

Problem set 2: Ordered fields, real numbers

Math 521 Section 001, UW-Madison, Spring 2024

January 29, 2024

Please solve the following problems in a clear, complete, and concise manner. You are welcome to work together, but your write-up must be your own. Use of outside internet resources is prohibited.

*Due on paper at the beginning of class on **Wednesday, Feb. 7th**.* Please be sure to staple your writeup.

0. Please rewrite any problems from Problem set 0 that were in need of correction. If you are doing so, please also hand in your original writeup again.
1. Rudin 1.1, 1.2, 1.4, 1.5.
2. Use the axioms for an ordered field to prove the following. Please indicate which axiom(s) you are using in each step of the proof. (For all questions after this one, you do not need to indicate which axioms you are using.)
 - (a) If $ax = a$ for some $a \neq 0$, then $x = 1$.
 - (b) If $a < b$, then $-b < -a$.
 - (c) If $0 \leq a < b$, then $a^2 < b^2$.
 - (d) If $a > 1$, then for each $n \in \mathbb{N}$, $a^{n+1} > a^n$.
3. For each of the following subset of \mathbb{Q} , determine (with proof) which ones are bounded above or bounded below. If the set is bounded above (resp. below), determine its supremum (resp. infimum).
 - (a) $\{p^2 \mid p \in \mathbb{Q}\}$
 - (b) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.
 - (c) $\{p \in \mathbb{Q} \mid p^2 < 4\}$.
4. Which of the following subsets of \mathbb{Q} are Dedekind cuts? (No written justification required.)

- (a) $\{p \in \mathbb{Q} \mid r \leq 1\}$
- (b) $\{p \in \mathbb{Q} \mid r^3 < 1\}$
- (c) $\{p \in \mathbb{Q} \mid -1 < r < 1\}$.
- (d) $\{p \in \mathbb{Q} \mid r < 2 \text{ and } r^2 \neq 2\}$.
- (e) $\{p \in \mathbb{Q} \mid r < 0 \text{ or } r^2 \leq 2\}$.

5. (Extra credit +1) Let α be a Dedekind cut. Define

$$\beta = \{p \in \mathbb{Q} \mid \exists r > 0 \text{ with } -p - r \notin \alpha\}.$$

Prove that β is also a Dedekind cut. What real number does β represent?