Review of set theory.

A **set** S is a collection of **elements**.

Ex: $T = \{0, 1, \pi, \text{tomato}\}$.

$$0 \in T$$
$$\text{carrot} \notin T.$$

Two sets are **equal** if they contain the same elements.

Ex: $\{0, 0, 1, 1, \pi, \text{tomato}\} = T$

We say that S is a **subset** of T, and write $S \subset T$, if every el't of S is also an el't of T.

Ex: $\{0, 1, \pi\} \subset T$, $T \subset T$, $\emptyset \subset T$.

Can define subsets as follows:

$$\{x \in T \mid x \text{ is an integer}\} = \{0, 1\}$$

"The set of x in T s.t. x is an integer"

$$\{x \in T \mid x \text{ is a vegetable}\} = \emptyset.$$

Note: the elements of a set can themselves be sets! $A = \{\{0,1\}, \emptyset\}$. "Collection of sets"

Q. $0 \in A$? No. Q. $\{0,1\} \in A$? Yes. Q. $\{0,1\} \subset A$? No. Q. $\emptyset \in A$? Yes.

$\{\{0,1\}\} \subset A$? Yes. Q. $\emptyset \subset A$? Yes.

Suppose $A \subset S$ and $B \subset S$ are subsets.

<u>Intersection</u>: $A \cap B = \{x \in S \mid x \in A \text{ and } x \in B\}$

<u>Union</u>: $A \cup B = \{x \in S \mid x \in A \text{ or } x \in B\}$

<u>Complement</u>: $A^c = \{x \in S \mid x \notin A\}$.

<u>Difference</u>: $A \setminus B = \{x \in A \mid x \in A \text{ and } x \notin B\}$

<u>Ex</u>: $A = \{0, 1, \ \}$, $B = \{1, \pi \}$, $S = T$.

$A \cap B = \{\pi\}$

$A \cup B = T$

$A^c = \{\pi, \text{tomato}\}$

$A \setminus B = \{0, 1\}$.

(Same) <u>Properties</u>: (1) $A \cup B = B \cup A$ $\quad A \cap B = B \cap A$ (Commutat.)

(2) $(A \cup B) \cup C = A \cup (B \cup C)$ $\quad (A \cap B) \cap C = A \cap (B \cap C)$ (assoc.)

(3) $A \cup \emptyset = A$ $\quad A \cap \emptyset = \emptyset$

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distrib.)

(5) $(A \cup B)^c = A^c \cap B^c$ $\quad (A \cap B)^c = A^c \cup B^c$.

Pf of (4) Suffices to show $\subseteq$ and $\supseteq$.

($\subseteq$) Let $x \in A \cup (B \cap C)$.

Then $x \in A$ or $x \in B \cap C$.

Case 1. $x \in A$.

Then $x \in A \cup B$ and $x \in A \cup C$.

$\therefore x \in (A \cup B) \cap (A \cup C)$ ✓.

Case 2. $x \in B \cap C$.

Then $x \in B$ and $x \in C$.

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$. ✓

($\supseteq$) Let $x \in (A \cup B) \cap (A \cup C)$.

Case 1. $x \in A$.

Then $x \in A \cup B$ and $x \in A \cup C$ ✓.

Case 2. $x \notin A$.

Then $x \in A \cup B \Rightarrow x \in B$.

$x \in A \cup C \Rightarrow x \in C$.

$\Rightarrow x \in B \cap C$ $\Rightarrow x \in A \cup (B \cap C)$. $\square$

(Other one on HW!)

Can go quicker.

Pf of (5). $x \in (A \cup B)^c$

$\iff x \notin A$ and $x \notin B$

$\iff x \in A^c$ and $x \in B^c$

$\iff x \in A^c \cap B^c$. $\square$

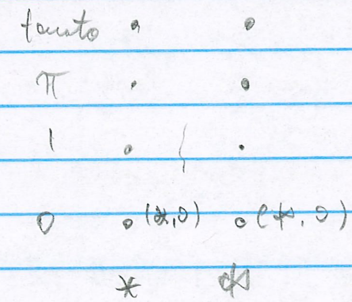
- The cartesian product $S \times T$ is the set $\{(x, y) \mid x \in S \text{ and } y \in T\}$

of all ordered pairs of el'ts in S and T.

Ex: $S = \{*, ✠\}$, T as above

$(*, \pi) \in S \times T$

$(\pi, *) \notin S \times T$.



- The power set $\mathcal{P}(S)$ is the collection of all subsets of S.

Ex: $\mathcal{P}(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$.

(~~Much~~ always a strictly bigger set — ~~~~ will formalize later).

domain $\quad$ codomain

- A **function** $f : S \to T$ is an assignment of **one** element of T to each element in S.

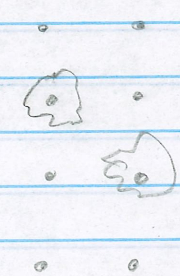
$$x \in S \;, \quad f(x) \in T$$

The **graph** of a function f is the subset

$$\mathrm{Graph}(f) = \{ (x, f(x)) \mid x \in S \} \subset S \times T .$$

Ex: $f : \{\star, \ast\} \to T$

$$\star \mapsto \pi$$
$$\ast \mapsto 1 .$$



A function and its graph are **equivalent**.

↳ must pass vertical line test.

$\forall x \in S, \exists !\, y \in T$ s.t. $(x, y) \in G$.

Given $A \subset S$, the **image** of A is

$$f(B) = \{ f(x) \in T \mid x \in A \} .$$

Given $B \subset T$, the **preimage** of B is

$$f^{-1}(B) = \{ x \in S \mid f(x) \in B \} .$$

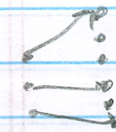
Note: image/preimage define maps

$$\mathcal{P}(S) \to \mathcal{P}(T) \qquad\qquad \mathcal{P}(T) \to \mathcal{P}(S)$$
$$A \mapsto f(A) \qquad\qquad B \mapsto f^{-1}(B) .$$

Ex: $f^{-1}(\{0, 1\}) = \{\ast\}$

$f^{-1}(\{0\}) = \emptyset .$

A function $f : S \to T$ is called:

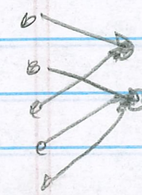


- **injective** if $f(x) = f(y) \Rightarrow x = y$.

  i.e. if $x \neq y$ then $f(x) \neq f(y)$ (contrapos.)

- **Surjective** if $f(S) = T$

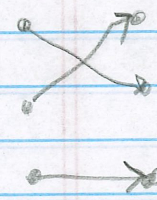
  i.e $\forall y \in T, \exists x \in S$ s.t. $f(x) = y$.



- **Bijective** if both injective and surjective.

  i.e. $\forall y \in T, \exists ! x \in S$ s.t. $f(x) = y$

  unique



A bijective function is sometimes also called a one-to-one correspondence.

A bijective function f has an inverse function f^{-1} (! notation.)

satisfying

$$f \circ f^{-1} = \mathbb{1} \quad , \quad f^{-1} \circ f = \mathbb{1}$$

identity function