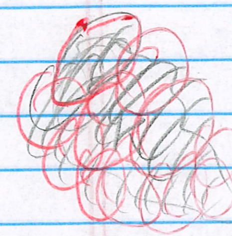
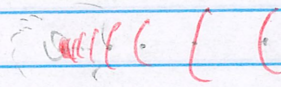


Recall: $K \subset X$ is called <u>compact</u> if for any open cover $\{U_\alpha\}$ of K, there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$.

<u>Saw:</u> • Finite sets are compact.

• $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is <u>not</u> compact.

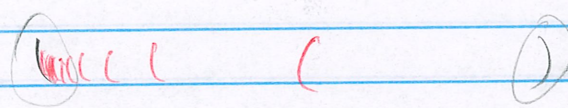


• $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ <u>is</u> cpt.

Similarly:

• (a,b) is <u>not</u> cpt.

(Consider $\{U_n\}$ where $U_n = (a+\frac{1}{n}, b)$.)



• Will see on Wednesday that $[a,b]$ <u>is</u> compact!

<u>Today:</u> General theorems abt cpt sets.

Theorem: Compact subsets are closed.

Pf: Will show K^c open.

Let $p \notin K$.

Then $d(p,q) > 0 \quad \forall q \in K$.

Define $\quad V_q = N_{\frac{1}{2}d(p,q)}(p), \quad W_q = N_{\frac{1}{2}d(p,q)}(q)$.

Then $\quad V_q \cap W_q = \phi$.

(Pf: $x \in V_q \Rightarrow d(p,x) < \frac{1}{2} d(p,q)$.

$$d(p,q) \overset{\Delta}{\leq} d(q,x) + d(p,x) < \tfrac{1}{2} d(p,q) + d(p,x)$$

$$\Rightarrow \tfrac{1}{2} d(p,q) < d(p,x) \quad \Rightarrow x \notin W_q.)$$

$\{W_q\}_{q \in K}$ is an open cover of K.

$\Rightarrow \exists$ finite subcover $W_{q_1}, \ldots, W_{q_n}$

$$K \subset \bigcup_{i=1}^{n} W_{q_i} = W.$$

Let $V = \bigcap_{i=1}^{n} V_{q_i}$. Since $V_{q_i} \cap W_{q_i} = \phi$

for each i, $\qquad V \cap W = \phi$.

But V is an open nbhd of p

$$V \cap W = \emptyset \Rightarrow V \cap K = \emptyset$$

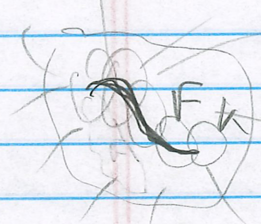
$$\Rightarrow V \subset K^c.$$

Since $p \in K^c$ was arb., K^c is open. $\square$

Theorem: Closed subsets of compact sets are compact.

closed, cpt.

Pf: Suppose $F \subset K \subset X$



Let $\{V_\alpha\}$ o.c. of F.

Consider $\{F^c\} \cup \{V_\alpha\}$.

This is an o.c. of K.

$\therefore$ exists finite subcover of K

If F^c belongs to the subcover, can remove it and still obtain open cover of F. $\square$

Corollary: If F is closed & K cpt, then $F \cap K$ is compact.

<u>Pf</u>: K closed $\Rightarrow$ $F \cap K$ closed

$F \cap K \subset K \Rightarrow$ cpt. $\square$

<u>Theorem</u>: If $\{K_\alpha\}$ is a collection of cpt subsets of X s.t. the intersection of any finite subcollection is nonempty, then

$$\bigcap_\alpha K_\alpha \neq \emptyset.$$

<u>Pf</u>: Let $G_\alpha = K_\alpha^c$

Fix $K_1 \in \{K_\alpha\}$

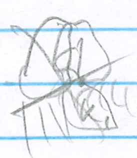
Assume that no point of K_1 belongs to every other K_α. Then $\{G_\alpha\}$ covers K_1

$\longrightarrow \exists$ finite subcover $G_{\alpha_1}, \ldots, G_{\alpha_n}$ s.t.

$$K_1 \subset \bigcup_{i=1}^{n} G_{\alpha_i}.$$

$$\Rightarrow \quad K_1^c \subset \left(\bigcap_{i=1}^{n} G_{\alpha_i}^c\right)^c = \left(\bigcap_{i=1}^{n} K_{\alpha_i}\right)^c.$$

$$\Rightarrow \quad K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset. \ \times \quad \square$$



Draw corollary! $\longrightarrow$

Corollary: If $\{K_n\}$ is a sequence of nonempty cpt sets s.t. $K_n \supset K_{n+1}$ $\forall n$,

then $\bigcap_{i=1}^{\infty} K_i \neq \phi$.

Pf: $\bigcap_{i=1}^{n} K_i = K_n \neq \phi$, so hypoth of thm is satisfied. $\square$

Note: The next theorem provides evidence for the fact that closed intervals are compact, and will be used to establish this fact.

Thm: If $I_n = [a_n, b_n]$, $a_n < b_n$, is a sequence of closed intervals in $X = \mathbb{R}^1$ s.t.

$I_n \supset I_{n+1}$ (i.e. $a_n \le a_{n+1}$ — and $b_n \ge b_{n+1}$ —)

then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Proof: Let $E = \{a_n \mid n \in \mathbb{N}\}$.

Then E is bdd above by b_1.

Let $x = \sup E$. Clearly $a_n \le x$.

Since for $m \ge n$, have

[[[|—| ...]]]]

$$a_n \le a_m \le b_m \le b_n,$$

each b_n is an u.b. for E.

$\Rightarrow \quad x \le b_n \ \forall n$.

$\Rightarrow \quad a_n \le x \le b_n$

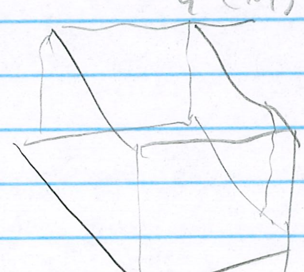
$\Rightarrow \quad x \in I_n \ \forall n$

$\Rightarrow \quad x \in \bigcap_{n=1}^{\infty} I_n$ ✓. $\square$

Definition: A k-cell in $\mathbb{R}^k$ is a product of intervals:

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k].$$

$$= \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid a_i \le x_i \le b_i \ \forall i\}.$$



Theorem. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of k-cells s.t. $I_n \supset I_{n+1}$ $\forall n$.

Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Let $I_n = \underset{\mathbb{R}^1}{\underset{\cap}{I_{n,1}}} \times \underset{\mathbb{R}^1}{\underset{\cap}{I_{n,2}}} \times \cdots \times \underset{\mathbb{R}^1}{\underset{\cap}{I_{n,k}}}$

Since $I_n \supset I_{n+1}$, $I_{n,j} \supset I_{n+1,j}$ $\forall j$.

$\Rightarrow \bigcap_{n=1}^{\infty} I_{n,j} \ni x_j \in \mathbb{R}$.

Let $\vec{x} = (x_1, \ldots, x_k)$.

Since $x_j \in I_{n,j}$ $\forall n, j$, $x \in \bigcap_{n=1}^{\infty} I_n$. $\square$

Theorem: Every k-cell is compact.

Pf: Next time.