Last time: Cardinality. $S \sim T$ if $\exists$ bijective map $f: S \xrightarrow{\sim} T$.

$J_n = \{1, 2, \dots, n\} \subset \mathbb{N}$.

- S is finite, with "$\# S = n$", if $S \sim J_n$ for some $n \in \mathbb{N}$.

  $\iff S = \{\overset{f(1)}{s_1}, \overset{f(2)}{s_2}, s_3, \dots, s_n\}$.

- S is countably infinite if $S \sim \mathbb{N}$.

  $\iff S = \{\overset{f(1)}{s_1}, \overset{f(2)}{s_2}, s_3, \dots\}$

  ↳ every element of S appears on the list exactly once.

- S is countable if finite or countably infinite.
  (Rudin: "at most countable")

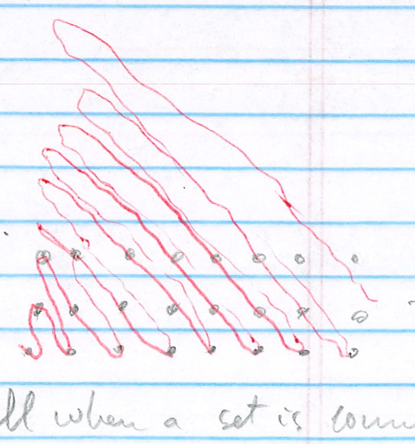
- S is uncountable if not countable.

Ex: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$= \{0, 1, -1, 2, -2, 3, -3, \dots\}$ countably infinite.

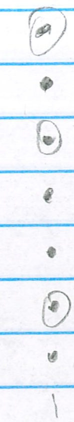
Ex: $\mathbb{N} \times \mathbb{N} = \{(m, n) \mid m, n \in \mathbb{N}\}$.

$= \{(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), (4,1), (3,2), (2,3), (1,4), \dots\}$.



countably infinite.

Now I'll give you some tools to tell when a set is countable.



<u>Theorem</u>: Any subset of a countable set is countable.

<u>Pf</u>: $A \subset S$, assume S is countable.

$$\Rightarrow S = \{s_1, s_2, s_3, s_4, s_5, \ldots\}$$

$$A = \{a_1, \quad a_2, \quad a_3, \ldots\} \qquad \square$$

<u>Ex</u>: $A =$ the set of prime numbers, $S = \mathbb{N}$.

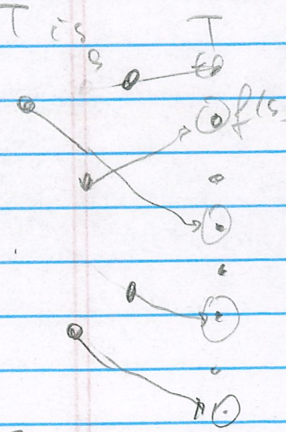
$= \{n \in \mathbb{N} \mid m \mid n \Rightarrow m = 1 \text{ or } n\}$.

Theorem $\Rightarrow$ countable.

Euclid: A is infinite.

$\therefore A$ is countably infinite.

<u>Thm</u>: Suppose $\exists$ injection $f: S \hookrightarrow T$ and T is countable. Then S is countable.



<u>Pf</u>: $f: S \xrightarrow{\sim} f(S) \subset T$ is a bijection.

$f(S)$ is countable

$\therefore S \sim f(S)$ also countable. $\square$

<u>Thm</u>: Suppose $\exists$ surjection $f: S \twoheadrightarrow T$ and S is countable. Then T is countable.

Pf: For each $y \in T$, choose one element $x_y \in f^{-1}(\{y\})$.

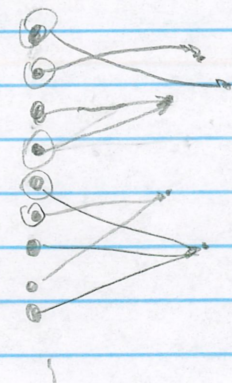
Let $Q = \{x_y \mid y \in T\} \subset S$.

Then $f|_Q : Q \xrightarrow{\sim} T$ is bijective

$\Rightarrow Q \sim T$

But $Q \subset S$ and S is countable, so Q is ctble.

$\therefore T \sim Q$ is also countable. $\square$



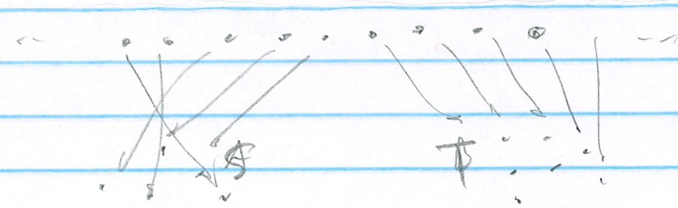
Theorem: If S and T are both countable, then $S \cup T$ is countable.

Pf: $S = \{s_1, s_2, \dots\}$, $T = \{t_1, t_2, \dots\}$

Define $f : \mathbb{Z} \to S \cup T$

$n \mapsto s_n \quad n > 0$

$n \mapsto t_{-n} \quad n < 0$



$\square$

$\mathbb{Z}$ is countable, and $f: \mathbb{Z} \to S \cup T$, so $S \cup T$ is countable. $\square$

Cor: Let $S_1, \ldots, S_n$ be a finite collection of countable sets. Then

$$S = S_1 \cup S_2 \cup \cdots \cup S_n \text{ is countable.}$$

Pf: induction. S_1 = ctble ✓

$$\begin{aligned} S &= S_1 \cup \cdots \cup S_n \\ &= (S_1 \cup \cdots \cup S_{n-1}) \cup S_n \\ &= \text{ctble} \cup \text{ctble} \\ &= \text{ctble} \end{aligned} \quad \square$$

Thm: S, T countable $\Rightarrow S \times T$ countable. Pf: $\mathbb{N} \times \mathbb{N} \twoheadrightarrow S \times T$. $\square$

What about a countable union of countable sets?

Thm: Let $\{S_m \mid m \in \mathbb{N}\} = \{S_1, S_2, S_3, \ldots\}$ be a collection of sets indexed by $m \in \mathbb{N}$, each of which is countable. Then

$$S = \bigcup_{m=1}^{\infty} S_m = \{x \mid x \in S_m \text{ for some } m \in \mathbb{N}\}$$

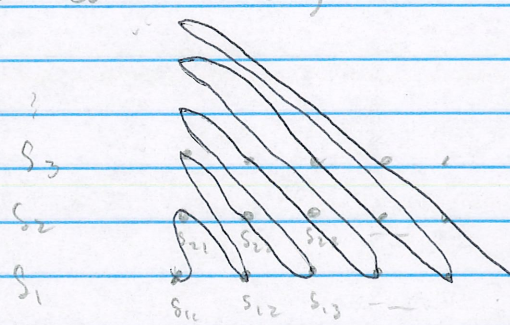
is countable.

Pf: Since S_m is countable, we may write $S_m = \{s_{m1}, s_{m2}, s_{m3}, \dots\}$.

Define a map $\mathbb{N}\times\mathbb{N} \to S$

$$(m,n) \mapsto s_{mn} \in S_m \subset S.$$

But $\mathbb{N}\times\mathbb{N}$ is countable, so S is countable. $\square$



What about a countable × product of sets?

Theorem $\{0,1\}^{\mathbb{N}} = \{(a_1, a_2, a_3, \dots) \mid a_n \in \{0,1\}, n \in \mathbb{N}\}$

is uncountable.

Pf: Suppose that S is countable.

Then we can label all the elements of $\{0,1\}^{\mathbb{N}}$ by $n \in \mathbb{N}$, as follows.

$$s_1 = (s_{11}, s_{12}, s_{13}, s_{14}, \dots)$$

$$s_2 = (s_{21}, s_{22}, s_{23}, s_{24}, \dots)$$

$$s_3 = (s_{31}, s_{32}, s_{33}, \dots)$$

$$\vdots$$

Define an element $t \in \{0,1\}^{\mathbb{N}}$ by

$$t_n = \begin{cases} 0 & \text{if } s_{nn} = 1 \\ 1 & \text{if } s_{nn} = 0. \end{cases}$$

t

$s_1 = \cancel{0}\,1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 0 .$

$s_2 = 1 \;\; \cancel{0}\,1 \;\; 1 \;\; \text{——}$

$s_3 = 1 \;\; 0 \;\; \cancel{}\,0 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; 1 \;\; \text{——}$

Then $t_n \neq s_{nn}$ for all $n \in \mathbb{N}$

$\Rightarrow t \neq s_n$ for all $n \in \mathbb{N}$.

$\Rightarrow t$ does not appear on the list of all elements of S

But $t \in S$ $\quad \times$

i.e. S is not countable. $\square$

"Cantor's second diagonal argument."

<u>Corollary</u>: The set $\mathbb{R}$ is uncountable.

<u>Pf:</u> Every number $x \in [0,1] \subset \mathbb{R}$ has a <u>binary expansion</u> $x = .0100\ldots110\ldots$.

$\Rightarrow \exists$ surjective map

$$B : \{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1] \subset \mathbb{R}$$

$$(a_1, a_2, \ldots) \mapsto .a_1a_2a_3a_4$$

$$= \sup\left\{ \sum_{i=1}^{n} \frac{a_i}{2^i} \,\middle|\, n \in \mathbb{N} \right\}.$$

But $x \in [0,1]$ can have <u>at most two</u> different binary expansions.

$$.a_1a_2a_3 0 1111111 = .a_1a_2a_3 100000\ldots$$

$\therefore$ for each $x \in [0,1]$, $\# B^{-1}(x) \leq 2$.

$$\{0,1\}^{\mathbb{N}} = \bigcup_{x \in [0,1]} B^{-1}(x)$$

would be countable if $[0,1]$ were. $\square$

$\therefore [0,1]$ is <u>uncountable</u>

$\cap$

$\therefore \mathbb{R}$ is uncountable. $\square$