Problem set 3 solutions

Math 521 Section 001, UW-Madison, Spring 2024

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1. Let $x \in \mathbb{R}$. Prove that
$$A = \sup\{q \in \mathbb{Q} \mid q < x\} = x.$$

   Solution. (i) By definition, $q < x$ for all $q \in A$, so x is an upper bound.

   (ii) Suppose $\gamma < x$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in Q$ with $\gamma < q < x$. But then $q \in A$, so γ is not an upper bound for A. $\square$

2. Rudin 1.6-9, 1.12-14.

   (1.6) We shall assume the following facts:

   - $b^n \cdot b^m = b^{m+n}$, for $m, n \in \mathbb{Z}$ (provable by induction)
   - $b^{m \cdot n} = (b^m)^n$, for $m, n \in \mathbb{Z}$ (provable by induction)
   - Since $b > 1$, $m < n \iff b^m < b^n$ (follows from Pset 2, problem 2c)
   - If $x, y > 0$, then $x^n = y^n \iff x = y$ (uniqueness of $\sqrt[n]{\cdot}$)
   - If $0 \le x < y$, then $0 \le x^n < y^n$ for all $n \in \mathbb{N}$ (similar to Pset 2, problem 2a)
   - The binomial formula:
   $$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \frac{n(n-1)}{2}x^2y^{n-2} + nxy^n + y^n.$$

   (a) First note that since $\frac{m}{n} = \frac{p}{q}$, we have $mq = np$. Denote $k := mq = np$.

   To prove the claimed equality, since both sides are positive, it suffices to show that the k'th powers of the LHS and the RHS are the same. We have:
$$\begin{aligned}(LHS)^k = \left((b^m)^{1/n}\right)^k &= \left((b^m)^{1/n}\right)^{np} \\ &= \left(\left((b^m)^{1/n}\right)^n\right)^p \\ &= (b^m)^p = b^{mp}.\end{aligned}$$

   On the other hand, we have
$$\begin{aligned}(RHS)^k = \left((b^p)^{1/q}\right)^k &= \left((b^p)^{1/q}\right)^{qm} \\ &= \left(\left((b^p)^{1/q}\right)^q\right)^m \\ &= (b^p)^m = b^{mp}.\end{aligned}$$

Since $LHS^k = RHS^k$, we must have $LHS = RHS$, as desired.

(b) Let $r = \frac{m}{n}$, $s = \frac{p}{q}$. Then

$$r + s = \frac{mq + np}{nq}.$$

We need only check that $LHS^{nq} = RHS^{nq}$. We have

$$LHS^{nq} = (b^{r+s})^{nq} = b^{mq+np},$$

whereas

$$\begin{aligned}(b^r b^s)^{nq} &= b^{rnq} b^{snq} \\ &= b^{mq} b^{pn} \\ &= b^{mq+np}.\end{aligned}$$

Before doing (c), we will prove the following:

Lemma 0.1. *If $b > 1$ and $s < t$, with both $s, t \in \mathbb{Q}$, then $b^s < b^t$.*

Proof. Taking a common denominator, we may write $s = \frac{m}{q}$ and $t = \frac{n}{q}$, with $q > 0$. Since $s < t$, we have $m < n$. If $b^s \geq b^t$, then since b^s, b^t and q are positive, we have $b^{sq} \geq b^{tq}$. But then $b^m \geq b^n$, which is impossible since $m < n$ and $b > 1$. □

(c) Let

$$B(x) = \{b^t \mid t \leq x, t \in \mathbb{Q}\}.$$

Suppose that $x \in \mathbb{Q}$. We need to show that $\sup B(x) = b^x$.

(i) If $t < x$ then $b^t < b^x$ by Lemma 0.1. If $t = x$ then $b^t = b^x$. So b^x is an upper bound.

(ii) Since $b^x \in B(x)$, it must be the least upper bound as usual.

Before proving (d), we will prove the following two Lemmas:

Lemma 0.2. *For any $x > 1$ and $y > 1$, there exists $n \in \mathbb{N}$ such that $x^{\frac{1}{n}} < y$.*

Proof. This is equivalent to choosing n such that $x < y^n$. Write $y = 1 + \alpha$, with $\alpha > 0$. By the binomial formula, we have

$$\begin{aligned}y^n &= (1 + \alpha)^n = 1 + n\alpha + \cdots + \alpha^n \\ &\geq 1 + n\alpha.\end{aligned}$$

Since $\alpha > 0$, by the Archimedean property, we can choose n large enough that $1 + n\alpha > x$. Then $y^n \geq 1 + n\alpha > x$, as desired. □

Lemma 0.3. *Let $\tilde{B}(x) = \{b^t \mid t < x, t \in \mathbb{Q}\}$. Then $\sup \tilde{B}(x) = \sup B(x)$.*

Proof. $(\leq)$ is clear because $\tilde{B}(x) \subset B(x)$.

We now show that strict inequality $<$ is impossible, by contradiction. Note first that strict inequality is only possible if $t \in \mathbb{Q}$, since otherwise $\tilde{B}(x) = B(x)$ are exactly the same sets. By (c), we have $\sup B(x) = b^x$. So we will assuming (for contradiction) that

$$\sup \tilde{B}(x) < b^x.$$

In particular,

$$1 < \frac{b^x}{\sup \tilde{B}(x)}.$$

Applying the previous Lemma, we may choose $n \in \mathbb{N}$ such that

$$1 < b^{\frac{1}{n}} < \frac{b^x}{\sup \tilde{B}(x)}.$$

But then

$$\sup \tilde{B}(x) < b^{x-\frac{1}{n}}.$$

Since $x \in \mathbb{Q}$ and $\frac{1}{n} \in \mathbb{Q}$, we have $x - \frac{1}{n} \in \mathbb{Q}$, so $b^{x-\frac{1}{n}} \in \tilde{B}(x)$. This is a contradiction. $\square$

(d) We need to show that $\sup B(x+y) = \sup B(x) \sup B(y)$. Let $L = \sup B(x+y)$, $M = \sup B(x)$ and $N = \sup B(y)$.

$(\leq)$ By Lemma 0.3, we have $\sup \tilde{B}(x+y) = \sup B(x+y) = L$. Let $r < x+y$. We claim that $r = t+s$ for some $t < x$ and $s < y$. To prove this, note that

$$r - y < x.$$

By density of $\mathbb{Q}$ in $\mathbb{R}$, we may choose $t \in \mathbb{Q}$ with $r - s < t < x$. Rearranging, we have $r - t < y$. Let $s = r - t \in \mathbb{Q}$. Then we have $t < x$, $s < y$, and $s + t = r$, as claimed.

We now have

$$b^r = b^{s+t} = b^s b^t \leq MN,$$

since $s \leq y$ and $t \leq x$. Since $r \in \tilde{B}(x+y)$ was arbitrary, we conclude that MN is an upper bound for $\tilde{B}(x+y)$, as desired.

We now assume for contradiction that strict inequality $(<)$ holds. Then

$$L < MN.$$

Note that for $n \in \mathbb{N}$, we have

$$\left(M - \frac{1}{n}\right)\left(N - \frac{1}{n}\right) = MN - \frac{1}{n}(M+N) + \frac{1}{n^2} \geq MN - \frac{M+N}{n}.$$

By the Archimedean property, we may choose n so that $M + N < n(MN - L)$. Then

$$MN - L > \frac{M+N}{n}$$

and

$$\left(M - \frac{1}{n}\right)\left(N - \frac{1}{n}\right) \geq MN - \frac{M+N}{n} > L.$$

Now, since $M = \sup \tilde{B}(x)$, we may choose a rational $t < x$ such that $M - \frac{1}{n} \leq b^t < M$. Similarly we may choose a rational $s < y$ such that $N - \frac{1}{n} \leq b^s < N$. This gives

$$L < \left(M - \frac{1}{n}\right)\left(N - \frac{1}{n}\right) \leq b^s b^t = b^{s+t} \in B(x+y).$$

But since $L = \sup B(x+y)$, this is a contradiction.

(1.7) (a) We have $b^n - 1 = (b-1)\left(b^{n-1} + b^{n-2} + \cdots + 1\right) \geq n(b-1)$, since there are n terms on the RHS and each is greater than or equal to one.

(b) This follows by applying (a) with $b^{1/n}$ in place of b.

(c) Since $t - 1 > 0$, we may cross-multiply to obtain $n(t-1) > b - 1 \geq n(b^{1/n} - 1)$. The result follows by cancelling n and adding 1 to both sides.

(d) We can either use Lemma 0.2 above or use (c) following Rudin's instructions.

(e) Same as (d).

(f) This follows the usual argument using (d) and (e).

(g) If suffices to show that if $x < z$, for $x, z \in \mathbb{R}$, then $b^x < b^z$. We have this for $t, s \in \mathbb{Q}$ by Lemma 0.1. Let $t, s \in \mathbb{Q}$ with $x \leq t < s \leq z$. Then $b^x = \sup B(x) \leq b^t < b^s \leq \sup B(y) = b^y$.

(1.8) We have $i^2 = -1$ in $\mathbb{C}$. But $-1 < 0$, so if $\mathbb{C}$ were an ordered field, this would contradict Proposition 1.18d.

(1.9) (i) We need to show that either $w < z, w = z$, or $w > z$. Either $a < c$, in which case $w < z$, or $a > c$, in which case $w > z$, or $a = c$. In case $a = c$, either $b > d$, in which case $w > z$, or $b < d$, in which case $w < z$, or $b = d$, in which case both $a = c$ and $b = d$, so $w = z$. So we're done.

(ii) Suppose $z < w < u$, and write $r = e + if$. In the first case, $a < c$. Then we must have $c \leq e$, in which case $a < e$, so $z < u$ and we're done. In the second case, $a = c$ and $b < d$. Then either $c < e$, in which case $a < e$, or $c = e$ and $d < f$, in which case $a = c = e$ and $a < d < f$, so $z < u$ and we're done.

We claim that the least-upper-bound property fails. We must exhibit a set E which is bounded above but does not have a least upper bound in $\mathbb{C}$ with the lexicographic order. Take the imaginary axis

$$E = \{iy \mid y \in \mathbb{R}\} \subset \mathbb{C}.$$

This is bounded above by $1 \in \mathbb{C}$.

Suppose that $z = a + bi$ is the least upper bound of E. Then we must have $a \geq 0$, and if $a > 0$ then $a/2 < a + bi$ is also an upper bound, so z is not the least upper bound. Hence $a = 0$, so $z = bi$. But then letting $y = b + 1$, we have $(b+1)i > z$ and $(b+1)i \in E$, so z is again not an upper bound. This is a contradiction.

(1.12) This follows by induction from Theorem 1.37(e), as follows. The base case $n = 1$ is trivial. Suppose that the inequality is true for $n-1$, we have

$$\begin{aligned}|z_1 + \cdots z_n| = |(z_1 + \cdots z_{n-1}) + z_n| &\leq |z_1 + \cdots + z_{n-1}| + |z_n| \\ &\leq |z_1| + \cdots + |z_{n-1}| + |z_n|.\end{aligned}$$

This completes the induction.

(1.13) Note that $y + (x - y) = x$, so by the triangle inequality (Theorem 1.37(e)), we have

$$|x| \leq |y| + |x - y|.$$

This gives

$$|x| - |y| \leq |x - y|.$$

Reversing the roles of x and y, we also have

$$|y| - |x| \leq |y - x| = |x - y|.$$

Overall, we obtain

$$||y| - |x|| \leq |x - y|$$

as desired.

(1.14) We calculate

$$\begin{aligned}|1 + z|^2 + |1 - z|^2 &= (1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) \\ &= 1 + 2z\bar{z} + |z|^2 + 1 - 2z\bar{z} + |z|^2 \\ &= 2 + 2|z|^2 = 4.\end{aligned}$$

3. Rudin 2.2-4.

(2.2) We need to assume the fact that for any such equation there are at most n distinct roots $z_1, \ldots, z_n \in \mathbb{C}$. (This follows from the "division algorithm," whereby one can factor any polynomial over $\mathbb{C}$ into the product of its roots.)

Now, the set of all polynomials of degree n is countable, since it is in bijection with $\mathbb{Z}^{n+1}$. Since each polynomial has at most n roots, the set of roots of n'th degree polynomials is a countable union of finite sets, hence countable. Taking the union over n, we obtain the set of all algebraic numbers as a countable union of countable sets, which is therefore countable.

(2.3) Since the set of algebraic numbers is countable, it must be a proper subset of $\mathbb{R}$, which is uncountable.

(2.4) Let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. If B were countable then since A is countable, $\mathbb{R} = A \cup B$ would also be countable. But $\mathbb{R}$ is uncountable.

4. (Extra credit) Write down an explicit bijection between the sets $\{0,1,2,3\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$.

Solution. We can make a bijection between $\{0,1,2,3\}$ and $\{0,1\} \times \{0,1\}$ by taking

$$\begin{aligned} \{0,1,2,3\} &\to \{0,1\} \times \{0,1\} \\ 0 &\mapsto (0,0) \\ 1 &\mapsto (0,1) \\ 2 &\mapsto (1,0) \\ 3 &\mapsto (1,1). \end{aligned}$$

To construct a bijection between $\{0,1,2,3\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}}$, we can do the following: a sequence

$$(3,1,0,2,\ldots)$$

gets mapped to the sequence

$$(1,0,0,1,0,0,1,0,\ldots).$$

To write this down formally, suppose that our bijection from $\{0,1,2,3\} \to \{0,1\} \times \{0,1\}$ is written as

$$a \mapsto (f(a), g(a)).$$

Then our map is given by

$$\begin{aligned} \{0,1,2,3\}^{\mathbb{N}} &\to \{0,1\}^{\mathbb{N}} \\ (a_1, a_2, a_3, \ldots) &\mapsto (f(a_1), g(a_1), f(a_2), g(a_2), f(a_3), g(a_3), \ldots). \end{aligned}$$

□