Last time: $\mathbb{N}$ – natural numbers $\{1, 2, \ldots\}$

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$\mathbb{Z}$ – integers $\{\ldots, -1, 0, 1, \ldots\}$

$\cap$

$\mathbb{Q}$ – rational numbers $\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}\}$

Today: Ordered fields

Definition: A set F is called a field if it is equipped with operations $+$ and $\cdot$ satisfying the following properties:

(A1) If $x \in F$ and $y \in F$ then $x + y \in F$

(A2) $x + y = y + x \quad \forall x, y \in F$

(A3) $x + (y + z) = (x + y) + z \quad \forall x, y, z \in F$

(A4) F contains an element 0 s.t. $0 + x = x \quad \forall x \in F$

(A5) To each $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.

(M1) If $x \in F$ and $y \in F$ then $x \cdot y \in F$

(M2) $x \cdot y = y \cdot x$

(M3) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M4) F contains an element $1 \neq 0$ s.t. $1 \cdot x = x \quad \forall x \in F$

(M5) If $x \in F$ and $x \neq 0$ then there exists an element $\frac{1}{x} \in F$ s.t. $x \cdot \frac{1}{x} = 1$.

(D) $(x+y) \cdot z = x \cdot z + y \cdot z$.

$(x+y)+z = x+y+z = x+(y+z)$

Notation: $x - y = x + (-y)$, $x \cdot \frac{1}{y} = \frac{x}{y}$, $x \cdot x = x^2$, —

Example: $\mathbb{Q}$ is a field!

(There are many other fields, but just being a field implies the following further properties:)

Proposition 1. (a) $x + y = x + z \Rightarrow y = z$

(b) $x + y = x \Rightarrow y = 0$

(c) $x + y = 0 \Rightarrow y = -x$

(d) $-(-x) = x$.

Pf: (a) $y \overset{(A4)}{=} 0 + y \overset{(A5)}{=} (-x + x) + y$

$\overset{(A3)}{=} -x + (x + y)$

$\overset{\text{assumption}}{=} -x + (x + z)$

$\overset{(A3)}{=} (-x + x) + z$

$\overset{(A5)}{=} 0 + z$

$\overset{(A4)}{=} z$ ✓

(b) $z = 0$ in (a).

(c) $x + y = 0 \overset{(A5)}{=} x + (-x)$

$\overset{(a)}{\Rightarrow} y = -x$ ✓.

(d) $0 \overset{(A5)}{=} x + (-x) \overset{(A1)}{=} (-x) + x$

$\overset{(c)}{\Rightarrow} x = -(-x)$ ✓. $\square$

Prop 2: Assume $x \neq 0$.

(a) $xy = xz \Rightarrow y = z$

(b) $xy = x \Rightarrow y = 1$.

(c) $xy = 1 \Rightarrow y = 1/x$

(d) $\dfrac{1}{(1/x)} = x$.

(a) $y = 1 \cdot y = (x \cdot \frac{1}{x}) \cdot y$
$= \frac{1}{x}(xy)$
$= \frac{1}{x} \cdot (xz)$
$= (\frac{1}{x} \cdot x) \cdot z$
$= 1 \cdot z$
$= z$ ✓.

Pf: Very similar to previous. $\square$

Prop 3: (a) $0 \cdot x = 0$

(b) $x \neq 0$ and $y \neq 0 \Rightarrow xy \neq 0$

(c) $(-x) \cdot y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$.

Pf: $0x + 0x \overset{(D)}{=} (0 + 0)x = 0x$

Prop. 1b $\Rightarrow 0 \cdot x = 0$ ✓.

(b). Assume $x \neq 0$ and $y \neq 0$ but $xy = 0$.

$$1 = 1 \cdot 1 = \left(\frac{1}{x} \cdot x\right) \cdot \left(\frac{1}{y} \cdot y\right)$$
$$= \left(\frac{1}{x} \cdot \frac{1}{y}\right) \cdot (x \cdot y)$$
$$= \left(\frac{1}{x} \cdot \frac{1}{y}\right) \cdot 0 \quad = 0$$

$1 = 0$. But this is a contradiction!

(c)
$$(-x) \cdot y + x \cdot y = (-x + x) y = 0 \cdot y = 0$$
$$\Rightarrow \quad (-x) \cdot y = -xy$$
$$x \cdot (-y) + x \cdot y = x(y - y) = 0$$
$$\Rightarrow \quad x \cdot (-y) = -x \cdot y \qquad \checkmark$$

(d) $(-x)(-y) \overset{(c)}{=} -(x(-y)) \overset{(c)}{=} -(-(xy)) \overset{(1d)}{=} xy$. $\square$

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subset of $S \times S$

Definition. Let S be a set. An <u>order</u> $<$ is a relation with the following properties:

(i) If $x \in S$ and $y \in S$, then one and only one of the following is true:

$$x < y, \quad x = y, \quad \text{or} \quad y < x.$$

(ii) If $x < y$ and $y < z$ then $x < z$.

We write $x \le y$ if $x < y$ or $x = y$.

$x > y$ if $y < x$

$x \ge y$ if $y < x$ or $y = x$.

An <u>ordered set</u> is a set on which an order is defined.

Suppose $E \subset S$. We say $\beta \in S$ is an <u>upper bound</u> for E if $x \le \beta$ for all $x \in E$. LATER

If an upper bound exists, we say E is <u>bounded above</u>.

<u>Example</u> $\mathbb{Z}$ is ordered by $m < n$ if $n - m \in \mathbb{N}$.

<u>Definition</u>: An <u>ordered field</u> is a field which is also an ordered set, satisfying

(i) $y < z \Rightarrow x + y < x + z$

(ii) $x > 0$ and $y > 0 \Rightarrow xy > 0$.

$x > 0$ is called positive

$x < 0$ " negative.

<u>Example</u>: For $r \in \mathbb{Q}$, define $r > 0$ if $r = \frac{p}{q}$ for some $p, q \in \mathbb{N}$, and $r < s$ if $s - r > 0$.

$\rightsquigarrow$ $\mathbb{Q}$ is an ordered field!

~~[illegible]~~

Proposition 4: The following are true in any ordered field:

(a) $x > 0 \iff -x < 0$

(b) $x > 0$ and $y < z \Rightarrow xy < xz$

(c) $x < 0$ and $y < z \Rightarrow xy > xz$

(d) $x \neq 0 \Rightarrow x^2 > 0$. In particular, $1 > 0$

(e) $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$.

Pf: (a) $x > 0 \Rightarrow 0 = -x + x > -x + 0$

$\Rightarrow 0 > -x$.

(b) $z > y \Rightarrow z + (-y) > y + (-y) = 0$

$\Rightarrow z - y > 0$

$\Rightarrow x \cdot (z - y) > 0$

$+xy \quad +xy$

$xz > xy$ ✓

(c) $-(x(z-y)) \overset{3c}{=} (-x)(z-y) > 0$

$+x(z-y) \quad +x(z-y)$

$x(z-y) < 0$

$\Rightarrow xz < xy$ ✓

(d) Case 1. $x > 0$. $x^2 = x \cdot x > 0$ by (ii)

Case 2. $x < 0$. $-x > 0$

$$\Rightarrow \underset{x^2}{\underbrace{(-x)^2}_{\overset{\parallel}{(3d)}}} > 0 \qquad \Rightarrow 1^2 > 0$$

$$\Rightarrow 1 > 0 \quad \checkmark$$

(e) $y > 0$. $y \cdot \frac{1}{y} = 1 > 0 \Rightarrow \frac{1}{y} > 0$.

(If $1/y < 0$ then $y \cdot \frac{1}{y} < 0$ by (b)

$\Rightarrow 1 < 0$ ✗ (d).)

also $x > 0 \Rightarrow \frac{1}{x} > 0$.

$\frac{1}{x} \cdot \frac{1}{y} \cdot [x < y]$

$\Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$ □.