Problem set 10 solutions

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1. This follows from the argument used in proving Theorem 6.11, as follows.

   Let $\phi(x) = \sqrt[p]{x}$. Since f is assumed to be in $\mathscr{R}(\alpha)$, it is bounded by $M \geq 0$. The function $\phi(x)$ is continuous on $[0, M]$, hence uniformly continuous.

   Let $\varepsilon > 0$. Since ϕ is uniformly continuous, there exists $\delta > 0$ with $\delta < \varepsilon$ such that $|s-t| \leq \delta \Rightarrow |\phi(s) - \phi(t)| < \varepsilon$. Since $\int |f|^p \, d\alpha = 0$, there exists a partition $P = \{x_0, \ldots, x_n\}$ such that
$$U(P, |f|^p, \alpha) < \delta^2.$$

   Let M_i be as usual for f^p, and let M_i^* be as usual for $\phi \circ f^p = f$. Divide the numbers $\{1, \ldots, n\}$ into two classes: $i \in A$ if $M_i < \delta$ and $i \in B$ if $M_i \geq \delta$.

   For $i \in A$, we have $M_i^* \leq \varepsilon$ by choice of δ.

   For $i \in B$, we still have $M_i^* \leq \sqrt[p]{M}$. By choice of our partition, we have
$$\sum_{i \in B} \delta \Delta\alpha_i \leq \sum_{i \in B} M_i \Delta\alpha_i < \delta^2.$$

   Cancelling δ, we get
$$\sum_{i \in B} \Delta\alpha_i < \delta.$$

   Combining these estimates, we have
$$\begin{aligned} U(P, |f|, \alpha) &= \sum_{i \in A} M_i^* \Delta\alpha_i + \sum_{i \in B} M_i^* \Delta\alpha_i. \\ &\leq \varepsilon\left(\alpha(b) - \alpha(a)\right) + K\delta < \varepsilon\left(\alpha(b) - \alpha(a) + K\right). \end{aligned}$$

   Since $\varepsilon > 0$ was arbitrary, this implies $\overline{\int}_a^b |f| \, d\alpha \leq 0$. But $|f| \geq 0$, so $\underline{\int}_a^b |f| \geq 0$. It follows that $\int |f| \, d\alpha = 0$.

2. (Rudin 6.12) Let M be an upper bound for $|f|$ on $[a, b]$. Choose a partition P such that $\sum (M_i - m_i)\Delta\alpha_i < \varepsilon/2M$. Note that the function in the hint satisfies $g(x_i) = f(x_i)$ for all i, is continuous, and attains its infima/suprema at the endpoints of each

interval. Hence, on each interval $[x_{i-1}, x_i]$, we have $m_i \le g(x) \le M_i$. In particular, $\sup_{x_{i-1}\le x\le x_i}(f(x)-g(x))^2 \le (M_i - m_i)^2$. We can now calculate:

$$\sum \sup_{x_{i-1}\le x\le x_i}(f(x)-g(x))^2\Delta\alpha_i \le \sum(M_i-m_i)^2\Delta\alpha_i \le 2M\sum(M_i-m_i)\Delta\alpha_i < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, if follows that $\int(f-g)^2\Delta\alpha_i = 0$, as desired.

3. (Rudin 6.15) Using integration by parts, we have

$$1 = \int_a^b f^2(x)dx = xf(x)^2\Big|_a^b - 2\int xf'(x)f(x)dx.$$

The first term vanishes since $f(a) = 0 = f(b)$. This gives the first identity. For the second, we apply Cauchy-Schwarz:

$$\frac{1}{4} = \left(\int xf'(x)f(x)dx\right)^2 \le \int (f'(x))^2\,dx \int x^2 f(x)^2\,dx.$$

Note that strict inequality holds if and only if the two pieces in Cauchy-Schwarz were equal up to a constant, i.e.

$$f'(x) = xf(x).$$

Letting $y = f(x)$, this comes to

$$\frac{dy}{y} = xdx.$$

The general solution is

$$y = f(x) = Ce^{x^2/2}.$$

But this only satisfies $f(a) = f(b) = 0$ if $C = 0$, which contradicts the assumption that $\int f^2dx = 1$. Hence in fact strict inequality holds.

4. (Rudin 6.16) (a) Since $[x] = \sum_{n=1}^{[x]} I(x-n)$, we have

$$\begin{aligned} s\int_1^N \frac{[x]}{x^{s+1}}\,dx &= s\sum_{n=1}^{n}\int_n^N \frac{1}{x^{s+1}}\,dx \\ &= \sum_{n=1}^{N}\left(\frac{1}{n^s} - \frac{1}{N^s}\right). \end{aligned}$$

Therefore

$$\zeta(s) - s\int_1^N \frac{[x]}{x^{s+1}}\,dx = N^{1-s}.$$

Since $s > 1$, this difference tends to zero as $N \to \infty$, which proves the identity.

(b) The identity is equivalent to

$$\frac{1}{s-1} = \int_1^\infty \frac{1}{x^s}\,dx,$$

which is clear. Note that $0 \le x - [x] \le 1$, hence the integrand is nonnegative and it suffices to check that the partial integrals are bounded above. (Since $\lim_{y\to\infty} \int_1^y \frac{x-[x]}{x^{s+1}}\,dx = \sup_y \int_1^y \frac{x-[x]}{x^{s+1}}\,dx$.) We have

$$\int_1^y \frac{x-[x]}{x^{s+1}}\,dx \le \int_1^y \frac{1}{x^{s+1}}\,dx = \frac{1}{s}\left(1 - \frac{1}{y^s}\right),$$

which is indeed bounded above.

5. (Rudin 7.1) Since f_n is uniformly convergent, for N sufficiently large, we have

$$\|f_N - f_n\|_{C^0} < 1.$$

Since f_N is bounded, we have $\|f_N\|_{C^0} \le M$. This gives

$$\|f_n\|_{C^0} \le \|f_N - f_n\|_{C^0} + \|f_N\|_{C^0} \le 1 + M$$

for $n \ge M$. As our upper bound, we can therefore take

$$\max\{\|f_1\|_{C^0}, \|f_2\|_{C^0}, \dots, \|f_{N-1}\|_{C^0}, M+1\}.$$

6. (Rudin 7.3) Let $E = [0,1]$. We can take $f_n = \frac{1}{x}$ for all n, and $g_n = \frac{1}{n}$. Then $f_n g_n = \frac{1}{nx}$, which converges to zero pointwise but not uniformly since $f_n g_n(\frac{1}{n}) = 1$.

7. (Rudin 7.4) The series converges absolutely (pointwise) for $x \ne 0$, since it is dominated by $\frac{1}{x}\sum \frac{1}{n^2}$.

   It converges uniformly for any interval of the form $[a,b]$, where both a and b are either positive or negative. This follows by the Weierstrass M-test with the series $\frac{1}{\min\{|a|,|b|\}}\sum \frac{1}{n^2}$.

   The series fails to converge uniformly on $(0,b]$ or $[a,0)$, since it is unbounded. (If the series converged uniformly, since the terms are uniformly bounded, $f(x)$ would be bounded by Exercise 7.1.) Indeed, given $N \in \mathbb{N}$, we have

$$\lim_{x\to 0}\sum_{n=1}^{N} \frac{1}{1+n^2x} = \sum_{n=1}^{N} \lim_{x\to 0} \frac{1}{1+n^2x} = N,$$

   since finite sums can always be exchanged with limits. But we have

$$N = \lim_{x\to 0}\sum_{n=1}^{N} \frac{1}{1+n^2x} \le \lim_{x\to 0}\sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

   Since N was arbitrary, this shows that $\lim_{x\to 0} f(x) = +\infty$, so $f(x)$ cannot be bounded.

   The function $f(x)$ is continuous where the series converges, i.e. for $x \ne 0$, because any such x is contained in one of the intervals mentioned above, on which the series converges uniformly to a continuous limit.

8. (Rudin 7.5) The sequence converges pointwise to zero since $f_n(x)$ is identically zero for $x \le 0$, and for $x > 0$, $f_n(x) = 0$ provided that $n > \frac{1}{x}$. However, we have

$$f_n(\frac{1}{n+\frac{1}{2}}) = \sin^2((n+\frac{1}{2})\pi) = 1,$$

so the convergence is not uniform.

We are also asked to show that $\sum f_n$ converges pointwise (absolutely) but not uniformly. For $x \le 0$, we have $\sum f_n(x) = 0$, and for $x > 0$, the sum is zero for all but one term, so is absolutely convergent. However, the series cannot be uniformly convergent since the terms do not tend to zero in the uniform norm.