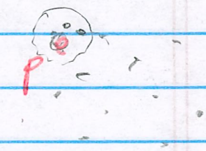
Ch. 3. Sequences and series

Recall. A *sequence* of points $\{p_n\}_{n=1}^{\infty}$ in a metric space X is an assignment of $p_n \in X$ to each $n \in \mathbb{N}$.



$p_1, p_2, p_3, \ldots$

Def. A sequence $p_n \in X$ is said to *converge to $p \in X$* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t.

$n \geq N \implies d(p_n, p) < \varepsilon.$

Write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

The *Range* of p_n is the subset $\{p_n\} \subset X$ (forget numbering).

Theorem: (a) p_n converges to $p \in X$ iff every open set containing p contains p_n for all but finitely many n.

(b) If $p \in X$ and $p' \in X$, and $p_n \to p$ and $p_n \to p'$, then $p = p'$

(c) If p_n converges then it is bounded.

(i.e. its range is a bounded set ⇒ $\exists\, q \in X$, $M > 0$ s.t.

$d(q, p_n) < M \ \forall n \in \mathbb{N}$).

(d) If $E \subset X$ and p is a lim. pt. of E, then there is a sequence $\{p_n\}$ in E s.t. $p = \lim_{n\to\infty} p_n$.

Pf (a) ($\Rightarrow$) $G \ni p$ open. $\exists\, \varepsilon > 0$ s.t. $N_\varepsilon(p) \subset G$.

$\Rightarrow \exists\, N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow p_n \in N_\varepsilon(p) \subset G$.

$\therefore p_n \notin G \Rightarrow n < N$. ✓.

($\Leftarrow$) Then $\forall\, \varepsilon > 0$, $N_\varepsilon(p)$ cont. all but fin. many p_n, namely, $\{p_{n_1}, \ldots, p_{n_k}\} = \{p_n\} \setminus N_\varepsilon(p)$.

Let $N = \max\{n_1, \ldots, n_k\} + 1$.

$n \geq N \Rightarrow p_n \notin \{p_{n_1}, \ldots, p_{n_k}\}$

$\Rightarrow p_n \in N_\varepsilon(p)$

$\Rightarrow d(p, p_n) < \varepsilon$ ✓.

(b) Suff. to show $d(p, p') = 0$, or $d(p, p') < \varepsilon \ \forall \varepsilon > 0$.

Let $\varepsilon > 0$. $\exists\, N, N' > 0$ s.t.

$n \geq N \Rightarrow d(p_n, p) < \varepsilon/2$

and $n \geq N' \Rightarrow d(p_n, p') < \varepsilon/2$.

Let $n \geq \max\{N, N'\}$.

Then $d(p,p') \le d(p,p_n) + d(p_n,p')$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arb., $d(p,p') = 0$

$$\therefore p = p'. \quad \square$$

(c) Suppose $p_n \to p$.

$\exists$ N s.t. $n \geq N \Rightarrow d(p,p_n) < 1$.

Let $M = \max\{1, d(p,p_1), d(p,p_2), \ldots, d(p,p_N)\}$.

Then $d(p,p_n) \le M$ for all $n \in \mathbb{N}$. ✓

(d) For each $n \in \mathbb{N}$, $\exists$ $p_n \in E$ s.t. $p_n \neq p$

and $d(p,p_n) < \frac{1}{n}$.

Consider the sequence p_n.

Given $\varepsilon > 0$, choose $N = \lceil \frac{1}{\varepsilon} \rceil + 1$

Then for $n \geq N$, $\frac{1}{n} \le \frac{1}{N} < \varepsilon$.

$\Rightarrow d(p,p_n) \le \frac{1}{n} < \varepsilon$ ✓. $\square$

<u>Examples.</u> $X = \mathbb{C}$ ($= \mathbb{R}^2$ as a metric space.)

(a) $s_n = \frac{1}{n}$, $s_n \xrightarrow{\text{convergent}} 0$, infinite range.

(b) $s_n = n^2$, divergent (unbounded)

(c) $s_n = 1 + \frac{(-1)^n}{n}$, $s_n \longrightarrow 1$.

(d) $s_n = i^n$, Divergent, bdd, finite range.

(Not a Cauchy sequence!)

next time.

<u>Thm.</u> $X = \mathbb{C}$. Suppose $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$.

(a) $t_n + s_n \longrightarrow t + s$

(b) $c s_n \longrightarrow c s$

(c) $s_n t_n \longrightarrow s t$

(d) $\frac{1}{s_n} \to \frac{1}{s}$ if $s \neq 0$.

<u>Pf:</u> (a) Let $\varepsilon > 0$. $\exists N_1, N_2$ s.t.

$$n \geq N_1 \Rightarrow |s_n - s| < \tfrac{\varepsilon}{2}$$

$$n \geq N_2 \Rightarrow |t_n - t| < \varepsilon/2 .$$

Let $N = \max\{N_1, N_2\}$

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon \quad \checkmark$$

(b) same

(c) $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$

$\downarrow 0 \qquad \downarrow 0$ by (a)-(b).

Need to show $\downarrow 0$.

Let N_1 s.t. $n \geq N_1 \Rightarrow |s_n - s| < \sqrt{\varepsilon}$

$N_2 \qquad n \geq N_2 \Rightarrow |t_n - t| < \sqrt{\varepsilon}$.

$n \geq N = \max\{N_1, N_2\} \Rightarrow |(s_n - s)(t_n - t)| < \varepsilon$. ✓

(d) $s \neq 0$. Choose m s.t. $n \geq m \Rightarrow |s_n - s| < \frac{1}{2}|s|$.

Then $|s_n| \geq |s| - |s_n - s| > \frac{1}{2}|s|$.

$\Rightarrow \frac{1}{|s_n|} < \frac{2}{|s|}$.

Choose $N \geq m$ s.t. $n \geq N$

$\Rightarrow |s_n - s| < \frac{1}{2}|s|^2 \varepsilon$.

Then $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s||s_n|}$

$< \frac{2}{|s|^2} \cdot |s - s_n| < \varepsilon$ $\square$.

Thm Suppose $X = \mathbb{R}^k$.

Let $x_n = (\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,k})$ be a sequence

(a) $x_n \to x \in \mathbb{R}^k \iff \alpha_{n,j} \to x_j \ \forall j$.

(b) Suppose $x_n \to x$, $y_n \to y$, $\beta_n \xrightarrow{\in \mathbb{R}} \beta$, then

$x_n + y_n \to x + y$, $x_n \cdot y_n \to x \cdot y$, $\beta_n x_n \to \beta \cdot x$

Pf (a) ($\Rightarrow$) $|\alpha_{n,j} - \alpha_j| \le |x_n - x| = \sqrt{\sum_{j=1}^{k} (\alpha_{n,j} - \alpha_j)^2}$.

($\Leftarrow$) Choose N s.t. $n \ge N$

$\Rightarrow |\alpha_{n,j} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}}$ for $j = 1, \ldots, k$

Then $|x_n - x| = \left(\sum |\alpha_{n,j} - \alpha_j|^2\right)^{1/2}$

$< \left(\cancel{k} \frac{\varepsilon^2}{\cancel{k}}\right)^{1/2} = \varepsilon$ $\square$

Defn: Given a sequence $\{p_n\}$, consider a sequence of pos. integers $\{n_i\}_{i=1}^{\infty}$ with

$n_1 < n_2 < n_3 < \cdots$

The sequence $\{p_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{p_n\}$.

If $p_{n_i} \to p$ as $i \to \infty$, p is called a subsequential limit of p_n

$X = \mathbb{C}$

<u>Ex:</u> $p_n = i^n$.

$n_1 = 1, \quad n_2 = 5, \quad n_3 = 9, \ \dots$

$n_j = 1 + 4j$.

$i^{n_j} = i \cdot i^{4j} = i$.

$\{p_{n_j}\} = \{i, i, i, \dots\}$

$\longrightarrow i$ as $j \to \infty$!

<u>Convergent subsequence.</u>

<u>Theorem.</u> (a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of p_n converges.

(b) Every <u>bounded</u> sequence in $\mathbb{R}^k$ contains a convergent subsequence.

<u>Pf</u>: (a) The range $E = \{p_n\}$ either finite or infinite.

$p_{n_1} = p_{n_2} = \dots = p$ for some subseq.

has a limit pt, p, since X is cpt!

choose n_i s.t. $d(p_{n_i}, p) < \frac{1}{i}$.

$\Rightarrow p_{n_i} \to p$ as $i \to \infty$.

(b) $\{p_n\} \subset I$, cpt, for some k-cell I!