Problem set 11: Sequences and series of functions, power series

Math 521 Section 001, UW-Madison, Spring 2024

April 29, 2024

Last HW! Due on last day of class (review day), Friday 5/3.

Please solve the following problems in a clear, complete, and concise manner. You are welcome to work together, but your write-up must be your own. Use of outside internet resources is prohibited.

Due on paper at the beginning of class on **Friday, May 3rd**. Please be sure to staple your writeup.

1. Rudin 7.6-9.

2. (Rudin 7.12, broken into parts) Suppose that $g : (0, \infty) \to \mathbb{R}$ is nonnegative and Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$. Suppose that the limit

$$\lim_{t \to 0} \lim_{T \to \infty} \int_t^T g(x) dx$$

exists, and denote it by

$$\int_0^\infty g(x)\, dx.$$

(a) Prove that $\lim_{T\to\infty} \lim_{t\to 0} \int_t^T g(x)\, dx = \int_0^\infty g(x)\, dx$.

(b) Suppose that $|f(x)| \le g(x)$ on $(0, \infty)$ and $f(x)$ is integrable on $[t, T]$ whenever $0 < t < T < \infty$. Prove that for any $t > 0$, the limit

$$\lim_{n\to\infty} \int_t^n f(x)\, dx$$

exists. (Hint: use the Cauchy criterion for sequences.)

(c) Prove that $\lim_{T\to\infty} \int_t^T f(x)\, dx =: \int_t^\infty f(x)\, dx$ exists and equals the limit in (b).

(d) Prove similarly that $\lim_{t\to 0} \int_t^\infty f(x)\, dx =: \int_0^\infty f(x)\, dx$ exists.

(e) Check that $\lim_{T\to\infty} \lim_{t\to 0} \int_t^T f(x)\, dx = \int_0^\infty f(x)\, dx$.

(f) Suppose that $f_n, n = 1, 2, 3, \ldots$, are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ for each $0 < t < T < \infty$, $|f_n| \le g$, and $f_n \to f$ uniformly on compact subsets of $(0, \infty)$. Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx.$$

(Hint: given $\varepsilon > 0$, break the integral into three segments.)

3. (Based on Rudin 7.13b) Suppose that $f_n : [a, b] \to \mathbb{R}$ is a monotonically increasing function for each n, with $0 \le f_n(x) \le 1$. If $f_n(x) \to f(x)$ pointwise on $[a, b]$ and $f(x)$ is continuous, prove that $f_n \to f$ uniformly on $[a, b]$.

4. Rudin 3.9

5. Rudin 8.1,2,4.

6. (Extra credit +1) Rudin 7.14.

7. (Extra credit +1) Prove the statement in Exercise 8.3 directly, without using Theorem 8.3 (as in class).