日期	科目	班级	姓名	学号
2022 年 10 月 27 日	偏微分方程	强基数学 002	吴天阳	2204210460

第二章第二次作业

题目 1. (19) 求解三维波动方程的 Cauchy 问题

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = 0, \\ u_t|_{t=0} = x^3 + y^2 z. \end{cases}$$

解答. 由 Kirchhoff 公式可知:

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi a^2 t}\iint_{B_{at}(x)} x^3 + y^2 z\,\mathrm{d}S \\ &= \frac{1}{4\pi a^2 t}\iint_{B_{at}} (x+x_1)^3 + (y+x_2)^2(z+x_3)\,\mathrm{d}S \\ &= \frac{1}{4\pi a^2 t}\iint_{B_{at}} 2x_1x^2 + x_3y^2\,\mathrm{d}S + (x_1^3 + x_2^2x_3)t \\ &= \frac{a^3t^4}{15}(2x_1 + x_3) + (x_1^3 + x_2^2x_3)t \end{aligned}$$

题目 2. 20 用降维法导出一维波动方程 Cauchy 问题的求解公式.

解答. 由二维 Cauchy 问题公式可知

$$\begin{aligned} &\iint_{\sum_{at}(x)} \frac{\varphi(y)}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\,\mathrm{d}y \\ =\ & 2\int_{x_1-at}^{x_1+at} \varphi(y_1)\int_{x_2}^{x_2+\sqrt{a^2t^2-(y_1-x_1)^2}} \frac{1}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\,\mathrm{d}y_2\mathrm{d}y_1 \end{aligned}$$

由于

$$\begin{aligned} \int_{x_2}^{x_2+\sqrt{b}} \frac{1}{\sqrt{b-(y_2-x_2)^2}}\,\mathrm{d}y_2 &\xlongequal{y=y_2-x_2} \int_0^{\sqrt{b}} \frac{1}{\sqrt{b-y^2}}\,\mathrm{d}y \\ \xlongequal{y=\sqrt{b}z} \int_0^1 \frac{1}{\sqrt{1-z^2}}\,\mathrm{d}z &\xlongequal{z=\cos\theta} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta = \frac{\pi}{2} \end{aligned}$$

则

$$\begin{aligned} \iint_{\sum_{at}(x)} \frac{\varphi(y)}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\,\mathrm{d}y &= \pi\int_{x_1-at}^{x_1+at} \varphi(y_1)\,\mathrm{d}y_1 \\ \iint_{\sum_{at}(x)} \frac{\psi(y)}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\,\mathrm{d}y &= \pi\int_{x_1-at}^{x_1+at} \psi(y_1)\,\mathrm{d}y_1 \\ \iint_{\sum_{a(t-\tau)}(x)} \frac{f(y,\tau)}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\,\mathrm{d}y &= \pi\int_{x_1-a(t-\tau)}^{x_1+a(t-\tau)} f(y_1,\tau)\,\mathrm{d}y_1 \end{aligned}$$

代入到二维 Cauchy 问题解的公式中

$$
\begin{aligned}
u(x,t)=&\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint_{\sum_{at}(x)}\frac{\varphi(y)}{\sqrt{a^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\,\mathrm{d}y\right]\\
&+\frac{1}{2\pi a}\iint_{\sum_{at}(x)}\frac{\psi(y)}{\sqrt{a^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\,\mathrm{d}y\\
&+\frac{1}{2\pi a}\int_0^t\iint_{\sum_{a(t-\tau)}(x)}\frac{f(y,\tau)}{\sqrt{a^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\,\mathrm{d}y
\end{aligned}
$$

即可得到一维解公式

$$
\begin{aligned}
u(x,t)=&\frac{1}{2}[\varphi(x+at)-\varphi(x-at)]+\frac{1}{2a}\int_{x_1-at}^{x_1+at}\psi(\xi)\,\mathrm{d}\xi\\
&+\frac{1}{2a}\int_0^t\mathrm{d}t\int_{x_1-a(t-\tau)}^{x_1+a(t-\tau)}f(\xi,\tau)\,\mathrm{d}\tau
\end{aligned}
$$

解答. 求解二维波动方程的 Cauchy 问题:

$$
\begin{cases}
u_{tt}=a^2(u_{xx}+u_{yy}),\\
u|_{t=0}=x^2(x+y),\\
u_t|_{t=0}=0.
\end{cases}
$$

解答. 代入到二维 Cauchy 问题解的公式中

$$
\begin{aligned}
u(x,t)&=\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint_{\sum_{at}}\frac{(y_1+x_1)^2(y_1+y_2+x_1+x_2)}{\sqrt{a^2t^2-y_1^2-y_2^2}}\,\mathrm{d}y_1\mathrm{d}y_2\right]\\
&=\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint_{\sum_{at}}\frac{(3x_1+x_2)y_1^2+(x_1+x_2)x_1^2}{\sqrt{a^2t^2-y_1^2-y_2^2}}\,\mathrm{d}y_1\mathrm{d}y_2\right]\\
&\xlongequal[y_2=r\sin\theta]{y_1=r\cos\theta}\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[(3x_1+x_2)\int_0^{at}\int_0^{2\pi}\frac{r^2\cos^2\theta}{\sqrt{a^2t^2-r^2}}r\,\mathrm{d}\theta\mathrm{d}r+2\pi(x_1+x_2)x_1^2\int_0^{at}\frac{r}{\sqrt{a^2t^2-r^2}}\,\mathrm{d}r\right]
\end{aligned}
$$

注意到以下积分

$$
\int_0^a\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\frac{\pi}{2},\ \int_0^a\frac{x}{\sqrt{a^2-x^2}}\,\mathrm{d}x=a,\ \int_0^a\frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\frac{\pi}{4}a^2,\ \int_0^a\frac{x^3}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\frac{2}{3}a^3.
$$

于是

$$
\begin{aligned}
u(x,t)&=\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[(3x_1+x_2)\frac{2\pi a^3t^3}{3}+2\pi(x_1+x_2)x_1^2at\right]\\
&=a^2t^2(3x_1+x_2)+(x_1+x_2)x_1^2
\end{aligned}
$$

题目 3. (22.(4)) 求解以下特征值问题的特征函数:

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(0) = X'(l) + hX(l) = 0 \ (h > 0\text{常数}). \end{cases}$$

解答. 求解常微分方程可得

$$X(x) = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda},\ X'(x) = C_1\sqrt{\lambda}\cos\sqrt{\lambda}x - C_2\sqrt{\lambda}\sin\sqrt{\lambda}x.$$

则

$$X(0) = C_2 = 0,\ X'(l) + hX(l) = C_1\sqrt{\lambda}\cos\sqrt{\lambda}l + hC_1 \sin\sqrt{\lambda}l = 0,$$

下求 $C_1 \neq 0$ 的非平凡解，当 $\cos\sqrt{\lambda}l = 0$ 时，则 $\lambda = \left(\dfrac{(2n+1)\pi}{2l}\right)^2$，$(n = 0, 1, 2, \cdots)$，当 $\cos\sqrt{\lambda}l \neq 0$ 时，则

$$\sqrt{\lambda} + h \cdot \tan\sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda} = h \cdot \tan\sqrt{\lambda}l$$

该方程为超越方程，只能求 λ 的近似解.

题目 4. 23.(2) 用分离变量法求解以下定解问题:

$$\begin{cases} Lu = 0, & (x,t) \in Q, \\ u|_{x=0} = u_x|_{x=l} = 0, & t \geqslant 0, \\ u|_{t=0} = x(x-2l),\ u_t|_{t=0} = 0, & 0 \leqslant x \leqslant l. \end{cases}$$

解答. 令 $u = X(x)T(t)$，则 $XT'' - a^2X''T = 0 \Rightarrow \dfrac{X''}{X} = \dfrac{T''}{a^2T} = -\lambda$ 于是

$$\begin{cases} X'' + \lambda X = 0, \\ T'' + a^2\lambda T = 0, \\ X(0) = X'(l) = 0. \end{cases}$$

求解常微分方程可得

$$X(x) = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda},\ X'(x) = C_1\sqrt{\lambda}\cos\sqrt{\lambda}x - C_2\sqrt{\lambda}\sin\sqrt{\lambda}x.$$

则

$$X(0) = C_2 = 0,\ X'(l) = \sqrt{\lambda}C_1 \cos\sqrt{\lambda}l = 0$$

于是 $\lambda = \left(\dfrac{(2n-1)\pi}{2l}\right)^2$，$(n = 1, 2, \cdots)$，则

$$X(x) = C \sin\frac{(2n-1)\pi}{2l}x$$
$$T(t) = A_n \sin\left(\frac{a(2n-1)\pi}{2l}t\right) + B_n \cos\left(\frac{a(2n-1)\pi}{2l}t\right)$$

则

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}(A_n \sin\left(\frac{a(2n-1)\pi}{2l}t\right) + B_n \cos\left(\frac{a(2n-1)\pi}{2l}t\right)) \sin\frac{(2n-1)\pi}{2l}x,\\
u|_{t=0} &= \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi}{2l}x = x^2 - 2lx,\\
u_t|_{t=0} &= \sum_{n=1}^{\infty} \frac{a(2n-1)\pi}{2l} A_n \sin\frac{(2n-1)\pi}{2l}x = 0,
\end{aligned}
$$

于是 $A_n = 0$，$B_n = \dfrac{2}{l}\displaystyle\int_0^l (x^2 - 2lx)\sin\frac{(2n-1)\pi}{2l}x\,\mathrm{d}x = \frac{16l^2}{(2n-1)^2\pi^2}\left((-1)^{n-1} - \frac{2}{(2n-1)\pi}\right)$，则

$$
u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{a(2n-1)\pi}{2l}t\right)\sin\frac{(2n-1)\pi}{2l}x
$$

题目 5. 设 $u(x,t)$ 适合定解问题：

$$
\begin{cases}
Lu = f(x,t), & (x,t)\in Q,\\
\left(-\dfrac{\partial u}{\partial x} + \alpha u\right)_{x=0} = \mu_1(t), & t \geqslant 0,\\
\left(\dfrac{\partial u}{\partial x} + \beta u\right)_{x=l} = \mu(t), & t \geqslant 0,\\
u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \psi(x), & 0 \leqslant x \leqslant t,
\end{cases}
$$

试引进辅助函数，把边界条件齐次化，设

(a)$\alpha > 0, \beta > 0$； (b)$\alpha = \beta = 0$.

解答. (a) 令 $v = \dfrac{l-x}{l}(-u_x + \alpha u - \mu_1(t)) + \dfrac{x}{l}(u_x + \beta u - \mu(t))$，则 $v|_{x=0} = v|_{x=l} = 0$.

(b) 令 $v = \dfrac{l-x}{l}(-u_x - \mu_1(t)) + \dfrac{x}{l}(u_x - \mu(t))$，则 $v|_{x=0} = v|_{x=l} = 0$.

题目 6. (26) 用分离变量法求解下列定解问题：

$$
\begin{cases}
Lu = -2b\dfrac{\partial u}{\partial t} + g, & (x,t)\in Q,\\
u|_{x=0} = u|_{x=l} = 0, & t \geqslant 0,\\
u|_{t=0} = u_t|_{t=0} = 0, & 0 \leqslant x \leqslant l.
\end{cases}
$$

题目 7. (27) 考虑定解问题：

$$
\begin{cases}
u_{tt} - u_{xx} = f(x,t), & (x,t)\in Q,\\
u|_{x=0} = u|_{x=l} = 0, & 0 \leqslant t \leqslant T,\\
u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \psi(x), & 0 \leqslant x \leqslant l.
\end{cases}
$$

试问对 φ, ψ, f 加什么条件才能保证由 Fourier 方法所得的解是古典解?

题目 8. (28) 用能量不等式证明一维波动方程带有第三边值条件的初边值问题解的唯一性.

27. 解答. 设 $\varphi(x) \in C^3[0,l]$, $\psi(x) \in C^2[0,l]$, $f(x,t) \in C^2(\bar{Q})$，类似**定理 4.2** 有如下相容性条件：$\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \psi(0) = \psi(l) = 0$，由 $f(x,t) = u_{tt} - u_{xx}$ 可知，$f(0,0) = 0 - \varphi''(0) = 0$, $f(l,0) = 0 - \varphi''(l) = 0$. 非齐次初值问题通解为 $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$，其中

$$u_n(x,t) = \left[A_n \cos\left(\frac{an\pi}{l}t\right) + B_n \sin\left(\frac{an\pi}{l}t\right) + \frac{l}{an\pi}\int_0^t f_n(\tau)\sin\left(\frac{an\pi}{l}(t-\tau)\right)\,\mathrm{d}\tau\right]\sin\left(\frac{n\pi}{l}x\right)$$

令 $g_n(x,t) = \left[\frac{l}{an\pi}\int_0^t f_n(\tau)\sin\left(\frac{an\pi}{l}(t-\tau)\right)\,\mathrm{d}\tau\right]\sin\left(\frac{n\pi}{l}x\right)$，由**定理 4.2** 可知，只需证明 $\{D^\alpha g_n\}$, $(\alpha = 0,1,2)$ 一致收敛即可.

由于 $f(0,0) = f(l,0) = 0$，则

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\left(\frac{n\pi}{l}x\right)\,\mathrm{d}x = \frac{2}{n\pi}\int_0^l f_x(x,t)\cos\left(\frac{n\pi}{l}x\right)\,\mathrm{d}x = -\frac{l^2}{n^2\pi^2}c_n(t)$$

其中 $c_n(t) = \frac{2}{l}\int_0^l f_{xx}(x,t)\sin\left(\frac{n\pi}{l}x\right)\,\mathrm{d}x$，则

$$|g_n(x,t)| \leqslant O\left(\frac{1}{n^3}\right),\ |Dg_n(x,t)| \leqslant O\left(\frac{1}{n^2}\right)$$

由于

$$\begin{aligned}
|D^2 g_n| &= O\left(\frac{1}{n}\left|\int_0^t c_n(\tau)\sin\left(\frac{an\pi}{l}(t-\tau)\right)\,\mathrm{d}\tau\right|\right) \\
&\leqslant O\left(\frac{1}{n^2}\right) + \left|\int_0^t c_n(\tau)\sin\left(\frac{an\pi}{l}(t-\tau)\right)\,\mathrm{d}\tau\right|^2 \\
&\leqslant O\left(\frac{1}{n^2}\right) + c_n^2(t_0), \quad (t_0 \in (0,t))
\end{aligned}$$

由 Bessel 不等式可知 $\sum_{n=1}^{\infty} c_n^2(t) \leqslant \frac{2}{l}\int_0^l |f_{xx}(x,t)|^2\,\mathrm{d}x < \infty$, $(\forall t \geqslant 0)$，于是 $|D^\alpha g_n|$, $(\alpha = 0,1,2)$ 均一致收敛，结合**定理 4.2** 则上述所有级数均在 $\bar{Q}$ 上一致收敛.