Recall:

Theorem. Let X be a metric space.

(a) For any collection $\{G_\alpha\}_{\alpha\in I}$ of open sets,

$$\bigcup_{\alpha\in I} G_\alpha \quad \text{is open.}$$

(b) For any collection $\{F_\alpha\}_{\alpha\in I}$ of closed sets,

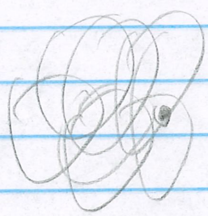
$$\bigcap_{\alpha\in I} F_\alpha \quad \text{is closed.}$$

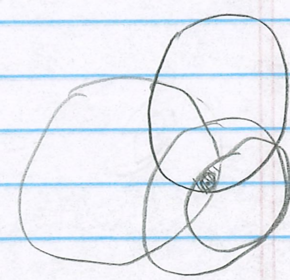
(c) For a finite collection $G_1, \dots, G_n$ of open sets,

$$\bigcap_{i=1}^{n} G_n \quad \text{is open.}$$

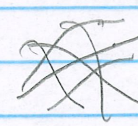
(d) For a finite collection $F_1, \dots, F_n$ of closed sets,

$$\bigcup_{i=1}^{n} F_i \quad \text{is closed.}$$

(a) 

(b)  ✓

(c) 

(d) 

$$\Big(\bigcap_{\alpha\in I} F_\alpha\Big)^c = \bigcup_{\alpha\in I} \overbrace{F_\alpha^c}^{\text{open}}$$

open

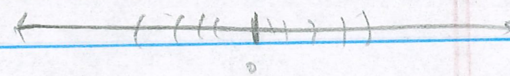
$$\Big(\bigcup_{i=1}^{n} F_i\Big)^c = \bigcap_{i=1}^{n} \overbrace{F_i^c}^{\text{open}}$$

open.

Counterexamples when "finite" is dropped from (c) or (d).

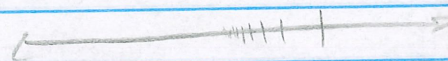
$X = \mathbb{R}$.

- Let $G_i = (-\frac{1}{i}, \frac{1}{i})$, open.



$$\bigcap_{i=1}^{\infty} G_i = \{0\}, \text{ \underline{not} open!}$$

- Let $F_i = \{\frac{1}{i}\}$, closed.



$$\bigcup_{i=1}^{\infty} F_i = \{\tfrac{1}{n} \mid n \in \mathbb{N}\}, \text{ \underline{not} closed!}$$

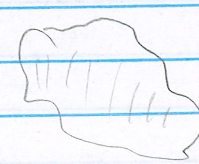
Note: Since points are closed, every subset $E \subset X$ is a union of closed sets!

$$E = \bigcup_{x \in E} \{x\}.$$

Question: Can you "make" a set open or closed?

Recall: $\mathring{E} = \{\text{interior points of } E\} \subset E$.

"interior" of E.



$\mathring{E} \subset E$ by definition.

$\mathring{E} = E \iff E$ is open, by definition.

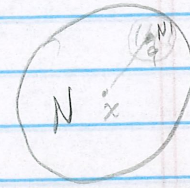
(Note: can have $\mathring{E} = \emptyset$!)

<u>Theorem</u>: (a) $\mathring{E}$ is open

(b) For every open set $G \subset E$, $G \subset \mathring{E}$.

<u>Pf</u>: Let $x \in \mathring{E}$.

Have $N_r \ni x$ s.t. $N_r(x) \subset E$.



By last class, N is open.

$\rightsquigarrow$ given $q \in N$, $\exists\ N' \ni q$ s.t. $N' \subset N$.

$\Rightarrow N' \subset E$.

$\Rightarrow q \in \mathring{E}$.

Since $q \in N$ was arb., $N \subset \mathring{E}$

$\Rightarrow \mathring{E}$ is open ✓

(b) Let $x \in G$.

Since $G \subset E$ is open, $\exists\ N \ni x$ s.t. $N \subset G \subset E$

$\Rightarrow x \in \mathring{E}$.

Since $x \in G$ was arb., $G \subset \mathring{E}$. $\square$

<u>Note</u>: (b) says that $\mathring{E}$ is the <u>largest</u> open set contained in E.

<u>Example</u>: $\mathring{[0,1)} = (0,1)$, $\{\frac{1}{n} \mid n \in \mathbb{Z}\}^{\circ} = \emptyset$.

<u>Definition:</u> The <u>closure</u> of E is given by $\overline{E} := E \cup E'$.

By defn, $E \subset \overline{E}$.

<u>Theorem:</u> (a) $\overline{E}$ is closed

(b) For every closed set F s.t. $E \subset F$, $\overline{E} \subset F$.

<u>Note:</u> (b) says that $\overline{E}$ is the <u>smallest</u> closed set containing E.

<u>Pf:</u> (a) <u>Claim:</u> $(\overline{E})^c = (E^c)^\circ$.

$x \notin E$ not a lim. pt. of $E \iff \exists\ N \ni x$ s.t. $N \cap E = \emptyset$, i.e. $N \subset E^c$.

$\iff x \in (E^c)^\circ$. ✓

(b) $F \supset E \Rightarrow F' \supset E'$

But F closed $\Rightarrow F \supset F'$

$\therefore F \supset E' \cup E = \overline{E}$ $\square$

Theorem

Theorem: Let $E \subset \mathbb{R}^{x}$ be nonempty & bdd above.

Then $\sup E \in \bar{E}$.

Pf: If $\sup E \in E$, done.

If $\sup E \in E' \setminus E$, then $\forall h > 0$

$\exists$ $x \in E$ with $\sup E - h < x < \sup E$.

$\Rightarrow$ $x \in N_h(\sup E)$, $x \neq \sup E$.

Since $h > 0$ was arb., $\sup E \in E' \subset \bar{E}$. $\square$