Problem set 1: Set-theory review

Math 521 Section 001, UW-Madison, Spring 2024

January 23, 2024

Please solve the following problems in a clear, complete, and concise manner. You are welcome to work together, but your write-up must be your own. Use of outside internet resources is prohibited.

Due on paper at the beginning of class on **Wednesday, Jan. 31st**. Please be sure to staple your writeup. For this homework, you will have the opportunity to rewrite and resubmit any problems together with your second homework on Wednesday, Feb. 7th.

1. Let $S = \{\{1, \{2\}\}, \{3\}\}$.
   (a) List all the elements of S.
   (b) List all the subsets of S.

2. True or false? No written justification required.
   (a) $\{\{\varnothing\}\} \cup \varnothing = \{\varnothing, \{\varnothing\}\}$
   (b) $\{\{\varnothing\}\} \cup \{\varnothing\} = \{\varnothing, \{\varnothing\}\}$
   (c) $\{\varnothing, \{\varnothing\}\} \cap \{\{\varnothing\}, \{\{\varnothing\}\}\} = \{\varnothing\}$

3. Let $S = \{*, \dagger, \#\}$ and $T = \{\&, @, \%\}$. Which of the following subsets of $S \times T$ is the graph of a function $f : S \to T$? Please say why or why not.
   (a) $\{(\dagger, @), (\#, \&)\}$
   (b) $\{(\#, @), (\dagger, @), (*, \%)\}$
   (c) $\{(*, @), (\#, \&), (\#, \%)\}$.

4. Write the negation of each of the following statements.
   (a) Either $x \in S$ or $x \notin T$.
   (b) Every even natural number greater than 2 is equal to a sum of two prime numbers.
   (c) For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon$ for all x with $|x| < \delta$.

5. Write the contrapositive of each of the following implications.

   (a) $x \in S \Rightarrow x \in Q$ or $x \in T$.

   (b) $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.

   (c) $\triangle BAC$ is a right triangle $\Rightarrow a^2 + b^2 = c^2$.

6. Fix an integer $x \neq 1$. Use induction to prove the formula:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

7. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

8. Give a counterexample to one of the following four formulas for images and inverse images of sets (the other three are true):

$$\begin{aligned} f(A \cup B) &= f(A) \cup f(B), \qquad & f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\ f(A \cap B) &= f(A) \cap f(B), \qquad & f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

9. (Extra credit) Give an example of an injective function $\tilde{S} : \mathbb{N} \hookrightarrow \mathbb{N}$ such that $\tilde{S}(\mathbb{N}) = \mathbb{N} \setminus \{1\}$ but for which the 3rd Peano axiom fails; i.e., there exists a subset $A \subset \mathbb{N}$ such that $1 \in A$ and $x \in A \Rightarrow \tilde{S}(x) \in A$, but $A \neq \mathbb{N}$.