<u>Recall:</u> $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

$\hookrightarrow$ ordered field!

<u>Defn:</u> An ordered set S has the <u>least upper bound</u> <u>property</u> if for every nonempty set $E \subset S$ which is bounded above, there exists a <u>least upper bound</u> $\beta = \sup E$ in S.

(i) β is an upper bd of E

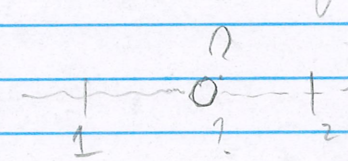
(ii) If $\gamma < \beta$ then γ is <u>not</u> an upper bd.

<u>Saw last time:</u> $\mathbb{Q}^{\le}$ does <u>not</u> have the LUB property!

<u>Counterexample:</u> $E = \{ p \in \mathbb{Q} \mid p^2 < 2 \}$

Has no largest element

and has no least upper bound!



<u>Showed:</u> If $\beta^2 < 2$ then β is not an upper bd.
((i) fails)

If $\beta^2 > 2$ then there <u>does</u> exist $\gamma < \beta$ s.t. γ is also an upper bound ((ii) fails)

<u>and:</u> $\beta^2 = 2$ is not possible if $\beta \in \mathbb{Q}$.

<u>Goal</u>: construct an ordered field with the <u>l.u.b. property</u>.

(Will get $\sqrt{2}$ for free!)

<u>Idea</u>: turn the "holes in $\mathbb{Q}$" into a whole field!

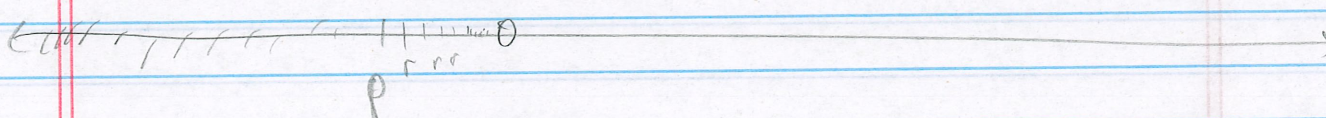
<u>Definition</u>: Let $\alpha \subset \mathbb{Q}$ be a subset. We say that α is a <u>Dedekind cut</u> if it has the following properties:

(I) α is nonempty and bounded above

(II) If $p \in \alpha$ and $q \in \mathbb{Q}$ with $q < p$, then $q \in \alpha$

(III) If $p \in \alpha$ then $p < r$ for some $r \in \alpha$.

(Contains no largest element)



<u>Example</u>: $\alpha = \{ p \in \mathbb{Q} \mid p < 0 \}$.

(I) $-1 \in \mathbb{Q}$, 0 is an upper bound.

(II) $p \in \alpha \Rightarrow p < 0$. $q < p < 0 \Rightarrow q < 0$
$\Rightarrow q \in \alpha$ ✓.

(III) $p < 0 \Rightarrow \frac{p}{2} < 0$

Take $r = \frac{p}{2} \in \alpha$

Then $p < \frac{p}{2} = r$ ✓.

Example: Given any rational # $t \in \mathbb{Q}$,

define $t^* = \{p \in \mathbb{Q} \mid t < q\}$.

This is called a rational Dedekind cut.

Last example: $\alpha = 0^*$.

We say that α is an irrational Dedekind cut if $\nexists\, t \in \mathbb{Q}$ s.t. $\alpha = t^*$.

Example: "$\sqrt{2}$" $= \{p \in \mathbb{Q} \mid p^2 < 2\} \cup 0^*$.

The vast majority of D. cuts are irrational.

Definition: We denote the collection of all Dedekind cuts by $\boxed{\mathbb{R}.}$

How to add?

Defn: $\alpha + \beta := \{p + q \mid p \in \alpha, q \in \beta\}$.

Is $\alpha + \beta \in \mathbb{R}$?

(I) Clearly $\alpha + \beta \neq \emptyset$. Suppose M and N are upper bounds for α and β, respectively.

Then $\forall p \in \alpha, q \in \beta, \quad p + q \leq M + M'$

$\Rightarrow M + M'$ is an upper bound for $\alpha + \beta$ ✓.

(II) Let $p + p' \in \alpha + \beta$.

Let $q < p + p'$.

Take $p'' = q - p$.

Then $p'' < p + p' - p = p'$

$\Rightarrow p'' \in \beta$

$\Rightarrow p + p'' = q \in \alpha + \beta$ ✓.

(III) Let $p + p' \in \alpha + \beta$.

$\exists\, r \in \beta$ with $p' < r$.

$\Rightarrow p + p' < p + r \in \alpha + \beta$. ✓.

We have shown:

(A1) $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha + \beta \in \mathbb{R}$.

(A2) $\alpha + \beta = \{p + p' \mid p \in \alpha, p' \in \beta\}$

$= \{p' + p \mid p \in \beta, p' \in \alpha\}$

$= \beta + \alpha$ ✓.

(A3) $(\alpha + \beta) + \gamma = \{(p + p') + p'' \mid p \in \alpha, p' \in \beta, p'' \in \gamma\} = \{p + (p' + p'') \mid p \in \alpha, p' \in \beta, p'' \in \gamma\}$

$= \alpha + (\beta + \gamma)$ ✓

(A4) similar

(A5) E.C. in homework.

We can define mult. similarly.

<u>Definition</u>: For $\alpha, \beta \in \mathbb{R}$, we say $\alpha \leq \beta$

if $\alpha \subset \beta$.

Is this an order?

(i) need to show $\alpha < \beta$, $\alpha = \beta$, xor $\alpha > \beta$.

$\hookrightarrow$ i.e. $\alpha \subset \beta$ but $\alpha \neq \beta$.

Suppose $\alpha \neq \beta$. Then either $\beta \setminus \alpha \neq \emptyset$

or $\alpha \setminus \beta \neq \emptyset$.

Suppose $\beta \setminus \alpha \neq \emptyset$ and let $p \in \mathbb{Q}$ s.t. $p \in \beta$ but $p \notin \alpha$.

Let $q \in \alpha$.

Then $q < p$ (If $p \leq q$ then $p \in \alpha$ by (II)).

$\Rightarrow q \in \beta$.

Since $q \in \alpha$ was arbitrary, we conclude $\alpha \subset \beta$.

$\Rightarrow \alpha < \beta$

Similarly, if $\alpha \setminus \beta \neq \emptyset$ then $\beta < \alpha$.

(ii) $\alpha < \beta$ and $\beta < \gamma$

$\Rightarrow \alpha \subset \beta$ and $\alpha \neq \beta$ and $\beta \subset \gamma$ $\Rightarrow$ $\alpha \subset \gamma$ and $\alpha \neq \gamma$ $\Rightarrow \alpha < \gamma$ ✓.

Theorem: $\mathbb{R}$ is an ordered field with the LUB property.

Pf: We checked (A1-A3) and that $<$ is an order; we'll now prove the LUB property. See Rudin Ch. 1, appendix for the rest.

LUB property: Let $E \subset \mathbb{R}$ be a subset which is bdd above by $\beta \in \mathbb{R}$.

i.e.: $\forall\ x \in E$, $x \leq \beta$

i.e. $x \subset \beta$.

Let $\alpha = \bigcup_{x \in E} x$.

$= \{ p \in \mathbb{Q} \mid p \in x \text{ for some } x \in E \}$.

Is $\alpha \in \mathbb{R}$?

(I) Let M be an upper bound for β.

Suppose $p \in \alpha$. Then $p \in x$ for some $x \in E$

$\Rightarrow p \in \beta$

$\Rightarrow p \leq M$.

$\therefore M$ is also an u.b. for α.

(II) $p \in \alpha$, $q < p$ ($q \in \mathbb{Q}$)

$\Rightarrow p \in x$ for some $x \in E$

$\Rightarrow q \in x$, by (II) since $x \in \mathbb{R}$

$\Rightarrow q \in x$

$\Rightarrow q \in \bigcup_{x \in E} x = \alpha$ ✓.

(III) $p \in \alpha \Rightarrow p \in x$

$\Rightarrow \exists\, r \in x$ s.t. $p < r$

$\Rightarrow r \in \alpha$ ✓.

Is α the LUB for E?

(i) $x \in E \Rightarrow x \subset \alpha \Rightarrow x \leq \alpha$ ✓.

(ii) Suppose $\gamma < \alpha$.

Then $\gamma \subset \alpha$ but $\gamma \neq \alpha$, so $\exists\, p \in \alpha$ with $p \notin \gamma$.

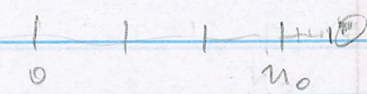
But also $p \in x$ for some $x \in E$

$\Rightarrow x \not\subset \gamma$.

$\Rightarrow \gamma$ is <u>not</u> an upper bound ✓.

What is a decimal expansion?

Let $\alpha \in \mathbb{R}$.



Let $n_0 \in (\mathbb{Z} \cap \alpha)$ be the largest integer in α.

Let $n_1 \in \{0, \dots, 9\}$ be the largest integer s.t.

$N_1 = n_0 + \frac{n_1}{10} \in \alpha$. must be, otw n_0 wouldn't have been the largest.

$\vdots$

$N_k = \sum_{i=0}^{k} \frac{n_i}{10^i}$.

$\{N_0 = n_0, N_1, N_2, \dots\} \subset \alpha$.

We write $\boxed{\alpha = n_0.n_1n_2n_3\dots}$

$\in \mathbb{Z}$, $\in \{0, \dots, 9\}$

Conversely, given $n_0, n_1, n_2, \dots$, can define the Dedekind cut $n_0.n_1n_2\dots = \{q \in \mathbb{Q} \mid q < N_k \text{ for some } k \in \mathbb{N}\}$.

Example: $.99999\ldots$

$a_0 = 0,\ a_i = 9 \ \forall\, i \in \mathbb{N}$

$$\sum_{i=0}^{k} \frac{a_i}{10^i} = 9 \cdot \sum_{i=1}^{k} \left(\frac{1}{10}\right)^i = \frac{9}{10} \cdot \sum_{i=0}^{k-1} \left(\frac{1}{10}\right)^i$$

$$= \frac{9}{10} \cdot \left(\frac{1 - \left(\frac{1}{10}\right)^{k+1}}{1 - \frac{1}{10}} \right)$$

$$= 1 - \left(\frac{1}{10}\right)^k$$

Given any $q < 1$, $\exists\, k$ s.t. $\frac{1}{10^k} < 1 - q$

$\Rightarrow q < \frac{1}{10^k}$

$\Rightarrow q < .99\ldots$

$\Rightarrow \boxed{.999\ldots = 1}$