Problem 1 (Problem 4.8, Baby Rudin)**.**

Proof. E is bounded, so $E \subset [a,b]$ where $a, b \in \mathbb{R}, \quad a<b$. Since f is uniformly continuous on E, there exists $\delta>0$ such that $|f(x)-f(y)|<1$ if $x, y \in E$ and $|x-y|<\delta$. For some positive integer n, we have $h=\dfrac{a-b}{n}<\delta$.

Let $a_0=a,\ a_k=a_0+hk$ for $k=1,2,\ldots,n$. Then $a_n=b$. If $k=1,2,\ldots,n$ and $[a_{k-1}, a_k] \cap E$ is nonempty, and $x_k \in [a_{k-1}, a_k] \cap E$, then for any $x \in [a_{k-1}, a_k] \cap E$, we have $|f(x)-f(x_k)|<1$ since $|x-x_k| \leqslant a_k-a_{k-1}=h<\delta$. Thus, $|f(x)| \leqslant |f(x_k)|+1$ for all $x \in [a_{k-1}, a_k] \cap E$.

If $[a_{k-1}, a_k] \cap E \neq \emptyset$, denote $M_k=|f(x_k)|+1$, and if $[a_{K-1}, a_k] \cap E=\emptyset$, set $M_k=0$. Then $|f(x)| \leqslant \max\{M_1, \ldots, M_n\}$ for all $x \in E$. Thus f is bounded on E.

If E is unbounded; then f does not necessarily bounded. Indeed, if $E=\mathbb{R}$, $f(x)=x$ is uniformly continuous but unbounded on E. $\square$

Problem 2 (Problem 4.10, Baby Rudin)**.**

Proof. By Theorem 2.37, $\{p_n\}$ has a subsequence which converges to $p \in X$. Replace $\{p_n\}$ with this subsequence and replace $\{q_n\}$ with the corresponding subsequence. Similarly, $\{q_n\}$ has a subsequence which converges to $q \in X$. Again replace $\{q_n\}$ with this subsequence and replace $\{p_n\}$ with the corresponding subsequence. Since $d_X(p_n, q_n)$ converges to 0 , we must have $p=q$. Hence by continuity, $f(p_n)$ and $f(q_n)$ must both converge to $f(p)=f(q)$, contradicting the assumption that $d_Y(f(p_n), f(q_n))>\varepsilon$. $\square$

Problem 3 (Problem 4.14, Baby Rudin)**.**

Proof. We consider $f(x) \in [0,1]$ for any $x \in [0,1]$. If $f(0)=0$, or $f(1)=1$ then we have proved.

Otherwise, we have $f(0)>0$ and $f(1)<1$. Let $g(t)=f(t)-t$. Then $g(t)$ is continuous on $[0,1]$, and

$$\begin{aligned} g(0) &= f(0)-0>0 \\ g(1) &= f(1)-1<0. \end{aligned}$$

Thus, by intermediate value Theorem 4.23, we have $g(x)=0$ for some $x \in (0,1)$. Thus $f(x)-x=0$, ie $f(x)=x$. $\square$

Problem 4 (Problem 4.18, Baby Rudin)**.**

Proof. We first prove that, for any $\alpha \in \mathbb{R}, \quad \lim\limits_{x \to \alpha} f(x)=0$. Let $\varepsilon>0$. We need to show that there exits $\delta>0$ such that $|f(x)|<\varepsilon$ for any $x \in (\alpha-\delta, \alpha+\delta)$, $x \neq \alpha$. We consider all numbers $\dfrac{m}{n} \in \left(\alpha-\dfrac{1}{2}, \alpha+\dfrac{1}{2}\right)$, where m, n are integer and $n>0, n \leqslant \dfrac{1}{\varepsilon}$.

Claim 1: Set of all such numbers is finite.

Indeed, foe each fixed n, there is at most n numbers $\dfrac{m}{n}$ in $\left(\alpha-\frac{1}{2}, \alpha+\frac{1}{2}\right)$, and we consider integer n satisfying $0<n \leq \dfrac{1}{2}$, ie finite set of n's.

Denote these numbers $\frac{m}{n}$ by $r_1, \ldots, r_s$. Let $\delta=\min(|r_i-\alpha|)$, where minimum is take over all $i=1, \ldots, s$, such that $r_i \neq \alpha$. Then $\delta>0$. Thus, in the interval $(\alpha-\delta, \alpha+\delta)$ there is no numbers $r_1, \ldots, r_s$ except possibly α. Thus for any rational $r=\dfrac{m}{n} \in (\alpha-\delta, \alpha+\delta)$, then $\dfrac{m}{n} \neq \alpha$, and we have $n>\frac{1}{\varepsilon}$. For fixed such r, choosing its form with m, n without common divisors, we get $f(r)=\frac{1}{n}$, that $0<f(r)<\varepsilon$. So, $|f(r)|<\varepsilon$ for any rational $r \in (\alpha-\delta, \alpha+\delta), r \neq \alpha$. If $y \in (\alpha-\delta, \alpha+\delta)$

is irrational, we have $f(y) = 0$. Thus, $|f(x)| < \varepsilon$ for any $x \in (\alpha - \delta, \alpha + \delta^{-}), x \neq \alpha$. Thus, $\lim_{x\to\infty} f(x) = 0$.

Now, if α is irrational, $f(\alpha) = 0 = \lim_{x\to\alpha} f(x)$, so f is continuous at α. If α is rational then $f(\alpha) \neq 0$; thus $f(\alpha) \neq \lim_{x\to\alpha} f(x)$ (where the limit exists), that is f has a simple discontinuity at α. □

Problem 5 (Problem 4.20, Baby Rudin)**.**

Proof. (a). First prove that $\rho_E(x) = 0$ if $x \in \bar{E}$. Since $E \subset X$, this means $x \in \bar{E} \subset X$. Then, since $x \in \bar{E}$, given $\epsilon > 0$ we can find a $z \in E$ such that $d(x, z) < \epsilon$. So, $\inf_{z\in E} d(x, z) = \rho_E(x) = 0$.

Now prove the converse, that if $\rho_E(x) = 0, x \in \bar{E}$. This means that $\inf_{z\in E} d(x, z) = 0$ which is only possible when $z = x$. However, for $z = x, x \in E$ must be true. Since $E \subset \bar{E}$, this also means $x \in \bar{E}$.
(b). From the definition of ρ_E, it follows that $\rho_E(x) \le d(x, z) \le d(x, y) + d(y, z)$, for any $z \in E$. Then, we get

$$\rho_E(x) = \inf_{z\in E} d(x, z) \le \inf_{z\in E}(d(x, y) + d(y, z)) = d(x, y) + \rho_E(y),$$

Therefore, we obtain

$$\rho_E(x) - \rho_E(y) \le d(x, y). \tag{1}$$

Moreover, we also have $\rho_E(y) \le d(y, z) \le d(y, x) + d(x, z)$, for any $z \in E$. Then,

$$\rho_E(y) = \inf_{z\in E}(d(y, z)) \le \inf_{z\in E}(d(x, y) + d(x, z)) = d(x, y) + \rho_E(x),$$

This implies that

$$\rho_E(y) - \rho_E(x) \le d(x, y). \tag{2}$$

Combining (1) and (2) have to be true, it follows that

$$\max\left(\rho_E(x) - \rho_E(y), \rho_E(y) - \rho_E(x)\right) = |\rho_E(x) - \rho_E(y)| \le d(x, y)$$

Now prove that ρ_E is a uniformly continuous on X. For every $\epsilon > 0$, let $\delta = \epsilon$, so $d(x, y) < \delta$ implies $d(x, y) < \epsilon$, so $|\rho_E(x) - \rho_E(y)| < d(x, y) < \epsilon$ for all $x, y \in X$. Thus, ρ_E is uniformly continuous on X. □

Problem 6 (Problem 4.21, Baby Rudin)**.**

Proof. Since K, F are disjoint and closed, it follows from Exercise 4.20 that $\rho_F(p) > 0$ for all $p \in K$ and ρ_F is continuous on K. Since K is compact, by Weierstrass' extreme value theorem, ρ_F attains its minimum at some $p_0 \in K$. Hence,

$$0 < \rho_F(p_0) \le \rho_F(p) = \inf_{q\in F} d(p, q) \le d(p, q), \quad \forall p \in K, q \in F$$

Choosing $\delta = \rho_F(p_0)$ completes the proof.

To see that the conlusion may fail if neither K nor F are compact, let our metric space be the rationals equipped with the Euclidean metric. Then $K := (\sqrt{2}, \infty)$ and $F := (-\infty, \sqrt{2})$ are closed in $\mathbb{Q}$, but neither is compact. However, $\inf_{p\in K, q\in F} d(p, q) = 0$ since points in K and F become arbitrarily close as they approach $\sqrt{2}$ from the right and left, respectively. □

Problem 7 (Problem 5.1, Baby Rudin)**.**

Proof. Starting at $f(0)$, we show that $f(x)=f(0)$ for all $x\in\mathbb{R}^1$. Consider the case $x>0$, and for $n=1,2\ldots$, let $\{x_0,x_1,x_2,\ldots,x_n\}$ be a partition of the interval $[0,x]$ with $x_0=0, x_n=x$, and $x_{j+1}-x_j=\frac{x}{n}$ for $j\in\{0,1,2,\ldots,n-1\}$. Then

$$|f(x)-f(0)|=\left|\sum_{j=0}^{n-1} f\left(x_{j+1}\right)-f\left(x_j\right)\right|\leq\sum_{j=0}^{n-1}\left|f\left(x_{j+1}\right)-f\left(x_j\right)\right|\leq\sum_{j=0}^{n-1}\left(x_{j+1}-x_j\right)^2=\sum_{j=0}^{n-1}\left(\frac{x}{n}\right)^2=\frac{x^2}{n}$$

where the third line of the above inequality follows by assumption. Since $\frac{x^2}{n}\to 0$ as $n\to\infty$, we conclude that $|f(x)-f(0)|=0$ or $f(x)=f(0)$. The other case $x<0$ is in a similar way. Hence f is constant. □

Problem 8 (Problem 5.4, Baby Rudin)**.**

Proof. Consider the polynomial defined by

$$p(x)=C_0x+\frac{C_1}{2}x^2+\cdots+\frac{C_n}{n+1}x^{n+1},\quad x\in[0,1].$$

Then, $p(1)=C_0+\frac{C_1}{2}+\cdots+\frac{C_{n-1}}{n}+\frac{C_n}{n+1}=0=p(0)$ and by the Mean Value Theorem, Theorem 5.10, we see that

$$C_0+C_1c+\cdots+C_{n-1}c^{n-1}+C_nc^n=p'(c)=\frac{p(1)-p(0)}{1-0}=0,\quad c\in(0,1),$$

hence c is a root of $C_0+C_1x+\cdots+C_{n-1}x^{n-1}+C_nx^n$. □

Problem 9 (Problem 5.5, Baby Rudin)**.**

Proof. Let $\varepsilon>0$. Choose x_0 such that $|f'(x)|<\varepsilon$ if $x>x_0$. Then for any $x\geq x_0$ there exists $x_1\in(x,x+1)$ such that

$$f(x+1)-f(x)=f'\left(x_1\right).$$

Since $\left|f'\left(x_1\right)\right|<\varepsilon$, it follows that $|f(x+1)-f(x)|<\varepsilon$, as required. □

Problem 10 (Problem 5.6, Baby Rudin)**.**

Proof. Since $g'(x)=\frac{xf'(x)-f(x)}{x^2}$ where $x>0$, by Theorem 5.11(a), it suffices to show that

$$xf'(x)-f(x)\geq 0$$

for all $x>0$. Since conditions (a), (b), and (c) hold, by the mean value theorem,

$$f(x)=f(x)-f(0)=f'(c)(x-0)=xf'(c)$$

for some $c\in(0,x)$. Since $c<x$ here, by condition (d), $xf'(c)\leq xf'(x)$. Therefore, we derive $f(x)\leq xf'(x)$ □

Problem 11 (Problem 5.15, Baby Rudin)**.**

Proof. (a) First of all, we see that M_0M_2 be of the form $0\cdot\infty$ is exceptional. Now we consider the following four cases.

Case 1: $M_0=\infty$ or $M_2=\infty$. The result is trivial.

Case 2: $M_0=0$. Then $f(x)=0$, and then $f'(x)=f''(x)=0$ for all $x\in(a,\infty)$. It follows that $M_1=M_2=0$, and the result is also trivial.

Case 3: $0<M_0<\infty$ and $M_2=0$. Then $f''(x)=0$, and then $f'(x)=c$ and $f(x)=cx+d$ for some constants $c,d\in R$, for all $x\in(a,\infty)$. Since M_0 is finite, we need $c=0$ and then $M_1=0$. The result therefore follows.

Case 4: $0 < M_0 < \infty$ and $0 < M_2 < \infty$. Following the hint, use Taylor's theorem, Theorem 5.15, for $x \in (a, \infty)$ and $h > 0$ there exists $\xi \in (x, x+2h)$ such that

$$f(x+2h) = f(x) + f'(x) \cdot 2h + \frac{f''(\xi)}{2} \cdot (2h)^2 = f(x) + f'(x) \cdot 2h + f''(\xi) \cdot 2h^2$$

Thus $f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$, and hence $|f'(x)| \le hM_2 + \frac{M_0}{h}$. By assumption, we may put $h = \sqrt{\frac{M_0}{M_2}}$ and the inequality becomes $|f'(x)| \le 2\sqrt{M_0 M_2}$, which implies $M_1 \le 2\sqrt{M_0 M_2}$ or $M_1^2 \le 4M_0 M_2$.

(b) By some calculation, we have a two cases:

Case 1: If $-1 < x < 0$, then $f'(x) = 4x$, which implies $f''(x) = 4$.

Case 2: If $0 < x < \infty$, then we get $f'(x) = \frac{4x}{(x^2+1)^2}$. This implies $f''(x) = \dfrac{-4\left(3x^2-1\right)}{\left(x^2+1\right)^3}$.

To find $f'(0)$ and $f''(0)$, calculate

$$\lim_{t \to 0^+} \frac{f(t) - f(0)}{t} = \lim_{t \to 0^+} \frac{\frac{t^2-1}{t^2+1} + 1}{t} = \lim_{t \to 0^+} \frac{2t}{t^2+1} = 0$$
$$\lim_{t \to 0^-} \frac{f(t) - f(0)}{t} = \lim_{t \to 0^-} \frac{\left(2t^2 - 1\right) + 1}{t} = \lim_{t \to 0^-} 2t = 0$$

Then $f'(0) = 0$. Next, we consider

$$\lim_{t \to 0^+} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0^+} \frac{\frac{4t}{(t^2+1)^2}}{t} = \lim_{t \to 0^+} \frac{4}{\left(t^2+1\right)^2} = 4$$
$$\lim_{t \to 0^-} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0^-} \frac{4t}{t} = 4$$

Then $f''(0) = 4$. Now, $-1 \le f(x) < 1$ for $x \in (-1, \infty)$, and $\lim\limits_{x \to \infty} f(x) = 1$, are both trivial. Hence $M_0 = 1$. Also, since $f''(x) = 4$ for $x \in (-1, 0]$, and $\lim\limits_{x \to 0^+} f''(x) = 4$, we see that f is twice-differentiable on $(-1, \infty)$. To show that $|f''(x)| < 4$ for all $x > 0$, we observe that

$$3x^2 - 1 < 3x^2 + 1 < x^6 + 3x^4 + 3x^2 + 1 = \left(x^2+1\right)^3$$

and that $\left|\frac{3x^2-1}{(x^2+1)^3}\right| = \frac{3x^2-1}{(x^2+1)^3} < 1$ for $x > 0$. Thus, we have $|f''(x)| = 4\left|\frac{3x^2-1}{(x^2+1)^3}\right| < 4$ for $x > 0$, and hence $M_2 = 4$.

Finally, since $0 \le f'(x) < 4$ for $x \in (-1, 0]$, and $2x \le 1 + x^2 < 1 + 2x^2 + x^4 = \left(x^2+1\right)^2$ for $x > 0$. We see that $|f'(x)| = \left|\dfrac{4x}{\left(x^2+1\right)^2}\right| = 2\left[\frac{2x}{(x^2+1)^2}\right] < 2 < 4$ for $x > 0$, and since $\lim\limits_{t \to (-1)^+} f'(x) = -4$, we conclude that $M_1 = 4$.

(c) The answer is yes. In $\mathbb{R}^k$, let $f(x) = (f_1(x), \ldots, f_k(x))$ where each $f_j (1 \le j \le k)$ is a twice-differentiable real function. Note that $M_0 M_2$ be of the form $0 \cdot \infty$ is exceptional, and we consider the following four cases.

Case 1: $M_0 = \infty$ or $M_2 = \infty$. The result is trivial.

Case 2: $M_0 = 0$. Then $f_j(x) = 0$, and then $f_j'(x) = f_j''(x) = 0$ for all $x \in (a, \infty)$ and for all j. It follows that $M_1 = M_2 = 0$, and the result is also trivial.

Case 3: $0 < M_0 < \infty$ and $M_2 = 0$. Then $f_j''(x) = 0$, and then $f_j'(x) = c_j$ and $f_j(x) = c_j x + d_j$ for some constants $c_j, d_j \in R$, for all $x \in (a, \infty)$ and for all j. Since M_0 is finite, we need $c_j = 0$ for each j, and then $M_1 = 0$. The result therefore follows.

Case 4: $0 < M_0 < \infty$ and $0 < M_2 < \infty$. If $M_1 = 0$ then we are done. If $M_1 > 0$, let $p \in \mathbb{R}^1$ be such that $0 < p < M_1$, and let $x_0 \in (a, \infty)$ be such that $|f'(x_0)| > p$. Define $u = \frac{f'(x_0)}{|f'(x_0)|}$. Consider the real-valued function $g(x) = u \cdot f(x)$ for $x \in (a, \infty)$, and note that g is twice-differentiable. Let N_0, N_1, N_2 be the least upper bounds of $|g(x)|, |g'(x)|, |g''(x)|$, respectively. Since $|u| = 1$, by Schwarz inequality, Theorem 1.37(d),

$$|g(x)| \leq |u||f(x)| = |f(x)|, \quad |g''(x)| \leq |u| \left|f''(x)\right| = \left|f''(x)\right|$$

for all $x \in (a, \infty)$. So that $N_0 \leq M_0$ and $N_2 \leq M_2$. Also, since

$$N_1 \geq g'(x_0) = u \cdot f'(x_0) = \left|f'(x_0)\right| > p$$

and since $N_1 \leq 4N_0N_2$, by part (a), we have $p < 4M_0M_2$. Since p is arbitrarily chosen such that $0 < p < M_1$, we conclude that $M_1 \leq 4M_0M_2$. □

Problem 12 (Extra credits Problem 4.15, Baby Rudin)**.**

Proof. Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be a continuous open mapping. Assume it is not monotonic. Then, there exists two cases: there exist $x_1 < x_2 < x_3$ such that $f(x_1) < f(x_2)$ and $f(x_3) < f(x_2)$, or there exist $x_1 < x_2 < x_3$ such that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$.

Case 1: By the extreme value theorem, $f([x_1, x_3])$ has a maximum attained in this interval; since $f(x_1) < f(x_2)$ and $f(x_3) < f(x_2)$, this maximum is attained in the open interval (x_1, x_3). Now assume that $f((x_1, x_3))$ forms an open set.

However, this is a contradiction since if this set were open, we could draw a neighborhood around $f(x_2)$ that is contained in $f((x_1, x_3))$, but this means that there is a value of f that is greater than $f(x_2)$ that is still in $f((x_1, x_3))$. This contradicts the fact that $f(x_2)$ was the maximum. So, the values for f on the interval (x_1, x_3) is not open, which contradicts that f is an open mapping.

Case 2: By the extreme value theorem, $f([x_1, x_3])$ has a minimum attained in this interval; since $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$, this minimum is attained in the open interval (x_1, x_3). Now assume that $f((x_1, x_3))$ forms an open set.

However, this is a contradiction since if this set were open, we could draw a neighborhood around $f(x_2)$ that is contained in $f((x_1, x_3))$, but this means that there is a value of f that is less than $f(x_2)$ that is still in $f((x_1, x_3))$. This contradicts the fact that $f(x_2)$ was the minimum. So, the values for f on the interval (x_1, x_3) is not open, which contradicts that f is an open mapping.

Thus, f must be monotonic. □ □

Problem 13 (Extra credits Problem 5.25, Baby Rudin)**.**

Proof. (a) Note that the tangent line of the graph of f passing through $(x_n, f(x_n))$ is of the form

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Thus $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the intersection of the tangent line and the x-axis.

(b) We first show that $x_{n+1} \leq x_n$. Since $f'' \geq 0$, the function f' is increasing. If $x_n = \xi$ for some n, then clearly $x_m = \xi$ for all $m \geq n$ and it is nothing to prove. If $x_n > \xi$, then by the mean value theorem there exists $c_n \in (\xi, x_n)$ such that

$$f(x_n) = f(x_n) - f(\xi) = f'(c_n)(x_n - \xi) \leq f'(x_n)(x_n - \xi)$$

Since $f' \geq \delta > 0$, this reveals that $\frac{f(x_n)}{f'(x_n)} \leq x_n - \xi$, and that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \geq x_n - (x_n - \xi) = \xi$$

Note that $x_1 > \xi$, so by induction, we conclude that $x_n \geq \xi$ for all n, and then $f(x_n) \geq 0$ for all n, since $f' > 0$ implies f is increasing. It follows that $\dfrac{f(x_n)}{f'(x_n)} \geq 0$, or equivalently $x_{n+1} \leq x_n$ for all n. We next show that $\lim_{n\to\infty} x_n = \xi$. Since $\{x_n\}$ is monotonically decreasing with a lower bound

ξ, $\lim\limits_{n\to\infty} x_n = x$ for some $x \geq \xi$. But then $x = x - \frac{f(x)}{f'(x)}$, implying $f(x) = 0$, so by the uniqueness, $x = \xi$.

(c) By Taylor's theorem, there exists $t_n \in (\xi, x_n)$ such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

Since $f(\xi) = 0$, we have

$$\begin{aligned} &x_{n+1} - \xi \\ &= x_n - \xi - \frac{f(x_n)}{f'(x_n)} = x_n - \xi + \frac{1}{f'(x_n)}\left[f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2\right] = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \end{aligned}$$

(d) By part (c), we have

$$x_{n+1} - \xi \leq A(x_n - \xi)^2 \leq A \cdot A^2 (x_{n-1} - \xi)^4 \ldots \leq A \cdot A^2 \cdots A^{2^{n-1}} (x_1 - \xi)^{2^n} = \frac{1}{A}\left[A(x_1 - \xi)\right]^{2^n}$$

(e) Since $g(x) = x$ if and only if $f(x) = 0$, the result then follows. Next, we compute

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

It follows that $g(x)$ tends to 0 as x near ξ.

(f) Given $x_n \in (-\infty, \infty)$ with $x_n \neq 0$, then $f'(x_n) = \frac{1}{3}x_n^{-\frac{2}{3}}$. Therefore, we get

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = -2x_n$$

and we see that $\{x_n\}$ oscillates and diverges. □