Last time: $\mathbb{R}$ is uncountable.

Today: Metric spaces.

Definition. Let X be a set. A **metric** on X is a function $d : X \times X \to \mathbb{R}$ satisfying

(a) $d(p,q) > 0$ if $p \neq q$, $d(p,p) = 0$. (nonnegativity)

(b) $d(p,q) = d(q,p)$. $\forall p,q \in X$ (symmetry)

(c) $d(p,q) \leq d(p,r) + d(r,q)$ $\forall p,q,r \in X$ (triangle ineq.)

A set X with a metric d is called a **metric space**.

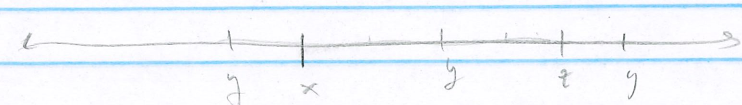
Elements $p \in X$ are called points.

Example: $X = \mathbb{R}$, $d(x,y) = |x-y|$.

(a) $x \neq y \Rightarrow x < y$ or $y < x \Rightarrow |x-y| > 0$.

(b) $|x-y| = |y-x|$ ✓.

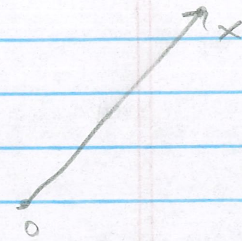
(c) $$|x-z| = \begin{cases} |x-y| + |y-z| & x \leq y \leq z \\ |z-y| - |x-z| & y < x \\ |y-x| - |y-z| & z < y \end{cases}$$



$$\leq |x-y| + |y-z| \quad ✓$$

<u>Recall:</u> $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}\times\dots\times\mathbb{R}$.

<u>Inner product:</u> $x \cdot y = \sum_{i=1}^{n} x_i y_i$



<u>Norm:</u> $|x| = \sqrt{x \cdot x}$

$= \sqrt{\sum_{i=1}^{n} x_i^2}$.

<u>Addition:</u> $x + y = (x_1 + y_1, x_2 + y_2, \dots)$

<u>Scalar mult:</u> $\alpha \in \mathbb{R}$, $\alpha x = (\alpha x_1, \dots, \alpha x_n)$.

<u>Theorem from Ch. 1.</u> $x, y, z \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

(a) $|x| \geq 0$, $|x| = 0 \iff x = 0$.

(b) $x \cdot y = y \cdot x$

(c) $|\alpha x| = |\alpha| |x|$.

(d) $|x + y| \leq |x| + |y|$

(e) $|x - z| \leq |x - y| + |y - z|$.

<u>Pf.</u> (a) $\sum x_i^2 = 0 \iff x_i = 0 \ \forall i$.

(b) obv.

(c) $|\alpha x| = \sqrt{\sum (\alpha x_i)^2} = \sqrt{\alpha^2 \sum x_i^2}$

$= |\alpha| \sqrt{\sum x_i^2}$.

(d) This is Cauchy-Schwarz!

(e) $|x+y|^2 = (x+y)\cdot(x+y)$

$= |x|^2 + 2x\cdot y + |y|^2$

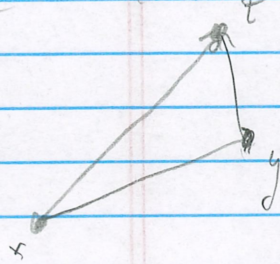
$\overset{(d)}{\le} |x|^2 + 2|x||y| + |y|^2 = (|x|+|y|)^2$. $\square$

(f) $|a+b| \le |a|+|b|$.

Let $a = x-y$, $b = y-z$.

$a+b = x-z$. $\square$

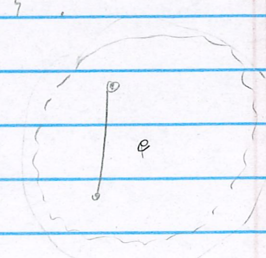
Definition: The *standard metric* on $\mathbb{R}^n$ is given by $d(x,y) = |x-y|$.



(a), (b), (f) $\Rightarrow$ d is a metric!

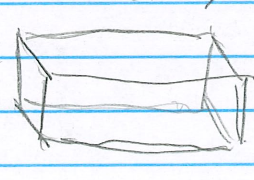
Defn. The open *ball* of radius r centered at $x \in \mathbb{R}^n$ is

$$B_r(x) = \{y \in \mathbb{R}^n \mid d(x,y) < r\}.$$



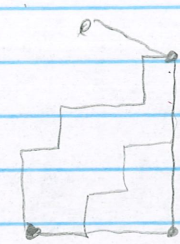
Defn: Let $a_i < b_i$ for $i = 1, \ldots, n$. An *n-cell* is the cartesian product

$$[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n] = \{(x_i) \mid a_i \le x_i \le b_i,\ x_i \in \mathbb{R}\} \subset \mathbb{R}^n$$



Example: The <u>taxicab metric</u> on $\mathbb{R}^n$

is given by $d_T(x,y) = \sum_{i=1}^{n} |y_i - x_i|$.



$(x-y)$ ✓

$$d_T(x,z) = \sum_{i=1}^{n} |z_i - x_i|$$
$$= \sum_{i=1}^{n} |z_i - y_i + y_i - x_i|$$
$$\leq \sum_{i=1}^{n} \left(|z_i - y_i| + |y_i - x_i|\right)$$
$$= d_T(x,y) + d_T(y,z) \quad ✓$$

How does the "ball" of radius r look?

$$B_{(d_T, r)}(x) = \{ y \in \mathbb{R}^n \mid \sum |x_i - y_i| < r \}.$$