Problem set 1 solutions

Math 521 Section 001, UW-Madison, Spring 2024

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1. Let $S = \{\{1, \{2\}\}, \{3\}\}$.

   (a) List all the elements of S.

   Answer. The elements of S are $\{1, \{2\}\}$ and $\{3\}$. $\square$

   (b) List all the subsets of S.

   Answer. The subsets are $\varnothing$, $\{\{1, \{2\}\}\}$, $\{\{3\}\}$, and S. $\square$

2. True or false? No written justification required.

   (a) $\{\{\varnothing\}\} \cup \varnothing = \{\varnothing, \{\varnothing\}\}$ (*False*)

   (b) $\{\{\varnothing\}\} \cup \{\varnothing\} = \{\varnothing, \{\varnothing\}\}$ (*True*)

   (c) $\{\varnothing, \{\varnothing\}\} \cap \{\{\varnothing\}, \{\{\varnothing\}\}\} = \{\varnothing\}$ (*False*)

3. Let $S = \{*, \dagger, \#\}$ and $T = \{\&, @, \%\}$. Which of the following subsets of $S \times T$ is the graph of a function $f : S \to T$? Please say why or why not.

   (a) $\{(\dagger, @), (\#, \&)\}$

   Answer. No. The element $* \in S$ is not assigned any value. $\square$

   (b) $\{(\#, @), (\dagger, @), (*, \%)\}$

   Answer. Yes. Each element of S is assigned exactly one value. $\square$

   (c) $\{(*, @), (\#, \&), (\#, \%)\}$.

   Answer. No. The element $\# \in S$ is assigned two different values. $\square$

4. Write the negation of each of the following statements.

   (a) Either $x \in S$ or $x \notin T$.

Answer. $x \notin S$ and $x \in T$. $\square$

(b) Every even natural number greater than 2 is equal to a sum of two prime numbers.

Answer. There exists a natural number $n \geq 2$ such that for all primes p and q, $p + q \neq n$. $\square$

(c) For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon$ for all x with $|x| < \delta$.

Answer. There exists $\varepsilon > 0$ such that for each $\delta > 0$, there exists $-\delta < x < \delta$ such that $|f(x) - f(0)| \geq \varepsilon$. $\square$

5. Write the contrapositive of each of the following implications.

(a) $x \in S \Rightarrow x \in Q$ or $x \in T$.

Answer. If $x \notin Q$ and $x \notin T$ then $x \notin S$. $\square$

(b) $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.

Answer. If $a \neq 0$ and $b \neq 0$ then $ab \neq 0$. $\square$

(c) $\triangle BAC$ is a right triangle $\Rightarrow a^2 + b^2 = c^2$.

Answer. If $a^2 + b^2 \neq c^2$ then $\triangle BAC$ is not a right triangle. $\square$

6. Fix an integer $x \neq 1$. Use induction to prove the formula:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Proof. Base case. For $n = 1$, the LHS of the formula is $1 + x$. The RHS is

$$\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{(x-1)} = 1 + x.$$

But this is just the LHS, so the formula is true.

Inductive step. Assume for induction that the result is true for n. Then for $n + 1$, the LHS of the formula is $1 + x + x^2 + \cdots + x^{n+1}$. Using the inductive hypothesis, we have

$$\begin{aligned} 1 + x + x^2 + \cdots + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1 + x^{n+1}(x-1)}{x - 1} \\ &= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1} \\ &= \frac{x^{n+2} - 1}{x - 1}. \end{aligned} \tag{0.1}$$

But this is just the RHS of the formula for $n+1$. This completes the inductive step.

By the principle of mathematical induction, we conclude that the formula is true for all $n \in \mathbb{N}$. $\square$

7. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. ($\subset$) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and x is either in B or C.

Case 1. If $x \in B$ then $x \in A$ and $x \in B$, so $x \in A \cap B$. Therefore $x \in (A \cap B) \cup (A \cap C)$.

Case 2. If $x \in C$ then $x \in A$ and $x \in C$, so $x \in A \cap C$. Therefore $x \in (A \cap B) \cup (A \cap C)$.

($\supset$) Let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$.

Case 1. If $x \in A \cap B$ then $x \in A$ and $x \in B$, so $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.

Case 2. If $x \in A \cap C$ then $x \in A$ and $x \in C$, so $x \in C \cup B = B \cup C$. Therefore $x \in A \cap (B \cup C)$. $\square$

8. Give a counterexample to one of the following four formulas for images and inverse images of sets (the other three are true):

$$
\begin{aligned}
f(A \cup B) &= f(A) \cup f(B), \qquad & f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\
f(A \cap B) &= f(A) \cap f(B), \qquad & f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B).
\end{aligned}
$$

Answer. If we try to prove all of them, we will succeed except in one case. Let's go ahead and do that.

$\underline{f(A \cup B) = f(A) \cup f(B)}$. ($\subset$) Given $y \in f(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$. Then either $x \in A$ or $x \in B$, so $f(x) \in f(A)$ or $f(x) \in f(B)$, i.e. $y = f(x) \in f(A) \cup f(B)$.

($\supset$) Given $y \in f(A) \cup f(B)$, there either exists $x \in A$ or $x \in B$ such that $f(x) = y$. But in either case, $x \in A \cup B$, so $f(x) = y \in f(A \cup B)$.

$\underline{f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)}$. ($\subset$) Given $x \in f^{-1}(A \cup B)$, we know that $f(x) \in A \cup B$, *i.e.* $f(x) \in A$ or $f(x) \in B$. But this just says that $x \in f^{-1}(A) \cup f^{-1}(B)$.

($\supset$) Given $x \in f^{-1}(A) \cup f^{-1}(B)$, we have $f(x) \in A$ or $f(x) \in B$, *i.e.* $f(x) \in A \cup B$. Therefore $x \in f^{-1}(A \cup B)$.

$\underline{f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)}$. ($\subset$) If $x \in f^{-1}(A \cap B)$ then $f(x) \in A \cap B$, *i.e.* $f(x) \in A$ and $f(x) \in B$. Therefore $x \in f^{-1}(A) \cap f^{-1}(B)$.

($\supset$) If $x \in f^{-1}(A) \cap f^{-1}(B)$, then $f(x) \in A$ and $f(x) \in B$. But then $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$.

$\underline{f(A \cap B) = f(A) \cap f(B)}$. ($\subset$) Let $y \in f(A \cap B)$. Then there exists $x \in A \cap B$ such that $f(x) = y$. But then $f(x) \in f(A)$ and $f(x) \in f(B)$, so $y = f(x) \in f(A) \cap f(B)$.

($\supset$) Given $y \in f(A) \cap f(B)$, there exist $a \in A$ and $b \in B$ such that $f(a) = y = f(b)$. But we need to find $x \in A \cap B$ such that $f(x) = y$. And such an x need not exist!

For instance, if $A \cap B = \varnothing$, then such an x can't exist. So for our counterexample, let's take

$$A = \{*\}, \quad B = \{\#\}, \quad C = \{\dagger\}.$$

Define the function

$$\begin{aligned} f : A \cup B &\to C \\ * &\mapsto \dagger \\ \# &\mapsto \dagger. \end{aligned} \tag{0.2}$$

Then $f(A) = f(B) = \{\dagger\} = C$, so $f(A) \cap f(B) = C$. But $A \cap B = \varnothing$, so $f(A \cap B) = \varnothing$. Therefore $f(A \cap B) \not\supseteq f(A) \cap f(B)$. □

9. (Extra credit +1) Give an example of an injective function $\tilde{S} : \mathbb{N} \hookrightarrow \mathbb{N}$ such that $\tilde{S}(\mathbb{N}) = \mathbb{N} \smallsetminus \{1\}$ but for which the 3rd Peano axiom fails; i.e., there exists a subset $A \subset \mathbb{N}$ such that $1 \in A$ and $x \in A \Rightarrow \tilde{S}(x) \in A$, but $A \neq \mathbb{N}$.

   Answer. We can take

$$\tilde{S}(n) = \begin{cases} 3 & n = 1 \\ 2 & n = 2 \\ n+1 & n \geq 3. \end{cases}$$

   The statement fails for $A = \mathbb{N} \smallsetminus \{2\}$. □