日期	科目	班级	姓名	学号
2022 年 12 月 5 日	偏微分方程	强基数学 002	吴天阳	2204210460

第三章作业

题目 1. (1) 求解 Fourier 变式：

$$(2)\ f(x)=\begin{cases}0, & |x|>a,\\ 1-\dfrac{|x|}{a}, & |x|\leqslant a;\end{cases}\qquad (4)\ f(x)=\mathrm{e}^{-a|x|},\quad (a>0);$$

解答. (2) 令 $g(x)=1-\dfrac{x}{a},\quad (0\leqslant x<a)$，则 $f(x)=g(x)-g(-x)$，由于

$$\hat{g}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^a\left(1-\frac{x}{a}\right)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}a\lambda^2}[1-\mathrm{i}a\lambda-\mathrm{e}^{-\mathrm{i}a\lambda}]$$

于是 $\hat{f}(\lambda)=\hat{g}(\lambda)+\widehat{g(-x)}(\lambda)=\hat{g}(\lambda)+\hat{g}(-\lambda)=\dfrac{1}{\sqrt{2\pi}a\lambda^2}\left(2-\mathrm{e}^{-\mathrm{i}a\lambda}-\mathrm{e}^{\mathrm{i}a\lambda}\right)$

(4) 令 $g(x)=\mathrm{e}^{-ax}\chi_{(0,\infty)}(x)$，则 $f(x)=g(x)+g(-x)$，由于

$$\hat{g}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-ax}\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}\lambda+a}$$

于是 $\hat{f}(\lambda)=\hat{g}(\lambda)+\widehat{g(-x)}(\lambda)=\hat{g}(\lambda)+\hat{g}(-\lambda)=\dfrac{1}{\sqrt{2\pi}}\dfrac{2a}{a^2+\lambda^2}$.

题目 2. (2) 求解下列函数的 Fourier 变式：

$$(3)\ f(x)=\begin{cases}\mathrm{e}^{\mu x}, & |x|<a,\\ 0, & |x|\geqslant a;\end{cases}\qquad f(x)=\begin{cases}\mathrm{e}^{\mathrm{i}\lambda_0 x}, & |x|<L,\\ 0, & |x|\geqslant L.;\end{cases}\qquad f(x)=\frac{x}{a^2+x^2}$$

解答. (3) 令 $g(x)=\mathrm{e}^{\mu x}\chi_{[0,a)}(x)$，则 $f(x)=g(x)+g(-x)$，由于

$$\hat{g}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^a \mathrm{e}^{\mu}\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\frac{1}{\mu-\mathrm{i}\lambda}\left(\mathrm{e}^{(\mu-\mathrm{i}\lambda)a}-1\right)$$

则 $\hat{f}(\lambda)=\hat{g}(\lambda)+\hat{g}(-\lambda)=\dfrac{1}{\sqrt{2\pi}}\left[\dfrac{\mathrm{e}^{(\mu-\mathrm{i}\lambda)a}}{\mu-\mathrm{i}\lambda}+\dfrac{\mathrm{e}^{(\mu+\mathrm{i}\lambda)a}}{\mu+\mathrm{i}\lambda}-\dfrac{2\mu}{\mu^2+\lambda^2}\right]$.

(5) 令 $g=\mathrm{e}^{\mathrm{i}\lambda_0 x}\chi_{[0,L)}(x)$，则 $f(x)=g(x)+g(-x)$，由于

$$\hat{g}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^L \mathrm{e}^{\mathrm{i}\lambda_0 x}\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_0^L \mathrm{e}^{\mathrm{i}(\lambda_0-\lambda)x}\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}(\lambda_0-\lambda)}\left(\mathrm{e}^{\mathrm{i}(\lambda_0-\lambda)L}-1\right)$$

则 $\hat{f}(\lambda)=\hat{g}(\lambda)+\hat{g}(-\lambda)=\dfrac{1}{\sqrt{2\pi}\mathrm{i}}\left[\dfrac{\mathrm{e}^{\mathrm{i}(\lambda_0-\lambda)L}}{\lambda_0-\lambda}+\dfrac{\mathrm{e}^{\mathrm{i}(\lambda_0+\lambda)L}}{\lambda_0+\lambda}-\dfrac{2\lambda_0}{\lambda_0^2+\lambda^2}\right]$.

(8) 令 $g(x)=\dfrac{1}{a^2+x^2}$，则 $\check{g}(\lambda)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\dfrac{\mathrm{e}^{\mathrm{i}\lambda x}}{a^2+x^2}\,\mathrm{d}x$，考虑复函数 $h(z)=\dfrac{\mathrm{e}^{\mathrm{i}|\lambda|z}}{a^2+z^2}$，半圆围道 $C=[-R,R]\cup\gamma_R$，其中 $\gamma_R=\{\mathrm{e}^{-i\theta}:\theta\in[0,\pi]\}$，则

$$\int_C f(z)\,\mathrm{d}z=\int_{-\mathbb{R}}^{R}\mathrm{d}x+\int_{\gamma_R}f(z)\,\mathrm{d}z$$

由于 $a\mathrm{i}$ 是 C 内的奇点，由留数定理可知：$\displaystyle\int_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\mathrm{Res}(f;a\mathrm{i})$，由于

$$\mathrm{Res}(f;a_i) = \lim_{z\to a\mathrm{i}}(z-a\mathrm{i})\frac{\mathrm{e}^{\mathrm{i}|\lambda|z}}{a^2+z^2} = \lim_{z\to a\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}|\lambda|z}}{z+a\mathrm{i}} = \frac{\mathrm{e}^{-|\lambda|a}}{2a\mathrm{i}}$$

又由 Jordan 引理可知 $\lim\limits_{R\to\infty} f_{\gamma_R} f(z)\,\mathrm{d}z = 0$，于是 $\displaystyle\int_C f(z)\,\mathrm{d}z = \frac{\pi\mathrm{e}^{-|\lambda|}}{a}$.

当 $\lambda < 0$ 时，$\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}(-\lambda)x}}{a^2+x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{a^2+x^2}\,\mathrm{d}x$，于是 $\hat{g}(\lambda) = \check{g}(-\lambda) = \dfrac{1}{\sqrt{2\pi}}\check{h}(-\lambda) = \sqrt{\dfrac{\pi}{2}}\dfrac{\mathrm{e}^{-|\lambda|a}}{a}$，则

$$\begin{aligned}\hat{f}(\lambda) = \widehat{xg}(\lambda) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\hat{g}(\lambda) &= \begin{cases}-\mathrm{i}\dfrac{\sqrt{\pi}2}{\mathrm{e}}^{-\lambda a}, & \lambda > 0,\\ \mathrm{i}\dfrac{\sqrt{\pi}2}{\mathrm{e}}^{\lambda a}, & \lambda < 0,\end{cases}\\ &= -\mathrm{i}\sqrt{\frac{\pi}{2}}\mathrm{e}^{-|\lambda|a}\mathrm{sign}(\lambda).\end{aligned}$$

题目 3. (3) 求以下函数的 Fourier 逆变换：

(1) $f(\lambda) = \mathrm{e}^{-a^2\lambda^2}t,\quad t > 0$为参数；

(2) $f(\lambda) = \mathrm{e}^{(-a^2\lambda^2+\mathrm{i}b\lambda+c)t},\quad a,b,c$为常数, $t > 0$为常数；

(3) $f(\lambda) = \mathrm{e}^{-|\lambda|y},\quad y > 0$为参数.

解答. (1) 由 (2) 的结论，取 $b = c = 0$，$\check{f}(x) = \dfrac{1}{\sqrt{2a^2t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}$.

(2)

$$\begin{aligned}\check{f}(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{(-a^2\lambda^2+\mathrm{i}b\lambda+c)t}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda = \frac{\mathrm{e}^{ct}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-a^t\left(\lambda-\frac{\mathrm{i}(bt+x)}{2a^2t}\right)^2}\mathrm{e}^{-\frac{(bt+x)^2}{4a^2t}}\,\mathrm{d}\lambda\\ &= \frac{\mathrm{e}^{ct}}{\sqrt{2\pi}}\sqrt{\frac{\pi}{a^2t}}\mathrm{e}^{-\frac{(bt+x)^2}{4a^2t}} = \frac{1}{\sqrt{2a^2t}}\mathrm{e}^{ct-\frac{(bt+x)^2}{4a^2t}}\end{aligned}$$

(3) 令 $g(\lambda) = \mathrm{e}^{-\lambda y}\chi_{[0,\infty)}(\lambda)$，则 $f(\lambda) = g(\lambda) + g(-\lambda)$，由于

$$\hat{g}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\lambda y}\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}\lambda = \frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}x+y}$$

则 $\check{f}(x) = \widehat{f(-\lambda)}(x) = \hat{g}(x) + \hat{g}(-x) = \dfrac{1}{\sqrt{2\pi}}\left[\dfrac{1}{\mathrm{i}x+y} + \dfrac{1}{\mathrm{i}x+y}\right] = \sqrt{\dfrac{2}{\pi}}\dfrac{y}{y^2+x^2}$.

题目 4. (4.1) 应用 Fourier 变换求解以下定解问题：

$$\begin{cases}u_t - a^2u_{xx} - bu_x - cu = f(x,t), & x\in\mathbb{R},\ t > 0,\\ u|_{t=0} = \varphi(x), & x\in\mathbb{R}.\end{cases}$$

解答. 对于上述两式对 x 的 Fourier 变换，则

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} + (a^2\lambda^2 - \mathrm{i}b\lambda - c)\hat{u} = \hat{f}(\lambda,t), \hat{u}|_{t=0} = \hat{\varphi}(\lambda).$$

求解第一个常微分方程：

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{u}\cdot\mathrm{e}^{(a^2\lambda^2-\mathrm{i}b\lambda-c)t}\right] = \hat{f}(\lambda,t)\mathrm{e}^{(a^2\lambda^2-\mathrm{i}b\lambda-c)t}\\ &\hat{u} = \int_0^t\hat{f}(\lambda,\tau)e^{(a^2\lambda^2-\mathrm{i}b\lambda-c)(t-\tau)}\,\mathrm{d}\tau + \hat{\varphi}(\lambda)\mathrm{e}^{-(a^2\lambda^2-\mathrm{i}b\lambda-c)t}\end{aligned}$$

令 $g(x,t)=\left(e^{(a^2\lambda^2-\mathrm{i}b\lambda-c)t}\right)^\vee \xlongequal{\text{由 3.(2) 可知}} \dfrac{1}{a\sqrt{2t}}\mathrm{e}^{ct-\frac{(bt+x)^2}{4a^2t}}$，则

$$
\begin{aligned}
u&=(\hat{\varphi}\hat{g})^\vee+\int_0^t\left(\hat{f}(\lambda,\tau)\hat{g}(\lambda,t-\tau)\right)^\vee=\frac{1}{\sqrt{2\pi}}\varphi*g+\frac{1}{\sqrt{2\pi}}\int_0^t f(x,\tau)*g(x,t-\tau)\,\mathrm{d}\tau\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}\varphi(\xi)g(x-\xi,t)\,\mathrm{d}\xi+\int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}f(\xi,\tau)g(x-\xi,t-\tau)\,\mathrm{d}\tau\right]
\end{aligned}
$$

题目 5. (5) 证明在 $D^*(\mathbb{R})$ 的意义下：

(2) $\varphi(x)\delta'(x)=-\varphi'(0)\delta(x)+\varphi(0)\delta'(x)$；

(4) $x^m\delta^{(m)}(x)=(-1)^m m!\delta(x)$.

解答. (2) $\forall\psi(x)\in D^*(\mathbb{R})$，则

$$
\begin{aligned}
\langle\varphi(x)\delta'(x),\psi(x)\rangle&=\langle\delta'(x),\varphi(x)\psi(x)\rangle=-\langle\delta(x),\varphi'(x)\psi(x)+\varphi(x)\psi'(x)\rangle\\
&=-\varphi'(0)\psi(0)-\varphi(0)\psi'(0)=\langle-\varphi'(0)\delta(x)+\varphi(0)\delta'(x),\psi(x)\rangle
\end{aligned}
$$

所以 $\varphi(x)\delta'(x)=-\varphi'(0)\delta(x)+\varphi(0)\delta'(x)$.

(4) 由于

$$
\begin{aligned}
\left\langle x^m\delta^{(m)}(x),\varphi(x)\right\rangle&=\left\langle\delta^{(m)}(x),x^m\varphi(x)\right\rangle=(-1)^m\left\langle\delta(x),\frac{\mathrm{d}^m(x^m\varphi(x))}{\mathrm{d}x^m}\right\rangle\\
&=(-1)^m\left\langle\delta(x),\sum_{k=0}^m\binom{m}{k}\varphi^{(k)}(x)(x^m)^{(m-k)}\right\rangle\\
&=(-1)^m\left\langle\delta(x),\sum_{k=0}^m\binom{m}{k}\frac{m!}{k!}\varphi^{(k)}(x)x^k\right\rangle\\
&=(-1)^m m!\varphi(0)=\langle(-1)^m m!\delta(x),\varphi(x)\rangle
\end{aligned}
$$

所以 $x^m\delta^{(m)}(x)=(-1)^m m!\delta(x)$.

题目 6. (6) 求解

(1) $|x|^{(m)},\quad(m\geqslant1)$；　　(3) $(H(x)\mathrm{e}^{ax})''$.

解答. (1) $|x|'=xH(x)-xH(-x)=(x(H(x)-H(-x)))'=H(x)-H(-x)+x(H(x)-H(-x))'$，由于

$$
-\langle H'(-x),\varphi(x)\rangle=-\int_{-\infty}^0\varphi'(-x)\,\mathrm{d}x=-\int_0^{\infty}\varphi'(x)\,\mathrm{d}x=\varphi(0)=\langle\delta(x),\varphi(x)\rangle
$$

于是 $|x|'=H(x)-H(-x)+2x\delta(x)=H(x)-H(-x)$，故

$$
|x|^{(m)}=[H(x)-H(-x)]^{(m-1)}=2\delta^{(m-2)}(x)
$$

(3) 由于

$$\begin{aligned}\langle[H(x)\mathrm{e}^{ax}]',\varphi(x)\rangle &= -\langle H(x),\varphi'(x)\mathrm{e}^{ax}\rangle=-\int_0^\infty\varphi'(x)\mathrm{e}^{ax}\,\mathrm{d}x\\ &= -\int_0^\infty \mathrm{e}^{ax}\,\mathrm{d}\varphi(x)=\varphi(0)+a\int_0^\infty \mathrm{e}^{ax}\varphi(x)\,\mathrm{d}x=\varphi(0)+a\int_0^\infty \mathrm{e}^{ax}\varphi(x)\,\mathrm{d}x\\ &= \langle\delta(x),\varphi(x)\rangle+a\langle H(x)\mathrm{e}^{ax},\varphi\rangle=\langle\delta(x)+aH(x)\mathrm{e}^{ax},\varphi(x)\rangle\\ \langle[\delta(x)+aH(x)\mathrm{e}^{ax}]',\varphi(x)\rangle &= -\langle\delta+aH(x)\mathrm{e}^{ax},\varphi'(x)\rangle\\ &= \langle\delta'(x),\varphi(x)\rangle-a\langle H(x),\varphi'(x)\mathrm{e}^{ax}\rangle\\ &= \left\langle\delta'+a\delta+a^2H(x)\mathrm{e}^{ax},\varphi(x)\right\rangle\end{aligned}$$

则 $[H(x)\mathrm{e}^{ax}]''=\delta'(x)+a\delta(x)+a^2H(x)\mathrm{e}^{ax}$.

题目 7. (7) 求广义导数 $f'(x)$

$$(1)f(x)=\begin{cases}\sin x, & x\geqslant 0,\\ 0, & x<0;\end{cases}\qquad (3)f(x)=\begin{cases}x^2, & |x|\leqslant 1,\\ 0, & |x|>1.\end{cases}$$

解答. (1) $f(x)=H(x)\sin x$，由于

$$\langle f'(x),\varphi(x)\rangle=-\langle H(x),\varphi'(x)\sin x\rangle=-\int_0^\infty\varphi'\sin x\,\mathrm{d}x=\int_0^\infty\varphi\cos x\,\mathrm{d}x=\langle H(x)\cos x,\varphi(x)\rangle$$

则 $f'(x)=H(x)\cos x$.

(3) 由于

$$\begin{aligned}\langle f'(x),\varphi(x)\rangle &= -\langle f(x),\varphi'(x)\rangle=-\int_{-1}^1 x^2\varphi'(x)\,\mathrm{d}x=-\int_{-1}^1 x^2\,\mathrm{d}\varphi(x)\\ &= -\varphi(1)+\varphi(-1)+\int_{-1}^1 2x\varphi(x)\,\mathrm{d}x=-\varphi(1)+\varphi(-1)+\langle g(x),\varphi(x)\rangle\\ &= \langle-\delta(x-1)+\delta(x+1)+g(x),\varphi(x)\rangle\end{aligned}$$

其中 $g(x)=2x\chi_{[-1,1]}(x)$，所以 $f'=-\delta(x-1)+\delta(x+1)+2x\chi_{[-1,1]}(x)$.

题目 8. (9) 用分离变量法求解下列混合问题：

$$(2)\begin{cases}u_t=a^2u_{xx}, & 0<x<\pi,t>0,\\ u|_{t=0}=\sin x, & 0\leqslant x\leqslant l,\\ u|_{x=0}=0,\quad u|_{x=l}=0, & t>0;\end{cases}$$

$$(4)\begin{cases}u_t=a^2u_{xx}, & 0<x<l,t>0,\\ u|_{t=0}=0, & 0\leqslant x\leqslant l,\\ u|_{x=0}=0,u|_{x=l}=At, & t>0;\end{cases}$$

$$(6)\begin{cases}u_t-a^2u_{xx}=0, & 0<x<l,t>0,\\ u|_{t=0}=0, & 0\leqslant x\leqslant l,\\ u_x|_{x=0}=0,u_x|_{x=l}=q, & t>0.\end{cases}$$

解答. (2) 令 $u(x,t)=X(x)T(t)$，则 $\begin{cases} X''+\lambda X=0, \\ T'+a^2\lambda T=0. \end{cases}$ 则

$$X(x)=c_1\sin\sqrt{\lambda}x+c_2\cos\sqrt{\lambda}x,\quad X'(x)=c_1\sqrt{\lambda}\cos\sqrt{\lambda}x-c_2\sqrt{\lambda}\sin\sqrt{\lambda}x$$

由于 $X'(0)=X'(\pi)=0$，则 $\begin{cases} c_1\sqrt{\lambda}=0 \\ -c_2\sqrt{\lambda}\sin\sqrt{\lambda}\pi=0 \end{cases}$ $\Rightarrow \lambda=n^2,\ (n=0,1,2,\cdots)$，则 $X_n(x)=c_2\cos nx,\ (n=0,1,2,\cdots)$. 求解可得 $T(t)=\mathrm{e}^{-a^2n^2t}$，则 $u=\sum\limits_{n\geqslant 0}A_n\mathrm{e}^{-a^2n^2t}\cos nx$，由于 $u|_{t=0}=\sum\limits_{n\geqslant 0}A_n\cos nx=\sin x$，则

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi}\sin\cos nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}\sin((n+1)x)-\sin((n-1)x)\,\mathrm{d}x \\ (n\geqslant 2\text{时})\quad &= \frac{4}{(1-n^2)\pi}[(-1)^n+1] \\ (n=1\text{时})\quad &= \frac{2}{\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x=0 \\ (n=0\text{时})\quad &= \frac{2}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x=\frac{4}{\pi} \end{aligned}$$

令 $n=2k,\ (k=1,2,3,\cdots)$ 时 $A_n\neq 0$，综上 $u=\dfrac{4}{\pi}+\sum\limits_{k\geqslant 1}\dfrac{8}{(1-4k^2)\pi}\mathrm{e}^{-4a^2k^2t}\cos 2kx.$

(4) 由于原方程边界条件不齐次，令 $v=X(x)T(t)$，则原方程转化为

$$\begin{cases} v_t-a^2v_{xx}=-\dfrac{A}{l}x=f, \\ v|_{t=0}=0, \\ v|_{x=0}=v|_{x=l}=0. \end{cases}$$

设 $u=X(x)T(t)$，则 $\begin{cases} T'+a^2\lambda T=0 \\ X''+\lambda X=0 \end{cases}$ ，于是 $X=c_1\sin\sqrt{\lambda}x+c_2\cos\sqrt{\lambda}x$，代入边界条件可得 $c_2=0,\ \lambda=\left(\dfrac{n\pi}{l}\right)^2,\ (n=1,2,\cdots)$，于是 $x_n=c\sin\dfrac{n\pi}{l}x.$

设 $v=\sum\limits_{n\geqslant 1}T_n\sin\dfrac{n\pi}{l}x,\ -\dfrac{A}{l}x=\sum\limits_{n\geqslant 1}f_n\sin\dfrac{n\pi}{l}x$，满足

$$\begin{cases} T_n'+\left(\dfrac{an\pi}{l}\right)^2T_n=f_n \\ T_n(0)=0 \end{cases}$$

由于 $f_n=\dfrac{2}{l}\displaystyle\int_0^l-\dfrac{A}{l}x\sin\dfrac{n\pi}{l}x\,\mathrm{d}x=\dfrac{2A(-1)^n}{n\pi}$，解上述常微分方程可得

$$T_n(t)=\int_0^t\frac{2A(-1)^n}{n\pi}\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)}\,\mathrm{d}\tau=\frac{2Al^2(-1)^n}{n^3\pi^3a^2}\left(1-\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2t}\right)$$

综上，$u = v + \frac{A}{l}xt = \sum_{n\geqslant 1} \frac{2Al^2(-1)^n}{n^3\pi^3a^2}\left(1-\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2 t}\right)\sin\frac{n\pi}{l}x + \frac{A}{l}xt.$

(6) 由于边界条件不齐次，令 $v = u - \frac{q}{2l}x^2$，则原方程等价求解以下齐次边界问题

$$\begin{cases} v_t - a^2 v_{xx} = \frac{a^2 q}{l} = f, \\ v|_{t=0} = -\frac{q}{2l}x^2 = \varphi, \\ v_x|_{x=0} = v_x|_{x=l} = 0. \end{cases}$$

类似 (4) 题结果，可知特征函数为 $X_n(x) = c\sin\frac{n\pi}{l}x$，设 $v = \sum_{n\geqslant 1} T_n \sin\frac{n\pi}{l}x$，满足

$$\begin{cases} T_n' + \left(\frac{an\pi}{l}\right)^2 T_n = f_n, \\ T_n(0) = \varphi_n. \end{cases}$$

由于

$$\begin{aligned} f_n &= \frac{2}{l}\int_0^l \frac{a^2 q}{l}\sin\frac{n\pi}{l}x\,\mathrm{d}x = \frac{2a^2q}{l^2}((-1)^{n-1}+1) \\ \varphi_n &= -\frac{2}{l}\int_0^l \frac{q}{2l}x^2\sin\frac{n\pi}{l}x\,\mathrm{d}x = \frac{ql}{n\pi}(-1)^n + \frac{2q}{n^2\pi^2}((-1)^{n-1}+1) \end{aligned}$$

求解常微分方程可得

$$\begin{aligned} T_n &= \varphi_n \mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2} + f_n\left(1-\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2 t}\right)\left(\frac{l}{n\pi a}\right)^2 \\ &= \left(\frac{ql}{n\pi}(-1)^n + \frac{2q}{n^2\pi^2}((-1)^{n-1}+1)\right)\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2} + \frac{2q}{n^2\pi^2}((-1)^{n-1}+1)\left(1-\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2}\right) \\ &= \frac{ql(-1)^n}{n\pi}\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2} + \frac{2q}{n^2\pi^2}((-1)^{n-1}+1) \\ &= (-1)^n\left(\frac{ql}{n\pi}\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2} - \frac{2q}{n^2\pi^2}\right) + \frac{2q}{n^2\pi^2} \end{aligned}$$

综上

$$u = v + \frac{q}{2l}x^2 = \sum_{n\geqslant 1}\left[(-1)^n\left(\frac{ql}{n\pi}\mathrm{e}^{-\left(\frac{n\pi a}{l}\right)^2} - \frac{2q}{n^2\pi^2}\right) + \frac{2q}{n^2\pi^2}\right]\sin\frac{n\pi}{l}x + \frac{q}{2l}x^2.$$

题目 9. (13) 设 $u \in C^{2,1}(\bar{Q}), u_t \in C^{2,1}(Q)$ 且满足以下定解问题

$$\begin{cases} u_t - u_{xx} = f(x,t), & (x,t)\in Q, \\ u|_{t=0} = \varphi(x), & 0\leqslant x\leqslant l, \\ u|_{x=0} = u|_{x=l} = 0, & 0\leqslant t\leqslant T, \end{cases}$$

则有以下估计

$$\max_{\bar{Q}} |u_t(x,t)| \leqslant C(||f||_{C^1(\bar{Q})} + ||\varphi''||_{C[0,l]}),$$

其中 C 仅依赖于 T.

解答. 对原式每个方程都对 t 求偏导，并令 $v=u_t$ 可得

$$\begin{cases} u_{tt}-u_{txx}=f_t(x,t), \\ u_{xx}|_{t=0}=\varphi''(x), \\ u_t|_{x=0}=u_t|_{x=l}=0. \end{cases} \Rightarrow \begin{cases} v_t-v_{xx}=f_t(x,t), \\ v|_{t=0}=u_t|_{t=0}=[u_{xx}+f(x,t)]|_{t=0}=\varphi''(x)+f(x,0), \\ v|_{x=0}=v|_{x=l}=0. \end{cases}$$

记 $F=||f||_{C^1(\bar{Q})}=\sup\limits_{\bar{Q}}|f|+\sup\limits_{\bar{Q}}|f_t|$, $B=||\varphi''||_{C[0,l]}=\sup\limits_{x\in[0,l]}|\varphi''|$，令 $w=F(t+1)+B$，要证 $\max\limits_{\bar{Q}}|v|\leqslant C(||f||_{C^1(\bar{Q})}+||\varphi''||_{C[0,l]})$，只需证 $w\geqslant 0$, $(x,t)\in\bar{Q}$，也就是证 $\begin{cases} Lw\geqslant 0, \\ w|_\Gamma\geqslant 0. \end{cases}$ 其中 Γ 为 Q 的抛物边界.

由于 $F\pm f_t\geqslant 0$，在 Γ 上有 $F+B\pm(\varphi''(x)+f(x,0))\geqslant 0$，则取 $C=T+1$，有

$$\max_{\bar{Q}}|u_t(x,t)|=\max_{\bar{Q}}|v|\leqslant(T+1)(||f||_{C^1(\bar{Q})}+||\varphi''||_{C[0,l]}).$$

题目 10. (15) 设 $u,u_x\in C(\bar{Q})\cap C^{2,1}(Q)$，$u$ 满足第三边值问题

$$\begin{cases} Lu=u_t-u_{xx}=f(x,t), & (x,t)\in Q, \\ u|_{t=0}=\varphi(x), & 0\leqslant x\leqslant l, \\ [-u_x+\alpha u]_{x=0}=g_1(t), & 0\leqslant t\leqslant T, \\ [u_x+\beta u]_{x=l}=g_2(t), & 0\leqslant t\leqslant T. \end{cases}$$

其中 $\alpha\geqslant 0,\beta\geqslant 0$，给出 $\max\limits_{\bar{Q}}|u_x|$ 的估计.

题目 11. (18) 设 $u\in C(\bar{Q})\cap C^{2,1}(Q)$ 且满足：

$$Lu=u_t-a^2u_{xx}+c(x,t)u\leqslant 0,\quad (x,t)\in Q,$$

其中 $c(x,t)$ 有界，且 $c(x,t)\geqslant 0$. 试证明：如果 u 在 $\bar{Q}$ 上存在非负最大值，则 u 必在抛物边界 Γ 上达到它在 $\bar{Q}$ 上的非负最大值.

解答. 令 $f(x,t)=Lu(x,t)$.

(1) 设 $f<0$ 时，反设 u 能在 $\bar{Q}\backslash\Gamma$ 上取到非负最大值 $P_0(x_0,t_0)\in\bar{Q}\backslash\Gamma$，使得 $u|_{P_0}=\max\limits_{\bar{Q}}u(x,t)\geqslant 0$，于是

$$u_x|_{P_0}=0,\quad u_{xx}|_{P_0}\leqslant 0,\quad u_t|_{P_0}=0\,(t_0<T),\quad u_t|_{P_0}\geqslant 0,(t_0=T).$$

则 $f(x_0,t_0)=[u_t-a^2u_{xx}+c(x,t)u]_{P_0}\geqslant 0$ 与 $f(x_0,t_0)<0$ 矛盾，故 u 在 Γ 上取到非负最大值.

(2) 设 $f\leqslant 0$，$\forall\varepsilon>0$，考虑辅助函数 $v(x,t)=u(x,t)-\varepsilon t$，则

$$Lv=Lu-\varepsilon-c(x,t)\varepsilon t=f-\varepsilon(1+c(x,t)t)<0$$

由 (1) 可知，v 在 Γ 上非负最大值，则

$$\max_{\bar{Q}} u(x,t) = \max_{\bar{Q}}(v+\varepsilon t) \leqslant \max_{\Gamma} v + \varepsilon T \leqslant \max_{\Gamma} u + \varepsilon T \leqslant \max_{\Gamma} u, \quad (\varepsilon \to 0)$$

故 u 在 Γ 上取到 $\bar{Q}$ 上的非负最大值.

题目 12. (21) 证明半无界问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t), & 0 < x, t > 0, \\ u|_{t=0} = \varphi(x), & 0 \geqslant 0, \\ u|_{x=0} = \mu(t), & t \geqslant 0, \\ \big(\text{或} - u_x + \alpha u|_{x=0} = \mu(t), \text{常数}\alpha > 0\big) \end{cases}$$

的有界解是唯一的.

证明. 令 u_1, u_2 为上述方程的解，令 $v = u_1 - u_2$，则

$$\begin{cases} Lv = v_t - a^2 v_{xx} = 0, & x > 0, t > 0, \\ v|_{t=0} = 0, & x > 0, \\ v|_{x=0} = 0, & t > 0. \end{cases}$$

令 $Q_L = \{(x,t) : 0 < x < L, 0 < t \leqslant T\}$，记 $K = \sup\limits_{Q} |v|$，过哦早 Q_L 上的辅助函数，$w(x,t) = g(x,t) \pm v(x,t)$，使得

$$\begin{cases} Lw = Lg \pm Lv = 0, \\ w|_{t=0} = g|_{t=0} \pm v|_{t=0} \geqslant 0, \\ w|_{x=0} = g|_{x=0} \pm v|_{x=0} \geqslant 0, \\ w|_{x=L} = g|_{x=L} \pm v|_{x=L} \geqslant K \pm v|_{x=L} \geqslant 0. \end{cases} \quad \text{即} \begin{cases} Lg = 0, \\ g|_{t=0} \geqslant 0, \\ g|_{x=0} \geqslant 0, \\ g|_{x=L} \geqslant K. \end{cases}$$

取 $g = \dfrac{K}{L^2}(x^2 + 2a^2 t)$ 即可满足上式，由比较定理可知 $w(x,t) \geqslant 0$, $(x,t) \in Q_L$，于是 $|v(x,t)| \leqslant g = \dfrac{K}{L^2}(x^2 + 2a^2 t) \to 0$, $(L \to \infty)$，于是 $v = 0 \Rightarrow u_1 = u_2$, $(x \in Q)$. □

题目 13. (22) 设 $u(x,t) \in C^{2,1}(\bar{Q})$ 是问题

$$\begin{cases} u_t - u_{xx} = f, & (x,t) \in Q, \\ u(x,0) = \varphi(x), & 0 \leqslant x \leqslant l, \\ u(0,t) = u(l,t) = 0, & 0 \leqslant t \leqslant T \end{cases}$$

的解，证明 u 满足以下估计

$$\sup_{0 \leqslant t \leqslant T} \int_0^l u_x^2 \,\mathrm{d}x + \int_0^T \int_0^l u_t^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant M \left[\int_0^l (\varphi'(x))^2 \,\mathrm{d}x + \int_0^T \int_0^l f^2(x,t) \,\mathrm{d}x \,\mathrm{d}t \right],$$

其中 M 只依赖于 T, l.

证明. 令 $Q_\tau = \{(x,t) : x \in [0,l], t \in [0,\tau]\}$，对热传导方程左右同乘 u_t，再在 Q_τ 上积分，得

$$\begin{aligned}
&\int_{Q_\tau} u_t^2 \,\mathrm{d}x\,\mathrm{d}t - \int_{Q_\tau} u_t u_{xx} \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_0^l f \cdot u_t \,\mathrm{d}x\,\mathrm{d}t \\
\Rightarrow &\int_{Q_\tau} u_t^2 \,\mathrm{d}t - \int_{Q_\tau} \left[\frac{\partial}{\partial x}(u_x u_t) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2)\right] \mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_{Q_\tau} f^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{Q_\tau} u_t^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

左式第二项使用 Green 公式可得 $I_2 = -\int_{\partial Q} u_x u_t \,\mathrm{d}t + \frac{1}{2}u_x^2 \,\mathrm{d}x$，由于 $u|_{x=0} = u|_{x=l} = 0$ 并注意符号，得 $I_2 = -\frac{1}{2}\int_0^l \varphi_x^2 \,\mathrm{d}x + \frac{1}{2}\int_0^l u_x^2(x,\tau)\,\mathrm{d}x$，于是原不等式转化为

$$\int_0^l u_x^2(x,\tau)\,\mathrm{d}x + \int_{Q_\tau} u_t^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^l \varphi_x^2 \,\mathrm{d}x + \int_{Q_\tau} f^2 \,\mathrm{d}x\,\mathrm{d}t$$

对 τ 在 $[0,T]$ 中取上确界可得

$$\sup_{0\leqslant\tau\leqslant T}\int_0^l u_x^2(x,\tau)\,\mathrm{d}x + \int_0^T \int_0^l u_t^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^l \varphi_x^2(x)\,\mathrm{d}x + \int_0^T \int_0^l f^2(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

□

题目 14. (23) 设 $u(x,t) \in C^{1,0}(\bar{Q}) \cap C^{2,1}(Q)$ 且满足以下定解问题

$$\begin{cases}
u_t - a^2 u_{xx} = f(x,t), & (x,t) \in Q, \\
u(x,0) = \varphi(x), & 0 \leqslant x \leqslant l, \\
[-u_x + \alpha u]_{x=0} = [u_x + \beta u]_{x=l} = 0, & 0 \leqslant t \leqslant T,
\end{cases}$$

其中 $\alpha \geqslant 0, \beta \geqslant 0$，证明

$$\sup_{0\leqslant t\leqslant T}\int_0^l u_x^2 \,\mathrm{d}x + \int_0^T \int_0^l u_t^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant M\left[\int_0^l \varphi^2(x)\,\mathrm{d}x + \int_0^T \int_0^l f^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right],$$

其中 M 只依赖于 T, a.

证明. 令 $Q_\tau = \{(x,t) : x \in [0,l], t \in [0,\tau]\}$，对热传导方程左右同乘 u，再在 Q_τ 上积分可得

$$\begin{aligned}
&\int_{Q_\tau} u u_t - a^2 u u_{xx} \,\mathrm{d}t = \int_{Q_\tau} f \cdot u \,\mathrm{d}x\,\mathrm{d}t \\
\Rightarrow\, &I_1 + I_2 = \int_Q \frac{1}{2}(u^2)_t - a^2 \int_{Q_\tau} u u_{xx} \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_{Q_\tau} f^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{Q_\tau} u^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

下面分别求解 I_1, I_2

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{Q_\tau} (u^2)_t \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^l u^2(x,\tau)\,\mathrm{d}x - \frac{1}{2}\int_0^l \varphi^2 \,\mathrm{d}x \\
I_2 &= \int_{Q_\tau} -a^2 u \,\mathrm{d}u_x \,\mathrm{d}t = -a^2\left[\int_0^\tau u u_x|_{x=l}\,\mathrm{d}t - \int_0^\tau u u_x|_{x=0}\,\mathrm{d}t\right] + a^2 \int_{Q_\tau} u_x^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= a^2\left(\frac{1}{\beta} + \frac{1}{\alpha}\right)\int_0^\tau u_x^2 \,\mathrm{d}t + a^2 \int_{Q_\tau} u_x^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

于是原不等式变为

$$\int_0^l u^2(x,\tau)\,\mathrm{d}x + 2a^2\int_{Q_\tau} u_x^2\,\mathrm{d}x\,\mathrm{d}t + 2a^2\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)\int_0^\tau u_x^2\,\mathrm{d}t \leqslant \int_{Q_\tau} f^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^l \varphi^2\,\mathrm{d}x + \int_{Q_\tau} u^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\Rightarrow \int_0^l u^2(x,\tau)\,\mathrm{d}x + 2a^2\int_{Q_\tau} u_x^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{Q_\tau} f^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^l \varphi^2\,\mathrm{d}x + \int_{Q_\tau} u^2\,\mathrm{d}x\,\mathrm{d}t$$

令 $G(\tau) = \int_{Q_\tau} u^2\,\mathrm{d}x\,\mathrm{d}t, F(\tau) = \int_{Q_\tau} f^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^l \varphi^2\,\mathrm{d}x$，则 $G(0) = 0$ 且 $G(\tau)$ 单调递增，则 $\frac{\mathrm{d}G(\tau)}{\mathrm{d}\tau} \leqslant G(\tau) + F(\tau)$，有 Gronwall 不等式可知，存在 $M > 0$ 使得 $G(\tau) \leqslant MF(\tau)$，于是

$$\int_0^l u^2(x,\tau)\,\mathrm{d}x + 2a^2\int_{Q_\tau} u_x^2\,\mathrm{d}x\,\mathrm{d}t \leqslant (1+M)\left[\int_{Q_\tau} f^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^l \varphi^2\,\mathrm{d}x\right]$$

对 τ 在 $[0,T]$ 中取上确界可得

$$\sup_{0\leqslant\tau\leqslant T}\int_0^l u_x^2(x,\tau)\,\mathrm{d}x + 2a^2\int_0^T\int_0^l u_t^2\,\mathrm{d}x\,\mathrm{d}t \leqslant M'\left[\int_0^T\int_0^l f^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^l \varphi^2(x)\,\mathrm{d}x\right],$$

其中 $M' = 1 + M$. □