Problem set 5 solutions

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(2.6) Let $p \in (E')'$. We must show that $p \in E'$.

Given $r > 0$, there exists $q \in E'$ with $q \neq p$ such that $q \in N_{r/2}(p)$. Let

$$s = \min\left\{\frac{r}{2}, d(p,q)\right\} > 0.$$

Since $q \in E'$, there also exists $x \in E$ with $x \neq q$ such that $x \in N_s(q)$. Since $d(x,q) < s \leq d(p,q)$, we must have $x \neq p$. We then have

$$d(x,p) \leq d(x,q) + d(q,p) < \frac{r}{2} + s < \frac{r}{2} + \frac{r}{2} = r.$$

Therefore $x \in N_r(p)$ and $x \neq p$. Since $r > 0$ was arbitrary, we conclude that p is a limit point of E, as desired.

We must show that $\bar{E}' = E'$. It is a general fact that $(A \cup B)' = A' \cup B'$. The containment $(\supset)$ is obvious, and the containment $(\subset)$ can be shown as follows. Suppose that p is *neither* a limit point of A nor of B. Then there exist $s, t > 0$ such that $N_s(p) \cap A = \varnothing$ or $\{p\}$ and $N_t(p) \cap B = \varnothing$ or $\{p\}$. Letting $r = \min\{s,t\}$, we obtain that $N_r(p)$ contains at most one point of $A \cup B$. Therefore p is not a limit point of $A \cup B$.

Since $\bar{E} = E \cup E'$, by what we've just shown, $\bar{E}' = (E \cup E')' = E' \cup E''$. But since E' is closed, we have $E'' \subset E'$, so finally $\bar{E}' = E'$.

It is not true that E and E' necessarily have the same limit points: take the usual set $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then as we've shown before, $E' = \{0\}$, but meanwhile $\{0\}' = \varnothing$.

(2.7) We showed in the last problem that $(A \cup B)' = A' \cup B'$. This implies

$$\overline{A \cup B} = (A \cup B) \cup (A \cup B)' = A \cup B \cup A' \cup B' = \bar{A} \cup \bar{B}.$$

(a) We can use induction. The case $n = 1$ is trivial. Supposing that the result has been shown for n, we have

$$\bar{B}_{n+1} = \overline{B_n \cup A_{n+1}} = \bar{B}_n \cup \bar{A}_{n+1}.$$

Using the induction hypothesis, this must equal $\cup_{i=1}^{n} \bar{A}_i \cup \bar{A}_{n+1}$, so we are done.

(b) Let $x \in \overline{\cup_{i=1}^{\infty} A_i}$. Then $x \in \bar{A}_i$ for some i. In particular either $x \in A_i$, and we're done, or else $x \in A_i'$. We claim that $x \in B'$. Let $r > 0$. Then there exists $q \in A_i$ with $q \neq p$ such that $d(p,q) < r$. But also $q \in B$. Since $r > 0$ was arbitrary, this shows that $x \in B'$, so we're done.

The inclusion can be proper: let $A_n = \left\{\frac{1}{n}\right\}$. Then B is the set above, and $\bar{A}_n = A_n$, so $\cup \bar{A}_n = B$ as well. However, $\bar{B} \ni 0$, while $B \not\ni 0$.

(2.8) For open sets, yes. If $p \in E$ and E is open, then there exists a neighborhood $N_r(p) \subset E$. For any $s < r$, also $N_s(p) \subset E$. Each neighborhood $N_s(p)$ is infinite, so p is certainly a limit point.

For closed sets, this is certainly false, for example any finite set has no limit points.

(2.14) We can take the open cover $\{(r,1)\}_{r \in (0,1)}$. A finite subcover is indexed by $r_1, \ldots, r_n > 0$, but the union

$$\cup_{i=1}^{n} (r_i, 1) = (\min r_i, 1).$$

Since $\min r_i > 0$, this is not equal to the whole interval $(0,1)$. Hence there is no finite subcover.

(2.15) We shall exhibit a family of nested closed (resp. bounded) subsets of $X = \mathbb{R}$ whose intersection is empty.

For closed sets, take $K_n = [n, +\infty)$. For a finite intersection, we have

$$K_{n_1} \cap \cdots \cap K_{n_k} = K_{\max n_i} \neq \varnothing.$$

However, $\cap_{n=1}^{\infty} K_n = \varnothing$ because for any $x \in \mathbb{R}$, if $n > x$, then $x \notin K_n$.

For bounded sets, take $B_n = \left(0, \frac{1}{n}\right)$ as in class. For a finite collection,

$$B_{n_1} \cap \cdots \cap B_{n_k} = B_{\max n_i} \neq \varnothing.$$

However, $\cap_{n=1}^{\infty} B_n = \varnothing$ because for any $x \in \mathbb{R}$, if $x > 0$, then for $1/n < x$ we have $x \notin B_n$. Meanwhile, if $x \leq 0$ then $x \notin B_n$ for all n.

(2.16) Note that

$$E = \left(\left(-\sqrt{3}, -\sqrt{2}\right) \cap \mathbb{Q}\right) \cup \left(\left(\sqrt{2}, \sqrt{3}\right) \cap \mathbb{Q}\right).$$

So E is clearly bounded as a subset of $\mathbb{R}$, hence also as a subset of $\mathbb{Q}$.

To show that E is closed in $\mathbb{Q}$, by Theorem 2.30 (which holds for closed sets by taking complements), it is equivalent to show that $E = F \cap \mathbb{Q}$ for a closed subset $F \subset \mathbb{R}$. Let

$$F = \left[-\sqrt{3}, -\sqrt{2}\right] \cup \left[\sqrt{2}, \sqrt{3}\right].$$

Since a union of finitely many closed sets is closed, F is closed. Now one can just observe that $F \cap \mathbb{Q} = E$, since $\pm\sqrt{2}, \pm\sqrt{3} \notin \mathbb{Q}$.

The set E is definitely also open in $\mathbb{Q}$ because it is an intersection of real open intervals with $\mathbb{Q}$.

To see that E is not compact, recall that compactness is independent of the ambient metric space, by Theorem 2.33. It therefore suffices to show that E is not compact as a subset of $\mathbb{R}$. Since E is bounded, by Heine-Borel (Theorem 2.41), it is compact if and only if it is closed. But it is clearly not closed, since $\bar{E} = F$ and $F \not\subset \mathbb{Q}$.

(2.19) (a) We have $\bar{A} \cap B = A \cap B = \varnothing$ and $A \cap \bar{B} = A \cap B = \varnothing$.

(b) Since B is open, B^c is closed. Since $A \subset B^c$, by Theorem 2.27c, $\bar{A} \subset B^c$ also. Hence $\bar{A} \cap B = \varnothing$. The same argument works the other way.

(c) We have the following Lemma: given any $p \in X$ and $\delta > 0$, the set

$$M_\delta(p) = \{x \in X \mid d(p,x) > \delta\}$$

is open. For, given $q \in M_\delta(p)$, let $r = d(q,p) - \delta > 0$. We claim that $N_r(q) \subset M_\delta(p)$. For, given $x \in N_r(q)$, we have

$$d(p,q) \leq d(p,x) + d(x,q) < d(p,x) + r = d(p,x) + d(p,q) - \delta.$$

Cancelling $d(p,q)$ and adding δ to both sides, we obtain

$$\delta < d(p,x).$$

Hence $x \in M_\delta(p)$. Since $x \in N_r(q)$ was arbitrary, we conclude that $N_r(q) \subset M_\delta(p)$. Therefore $M_\delta(p)$ is open.

Now, Rudin asks us to show that $A = N_\delta(p)$ and $B = M_\delta(p)$ are separated. They are both open, so by (b), it is sufficient to show that they are disjoint. But this is obvious because for any x, either $d(p,x) < \delta$ (and $x \in A$), or $d(p,x) = \delta$ (and x is in neither A nor B), or $d(p,x) > \delta$ (and $x \in B$), and these cases are mutually exclusive.

(d) Let $p \neq q$ be the two points of the metric space X, and let $r = d(p,q)$. Since X is connected by assumption, for any $\delta \in [0,r]$, there must be a point $x \in X$ with $d(p,x) = \delta$. For, if this fails for some $\delta > 0$, then $\{x \in X \mid d(p,x) = \delta\} = \varnothing$. Then the two disjoint open sets $A = N_\delta(p)$ and $B = M_\delta(p)$ cover X and are each nonempty (since the former contains p and the latter contains q). Since X is connected, this is impossible.

For each $\delta \in [0,r]$, we can now choose a point $x_\delta \in X$ with $d(p,x_\delta) = \delta$. This gives us an injective map from $[0,r]$ to X which sends $\delta \mapsto x_\delta$. (It is injective since if $\delta \neq \epsilon$, then $x_\delta \neq x_\epsilon$ because they have different distances to p). But this would imply that the interval $[0,r]$ is countable, which is false.

(2.20) The closure of a connected set is always connected. For, if $\bar{E}$ were separated, there would exist separated sets A and B such that $\bar{E} = A \cup B$. Then $E = (A \cap E) \cup (B \cap E)$. We have $\overline{A \cap E} = \bar{A} \cap \bar{E} = \bar{A}$ since $A \subset \bar{E}$ and so $\bar{A} \subset \bar{E}$. Hence

$$\overline{A \cap E} \cap (B \cap E) = \bar{A} \cap (B \cap E) \subset \bar{A} \cap B = \varnothing.$$

Similarly, $(A \cap E) \cap \overline{B \cap E} = \varnothing$. Hence E is separated. By contrapositive, if E is connected then so is $\bar{E}$.

The interior of a connected set may not be connected (although this is true in $\mathbb{R}^1$, as one can show using Theorem 2.47). In $X = \mathbb{R}^2$, let

$$B_{-1} = \{x \in \mathbb{R}^2 \mid d(x, -1) \leq 1\},$$

$$B_1 = \{x \in \mathbb{R}^2 \mid d(x, 1) \leq 1\},$$

and

$$E = B_{-1} \cup B_1.$$

This is the union of two closed balls of radius 1, centered at -1 and 1. These are each convex, hence connected (by Exercise 2.21c, below). The subset E is also connected, intuitively, because the balls touch at the point $(0, 0)$. To prove this, we must show that E is not separated.

Suppose that $A \cup B = E$ with A and B separated. The intersections $A \cap B_1$ and $B \cap B_1$ are again separated. Since B_1 is connected, we must have either $A \cap B_1 = \varnothing$ or $B \cap B_1 = \varnothing$. Similarly, we must have either $A \cap B_{-1} = \varnothing$ or $B \cap B_{-1} = \varnothing$.

Suppose without loss of generality that $A \cap B_1 = \varnothing$. Then we must have $B_1 \subset B$, so in particular $(0, 0) \subset B$. But then $B \cap B_{-1} \neq \varnothing$, so we must have $A \cap B_{-1} = \varnothing$. But then $A \cap E = A = \varnothing$. Hence E can only equal a union of separated sets if one of them is empty; therefore E is connected.

The interior of E, however, is the union of the interiors of E_1 and E_{-1}, which is $N_1(1) \cup N_1(-1)$. In particular it is a disjoint union of two nonempty open sets, which by (b), is separated.

2.21. (a) Since $\mathbf{a} \in A$, $0 \in A_0$, while since $\mathbf{b} \in B$, $1 \in B_0$, hence both are nonempty. We claim that

$$\bar{A}_0 \subset \mathbf{p}^{-1}(\bar{A}).$$

Given $t_0 \in \bar{A}_0$, we must show that $\mathbf{p}(t_0) \in \bar{A}$. We can assume without loss of generality that $t_0 \in A_0'$.

Let $r > 0$. Since $t_0 \in A_0'$, there exists $t \in A_0$ with $|t - t_0| < r/|\mathbf{a} - \mathbf{b}|$. By definition, $\mathbf{p}(t) \in A$, and we have

$$\begin{aligned} \mathbf{p}(t) - \mathbf{p}(t_0) &= (1-t)\mathbf{a} + t\mathbf{b} - ((1-t)\mathbf{a} + t\mathbf{b}) \\ &= (t_0 - t)\mathbf{a} + (t - t_0)\mathbf{b} \\ &= (t - t_0)(\mathbf{b} - \mathbf{a}). \end{aligned}$$

Hence

$$|\mathbf{p}(t) - \mathbf{p}(t_0)| \leq |t - t_0| \, |\mathbf{b} - \mathbf{a}| < \frac{r}{|\mathbf{a} - \mathbf{b}|} |\mathbf{b} - \mathbf{a}| = r.$$

Since $r > 0$ was arbitrary, we have shown that $\mathbf{p}(t_0) \in A'$.

We have shown that $\bar{A}_0 \subset \mathbf{p}^{-1}(\bar{A})$. Since $\bar{A}$ and B are disjoint and functions take unique values, $\mathbf{p}^{-1}(\bar{A})$ and $\mathbf{p}^{-1}(B) = B_0$ are also disjoint. Hence $\bar{A}_0$ and B are disjoint. One can show in the same way that $\bar{B}_0$ and A_0 are disjoint. Hence A_0 and B_0 are separated.

(b) Since $(0,1)$ is connected (by Theorem 2.47), and $A_0 \cap (0,1)$ and $B_0 \cap (0,1)$ are separated, we cannot have $(0,1) = A_0 \cup B_0$. So there exists $t_0 \in (0,1)$ such that $t_0 \notin A_0 \cup B_0$. By definition, $\mathbf{p}(t_0) \notin A$ and $\mathbf{p}(t_0) \notin B$, as desired.

(c) Let E be a convex set, and suppose for contradiction that $E = A \cup B$ is a union of nonempty separated sets. Pick $\mathbf{a} \in A$ and $\mathbf{b} \in B$ and consider the map $\mathbf{p}(t)$ given above. Since E is convex, $\mathbf{p}(t) \in E$ for each $t \in [0,1]$. But by (b), there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$. Hence $E \not\subset A \cup B$, a contradiction.

(2.13) (Extra credit) Let $E_x = \{x + \frac{1}{n} \mid n \in \mathbb{N}\}$. We can take

$$E = \{0\} \cup \bigcup_{m \in \mathbb{N} \cup \{0\}} E_{\frac{1}{m}} = \{0\} \cup E_0 \cup \left\{ \frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N} \right\}.$$

This has $E' = E_0 \cup \{0\}$, because for any $m \in \mathbb{N}$, a neighborhood of $\frac{1}{m}$ in E coincides with $E_{1/m}$. Since E is closed and bounded, it is compact.