Problem set 2 solutions

Math 521 Section 001, UW-Madison, Spring 2024

February 7, 2024

Please solve the following problems in a clear, complete, and concise manner. You are welcome to work together, but your write-up must be your own. Use of outside internet resources is prohibited.

Due on paper at the beginning of class on **Wednesday, Feb. 7th**. Please be sure to staple your writeup.

1. Rudin problems:

   1.1) If r and $r + x$ were both rational, then $x = (r + x) - r$ would also be rational. Similarly, if rx were rational, then $x = \frac{rx}{r}$ would also be rational. □

   1.2) Since $\sqrt{12} = 2\sqrt{3}$, by the previous problem, we need only prove that $\sqrt{3}$ is irrational.

   Suppose, for the sake of contradiction, that m and n are integers such that $\left(\frac{m}{n}\right)^2 = 3$. By cancelling common factors, we may assume that m and n are not both divisible by 3. By the assumption, we have

   $$m^2 = 3n^2.$$

   Therefore m^2 is divisible by 3, and so m must also be divisible by 3. Hence $m = 3m'$, and the above equation becomes

   $$9(m')^2 = 3n^2.$$

   Dividing both sides by 3, we have

   $$3(m')^2 = n^2.$$

   Hence n^2 is divisible by 3, and so n must be divisible by 3 as well. But this contradicts our assumption. □

1.4) Since E is nonempty, there exists at least one element $x \in E$. Since α is a lower bound, we have $\alpha \leq x$. Since β is an upper bound, we have $x \leq \beta$. By property (ii) of an ordered set, the chain of inequalities

$$\alpha \leq x \leq \beta$$

implies that $\alpha \leq \beta$, as desired. □

1.5) First note that $-A$ is bounded above, for the following reason: if β is any lower bound of A, then $\beta \leq x$ for all $x \in A$. But then $-x \leq -\beta$ for $x \in A$. Since $x \in A$ was arbitrary, we have $y \leq -\beta$ for all $y \in -A$. This shows that $-A$ is bounded above.

Let $\alpha = -\sup(-A)$. We need to show that α is the greatest lower bound of A. There are two properties to check:

(i) α is a lower bound. Since $-\alpha = -(-\sup(-A)) = \sup(-A)$ is an upper bound for $-A$, we have $y \leq -\alpha$ for all $y \in -A$. Rearranging, we have

$$\alpha \leq -y.$$

But for any $x \in A$, we may take $y = -x$. We then have

$$\alpha \leq -y = -(-x) = x$$

as desired.

(ii) If $\gamma > \alpha$ then γ is not a lower bound. We have

$$-\gamma < -\alpha.$$

Since $-\alpha = \sup(-A)$, by (ii) in the definition of sup, $-\gamma$ is not an upper bound for $-A$. Hence there exists $y \in -A$ such that

$$-\gamma < y.$$

But $y = -x$ for some $x \in A$. Hence

$$-\gamma < -x$$

and

$$x < \gamma.$$

Therefore γ is not a lower bound for A, as desired. □

2. Use the axioms for an ordered field to prove the following. Please indicate which axiom(s) you are using in each step of the proof. (For all questions after this one, you do not need to indicate which axioms you are using.)

(a) If $ax = a$ for some $a \neq 0$, then $x = 1$.

(b) If $a < b$, then $-b < -a$.

(c) If $0 \leq a < b$, then $a^2 < b^2$.

(d) If $a > 1$, then for each $n \in \mathbb{N}$, $a^{n+1} > a^n$.

Solutions. (a) We have $ax = a = a \cdot 1$, by (M4). Since $a \neq 0$, by Proposition 1.15*b*, this implies $x = 1$.

(b) By axiom (i) for an ordered field, we may add $-a-b$ to both sides of $a < b$, to obtain

$$\begin{aligned} a + (-a-b) &< b + (-a-b) \\ (a-a) - b &< (b-b) + a \qquad (A3) \\ 0 - b &< 0 - a \qquad (A5) \\ -b &< -a \qquad (A4). \end{aligned}$$

(c) By Proposition 1.18*b*, we may multiply $0 \leq a < b$ by $a \geq 0$, to obtain

$$0 \leq a^2 \leq ab.$$

But we may also multiply $a < b$ by $b > 0$, to obtain

$$ab < b^2.$$

By the axiom (ii) for an ordered set, we may combine these inequalities, to obtain

$$0 \leq a^2 < b^2.$$

(d) (*First solution*) We use induction. For the base case ($n = 1$), since $a > 1 > 0$, we can multiply $a > 1$ by a (by Proposition 1.18*b*), to get

$$a^2 > a.$$

For induction, assume that $a^{n+1} > a^n$. Again, multiply by a (by Proposition 1.18*b*), to obtain

$$a^{n+2} > a^{n+1}.$$

This completes the induction.

(*Second solution*) We have $1 < a$ by assumption. Multiplying both sides by a^n, by Proposition 1.18*b*, we obtain

$$1 \cdot a^n < a \cdot a^n$$

and, by (M4),

$$a^n < a^{n+1},$$

as desired. □

3. Determine (with proof) which of the following subsets of $\mathbb{Q}$ are bounded above or bounded below. If the set is bounded above (resp. below), determine its supremum (resp. infimum).

(a) $A = \{p^2 \mid p \in \mathbb{Q}\}$

(b) $B = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.

(c) $C = \{p \in \mathbb{Q} \mid p^2 < 4\}$.

Solutions. (a) Since $\mathbb{N} \subset \mathbb{Q}$, we have $\{n^2 \mid n \in \mathbb{Z}\} \subset A$. By the proof from class, the former is unbounded above, so A is as well.

Since $\mathbb{Q}$ is an ordered field, we have $0 \le p^2$ for all $p \in \mathbb{Q}$ by Proposition 1.18*d*, so 0 is a lower bound. Hence A is bounded below. Since $0 \in A$, this must be the infimum.

(b) We have $n < n+1$ and $n+1 > 0$ for $n \in \mathbb{N}$. Dividing by $n+1$, we have

$$\frac{n}{n+1} < 1.$$

Hence 1 is an upper bound for B.

To show that 1 is the least upper bound, let $\gamma < 1$. By the Archimedean property, we may choose n such that $1 < (1-\gamma)(n+1)$, which gives

$$\frac{1}{n+1} < 1 - \gamma.$$

Rearranging, we have

$$\gamma < 1 - \frac{1}{n+1} = \frac{(n+1)-1}{n+1} = \frac{n}{n+1}.$$

But $\frac{n}{n+1} \in B$, so this implies that γ is not an upper bound, as desired.

Next, we claim that $\frac{1}{2}$ is a lower bound. Since $n \ge 1$, we have $n+1 \le 2n$. Inverting, we obtain

$$\frac{1}{2n} \le \frac{1}{n+1}.$$

Multiplying by n, we have

$$\frac{1}{2} = \frac{n}{2n} \le \frac{n}{n+1},$$

as desired. Since $\frac{1}{2} = \frac{1}{1+1} \in B$, this must be the greatest lower bound.

(c) We claim that $2 = \sup C$ and $-2 = \inf C$.

To show that 2 is an upper bound for C, we can prove the contrapositive, namely, that if $p > 2$ then $p^2 > 4$, i.e., $p \notin C$. But this is clear.

Similarly, to show that -2 is a lower bound, we may let $p < -2$. Then $2 < -p$ and $(-p)^2 < 4$. But then $p^2 = (-p)^2 \ge 4$, so $p \notin C$, which is the contrapositive.

To show that 2 is the least upper bound, let $\gamma < 2$. By density of $\mathbb{Q}$ in $\mathbb{R}$, we can choose a rational number p with $\gamma < p < 2$. But then we have $p^2 < 4$, so $p \in C$, hence γ is not an upper bound. The proof that -2 is the greatest lower bound is the same. □

4. Which of the following subsets of $\mathbb{Q}$ are Dedekind cuts? (No written justification required.)

   (a) $\{p \in \mathbb{Q} \mid p \le 1\}$ (No. (III) fails when $p = 2$.)

   (b) $\{p \in \mathbb{Q} \mid p^3 < 1\}$ (Yes. This is 1^*.)

(c) $\{p \in \mathbb{Q} \mid -1 < p < 1\}$. (No. (II) fails.)

(d) $\{p \in \mathbb{Q} \mid p < 2 \text{ and } p^2 \neq 2\}$. (Yes. This is 2^*.)

(e) $\{p \in \mathbb{Q} \mid p < 0 \text{ or } p^2 \leq 2\}$. (Yes. This is "$\sqrt{2}$.")

5. (Extra credit +1) Let α be a Dedekind cut. Define

$$\beta = \{p \in \mathbb{Q} \mid \exists r > 0 \text{ with } -p-r \notin \alpha\}.$$

Prove that β is also a Dedekind cut. What real number does β represent?

Solution. The Dedekind cut β represents $-\alpha$. See Rudin, p. 19. $\square$