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偏微分方程

强基数学 002

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第二章第一次作业

题目 1. 用特征线法求解下述 Cauchy 问题:

$$\begin{cases} u_t + 2u_x + u = xt, & t > 0, x \in \mathbb{R}, \\ u|_{t=0} = 2 - x, & x \in \mathbb{R}. \end{cases}$$

解答. 令
$$\begin{cases} \frac{dx}{dt}=2,\\ x(0)=c. \end{cases}$$
 则 $x=2t+c,\ c=x-2t,\$ 于是原问题等价于

$$\begin{cases} \frac{du}{dt} + u = xt = (2t+c)t, \\ u(0) = u(x(0), 0) = u(c, 0) = 2 - c. \end{cases}$$

求解一阶线性常微分方程
$$\frac{d}{dt}(ue^t) = e^t(2t+c)t \Rightarrow ue^t = \int_0^t e^\tau(2\tau+c)\tau d\tau + c$$
,可得
$$u = 2t^2 + (c-4)t + 4 - c + (2c-4)e^{-t}.$$

代入c = x - 2t, 得到原方程解

$$u = 2t^{2} + (x - 2t - 4)(t - 1) + 2(x - 2t - 2)e^{-t}.$$

题目 2. 试证明 Cauchy 问题

$$\begin{cases} u_{tt} - u_{xx} = 6(x+t), & x \in \mathbb{R}, t > x, \\ u|_{t=x} = 0, u_t|_{t=x} = u_1(x), & x \in \mathbb{R}. \end{cases}$$

有解的充要条件是 $u_1(x)-3x^2=$ const(经尝试只能得到 $u_1(x)-6x^2=$ const),如果有解,解不唯一. 试问:若把初值给定在直线 t=ax 上,为什么在 $a=\pm 1$ 与 $a\neq \pm 1$ 的情况,关于存在唯一性的结论不一样?

解答. 令
$$\begin{cases} \xi = x - t, \\ \eta = x + t \end{cases}, \quad \mathcal{M} u_{tt} - u_{xx} = -2u_{\xi\eta} = 6\eta \Rightarrow u_{\xi\eta} = -3\eta, \quad \text{解得} \end{cases}$$
$$u = -\frac{3}{2}\xi\eta^2 + F(\xi) + G(\eta) = -\frac{3}{2}(x - t)(x + t)^2 + F(x - t) + G(x + t).$$

于是

$$\begin{cases} u|_{t=x} = F(0) + G(2x) = 0 \Rightarrow G'(2x) = 0, \\ u_t = -\frac{3}{2}(x - 3t)(x + t) - F'(x - t) + G'(x + t), \\ u_t|_{t=x} = 6x^2 - F'(0) + G'(2x) = u_1(x) \end{cases}$$

则有解的充要条件为 $u_1(x) - 6x^2 = F'(0) = \text{const}$,由于无法确定 F 的具体表达式,故解不唯一。一般的,令初值为 t = ax,可得

$$\begin{cases}
 u|_{t=ax} = \frac{3}{2}(a-1)(a+1)^2x^3 + F((1-a)x) + G((1+a)x) = 0, \\
 u_t|_{t=ax} = \frac{3}{2}(3a-1)(a+1)x^2 - F'((1-a)x) + G'((1+a)x) = u_1(x).
\end{cases}$$
(1)

当 a = -1 时,可得 F'(2x) = 0, $-F'(2x) + G'(0) = u_1(x) \Rightarrow u_1(x) = G'(0)$,所以无法确定 G 的具体表达式,故解不唯一.

当 $a \neq \pm 1$ 时,则 (1) 式中函数 F, G 均与 x 相关,可通过对 (1) 中第二式进行积分,然后与第一式进行联立求解得到函数 F, G,所以解唯一存在.

题目 3. 若 u = u(x, y, z, t) 是波动方程初值问题

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy} + u_{zz}) = 0, \\ u|_{t=0} = f(x) + g(y), \\ u_t|_{t=0} = \varphi(y) + \psi(z) \end{cases}$$

的解, 式求解的表达式.

解答. 设 $x = (x_1, x_2, x_3)$,则由三维波动方程解的一般表达式可知

题目 4. 试利用<mark>唯一性结果</mark>直接证明: 当初值 $\varphi(x)$, $\psi(x)$ 是偶函数,非齐次项 f(x,t) 是 x 的偶函数时,非齐次波动方程初值问题的解 u(x,t) 关于 x 也是偶函数.

根据以上事实,用延拓法求解半无界问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x, t), & x > 0, t > 0, \\ u_{t=0} = u_t|_{t=0} = 0, & x \geqslant 0, \\ u_x|_{x=0} = 0, & t \geqslant 0. \end{cases}$$

并说明当 f(x,t) 满足什么条件时,导出的公式确实是问题的解.

解答. 设 φ, ψ, f 均为关于 x 的偶函数,则一维波动方程初值问题的解满足

$$\begin{split} u(-x,t) &= \frac{1}{2}(\varphi(-x+at) + \varphi(-x-at)) + \frac{1}{2a} \int_{-x-at}^{-x+at} \psi(-\xi) \, \mathrm{d}\xi + \frac{1}{2a} \int_{0}^{t} \mathrm{d}\tau \int_{-x-a(t-\tau)}^{-x+a(t-\tau)} f(-\xi,\tau) \, \mathrm{d}\tau \\ &= \frac{1}{2}(\varphi(x+at) + \varphi(x-at)) + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) \, \mathrm{d}\xi + \frac{1}{2a} \int_{0}^{t} \, \mathrm{d}\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) \, \mathrm{d}\xi = u(x,t), \end{split}$$

所以 u(x,t) 也为关于 x 的偶函数.

令 \bar{f} 为 f 关于 x 的偶延拓,则对应波动方程初值问题的解 \bar{u} 也为关于 x 的偶函数,于是 $\bar{u}_x|_{x=0}=0$,满足题目要求,于是半无界问题的解为 $u=\bar{u}|_{x\geqslant 0}$.

$$\bar{u}_{x}|_{x=0}=0$$
,满足题目要求,于是半无界问题的解为 $u=\bar{u}|_{x\geqslant 0}$. 当 $x\geqslant at$ 时,有 $u(x,t)=\frac{1}{2a}\int_{0}^{t}\mathrm{d}\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)\,\mathrm{d}\xi$. 当 $0\leqslant x\leqslant at$ 时,有

$$u(x,t) = \frac{1}{2a} \int_{t-\frac{x}{a}}^{t} d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi + \frac{1}{2a} \int_{0}^{t-\frac{x}{a}} d\tau \int_{0}^{x+a(t-\tau)} f(\xi,\tau) d\xi + \frac{1}{2a} \int_{0}^{t-\frac{x}{a}} d\tau \int_{0}^{x+a(t-\tau)-x} f(\xi,\tau) d\xi$$

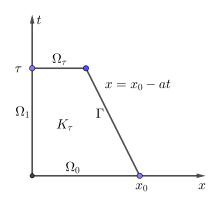
当 $f \in C^2([0,\infty)^2)$ 时,且 $\lim_{(x,t)\to(0,0)}(\Box u - f) = 0 \Rightarrow f(0,0) = 0$ 导出的公式是原问题的解.

题目 5. 证明半无界问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x, t), & x > 0, t > 0, \\ u|_{t=0} = \varphi(x), \ u_t|_{t=0} = \psi(x), & x \geqslant 0, \\ u|_{x=0} = \mu(t), & t \geqslant 0 \end{cases}$$

解的唯一性.

解答. 在区域 $\{0 \le x \le x_0 - at, 0 \le t \le \tau\}$, $(0 < \tau \le \frac{x_0}{a})$ 上考虑唯一性问题,如下图所示, $\Omega_0, \Omega_1, \Gamma, \Omega_\tau$ 构成梯形的边界, K_τ 表示梯形边界及其内部,梯形斜边为 $x = x_0 - at$.



考虑对波动方程进行变换

$$\int_{K_{\tau}} \frac{\partial u}{\partial t} \left(\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} \right) dx dt = \int_{K_{\tau}} \frac{\partial u}{\partial t} f(x, t) dx dt.$$

对左式进行化简

左式 =
$$\int_{K_{\tau}} \left\{ \frac{1}{2} \frac{\partial}{\partial t} \left((\frac{\partial u}{\partial t})^2 + a^2 (\frac{\partial u}{\partial x})^2 \right) - a^2 \frac{\partial}{\partial x} (\frac{\partial u}{\partial x} \frac{\partial u}{\partial t}) \right\} dx dt$$

$$\frac{\text{Green 公式}}{\frac{\partial u}{\partial x}} - \int_{\partial K_{\tau}} \frac{1}{2} \left((\frac{\partial u}{\partial t})^2 + a^2 (\frac{\partial u}{\partial x})^2 \right) dx + a^2 (\frac{\partial u}{\partial x} \frac{\partial u}{\partial t}) dt$$

$$= \int_{\Omega} \frac{1}{2} (u_t^2 + a^2 u_x^2) dx - \int_{\Omega_2} \frac{1}{2} (\psi^2 + a^2 \varphi_x^2) dx + \int_{\Omega_1} a^2 \mu_x \mu_t dt - \int_{\Gamma} \frac{1}{2} (u_t^2 + a^2 u_x^2) dx + a^2 (u_x u_t) dt.$$

を Γ 上有 dx =
$$-a$$
 dt,则右式第四项等于 $\int_0^\tau \frac{a}{2}(u_t + au_x)^2 dt \ge 0$.
由于 $\int_{K_\tau} 2u_t f \, dx \, dt \le \int_{K_\tau} u_t^2 \, dx \, dt + \int_{K_\tau} f^2 \, dx \, dt$,于是有
$$\int_{\Omega_\tau} (u_t^2 + a^2 u_x^2) \, dx \le \int_{\Omega_0} (\psi^2 + a^2 \varphi_x^2) \, dx - 2 \int_{\Omega_1} a^2 \mu_x \mu_t \, dt + \int_{K_\tau} f^2 \, dx \, dt + \int_{K_\tau} u_t^2 \, dx \, dt$$
 令 $G(\tau) = \int_{K_\tau} (u_t^2 + a^2 u_x^2) \, dx = \int_0^\tau \int_{\Omega_\tau} (u_t^2 + a^2 u_x^2) \, dx \, dt$,所以有 $G(0) = 0$ 且.
$$\frac{dG(\tau)}{d\tau} \le G(\tau) + F(\tau)$$

其中 $F(\tau)=\int_{\Omega_0}(\psi^2+a^2\varphi_x^2)\,\mathrm{d}x-2\int_{\Omega_1}a^2\mu_x\mu_t\,\mathrm{d}t+\int_{K_\tau}f^2\,\mathrm{d}x\,\mathrm{d}t.$ 由 Gronwall 不等式可知 $G(\tau)\leqslant MF(\tau)$,于是可得到类似能量不等式结果

$$\int_{\Omega_{\tau}} (u_t^2 + a^2 u_x^2) \, \mathrm{d}x \leqslant M_1 \left(\int_{\Omega_0} (\psi^2 + a^2 \varphi_x^2) \, \mathrm{d}x - 2 \int_{\Omega_1} a^2 \mu_x \mu_t \, \mathrm{d}t + \int_{K_{\tau}} f^2 \, \mathrm{d}x \, \mathrm{d}t \right)$$

进一步,可得到能量模估计

$$\int_{\Omega_{\tau}} u^2(x,\tau) \, \mathrm{d}x \leq M_2 \left(\int_{\Omega_0} (\varphi^2 + \psi^2 + a^2 \varphi_x^2) \, \mathrm{d}x - 2 \int_{\Omega_1} a^2 \mu_x \mu_t \, \mathrm{d}t + \int_{K_{\tau}} f^2 \, \mathrm{d}x \, \mathrm{d}t \right)$$

不难得到,当 $\varphi = \psi = \mu = f = 0$ 时,该问题只有零集,所以上式说明了该问题解的唯一性. 题目 **6.** 证明以下 Cauchy 问题

$$\begin{cases} u_{tt} - a^2 u_{xx} + b(x, t) u_x + c(x, t) u_t = f(x, t), & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = \varphi(x), \ u_t|_{t=0} = \psi(x), & x \in \mathbb{R} \end{cases}$$

解的唯一性,其中 b(x,t), c(x,t) 都是有界连续函数.

解答. 类似于能量不等式证明,首先对第一式左右同乘 u_t ,再在 K_{τ} 上进行积分可得

$$\int_{K_\tau} \left\{ \frac{1}{2} \frac{\partial}{\partial t} (u_t^2 + a^2 u_x^2) - a^2 \frac{\partial}{\partial x} (u_x u_t) \right\} \, \mathrm{d}x \, \mathrm{d}t + \int_{K_\tau} b(x,t) u_x u_t + c(x,t) u_t^2 \, \mathrm{d}x \, \mathrm{d}t = \int_{K_\tau} u_t f \, \mathrm{d}x \, \mathrm{d}t.$$

对左式第一项进行变化并使用 Green 公式转化为第二型曲线积分,再证明 $J_3\geqslant 0$,于是可得

$$\int_{\Omega_{\tau}} (u_t^2 + a^2 u_x^2) \, \mathrm{d}x \leqslant \int_{\Omega_0} (\psi^2 + a^2 \varphi_x^2) \, \mathrm{d}x - 2 \int_{K_{\tau}} (b(x,t) u_x u_t + c(x,t) u_t^2) \, \mathrm{d}x \, \mathrm{d}t + \int_{K_{\tau}} u_t^2 \, \mathrm{d}x \, \mathrm{d}t + \int_{K_{\tau}} f^2 \, \mathrm{d}x \, \mathrm{d}t.$$

由于
$$b(x,t),\ c(x,t)$$
 有界,不妨令 $C=\sup_{(x,t)\in K_{\tau}}2|b(x,t)|+2|c(x,t)|$, $C_0=\max\{\frac{C}{a^2},2C\}$,于是

$$\begin{split} -2 \int_{K_{\tau}} (b(x,t) u_x u_t + c(x,t) u_t^2) \, \mathrm{d}x \, \mathrm{d}t \leqslant C \int_{K_{\tau}} (u_x u_t + u_t^2) \, \mathrm{d}x \, \mathrm{d}t \leqslant \int_{K_{\tau}} C(2 u_t^2 + u_x^2) \, \mathrm{d}x \, \mathrm{d}t \\ \leqslant \int_{K_{\tau}} C_0 (u_t^2 + a^2 u_x^2) \, \mathrm{d}x \, \mathrm{d}t. \end{split}$$

记
$$G(\tau) = \int_{K_{\tau}} (u_t^2 + a^2 u_x^2) \, \mathrm{d}x \, \mathrm{d}t, \ F(\tau) = \int_{\Omega_0} (\psi^2 + a^2 \varphi_x^2) \, \mathrm{d}x + \int_{K_{\tau}} f^2 \, \mathrm{d}x \, \mathrm{d}t, \$$
于是有
$$\frac{\mathrm{d}G(\tau)}{\mathrm{d}\tau} \leqslant (C_0 + 1)G(\tau) + F(\tau)$$

由 Gronwall 不等式可知 $G(\tau) \leq MF(\tau)$,于是可以得到能量不等式,进一步,可以得到能量模不等式,从而说明该 Cauchy 问题具有唯一解.

题目 7. 试问下述半无界问题

$$\begin{cases} u_{tt} - u_{xx} + u_t + u_x = 0, & x > 0, t > 0, \\ u|_{x=0} = 0, & t \geqslant 0, \\ u|_{t=0} = \varphi(x), \ u_t|_{t=0} = \psi(x), & x \geqslant 0 \end{cases}$$

能否直接用对称开拓法求解? 为什么? 试用特征线法求解

解答. 通过下属求解解的表达式可知,该问题的解无法保证与 φ , ψ 保持相同的奇偶性,因此无法使用对称开拓法进行求解. 下面使用特征线法进行求解.

由于
$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + 1\right) \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$
. $\diamondsuit u_t + u_x = v$,则上述半平

面问题等价于求解

$$\begin{cases} u_t + u_x = v, \\ u|_{x=0} = 0, \\ u(x,0) = \varphi(x). \end{cases}$$
 (1)
$$\begin{cases} v_t - v_x = -v, \\ v(x,0) = u_t(x,0) + u_x(x,0) = \psi(x) + \varphi'(x). \end{cases}$$
 (2)

先求解
$$(2)$$
 式,令
$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -1, \\ x(0) = c. \end{cases}$$
 则 $x = -t + c, \ c = x + t, \$ 方程 (2) 转化为 $\frac{\mathrm{d}v}{\mathrm{d}t} = -v$,于是

$$v = (\psi(c) + \varphi'(c))e^{-t} = (\psi(x+t) + \varphi'(x+t))e^{-t},$$

再求解方程 (1),令 $\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = 1, \\ x(0) = c. \end{cases}$ 则 $x = t + c, \ c = x - t, \ 方程 (1) 转化为 <math>\frac{\mathrm{d}u}{\mathrm{d}t} = (\psi(x + t) + t)$

 $\varphi'(x+t)$) e^{-t} ,由于x>0,t>0,分两类情况讨论,当 $x\geqslant t$ 时

$$\begin{split} u &= \varphi(x(0)) + \int_0^t \{\psi(x(\tau) + \tau) + \varphi'(x(\tau) + \tau)\} \mathrm{e}^{-\tau} \, \mathrm{d}\tau \\ &= \varphi(c) + \int_0^t \{\psi(c + 2\tau) + \varphi'(c + 2\tau)\} \mathrm{e}^{-\tau} \, \mathrm{d}\tau \\ &\stackrel{\underline{\xi = c + 2\tau}}{=} \varphi(c) + \frac{1}{2} \int_c^{c + 2t} \{\psi(\xi) + \varphi'(\xi)\} \mathrm{e}^{-\frac{\xi - c}{2}} \, \mathrm{d}\xi \\ &= \varphi(x - t) + \frac{1}{2} \int_{x - t}^{x + t} \{\psi(\xi) + \varphi'(\xi)\} \mathrm{e}^{-\frac{\xi - x + t}{2}} \, \mathrm{d}\xi \\ &= \frac{1}{2} \varphi(x - t) + \frac{1}{2} \varphi(x + t) \mathrm{e}^{-t} + \frac{1}{2} \int_{x - t}^{x + t} \{\psi(\xi) + \varphi(\xi)\} \mathrm{e}^{-\frac{\xi - x + t}{2}} \, \mathrm{d}\xi \end{split}$$

当 $0 \leqslant x < t$ 时

$$\begin{split} u &= \int_{t-x}^t \{\psi(x(\tau) + \tau) + \varphi'(x(\tau) + \tau)\} \mathrm{e}^{-\tau} \, \mathrm{d}\tau \\ &= \int_{t-x}^t \{\psi(c + 2\tau) + \varphi'(c + 2\tau)\} \mathrm{e}^{-\tau} \, \mathrm{d}\tau \\ &= \frac{1}{2} \int_{c+2t-2x}^{c+2t} \{\psi(\xi) + \varphi'(\xi)\} \mathrm{e}^{-\frac{\xi-c}{2}} \, \mathrm{d}\xi \\ &= \frac{1}{2} \int_{t-x}^{t+x} \{\psi(\xi) + \varphi'(\xi)\} \mathrm{e}^{-\frac{\xi-x+t}{2}} \, \mathrm{d}\xi \\ &= \frac{1}{2} \varphi(x+t) \mathrm{e}^{-t} - \frac{1}{2} \varphi(t-x) \mathrm{e}^{-t+x} + \frac{1}{2} \int_{t-x}^{x+t} \{\psi(\xi) + \varphi(\xi)\} \mathrm{e}^{-\frac{\xi-x+t}{2}} \, \mathrm{d}\xi \end{split}$$

题目 8. 求解古尔沙问题

$$\begin{cases} u_{tt} = u_{xx}, & t > |x|, \\ u|_{t=-x} = \varphi(x), & x \leq 0, \\ u|_{t=x} = \psi(x), & x \geq 0, \end{cases}$$

其中 $\varphi(0)=\psi(0)$. 如果 $\varphi(x)$ 给定在 (-a,0], $\psi(x)$ 给定在 [0,b],指出此定解条件的决定区域.

解答. 使用行波法求解,令 $\begin{cases} \xi = x - t, \\ \eta = x + t. \end{cases}$ 则 $u_{tt} = u_{xx} \Rightarrow u_{\xi\eta} = 0 \Rightarrow u = F(\xi) + G(\eta) = 0$

$$F(x-t) + G(x+t)$$

当
$$x \leqslant 0$$
 时, $u|_{t=-x} = u(x, -x) = F(2x) + G(0) = \varphi(x).$

当
$$x \geqslant 0$$
 时, $u|_{t=x} = F(0) + G(2x) = \psi(x)$. 且 $F(0) + G(0) = \varphi(0) = \psi(0)$.

于是
$$\begin{cases} F(x) = \varphi(\frac{x}{2}) - G(0), \\ G(x) = \psi(\frac{x}{2}) - F(0). \end{cases} \Rightarrow u = \varphi(\frac{x-t}{2}) + \psi(\frac{x+t}{2}) - \varphi(0).$$

当给定 φ 在 (-a,0], ψ 在 [0,b], 于是决定区域为

$$\begin{cases}
-a < \frac{x-t}{2} \leqslant 0, \\
0 \leqslant \frac{x+t}{2} \leqslant b.
\end{cases} \Rightarrow \begin{cases}
x \leqslant t < x+2a, \\
-x \leqslant t \leqslant 2b-x.
\end{cases}$$