Problem set 11 solutions

Math 521 Section 001, UW-Madison, Spring 2024

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1. (Rudin 7.6) *First solution.* The series is equal to
$$\sum_{n=1}^{\infty}\frac{x^n}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n}.$$
Consider a bounded interval $[a,b]$, with $\max\{|a|,|b|\}=M$. Then $x^2\le M^2$ on $[a,b]$. The first series converges uniformly by comparison the Weierstrass M-test, where $M_n=M^2/n^2$. The second series converges uniformly, since it converges and is independent of x. Hence, their sum converges uniformly (by Exercise 7.2).

   Second solution. This is an alternating series, so we can use a version of the alternating series test for uniform convergence.

   Lemma. *Suppose $f_n\ge f_{n+1}\ge\cdots\ge 0$ is a decreasing sequence of nonnegative, bounded functions. Then $\sum(-1)^n f_n$ converges uniformly if and only $\|f_n\|_{C^0}\to 0$.*

   This can be proven exactly as in Problem set 7, question 1, using $\|f_n\|_{C^0}$ in place of $|b_n|$.

2. (Rudin 7.7) We claim that $f_n(x)\to 0$ uniformly. Using the inequality $ab\le\frac{1}{2}(a^2+b^2)$ with $a=1$ and $b=|x|\sqrt{n}$, we have
$$|x|\sqrt{n}=1\cdot x\sqrt{n}\le 1+nx^2.$$
Inverting, we get
$$\frac{1}{1+nx^2}\le\frac{1}{|x|\sqrt{n}}.$$
This gives
$$|f(x)|\le\frac{|x|}{1+nx^2}\le\frac{|x|}{|x|\sqrt{n}}=\frac{1}{\sqrt{n}}.$$
Hence $f(x)\to 0$ uniformly. On the other hand,
$$f_n'(x)=\frac{1-nx^2}{(1+nx^2)^2}.$$
This goes as n/n^2 if $x\ne 0$, so tends to zero. But $f_n'(0)=1$ for all n, which does not tend to zero.

3. (Rudin 7.8) We use the Weierstrass M-test. We have $M_n = |c_n|$, whose sum converges, so the series converges uniformly. If $x \neq x_n$ for all n, then $I(x - x_n)$ is continuous at x for all n, as are the partial sums (which are finite sums). By the uniform limit theorem, $f(x)$ is also continuous at x.

4. (Rudin 7.9) Let $\varepsilon > 0$. Since f is continuous, we can choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for $|y - x| < \delta$. Since $x_n \to x$, we can choose N_1 such that $|x_n - x| < \delta$ for $n \geq N$. Since $f_n \to f$ uniformly, we can choose N_2 such that $|f_n(y) - f(y)| < \varepsilon/2$ for $n \geq N_2$, and for all $y \in E$. Now, for $n \geq \max\{N_1, N_2\}$, we have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown that $f_n(x_n) \to f(x)$.

The converse is false: consider the functions $f_n(x) = x/n$ on $E = \mathbb{R}$. Then $f_n(x) \to 0$ uniformly on every compact interval. Since a given point $x \in \mathbb{R}$ is contained in some compact interval, the statement holds true by what we've just shown. However, $f_n(n) = 1 \not\to 0$, so the convergence is not uniform on $\mathbb{R}$.

(Notice that if we assume further that E is compact, then the converse is in fact true, since any contradicting sequence will sub-converge to some point x.)

5. (Rudin 7.12, broken into parts)

(a) We have

$$\begin{aligned}\lim_{T\to\infty}\lim_{t\to 0}\int_t^T g(x)\,dx &= \lim_{T\to\infty}\lim_{t\to 0}\left(\int_t^1 g(x)dx + \int_1^T g(x)dx\right)\\ &= \lim_{T\to\infty}\int_1^T g(x)\,dx + \lim_{t\to 0}\int_t^1 g(x)\,dx\\ &= \lim_{t\to 0}\lim_{T\to\infty}\int_t^T g(x)\,dx.\end{aligned}$$

(b) We show that $a_n = \int_t^n f(x)dx$ fulfills the Cauchy criterion. Let $\varepsilon > 0$. Choose N large enough that $\int_N^\infty g(x)dx < \varepsilon$. For $n \geq m \geq N$, we have $|a_n - a_m| = |\int_m^n f(x)\,dx| \leq \int_m^n |f(x)|\,dx \leq \int_m^n g(x)\,dx < \varepsilon$.

(c) Let L be the limit from (b). Let $\varepsilon > 0$, let N be as in (b). Given $y \geq N$, let $n = [y]$. We have

$$|a_n - \int_t^y f(x)dx| \leq |\int_n^y g(x)dx| < \varepsilon.$$

This gives

$$|\int_t^y f(x)dx - L| \leq |\int_t^y f(x)dx - a_n| + |a_n - L| < 2\varepsilon.$$

(d) We can set $b_n = \int_{1/n}^\infty f(x)\,dx$ as in (b), and show this limit exists. Then we can argue as in (c) that it equals the $\lim_{t\to 0}\int_t^\infty f(x)\,dx$.

(e) Since we know that the relevant limits all exist, this can be shown exactly as in (a).

(f) We have shown above that $\int_0^\infty f\,dx$ exists, so it remains to prove that this equals the LHS.

Let $\varepsilon > 0$. Choose t and T small and large enough, respectively, so that

$$\int_0^t g(x)\,dx + \int_T^\infty g(x)\,dx < \varepsilon.$$

This gives

$$|\int_0^t f(x)\,dx| + |\int_T^\infty f(x)\,dx| < \varepsilon$$

as well as

$$|\int_0^t f_n(x)\,dx| + |\int_T^\infty f_n(x)\,dx| < \varepsilon$$

for all n. Now, consider the difference

$$A_n = \int_0^\infty f(x)dx - \int_0^\infty f_n(x)dx = \int_0^\infty (f(x) - f_n(x))\,dx.$$

This is equal to

$$A_n = \int_0^t (f(x) - f_n(x))\,dx + \int_{t^T} (f(x) - f_n(x))\,dx + \int_T^\infty (f(x) - f_n(x)\,dx.$$

Taking absolute values, we have

$$\begin{aligned}|A_n| &\le |\int_0^t (f(x) - f_n(x))\,dx| + |\int_t^T (f(x) - f_n(x))\,dx| + |\int_T^\infty (f(x) - f_n(x)\,dx| \\ &\le |\int_t^T (f(x) - f_n(x))\,dx| + 2\varepsilon.\end{aligned}$$

By uniform convergence, we know that the integral tends to zero as $n \to \infty$, giving

$$\limsup_{n\to\infty} |A_n| \le 2\varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so $A_n \to 0$.

6. (Based on Rudin 7.13b) Let $\varepsilon > 0$.

Since $f(x)$ is continuous and $[a, b]$ is compact, it is uniformly continuous. There exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon/3$$

for $|x - y| < \delta$, $a \le x, y \le b$. Choose a partition P with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, for which $x_i - x_{i-1} < \delta$. Since f_n converges pointwise to f and P is a finite set, we can choose N large enough that for $n \ge N$, we have

$$|f_n(x_i) - f(x_i)| < \varepsilon/3$$

for each $x_i \in P$. Notice that by monotonicity, if $x_{i-1} \le x \le x_i$, we have

$$|f_n(x) - f_n(x_i)| \le |f_n(x_i) - f_n(x_{i-1})| < \varepsilon/3.$$

We claim that this N works for uniform convergence. Given any $x \in [a,b]$, we have $x \in [x_{i-1}, x_i]$ for some i. Applying the triangle inequality, we have

$$\begin{aligned} |f_n(x) - f(x)| &\le |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

7. (Rudin 3.9) (a) $\limsup_{n\to\infty} \sqrt[n]{n^3} = (\lim_{n\to\infty} \sqrt[n]{n})^3 = 1 = R$.
   (b) $\limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n!}} = 2 \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0$, hence $R = \infty$.
   (c) $R = 1/2$.
   (d) $R = 3$.

8. (Rudin 8.1) From lecture, it remains only to prove that

$$\left(\frac{d}{dx}\right)^n e^{-1/x^2} = p_n\left(\frac{1}{x}\right) e^{-1/x^2}$$

   for some polynomial $p_n(u)$. We use induction. The base case $n = 0$ is clear. For the induction step, note that if $u = \frac{1}{x}$, then $\frac{du}{dx} = -u^2$. We have

$$\begin{aligned} \left(\frac{d}{dx}\right)^{n+1} e^{-1/x^2} &= \frac{d}{dx}\left(\frac{d}{dx}\right)^n e^{-1/x^2} \\ &= \frac{d}{dx}\left(p_n(u)e^{-u^2}\right) \\ &= \frac{du}{dx}\left(p_n'(u) - 2up_n(u)\right)e^{-u^2} \\ &= -u^2\left(p_n'(u) - 2up_n(u)\right)e^{-u^2}. \end{aligned}$$

   We can take $p_{n+1}(u) = u^2\left(2up_n(u) - p_n'(u)\right)$, which is again a polynomial, so the claim is true.

9. (Rudin 8.2) The sum of the i'th column is

$$-2 + \frac{1}{2}\sum_{j=0}^{i-1} 2^{-j} = -2 + \frac{1 - \frac{1}{2^i}}{1 - \frac{1}{2}} = -\frac{1}{2^{i-1}}.$$

   These sum to -2, as claimed. However, the sum of each j'th row is zero, so their total is also zero.

10. (Rudin 8.4) (a) The LHS is the derivative at zero of $b^x = e^{(\log b)x}$.

(b) The LHS is the derivative of $\log(1+x)$ at $x = 0$.

(c) The LHS is $\lim_{x\to 0} \operatorname{Exp}\left(\frac{\operatorname{Log}(1+x)}{x}\right)$. Since $\operatorname{Exp}(u)$ is continuous at $u = 1$, the limit is equal to

$$\operatorname{Exp}\left(\lim_{x\to 0} \frac{\operatorname{Log}(1+x)}{x}\right) = \operatorname{Exp}(1) = e.$$

(d) Letting $y_n = \frac{x}{n}$, we have

$$\left(1 + \frac{x}{n}\right)^n = (1+y_n)^{\frac{x}{y_n}} = \left((1+y_n)^{\frac{1}{y_n}}\right)^x.$$

Since $y_n \to 0$ as $n \to \infty$, by (c), the limit equals e^x.