

第一次作业

题目 1. 证明下面恒等式:

$$1. (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$$

$$2. \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

$$3. (\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})^2$$

证明. 1.

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{d}))$$

$$\stackrel{\text{二重向量积}}{=} \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d})$$

$$= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$2. \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$

$$\stackrel{\text{二重向量积}}{=} (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

$$\stackrel{\text{内积交换律}}{=} 0$$

$$3. (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})$$

$$= (\mathbf{a} \times \mathbf{b}) [((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a})\mathbf{c} - ((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}) \cdot \mathbf{a}]$$

$$\stackrel{\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0}{=} [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}][(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$$

$$= (\mathbf{b} \times \mathbf{c} \cdot \mathbf{a})^2 = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})^2$$

□

题目 2. 采用向量方法证明:

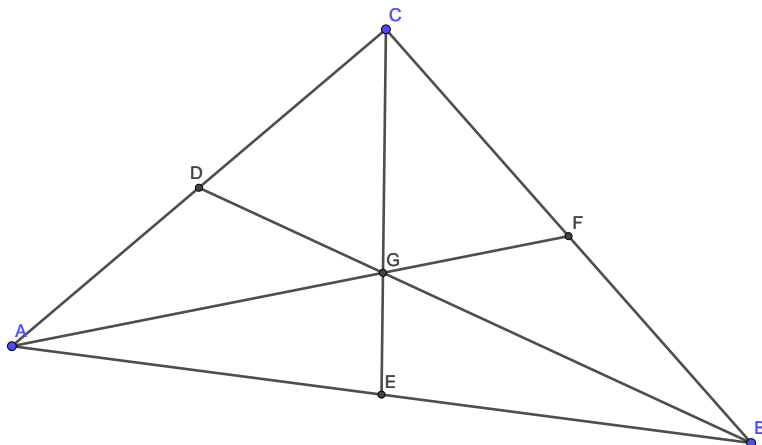
1. 平面上三角形的三条中线交于一点, 这点分中线为 1:2 两部分, 称为三角形的重心.

2. 空间中四面体的顶点到对面三角形的重心的连线称为四面体的中线. 证明四条中线相交于同一个点, 称为四面体的重心, 并且重心分中线为 1:3 两个部分.

解答.

1. 以右图三角形 $\triangle ABC$ 为例, D, E, F 为三边的中点, 则 G 为三条重心, 设 $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AC} = \mathbf{b}$, 则 $\overrightarrow{BC} = \mathbf{b} - \mathbf{a}$. 下证 G 为 AF 的三等分点, 由于 $\overrightarrow{AF} = \frac{\mathbf{a} + \mathbf{b}}{2}$, $\overrightarrow{CE} = \frac{\mathbf{a} - 2\mathbf{b}}{2}$, 设 $|AG| : |AF| = \alpha$, $|CG| : |CE| = \beta$, 于是

$$\alpha \overrightarrow{AF} + \beta \overrightarrow{EC} = \overrightarrow{AC}$$



也就是

$$\alpha \frac{\mathbf{a} + \mathbf{b}}{2} + \beta 2\mathbf{b} - \mathbf{a} = \mathbf{b} \Rightarrow \begin{cases} \alpha - \beta = 0, \\ \alpha + 2\beta = 2 \end{cases} \Rightarrow \alpha = \beta = \frac{2}{3}$$

于是 G 是 AF 的三等分点, 类似地, 可以证明 G 是 BD, CE 的三等分点.

2. 以右图四面体 $ABCD$ 为例, G_1 为 $\triangle BCD$ 的重心, G_2 为 $\triangle ACD$ 的重心, 设

$$\overrightarrow{AB} = \mathbf{b}, \overrightarrow{AC} = \mathbf{c}, \overrightarrow{AD} = \mathbf{d}$$

则 $\overrightarrow{BD} = \mathbf{d} - \mathbf{b}, \overrightarrow{CD} = \mathbf{d} - \mathbf{c}$, 于是

$$\overrightarrow{DG_1} = \frac{1}{3}(\overrightarrow{DB} + \overrightarrow{DC}) = \frac{1}{3}(\mathbf{b} + \mathbf{c} - 2\mathbf{d})$$

$$\overrightarrow{DG_2} = \frac{1}{3}(\overrightarrow{DA} + \overrightarrow{DC}) = \frac{1}{3}(\mathbf{c} - \mathbf{d})$$

则

$$\overrightarrow{AG_1} = \overrightarrow{AD} + \overrightarrow{DG_1} = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d}), \quad \overrightarrow{BG_2} = \overrightarrow{BD} + \overrightarrow{DG_2} = -\mathbf{b} + \frac{1}{3}(\mathbf{c} + \mathbf{d})$$

假设存在 $\alpha, \beta \in \mathbb{R}$, 使得 $\overrightarrow{AB} = \alpha \overrightarrow{AG_1} + \beta \overrightarrow{BG_2}$, 于是

$$\frac{\alpha}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d}) + \beta \mathbf{b} - \frac{\beta}{3}(\mathbf{c} + \mathbf{d}) = \mathbf{b} \Rightarrow \begin{cases} \alpha/3 + \beta = 1 \\ \alpha/3 = \beta/3 \end{cases} \Rightarrow \alpha = \beta = \frac{3}{4}$$

故四面体重心是其中线的四等分点.

