} \frac{\lfloor n-1 \rceil!}{\lfloor n-k-1 \rceil!} \lambda_{n-k}^{(t)}(x). \end{aligned}$$

Thus, for $n \neq 0$, 1, we obtain the following remarkable identity relating the Gould graded sequence to the lower factorial graded sequence:

$$G_n^{(t)}(x) = \sum_{\substack{k \geqslant 0 \\ k \neq n}} (-1)^{n+k} \binom{na}{k} \frac{\lfloor n-1 \rceil!}{\lfloor n-k-1 \rceil!} \langle x \rangle_{n-k}^{(t)}.$$

In particular, for $t=1$, we obtain an explicit formula for $G_n^{(t)}(x)$.

For $n=0$, Eq. (57) reduces to

$$\begin{aligned} r_0^{(t)}(x) &= ((a-1)\mathbf{D}+\mathbf{I})(\mathbf{I}-\mathbf{D})^{-1} \lambda_0^{(t)}(x) \\ &= (1 + a\mathbf{D} + a\mathbf{D}^2 + a\mathbf{D}^3 + \cdots) \lambda_0^{(t)}(x) \\ &= \lambda_0^{(t)}(x) + a \sum_{k>0} \lfloor -k \rceil!^{-1} \lambda_{-k}^{(t)}(x). \end{aligned}$$

Thus, the Gould series of degree zero are given by the elegant expression

$$G_0^{(t)}(x) = \langle x \rangle_0^{(t)} + a \sum_{k>0} \lfloor -k \rceil!^{-1} \langle x \rangle_{-k}^{(t)}.$$

Thus,

$$G_0^{(1)}(x)=\log(x+1)+\frac{B_1}{x+1}-\frac{2!\,B_2}{(1+x)^2}+\cdots$$

$$+\frac{a}{x+1}-\frac{a}{(x+1)(x+2)}+\frac{a}{2!\,(x+1)(x+2)(x+3)}+\cdots.$$

Finally, from Proposition 7.2.10, we obtain

$$\begin{aligned} r_1^{(t)}(x) &= -(\mathbf{I}-\mathbf{D})^a\,\lambda_1^{(t)}(x)-a(\mathbf{I}-\mathbf{D})^{a-1}\,\lambda_0^{(t)}(x)\\ &=\lambda_1^{(t)}(x)+\sum_{k\geqslant 2}\left(\binom{a}{k}+a\binom{a-1}{k-1}\right)\lfloor 1-k\rceil!^{-1}\,\lambda_{1-k}^{(t)}(x)\\ &=\lambda_1^{(t)}(x)+\sum_{k\geqslant 2}\left(\binom{a}{k}+a\binom{a-1}{k-1}\right)(-1)^k\,(k-2)!\,\lambda_{1-k}^{(t)}(x), \end{aligned}$$

so

$$G_1^{(t)}(x)=\langle x\rangle_1^{(t)}+\sum_{k\geqslant 2}\left(\binom{a}{k}+a\binom{a-1}{k-1}\right)(-1)^k\,(k-2)!\,\langle x\rangle_{1-k}^{(t)}$$

$$G_1^{(1)}(x)=x\log(x)-x+\sum_{k\geqslant 2}B_k^{(2)}\begin{bmatrix}2\\k\end{bmatrix}x^{1-k}$$

$$+\sum_{k\geqslant 2}\frac{(-1)^k\,(\binom{a}{k}+a\binom{a-1}{k-1})(k-2)!}{(x-1)\cdots(x-k+1)}.$$

8.2.7. *Lower Factorial to Gould*

Similarly, here we are interested in the operator $f(\mathbf{D})=\mathbf{D}(\mathbf{I}+\mathbf{D})^a$. Its associated graded sequence is

$$\begin{aligned} r_n^{(t)}(x) &= f'(\mathbf{D})\left(\frac{f(\mathbf{D})}{\mathbf{D}}\right)^{-n-1}\lambda_n^{(t)}(x)\\ &=((a+1)\mathbf{D}+\mathbf{I})(\mathbf{I}+\mathbf{D})^{na-1}\,\lambda_n^{(t)}(x). \end{aligned}$$

For $n\neq 0$, 1,

$$\begin{aligned} r_n^{(t)}(x) &= \sigma(\mathbf{I}+\mathbf{D})^{-na}\,\lambda_{n-1}^{(t)}(x)\\ &=\sum_{\substack{k\geqslant 0\\ k\neq n}}\binom{na}{k}\frac{\lfloor n-1\rceil!}{\lfloor n-k-1\rceil!}\,\lambda_{n-k}^{(t)}(x). \end{aligned}$$

Thus, for $n \neq 0$, 1,

$$G_n^{(t)}(x) = \sum_{\substack{k \geqslant 0 \\ k \neq n}} \binom{-na}{k} \frac{\lfloor n-1 \rceil!}{\lfloor n-k-1 \rceil!} \langle x \rangle_{n-k}^{(t)}.$$

Now, for $n = 0$,

$$\begin{aligned} r_0^{(t)}(x) &= ((a+1)\mathbf{D} + \mathbf{I})(\mathbf{I} + \mathbf{D})^{-1} \lambda_0^{(t)}(x) \\ &= (1 + a\mathbf{D} - a\mathbf{D}^2 + a\mathbf{D}^3 - \cdots) \lambda_0^{(t)}(x) \\ &= \lambda_0^{(t)}(x) + a \sum_{k>0} (k-1)! \lambda_{-k}^{(t)}(x), \end{aligned}$$

so that again

$$G_0^{(t)}(x) = \langle x \rangle_0^{(t)}(x) + a \sum_{k>0} (k-1)! \langle x \rangle_{-k}^{(t)}.$$

For $n = 1$,

$$\begin{aligned} r_1^{(t)}(x) &= (\mathbf{I} + \mathbf{D})^{-a} \lambda_1^{(t)}(x) - a(\mathbf{I} + \mathbf{D})^{-a-1} \lambda_0^{(t)}(x) \\ &= \lambda_1^{(t)}(x) + \sum_{k \geqslant 2} \left(\binom{-a}{k} + a(k-1) \binom{-a-1}{k-1} \right) k! \lambda_{1-k}^{(t)}(x), \end{aligned}$$

hence another remarkable expansion for a residual series:

$$G_{-1}^{(t)}(x) = \sum_{k \geqslant 0} \left(\binom{a}{k} + a \binom{a-1}{k-1} \right) k! \langle x \rangle_{-1-k}^{(t)}.$$

We hope the preceding examples display the usefulness of the theory of formal power series of logarithmic type.

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