Recall: A metric space is a set X with a "distance function" (a.k.a. metric) $d: X \times X \to \mathbb{R}$ satisfying

(1) $d(x,y) \geq 0$, $d(x,y) = 0 \iff x = y$. (nonneg.)

(2) $d(x,y) = d(y,x)$ (symmetric)

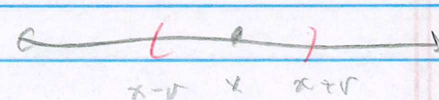
(3) $d(x,z) \leq d(x,y) + d(y,z)$ (Δ ineq.)

These turn out to be very powerful.

Definition. Let $p \in X$. A neighborhood (a.k.a. ball) of radius r in X (w.r.t. d) is:

$$N_r(p) = \{ x \in X \mid d(p,x) < r \}.$$

Ex: $X = \mathbb{R}$, $d(x,y) = |x-y|$.

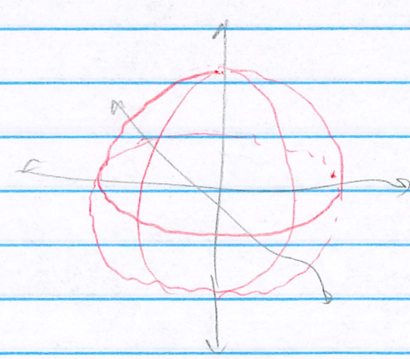


$N_r(p) = (p-r, p+r)$

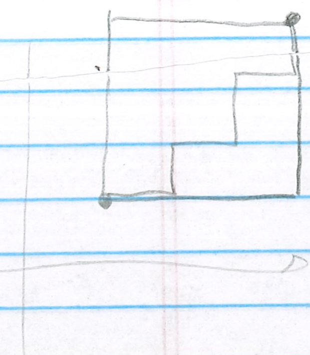
Ex: $X = \mathbb{R}^n$, $d(x,y) = |x-y| = \sqrt{\sum (x_i - y_i)^2}$

$$N_r(p) = \{ x \in \mathbb{R}^n \mid \sum (x_i - p_i)^2 < r^2 \}$$

Ball of radius r.



Ex: $X = \mathbb{R}^n$, <u>taxicab distance</u>



$$d_T(x,y) = \sum |x_i - y_i|.$$

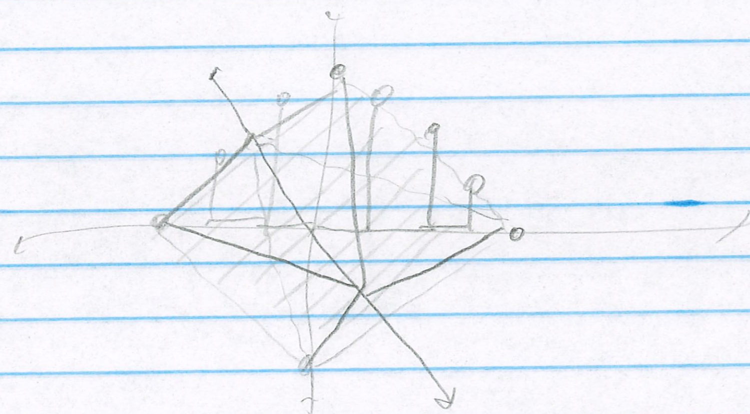
Metric?

(1) ✓ (2) ✓ (3) $\sum |x_i - z_i| = \sum |x_i - y_i + y_i - z_i|$

$$\le \sum |x_i - y_i| + |y_i - z_i|$$

$$= d_T(x,y) + d_T(y,z) \quad ✓.$$

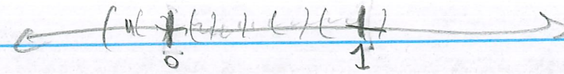
$N_r(p) =$



<u>Definition:</u> Let $E \subset X$ be a subset.

- $p \in X$ is called a <u>limit point</u> of E if every neighborhood of p contains a point $q \neq p$ with $q \in E$.
- E is called <u>closed</u> (as a subset of X) if every limit point of E belongs to E.

Ex: $X = \mathbb{R}$, $d(x,y) = |x-y|$.

$E = [0,1]$. 

Limit points of E?

$p \in E \iff 0 \le p \le 1$.

$N_r(p) = (p-r, p+r)$ ⇝ p is a limit point.

$p \notin E \iff p < 0$ or $p > 1$.

Let $r = |p|$. not a lim pt.

$\Rightarrow N_r(p) = (2p, 0) \cap E = \emptyset$

$\Rightarrow p$ not a lim. pt.

$\therefore$ p is a lim. pt of $E \Rightarrow p \in E$

$\therefore$ E is closed. ⇝ $[a,b]$ = "closed interval"

Ex: $E = (0,1)$.

$0, 1$ are limit pts

but $0, 1 \notin E \Rightarrow E$ is not closed

Ex: $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

$p = 0$: for any $r > 0$, choose $n > \frac{1}{r}$

$\Rightarrow \frac{1}{n} < r$

But $0 \notin E \Rightarrow E$ is not closed. $\in E \cap N_r(0)$

<u>Theorem</u>: If p is a lim. pt of E, then every nbhd of p contains ∞ly many points of E.

<u>Pf</u>: Suppose N is a nbhd containing only fin. many pts $q_1, \dots, q_n \in N \cap E$, $q_i \neq p$ $\forall i$.

Let $r = \min_{1 \le i \le n} d(p, q_i) > 0$.

$\Rightarrow d(p, q_i) \geq r$ $\forall i$
$\not< r$

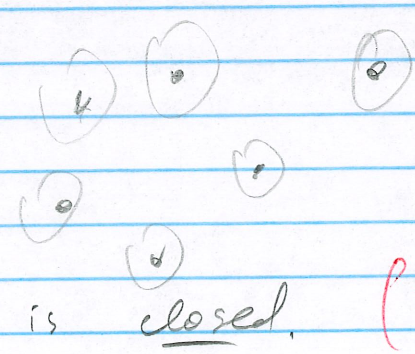
$\Rightarrow q_i \notin N$ $\forall i$

$\Rightarrow N \cap E = p$

$\Rightarrow p$ is not a lim. pt.

Contrapos: desired statement. $\square$

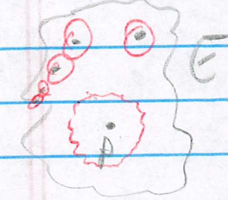
<u>Corollary</u>: A finite set has <u>no</u> limit points.



<u>Corollary</u>: A finite set is <u>closed</u>. (Typical closed set)

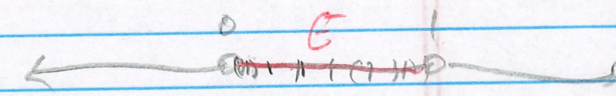
Definition: Let $E \subset X$ be a subset.

- $p \in E$ is called an <u>interior point</u> if there exists a neighborhood $N \ni p$ s.t. $N \subset E$.
- E is called <u>open</u> if every point of E is an interior point.



Ex: $X = \mathbb{R}$.

- $E = (0,1) = \{0 < x < 1 \mid x \in \mathbb{R}\}$.



Let $p \in E$. $0 < p < 1$.

Let $r = \min\{p, 1-p\}$.

$$N_r(p) = (\underset{\substack{\wedge \\ p-p=0}}{p-r}, \underset{\substack{\vee \\ p+1-p=1}}{p+r}) \subset (0,1) \quad \checkmark.$$

(a,b) is called an <u>open</u> interval.

- $E = [0,1]$. (not closed)

$p = 0$? For any $r > 0$, $N_r(0)$ contains $-\#$'s

$\Rightarrow N_r(0) \not\subset E \quad \forall\, r > 0$.

$\Rightarrow E$ is not open.

- $E =$ [illegible] (is closed)

• $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

Let $p = \frac{1}{n} \in E$.

Then $m > n \Rightarrow m \geq n+1$

$$\Rightarrow \frac{1}{m} \leq \frac{1}{n+1}$$

$$\Rightarrow \left(\frac{1}{n+1}, \frac{1}{n}\right) \cap E = \emptyset$$

Let $r = \frac{1}{n(n+1)}$.

Then $\frac{1}{n} - r = \frac{1}{n+1}$

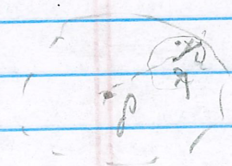
$$\Rightarrow \underset{\displaystyle N_r(\tfrac{1}{n})}{\underset{\cap}{\left(\tfrac{1}{n} - r,\ r\right)}} \cap E = \emptyset.$$

$$\Rightarrow N_r\left(\frac{1}{n}\right) \not\subset E.$$

E is $\boxed{\text{neither open nor closed}}$.

Thm: Every neighborhood is an open set.

Pf: Consider $E = N_r(p)$, $p \in X$, $r > 0$



Let $q \in E$. Since $d(p,q) < r$, let

$$h = r - d(p,q) > 0.$$

If $x \in N_h(q)$ then $d(q,x) < h$.

$$\Rightarrow d(q,x) + d(p,q) < r$$

$$\Rightarrow d(p,x) \leq d(q,x) + d(p,q) < r \quad \text{by } \triangle \text{ ineq.}$$

$\Rightarrow x \in N_r(p)$. Since x was arb., $N_h(q) \subset E$. □