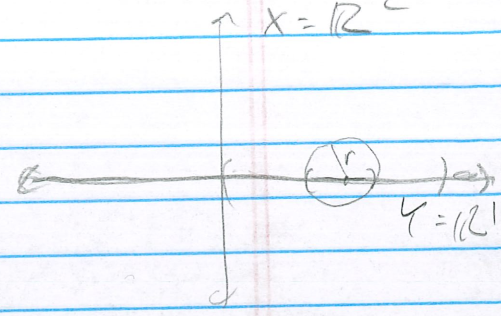


Recall: X metric space w/ metric d.

$Y \subset X$, any subset

$\rightarrow$ d defines a metric on Y!

$y_1, y_2 \in Y \subset X \rightsquigarrow d(y_1, y_2)$.

$$N_r^Y(p) = \{ y \in Y \mid d(y,p) < r \} = N_r^X(p) \cap Y.$$

Defn. Let $E \subset Y \subset X$. We say E is open relative to Y if it is open when considered as a subset of Y; i.e., for each $p \in Y$, $\exists\, r > 0$ s.t.

$$N_r^Y(p) \subset Y.$$

Ex: $X = \mathbb{R}^2$, $Y = \mathbb{R}$

$E = (0,1) \times \{0\}$.

Theorem: G is open relative to Y

$\Leftrightarrow$ $\exists$ open subset $\tilde{G} \subset X$ s.t. $\tilde{G} \cap Y = E$.

Pf: ($\Rightarrow$) For each $p \in E$, $\exists\, r_p > 0$ s.t. $N_{r_p}^Y(p) \subset G$

$\Leftrightarrow$ $N_{r_p}^X(p) \cap Y \subset G$.

Let $\tilde{G} = \bigcup_{p \in E} N_{r_p}^X(p)$.

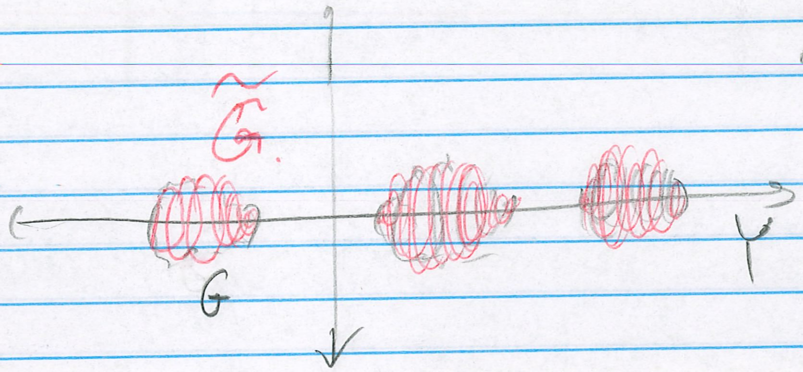
Since $N^X_{r_p}(p)$ is open (in X), and union of opens is open, $\tilde{G}$ is open (in X).

Hence: $\tilde{G} \cap Y = \left(\bigcup_{p \in E} N^X_{r_p}(p) \right) \cap Y$

$= \bigcup_{p \in E} \left(N^X_{r_p}(p) \cap Y \right)$ by distrib. prop. (day 1)

$= \bigcup_{p \in E} N^Y_{r_p}(p) \supset G$ by defn

$\subset G$ since $N^Y_{r_p}(p) \subset G$ by choice of r_p. ✓



($\Leftarrow$) If $\hat{G} \subset X$ is open, then for $p \in G = \hat{G} \cap Y$

$\exists\ r > 0$ s.t. $N^X_r(p) \subset \hat{G}$

$\Rightarrow N^Y_r(p) = N^X_r(p) \cap Y \subset \hat{G} \cap Y = G$.

$\therefore G \subset Y$ is open rel. Y. $\square$

Also have relatively closed: every limit pt. $p \in Y$ belongs to E.

Ex: $X = \mathbb{R}$, $Y = (0,1)$. $E = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$.

$E' = \{0\} \notin Y$ ... so relatively closed in Y.

There is a stronger notion than closedness which is intrinsic, i.e. turns out not to depend on the "ambient" metric space.

Defn. Let X be a metric space, $E \subset X$.

- A collection $\{U_\alpha\}_{\alpha \in I}$ of open sets $U_\alpha \subset X$ is called an open cover of E if

$$\bigcup_{\alpha \in I} U_\alpha \supset E.$$

- A subset $K \subset X$ is called compact if every open cover of K contains a finite subcover.

i.e., for any open cover $\{U_\alpha\}$ of E, there exist $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} \in \{U_\alpha\}$ s.t.

$$\bigcup_{i=1}^{n} U_{\alpha_i} \supset K.$$

Example. • Every finite set is compact.

$K = \{x_1, \dots, x_n\}$.

Let $\{U_\alpha\}$ open cover.

For each i, choose α_i s.t. $x_i \in U_{\alpha_i}$. Then $\bigcup_{i=1}^{n} U_{\alpha_i} \supset \bigcup_{i=1}^{n} \{x_i\} = K$ ✓.

2. Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Claim: E is not compact.

Pf: Need to show some open cover w/o finite subcover.

Let $A_n = (\frac{1}{n}, +\infty)$. Consider the open cover $\{A_n\}_{n \in \mathbb{N}}$.

Since $\frac{1}{n} \in A_{n+1}$, $\bigcup_{n=1}^{\infty} A_n \supset E$.

$\Rightarrow$ this is an open cover.

But suppose that $A_{n_1}, A_{n_2}, \dots, A_{n_k}$ is a finite subcover. Then $\bigcup_{i=1}^{k} A_{n_i} = A_m$,

where $m = \max_{i=1}^{k} n_i$.

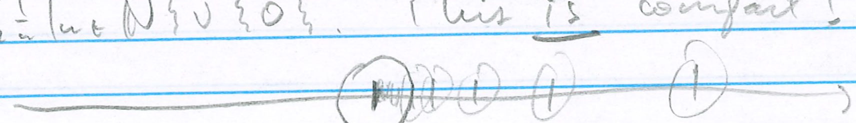
$$\frac{1}{m} \notin A_m = \bigcup_{i=1}^{k} A_{n_i}$$

$\Rightarrow A_{n_1}, \dots, A_{n_k}$ is not an open cover!

Since this was an arbitrary finite subset of $\{A_n\}_{n \in \mathbb{N}}$, we conclude that no finite subset can cover E, i.e. $\nexists$ finite subcover of $\{A_n\}_{n \in \mathbb{N}}$.

$\therefore$ E is not compact!

• Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. This is compact! (HW.)



<u>Theorem</u>: Suppose $K \subset Y \subset X$.

K is cpt as subset of Y

$\iff$ " " X.

<u>Pf.</u> ($\Rightarrow$) Let $\{\hat{G}_\alpha\}$ be coll. of open subsets of X covering X. Then $G_\alpha = \hat{G}_\alpha \cap Y$ is open, and

$$\bigcup_\alpha G_\alpha = \bigcup_\alpha (\hat{G}_\alpha \cap Y)$$

$$= \left(\bigcup_\alpha \hat{G}_\alpha\right) \cap Y \supset E \cap Y = E$$

$\to \{G_\alpha\}$ covers E.

By cptness of $K \subset Y$, $\exists$ finite subcover

$G_{\alpha_1}, \ldots, G_{\alpha_n}$, w. $\bigcup_{i=1}^n G_{\alpha_i} \supset E$

$$\Rightarrow \bigcup_{i=1}^n \hat{G}_{\alpha_i} \supset E \quad \checkmark$$

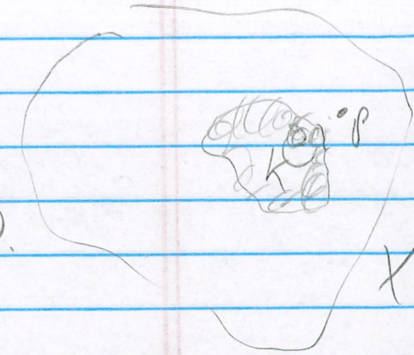
$\square$

($\Leftarrow$) very similar.

<u>Note</u>: In view of the theorem, we can say "K is compact" without specifying an ambient space! In partic., if $K = X$, we'd just say "X is a compact metric space".

Theorem. Compact subsets of metric spaces are closed.

Pf. Let $p \in X \setminus K$. Will show p is *not* a limit pt.



For each $q \in K$, let $r_q = \frac{1}{2} d(p,q) > 0$.

$$V_q = N_{r_q}(q)$$

$$\Rightarrow \{V_q\}_{q \in K} \text{ covers } K.$$

Since K is cpt, $\exists$ finite subcover.

$$V_{q_1}, \dots, V_{q_n} \text{ s.t. } \bigcup_{i=1}^{n} V_{q_i} \supset K.$$

Let $W_{q_i} = N_{r_{q_i}}(p)$ $\qquad W_{q_i} \cap V_{q_i}$ ″

Since $d(p,q_i) = 2r_{q_i}$, $N_{r_{q_i}}(p) \cap N_{r_{q_i}}(q) = \emptyset$.

(Pf: $x \in N_{r_i}(p) \Rightarrow d(x,p) < r_i$

$2r_i = d(p,q_i) \le d(x,q) + d(x,p) < d(x,q) + r_i$

$\Rightarrow r_i < d(x,q) \Rightarrow q \notin N_{r_{q_i}}(q)$ ✓.)

Then $\bigcap_{i=1}^{n} W_{q_i} \cap \underset{\supset}{\bigcup_{i=1}^{n}} V_{q_i} = \emptyset$

$\Rightarrow \bigcap_{i=1}^{n} W_{q_i} \cap K = \emptyset$.

But $\bigcap_{i=1}^{n} W_{q_i}$ is an open nbhd of p! $\Rightarrow K^c$ is open. $\square$

$= N_{\min\{r_{q_i}\}}(p)$