Last time: ordered fields

Today: The least-upper-bound property.

Recall: A set S is ordered if it has a relation $<$ such that

(i) $\forall x, y \in S$, exactly one of the following is true:

$x < y$, $x = y$, or $y < x$

(ii) If $x, y, z \in S$ with $x < y$ and $y < z$, then $x < z$.

Notation: $x \leq y$ if $x < y$ or $x < y$

$x > y$ if $y < x$.

Notice: $x \leq y$ and $y \leq x \iff x = y$

$x \leq y$ and $y \leq z \Rightarrow x \leq z$

$x \leq y$ and $y < z \Rightarrow x < z$.

Definition: Suppose $E \subset S$ is a subset.

We say $\beta \in S$ is an upper bound for E if

$x \leq \beta \quad \forall x \in S$.

E is bounded above if such a β exists.

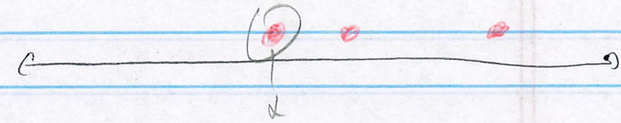
- We say α is a lower bound for E if

$$\alpha \leq x \quad \forall \; x \in E.$$

- E is bounded below if such an α exists.
- E is bounded if both bdd above & below.

Examples. $S = \mathbb{Q}$.

- $A = \{n^2 \mid n \in \mathbb{Z}\}$.



Bdd above? No.

Pf: Need to show $\nexists$ upper bound for E in $\mathbb{Q}$.

i.e. given any $M \in \mathbb{Q}$, $\exists \; x \in S$ s.t. $M < x$.

Let $\beta = \frac{m}{n} \in \mathbb{Q}$.

Take $x = m + 1 \geq 1$.

Then $m < m + 1 \leq (m+1)^2$ by hw.

$$\beta = \frac{m}{n} \leq m < (m+1)^2 = x \quad ✓.$$

Bdd below? Yes.

Take $\alpha = 0$.

$$\alpha = 0 \leq n^2 \quad \forall \; n \in \mathbb{Z}. \quad ✓.$$

Notice: $\alpha \in E$! (A contains a "smallest element")

- $B = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Bdd above? Yes.

Take $\beta = 1$.

$1 \le n \ \forall n \in \mathbb{N}$

$\Rightarrow \frac{1}{n} \le 1 = \beta$ ✓. Notice: $\beta = \frac{1}{1} \in B$.

(B contains a largest element.)

Bdd below? Yes.

Take $\alpha = 0$.

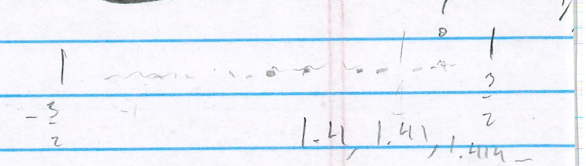
$0 < 1 \le n$

$\Rightarrow 0 < \frac{1}{n}$. ✓. Notice: $0 \notin B$!

(B does not contain a smallest element!)

α''

- $C = \{p \in \mathbb{Q} \mid p^2 \le 2\}$.



Bdd above? yes.

Take $\beta = \frac{3}{2}$. Need: $p \in C \Rightarrow p \le \frac{3}{2}$.

Contrapos: $p > \frac{3}{2} \Rightarrow p \notin C$.

$\Rightarrow p^2 > \frac{3}{2}p > \frac{9}{4} > \frac{8}{4} = 2 \Rightarrow p^2 > 2$ or $p \notin C$. ✓.

Bdd below? Yes.

Take $\alpha = -2$. $p < -\frac{3}{2} \Rightarrow p^2 > \frac{9}{4} > 2$. ✓.

Notice: $\frac{3}{2} \notin A$. In fact, not optimal!

<u>Definition</u>: Suppose E is bounded above.

- If there exists $\beta \in \mathbb{Q}$ s.t.

(i) β is an upper bound for E

(ii) If $\gamma < \beta$ then γ is <u>not</u> an upper bd for E,

then we say that α is the <u>least upper bound</u>, or <u>supremum</u>, of E. We write

$$\beta = \sup E.$$

- We write $\sup E = \infty$ if E is <u>not</u> bounded above in S.
- The greatest lower bound, or <u>infimum</u>, is defined similarly and called <u>$\inf E$</u>.
- An ordered set S has the <u>least-upper-bound property</u> if: for any nonempty subset $E \subset S$ which is bounded above, $\sup E$ exists in S.

<u>Examples</u>: $S = \mathbb{Q}$.

$\sup A = \infty$

<u>Claim</u>: $\inf A = 0$. (i) ✓.

(ii) If $0 < \gamma$ then since $0 \in A$, γ is not a lower bound!

- $\sup B = 1$ (since $1 \in B$).

<u>Claim</u>: $\inf B = 0$ (i) ✓.

(ii) Let $0 < \gamma \in \mathbb{Q}$

$\Rightarrow \gamma = \frac{m}{n}$, $m, n > 0$.

$\Rightarrow \frac{1}{n+1} < \frac{1}{n} \le \frac{m}{n} = \gamma$

But $\frac{1}{n+1} \in B$

$\Rightarrow \gamma$ is not a lower bound ✓.

<u>Notice</u>: $\inf B \notin B$ but still $\inf B \in \mathbb{Q}$.

<u>However</u>:

<u>Proposition</u>: C has no supremum or infimum in $\mathbb{Q}$.

<u>Pf</u>: <u>Claim 1</u>. We may replace $\le$ by $<$ in the definition:

$$A = \{ p \in \mathbb{Q} \mid p^2 < 2 \}.$$

<u>Pf of Claim 1</u>: $p \in A \Rightarrow p^2 < 2$ or $p = 2$.

Suffices to show $\nexists\, p \in \mathbb{Q}$ s.t. $p^2 = 2$.

Suppose that $p^2 = 2$ with $p \in \mathbb{Q}$.

We may let $p = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ are <u>not both even</u>. (Cancel common factors of 2).

Then $p^2 = \frac{m^2}{n^2} = 2$.

$\Rightarrow m^2 = 2n^2$

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even (odd . odd = odd)

$\Rightarrow m = 2m'$ for some $m' \in \mathbb{Z}$.

$m^2 = (2m')^2 = 2n^2$

$4m'^2 = 2n^2$

$2m'^2 = n^2$

$\Rightarrow n^2$ is even

$\Rightarrow n$ is even

$\Rightarrow$ both m and n are even

But this contradicts our assumption. $\times$ ✓

<u>Claim</u>: $p \in \mathbb{Q}$ is an upper bd for C

$\Rightarrow p^2 \geq 2$.

<u>Pf</u>: Suppose $p^2 \leq 2$. Then $p^2 < 2$.

Let $q = p + \frac{2-p^2}{p+2} > p$.

$$q = \frac{p^2 + 2p - p^2 + 2}{p+2}$$

$$= \frac{2(p+1)}{p+2}$$

$$q^2 - 2 = \frac{4(p^2+2p+1) - 2(p^2+4p+4)}{(p+2)^2}$$

$$= \frac{2p^2 - 4}{(p+2)^2} = \frac{2(p^2-2)}{p+2} < 0$$

$$\Rightarrow q^2 < 2 \Rightarrow q \in C$$

But $p < q \Rightarrow p$ is not an u.b. ✓.

<u>Claim 3</u>. p is an upper bd for C

$\Rightarrow p$ is <u>not</u> the least upper bd.

<u>Pf</u>: Saw: $p^2 > 2$.

Let $q = p - \frac{p^2-2}{p+2} < p$ as before.

$$q^2 - 2 = \frac{2(p^2-2)}{p+2} > 0$$

$$\Rightarrow q^2 > 2$$