5.1 求下列给定区间上关于权函数 $\omega(x)$ 的正交多项式 $g_0(x), g_1(x), g_2(x)$：

$$[0,1],\ \omega(x)=\sqrt{x}$$

解答. 令 $g_0(x)=1$，则 $\gamma_0=(1,1)=\int_0^1\sqrt{x}\,dx=\frac{2}{3}, \beta_0=(x,1)=\int_0^1 x\sqrt{x}\,dx=\frac{2}{5}$，通过三项递推式可得

$$g_1(x)=(x-\frac{\beta_0}{\gamma_0})g_0(x)=x-\frac{3}{5}$$

则

$$\begin{aligned}\gamma_1&=(x-\frac{3}{5},x-\frac{3}{5})=\int_0^1\sqrt{x}(x-\frac{3}{5})(x-\frac{3}{5})=\frac{2^3}{5^2\cdot 7}\\ \beta_1&=(x(x-\frac{3}{5}),x-\frac{3}{5})=\int_0^1 x\sqrt{x}(x-\frac{3}{5})(x-\frac{3}{5})=\frac{2^3\cdot 23}{3^2\cdot 5^3\cdot 7}\end{aligned}$$

通过递推式可得

$$\begin{aligned}g_2(x)=(x-\frac{\beta_1}{\gamma_1})g_1(x)-\frac{\gamma_1}{\gamma_0}g_0(x)&=\left(x-\frac{2^3\cdot 23}{3^2\cdot 5^3\cdot 7}\cdot\frac{5^2\cdot 7}{2^3}\right)(x-\frac{3}{5})-\frac{8}{5^2\cdot 7}\cdot\frac{3}{2}\\ &=x^2-\frac{10}{9}x+\frac{5}{21}\end{aligned}$$

综上，

$$g_0(x)=1,\ g_1(x)=x-\frac{3}{5},\ g_2(x)=x^2-\frac{10}{9}x+\frac{5}{21}$$

5.2 给定数据如下表中所示，求其最小二乘拟合函数 $p(x)$：

(1) $p(x)=c_0+c_1x+c_2x^2$.

x_i	1	3	4	5	6	7	8	9	10
y_i	2	7	8	10	11	11	10	9	8

解答. 取 $\phi_0(x)=1,\phi_1(x)=x,\phi_2(x)=x^2$，则正规方程组为

$$\begin{bmatrix}9&53&381\\53&381&3017\\381&3017&25317\end{bmatrix}\begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}=\begin{bmatrix}76\\489\\3547\end{bmatrix}$$

解得

$$c_0=-1.45966,\quad c_1=3.60531,\quad c_2=-0.26757$$

所求的最小二乘拟合函数为

$$p(x) = -1.45966 + 3.60531x - 0.26757x^2$$

5.3 求下列函数在指定区间上的最优平方逼近一次多项式:

(1) $y = \sqrt{x},\ [0,1]$; (2) $y = e^x,\ [-1,1]$.

解答. 设 $f = \sqrt{x}, p(x) = c_0 + c_1 x$, 则 $\phi_0(x) = 1, \phi_1(x) = x$,

$$(\phi_i, \phi_j) = \int_0^1 x^{i+j}\,dx = \frac{1}{i+j+1}, \quad (\phi_i, f) = \int_0^1 x^i \sqrt{x} = \frac{2}{2i+3}$$

由正规方程组有

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/5 \end{bmatrix}$$

解得

$$c_0 = \frac{4}{15}, \quad c_1 = \frac{4}{5}$$

所求的最优平方逼近一次多项式为

$$p(x) = \frac{4}{15} + \frac{4}{5}x$$

(2) 设 $f = e^x, p(x) = c_0 + c_1 x$, 则 $\phi_0(x) = 1, \phi_1(x) = x$,

$$(\phi_i, \phi_j) = \int_{-1}^1 x^{i+j}\,dx = \frac{1 + (-1)^{i+j}}{i+j+1}, \quad (\phi_0, f) = \int_{-1}^1 e^x\,dx = e - \frac{1}{e}, \quad (\phi_1, f) = \int_{-1}^1 xe^x\,dx = \frac{2}{e}$$

由正规方程组有

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} e - 1/e \\ 2/e \end{bmatrix}$$

解得

$$c_0 = \frac{e}{2} - \frac{1}{2e}, \quad c_1 = \frac{3}{e}$$

所求的最优平方逼近一次多项式为

$$p(x) = \frac{e}{2} - \frac{1}{2e} + \frac{3}{e}x$$

5.4 利用正交多项式求下列函数的最优平方逼近二次多项式：

$$y = \arcsin x,\ [0,1].$$

解答. 先计算几个积分的值为后面求值准备：

$$\begin{aligned}
I_0 &= \int_0^1 \arcsin x\,dx \xlongequal{x=\sin\theta} \int_0^{\pi/2} \theta\,d\sin\theta = \frac{\pi}{2} - 1 \\
I_1 &= \int_0^1 x\arcsin x\,dx \xlongequal{x=\sin\theta} \frac{1}{2}\int_0^{\pi/2} \theta\sin 2\theta\,d\theta = -\frac{1}{4}\int_0^{\pi/2} \theta\,d\cos 2\theta = \frac{\pi}{8} \\
I_2 &= \int_0^1 x^2\arcsin x\,dx \xlongequal{x=\sin\theta} \int_0^{\pi/2} \theta\sin^2\theta\cos\theta\,d\theta = \int_0^{\pi/2} \frac{\theta}{2}(\cos\theta - \cos\theta\cos 2\theta)\,d\theta \\
&= \int_0^{\pi/2} \frac{\theta}{4}(\cos\theta - \cos 3\theta)\,d\theta = \frac{1}{4}\left(\int_0^{\pi/2} \theta\,d\sin\theta - \frac{1}{3}\int_0^{\pi/2} \theta\,d\sin 3\theta\right) = \frac{\pi}{6} - \frac{2}{9}
\end{aligned}$$

通过三项递推的方式求解 $[0,1]$ 上的二次正交多项式，令 $g_0(x) = 1, g_1(x) = x - \dfrac{\beta_0}{\gamma_0}, g_2(x) = (x - \dfrac{\beta_1}{\gamma_1})g_1 - \dfrac{\gamma_1}{\gamma_0}$，由于

$$\beta_0 = (x,1) = \int_0^1 x\,dx = \frac{1}{2}, \quad \gamma_0 = (1,1) = \int_0^1 dx = 1$$

可知

$$\begin{aligned}
g_1(x) = x - \frac{1}{2}, \quad \beta_1 &= (x(x-\frac{1}{2}), x-\frac{1}{2}) = \int_0^1 x(x-\frac{1}{2})^2\,dx = \frac{1}{24}, \\
\gamma_1 &= (x-\frac{1}{2}, x-\frac{1}{2}) = \int_0^1 (x-\frac{1}{2})^2\,dx = \frac{1}{12}.
\end{aligned}$$

则

$$g_2(x) = (x - \frac{1}{24}\cdot 12)(x - \frac{1}{2}) - \frac{1}{12} = x^2 - x + \frac{1}{6}, \quad \gamma_2 = \int_0^1 (x^2 - x + \frac{1}{6})^2\,dx = \frac{1}{180}$$

设最优平方逼近二次多项式为 $p(x) = c_0g_0(x) + c_1g_1(x) + c_2g_2(x)$，利用正交性，知 $c_i = \dfrac{(f,g_i)}{(g_i,g_i)}$，从而得

$$c_0 = \frac{I_0}{\gamma_0} = \frac{\pi}{2} - 1, \quad c_1 = \frac{I_1}{\gamma_1} = \frac{3\pi}{2}, \quad c_2 = \frac{I_2}{\gamma_2} = 30\pi - 40$$

所求的最优平方逼近二次多项式为：

$$\begin{aligned}
p(x) &= \frac{\pi}{2} - 1 + \frac{3\pi}{2}(x - \frac{1}{2}) + (30\pi - 40)(x^2 - x + \frac{1}{6}) \\
&= \frac{19\pi}{4} - \frac{23}{3} + (40 - \frac{57\pi}{2})x + (30\pi - 40)x^2
\end{aligned}$$

5.5 取基函数为勒让德多项式，求函数 $f(x)=\sin\frac{\pi}{2}x$ 在区间 $[-1,1]$ 上的最优平方逼近三次多项式。

解答. 通过三项递推式可求出 Legendre 多项式

$$p_0(x)=1,\quad p_1(x)=x,\quad p_2(x)=\frac{1}{2}(3x^2-1),\quad p_3(x)=\frac{1}{2}(5x^3-3x)$$

设最优平方逼近三次多项式为：$p(x)=c_0p_0(x)+c_1p_1(x)+c_2p_2(x)+c_3p_3(x)$，由 Legendre 多项式性质可知 $(p_k,p_k)=\dfrac{2}{2k+1}$，则

$$c_k=\frac{(f,p_k)}{(p_k,p_k)}=\frac{(f,p_k)\cdot(2k+1)}{2}$$

下面求解几个积分的值为后面计算准备：

$$\begin{aligned}
I_0&=\int_{-1}^{1}\sin\frac{\pi}{2}x\,dx=0,\quad I_2=\int_{-1}^{1}x^2\sin\frac{\pi}{2}x\,dx=0\\
I_1&=\int_{-1}^{1}x\sin\frac{\pi}{2}x\,dx=-\frac{2}{\pi}\int_{-1}^{1}x\,d\cos\frac{\pi}{2}x=\frac{8}{\pi^2}\\
I_3&=\int_{-1}^{1}x^3\sin\frac{\pi}{2}\,dx=-\frac{2}{\pi}\int_{-1}^{1}x^3\,d\cos\frac{\pi}{2}x=\frac{4}{\pi^2}\int_{-1}^{1}3x^2\,d\sin\frac{\pi}{2}x=\frac{24}{\pi^2}(1-I_1)=\frac{24(\pi^2-8)}{\pi^4}
\end{aligned}$$

则

$$c_0=\frac{I_0}{2}=0,\quad c_1=\frac{3}{2}I_1=\frac{12}{\pi^2},\quad c_2=\frac{5(3I_2-I_0)}{4}=0,\quad c_3=\frac{7(5I_3-3I_1)}{4}=\frac{168(\pi^2-10)}{\pi^4}$$

所求的最优平方逼近三次多项式为：

$$p(x)=\frac{12}{\pi^2}x+\frac{168(\pi^2-10)}{\pi^4}(\frac{1}{2}(5x^3-3x))=\frac{-240\pi^2+252}{\pi^4}x+\frac{420(\pi^2-10)}{\pi^4}x^3$$

5.6 求下列函数在指定区间上的最优一致逼近一次多项式：

$$y=\sqrt{x},\quad [\frac{1}{4},1]$$

解答. 设 $f(x)=y=\sqrt{x}$，则 $f'(x)=\dfrac{1}{2\sqrt{x}}$，$f''(x)=-\dfrac{1}{4}x^{-3/2}<0$ $(x\in(1/4,1))$，令最优一致逼近一次多项式为：$p(x)=c_0+c_1x$，取偏差点 $\tilde{x}_0=1/4,\tilde{x}_2=1$，则

$$\begin{cases}\dfrac{1}{2}-c_0-\dfrac{1}{4}c_1=\mu\\ \sqrt{\tilde{x}_1}-c_0-c_1\tilde{x}_1=-\mu\\ 1-c_0-c_1=\mu\\ \dfrac{1}{2\sqrt{\tilde{x}_1}}-c_1=0\end{cases}\quad \text{解得}\begin{cases}c_1=\dfrac{2}{3},\quad \tilde{x}_1=\dfrac{1}{4c_1^2}=\dfrac{9}{16}\\ c_0=\dfrac{1}{2}(\dfrac{1}{2}+\sqrt{\tilde{x}_1}-(\dfrac{1}{4}+\tilde{x}_1)c_1)=\dfrac{17}{48}\\ \mu=1-c_0-c_1=-\dfrac{1}{48}\end{cases}$$

则 $y=\sqrt{x}$ 在 $[1/4,1]$ 上的最优一致逼近一次多项式为

$$p(x)=\frac{17}{48}+\frac{2}{3}x\approx 0.35417+0.66667x$$

最大误差 $E=-\mu=\dfrac{1}{48}\approx 0.02083$。

5.10 用两种方法求下列函数在指定区间上的近似最优一致逼近一次式，并求其偏差：

$$y=e^{-x},\quad [-1,1].$$

解答. 法一：Chebyshev 插值多形式，$T_2(x)$ 的两个零点为

$$x_0=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2},\ x_1=\cos\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$$

利用 Newton 插值多项式可知，近似最优一致逼近一次式为

$$N_1(x)=y_0+\frac{y_1-y_0}{x_1-x_0}(x-x_0)=e^{-\frac{\sqrt{2}}{2}}+\frac{e^{\frac{\sqrt{2}}{2}}-e^{-\frac{\sqrt{2}}{2}}}{-\sqrt{2}}(x-\frac{\sqrt{2}}{2})\approx 1.2606-1.0854x$$

由 Chebyshev 插值多项式余项估计

$$\max_{-1\leqslant x\leqslant 1}|R_1(x)|\leqslant\frac{M_2}{2!}\cdot\frac{1}{2}=\frac{e}{4}\approx 0.67957$$

其中 $M_2=\max\limits_{-1\leqslant x\leqslant 1}|y''|=\max\limits_{-1\leqslant x\leqslant 1}e^{-x}=e$。

法二：缩短幂级数法，将目标函数 Tyalor 展开至 2 次项

$$e^{-x}=p_2(x)=1-x+\frac{1}{2}x^2+R_2(x)$$

且

$$\max_{-1\leqslant x\leqslant 1}|R_2(x)|=\max_{-1\leqslant x\leqslant 1}\left|\frac{-e^{-x}}{3!}x^3\right|=\frac{e}{6}$$

由缩短幂级数法知，近似最优一致逼近一次式为

$$p_1(x)=p_2(x)-\frac{1}{2}\cdot\frac{T_2(x)}{2}=1-x+\frac{1}{2}x^2-\frac{1}{4}T_2=\frac{5}{4}-x=1.25-x$$

偏差估计为

$$\max_{-1\leqslant x\leqslant 1}|R_2(x)|+\sup_{-1\leqslant x\leqslant 1}\frac{1}{4}T_2(x)\leqslant\frac{e}{6}+\frac{1}{4}\approx 0.70305$$

5.12 定义 $T_k^*() = T_k(2x-1)$.

(1) 求 $T_k^*(x)$ $(k = 0, 1, 2\cdot, 4)$;

(2) 证明 $T_k^*(x)$ 在区间 $[0,1]$ 上关于权函数 $\omega(x) = 1/\sqrt{x-x^2}$ 正交;

(3) 证明 $T_k^*(x^2) = T_{2k}(x)$。

解答. (1)

$$
\begin{aligned}
&T_0^*(x) = T_0(2x-1) = 1\\
&T_1^*(x) = T_1(2x-1) = 2x-1\\
&T_2^*(x) = T_2(2x-1) = 2(2x-1)^2 - 1 = 8x^2 - 8x + 1\\
&T_3^*(x) = 4(2x-1)^3 - 3(2x-1) = 32x^3 - 48x^2 + 18x - 1\\
&T_4^*(x) = 8(2x-1)^4 - 8(2x-1)^2 + 1 = 128x^4 - 256x^3 + 160x^2 - 32x + 1
\end{aligned}
$$

(2) $\forall k, j \in \mathbb{N},\ k \neq j$，则

$$
\begin{aligned}
\big(T_k^*(x), T_j^*(x)\big) &= \int_0^1 \frac{\cos(k\ \arccos(2x-1))\cos(j\ \arccos(2x-1))}{\sqrt{x-x^2}}\,dx\\
&\xlongequal{2x-1=\cos t} \int_0^\pi \frac{\cos(kt)\cos(jt)}{\sqrt{\frac{1+\cos t}{2} - \frac{(1+\cos t)^2}{4}}}\left(\frac{\sin t}{2}\right)dt\\
&= \int_0^\pi \cos(kt)\cos(jt)\,dt = 0
\end{aligned}
$$

(3) 设 $x = \cos\theta, 2x^2 - 1 = \cos\varphi$，则

$$
\begin{aligned}
&\cos\varphi = 2\cos^2\theta - 1 = \cos 2\theta\\
\Rightarrow\ &\varphi = \pm 2\theta + 2t\pi \quad (t \in \mathbb{Z})\\
\Rightarrow\ &k\arccos(2x^2-1) = \pm 2k\arccos x + 2t\pi \quad (t \in \mathbb{Z})\\
\Rightarrow\ &\cos(k\arccos(2x^2-1)) = \cos(2k\arccos x)\\
\Rightarrow\ &T_k^*(x^2) = T_{2k}(x)
\end{aligned}
$$