日期	科目	班级	姓名	学号
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第二次作业

题目 1. 设 $X^{(1)}$ 和 $X^{(2)}$ 均为 p 维随机向量，已知

$$X = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} \sim N_{2p}\left(\begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{bmatrix}\right)$$

其中 $\mu^{(i)}(i=1,2)$ 为 p 维向量，$\Sigma_i(i=1,2)$ 是 p 阶矩阵.

1. 证明 $X^{(1)}+X^{(2)}$ 和 $X^{(1)}-X^{(2)}$ 相互独立；
2. 求 $X^{(1)}+X^{(2)}$ 和 $X^{(1)}-X^{(2)}$ 的分布.

解答. 1. 设 I_p 为 p 阶单位阵，由于

$$\begin{bmatrix} X^{(1)}+X^{(2)} \\ X^{(1)}-X^{(2)} \end{bmatrix} = \begin{bmatrix} I_p & I_p \\ I_p & -I_p \end{bmatrix} X \sim N_{2p}\left(\begin{bmatrix} \mu_1+\mu_2 \\ \mu_1-\mu_2 \end{bmatrix}, \begin{bmatrix} 2(\Sigma_1+\Sigma_2) & 0 \\ 0 & 2(\Sigma_1-\Sigma_2) \end{bmatrix}\right)$$

于是 $X^{(1)}+X^{(2)}$ 与 $X^{(1)}-X^{(2)}$ 独立.

2. 由上一问可知

$$\begin{aligned} X^{(1)}+X^{(2)} &= \begin{bmatrix} I_p & I_p \end{bmatrix} X \ \sim N_p(\mu_1+\mu_2, 2(\Sigma_1+\Sigma_2)), \\ X^{(1)}-X^{(2)} &= \begin{bmatrix} I_p & -I_p \end{bmatrix} X \ \sim N_p(\mu_1-\mu_2, 2(\Sigma_1-\Sigma_2)). \end{aligned}$$

题目 2. 设 $X \sim N_3(\boldsymbol{\mu}, \Sigma)$，其中

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)', \quad \Sigma = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix} \quad (0<\rho<1)$$

1. 求条件分布 $(X_1, X_2|X_3)$ 和 $(X_1|X_2, X_3)$.
2. 给定 $X_3 = x_3$ 时，求出 X_1 和 X_2 的条件协方差.

解答. 1. 由条件分布计算公式可知

$$\begin{aligned} (X_1, X_2|X_3=x_3) &\sim N_2\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \rho \\ \rho \end{bmatrix}(x_3-\mu_3), \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} \rho \\ \rho \end{bmatrix}\begin{bmatrix} \rho & \rho \end{bmatrix}\right) \\ &\sim N_2\left(\begin{bmatrix} \mu_1+\rho(x_3-\mu_3) \\ \mu_2+\rho(x_3-\mu_3) \end{bmatrix}, \begin{bmatrix} 1-\rho^2 & \rho-\rho^2 \\ \rho-\rho^2 & 1-\rho^2 \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned} (X_1|X_2=x_2, X_3=x_3) &\sim N_1\left(\mu_1 + \begin{bmatrix} \rho & \rho \end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix}\right), 1 - \begin{bmatrix} \rho & \rho \end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1}\begin{bmatrix} \rho \\ \rho \end{bmatrix}\right) \\ &\sim N_1\left(\mu_1 + \frac{\rho}{1+\rho}(x_2+x_3-\mu_2-\mu_3), 1-\frac{2\rho^2}{1+\rho}\right) \end{aligned}$$

2. 由第一问可知，X_1, X_2 在给定 $X_3=x_3$ 下的条件协方差均为 $1-\rho^2$.

题目 3. 设 $X_1 \sim N(0,1)$，$X_2 = \begin{cases} -X_1, & -1 \leqslant X_1 \leqslant 1, \\ X_1, & \text{否则}. \end{cases}$

1. 证明：$X_2 \sim N(0,1)$.
2. 证明 (X_1, X_2) 的联合分布不是正态分布.

证明. 1. 当 $x \in [-1,1]$ 时，$f_{X_2}(x) = f_{X_1}(-x) = \dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$；当 $x \notin [-1,1]$ 时，$f_{X_2}(x) = f_{X_1}(x) = \dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$. 综上 $f_{X_1} = f_{X_2}$，所以 $X_2 \sim N(0,1)$.

2. 当 $x_2 \in [-1,1]$ 时，X_1, X_2 的联合分布函数满足

$$F_{X_1,X_2}(x_1,x_2) = \mathrm{P}[X_1 \leqslant x_1, X_2 \leqslant x_2] = \mathrm{P}[X_1 \leqslant x_1, -X_1 \leqslant x_2] = P[-x_2 \leqslant X_1 \leqslant x_1]$$

所以 $F_{X_1,X_2}(x_1,x_2)$ 不是正态分布. □

题目 4. 设 $X \sim N_p(\mu, \Sigma)$，A 为对称阵，证明：

(1). $\mathrm{E}(XX') = \Sigma + \mu\mu'$；

(2). $\mathrm{E}(X'AX) = \mathrm{tr}(\Sigma A) + \mu' A \mu$；

(3). 当 $\mu = a\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} =: a\mathbf{1}_p$, $A = I_p - \dfrac{1}{p}\mathbf{1}_p\mathbf{1}_p'$, $\Sigma = \sigma^2 I_p$ 时，试利用 (1) 和 (2) 的结果证明 $\mathrm{E}(X'AX) = \sigma^2(p-1)$.

若记 $X = (X_1, \cdots, X_p)'$ 此时 $X'AX = \sum_{i=1}^{p}(X_i - \bar{X})^2$，则

$$\mathrm{E}\left[\sum_{i=1}^{p}(X_i - \bar{X})^2\right] = \sigma^2(p-1).$$

解答. (1).

$$\begin{aligned}
&\int_{\mathbb{R}^p} \frac{xx'}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} \mathrm{d}x \\
\xlongequal{x \leftarrow x-\mu}\ & \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} (x+\mu)(x+\mu)' \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \mathrm{d}x \\
&= \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} (xx' + 2\mu x' + \mu\mu') \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \mathrm{d}x \\
&= \mu\mu' - \frac{\Sigma}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} (x+2\mu)\,\mathrm{d}\exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \\
&= \mu\mu' + \frac{\Sigma}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \mathrm{d}x \\
&= \Sigma + \mu\mu'
\end{aligned}$$

(2).

$$\begin{aligned}
&\int_{\mathbb{R}^p} \frac{x'Ax}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} \mathrm{d}x \\
\xlongequal{x\leftarrow x-\mu}\ & \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} (x+\mu)'A(x+\mu) \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \mathrm{d}x \\
\xlongequal{A\text{为对称阵}}\ & \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} (x'Ax+2\mu'Ax+\mu'A\mu) \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \mathrm{d}x \\
&= \mu'A\mu - \frac{\Sigma A}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} (x+2\mu)\, \mathrm{d} \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \\
&= \mu\mu' + \frac{\Sigma A}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^p} \exp\left\{-\frac{1}{2}x'\Sigma^{-1}x\right\} \mathrm{d}x \\
&= \mathrm{tr}(\Sigma A) + \mu'A\mu
\end{aligned}$$

(3). 由 (2) 可知: $\mathrm{E}(X'AX) = \mathrm{tr}(\Sigma A) + \mu'A\mu = p\sigma^2(1-1/p) + a^2 \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{1}_p = \sigma^2(p-1)$. 由于

$$\begin{aligned}
X'AX &= \begin{bmatrix} X_1, \cdots, X_p \end{bmatrix} \begin{bmatrix} 1-1/p & -1/p & \cdots & -1/p \\ -1/p & 1-1/p & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1/p \\ -1/p & \cdots & -1/p & 1-1/p \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} \\
&= \begin{bmatrix} X_1-\bar{X} & X_2-\bar{X} & \cdots & X_n-\bar{X} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} \\
&= \sum_{i=1}^{p} X_i^2 - X_i\bar{X} = \sum_{i=1}^{P} X_i^2 - 2X_i\bar{X} + \sum_{i=1}^{p} X_i\bar{X} \\
&= \sum_{i=1}^{p} (X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \sum_{i=1}^{p} (X_i-\bar{X})^2
\end{aligned}$$

故

$$\mathrm{E}\left[\sum_{i=1}^{p}(X_i-\bar{X})^2\right] = \sigma^2(p-1).$$

题目 5. 2-3 练习 1 设 $X \sim N_n(\boldsymbol{\mu}, \sigma^2 I_n)$, A 为 n 阶对撑幂等矩阵, 且 $\mathrm{rank}(A) = r(r \leqslant n)$, 证明: $\frac{1}{\sigma^2}X'AX \sim \chi^2(r,\delta)$, 其中 $\delta = \frac{1}{\sigma^2}\boldsymbol{\mu}'A\boldsymbol{\mu}$.

证明. 由于 $X \sim N_n(\boldsymbol{\mu}, \sigma^2 I_n)$, 则 $AX \sim N_n(A\boldsymbol{\mu}, \sigma^2 A'A) = N_n(A\boldsymbol{\mu}, \sigma^2 A)$, 于是

$$(AX)'(\sigma^2 A)^{-1}(AX) = \frac{1}{\sigma^2}X'AX \sim \chi^2(r, (A\boldsymbol{\mu})'(\sigma^2 A)^{-1}(A\boldsymbol{\mu})) = \chi^2(r, \frac{1}{\sigma^2}\boldsymbol{\mu}'A\boldsymbol{\mu})$$

□

题目 6. 2-3 练习 2 设 $X \sim N_n(\boldsymbol{\mu}, \sigma^2 I_n)$，$A, B$ 为 n 阶对称矩阵，若 $AB = O$，证明：$X'AX$ 与 $X'BX$ 相互独立.

证明. 由于 $AB = O$，由高代知识可知 $r(AB) \geqslant r(A) + r(B) - n \Rightarrow r(A) + r(B) \leqslant n$，令 $r(A) = p, r(B) = q$，于是 $p + q \leqslant n$，又由于 A, B 为对称阵，则存在正交阵 P, Q 使得 $P'AP = \begin{bmatrix} A_\lambda & 0 \\ 0 & 0 \end{bmatrix}, Q'BQ = \begin{bmatrix} B_\lambda & 0 \\ 0 & 0 \end{bmatrix}$其中 $A_\lambda = \text{diag}(\lambda_{11}, \cdots, \lambda_{1p}), B_\lambda = \text{diag}(\lambda_{21}, \cdots, \lambda_{2q})$，其中 $\lambda_{1i}, \lambda_{2j}$, $(1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q)$ 分别为 A, B 的特征值.

由于$AB = P \begin{bmatrix} A_\lambda & 0 \\ 0 & 0 \end{bmatrix} P'Q \begin{bmatrix} B_\lambda & 0 \\ 0 & 0 \end{bmatrix} Q'$，令 $P'Q = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ 其中 $C_{11} \in \mathbb{R}^{p \times q}, C_{12} \in \mathbb{R}^{p \times (n-q)}, C_{21} \in \mathbb{R}^{(n-p) \times q}, C_{22} \in \mathbb{R}^{(n-p) \times (n-q)}$，则

$$AB = P \begin{bmatrix} A_\lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} B_\lambda & 0 \\ 0 & 0 \end{bmatrix} Q' = P \begin{bmatrix} A_\lambda C_{11} B_\lambda & 0 \\ 0 & 0 \end{bmatrix} Q' = O$$

则 $C_{11} = 0$. 令 $Y_1 = P'X = (Y_{1i})_1^n, Y_2 = Q'X = (Y_{2i})_1^n$，于是

$$X'AX = (P'X)'P'AP(P'X) = Y_1' \begin{bmatrix} A_\lambda & 0 \\ 0 & 0 \end{bmatrix} Y_1 = \sum_{i=1}^{p} \lambda_{1i} Y_{1i}^2$$

$$X'BX = (Q'X)'Q'AQ(Q'X) = Y_2' \begin{bmatrix} B_\lambda & 0 \\ 0 & 0 \end{bmatrix} Y_2 = \sum_{i=1}^{q} \lambda_{2i} Y_{2i}^2$$

且$Y_1 = P'QY_2 = \begin{bmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} Z_q \\ Z_{n-q} \end{bmatrix} = \begin{bmatrix} C_{12} Z_{n-q} \\ C_{21} Z_q + C_{22} Z_{n-q} \end{bmatrix}$，其中 $Z_q = (Y_{21}, \cdots, Y_{2q})', Z_{n-q} = (Y_{2,q+1}, \cdots, Y_{2n})'$，由于 $p \leqslant n - q$，于是 $Y_{11}, \cdots, Y_{1p}$ 是 $Y_{2,q+1}, \cdots, Y_{2n}$ 的线性组合，又由于 $Y_2 = Q'X = N(Q'\boldsymbol{\mu}, \sigma^2 I_n)$，则 Y_{2i} 与 Y_{2j}, $(i \neq j)$ 独立，所以 $\{Y_{11}, \cdots, Y_{1p}\}$ 与 $\{Y_{21}, \cdots, Y_{2q}\}$ 独立，故 $X'AX$ 与 $X'BX$ 独立. □

题目 7. 2-3 练习 3 设 $X \sim N_p(\boldsymbol{\mu}, \Sigma), \Sigma > 0$，$A, B$ 为 p 阶对称阵，证明：$(X - \boldsymbol{\mu})'A(X - \boldsymbol{\mu})$ 与 $(X - \boldsymbol{\mu})'B(X - \boldsymbol{\mu})$ 独立，当且仅当，$\Sigma A \Sigma B \Sigma = O_{p \times p}$.

证明. 下面证明充分性，令 $r(\Sigma) = p$，特征值为 $\lambda_1, \cdots, \lambda_p$，由于 $\Sigma > 0$，则存在正交阵 P，使得 $P'\Sigma P = \text{diag}(\lambda_1, \cdots, \lambda_p)$. 令 $\Sigma^{\frac{1}{2}} = P\text{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_2})P'$，则 $\Sigma = \Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}$.

令 $Y = \Sigma^{\frac{1}{2}}(X - \mu)$，于是 $Y \sim N_p(0, (\Sigma^{\frac{1}{2}})'\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}) = N_p(0, I_p)$，于是

$$(X - \boldsymbol{\mu})'A(X - \boldsymbol{\mu}) = Y'\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}} Y = Y'C_1 Y$$

$$(X - \boldsymbol{\mu})'B(X - \boldsymbol{\mu}) = Y'\Sigma^{\frac{1}{2}} B \Sigma^{\frac{1}{2}} Y = Y'C_2 Y$$

其中 $C_1 = \Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}, C_2 = \Sigma^{\frac{1}{2}} B \Sigma^{\frac{1}{2}}$ 且均为对称阵，由上题可知，若 $C_1 C_2 = O$，则 $Y'C_1 Y$ 与 $Y'C_2 Y$ 独立，于是 $\Sigma A \Sigma B \Sigma = O \Leftrightarrow \Sigma^{\frac{1}{2}} A \Sigma B \Sigma^{\frac{1}{2}} = O \Leftrightarrow C_1 C_2 = O \Rightarrow Y'C_1 Y$ 与$Y'C_2 Y$ 独立， □

题目 8. 2-3 练习 4 证明 Wishart 分布的性质 4：设 $X_{(a)} \sim N_p(\mathbf{0}, \Sigma)(a = 1, 2, \cdots, n)$ 相互独立，其中 $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$，其中 $\Sigma_{11} \in \mathbb{R}^{r\times r}, \Sigma_{22} \in \mathbb{R}^{(p-r)\times(p-r)}$，且已知

$$W = \sum_{a=1}^{n} X_{(a)} X_{(a)}' = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \sim W_p(n, \Sigma)$$

其中 $W_{11} \in \mathbb{R}^{r\times r}, W_{22} \in \mathbb{R}^{(p-r)\times(p-r)}$，则有以下结论：

(1) $W_{11} \sim W_r(n, \Sigma_{11}), W_{22} \sim W_{p-r}(n, \Sigma_{22})$.

(2) 当 $\Sigma_{12} = O$ 时，W_{11} 与 W_{22} 相互独立.

证明. (1) 令 $X = (X_{(1)}, X_{(2)}, \cdots, X_{(n)}) = (X_1, X_2)$，其中 $X_1 \in \mathbb{R}^{n\times r}, X_2 \in \mathbb{R}^{n\times(p-r)}$，且 $X_1 \sim N(\mathbf{0}, \Sigma_{11}), X_2 \sim N(\mathbf{0}, \Sigma_{22})$ 由于 $W = X'X$，则 $W_{11} = X_1'X_1, W_{22} = X_2'X_2$，于是 $W_{11} \sim W_r(n, \Sigma_{11}), W_{22} \sim W_{p-r}(n, \Sigma_{22})$.

(2) 由于 $\Sigma_{12} = \Sigma_{21}' = 0$，则 X_1 与 X_2 独立，由于 W_{11}, W_{22} 分别由 X_1 和 X_2 表出，所以 W_{11} 与 W_{22} 也独立. □

题目 9. 2-3 练习 5 对单个 p 元正态总体 $N_p(\boldsymbol{\mu}, \Sigma)$ 的均值向量的检验问题，试用似然比原理导出检验 $H_0 : \mu = \mu_0$(Σ_0已知) 的似然比统计量及其分布.

解答. 似然比函数分子为 $L_1 = L(\boldsymbol{\mu}_0, \Sigma_0) = (2\pi)^{-\frac{np}{2}} |\Sigma_0|^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (X_i - \boldsymbol{\mu})' \Sigma_0^{-1} (X_i - \boldsymbol{\mu}) \right\}$,

分母为 $L_2 = \max_{\boldsymbol{\mu}} L(\boldsymbol{\mu}, \Sigma_0) = L(\hat{\boldsymbol{\mu}}, \Sigma_0) \xlongequal{\hat{\boldsymbol{\mu}}=\bar{X}} (2\pi)^{-\frac{np}{2}} |\Sigma_0|^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (X_i - \bar{X})' \Sigma_0^{-1} (X_i - \bar{X}) \right\}$,
于是广义似然比为

$$\lambda = \frac{L_1}{L_2} = \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} [(X_i - \boldsymbol{\mu})' \Sigma_0^{-1} (X_i - \boldsymbol{\mu}) - (X_i - \bar{X})' \Sigma_0^{-1} (X_i - \bar{X})] \right\}$$

由于

$$\begin{aligned}
&\sum_{i=1}^{n} \left\{ (X_i - \boldsymbol{\mu})' \Sigma_0^{-1} (X_i - \boldsymbol{\mu}) - (X_i - \bar{X})' \Sigma_0^{-1} (X_i - \bar{X}) \right\} \\
&= \sum_{i=1}^{n} \left\{ X_i' \Sigma_0^{-1} X_i - 2X_i' \Sigma_0^{-1} \boldsymbol{\mu} + \boldsymbol{\mu}' \Sigma_0^{-1} \boldsymbol{\mu} - \left(X_i' \Sigma_0^{-1} X_i - 2X_i' \Sigma_0^{-1} \bar{X} + \bar{X}' \Sigma_0^{-1} \bar{X} \right) \right\} \\
&= \sum_{i=1}^{n} \left\{ \boldsymbol{\mu}' \Sigma_0^{-1} \boldsymbol{\mu} + 2(X_i' \Sigma_0^{-1} \bar{X} - X_i' \Sigma_0^{-1} \boldsymbol{\mu}) - \bar{X}' \Sigma_0^{-1} \bar{X} \right\} \\
&= n\boldsymbol{\mu}' \Sigma_0^{-1} \boldsymbol{\mu} + 2n(\bar{X}' \Sigma_0^{-1} \bar{X} - \bar{X}' \Sigma_0^{-1} \boldsymbol{\mu}) - n\bar{X}' \Sigma_0^{-1} \bar{X} \\
&= n\left(\boldsymbol{\mu}' \Sigma_0^{-1} \boldsymbol{\mu} - 2\bar{X}' \Sigma_0^{-1} \boldsymbol{\mu} + \bar{X}' \Sigma_0^{-1} \bar{X} \right) \\
&= n(\bar{X} - \boldsymbol{\mu})' \Sigma_0^{-1} (\bar{X} - \boldsymbol{\mu})
\end{aligned}$$

于是 $\lambda = \exp\left\{-\frac{1}{2}(\bar{X}-\boldsymbol{\mu})'(\Sigma_0/n)^{-1}(\bar{X}-\boldsymbol{\mu})\right\}$，于是似然比统计量为

$$\Lambda = (2\pi)^{-\frac{p}{2}}|\Sigma_0/n|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(\bar{X}-\boldsymbol{\mu})(\Sigma_0/n)^{-1}(\bar{X}-\boldsymbol{\mu})\right\} \sim N_p(\boldsymbol{\mu}, \Sigma_0/n)$$

服从均值为 $\boldsymbol{\mu}$，协方差矩阵为 Σ_0/n 的 p 元正态分布.

题目 10. 2-3 练习 6 两个多元正态总体均值向量检验，样本量 $n = 10$，每个样本来自 $X \sim N_4(\boldsymbol{\mu}_1, \Sigma_1), Y \sim N_4(\boldsymbol{\mu}_2, \Sigma_2)$，检验均值向量是否相同.

解答.

```r
setwd("C:/Users/99366/Documents/GitHub/LaTex-Projects/Data Analysis/3,4,5,6")
data <- read.table(file = "data.csv", sep = ",")
X <- data[1:10, ]
Y <- data[11:20, ]
print(t.test(X, Y, var.equal = FALSE, paired = FALSE, mu = 0))
# var.equal 为 FALSE 表示假设两个总体方差不相等，paired 为 FALSE 表示假设两个样本是独
↪ 立的，mu 为 0 表示检验两个总体均值是否相等
```

输出结果

```
        Welch Two Sample t-test

data:  X and Y
t = -0.96819, df = 74.917, p-value = 0.3361
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -9.55497  3.30497
sample estimates:
mean of x mean of y
   50.125    53.250
```

由于 p-value 大于 0.05，所以接受原假设，即均值向量相同.