Problem 1 (Problem 6.1, Baby Rudin)**.**

Proof. First assume $x_0 \in (a, b)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \varepsilon$ if $x \in [a, b]$ and $|x - x_0| < \delta$. By reducing δ if necessary, can assume $a < x_0 - \frac{\delta}{2} < x_0 < x_0 + \frac{\delta}{2} < \delta$. Choose partition $P = \{z_0, z_1, z_2, z_3\}$ with $z_0 = a, z_1 = x_0 - \frac{\delta}{2}, z_2 = x_0 + \frac{\delta}{2}, z_3 = 6/b$. Then we have

$$M_1 = m_1 = M_3 = m_3 = 0, \quad M_2 = 1, m_2 = 0.$$

Thus $L(P, f, \alpha) = 0$, and

$$\begin{aligned} U(P, f, \alpha) &= M_2 \Delta\alpha_2 = 1 \cdot \left[\alpha\left(x_0 + \frac{\delta}{2}\right) - \alpha\left(x_0 - \frac{\delta}{2}\right)\right] \\ &= \left[\alpha\left(x_0 + \frac{\delta}{2}\right) - \alpha(x_0)\right] + \left[\alpha(x_0) - \alpha\left(x_0 - \frac{\delta}{2}\right)\right] < 2\varepsilon. \end{aligned}$$

Thus $U(P, f, \alpha) - L(P, f, \alpha) < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $f \in \mathcal{R}(\alpha)$. Also, then, for any $\varepsilon > 0$, we have

$$0 = L(P, f, \alpha) \leqslant \int_a^b f d\alpha \leqslant U(P, f, \alpha) < 2\varepsilon.$$

Thus, we conclude that $\displaystyle\int_a^b f d\alpha = 0$. □

Problem 2 (Problem 6.2, Baby Rudin)**.**

Proof. Suppose there exists $x_0 \in [a, b]$ such that $f(x_0) > 0$, then since f is continuous on $[a, b]$, there exists $\delta > 0$ such that $x \in [a, b]$ and $|x - x_0| \leq \delta$ implies $f(x) > \frac{f(x_0)}{2}$.

Consider the interval $I = [a, b] \cap [x_0 - \delta, x_0 + \delta]$, it is easy to see that $I = [c, d]$, where $a \leq c < d \leq b$. So if we let P be a partition in such a way that $c, d \in P$,

$$L(P, f) \geq \left[\inf_{x \in [c,d]} f(x)\right](d - c) \geq \frac{f(x_0)}{2}(d - c) > 0$$

where the first inequality shown above follows by the fact $f \geq 0$ on $[a, b]$. It follows that

$$0 = \int_a^b f(x)dx = \sup L(P, f) > 0$$

a contradiction. □

Problem 3 (Problem 6.4, Baby Rudin)**.**

Proof. Let $P = \{x_0, \ldots, x_n\}$ be any partition. By eliminating repeating points (which do not affect upper and lower sums), can assume $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Then on any interval $I_k = [x_{k-1,}, x_k]$ there exists rational $y_k \in I_k$ and irrational $z_k \in I_k$. Thus $M_k = 1, m_k = 0$ for $k = 1, \ldots, n$. This implies that

$$L(P, f) = 0, \quad U(P, f) = \sum_{k=1}^{n}(x_k - x_{k-1}) = x_n - x_0 = b - a > 0.$$

Therefore, we get $U(P, f) - L(P, f) = b - a > 0$ for any partition P. This implies that $f \notin \mathcal{R}$ by Theorem 6.6. □

Problem 4 (Problem 6.5, Baby Rudin)**.**

Proof. We give example $f \notin R$ but $f^2 \in R$. Let $f(x) = \begin{cases} \frac{1}{2} & x \text{ rational} \\ \frac{-1}{2} & x \text{ irrational.} \end{cases}$.

Then $f(x) + \frac{1}{2}$ is the function from Ex.6.4, and we therefore proved in Ex.6.4 that $f + \frac{1}{2} \notin \mathcal{R}$ on $[a, b]$. Since $f = \left(f + \frac{1}{2}\right) - \frac{1}{2}$, then $f \notin \mathcal{R}$ on $[a, b]$. However, we have $f^2(x) = \frac{1}{4}$ for all x, so $f \in \mathcal{R}$ on $[a, b]$.

Now, if $f^3 \in \mathcal{R}$ then $f \in \mathcal{R}$. Indeed, $f(x) = \sqrt[3]{f^3(x)}$ and the function $g(z) = \sqrt[3]{z}$ is defined and continuous on $\mathbb{R}$. Thus, by Theorem 6.11, $f \in \mathcal{R}$.

Note that this argument does not work for f^2. Indeed, $f(x) = \pm\sqrt{f^2(x)}$, i.e. it is not true that $f(x) = \sqrt{f^2(x)}$ in several. □

Problem 5 (Problem 6.8, Baby Rudin)**.**

Proof. Suppose $\sum_{n=1}^{\infty} f(n)$ converges. Since $f \geqslant 0$, then, denoting $S_k = \sum_{k=1}^{k} f(n)$, we get $S_1 \leqslant S_2 \leqslant \cdots \leqslant S$ and $\lim_{k\to\infty} S_k = S$. Also, since $f \geqslant 0$, the function $b \longmapsto \int_1^b f(x)dx$ is monotonically increasing. Thus

$$\lim_{b\to\infty} \int_1^b f dx = \sup_{b\in(1,\infty)} \int_1^b f dx$$

Thus it is sufficient to show $\sup_{b\in(1,\infty)} \int_1^b f dx < \infty$. Since f decreases inonotorically, $f(x) \leqslant f(k)$ for all $x \in [k, k+1]$. This implies that

$$0 \leqslant \int_1^n f dx = \sum_{k=1}^{n-1} \int_k^{k+1} f(x)dx \leqslant \sum_{k=1}^{n-1} f(k) \cdot 1 = S_{n-1} \leqslant S$$

For any $b > 1$, thee e exists integer $n > b$, thus $\int_1^6 f dx \leqslant \int_1^n f dx \leqslant S_{n-1} \leqslant S$. Thus, we get $\sup_{b\in(1,\infty)} \int_1^b f dx < \infty$, which implies $\lim_{b\to\infty} \int_1^b f dx$ exits.

Conversely, if $J = \lim_{b\to\infty} \int_1^b f dx$ exists, and $J < x_1$ then $\int_1^b f dx \leqslant J$ for all $b \in (1\infty)$, since $f \geqslant 0$. Also, $f(x) \geqslant f(k+1)$ for all $x \in [k, k+1]$. Thus we have

$$J \geqslant \int_1^n f dx = \sum_{k=1}^{n-1} \int_k^{k+1} f dx \geqslant \sum_{k=2}^{n} f(k) = S_n - f(1)$$

i.e. $S_n \leqslant J + f(1)$ for all $n = 1, 2, \ldots$.

Since $\sum_{n=1}^{\infty} f(n)$ is a series of positive terms, it boundedness of sequence of partial sums implies convergence of the series. □

Problem 6 (Problem 6.13, Baby Rudin)**.**

Proof. a) Following the hint (substituting u for t^2, integration by parts using $\sin u$ and $1/(2\sqrt{u})$, replacing $|\cos u|$ with 1), we get

$$f(x) = \int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}} du = -\frac{\cos(x+1)^2}{2(x+1)} + \frac{\cos x^2}{2x} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$$

and then we have

$$|f(x)| \le \frac{\left|\cos(x+1)^2\right|}{2(x+1)} + \frac{\left|\cos x^2\right|}{2x} + \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{3/2}} du < \frac{1}{2(x+1)} + \frac{1}{2x} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du = \frac{1}{x}$$

where the strict inequality is due to the fact that $|\cos u|$ will not be constantly equal to its maximum possible value 1 throughout the interval of integration.

b) In the previous problem we used integration by parts to determine that

$$f(x) = \frac{-\cos(x+1)^2}{2(x+1)} + \frac{\cos x^2}{2x} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}$$

Multiplying by $2x$ gives us

$$2xf(x) = \frac{-x\cos(x+1)^2}{x+1} + \cos x^2 - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$$

which is algebraically equivalent to

$$2xf(x) = \cos x^2 - \cos\left[(x+1)^2\right] + \frac{\cos(x+1)^2}{x+1} - 2x \int^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$$

Letting $r(x)$ be defined as

$$r(x) = \frac{\cos(x+1)^2}{x+1} - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$$

we have, by the triangle inequality,

$$|r(x)| \le \left|\frac{\cos(x+1)^2}{x+1}\right| + \left|2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du\right| \le \left|\frac{1}{x+1}\right| + \left|2x \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du\right|$$
$$= \frac{2}{x+1} < \frac{2}{x}$$

So we see that

$$2xf(x) = \cos x^2 - \cos\left[(x+1)^2\right] + r(x)$$

where $|r(x)| < 2/x$.

c) From part (b), we get that as $x \to \infty$,

$$xf(x) \approx \frac{\cos x^2 - \cos(x+1)^2}{2}.$$

Hence $xf(x)$ lies (more or less) between $(-1-1)/2 = -1$ and $(1+1)/2 = 1$.

d) Note that $\sin t^2$ is positive between $\sqrt{n\pi}$ and $\sqrt{(n+1)\pi}$ for any even integer n and negative for any odd integer n. Hence $\int_0^\infty \sin t^2 dt$ can be reduced to an alternating series, whose terms satisfy

$$\left|\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin t^2 dt\right| < \sqrt{n\pi} - \sqrt{(n+1)\pi} = \frac{\sqrt{\pi}}{\sqrt{n}+\sqrt{n+1}}$$

which goes to 0 as $n \to \infty$. Hence, by Theorem 3.43, the integral converges. □

Problem 7 (Extra credits Problem 6.6, Baby Rudin)**.**

Proof. Following the hint, recall that $P = \bigcup E_n$ where E_n is a set of 2^n disjoint close intervals of length 3^{-n} obtained by removing the middle thirds of the intervals in E_{n-1}. The total length of the intervals of E_n is $\left(\frac{2}{3}\right)^n$ and so $\to 0$ as $n \to \infty$. We can replace the closed intervals of E_n, $[a_{i,n}, b_{i,n}]$ with slightly larger open intervals $\left(a_{i,n} - \frac{\delta}{2^n}, b_{i,n} + \frac{\delta}{2^n}\right)$, with total length $\left(\frac{2}{3}\right)^n + \delta$, so that we can cover P with a set of disjoint open intervals with total length as small as possible.

Now proceeding as in the proof of Theorem 6.10, let $M = \sup |f(x)|$, and cover P with a collection of open intervals (u_j, v_j) with total length less than ε. The complement K of the union of the open intervals in $[a, b]$ is compact. Then f is uniformly continuous on K and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ if $s, t \in K, |s - t| < \delta$.

Form a partition $Q = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that each u_j and v_j occurs in Q, no point of any segment (u_j, v_j) occurs in Q, and if x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $M_i - m_i \leq 2M$ for every i, and that $M_i - m_i \leq \varepsilon$ if x_{i-1} is not one of the u_j. Hence

$$\begin{aligned} U(Q, f) - L(Q, f) = \sum_i (M_i - m_i)\,\Delta x_i &= \sum_{x_{i-1} \in \{u_j\}} (M_i - m_i)\,\Delta x_i + \sum_{x_{i-1} \notin \{u_j\}} (M_i - m_i)\,\Delta x_i \\ &\leq 2M\varepsilon + \varepsilon(b - a) = (2M + b - a)\varepsilon. \end{aligned}$$

Hence $f \in \mathscr{R}$ by Theorem 6.6. $\square$