Last time: End of ch. 1 of Andreescu.

Today: Ch. 2

Recall: $f : S \to T$ is called:

- Injective if $f(x) = f(x') \Rightarrow x = x'$, i.e. $\forall y \in T$ $\exists$ at most one $x \in S$ s.t. $f(x) = y$.
- Surjective if $f(S) = T$, i.e. $\forall y \in T$ $\exists$ at least one $x \in S$ s.t. $f(x) = y$
- Bijective if both inj. & surj., i.e. $\forall y \in T$ $\exists!$ $x \in S$ s.t. $f(x) = y$.

Inj: $\hookrightarrow$

Surj: $\twoheadrightarrow$

Bij: $\xrightarrow{\sim}$

Lemma: Suppose that $g : S \to U$ and $f : U \to T$ are both injective (resp. surjective). Then

$$f \circ g : S \to T$$

is also injective (resp. surj.).

$S \quad T \quad U$

Pf: Inj: $f(g(x)) = f(g(x')) \Rightarrow g(x) = g(x') \Rightarrow x = x'$ ✓.

Surj: Given $y \in T$, $\exists$ $u \in U$ s.t. $f(u) = y$. $\Rightarrow f(g(x)) = y$ ✓
$\exists$ $x \in S$ s.t. $g(x) = u$.

$\square$

<u>Theorem:</u> $f : S \to T$ is bijective iff $\exists$ $g : T \to S$, called the "inverse" function, s.t.

$$f \circ g = Id \quad \text{and} \quad g \circ f = Id.$$

i.e. $f(g(y)) = y \ \forall y \in T$, $\quad g(f(x)) = x \ \forall x \in S$.

Pf: ($\Rightarrow$) Given $y \in T$, $\exists! x$ s.t. $f(x) = y$.

Define $g(y) = x$. ✓.

($\Leftarrow$) Need to check f is inj & surj.

<u>Inj:</u> $f(x) = f(x')$

$$\Rightarrow \underbrace{g(f(x))}_{=x} = \underbrace{g(f(x'))}_{=x'} \quad \Rightarrow x = x' \quad ✓.$$

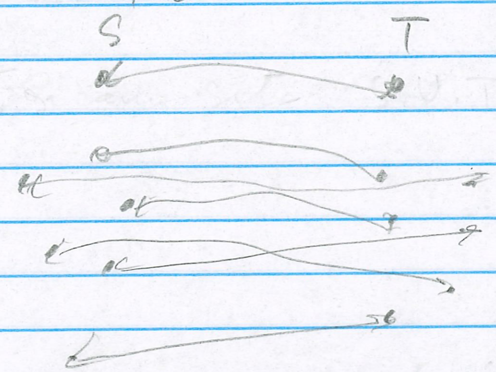
<u>Surj:</u> Given $y \in T$, let $x = g(y)$.

Then $f(x) = f(g(y)) = y$ ✓. □

<u>Notation:</u> $g(y) =$ "$f^{-1}(y)$".

⚠ Not same as inverse image of a set: $f^{-1}(E)$.

Definition: We write $S \sim T$ and say "S and T have the same cardinality" if $\exists$ a bijective map $S \to T$.



Properties of $\sim$:

(a) $S \sim S$ (reflexive)

(b) $S \sim T \Rightarrow T \sim S$ (symmetric)

(c) $S \sim U$ and $U \sim T \Rightarrow S \sim T$ (transitive).

Note: "$\sim$" is called an "equivalence relation".

Pf: (a) $Id : S \xrightarrow{\sim} S$, $x \mapsto x$ ✓ (b) $f : S \xrightarrow{\sim} T$, $f^{-1} : T \xrightarrow{\sim} S$ (c) $f, g \rightsquigarrow f \circ g : S \xrightarrow{\sim} T$.

Defn: Given $n \in \mathbb{N}$, let $J_n = \{1, \dots, n\} \subset \mathbb{N}$.

A set S is called (Rudin: $J = \mathbb{N}$).

finite if $S \sim J_n$ for some $n \in \mathbb{N}$.

We write $|S| = n$.

- S is called <u>infinite</u> otherwise.

- S is called <u>countably infinite</u> if $S \sim \mathbb{N}$

  (Rudin: "countable")

- S is called <u>countable</u> if either finite or countably infinite.

  (Rudin: "at most countable")

- S is called <u>uncountable</u> if it is not countable.

<u>Examples</u>: • $A = \{*, ✠, @, 1, \text{Biden}\}$.

Define
$$1 \mapsto *$$
$$2 \mapsto @$$
$$3 \mapsto \text{Biden}$$
$$4 \mapsto 1$$
$$5 \mapsto ✠.$$

$\Rightarrow A \sim J_5 \Rightarrow A$ is finite with $|A| = 5$.

- $B = \{n \in \mathbb{N} \mid n \text{ is even}\}$.

Define $\mathbb{N} \xrightarrow{\sim} B$ $\Rightarrow$ B is countably infinite.
$n \mapsto 2n$

$B = \{2, 4, 6, 8, \ldots\}$.

• $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N} \to \mathbb{Z}$

Define
$1 \mapsto 0$
$2 \mapsto -1$
$3 \mapsto 1$
$4 \mapsto -2$
$5 \mapsto 2$
$\vdots$
etc.

$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3\}$

✓

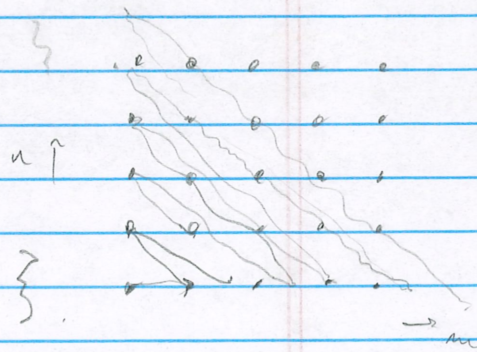
$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ odd} \\ -\frac{n}{2} & n \text{ even.} \end{cases}$$

Note: defining $f: \mathbb{N} \to A$ same as "indexing" or "listing" the elements of S.

$$A = \{f(1), f(2), f(3), \ldots\}$$
$$= \{a_1, a_2, a_3, \ldots\}$$

• $\mathbb{N} \times \mathbb{N} = \{(m,n) \mid m, n \in \mathbb{N}\}$.

$= \{(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), (4,1) \ldots\}$.



$\Rightarrow$ $\mathbb{N} \times \mathbb{N}$ is countably infinite!

("Cantor's first diagonal argument").

<u>Definition</u>: Let S and T be sets.

$$T^S \subset \mathcal{P}(S \times T)$$

denotes the set of all functions from S to T.

since a function is uniquely defined by its graph.

<u>Ex</u>: $S = \mathbb{N}$. a function $f: \mathbb{N} \to T$ is the same thing as a <u>sequence</u>

$$\overset{f(1)}{t_1}, \overset{f(2)}{t_2}, \overset{f(3)}{t_3}, \ldots$$

of elements of T, <u>indexed</u> by $n \in \mathbb{N}$.

<u>Ex</u>: $\{0,1\}^{\mathbb{N}} = \{(t_1, t_2, \ldots) \mid t_n = 0 \text{ or } 1 \text{ for } n \in \mathbb{N}\}$.

Some elements:

$(0,1,0,1,0,1, \ldots) \in \{0,1\}^{\mathbb{N}}$

$(0,1,1,0,0,0,1,1,1,1, \ldots) \in \{0,1\}^{\mathbb{N}}$

$\left(\begin{matrix}\text{bits transmitted over the} \\ \text{internet since its inception}\end{matrix}\right) \in \{0,1\}^{\mathbb{N}}$.

<u>Next time</u>:

<u>Theorem</u>: $\{0,1\}^{\mathbb{N}}$ is uncountable.

<u>Theorem</u>: $\mathbb{R}$ is uncountable.