

第二章第二次作业

题目 1. (19) 求解三维波动方程的 Cauchy 问题

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = 0, \\ u_t|_{t=0} = x^3 + y^2z. \end{cases}$$

解答. 由 Kirchhoff 公式可知:

$$\begin{aligned} u(x, t) &= \frac{1}{4\pi a^2 t} \iint_{B_{at}(x)} x^3 + y^2 z \, dS \\ &= \frac{1}{4\pi a^2 t} \iint_{B_{at}} (x + x_1)^3 + (y + x_2)^2 (z + x_3) \, dS \\ &= \frac{1}{4\pi a^2 t} \iint_{B_{at}} 2x_1 x^2 + x_3 y^2 \, dS + (x_1^3 + x_2^2 x_3) t \\ &= \frac{a^3 t^4}{15} (2x_1 + x_3) + (x_1^3 + x_2^2 x_3) t \end{aligned}$$

题目 2. 20 用降维法导出一维波动方程 Cauchy 问题的求解公式.

解答. 由二维 Cauchy 问题公式可知

$$\begin{aligned} &\iint_{\Sigma_{at}(x)} \frac{\varphi(y)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \, dy \\ &= 2 \int_{x_1 - at}^{x_1 + at} \varphi(y_1) \int_{x_2}^{x_2 + \sqrt{a^2 t^2 - (y_1 - x_1)^2}} \frac{1}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \, dy_2 \, dy_1 \end{aligned}$$

由于

$$\begin{aligned} &\int_{x_2}^{x_2 + \sqrt{b}} \frac{1}{\sqrt{b - (y_2 - x_2)^2}} \, dy_2 \stackrel{y=y_2-x_2}{=} \int_0^{\sqrt{b}} \frac{1}{\sqrt{b - y^2}} \, dy \\ &\stackrel{y=\sqrt{b}z}{=} \int_0^1 \frac{1}{\sqrt{1 - z^2}} \, dz \stackrel{z=\cos \theta}{=} \int_0^{\frac{\pi}{2}} \, d\theta = \frac{\pi}{2} \end{aligned}$$

则

$$\begin{aligned} &\iint_{\Sigma_{at}(x)} \frac{\varphi(y)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \, dy = \pi \int_{x_1 - at}^{x_1 + at} \varphi(y_1) \, dy_1 \\ &\iint_{\Sigma_{at}(x)} \frac{\psi(y)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \, dy = \pi \int_{x_1 - at}^{x_1 + at} \psi(y_1) \, dy_1 \\ &\iint_{\Sigma_{a(t-\tau)}(x)} \frac{f(y, \tau)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \, dy = \pi \int_{x_1 - a(t-\tau)}^{x_1 + a(t-\tau)} f(y_1, \tau) \, dy_1 \end{aligned}$$

代入到二维 Cauchy 问题解的公式中

$$\begin{aligned} u(x, t) = & \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{\Sigma_{at}(x)} \frac{\varphi(y)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy \right] \\ & + \frac{1}{2\pi a} \iint_{\Sigma_{at}(x)} \frac{\psi(y)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy \\ & + \frac{1}{2\pi a} \int_0^t \iint_{\Sigma_{a(t-\tau)}(x)} \frac{f(y, \tau)}{\sqrt{a^2 t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} dy \end{aligned}$$

即可得到一维解公式

$$\begin{aligned} u(x, t) = & \frac{1}{2} [\varphi(x + at) - \varphi(x - at)] + \frac{1}{2a} \int_{x_1 - at}^{x_1 + at} \psi(\xi) d\xi \\ & + \frac{1}{2a} \int_0^t dt \int_{x_1 - a(t-\tau)}^{x_1 + a(t-\tau)} f(\xi, \tau) d\tau \end{aligned}$$

解答. 求解二维波动方程的 Cauchy 问题:

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}), \\ u|_{t=0} = x^2(x + y), \\ u_t|_{t=0} = 0. \end{cases}$$

解答. 代入到二维 Cauchy 问题解的公式中

$$\begin{aligned} u(x, t) = & \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{\Sigma_{at}} \frac{(y_1 + x_1)^2(y_1 + y_2 + x_1 + x_2)}{\sqrt{a^2 t^2 - y_1^2 - y_2^2}} dy_1 dy_2 \right] \\ = & \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{\Sigma_{at}} \frac{(3x_1 + x_2)y_1^2 + (x_1 + x_2)x_1^2}{\sqrt{a^2 t^2 - y_1^2 - y_2^2}} dy_1 dy_2 \right] \\ \stackrel{\substack{y_1=r \cos \theta \\ y_2=r \sin \theta}}{=} & \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[(3x_1 + x_2) \int_0^{at} \int_0^{2\pi} \frac{r^2 \cos^2 \theta}{\sqrt{a^2 t^2 - r^2}} r d\theta dr + 2\pi(x_1 + x_2)x_1^2 \int_0^{at} \frac{r}{\sqrt{a^2 t^2 - r^2}} dr \right] \end{aligned}$$

注意到以下积分

$$\int_0^a \frac{1}{\sqrt{a^2 - x^2}} dx = \frac{\pi}{2}, \quad \int_0^a \frac{x}{\sqrt{a^2 - x^2}} dx = a, \quad \int_0^a \frac{x^2}{\sqrt{a^2 - x^2}} dx = \frac{\pi}{4} a^2, \quad \int_0^a \frac{x^3}{\sqrt{a^2 - x^2}} dx = \frac{2}{3} a^3.$$

于是

$$\begin{aligned} u(x, t) = & \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[(3x_1 + x_2) \frac{2\pi a^3 t^3}{3} + 2\pi(x_1 + x_2)x_1^2 at \right] \\ = & a^2 t^2 (3x_1 + x_2) + (x_1 + x_2)x_1^2 \end{aligned}$$

题目 3. (22.(4)) 求解以下特征值问题的特征函数:

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(0) = X'(l) + hX(l) = 0 \quad (h > 0 \text{ 常数}). \end{cases}$$

解答. 求解常微分方程可得

$$X(x) = C_1 \sin \sqrt{\lambda}x + C_2 \cos \sqrt{\lambda}x, \quad X'(x) = C_1 \sqrt{\lambda} \cos \sqrt{\lambda}x - C_2 \sqrt{\lambda} \sin \sqrt{\lambda}x.$$

则

$$X(0) = C_2 = 0, \quad X'(l) + hX(l) = C_1 \sqrt{\lambda} \cos \sqrt{\lambda}l + hC_1 \sin \sqrt{\lambda}l = 0,$$

下求 $C_1 \neq 0$ 的非平凡解, 当 $\cos \sqrt{\lambda}l = 0$ 时, 则 $\lambda = \left(\frac{(2n+1)\pi}{2l}\right)^2$, ($n = 0, 1, 2, \dots$), 当 $\cos \sqrt{\lambda}l \neq 0$ 时, 则

$$\sqrt{\lambda} + h \cdot \tan \sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda} = h \cdot \tan \sqrt{\lambda}l$$

该方程为超越方程, 只能求 λ 的近似解.

题目 4. 23.(2) 用分离变量法求解以下定解问题:

$$\begin{cases} Lu = 0, & (x, t) \in Q, \\ u|_{x=0} = u_x|_{x=l} = 0, & t \geq 0, \\ u|_{t=0} = x(x-2l), \quad u_t|_{t=0} = 0, & 0 \leq x \leq l. \end{cases}$$

解答. 令 $u = X(x)T(t)$, 则 $XT'' - a^2X''T = 0 \Rightarrow \frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$ 于是

$$\begin{cases} X'' + \lambda X = 0, \\ T'' + a^2\lambda T = 0, \\ X(0) = X'(l) = 0. \end{cases}$$

求解常微分方程可得

$$X(x) = C_1 \sin \sqrt{\lambda}x + C_2 \cos \sqrt{\lambda}x, \quad X'(x) = C_1 \sqrt{\lambda} \cos \sqrt{\lambda}x - C_2 \sqrt{\lambda} \sin \sqrt{\lambda}x.$$

则

$$X(0) = C_2 = 0, \quad X'(l) = \sqrt{\lambda}C_1 \cos \sqrt{\lambda}l = 0$$

于是 $\lambda = \left(\frac{(2n-1)\pi}{2l}\right)^2$, ($n = 1, 2, \dots$), 则

$$\begin{aligned} X(x) &= C \sin \frac{(2n-1)\pi}{2l}x \\ T(t) &= A_n \sin \left(\frac{a(2n-1)\pi}{2l}t\right) + B_n \cos \left(\frac{a(2n-1)\pi}{2l}t\right) \end{aligned}$$

则

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \sin \left(\frac{a(2n-1)\pi}{2l} t \right) + B_n \cos \left(\frac{a(2n-1)\pi}{2l} t \right) \right) \sin \frac{(2n-1)\pi}{2l} x,$$

$$u|_{t=0} = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi}{2l} x = x^2 - 2lx,$$

$$u_t|_{t=0} = \sum_{n=1}^{\infty} \frac{a(2n-1)\pi}{2l} A_n \sin \frac{(2n-1)\pi}{2l} x = 0,$$

于是 $A_n = 0$, $B_n = \frac{2}{l} \int_0^l (x^2 - 2lx) \sin \frac{(2n-1)\pi}{2l} x \, dx = \frac{16l^2}{(2n-1)^2 \pi^2} \left((-1)^{n-1} - \frac{2}{(2n-1)\pi} \right)$,

则

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cos \left(\frac{a(2n-1)\pi}{2l} t \right) \sin \frac{(2n-1)\pi}{2l} x$$

题目 5. 设 $u(x, t)$ 适合定解问题:

$$\begin{cases} Lu = f(x, t), & (x, t) \in Q, \\ \left(-\frac{\partial u}{\partial x} + \alpha u \right)_{x=0} = \mu_1(t), & t \geq 0, \\ \left(\frac{\partial u}{\partial x} + \beta u \right)_{x=l} = \mu(t), & t \geq 0, \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), & 0 \leq x \leq l, \end{cases}$$

试引进辅助函数, 把边界条件齐次化, 设

(a) $\alpha > 0, \beta > 0$; (b) $\alpha = \beta = 0$.

解答. (a) 令 $v = \frac{l-x}{l}(-u_x + \alpha u - \mu_1(t)) + \frac{x}{l}(u_x + \beta u - \mu(t))$, 则 $v|_{x=0} = v|_{x=l} = 0$.

(b) 令 $v = \frac{l-x}{l}(-u_x - \mu_1(t)) + \frac{x}{l}(u_x - \mu(t))$, 则 $v|_{x=0} = v|_{x=l} = 0$.

题目 6. (26) 用分离变量法求解下列定解问题:

$$\begin{cases} Lu = -2b \frac{\partial u}{\partial t} + g, & (x, t) \in Q, \\ u|_{x=0} = u|_{x=l} = 0, & t \geq 0, \\ u|_{t=0} = u_t|_{t=0} = 0, & 0 \leq x \leq l. \end{cases}$$

题目 7. (27) 考虑定解问题:

$$\begin{cases} u_{tt} - u_{xx} = f(x, t), & (x, t) \in Q, \\ u|_{x=0} = u|_{x=l} = 0, & 0 \leq t \leq T, \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), & 0 \leq x \leq l. \end{cases}$$

试问对 φ, ψ, f 加什么条件才能保证由 Fourier 方法所得的解是古典解?

题目 8. (28) 用能量不等式证明一维波动方程带有第三边值条件的初边值问题解的唯一性.