Recall: $\mathbb{R}$ is an ordered field with L.U.B. property (sup), G.L.B. property (inf), Archimedean property, $\mathbb{Q} \subset \mathbb{R}$ is "dense," n'th roots of nonneg. #'s exist.

↳ what about negative #'s?

Definition: A complex number is an ordered pair (a,b) of real #'s. $\neq (b,a)$

Suppose $x = (a,b)$, $y = (c,d)$ are complex #'s.

We say x and y are equal, $x = y$, iff $a = c$ and $b = d$.

Define $x + y = (a+c, b+d)$

$x \cdot y = (ac - bd, ad + bc)$

We write $\mathbb{C}$ $= \mathbb{R} \times \mathbb{R}$ for the set of all complex #'s endowed with the operations $+$ and $\cdot$.

Theorem: $\mathbb{C}$ is a field, with $0 = (0,0)$ and $1 = (1,0)$.

Pf: (A1) ok

(A2) $x + y = (a+c, b+d)$

$= (c+a, d+b)$

$= y + x$ ✓.

(A3) $(x+y)+z \overset{(e,f)}{=} (a+c, b+d) + (e,f)$

$= ((a+c)+e, (b+d)+f)$

$= (a+(c+e), b+(d+f))$

$= x+(y+z)$ ✓.

(A4) ✓

(A5) Let $-x = (-a, -b)$ ✓.

(M1) ✓

(M2) $x \cdot y = (ac-bd, ad+bc)$

$= (ca-db, da+cb)$

$= y \cdot x$. ✓

(M3-4) similar.

(M5) $x \neq 0 \iff a^2+b^2 > 0$.

Define $\frac{1}{x} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

$x \cdot \frac{1}{x} = \left(\frac{a \cdot a - b(-b)}{a^2+b^2}, \frac{a(-b)+b \cdot a}{a^2+b^2}\right)$

$= (1,0) = 1$ ✓

(D) similar. $\square$.

We consider $\mathbb{R} = \{(a,0) \mid a \in \mathbb{R}\} \subset \mathbb{C}$

$$\overset{\in \mathbb{R}}{(a,0)} + \overset{\in \mathbb{R}}{(b,0)} = (a+b,0) \in \mathbb{R} \qquad (A1)$$

$$(a,0)\cdot(b,0) = (ab,0) \in \mathbb{R} \qquad (M1)$$

"Subfield" of $\mathbb{C}$.

If $a \in \mathbb{R}$ we write $a = (a,0)$.

__Definition:__ $i = (0,1)$.

__Theorem:__ $i^2 = -1$

__Pf:__ $i^2 = (0,1)^2 = (-(1)^2, 0)$

$= (-1,0)$

$= -(1,0) = -1$ ✓.

__Note:__ $\frac{1}{i} = (0,-1) = -i$.

__Theorem__ If $a,b \in \mathbb{R}$, then $(a,b) = a + b\cdot i$.

__Pf:__ $a + b\cdot i = (a,0) + (b,0)\cdot(0,1)$

$= (a,0) + (0,b)$

$= (a,b)$ $\square$.

__Note:__ $(a+bi)\cdot(c+di) = ac + bci + adi + bd(i)^2$

$(a,b)\cdot(c,d)$ " $= ac - bd + (bc+ad)i = (ac-bd, bc+ad)$ ✓.

Defn: For $z = a + bi$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, the <u>complex conjugate</u> is

$$\bar{z} = a - bi$$

$$\Rightarrow a = \frac{z + \bar{z}}{2} =: \operatorname{Re} z \quad \text{"Real part"}$$

$$b = \frac{z - \bar{z}}{2i} =: \operatorname{Im} z \quad \text{"Imaginary part"}$$

Theorem: $z, w \in \mathbb{C}$

(a) $\overline{z + w} = \bar{z} + \bar{w}$

(b) $\overline{zw} = \bar{z}\bar{w}$

(c) $z + \bar{z} = 2\operatorname{Re} z$, $z - \bar{z} = 2i \operatorname{Im} z$

(d) $z\bar{z}$ is a positive real # $\Leftrightarrow z \neq 0$.

Pf: (a–c) ok.

(d) $z\bar{z} = (a,b)(a,-b) = (a^2 - b(-b), a\cancel{(-b)} + \cancel{ab})$

$$= (a^2 + b^2, 0) = a^2 + b^2. \quad \checkmark.$$

Defn: The "absolute value" of $z \in \mathbb{C}$ is

$$|z| := \sqrt{z\bar{z}} \longrightarrow \text{pos. sqrt.}$$

$$= \sqrt{a^2 + b^2}.$$

5

Thm: (a) $|z| > 0$ unless $z = 0$.

(b) $|\bar{z}| = |z|$

(c) $|zw| = |z||w|$

(d) $|\operatorname{Re} z| \leq |z|$

(e) $|z+w| \leq |z| + |w|$

(f) $2\operatorname{Re}(z\bar{w}) \leq |z|^2 + |w|^2$, with $=$ iff $z = w$.

Pf. (a-b) ok

(c) $|zw|^2 = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2$

$\sqrt{}$ $\sqrt{}$ ✓.

(d) $(\operatorname{Re} z)^2 = a^2 \leq a^2 + b^2 = |z|^2$ ✓.

(e) $|z+w|^2 = (z+w)(\bar{z}+\bar{w})$

$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$

$= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2$

$\leq |z|^2 + 2|z||w| + |w|^2$

$= (|z| + |w|)^2$ ✓.

(f) $0 \leq |z-w|^2 = (z-w)(\bar{z}-\bar{w})$

$= |z|^2 - \underbrace{z\bar{w} - \bar{z}w}_{2\operatorname{Re} z\bar{w}} + |w|^2$ ✓

Theorem (Cauchy-Schwarz Inequality)

Let $x_1, \dots, x_n \in \mathbb{C}$ and $y_1, \dots, y_n \in \mathbb{C}$.

$$\left|\sum_{i=1}^n x_i \bar{y}_i\right|^2 \le \left(\sum_{i=1}^n |x_i|^2\right)\left(\sum_{j=1}^n |y_j|^2\right)$$

Pf: LHS $= \left(\sum_{i=1}^n x_i \bar{y}_i\right) \cdot \overline{\left(\sum_{j=1}^n x_j \bar{y}_j\right)}$

$$= \sum_{i,j=1}^n x_i \bar{y}_i \bar{x}_j y_j$$

$$= \sum_{i=1}^n x_i \bar{y}_i \bar{x}_i y_i + \sum_{i<j} x_i \bar{y}_i \bar{x}_j y_j + x_j \bar{y}_j \bar{x}_i y_i$$

Let $z = x_i y_j$, $w = x_j y_i$.

$$2\,\mathrm{Re}\, z\bar{w} = x_i \bar{y}_i \bar{x}_j y_j + x_j \bar{y}_j \bar{x}_i y_i$$

$$\le |z|^2 + |w|^2 = |x_i|^2 |y_j|^2 + |x_j|^2 |y_i|^2$$

$$\le \sum_{i=1}^n |x_i|^2 |y_i|^2 + \sum_{i<j} |x_i|^2 |y_j|^2 + |x_j|^2 |y_i|^2$$

$$= \sum_{i=1}^n |x_i|^2 |y_i|^2 + \sum_{i \ne j} |x_i|^2 |y_j|^2$$

$$= \sum_{i,j=1}^n |x_i|^2 |y_j|^2 = \left(\sum_{i=1}^n |x_i|^2\right)\left(\sum_{j=1}^n |y_j|^2\right) = \text{RHS}. \quad \square$$

Definition: $\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n \text{ times}}$

= set of all n-tuples $(x_1, \ldots, x_n)$ of real #'s.

$$\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n) \qquad \text{(add'n)}$$

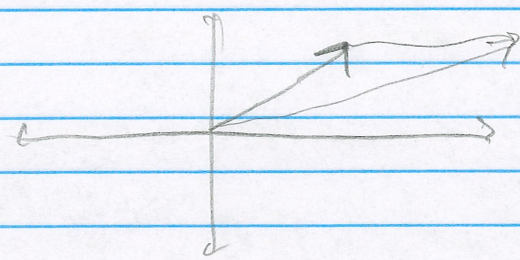
$\alpha \in \mathbb{R}$: $\alpha \vec{x} = (\alpha x_1, \ldots, \alpha x_n)$ (scalar mult.)

$$\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}.$$

"Vector space" over $\mathbb{R}$.

$\mathbb{R}^1 = \mathbb{R}$

$\mathbb{R}^2$



$= \mathbb{C}$.

+ agrees

· by $\mathbb{R}$ agrees.

Defn:

inner product $\vec{x} \cdot \vec{y} := \sum_{i=1}^{n} x_i y_i$.

Norm: $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\sum_{i=1}^{n} (x_i^2)}$

"Euclidean space."

Theorem: $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$.

(a) $|\vec{x}| \geq 0$

(b) $|\vec{x}| = 0 \iff \vec{x} = 0$.

(c) $|\alpha \vec{x}| = |\alpha||\vec{x}|$

(d) $|\vec{x} \cdot \vec{y}| \leq |x||y|$ $\rightsquigarrow$ Cauchy Schwarz !

(e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$

(f) $|\vec{x} - \vec{z}| \leq |x - y| + |y - z|$ "Triangle inequality"

Pf. (e)

$$
\begin{aligned}
|\vec{x} + \vec{y}|^2 &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\
&\leq |x|^2 + 2|\vec{x}||\vec{y}| + |y|^2 \\
&\leq (|\vec{x}| + |\vec{y}|)^2
\end{aligned}
$$

(f) Replace $\vec{x} \rightsquigarrow \vec{x} - \vec{y}$

$\vec{y} \rightsquigarrow \vec{y} - \vec{z}$

$\vec{x} + \vec{y} \rightsquigarrow \vec{x} - \cancel{\vec{y}} + \cancel{\vec{y}} - \vec{z} = \vec{x} - \vec{z}$ ✓.