Problem set 4 solutions

Math 521 Section 001, UW-Madison, Spring 2024

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1. Suppose that $g : A \to B$ is surjective and $f : B \to C$ is *not* injective. Show that $f \circ g$ is also *not* injective. (*Hint*: It may be helpful to start by drawing a picture.)

   Proof. Since $f : B \to C$ is not injective, there exist two elements $w_1 \neq w_2 \in B$ such that $f(w_1) = f(w_2)$. But since g is surjective, there exist two elements $x_1, x_2 \in A$ such that $g(x_1) = w_1$ and $g(x_2) = w_2$. Because $w_1 \neq w_2$, we must have $x_1 \neq x_2$ (because a function takes only one value per element). Therefore $x_1 \neq x_2$, and these elements satisfy

   $$(f \circ g)(x_1) = f(g(x_1)) = f(w_1) = f(w_2) = f(g(x_2)) = (f \circ g)(x_2).$$

   This shows that $f \circ g$ is not injective, as desired. □

2. Let $f : S \to T$ be a function. Suppose that S is uncountable and for each $y \in T$, $f^{-1}(\{y\})$ is countable. Prove that T is uncountable.

   Proof. If T were countable, then $S = \cup_{y \in T} f^{-1}(\{y\})$ would be a countable union of countable sets, hence countable. □

3. Consider the following subsets of $\mathbb{R}^2$. Which ones are open? Which ones are closed? Please justify each claim briefly; you do not have to give a full proof. (*Note*: For this problem, you are welcome to use drawings as justification.)

   (a) $A = [0,1] \times (0,1)$

   (b) $B = \{0\} \times (0,1)$

   (c) $C = \{(n, n^2) \mid n \in \mathbb{N}\}$

   Solution. (a) Neither. For, $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ belong to $A' \setminus A$, so A is not closed. Meanwhile $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$ belong to $A \setminus \mathring{A}$, so A is not open.

   (b) Neither. For, $(0,0)$ and $(0,1)$ belong to $B' \setminus B$, so B is not closed. Meanwhile, for any point $p = (0,b) \in B$, the neighborhood $N_r(p)$ contains points which do not belong to B : for instance, $(r/2, b) \in N_r(p) \setminus B$. Hence $N_r(p) \notin B$ for all $r > 0$, and B is not open.

(c) Closed but not open. The set C has no limit points: given any $x = (n, n^2) \in C$, we have $N_1(x) \cap C = \{x\}$, so x is not a limit point. Hence C is closed (vacuously). The set C is not open because for any $r > 0$, $N_r(x)$ contains points such as $(n + \frac{\min\{1,r\}}{2}, n^2)$ which do not belong to C. $\square$

4. Rudin 2.5, 10 (except the "...compact?" part), 11.

(2.5) Recall from class that the set $E_0 = \{1/n \mid n \in \mathbb{N}\}$ has only one limit $\{0\}$. By the same argument, the sets $E_a = \{a + 1/n \mid n \in \mathbb{N}\}$ also each have only one limit point, a. We can take

$$E = E_0 \cup E_1 \cup E_2.$$

Then $E' = E_0' \cup E_1' \cup E_2' = \{0, 1, 2\}$ (since this is a finite union).

(2.10) This is called the "discrete metric." Nonnegativity and symmetry are obvious. To check the triangle inequality, let $p, q, r \in X$. We must show

$$d(p, r) \leq d(p, q) + d(q, r).$$

Case 1. $p = r$. The LHS is zero and the RHS is nonnegative, so we're done.

Case 2. $p \neq r$. The LHS is 1. Now either $q = p$ or $q \neq p$. If $q = p$, then since $p \neq r$, $q \neq r$. Therefore $d(q, r) = 1$, so the RHS is a least 1 and the inequality holds. If $q \neq p$, then $d(p, q) = 1$, so the inequality also holds.

Let $E \subset X$ be a subset. First note that for any point $p \in X$, $N_{1/2}(p) = \{p\}$. Hence p is not a limit point of E. Therefore $E' = \varnothing$, so E is trivially closed. Since E was arbitrary, we conclude that *all* subsets of X are closed. By complementarity, all subsets of X are also open.

(2.11) (1) Not a metric: triangle inequality fails. Let $x = 0, y = 1, z = 2$. Then $d(x, z) = 4$ but $d(x, y) + d(y, z) = 1 + 1 = 2$.

(2) This is a metric. Nonnegativity and symmetry are clear. For the triangle inequality, we have from the ordinary triangle inequality

$$|x - z| \leq |x - y| + |y - z|.$$

Adding $2\sqrt{|x - y||y - z|}$ to the RHS, we have

$$|x - z| \leq \left(\sqrt{|x - y|} + \sqrt{|y - z|}\right)^2$$

The triangle inequality for d_2 follows by taking square roots.

(3) Not a metric: nonnegativity fails since $d(1, -1) = 0$ but $1 \neq -1$. (Note: this would be a metric if restricted to nonnegative numbers.)

(4) Not a metric: nonnegativity and symmetry fail: $d(1,1) = 1 \neq 0$, and $d(1,0) = 1 \neq 2 = d(0,1)$.

(5) This is a metric. Nonnegativity and symmetry are clear. For the triangle inequality, we first prove:

<u>Claim 1.</u> For $0 \leq a \leq b$, we have $\frac{a}{1+a} \leq \frac{b}{1+b}$.

Proof. We calculate:

$$\begin{aligned} a &\leq b \\ a + ab &\leq b + ab \\ a(1+b) &\leq b(1+a). \end{aligned}$$

The claim follows by dividing both sides by $(1+a)(1+b)$. $\square$

<u>Claim 2.</u> For $A, B \geq 0$, we have $\frac{A+B}{1+A+B} \leq \frac{A}{1+A} + \frac{B}{1+B}$.

Proof.

$$\begin{aligned} \frac{A+B}{1+A+B} &= \frac{A}{1+A+B} + \frac{B}{1+A+B} \\ &\leq \frac{A}{1+A} + \frac{B}{1+B}, \end{aligned} \tag{0.1}$$

since $\frac{1}{1+A+B} \leq \frac{1}{1+A}$ and $\frac{1}{1+A+B} \leq \frac{1}{1+B}$. $\square$

To prove the triangle inequality, we let $A = |x-y|$, $B = |y-z|$, and $C = |x-z|$. Since $C \leq A + B$, applying claim 1, we have

$$d(x,z) = \frac{C}{1+C} \leq \frac{A+B}{1+A+B}.$$

We then apply claim 2 to obtain

$$d(x,z) \leq \frac{A}{1+A} + \frac{B}{1+B} = d(x,z) + d(y,z)$$

as desired.

5. (Extra credit + 1) Let S be a nonempty set. Prove that S and its power set $\mathcal{P}(S)$ do not have the same cardinality.

First proof. We will show that any map $F : S \to \mathcal{P}(S)$ is not surjective; hence, it is impossible to have a bijective map.

We must define a subset E such that $E \neq F(x)$ for all $x \in S$. Let

$$E = \{x \in S \mid x \notin F(x)\}.$$

Then by definition, $x \notin E$ if $x \in F(x)$, whereas $x \in F(x)$ if $x \notin F(x)$. In particular, for each $x \in S$, E and $F(x)$ do not have the same elements. Hence $E \neq F(x) \forall x \in S$, so $E \notin F(S)$. Hence F is not surjective. $\square$

Second proof. Recall that we may identify $\mathcal{P}(S)$ with the set $\{0,1\}^S$ of all functions from S to $\{0,1\}$, as follows. Given $E \subset S$, define the function

$$\begin{aligned} f &: S \to \{0,1\} \\ f(x) &= \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}. \end{aligned} \tag{0.2}$$

Such an f determines and is uniquely determined by E, so this is a 1-1 correspondence. Hence, it suffices to prove that $\{0,1\}^S$ cannot be put in bijection with S.

As above, we will show that any map $F : S \to \{0,1\}^S$ is not surjective. For each $x \in S$, $F(x)$ is a function which we denote by

$$F(x) = f_x : S \to \{0,1\}.$$

We must define a function $g : S \to \{0,1\}$ such that $g \neq f_x$ for all $x \in S$. Let

$$g(x) = \begin{cases} 0 & f_x(x) = 1 \\ 1 & f_x(x) = 0. \end{cases}$$

□