日期	科目	班级	姓名	学号
2023 年 4 月 25 日	微分几何	强基数学 002	吴天阳	2204210460

第六次作业

题目 1. 4.3 练习 1. 证明定义 4.8 中定义的 $\partial_u(A)f := \left.\dfrac{\partial(f\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)}$ 的确是 A 点的导算子.

证明. 1. 局部性：f, g 为定义在 A 邻域中的两个光滑函数，且存在邻域 U，使得在 U 上有 $f=g$ 则

$$\frac{\partial(f\circ\varphi)}{\partial u}(x) = \frac{\partial(g\circ\varphi)}{\partial u}(x),\quad (x\in\varphi^{-1}(U))$$

于是

$$\partial_u f = \left.\frac{\partial(f\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} = \left.\frac{\partial(g\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} = \partial_u g$$

2. 线性性：$\forall\alpha,\beta\in\mathbb{R}$ 有

$$\partial_u(\alpha f+\beta g) = \left.\frac{\partial(\alpha f\circ\varphi+\beta g\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} = \alpha\left.\frac{\partial(f\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} + \beta\left.\frac{\partial(g\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} = \alpha\partial_u f+\beta\partial_u g$$

3. Leibniz 公式

$$\partial_u(fg) = \left.\frac{\partial(f\circ\varphi\cdot g\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} = \left.\frac{\partial(f\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} g(A) + \left.\frac{\partial(g\circ\varphi)}{\partial u}\right|_{\varphi^{-1}(A)} f(A) = g(A)\partial_u f+f(A)\partial_u g$$

□

题目 2. 4.3 练习 2. 考虑球面上的参数化：

$$x^1=\sin\theta\cos\varphi,\ x^2=\sin\theta\sin\varphi,\ x^3=\cos\theta$$

写出这个参数化的局部参数标架场，在 $\mathbb{R}^3$ 中的 $\{\boldsymbol{e}_1=(1,0,0),\boldsymbol{e}_2(0,1,0),\boldsymbol{e}_3(0,0,1)\}$ 下表出.

解答. 设任意的光滑函数 $f(x_1,x_2,x_3)=f(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$，于是

$$\begin{aligned}\partial_\theta f &= \partial_1 f\cdot\cos\theta\cos\varphi+\partial_2 f\cdot\cos\theta\sin\varphi-\partial_3 f\cdot\sin\theta\\ \partial_\varphi f &= \partial_1 f\cdot(-\sin\theta\sin\varphi)+\partial_2 f\cdot\sin\theta\cos\varphi\end{aligned}$$

于是 $\partial_\theta=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta),\partial_\varphi=(-\sin\theta\sin\varphi,\sin\theta\cos\varphi,0)$.

题目 3. 4.3 练习 3. 考虑 $\mathbb{R}^3$ 中被表示成函数图像的一部分的曲面 $S=\{(x^1,x^2,f(x^1,x^2)),-\varepsilon<x^1,x^2<\varepsilon\}$，求 S 上任一点的切空间，在 $\mathbb{R}^3$ 中的 $\{\boldsymbol{e}_1=(1,0,0),\boldsymbol{e}_2(0,1,0),\boldsymbol{e}_3(0,0,1)\}$ 下表出.

解答. 设曲面上的任意光滑函数 g，由参数 x_1,x_2 表出为 $g(x_1,x_2,f(x_1,x_2))$，则有

$$\partial_{x^1}g=\partial_1 g,\qquad \partial_{x^2}g=\partial_2 g$$

于是 $\partial_{x^1}=(1,0,0),\partial_{x^2}=(0,1,0)$，故任意点的切空间为 $\mathrm{span}\{(1,0,0),(0,1,0)\}$.

题目 4. 5.1 练习 1. 我们考虑 $\mathbb{E}^3$ 上的标准正交坐标系 $\{O,e_i\}$，考虑标准圆柱面

$$x^1=\cos\theta,\ x^2=\sin\theta,\ x^3=x^3$$

考虑参数坐标下，质点 P 在 $t=0$ 时位置是 $\theta(0)=x^3(0)=0$，初速度是 $\partial_\theta+\partial_3$. 设该自由质点只收到柱面的约束力，计算该质点的运动方程 $(\theta(t),x^3(t))$.

解答. 设 f 为曲面上的光滑函数，则

$$\partial_\theta f=\frac{\partial(f(\cos\theta,\sin\theta,x^3))}{\partial\theta}=-\sin\theta\cdot\partial_1 f+\cos\theta\cdot\partial_2 f+\partial_3 f$$

于是 $\partial_\theta=(-\sin\theta,\cos\theta,0),\partial_1=(0,0,1)$，则

$$\boldsymbol{g}=\begin{bmatrix} g_{\theta\theta} & g_{\theta 3}\\ g_{3\theta} & g_{33}\end{bmatrix}=\begin{bmatrix}(\partial_\theta,\partial_\theta) & (\partial_\theta,\partial_3)\\ (\partial_3,\partial_\theta) & (\partial_3,\partial_3)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I=\boldsymbol{g}^{-1}$$

则 $\Gamma^l_{ab}=\dfrac{1}{2}g^{lc}(\partial_a g_{bc}+\partial_b g_{ac}-\partial_c g_{ab})=0$.

设测地线为 $y(\theta(t),x^3(t))$，由初值条件 $y(0)=(\theta(0),x^3(0))=(0,0)$, $\hat{y}(0)=\partial_\theta+\partial_3=(1,1)$，可知

$$\begin{cases}\ddot{y}^1(t)=0,\\ \ddot{y}^2(t)=0.\end{cases}\Rightarrow\begin{cases}\theta(t)=y^1(t)=y^1(0)+\dot{y}^1(0)t+\dfrac{1}{2}t^2=t+\dfrac{1}{2}t^2,\\ x^3(t)=y^2(t)=y^2(0)+\dot{y}^2(0)t+\dfrac{1}{2}t^2=t+\dfrac{1}{2}t^2.\end{cases}$$

故测地线方程为 $\left(t+\dfrac{1}{2}t^2,t+\dfrac{1}{2}t^2\right)$.

题目 5. 我们考虑 $\mathbb{E}^3$ 上的标准正交坐标系 $\{O,e_i\}$，考虑以 O 为球心的单位球面. 考虑点 $(1,0,0)$ 附近的参数化：

$$x^1=\cos\theta\sin(\pi/2+\varphi),\ x^2=\sin\theta\sin(\pi/2+\varphi),\ x^3=\cos(\pi/2+\varphi)$$

参数 (θ,φ) 的变化区域为 $(-\pi/2,\pi/2)\times(-\pi/2,\pi/2)$. 设自由质点 P 的初始位置为 $\theta(0)=\varphi(0)=0$，初速度为 $\partial\theta+\partial\varphi$，请计算质点的运动方程 $(\theta(t),\varphi(t))$.

解答. 设曲面上的光滑函数为 f，则

$$\begin{aligned}\partial_\theta f&=\frac{\partial f(\cos\theta\sin(\frac{\pi}{2}+\varphi),\sin\theta\sin(\frac{\pi}{2}+\varphi),\cos(\frac{\pi}{2}+\varphi))}{\partial\theta}\\ &=\partial_1 f\cdot(-\sin\theta\sin(\pi/2+\varphi))+\partial_2 f\cdot(\cos\theta\sin(\pi/2+\varphi))\\ \partial_\varphi f&=\frac{\partial f(\cos\theta\sin(\frac{\pi}{2}+\varphi),\sin\theta\sin(\frac{\pi}{2}+\varphi),\cos(\frac{\pi}{2}+\varphi))}{\partial\varphi}\\ &=\partial_1 f\cdot(\cos\theta\cos(\pi/2+\varphi))+\partial_2 f\cdot(\sin\theta\cos(\pi/2+\varphi))+\partial_3 f\cdot(-\sin(\pi/2+\varphi))\end{aligned}$$

于是

$$\partial_\theta=(-\sin\theta\sin(\pi/2+\varphi),\cos\theta\sin(\pi/2+\varphi),0),$$
$$\partial\varphi=(\cos\theta\cos(\pi/2+\varphi),\sin\theta\cos(\pi/2+\varphi),-\sin(\pi/2+\varphi))$$
$$(\partial_\theta,\partial_\theta)=\ \sin^2(\pi/2+\varphi),\quad(\partial\theta,\partial\varphi)=(\partial_\varphi,\partial_\theta)=0,\quad(\partial\varphi,\partial\varphi)=1$$

则

$$\boldsymbol{g}=\begin{bmatrix}\sin^2(\pi/2+\varphi) & 0\\ 0 & 1\end{bmatrix},\quad \boldsymbol{g}^{-1}\begin{bmatrix}\dfrac{1}{\sin^2(\pi/2+\varphi)} & 0\\ 0 & 1\end{bmatrix}$$

于是

$$\begin{aligned}
&\Gamma^1_{ab}=\frac{1}{2}g^{11}(\partial_a g b1+\partial_b g_{a1})\\
\Rightarrow\ &\Gamma^1_{12}=\Gamma^1_{21}=\frac{1}{\tan(\pi/2+\varphi)}\text{ 且 }\Gamma^1_{ab}=0,\ (a,b)\notin\{(1,2),(2,1)\}\\
&\Gamma^2_{ab}=\frac{1}{2}g^{22}(\partial_a g_{b2}+\partial_b g_{a2}-\partial_\varphi g_{ab})\\
\Rightarrow\ &\Gamma^2_{11}=-\sin(\pi/2+\varphi)\cos(\pi/2+\varphi)=-\frac{1}{2}\sin(\pi+2\varphi)\text{ 且 }\Gamma^2_{ab}=0,\ (a,b)\neq(1,1)
\end{aligned}$$

设测地线为 $y(\theta(t),\varphi(t))$，满足一下初值条件：

$$\begin{cases}\tan(\pi/2+\varphi)\ddot{y}^1+2\dot{y}^1\dot{y}^2=0\\ \ddot{y}^2-\dfrac{1}{2}\sin(\pi+2\varphi)\dot{y}^1\dot{y}^1=0\end{cases},\quad \text{初值条件}\begin{cases}y(0)=(0,0),\\ \hat{y}(0)=(1,1).\end{cases}$$