Recall: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$+, \cdot, <$ $\quad - \quad$ $1/(\cdot)$ $\quad$ sup

Last time we constructed $\mathbb{R}$ using Dedekind cuts, proving:

Theorem: There exists an ordered field $\mathbb{R} \supset \mathbb{Q}$ with the least upper bound property.

Today: a few more properties of $\mathbb{R}$, confirming our intuition.

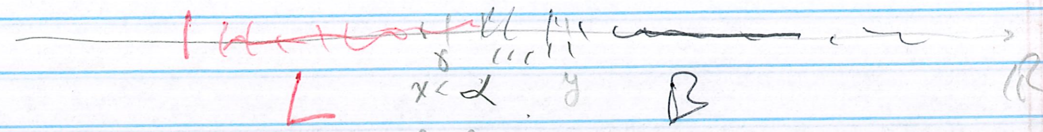
Note: These are actually easier to prove directly using the definition as D.-cuts, but we'll prove them just using ordered field axioms & L.U.B. property (as in Rudin).

Theorem: $\mathbb{R}$ also has the greatest-lower-bound property.

Pf: Let $\overset{\neq \emptyset}{B} \subset \mathbb{R}$ be a bounded below, nonempty set.

Must show B has a g.l.b.

Let $L \subset \mathbb{R}$ be the set of all lower bounds of B.



$L \neq \emptyset$ since B is bdd below, and any $x \in B$ is an upper bd for L.

By L.U.B. property of $\mathbb{R}$, we may let

$$\alpha = \sup L.$$

To show $\alpha = \inf B$, need to prove:

(i) α is a lower bd

(ii) $\alpha < \gamma \Rightarrow \gamma$ not a lower bd.

Pf of (i). $\alpha > x \Rightarrow x$ is not an upper bd for L

(By (ii) of L.U.B. property)

$\Rightarrow \exists \beta \in L$ s.t. $\beta > x$

$\Rightarrow y \geq \beta > x \ \forall y \in B$

$\Rightarrow x \notin B$

Contrapos: $x \in B \Rightarrow \alpha \leq x$ ✓.

Pf of (ii). $\gamma > \alpha \Rightarrow \gamma \notin L$ (by (i) of L.U.B. property)

$\Rightarrow \gamma$ not a lower bd of B $\square$

Theorem: (a) $\mathbb{R}$ has the "archimedean property":

If $x, y \in \mathbb{R}$ and $x > 0$ then there is a pos. integer n such that $nx > y$.

(b) $\mathbb{Q}$ is "dense in $\mathbb{R}$":

If $x, y \in \mathbb{R}$ and $x < y$, $\exists\, p \in \mathbb{Q}$ s.t. $x < p < y$.

Pf (a) Suppose that (a) is false.

Let $A = \{nx \mid n \in \mathbb{N}\}$.

Then A is bounded above by y.

Let $\alpha = \sup A$.

Then $\alpha - x < \alpha$, since $x > 0 \Rightarrow -x < 0$.

Since α is the LUB, $\alpha - x$ is not an u.b. for A.

$\therefore \exists\, m \in \mathbb{N}$ s.t. $\alpha - x < m \cdot x$.

But then $\alpha < mx + x = (m+1)x$

$\Rightarrow \alpha$ is not an u.b. for A! $\times$.

(b) $x < y \Rightarrow y - x > 0$.

By (a), may choose $n \in \mathbb{N}$ s.t. $n(y-x) > 1$.

$\Rightarrow nx + 1 < ny$

Also choose m_1, m_2 s.t. $m_1 > nx$,

$m_2 < -nx$

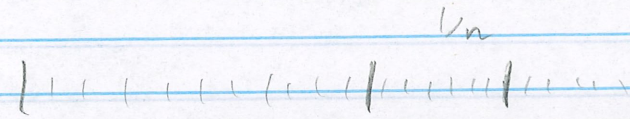
$$\Rightarrow \quad -m_2 < nx < m_1$$

Choose $m \in \{-m_2, \dots, m_1\}$ s.t.

$$m - 1 \le nx < m.$$

$$\Rightarrow \quad nx < m \le 1 + nx < ny$$

$$\Rightarrow \quad x < \frac{m}{n} < y \qquad \checkmark.$$



Theorem: For every real $x > 0$ and positive integer $n \in \mathbb{N}$, there is one and only one pos. real y s.t. $y^n = x$.

Notation: $y = \sqrt[n]{x}$.

Pf: at most one: $0 < y_1 < y_2 \Rightarrow 0 < y_1^n < y_2^n$ by induction.

Let $E = \{t \in \mathbb{R} \mid t > 0,\ t^n < x\}$.

Claim: E is nonempty & bounded.

Nonempty: Let $t = \frac{x}{1+x}$. Then $t < x$ and $t < 1$.

$$\Rightarrow t^n \le t < x$$

$$\Rightarrow t \in E \quad \checkmark$$

Bdd above. Suppose $t > x+1$.

Then $t > 1$ and $t > x$, so

$$t^n \geq t > x \implies t \notin E$$

$\therefore$ $x+1$ is an u.b. for E.

Let $y = \sup E$.

Claim 2: $y^n = x$.

Will show $y^n < x$ and $y^n > x$ both impossible.

Have: $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$.

$$= b^{n-1}\cdot b - b^{n-1}\cdot a + a(b-a)(b^{n-2} + \cdots + a^{n-2})$$

$$= b^n - b^{n-1}a + a(b^{n-1} - a^{n-1}) \quad \checkmark$$

If $0 < a < b$ then this implies

$$b^n - a^n < (b-a)\cdot n\, b^{n-1}$$

Assume $y^n < x$. Choose $0 < h < 1$ s.t.

$$h < \frac{x - y^n}{n(y+1)^{n-1}}$$

Take $a = y$, $b = y+h$.

$$\implies b^n - a^n = (y+h)^n - y^n < h\cdot n\cdot (y+h)^{n-1} <$$

$$< h \cdot n \cdot (y+1)^{n-1}$$

$$< x - y^n \quad \text{by choice of } h.$$

Cancel $-y^n$:

$$(y+h)^n < x.$$

$$\Rightarrow \quad y + h \in E.$$

But then y was not an upper bound! $\times$.

Assume $y^n > x$. Take

$$k = \frac{y^n - x}{n y^{n-1}} < \frac{y^n}{n y^{n-1}} = \frac{y}{n} < y$$

$$\Rightarrow \quad y - k > 0.$$

Suppose $t \geq y - k$.

Then $y^n - t^n \leq y^n - (y-k)^n < k n y^{n-1}$

$$= y^n - x$$

by choice of k.

$$\Rightarrow \quad x < t^n.$$

$$\Rightarrow \quad t \notin E$$

$\Rightarrow$ $y - k$ is an upper bd. of E $\times$ $\square$

Corollary: $a, b > 0$, $n \in \mathbb{N}$.

$$(ab)^{1/n} = a^{1/n} \cdot b^{1/n}$$

Pf:
$$ab = \left(a^{1/n}\right)^n \cdot \left(b^{1/n}\right)^n$$
$$= \left(a^{1/n} \cdot b^{1/n}\right)^n$$

Uniqueness of $\sqrt[n]{\ }$ $\Rightarrow$ $(ab)^{1/n} = a^{1/n} \cdot b^{1/n}$. $\square$.

Extended real #'s: $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

with order $-\infty < x < +\infty$ for every $x \in \mathbb{R}$.

This is an ordered set w/ L.U.B. property.

$\sup E = +\infty$ if E is unbounded above

$\inf E = -\infty$ " below.

Not an ordered field, but we sometimes write

for $x \in \mathbb{R}$. (a) $x + \infty = +\infty$ $\quad x - \infty = -\infty$

$$\frac{x}{+\infty} = \frac{x}{-\infty} = 0$$

(b) $x > 0 \Rightarrow x \cdot (+\infty) = +\infty \quad x \cdot (-\infty) = -\infty$

(c) $x < 0 \Rightarrow x \cdot (+\infty) = -\infty, \quad x \cdot (-\infty) = +\infty$

Careful not to write $\infty \cdot 0$, $\frac{\infty}{\infty}$ etc., since these are undefined.