日期	科目	姓名，学号
2022 年 3 月 23 日	复变函数	吴天阳 2204210460

第一章

1. 求复数 $1+i, 2-3i, 1+\cos\theta+i\sin\theta\ (-\pi\leqslant\theta<\pi)$ 的模和辐角主值。

解答. 取辐角主值在 $[-\pi,\pi)$。

(1). 令 $z=1+i$，则

$$|z| = \sqrt{2}$$
$$\arg\, z = \frac{\pi}{4}$$

(2). 令 $z=2-3i$，则

$$|z| = \sqrt{2^2+3^2} = \sqrt{13}$$
$$\arg\, z = \arctan\frac{-3}{2} = -\arctan\frac{3}{2}$$

(3). 令 $z=1+\cos\theta+i\sin\theta$，则

$$|z| = \sqrt{(1+\cos\theta)^2+\sin^2\theta} = \sqrt{2+2\cos\theta} = 2\cos\frac{\theta}{2}$$
$$\arg\, z = \arctan\frac{\sin\theta}{1+\cos\theta} = \arctan\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\theta}{2}$$

5. 证明:

(1) $|1-\bar{z}_1z_2|^2-|z_1-z_2|^2=(1-|z_1|^2)(1-|z_2|^2)$;

(2) 当 $|z_1|<1$，$|z_2|<1$ 之一成立时，$\left|\dfrac{z_1-z_2}{1-\bar{z}_1z_2}\right|<1$;

(3) 当 $|z_1|=1$ 或 $|z_2|=1$ 之一成立时，$\left|\dfrac{z_1-z_2}{1-\bar{z}_1z_2}\right|<1$.

证明. (1)

$$\begin{aligned}
|1-\bar{z}_1z_2|^2-|z_1-z_2|^2 &= (1-|z_1|^2)(1-|z_2|^2)\\
&= (1-\bar{z}_1z_2)(1-z_1\bar{z}_2)-(z_1-z_2)(\bar{z}_1-\bar{z}_2)\\
&= 1-2\mathrm{Re}z_1\bar{z}_2+|z_1|^2|z_2|^2-|z_1|^2+2\mathrm{Re}z_1\bar{z}_2-|z_2|^2\\
&= 1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2\\
&= (1-|z_1|^2)(1-|z_2|^2)
\end{aligned}$$

(2) 当 $|z_1|<1, |z_2|<1$，由 (1) 可知

$$\begin{aligned} &|1-\bar{z}_1 z_2|^2 > |z_1-z_2|^2 \\ \Rightarrow\ &\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|^2 < 1 \\ \Rightarrow\ &\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| < 1 \end{aligned}$$

(3) 当 $|z_1|=1$ 或 $|z_2|=1$ 之一成立时，由 (1) 可知

$$\begin{aligned} &|1-\bar{z}_1 z_2|^2 = |z_1-z_2|^2 \\ \Rightarrow\ &|1-\bar{z}_1 z_2| = |z_1-z_2| \\ \Rightarrow\ &\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| = 1 \end{aligned}$$

$\square$

11. 求出关于虚轴和圆周 $|z-2|=1$ 的公共对称点。

解答. 设 z_1, z_2 为满足题意的公共对称点，关于虚轴对称可知

$$z_2 = -\bar{z}_1$$

推导圆周的一般表达式

$$\begin{aligned} |z-2|=1 &\Rightarrow |z-2|^2=1 \\ &\Rightarrow (z-2)(\bar{z}-2)=1 \\ &\Rightarrow z\bar{z}-2z-2\bar{z}+3=0 \end{aligned}$$

由于 z_1, z_2 关于圆周对称，则

$$\begin{aligned} &z_2\bar{z}_1-2z_2-2\bar{z}_1+3=0 \\ \Rightarrow\ &-\bar{z}_1^2+2\bar{z}_1-2\bar{z}_1+3=0 \\ \Rightarrow\ &\bar{z}_1=\sqrt{3}=z_1 \end{aligned}$$

综上，满足题意的公共对称点为 $\sqrt{3}, -\sqrt{3}$。

第二章

3. 证明：(1) $\lim\limits_{n\to\infty}\left(1+\dfrac{x+iy}{n}\right)^n = e^{x+iy}$;

(2) 设 $z\neq 0, \lim\limits_{n\to\infty} n(\sqrt[n]{z}-1) = \log|z| + i\arg z + 2\pi i k (k=0,1,2,\cdots)$.

证明. (1)

$$\begin{aligned}
\left|\left(1+\frac{x+iy}{n}\right)^n\right| &= \left|1+\frac{x+iy}{n}\right|^n \\
&= \left((1+\frac{x}{n})^2+(\frac{y}{n})^2\right)^{n/2} \\
&= \left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{n/2} \\
&= \left(\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n}\right)^{n/2x}\right)^x \\
&= e^x \quad (n\to\infty)
\end{aligned}$$

$$\begin{aligned}
\arg\left(1+\frac{x+iy}{n}\right)^n &= n\,\arg\left(1+\frac{x+iy}{n}\right) \\
&= n\arctan\frac{y/n}{1+x/n} \\
&= n\frac{y/n}{1+x/n} \\
&= \frac{y}{1+x/n} \\
&= y \quad (n\to\infty)
\end{aligned}$$

则

$$\lim_{n\to\infty}\left(1+\frac{x+iy}{n}\right)^n = e^x\cdot e^{iy} = e^{x+iy}$$

(2) 令 $\arg z=\theta$，则 $z=|z|e^{i\theta+2\pi ik}$ $(k=0,1,2,\cdots)$，可得

$$\begin{aligned}
n(\sqrt[n]{z}-1) &= n(|z|^{1/n}e^{(i\theta+2\pi ik)/n}-1) \\
&= n\left(|z|^{1/n}\cos\frac{\theta+2k\pi}{n}-1+i|z|^{1/n}\sin\frac{\theta+2k\pi}{n}\right)
\end{aligned}$$

其中

$$\begin{aligned}
n\left(|z|^{1/n}\cos\frac{\theta+2k\pi}{n}-1\right) &= n\log\left(|z|^{1/n}\cos\frac{\theta+2k\pi}{n}\right) \\
&= \log|z|+n\log\cos\frac{\theta+2k\pi}{n} \\
&= \log|z| \quad (n\to\infty)
\end{aligned}$$

$$\begin{aligned}
in|z|^{1/n}\sin\frac{\theta+2k\pi}{n} &= in|z|^{1/n}\frac{\theta+2k\pi}{n} \\
&= i\theta+2\pi ik \quad (n\to\infty)
\end{aligned}$$

综上

$$\begin{aligned}
\lim_{n\to\infty} n(\sqrt[n]{z}-1) &= \log|z|+i\theta+2\pi ik \\
&= \log|z|+i\arg z+2\pi ik
\end{aligned}$$

□

5. 设 $f(t)$ 为 $[\alpha,\beta]$ 上复值连续函数，$c\in\mathbb{C}$，则

$$c\int_\alpha^\beta f(t)\,dt=\int_\alpha^\beta cf(t)\,dt$$

证明. 设 $c=a+ib$，$f(t)=x(t)+iy(t)$，则

$$\begin{aligned}c\int_\alpha^\beta f(t)\,dt&=(a+ib)\int_\alpha^\beta x(t)\,dt+i(a+ib)\int_\alpha^\beta y(t)\,dt\\&=\int_\alpha^\beta(ax(t)-by(t))\,dt+i\int_\alpha^\beta(ay(t)+bx(t))\,dt\\&=\int_\alpha^\beta(ax(t)-by(t)+i(ay(t)+bx(t)))\,dt\\&=\int_\alpha^\beta(a+ib)(x(t)+iy(t))\,dt\\&=\int_\alpha^\beta cf(t)\,dt\end{aligned}$$

□

第三章

2. 验证函数 $f(z)=f(x+iy)=\sqrt{|xy|}$ 在 $z=0$ 点满足 C-R 方程，$f(z)$ 在 $z=0$ 可导么?

解答. 设 $f(x+iy)=u(x,y)+iv(x,y)$，则 $u(x,y)=\sqrt{|xy|},v(x,y)=0$，有

$$\frac{\partial u}{\partial x}=\lim_{h\to0}\frac{\sqrt{|h\cdot0|}-0}{|h|}=0=\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=\quad\lim_{h\to0}\frac{\sqrt{|0\cdot h|}-0}{|h|}=0=-\frac{\partial v}{\partial x}$$

则 $f(z)$ 在 $z=0$ 点满足 C-R 方程，由于

$$\lim_{n\to\infty}\frac{\sqrt{\frac{1}{n}\cdot\frac{1}{n}}-0}{\frac{1}{n}+\frac{i}{n}}=\frac{1}{1+i}$$
$$\lim_{n\to\infty}\frac{\sqrt{\frac{4}{n}\cdot\frac{1}{n}}-0}{\frac{4}{n}+\frac{i}{n}}=\frac{2}{4+i}$$

则 $f(z)$ 在 $z=0$ 处不可导。

5. 若函数 $f(z)=u(z)+iv(z)$ 在区域 D 内解析，且 $u(z)=v^2(z)$，则 $f(z)$ 在 D 内为常数。

证明. 由于 $f(z)$ 在 D 内解析，则满足 C-R 方程，有

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v^2}{\partial x} = 2v\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= \frac{\partial v^2}{\partial y} = 2v\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} \\
\Rightarrow & (4v^2+1)\frac{\partial v}{\partial x} = 0 \\
\Rightarrow & \frac{\partial v}{\partial x} = 0 \\
\Rightarrow & \frac{\partial v}{\partial y} = 0
\end{aligned}
$$

所以 $v(z) = C$ 为常值函数，则 $u(z) = C^2$，综上 $f(z)$ 在 D 内为常数。 □

9. 若函数 $f(z), g(z)$ 在点 z_0 解析，且 $f(z_0) = g(z_0) = 0$，$g'(z_0) \neq 0$，则

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

证明.

$$
\begin{aligned}
\frac{f'(z_0)}{g'(z_0)} &= \lim_{\Delta z\to 0} \frac{\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}}{\frac{g(z_0+\Delta z)-g(z_0)}{\Delta z}} \\
&= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)}{g(z_0+\Delta z)} \\
&= \lim_{z\to z_0} \frac{f(z)}{g(z)}
\end{aligned}
$$

□

10. 设 $f(z)$ 在区域 D 内解析，且 $f(z) \neq 0$，求证：

(1) $4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)|^2 = 4|f'(z)|^2$;

(2) $4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)| = |f'(z)|^2/|f(z)|$;

(3) $4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2, p \in \mathbb{N}$.

证明. 设 $f(z) = u(z) + iv(z)$，则

$$
\begin{aligned}
\frac{\partial |f|^2}{\partial z} = \frac{\partial (f\cdot \bar{f})}{\partial z} = \frac{\partial (u^2+v^2)}{\partial z} \\
&= \frac{1}{2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} - i\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)\right) \\
&= (u-iv)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) \\
&= \bar{f}\cdot 2\frac{\partial u}{\partial z} = \bar{f}\cdot\frac{\partial f}{\partial z} \\
&= \bar{f}\cdot f'(z)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\frac{\partial |f|^2}{\partial \bar{z}} = \frac{\partial (f\cdot \bar{f})}{\partial \bar{z}} &= \frac{\partial (u^2+v^2)}{\partial \bar{z}} \\
&= \frac{1}{2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} + i\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)\right) \\
&= (u+iv)\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \qquad (2) \\
&= f\cdot 2\overline{\frac{\partial u}{\partial z}} = f\cdot \overline{\frac{\partial f}{\partial z}} \\
&= f\cdot \overline{f'(z)}
\end{aligned}
$$

$$
\frac{\partial \bar{f}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \overline{f'(z)} \qquad (3)
$$

$$
\frac{\partial}{\partial z}|f|^p = \frac{\partial}{\partial z}|f\cdot \bar{f}|^{p/2} = \frac{p}{2}|f\cdot\bar{f}|^{p/2-1}\cdot \frac{\partial |f|^2}{\partial z} \xlongequal{(1)} \frac{p}{2}|f\cdot\bar{f}|^{p/2-1}\cdot \bar{f}\cdot\frac{\partial f}{\partial z} \qquad (4)
$$

则原式

$$
\begin{aligned}
4\frac{\partial^2}{\partial z\partial\bar{z}}|f|^p &\xlongequal{(4)} 4\frac{\partial}{\partial \bar{z}}\left(\frac{p}{2}|f\cdot\bar{f}|^{p/2-1}\cdot\bar{f}\cdot\frac{\partial f}{\partial z}\right) \\
&= 4\left(\frac{p}{2}(\frac{p}{2}-1)|f|^{p-4}\cdot\frac{\partial |f|^2}{\partial\bar{z}}\cdot\bar{f}\cdot\frac{\partial f}{\partial z} + \frac{p}{2}|f|^{p-2}\cdot\frac{\partial\bar{f}}{\partial\bar{z}}\cdot\frac{\partial f}{\partial z}\right) \\
&\xlongequal{(2),(3)} p(p-2)|f|^{p-4}|f|^2|f'|^2 + 2p|f|^{p-2}|f'|^2 \\
&= p^2|f|^{p-2}|f'|^2
\end{aligned}
$$

分别取 $p=2,1$ 时，(1), (2) 得证。 □

13. 设 $f(z)=R(r,\theta)e^{i\Phi(r,\theta)}, z=re^{i\theta}$，则 C-R 方程为

$$
\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial\Phi}{\partial\theta}, \quad \frac{\partial R}{\partial\theta} = -Rr\frac{\partial\Phi}{\partial r}
$$

证明. 设 $g(z)=u(r,\theta)+iv(r,\theta), z=re^{i\theta}$，则通过两种趋近方式，可得

$$
\begin{aligned}
g'(z) &= \lim_{\Delta r\to 0}\frac{g((r+\Delta r)e^{i\theta}) - g(re^{i\theta})}{\Delta r e^{i\theta}} \\
&= \frac{1}{e^{i\theta}}\lim_{\Delta r\to 0}\left(\frac{u(r+\Delta r,\theta)-u(r,\theta)}{\Delta r} + i\frac{v(r+\Delta r,\theta)-v(r,\theta)}{\Delta r}\right) \\
&= \frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \\
g'(z) &= \lim_{\Delta\theta\to 0}\frac{g(r(e^{i(\theta+\Delta\theta)} - e^{i\theta})) - g(re^{i\theta})}{re^{i\theta}(e^{i\Delta\theta}-1)} \\
&= \frac{1}{re^{i\theta}}\lim_{\Delta\theta\to 0}\frac{\Delta\theta}{e^{i\Delta\theta}-1}\left(\frac{u(r,\theta+\Delta\theta)-u(r,\theta)}{\Delta\theta} + i\frac{v(r,\theta+\Delta\theta)-v(r,\theta)}{\Delta\theta}\right) \\
&= \frac{1}{re^{i\theta}}\lim_{\Delta\theta\to 0}\frac{1}{ie^{i\Delta\theta}}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right) \\
&= \frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right)
\end{aligned}
$$

通过对比实部和虚部可得

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{cases}$$

由于 $\log f(z) = \log(R) + i\Phi = u + iv$，代入上式可得

$$\begin{cases} \dfrac{1}{R}\dfrac{\partial R}{\partial r} = \dfrac{1}{r}\dfrac{\partial \Phi}{\partial \theta} \\ \dfrac{\partial \Phi}{\partial r} = -\dfrac{1}{rR}\dfrac{\partial R}{\partial \theta} \end{cases} \quad \Rightarrow \begin{cases} \dfrac{\partial R}{\partial r} = \dfrac{R}{r}\dfrac{\partial \Phi}{\partial \theta} \\ \dfrac{\partial R}{\partial \theta} = -Rr\dfrac{\partial \Phi}{\partial r} \end{cases}$$

□

29. 求 i^i 的主值并求 $|i^i|$ 与 $|i|^i$.

解答.

$$i^i = e^{i\mathrm{Log} i} = e^{i(\pi/2+2k\pi)i} = e^{-\pi/2+2k\pi} \quad (k \in \mathbb{Z})$$

当主值范围取 $(-\pi, \pi)$ 时，i^i 的主值为 $e^{-\pi/2}$，则 $|i^i| = e^{-\pi/2}$，且

$$|i|^i = 1^i = e^{i\mathrm{Log}1} = e^{i\cdot 2k\pi} = 1 \quad (k \in \mathbb{Z})$$