Problem 1 (Rudin 3.1)**.**

Proof. We show that convergence $s_n \to s$ implies convergence of $|s_n| \to |s|$. Let $\varepsilon > 0$. Convergence $s_n \to s$ implies existence of $N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon$ for all $n \geq N$. Then using triangle inequality

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon \quad \text{for all } n \geq N$$

which proves convergence of $|s_n| \to |s|$. Next we show: convergence of $|s_n|$ does not imply convergence of s_n. Indeed, let $s_n = (-1)^n$. Then s_n diverge, but $|s_n| = 1$ for all n, so $|s_n|$ converge. □

Problem 2 (Rudin 3.2)**.**

Proof. We compute

$$\begin{aligned}
\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) &= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}.
\end{aligned}$$

□

Problem 3 (Rudin 3.3)**.**

Proof. $s_1 = \sqrt{2} < 2$. Suppose inductively that $s_n < 2$. Then,

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < \sqrt{4} = 2.$$

Thus we see that $s_n < 2$ for all $n \geq 1$. We also have that

$$s_2 = \sqrt{2+\sqrt{\sqrt{2}}} > \sqrt{2} = s_1.$$

Suppose inductively that $s_n > s_{n-1}$. Then

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} > \sqrt{2+\sqrt{s_{n-1}}} = s_n$$

and therefore we have that $s_n > s_{n-1}$ for all $n \geq 1$ by induction. Thus the sequence is increasing and since $\cdots > s_n > s_{n-1} > \cdots > s_1 > 0$ we have an increasing sequence of nonnegative terms which is bounded above. Hence by Theorem 3.24, $\{s_n\}$ converges. □

Problem 4 (Rudin 3.4)**.**

Proof. Notice that

$$s_{2n+1} = \frac{1}{2} + s_{2n} = \frac{1}{2} + \frac{s_{2n-1}}{2}.$$

Notice that 1 is a fixed point of this map. Subtracting 1 from both sides, we get

$$s_{2(n+1)+1} - 1 = \frac{1}{2}\left(s_{2n+1} - 1\right).$$

Since $s_1 = 0$, we get

$$s_{2n+1} - 1 = -\left(\frac{1}{2}\right)^n$$

and

$$s_{2n+1} = 1 - \frac{1}{2^n}.$$

This also gives

$$s_{2n+2} = \frac{1}{2} s_{2n+1} = \frac{1}{2}\left(1 - \frac{1}{2^n}\right).$$

Hence, the odd terms approach 1 and the even terms approach $\frac{1}{2}$. These are the upper and lower limits, respectively.

□

Problem 5 (Rudin 3.5)**.**

Proof. If either lim sup on the right side above is equal to positive infinity, we are done, so we can assume that both are finite or equal to $-\infty$. Let

$$A_N = \sup\{a_n \mid n \geq N\}, \quad B_N = \sup\{b_n \mid n \geq N\},$$

and

$$C_N = \sup\{a_n + b_n \mid n \geq N\}.$$

By the definition from class, we have $\limsup a_n = \lim_{N\to\infty} A_N$, $\limsup b_n = \lim_{N\to\infty} B_N$, and $\limsup a_n + b_n = \lim_{n\to\infty} C_N$. But notice that for $n \geq N$, we have

$$a_n + b_n \leq \sup_{m\geq N} a_m + \sup_{m\geq N} b_m = A_N + B_N.$$

Hence $C_N \leq A_N + B_N$. This inequality is preserved under taking the limit $N \to \infty$. □

Problem 6 (Rudin 3.16)**.**

Proof. (a) First we show that $x_n > \sqrt{\alpha}$ for all $n \geq 1$ by induction. The $n = 1$ case is obvious since $x_1 > \sqrt{\alpha}$ by construction. So then assume that $x_n > \sqrt{\alpha}$. We then have

$$\begin{aligned}
&x_n > \sqrt{\alpha} \\
&x_n - \sqrt{\alpha} > 0 \\
&\left(x_n - \sqrt{\alpha}\right)^2 > 0 \quad \text{(since both sides } \geq 0\text{)} \\
&x_n^2 - 2x_n\sqrt{\alpha} + \alpha > 0 \\
&x_n^2 + \alpha > 2x_n\sqrt{\alpha} \\
&\frac{x_n^2 + \alpha}{x_n} > 2\sqrt{\alpha} \quad \left(\text{since } x_n > \sqrt{\alpha} > 0\right) \\
&x_n + \frac{\alpha}{x_n} > 2\sqrt{\alpha} \\
&\frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > \sqrt{\alpha} \\
&x_{n+1} > \sqrt{\alpha}
\end{aligned}$$

by definition. Therefore $x_n^2 > \alpha$ for any $n \geq 1$. From this we can easily show that $\{x_n\}$ is decreasing monotonically:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) < \frac{1}{2}\left(x_n + \frac{x_n^2}{x_n}\right) = \frac{1}{2}(x_n + x_n) = x_n.$$

Now since $\{x_n\}$ is monotonic and bounded (bounded above by x_1 and bounded below by $\sqrt{\alpha}$), it converges. So suppose that $\lim_{n\to\infty} x_n = x$. Then clearly it must be that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$ also. So we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\lim_{n\to\infty} x_n + \frac{\alpha}{\lim_{n\to\infty} x_n}\right) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right) = x,$$

where we have used [**?**, Theorem 3.3]. Solving this simple equation for x results in $x = \lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(b) We have simply

$$\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{1}{2x_n}\left(x_n^2 + \alpha\right) - \sqrt{\alpha} = \frac{1}{2x_n}\left(x_n^2 + \alpha - 2x_n\sqrt{\alpha}\right)$$
$$\frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

since $x_n > \sqrt{\alpha}$. So then, letting $\beta = 2\sqrt{\alpha}$, we claim that

$$\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$$

for all $n \geq 1$. A simple proof by induction shows that this is indeed the case. For $n = 1$ we have

$$\epsilon_{n+1} = \epsilon_2 < \frac{\epsilon_1^2}{\beta} = \beta\frac{\epsilon_1^2}{\beta^2} = \beta\left(\frac{\epsilon_1}{\beta}\right)^2 = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^1} = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}.$$

Now assume that

$$\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}.$$

Then we have

$$\epsilon_{n+2} < \frac{\epsilon_{n+1}^2}{\beta} < \frac{1}{\beta}\left[\beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}\right]^2 = \frac{1}{\beta}\left[\beta^2\left(\frac{\epsilon_1}{\beta}\right)^{2^n\cdot 2}\right] = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^{n+1}}.$$

(c) For $\alpha = 3$ and $x_1 = 2$ we first show that $\epsilon_1/\beta < 1/10$ without using decimal approximations for $\sqrt{\alpha} = \sqrt{3}$, starting with the obvious:

$$\begin{gathered}
100 < 108 = 36\cdot 3 \\
\sqrt{100} < \sqrt{36}\sqrt{3} \\
10 < 6\sqrt{3} \\
10 - 5\sqrt{3} < \sqrt{3} \\
20 - 10\sqrt{3} < 2\sqrt{3} \\
10(2 - \sqrt{3}) < 2\sqrt{3} \\
\frac{2 - \sqrt{3}}{2\sqrt{3}} < \frac{1}{10} \\
\frac{x_1 - \sqrt{\alpha}}{2\sqrt{\alpha}} < \frac{1}{10} \\
\frac{\epsilon_1}{\beta} < \frac{1}{10}.
\end{gathered}$$

We also have

$$\begin{aligned}
&12 < 16\\
&4 \cdot 3 < 16\\
&\sqrt{4}\sqrt{3} < \sqrt{16}\\
&2\sqrt{3} < 4\\
&\beta < 4,
\end{aligned}$$

from which it follows that

$$\epsilon_5 < \beta \left(\frac{\epsilon_1}{\beta}\right)^{2^4} < 4 \left(\frac{\epsilon_1}{\beta}\right)^{16} < 4 \left(\frac{1}{10}\right)^{16} = 4 \cdot 10^{-16}$$

Likewise

$$\epsilon_6 < \beta \left(\frac{\epsilon_1}{\beta}\right)^{2^5} < 4 \left(\frac{\epsilon_1}{\beta}\right)^{32} < 4 \left(\frac{1}{10}\right)^{32} = 4 \cdot 10^{-32}$$

□

Problem 7 (Rudin 3.20)**.**

Proof. Fix $\varepsilon > 0$. Since $\{p_{n_i}\}$ converges to p, there exists $K \in \mathbb{N}$ such that

$$d\left(p_{n_i}, p\right) < \frac{\varepsilon}{2} \quad \text{if } i \geq K.$$

Since $\{p_n\}$ is a Cauchy sequence, there exists integer N such that

$$d\left(p_n, p_m\right) < \frac{\varepsilon}{2} \text{ if } n, m \geq N$$

Since positive integer n_i satisfy $n_1 < n_2 < \ldots$, there exists $j > K$ such that $n_j > N$. Thus, from last two displayed inequalities, for any $n \geq N$

$$d\left(p_n, p\right) \leq d\left(p_n, p_{n_j}\right) + d\left(p_{n_j}, p\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is, $p_n \to p$. □

Problem 8 (Rudin 3.21)**.**

Proof. Denote $E = \bigcap_{n=1}^{\infty} E_n$. Suppose E consists of at least two points, then diam $E > 0$. But since $E \subset E_n$ for all n, we have $\operatorname{diam} E \leq \operatorname{diam} E_n$ for all n, which contradicts the assumption that $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$. Next, we show that $E \neq \varnothing$. Since $E_n \neq \varnothing$ for all n, we can choose a point $x_n \in E_n$ for each n, and form the sequence $\{x_n\}$.

> **Claim 1**: $\{x_n\}$ is Cauchy.
> Given $\varepsilon > 0$, then since $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$, there exists a positive integer N such that $n \geq N$ implies diam $E_n < \varepsilon$. For any $m \geq N$ and $n \geq N$, by assumption, $x_m \in E_m \subset E_N$ and $x_n \in E_n \subset E_N$. So $d\left(x_m, x_n\right) \leq \operatorname{diam} E_N < \varepsilon$.

Now, since X is complete and $\{x_n\}$ is Cauchy, we have $\lim\limits_{n\to\infty} x_n = x$ for some $x \in X$. Since E_n is closed for each n, we must have $x \in E_n$ for each n, hence $x \in E$. □

Problem 9 (Extra credit, Rudin 2.18)**.**

Proof. Yes. We imitate the construction of the Cantor set (which unfortunately we didn't have time to do in class), except we try to remove every rational number while making sure the "endpoints" of the open intervals that we remove are irrational.

Arrange the set of rational numbers between $\sqrt{2}$ and $\sqrt{2}+1$ in a sequence: $\{r_1, r_2, r_3, \ldots\}$. Let $E_1 = \left[\sqrt{2}, \sqrt{2}+1\right]$. We will choose a sequence of sets $E_n \supset E_{n+1} \cdots$ inductively, subject to the following hypotheses:

- Each E_n is a finite union of closed intervals with irrational endpoints
- E_n is nonempty
- $\{r_1, \ldots, r_{n-1}\} \subset E_n^c$
- $E_n \supset E_{n+1}$
- The set of endpoints of E_n is contained in the set of endpoints of E_{n+1}.

Having chosen E_n, if $r_n \notin E_n$, let $E_{n+1} = E_n$. If $r_n \in E_n$, r_n must be an interior point since E_n is a union of intervals with irrational endpoints. Choose $\delta_n > 0$, irrational, such that $(r_n - \delta_n, r_n + \delta_n) \subset E_n$, and let $E_{n+1} = E_n \setminus (r_n - \delta, r_n + \delta)$. This again satisfies the hypotheses, since we have only created new endpoints (by splitting up an interval).

Now, let $E = \cap_{n=1}^{\infty} E_n$. Since E_n are nested, compact, and nonempty, this is compact and nonempty. Since we have removed all rationals, we have $E \cap \mathbb{Q} = \emptyset$. Finally, to see that E is perfect, let $x \in E$. Choose any sequence of rational numbers $r_{n_1}, r_{n_2}, \ldots$ approaching x. We removed intervals with irrational endpoints around r_{n_i}, for infinitely many of these r_{n_i}'s. The endpoints of these intervals must approach x. By construction, the endpoints of each E_n remain inside E, so this shows that x is a limit point of E. $\square$