日期	科目	班级	姓名	学号	序号
2022 年 5 月 13 日	数值分析	强基数学 002	吴天阳	2204210460	59

9.1 取 $h=0.1$, 用欧拉法、后退欧拉法、中点法、梯形法求解初值问题:

$$y'=1-y,\qquad y(0)=0,\qquad 0\leqslant x\leqslant 1.$$

解答. 欧拉法: $y_{i+1}=y_i+hf(x_i,y_i)=0.9y_i+0.1$, 根据递推式可得

$$\begin{aligned}&y_0=0.000,\quad y_1=0.100,\quad y_2=0.190,\quad y_3=0.271,\quad y_4=0.344,\quad y_5=0.410,\\&y_6=0.469,\quad y_7=0.522,\quad y_8=0.570,\quad y_9=0.613,\quad y_{10}=0.652.\end{aligned}$$

后退欧拉法: $y_{i+1}=y_i+hf(x_{i+1},y_{i+1})\Rightarrow y_{i+1}=\dfrac{10}{11}y_i+\dfrac{1}{11}$, 根据递推式可得

$$\begin{aligned}&y_0=0.000,\quad y_1=0.091,\quad y_2=0.174,\quad y_3=0.249,\quad y_4=0.317,\quad y_5=0.379,\\&y_6=0.435,\quad y_7=0.486,\quad y_8=0.533,\quad y_9=0.575,\quad y_{10}=0.614.\end{aligned}$$

中点法: $y_{i+1}=y_{i-1}+2hf(x_i,y_i)=y_{i-1}-0.2y_i+0.2$, 根据线性微分方程解法, 原初值问题的解为 $y=1-e^{-x}$, 中点法的启动值取为 $y_1=1-e^{-0.1}\approx 0.095$, 根据递推式可得

$$\begin{aligned}&y_0=0.000,\quad y_1=0.095,\quad y_2=0.181,\quad y_3=0.259,\quad y_4=0.329,\quad y_5=0.393,\\&y_6=0.450,\quad y_7=0.503,\quad y_8=0.549,\quad y_9=0.593,\quad y_{10}=0.630.\end{aligned}$$

梯形法: $y_{i+1}=y_i+\dfrac{h}{2}(f(x_i,y_i)+f(x_{i+1},y_{i+1}))\Rightarrow y_{i+1}=\dfrac{2}{21}+\dfrac{19}{21}y_i$, 根据递推式可得

$$\begin{aligned}&y_0=0.000,\quad y_1=0.095,\quad y_2=0.181,\quad y_3=0.259,\quad y_4=0.330,\quad y_5=0.394,\\&y_6=0.452,\quad y_7=0.504,\quad y_8=0.551,\quad y_9=0.594,\quad y_{10}=0.633.\end{aligned}$$

9.5 利用标准的 4 级 4 阶 R-K 法求解习题 9.1.

解答. 经典 4 级 4 阶 R-K 法递推式为

$$\begin{cases}y_{i+1}=y_i+\dfrac{1}{6}(K_1+2K_2+2K_3+K_4),\\ K_1=hf(x_i,y_i),\\ K_2=hf\left(x_i+\dfrac{1}{2}h,y_i+\dfrac{1}{2}K_1\right),\\ K_3=hf\left(x_i+\dfrac{1}{2}h,y_i+\dfrac{1}{2}K_2\right),\\ K_4=hf(x_i+h,y_i+K_3).\end{cases}\Rightarrow\begin{cases}y_{i+1}=y_i+\dfrac{1}{6}(K_1+2K_2+2K_3+K_4),\\ K_1=-0.1y_i+0.1,\\ K_2=0.1-0.1(y_i+0.5K_1),\\ K_3=0.1-0.1(y_i+0.5K_2),\\ K_4=0.1-0.1(y_i+K_3).\end{cases}$$

根据递推式可得

$$\begin{aligned}&y_0=0.000,\quad y_1=0.095,\quad y_2=0.181,\quad y_3=0.259,\quad y_4=0.330,\quad y_5=0.394,\\&y_6=0.452,\quad y_7=0.504,\quad y_8=0.551,\quad y_9=0.594,\quad y_{10}=0.633.\end{aligned}$$

9.7 利用待定系数法确定以下求解公式中的系数, 使其阶数尽可能高, 并导出截断误差表达式:

$$y_{i+1} = \alpha_0 y_i + \alpha_1 y_{i-1} + \beta h f_{i+1}.$$

解答. 由题可知, 截断误差为

$$\begin{aligned} R[y] = y(x_{i+1}) - y_{i+1} &= y(x_{i+1}) - \alpha_0 y(x_i) - \alpha_1 y(x_{i-1}) - \beta h y'(x_{i+1}) \\ &= y(x_i + h) - \alpha_0 y(x_i) - \alpha_1 y(x_i - h) - \beta h y'(x_i + h), \end{aligned}$$

取 $x_i = 0$, 令 $R[x^k] = 0\ (k = 0, 1, 2)$ 得

$$\begin{cases} 1 - \alpha_0 - \alpha_1 = 0, \\ h(1 + \alpha_1 - \beta) = 0, \\ h^2(1 - \alpha_1 - 2\beta) = 0. \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha_0 = \dfrac{4}{3}, \\ \alpha_1 = -\dfrac{1}{3}, \\ \beta = \dfrac{2}{3}. \end{cases}$$

则 $y_{i+1} = \dfrac{1}{3}(4y_i - y_{i-1} + 2hf_{i+1})$, 于是 $R[y] = y(x_i + h) - \dfrac{4}{3}y(x_i) + \dfrac{1}{3}y(x_i - h) - \dfrac{2}{3}hy'(x_i + h)$, 取 $x_i = 0$, 令 $y = x^3$, 则

$$R[x^3] = h^3 - \frac{1}{h^3} - 2h^3 = -\frac{4}{3}h^3 \neq 0,$$

所以该公式的代数精度 $m = 2$, 取 $e(x) = \dfrac{1}{3!}y^{(3)}(\xi)(x - x_{i+1})(x - x_i)(x - x_{i-1})$, 由广义 Peano 定理知, 截断误差表示式为

$$R[y] = R[e(x)] = -\frac{2}{3}h \cdot \frac{1}{3!}y^3(\xi)(x_{i+1} - x_i)(x_{i+1} - x_{i-1}) = -\frac{2}{9}h^3 y^3(\xi).$$

9.11 将以下高阶微分方程化为一阶微分方程组初值问题:

$$\begin{cases} y'' = y'(1 - y^2) - y, \\ y(x_0) = y_0,\ y'(x_0) = y_0'. \end{cases}$$

解答. 令 $y_1 = y,\ y_2 = y'$, 则原高阶方程等价于求解以下一阶微分方程组

$$\begin{cases} y_1' = y_2, \\ y_2' = y_2(1 - y_1^2) - y_1, \\ y_1(x_0) = y_0,\ y_2(x_0) = y_0'. \end{cases}$$