Midterm 1 Solutions

Math 521 Section 001, UW-Madison, Spring 2024

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1. (8 points) The statement is *true.* ($\subset$) If $x \in f^{-1}(A \cap B)$ then $f(x) \in A \cap B$, *i.e.* $f(x) \in A$ and $f(x) \in B$. Therefore $x \in f^{-1}(A) \cap f^{-1}(B)$.

   ($\supset$) If $x \in f^{-1}(A) \cap f^{-1}(B)$, then $f(x) \in A$ and $f(x) \in B$. But then $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$.

2. (8 points) Let $y \in B$. We must show that there exists $x \in A$ such that $g(x) = y$.

   Let $z = f(y) \in C$. Since $f \circ g$ is surjective, there exists $x \in A$ such that $f(g(x)) = z$. But then $f(g(x)) = f(y)$. Since f is injective, this implies $g(x) = y$, as desired.

3. (12 points) (a) (I) $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$ (alternatively, α is bounded above).

   (II) If $p \in \alpha$ and $q < p$, with $q \in \mathbb{Q}$, then $q \in \alpha$

   (III) If $p \in \alpha$ then there exists $r \in \alpha$ with $p < r$.

   (b) $\alpha < \beta$ iff $\alpha \subset \beta$ and $\alpha \neq \beta$.

   (c) Let α and β be Dedekind cuts. We must show that exactly one of $\alpha < \beta$, $\alpha = \beta$, and $\beta < \alpha$ is true.

   If $\alpha = \beta$ then we are done.

   If $\alpha \neq \beta$, then either $\beta \setminus \alpha \neq \varnothing$ or $\alpha \setminus \beta \neq \varnothing$.

   <u>Case 1.</u> $\beta \setminus \alpha \neq \varnothing$. We may pick $p \in \beta \setminus \alpha$. Let $q \in \alpha$ be arbitrary; we will show $q \in \beta$.

   Suppose that $q > p$. Then since α is a Dedekind cut and $q \in \alpha$, by (II), we must have $p \in \alpha$. This contradicts the fact that $p \in \beta \setminus \alpha$. Therefore $q < p$. But since β is a Dedekind cut, by (II), we must have $q \in \beta$. We conclude that $\alpha \subset \beta$. Since $\beta \setminus \alpha \neq \varnothing$, we certainly have $\alpha \neq \beta$. Therefore $\alpha < \beta$, as desired.

   <u>Case 2.</u> $\alpha \setminus \beta \neq \varnothing$. Reversing the roles of α and β, this is exactly the same as Case 1, and we conclude that $\beta < \alpha$.

   Finally, note that Case 1 and Case 2 are mutually exclusive, because if $\alpha \subset \beta$ and $\beta \subset \alpha$ then $\alpha = \beta$.

(d) $\sup E = \cup_{\alpha \in E} \alpha$.

4. (12 points) (a) E is bounded above by 2. For, since $n \in \mathbb{N}$, we have

$$n + 1 \leq n + n = 2n.$$

Dividing by n^2, we obtain

$$\frac{n+1}{n^2} \leq \frac{2n}{n^2} = \frac{2}{n}.$$

Therefore

$$\frac{n+1}{n^2} \leq \frac{2}{n} \tag{0.1}$$

for each $n \in \mathbb{N}$. In particular, $\frac{n+1}{n^2} \leq 2$, so 2 is an upper bound. Since

$$\frac{1+1}{1^2} = 2 \in E,$$

it is the least upper bound. (For any $\gamma < 2$, γ cannot be an upper bound because $2 \in E$).

(b) E is bounded below by 0, since $\frac{n+1}{n^2} > 0$. We claim that 0 is the greatest lower bound. Given $\gamma > 0$, choose n large enough that $2 < \gamma n$, by the Archimedean property. Then

$$\frac{2}{n} < \gamma.$$

But by (0.1), this implies

$$\frac{n+1}{n^2} < \gamma.$$

Therefore γ is not a lower bound for E.

(c) From class/book, one always has $\inf E \in \bar{E}$. If E were closed then we would have $E = \bar{E}$. But by (b), $0 = \inf E \notin E$, so we must have $E \neq \bar{E}$. Therefore E is not closed.

(d) E is certainly not open: it is nonempty, but no point of E is an interior point. Let $p \in E$. Then for any $r > 0$, the neighborhood $N_r(p) = (p - r, p + r)$ contains infinitely many points of E^c. Indeed, if $r < p$, then $N_r(p)$ contains only finitely many points of E, whereas $N_r(p)$ itself is infinite. (This would already give full credit.) To show this, note that $N_r(p) = (p - r, p + r)$, and $p - r > 0$ if $r < p$. For $\frac{2}{n} \leq p - r$, we have

$$\frac{n+1}{n^2} \leq \frac{2}{n} \leq p - r,$$

and $\frac{n+1}{n^2} \notin N_r(p)$. Hence, all but finitely many points of E are not contained in $N_r(p)$.