1. 证明(略)

2. ① $E(X) = (-2)\times 0.2 + 0\times 0.5 + 2\times 0.3 = 0.2$

$E(X^2) = 4\times 0.2 + 0\times 0.5 + 4\times 0.3 = 2$

$E(3X^2+5) = 17\times 0.2 + 5\times 0.5 + 17\times 0.3 = 11$

② $E\left(\frac{1}{X+1}\right) = \sum_{k=0}^{\infty} \frac{1}{k+1} P(X=k) = \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k+1)!} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} = \frac{e^{-\lambda}}{\lambda}(e^{\lambda}-1) = \frac{1-e^{-\lambda}}{\lambda}$

3. ① 对称性: $E\left(\frac{X^2}{X^2+Y^2}\right) = E\left(\frac{Y^2}{X^2+Y^2}\right)$

而 $E\left(\frac{X^2}{X^2+Y^2}\right) + E\left(\frac{X^2}{X^2+Y^2}\right) = 1$

$\Rightarrow E\left(\frac{X^2}{X^2+Y^2}\right) = \frac{1}{2}$

② 记 R 为点 (X,Y) 到原点的距离. $R = \sqrt{X^2+Y^2}$

$f(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$

$E(R) = E(\sqrt{x^2+y^2}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sqrt{x^2+y^2} \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} dx\,dy$

$= \int_0^{2\pi} d\theta \int_0^{\infty} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr = \frac{1}{\sigma^2}\int_0^{+\infty} r^2 e^{-r^2/2\sigma^2} dr$

$= \sigma\sqrt{\frac{\pi}{2}}$

4. $f(x) = \begin{cases} \frac{1}{5} & 0<x<5 \\ 0 & \end{cases}$ $\qquad f(y) = \begin{cases} \frac{1}{5} & 0<y<5 \\ 0 & 0 \end{cases}$

① $E(XY) = E(X)E(Y) = 2.5\times 2.5 = 6.25$

$E(X/Y)$ 不存在. 因为 $\int_0^5\int_0^5 \frac{x}{y} dx\,dy$ 发散

$E(\ln(XY)) = \int_0^5\int_0^5 (\ln x + \ln(y)) \frac{1}{25} dx\,dy = 2\int_0^5\int_0^5 \frac{1}{5}\ln x\, dx\,dy = 10(\ln 5 - 1)$

$E(|Y-X|) = 2\int_0^5\int_x^5 (y-x)dx\,dy = \frac{125}{3}$

② $A = XY \qquad C = 2(X+Y)$

$Cov(A,C) = E(AC) - E(A)E(C) = E(2X^2Y + 2XY^2) - E(XY)E(2X+2Y)$

$E(X^2) = E(Y^2) = D(X) + [E(X)]^2 = \frac{25}{12} + \left(\frac{5}{2}\right)^2 = \frac{25+25\times 3}{12} = \frac{25}{3}$

$E(AC) = E(2X^2Y) + E(2XY^2) = 2E(X^2)E(Y) + 2E(X)E(Y^2) = 2\left(\frac{25}{3}\times\frac{5}{2}\right)\times 2$

$E(A) = E(XY) = \frac{25}{4} \qquad E(C) = E(2X+2Y) = 5+5 = 10 \qquad = \frac{250}{3}$

$Cov(A,C) = \frac{250}{3} - \frac{25}{4}\times 10 = \frac{500}{6} - \frac{375}{6} = \frac{125}{6}$

$D(A) = D(XY) = E(X^2Y^2) - E^2(XY) = \frac{4375}{144}$

$D(C) = D(2X+2Y) = 4D(X) + 4D(Y) = \frac{50}{3}$

$\rho_{AC} = \frac{Cov(A,C)}{D(A)D(C)} = \frac{\sqrt{42}}{7}$

5. $\mu_1 = \mu_2 = 0 \quad \sigma_1 = 1 \quad \sigma_2 = 2 \quad \rho = -\frac{1}{4}$

$$f(x,y) = \frac{1}{4\sqrt{1-\frac{1}{16}}} \exp\left[\left(\frac{-1}{2\sqrt{1-\frac{1}{16}}}\right)\left(x^2 + \frac{y^2}{4} + \frac{xy}{4}\right)\right.$$

$$= \frac{1}{\sqrt{15}} \exp\left[-\frac{8}{15}\left(x^2 + \frac{y^2}{4} + \frac{xy}{4}\right)\right.$$