# Midterm 2 Solutions

Math 521 Section 001, UW-Madison, Spring 2024

April 6, 2024

Each item is worth 5 points, for a total of 45.

1. The space $X$ is compact if every open cover of $X$ admits a finite subcover.

2. (a) The sequence $p_n \to p$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(p_n, p) < \varepsilon$.

   (b) Let $\varepsilon > 0$. Since $p_n \to p$, there exists $N_1$ such that $n \geq N_1 \Rightarrow d(p_n, p) < \varepsilon/2$. Since $p_n \to q$, there exists $N_2$ such that $n \geq N_1 \Rightarrow d(p_n, q) < \varepsilon/2$. Hence, for $n \geq N = \max\{N_1, N_2\}$, by the triange linequality, we have

   $$d(p,q) \leq d(p,p_n) + d(p_n,q) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   Since $\varepsilon > 0$ was arbitrary and $d(p,q) \geq 0$, we must have $d(p,q) = 0$. This implies that $p = q$, as desired.

   (c) It has a convergent subsequence.

3. (a) Let $t_n = \sum_{k=n}^{2n} a_k$. We claim that $t_n \to 0$. Given $\varepsilon > 0$, by the Cauchy criterion for series, there exists $N$ such that $m \geq n \geq N$ implies $|\sum_{k=n}^{m} a_k| < \varepsilon$. Applying this with $m = 2n$, we obtain $|t_n| = |t_n - 0| < \varepsilon$ for $n \geq N$. Since $\varepsilon > 0$ was arbitrary, we are done.

   (b) The series $\sum(-1)^n a_n/n$ need not converge. For a counterexample, we take $a_n = (-1)^n/\log n$ for $n \geq 2$. Since $\log n$ is monotonically increasing and tends to infinity, $1/\log n$ is monotonically decreasing and tends to zero. By the alternating series test (on HW), the series $\sum a_n$ converges. However, $\sum(-1)^n a_n/n = \sum 1/(n \log n)$ diverges, by class / textbook.

4. (a) Let $q \in X$. We shall use the following variant of the triangle inequality:

   $$|d(p,x) - d(p,q)| \leq d(x,q). \tag{0.1}$$

   To prove the inequality, first note that

   $$d(p,x) \leq d(p,q) + d(q,x) = d(p,q) + d(x,q)$$

by the triangle inequality and symmetry of $d(\cdot,\cdot)$. Subtracting $d(p,q)$ from both sides, we obtain

$$d(p,x) - d(p,q) \le d(x,q). \tag{0.2}$$

We also have

$$d(p,q) \le d(p,x) + d(x,q).$$

Subtracting $d(p,x)$ from both sides, we get

$$d(p,q) - d(p,x) \le d(x,q). \tag{0.3}$$

Combining the two inequalities (0.2-0.3), we obtain (0.1).

Now, to show continuity, let $\varepsilon > 0$. We take $\delta = \varepsilon$. Then for $d(q,x) < \delta$, we have

$$|f(x) - f(q)| = |d(p,x) - d(p,q)| \le d(x,q) < \delta = \varepsilon,$$

by (0.1).

(b) The *extreme value theorem:* given a continuous function $f : X \to \mathbb{R}$, where $X$ is compact, $f$ is bounded and attains its supremum, i.e. there exists $q \in X$ such that

$$f(q) = \sup_{x \in X} f(x).$$

(c) Let $r = d(p,q) = \sup_{x\in X} d(p,x)$. We claim that $f(X) = [0,r]$.

Since $p \in X$, we know that $f(p) = d(p,p) = 0$, hence $0 \in f(X)$. By (b), we know that $r = f(q) \in f(X)$. But $f$ is continuous, by (a), and since $X$ is connected, $f(X)$ is also connected. By a theorem from class/textbook, for any $0 < c < r$, we must have $c \in f(X)$. Hence $[0,r] \subset f(X)$. Meanwhile, $f(X)$ contains only nonnegative numbers less than or equal to $r = \sup_{x\in X} f(x)$, so clearly $f(X) \subset [0,r]$. We are done.