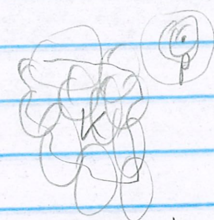
<u>Recall</u>: $K \subset X$ <u>compact</u> if for any open cover

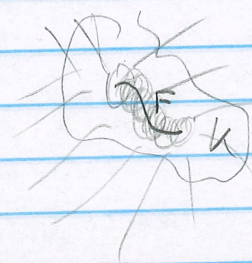
$\{U_\alpha\}$ of K, $\exists$ finite subcover

$U_{\alpha_1}, \dots, U_{\alpha_n}$

<u>Thm</u>: Cpt $\Rightarrow$ closed.

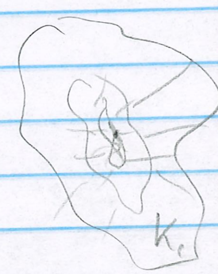


<u>Thm</u>: Closed subsets of cpt sets are cpt.



<u>Thm</u>: $\{K_n\}$ coll. of cpt sets with $K_n \supset K_{n+1}$, $\forall n$

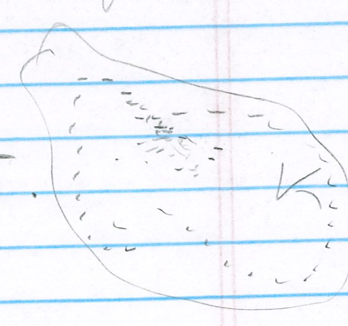
$\Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$



<u>Today</u>: More thms abt compactness.

<u>Theorem</u>: If E is an infinite subset of a compact set K, then E has a limit point in K.

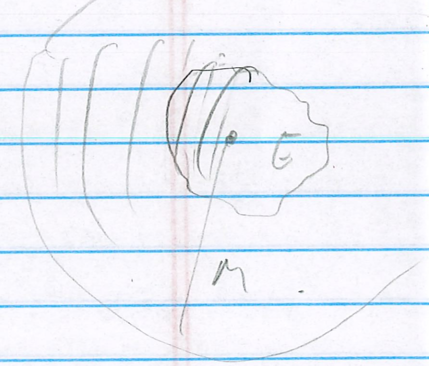
<u>Pf</u>: Suppose, on the contrary, that no point of K is a limit pt. of E.



Then for each $q \in K$, $\exists$ nbhd $V_q \ni q$
containing at most one point of E.
Since E is infinite, $\{V_q\}_{q \in K}$ has no finite subcover of E.

$\Rightarrow$ no finite subcover of K. ⨳ $\square$

<u>Definition</u>: $E \subset X$ is <u>bounded</u> if $\exists$ $p \in E$ and $M > 0$ s.t. $d(p, x) \leq M$ $\forall$ $x \in E$.



<u>Theorem</u>: Compact sets are <u>bounded</u>.

<u>Pf</u>: Let $p \in K$, cpt.
Consider $V_n = N_n(p)$, for $n \in \mathbb{N}$.
Since $d(p, x) < \infty$ for all $x \in E$, $x \in V_n$ for n suff. large.

$\Rightarrow$ $\{V_n\}$ is an o.c. of E.

$\Rightarrow$ has finite subcover

$V_{n_1}, \ldots, V_{n_k}$

Let $m = \max \{ n_i \}_{i=1}^{k}$.

$\Rightarrow V_{n_i} \subset V_m \;\; \forall i$

$\Rightarrow E \subset \cup V_{n_i} \subset V_m$

$\Rightarrow d(p,x) < m \;\; \forall x \in E$ $\square$

Have shown:

Compact $\Longrightarrow$ I Closed

II Bounded

III Every infinite subset has a limit pt.

?

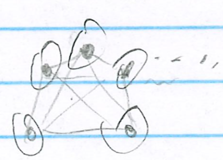
$\Longleftarrow$

Rudin ex. 2.10.

Example: Z = any infinite set.

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

"discrete metric".



Saw on HW: Every subset $E \subset \mathbb{Z}$ is both closed & open. (b/c $N_{\frac{1}{2}}(p) = \{p\}$).

Which sets $E \subset \mathbb{Z}$ are bounded?

$$M = 2 : \quad d(x,y) = \begin{cases} 0 \\ 1 \end{cases} \leq 2 \quad \forall x, y \in \mathbb{Z}$$

$\Rightarrow$ any set $E \subset \mathbb{Z}$ is bounded.

Every set $E \subset \mathbb{Z}$ is closed, open, and bounded.

However: $K \subset \mathbb{Z}$ is compact

$\iff$ K is finite.

Pf: ($\Rightarrow$) Suppose K is infinite. Then $\{N_{\frac{1}{2}}(p)\}_{p \in E}$ is an open cover, but each open contains only one pt.

$\Rightarrow$ no finite subcover.

($\Leftarrow$) class last Fri.

So: (I) closed & (II) bdd $\not\Rightarrow$ compact

in general metric spaces X. We will see that is true for $X = \mathbb{R}^n$! This is called the Heine-Borel Theorem.

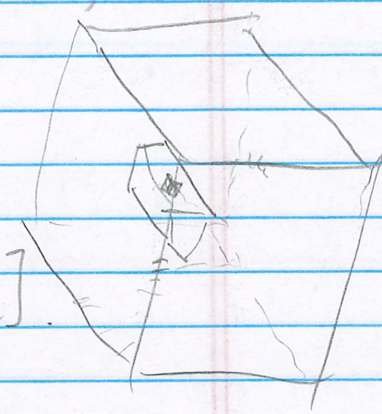
Also: it is true that (III) $\Rightarrow$ compact! (Honors problem)

Compactness in $X = \mathbb{R}^k$.

Recall:

Thm: If $\{I_n\}_{n=1}^{\infty}$ is a sequence of nested k-cells in $\mathbb{R}^k$, $I_n \supset I_{n+1}$ $\forall n \in \mathbb{N}$, then

$$\bigcap_{n=1}^{\infty} I_n \neq \phi.$$



Pf: Write

$$I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}].$$

$$I_n \supset I_{n+1} \Rightarrow a_{n,j} \leq a_{n+1,j} \leq \cdots$$

$$b_{n,j} \geq b_{n+1,j} \geq \cdots$$

Let $x_j^* = \sup_n a_{n,j}$.

Saw last time: $x_j^* \in \bigcap_{n=1}^{\infty} I_{n,j}$

$$\Rightarrow x^* = (x_1^*, \ldots, x_k^*) \in \bigcap_{n=1}^{\infty} I_n \quad . \quad \square$$

So k-cells behave a lot like cpt sets.

In fact:

Theorem: Every k-cell is compact.

Pf: Let $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$.

Let $\delta = \left(\sum_{i=1}^{k} (b_i - a_i)^2 \right)^{1/2}$

For $x, y \in I$, $|x-y| = \left(\sum_{i=1}^{k} (x_i - y_i)^2 \right)^{1/2} \leq \delta$.

Suppose, for contradiction, that there exists an open cover $\{G_\alpha\}$ of I with no finite subcover.

Let $c_j = \frac{(a_j + b_j)}{2}$.

Divide I into 2^k k-cells with sides

$[a_j, c_j]$, $[c_j, b_j]$ etc.

At least one of these cannot be covered by a finite subcollection of $\{G_\alpha\}$

(otw I could.)

Call this k-cell I_1.

Have $|x-y| \leq \frac{1}{2}\delta$ for all $x, y \in I_1$

Next, subdivide I_1, and continue the process to obtain:

(a) $I \supset I_1 \supset I_2 \supset \cdots$, nested k-cells

(b) I_n is not covered by fin. many G_α's

(c) $x, y \in I_n \Rightarrow |x-y| \leq 2^{-n}\delta$.

By (a) and previous theorem,

$$\bigcap_{n=1}^{\infty} I_n \ni x^*.$$

Now $x^* \in G_\alpha$ for some α.

But since G_α is open, $\exists\, r > 0$ s.t.

$N_r(x^*) \subset G_\alpha$.

Choose n s.t. $2^{-n}\delta < r$.

$y \in I_n \Rightarrow |x^* - y| < 2^{-n}\delta < r$

$\Rightarrow y \in N_r(x^*) \subset G_\alpha$.

$\Rightarrow I_n \subset G_\alpha$, which contradicts (b). $\square$