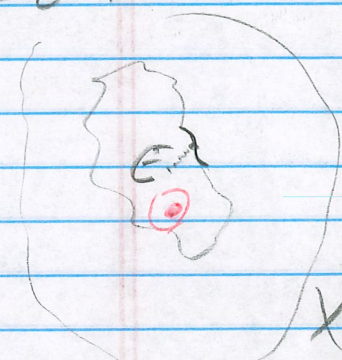
Let X be a metric space w/ metric d.

$E \subset X$ a subset.

Recall: • $p \in X$ is a limit point (or accumulation pt.) of E if every nbhd of p contains a point $q \neq p$ with $q \in E$. (i.e. $N_r(p) \cap (E \setminus \{p\}) \neq \emptyset \ \forall r > 0$).

• Let E' denote the set of all limit points of E.

• E is called closed if $E' \subset E$, i.e., every limit pt of E belongs to E.



E.g. $[0,1]$, finite sets, are closed

$[0,1)$ not closed. $(1 \in E' \setminus E)$.

• $p \in E$ is called an interior point of E if $\exists\, r > 0$ s.t. $N_r(p) \subset E$.

• Let $\mathring{E}$ denote the set of all interior pts of E, aka the interior of E.

• E is called open if $\mathring{E} = E$, i.e., every pt of E is an interior point.

E.g. $(0,1)$, $N_r(p)$ for any $p \in X$, $r > 0$ are open

$[0,1)$ not open. $(0 \in E \setminus \mathring{E})$.

<u>More examples.</u>

- $E = X$? $E = \phi$?

Both open & closed.

- $X = \mathbb{R}$

— $E = (0, \infty)$. open, not closed.
— $E = (-\infty, 0]$. not open, closed.
— $E = \mathbb{N}$. not open, closed
— $E = \mathbb{Q}$. <u>neither</u>. $(\mathbb{Q}' = \mathbb{R}, \ \mathring{\mathbb{Q}} = \phi !)$

$\hookrightarrow$ $\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$!

<u>Theorem</u>: $E \subset X$ is open $\iff E^c$ is closed.

<u>Pf</u>: $(\Leftarrow)$ Suppose E^c is closed.

Let $x \in E$. Since E^c is closed, $x \notin (E^c)'$.

$\therefore \exists$ nbhd $N \ni x$ s.t. $N \cap E^c = \phi$

$\Rightarrow N \subset E$ ✓.

$(\Rightarrow)$ Suppose E is open.

Equiv. to show contrapos: $x \notin E^c \Rightarrow x \notin (E^c)'$.

$\Downarrow$
$x \in E$.

Since E is open, $\exists N \ni x$ s.t. $N \subset E$.

$\Rightarrow N \cap E^c = \phi$

$\Rightarrow x \notin (E^c)'$ $\square$

<u>Corollary</u>. $E \subset X$ is closed $\Leftrightarrow E^c$ is open.

<u>Pf</u>: $(E^c)^c = E$ — reverse roles of E and E^c. $\square$

Let I be a set. A collection of sets

$\underline{E} = \{E_\alpha\}_{\alpha \in I} \subset \mathcal{P}(X)$ is <u>indexed</u> by I if

the elements E_α are in 1-1 corresp. with $\alpha \in I$.

<u>Defn</u>. $\bigcup_{\alpha \in I} E_\alpha = \{x \in X \mid x \in E_\alpha \text{ for some } \alpha \in I\}$

$\bigcap_{\alpha \in I} E_\alpha = \{x \in X \mid x \in E_\alpha \ \forall \alpha \in I\}$.

<u>Lemma</u>: $\left(\bigcup_{\alpha \in I} E_\alpha\right)^c = \bigcap_{\alpha \in I} E_\alpha^c$.

<u>Pf</u>: ($\subset$) Suppose $x \overset{\alpha}{}$. Then $x \notin E_\alpha \ \forall \alpha$. ✓.

($\supset$) Suppose $x \in \bigcap_{\alpha \in I} E_\alpha^c$. Then $x \in E_\alpha^c \ \forall \alpha$

$\Rightarrow x \notin E_\alpha \ \forall \alpha$

$\Rightarrow x \notin \bigcup_{\alpha \in I} E_\alpha$ $\square$

__Theorem__: (a) For __any__ collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For __any__ collection $\{F_\alpha\}$ of closed sets, $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(c) For any __finite__ collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open.

(d) For any __finite__ collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^{n} F_i$ is closed.

__Pf__: (a) Let $x \in \bigcup G_\alpha$.

Then $x \in G_\alpha$ for some α.

$\Rightarrow \exists\, N \ni x$ w/ $N \subset G_\alpha$

$\Rightarrow N \subset \bigcup_\alpha G_\alpha$ ✓.

(c) Let $x \in \bigcap_{i=1}^{n} G_i$. Then for each i, $\exists\, N_i = N_{r_i}(x)$, $r_i > 0$, s.t. $x \in N_{r_i} \subset G_i$.

Let $r = \min\{r_1, \ldots, r_n\}$.

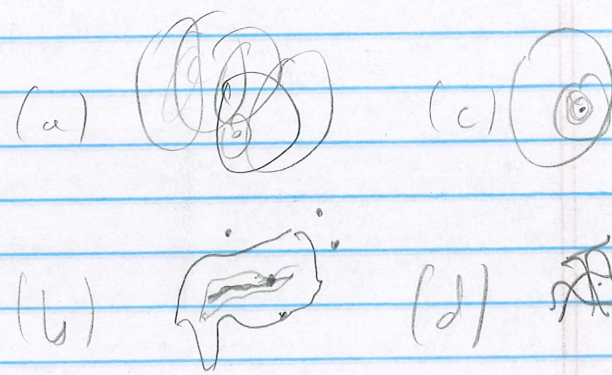
$\Rightarrow N_r(x) \subset N_{r_i}(x) \subset G_i \;\forall\, i \Rightarrow x \in N_r(x) \subset \bigcap_{i=1}^{n} G_i$ ✓

(b) $\left(\bigcap_{\alpha} F_\alpha\right)^c = \bigcup_{\alpha} F_\alpha^c$ = union of open sets, open

(a) $\Rightarrow$ $\bigcap F_\alpha$ closed. ✓

(d) $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$, open by (a).

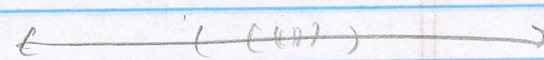
$\Rightarrow$ $\bigcup_{i=1}^{n} F_i$ closed. □



Counterexamples when "finite" is dropped from (c-d).

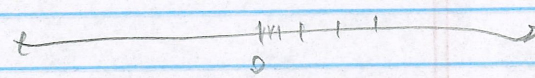
$X = \mathbb{R}$.

- Let $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.



$\Rightarrow \bigcap_{n=1}^{\infty} G_n = \{0\}$, not open!

- Let $F_n = \left\{\frac{1}{n}\right\}$



$\bigcup_{n=1}^{\infty} F_n = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, not closed.

(Note: since points are closed, every set is a union of closed sets!

$$E = \bigcup_{x \in E} \{x\}.$$
)