| Data | Subject | Name, ID |
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| March 12, 2022 | Complex Analysis | 吴天阳, 2204210460 |

**1.** Find the real and imaginary parts of each of the following:

$$\frac{1}{z}, z^3, \frac{3+5i}{1+7i}, \left(\frac{-1-i\sqrt{3}}{2}\right)^6.$$

**Solution.** Let $z = a + ib$, then

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \tag{1}$$

$$\text{real part} : \frac{a}{a^2+b^2} \quad \text{imaginary part} : -\frac{b}{a^2+b^2}$$

$$\begin{aligned} z^3 = (a+ib)^3 &= a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3 \\ &= a^3 + 3ia^2b - 3ab^2 - ib^3 \\ &= a^3 - 3ab^2 + i(3a^2b - b^3) \end{aligned} \tag{2}$$

$$\text{real part} : a^3 - 3ab^2 \quad \text{imaginary part} : 3a^2b - b^3$$

$$\frac{3+5i}{1+7i} = \frac{(3+5i)(1-7i)}{50} = \frac{19}{25} - \frac{8}{25}i \tag{3}$$

$$\text{real part} : \frac{19}{25} \quad \text{imaginary part} : -\frac{8}{25}$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^6 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^6 = \left(e^{2\pi i/6}\right)^6 = e^{2\pi i} = 1 \tag{4}$$

$$\text{real part} : 1 \quad \text{imaginary part} : 0$$

**2.** Prove

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

for complex numbers $z$ and $w$.

*Proof.* Let $z = a + ib, w = c + id$, then

$$\begin{aligned} |z+w|^2 + |z-w|^2 &= |(a+c) + i(b+d)|^2 + |(a-c) + i(b-d)|^2 \\ &= (a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2 \\ &= 2(a^2 + b^2 + c^2 + d^2) \\ &= 2(|z|^2 + |w|^2) \end{aligned}$$

$\square$

**3.** Find the sixth roots of unity.

**Solution.** The sixth roots of unity are the roots of the equation $x^6 = 1$ in $\mathbb{C}$.

Let $\xi = e^{2\pi i/6} = e^{\pi i/3}$, we have six different values

$$\xi^0, \xi^1, \xi^2, \xi^3, \xi^4, \xi^5$$

Since $(\xi^n)^6 = e^{2\pi i n} = 1$, hence, $\xi^n$ $(n = 0, 1, \cdots, 5)$ are the sixth roots of unity.

**4.** Show the limits

$$\lim_{z\to 0} \frac{z}{|z|}, \quad \lim_{z\to 0} e^{1/z}$$

do not exist.

*Proof.* (1) Suppose the limit exits. Let $z = x$ be purely real, then

$$\begin{aligned} \lim_{z\to 0} \frac{z}{|z|} &= \lim_{x\to 0^+} \frac{x}{x} = 1 \\ &= \lim_{x\to 0^-} \frac{x}{-x} = -1 \end{aligned}$$

$\lim_{z\to 0} \frac{z}{|z|} = 1 = -1$ is contradictive. Hence, $\lim_{z\to 0} \frac{z}{|z|}$ do not exits.
(2) Suppose the limit exits. Let $z = x$ be purely real, then

$$\begin{aligned} \lim_{z\to 0} e^{1/z} &= \lim_{x\to 0^+} e^{1/x} = +\infty \\ &= \lim_{x\to 0^-} e^{1/x} = 0 \end{aligned}$$

$\lim_{z\to 0} e^{1/z} = +\infty = 0$ is contradictive. Hence, $\lim_{z\to 0} e^{1/z}$ do not exits. $\square$

**5.** Assume that the function $f$ does not vanish on a deleted neighborhood of $z_0 \in \mathbb{C}$. Show that $\lim_{z\to z_0} f(z) = 0$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = \infty$.

*Proof.* "$\Rightarrow$": For all $M > 0$ exists $\delta > 0$, such that

$$|f(z_0 + h)| < \frac{1}{M} \Rightarrow \frac{1}{|f(z_0 + h)|} > M \quad (h \in D_\delta(z_0))$$

Hence, $\lim_{z\to z_0} \frac{1}{f(z)} = \infty$.
"$\Leftarrow$": Forall $\epsilon > 0$ exists $\delta > 0$, such that

$$\frac{1}{|f(z_0 + h)|} > \frac{1}{\epsilon} \Rightarrow |f(z_0 + h)| < \epsilon \quad (h \in D_\delta(z_0))$$

Hence, $\lim_{z\to z_0} f(z) = 0$. $\square$

**6.** Prove the Chain Rule of analytic functions.

*Proof.* We define two holomorphic functions $g : U \to V, f : V \to W$ and $U, V, W$ are open subset of $\mathbb{C}$. Assert that $f \circ g$ is holomorphic and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0) \quad (z_0 \in U)$$

Since $g, f$ are differentiable at $z_0, g(z_0)$, respectively, we write

$$\begin{aligned} \frac{g(z_0 + h_1) - g(z_0)}{h_1} &= g'(z_0) + \psi_1(h_1) \\ \frac{f(g(z_0) + h_2) - f(g(z_0))}{h_2} &= f'(g(z_0))h_2 + \psi_2(h_2)h_2 \end{aligned}$$

where $\psi_j(h) \to 0$ $(j = 1, 2)$ as $h \to 0$.

Let $h_2 = g(z_0 + h_1) - g(z_0)$, dividing by $h_1$, we obtain

$$\begin{aligned} \frac{f(g(z_0 + h_1)) - f(g(z_0))}{h_1} &= f'(g(z_0))\frac{g(z_0 + h_1) - g(z_0)}{h_1} + \psi_2(g(z_0 + h_1) - g(z_0))\frac{g(z_0 + h_1) - g(z_0)}{h_1} \\ &= f'(g(z_0))g'(z_0) + \psi_1(h_1)f'(g(z_0)) \\ &\qquad + \psi_2((g'(z_0) + \psi_1(h_1))h_1)(g'(z_0) + \psi_1(h_1)) \end{aligned}$$

Let $\psi_3(h_1) = \psi_1(h_1)f'(g(z_0)) + \psi_2((g'(z_0) + \psi_1(h_1))h_1)(g'(z_0) + \psi_1(h_1))$. As $h_1 \to 0$, $\psi_3(h_1) \to 0$, then

$$\frac{f(g(z_0 + h)) - f(g(z_0))}{h_1} = f'(g(z_0))g'(z_0) + \psi_3(h_1)$$

We conclude that

$$\lim_{h_1 \to 0} \frac{f(g(z_0 + h)) - f(g(z_0))}{h_1} = f'(g(z_0))g'(z_0)$$

as asserted by the Chain Rule. □