Problem 1. Give a direct proof (without using Theorems 3.41-43) of the alternating series test: suppose that b_n is a sequence of nonnegative reals with $b_n \geq b_{n+1}$ for all $n \in \mathbb{N}$, and let $a_n = (-1)^n b_n$. Then $\sum a_n$ converges iff $b_n \to 0$.

Proof. Let $s_n := \sum_{k=1}^{n} (-1)^{k-1} b_k, n \in \mathbb{N}$. We claim that $(s_{2n})_{n=1,2,\dots}$ is an increasing sequence. Indeed, we have

$$s_{2n+2} - s_{2n} = (-1)^{2n} b_{2n+1} + (-1)^{2n+1} b_{2n+2} = b_{2n+1} - b_{2n+2} \geq 0.$$

Moreover, $(s_{2n+1})_{n=1,2,\dots}$ is a decreasing sequence, because

$$s_{2n+3} - s_{2n+1} = (-1)^{2n+1} b_{2n+2} + (-1)^{2n+2} b_{2n+3} = -b_{2n+2} + b_{2n+3} \leq 0.$$

Further, $s_{2n+1} - s_{2n} = (-1)^{2n} b_{2n+1} = b_{2n+1} \geq 0$, so $s_{2n} \leq s_{2n+1}$ for all $n \in \mathbb{N}$.

By Theorem 3.14, both sequences $(s_{2n})_{n=1,2,\dots}$ and $(s_{2n+1})_{n=1,2,\dots}$ converge. But

$$\lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} b_{2n+1} = 0.$$

Thus, there exists a real number s such that $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = s$. Consequently, $\lim_{n\to\infty} s_n = s$. This shows that the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

Note that $s_{2n} \leq s \leq s_{2n+1}$ for all $n \in \mathbb{N}$. It follows that

$$0 \leq s - s_{2n} \leq s_{2n+1} - s_{2n} = b_{2n+1} \quad \text{and} \quad 0 \leq s_{2n+1} - s \leq s_{2n+1} - s_{2n+2} = b_{2n+2}.$$

Thus we get the following error estimate:

$$|s - s_n| \leq b_{n+1} \quad \forall n \in \mathbb{N}.$$

□

Problem 2 (Rudin 3.6)**.**

Proof. (a) We have

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{2\sqrt{n+1}}.$$

By denoting $n+1 = k$, we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$, and the last series diverges by Theorem 3.28. Thus the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges, and so does $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}}$ by Theorem 3.47 (because if the last series converges, then the previous series also converges by Theorem 3.47. Now $\sum a_n$ diverges by the comparison test, Theorem 3.25.

(b) We have, following the calculation from part (a)

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{n(\sqrt{n+1} + \sqrt{n})} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Theorem 3.28. Now $\sum a_n$ converges by the comparison test, Theorem 3.25.

c) For $n \geq 1$, we can show that $0 \leq (\sqrt[n]{n} - 1) < 1$:

$$\begin{aligned}
&\mapsto (\forall n \in \mathbb{N})\,(1 \leq n < 2^n) && \text{the } < 2^n \text{ can easily be proven via induction} \\
&\to (\forall n \in \mathbb{N})(1 \leq \sqrt[n]{n} < 2) && \text{take the } n^{\text{th}} \text{ root of each term} \\
&\to (\forall n \in \mathbb{N})(0 \leq \sqrt[n]{n} - 1 < 1) && \text{subtract 1 from each term}
\end{aligned}$$

We know that the series $\sum x^n$ converges when $0 \leq x < 1$, so by the comparison Theorem 3.25 we know that $\sum a_n$ is convergent.

(d) If $|z| \leq 1$ then $|z|^n \leq 1$, so $|1 + z^n| \leq 1 + |z|^n \leq 2$. Hence $\left|\frac{1}{1+z^n}\right| = \frac{1}{|1+z^n|} \geq \frac{1}{2}$. Hence the terms do not tend to zero and the series diverges.

Assuming $|z| > 1$, we apply the ratio test:

$$\left|\frac{1 + z^n}{1 + z^{n+1}}\right| = \left|\frac{z^{-n} + 1}{z^{-n} + z}\right| \to \frac{1}{|z|} < 1$$

as $n \to \infty$. □

Problem 3 (Rudin 3.7)**.**

Proof. First solution. Using the inequality $xy \leq \dfrac{x^2 + y^2}{2}$, we get $0 \leq \dfrac{\sqrt{a_n}}{n} \leq \dfrac{1}{2}\left(a_n + \dfrac{1}{n^2}\right)$ for $n = 1, 2, \ldots$ The series $\sum a_n$ converges by the assumption in the problem. The series $\sum \dfrac{1}{n^2}$ converges by Theorem 3.2] Then the series $\sum \dfrac{1}{2}\left(a_n + \dfrac{1}{n^2}\right)$ converges by Theorem 3.47. Now $\sum \dfrac{\sqrt{a_n}}{n}$ converges by the comparison test, Theorem 3.25.

Second solution. We use the Cauchy criterion; let $\epsilon > 0$. Since both $\sum a_n$ and $\sum \frac{1}{n^2}$ converge, there exists $N > 0$ such that $m \geq n \geq N$ implies both $|\sum_{k=n}^m a_n| = \sum_{k=n}^m a_n < \epsilon$ and $\sum_{k=n}^m \frac{1}{k^2} < \epsilon$. Using Cauchy-Schwarz, we have

$$\left|\sum_{k=n}^{m} \frac{\sqrt{a_n}}{n}\right| \leq \sqrt{\sum_{k=n}^{m} a_n}\sqrt{\sum_{k=n}^{m} \frac{1}{k^2}} < \sqrt{\epsilon}^2 = \epsilon.$$

□

Problem 4 (Rudin 3.8)**.**

Solving this problem requires summation by parts, which we didn't cover, so it won't be graded.

Problem 5 (Rudin 3.11)**.**

Proof. a) Proof by contrapositive. We will assume that $\sum \dfrac{a_n}{1 + a_n}$ converges and show that this implies that $\sum a_n$ converges.

Since the series is convergent, the terms $\frac{a_n}{1+a_n}$ must tend to zero. We may choose N sufficiently large that $\frac{a_n}{1+a_n} < \frac{1}{2}$. This gives us

$$\begin{aligned}
2a_n &< 1 + a_n \\
a_n &< 1 \\
1 + a_n &< 2 \\
\frac{1}{2} &< \frac{1}{1 + a_n}.
\end{aligned}$$

We then have

$$|a_n| = 2\frac{1}{2}a_n < 2\frac{a_n}{1+a_n}$$

for $n \geq N$. By the comparison test, $\sum a_n$ converges.

b) From the definition of s_n, we can see that

$$s_{n+1} = a_1 + \ldots + a_{n+1} = s_n + a_{n+1}$$

and we're told that every $a_n > 0$, so we know that $s_{n+1} > s_n$. This implies that $s_n \geq s_m$ whenever $n \geq m$. We calculate:

$$\begin{aligned} 1 - \frac{s_N}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} &= \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\ &= \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &\leq \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \end{aligned}$$

which is the desired inequality.

We can now show that $\sum a_n/s_n$ fails the Cauchy criterion. Let $\epsilon = \frac{1}{2}$. Given any $N \in \mathbb{N}$, since the sequence s_n diverges to $+\infty$, we can choose k sufficiently large that $s_{N+k} > 2s_N$. This implies that $1 - \frac{s_N}{s_{N+k}} > \frac{1}{2}$. By the above inequality, we have

$$\sum_{i=N+1}^{N+k} \frac{a_i}{s_i} > \frac{1}{2}.$$

But this shows that N does not fulfill the Cauchy criterion (taking $n = N+1$, $m = k+1$). Since N was arbitrary, the series diverges.

c) We have $s_n - s_{n-1} = a_n$. Dividing both sides by $s_n s_{n-1}$, we obtain

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{a_n}{s_{n-1}s_n}.$$

Since $\frac{1}{s_{n-1}} \geq \frac{1}{s_n}$, this implies

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} \geq \frac{a_n}{s_n^2},$$

as desired.

Now, since the series $\sum a_n$ is divergent with positive terms, $s_n \to +\infty$ as $n \to \infty$, so $\frac{1}{s_n} \to 0$. The above formula telescopes to give the estimate

$$\sum_{k=n}^{m} \frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_m}.$$

Since for $m \geq n \geq N$ sufficiently large, the RHS is arbitrary small, the Cauchy criterion is satisfied.

d) The series $\sum \dfrac{a_n}{(1+n^2a_n)}$ always converges. From the fact that $a_n > 0$, we can establish the following chain of inequalities:

$$\frac{a_n}{1+n^2a_n} = \frac{\frac{1}{a_n}}{\frac{1}{a_n}}\frac{a_n}{1+n^2a_n} = \frac{1}{\frac{1}{a_n} + n^2} < \frac{1}{n^2}$$

We know that $\sum \dfrac{1}{n^2}$ converges (Theorem 3.28), and therefore $\sum \dfrac{a_n}{(1+n^2a_n)}$ converges by the comparison test of Theorem 3.25.

The series $\sum \frac{a_n}{(1+na_n)}$ may or may not converge. If $a_n = \frac{1}{n}$, for instance, the summation becomes

$$\sum \frac{a_n}{1+na_n} = \sum \frac{\frac{1}{n}}{2} = \frac{1}{2}\sum \frac{1}{n}$$

which is divergent by Theorem 3.28. To construct a convergent series, let a_n be defined as

$$a_n = \begin{cases} 1 & \text{if } n = 2^m - 1 (m \in \mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

The series $\sum a_n$ is divergent, since there are infinitely many integers of the form $2^m - 1$. But the series $\sum \frac{a_n}{(1+na_n)}$ is convergent:

$$\sum_{n=0}^{\infty} \frac{a_n}{1+na_n} = \sum_{m=0}^{\infty} \frac{1}{2^m} = \sum \left(\frac{1}{2}\right)^m$$

This series is convergent to 2 by Theorem 3.26.

Note: by (b), it is actually not possible to construct a counterexample for which a_n is monotonically decreasing! □

Problem 6 (Rudin 4.1)**.**

Proof. Consider the function

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

This function satisfies the condition that

$$\lim_{h\to\infty} [f(x+h) - f(x-h)] = 0$$

for all x, but the function is not continuous at $x = 0:$ We can choose $\epsilon < 1$, and every neighborhood $N_\delta(0)$ will contain a point p for which

$$d(f(p), f(0)) = 1$$

Therefore we can't pick δ such that $d(p, 0) < \delta \to d(f(p), f(0)) < 1$, which means that f is not continuous by Definition 4.5. □

Problem 7 (Rudin 4.2)**.**

Proof. Let X be a metric space, let E be an arbitrary subset of X, and let $\bar{E}$ represent the closure of E. We want to prove that $f(\overline{E}) \subseteq \overline{f(E)}$. To do this, assume $y \in f(E)$. This means that $y = f(e)$ for some $e \in (E \cup E')$.

Case 1: $e \in E$. If $e \in E$, then $y \in f(E)$ and therefore $y \in \overline{f(E)}$.

Case 2: $e \in E'$. If $e \in E'$, then every neighborhood of e contains infinitely many points of E. Choose an arbitrarily small neighborhood $N_\epsilon(y)$. We're told that f is continuous so, by Definition 4.1, we're guaranteed the existence of δ such that $f(x) \in N_\epsilon(y)$ whenever $x \in N_\delta(e)$. But there are infinitely many elements of E in the neighborhood $N_\delta(e)$, so there are infinitely many elements of $f(E)$ in $N_\epsilon(y)$. This means that y is a limit point of $f(E)$. We've shown that every arbitrary element $y \in f(\bar{E})$ is either a member of $f(E)$ or a limit point of $f(E)$, which means that $y \in (f(E) \cup f(E)') = \overline{f(E)}$. This proves that $f(\bar{E}) \subseteq \overline{f(E)}$.

A function f for which $f(\bar{E})$ is a proper subset $\overline{f(E)}$ Let X be the metric space consisting of the interval $(0,1)$ with the standard distance metric. Let Y be the metric space $\mathbb{R}^1$. Define the function $f : X \to Y$ as $f(x) = x$. The interval $(0,1)$ is closed in X but open in Y, so we have

$$f(\bar{X}) = f(X) = (0,1) \neq \overline{(0,1)}$$

□

Problem 8 (Rudin 4.3)**.**

Proof. If we consider the image of $Z(f)$ under f, we have $f(Z(f)) = \{0\}$. This range is a finite set, and is therefore a closed set. By the corollary of Theorem 4.8, we know that $f^{-1}(\{0\}) = Z(f)$ must also be a closed set. □

Problem 9 (Rudin 4.4)**.**

Proof. a) $f(E)$ is dense in $f(X)$. To show that $f(E)$ is dense in $f(X)$ we must show that every element of $f(X)$ is either an element of $f(E)$ or a limit point of $f(E)$. Assume $y \in f(X)$. Then $p = f^{-1}(y) \in X$. We're told that E is dense in X, so either $p \in E$ or $p \in E'$.

Case 1: $p \in E$. If $p \in E$, then $y = f(p) \in f(E)$.

Case 2: $f^{-1}(y) \in E'$. If p is a limit point of E, then there is a sequence $\{e_n\}$ of elements of E such that $e_n \neq p$ and $\lim\limits_{n\to\infty} e_n = p$. We're told that f is continuous, so by Theorem 4.2 we know that $\lim\limits_{n\to\infty} f(e_n) = f(p) = y$. Using Definition 4.2 again, we know that there is a sequence $\{f(e_n)\}$ of elements of $f(E)$ From Theorem 4.2, this tells us that $\lim\limits_{x\to p} f(x) = f(p) = y$. Therefore y is a limit point of $f(E)$. We've shown that every element $y \in f(X)$ is either an element of $f(E)$ or a limit point of $f(E)$. By definition, this means that $f(X)$ is dense in $f(E)$.

b) Choose an arbitrary $p \in X$. We're told E is dense in X, so p is either an element of E or a limit point of E.

Case 1: $p \in E$. If $p \in E$, then we're told that $f(p) = g(p)$.

Case 2: $p \in E'$. If p is a limit point of E, then there is a sequence $\{e_n\}$ of elements of E such that $e_n \neq p$ and $\lim\limits_{n\to\infty} e_n =$ p. We're told that f and g are continuous, so by Theorem 4.2 we know that $\lim\limits_{n\to\infty} f(e_n) = f(p)$ and $\lim\limits_{n\to\infty} g(e_n) = g(p)$. But each e_n is an element of E, so we have $f(e_n) = g(e_n)$ for all n. This tells us that

$$g(p) = \lim_{n\to\infty} g(e_n) = \lim_{n\to\infty} f(e_n) = f(p)$$

We see that $f(p) = g(p)$ in either case. This proves that $f(p) = g(p)$ for all $p \in X$. □