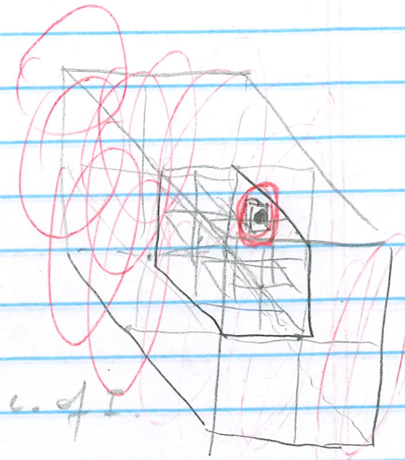


Recall: Want to characterize compact subsets of $\mathbb{R}^k$.

Thm: k-cells are compact.

Pf: Contradiction. Suppose $\{G_\alpha\}$ is o.c. of I.

Create nested k-cells $I_1 \supset I_2 \supset I_3 \supset \cdots$

each not covered by finite subcover.

$$\bigcap_{n=1}^{\infty} I_n = x^* \in G_\alpha \text{ for some } \alpha.$$

$\Rightarrow$ $I_n \subset G_\alpha$ for n suff. large

Contradiction, I_n is covered by finite subcover. $\times$ $\square$.

Theorem (Heine-Borel)

For a set $E \subset \mathbb{R}^k$, TFAE:

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E.

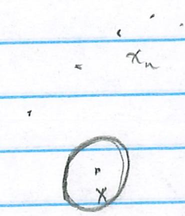
Pf: We showed that (b) $\Rightarrow$ (c) in any metric space. (also (b) $\Rightarrow$ (a))

any metric space. Remains to show (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) because if E is bounded then $E \subset I$ for some (large) k-cell I, and closed subsets of cpt sets are cpt.

(c) $\Rightarrow$ (a).

Bounded. If E is not bounded, then E contains points $\{x_n\}$ with $|x_n| > n$, for $n = 1, 2, \ldots$



$\rightsquigarrow$ has no limit pt. in $\mathbb{R}^n$

(b/c $n < d(0, x_n) \leq d(0, x) + d(x, x_n)$

$\Rightarrow \quad n - d(0,x) \leq d(x, x_n)$

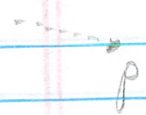
$\Rightarrow \quad 1 < d(x, x_n) \quad$ for $n \geq \lfloor d(0,x) \rfloor + 1$

$\Rightarrow$ only finitely many pts of $\{x_n\}$ lie in $N_1(x)$

$\Rightarrow$ x is not a lim. pt.)

Closed. If E is not closed, $\exists\ p \in E'$ s.t. $p \notin E$. $\Rightarrow$ for each n, $\exists\ x_n \overset{\neq p}{} \in E$ s.t. $|x_n - p| < \frac{1}{n}$.

Let $S = \{x_n\}_{n=1}^{\infty}$.



Then S is infinite, $S \subset E$, but

S has only one limit pt. in $\mathbb{R}^k$, namely, p.

(For if $y \neq p$ then

$$|x_n - y| \geq |p-y| - |x_n - p|$$

$$\geq |p-y| - \frac{1}{n}$$

$$\geq \frac{1}{2}|p-y| \quad \text{for } n \text{ suff. large.}$$

$\Rightarrow N_{\frac{1}{2}|p-y|}(y)$ contains only finitely many points of S.)

But $p \notin E$, so (c) is false.

Contrapositively, (c) $\Rightarrow$ E is closed.

& (c) $\Rightarrow$ (a). $\square$

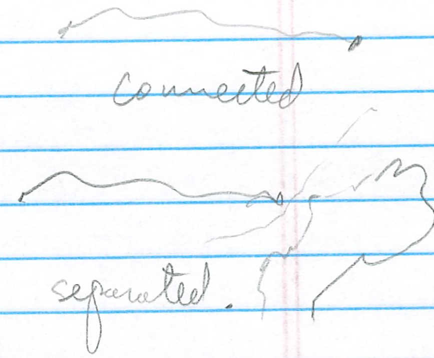
<u>Corollary</u> (Weierstrass) Every bounded, infinite subset of $\mathbb{R}^k$ has a limit point.

<u>Pf</u>: $E \subset I$ for some k-cell I, which is compact.

(c) $\Rightarrow$ E has a limit pt. in I. $\square$

Connectedness.

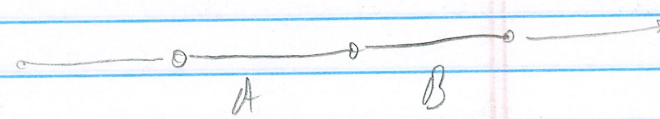
More intuitive property than cptness.



Defn: Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty

~~$E \subset X$ is called connected if E is not a union of two nonempty separated sets.~~

Ex: $A = (0,1)$, $B = (1,2)$



$\bar{A} = [0,1]$ $\quad \bar{B} = [1,2]$.

$A \cap \bar{B} = \phi = \bar{A} \cap B$.

More generally: disjoint union of two open sets or two closed sets.

Separated.

$B = \{\pi\}$.

Ex: $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$.

A is dense in $\mathbb{R}$, i.e. $\bar{A} = \mathbb{R}$

$\Rightarrow \bar{A} \cap B \neq \phi$.

Not separated.

Ex: $A = [0,1]$, $B = (1,2)$. Not separated.

<u>Defn</u>: $E \subset X$ is <u>connected</u> if E is not the union of two nonempty separated sets.

<u>Theorem</u>: A subset $E \subset \mathbb{R}^1$ is <u>connected</u> if and only if it has the following property:

If $x \in E$, $y \in E$ and $x < y < z$, then $z \in E$.

<u>Pf</u>: ($\Rightarrow$) Suppose $x < z < y$ and $z \notin E$.

Then $E = A_z \cup B_z$, where $A_z = (-\infty, z) \cap E \ni x$

$B_z = (z, +\infty) \cap E \ni y$

$\Rightarrow$ both nonempty.

Since $A_z \subset (-\infty, z]$, which is closed,

$\bar{A}_z \subset (-\infty, z]$.

$\Rightarrow \bar{A}_z \cap B_z = \phi$.

similarly, $\bar{B}_z \cap A_z = \phi$

Since $E = A_z \cup B_z$, and these are separated,

E is not connected.

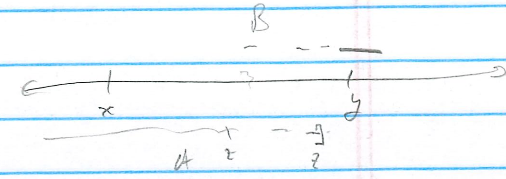
($\Leftarrow$) Suppose E is not connected.

Then $E = A \cup B$, where $\bar{A} \cap B = \phi = A \cap \bar{B}$.

"without loss of generality"

Pick $x \in A$, $y \in B$, and assume wlog $x < y$

Define $z = \sup(A \cap [x, y])$.



Then $z \in \overline{A}$, so $z \notin B$.

$\Rightarrow x \leq z < y$.

If $z \notin A$, then $z \notin E$ and $z \neq x$ so $x < z < y$, and we're done.

If $z \in A$, then $z \notin \overline{B}$, so $\exists\, z_1 \notin B$ s.t. $z < z_1 < y$

$\Rightarrow x < z_1 < y$.

Then also $z_1 \notin A$, so $z_1 \notin E$. $\square$

Corollary: (a,b), $[a,b)$, $[a,b]$ are all connected!

Ex: $\mathbb{Q}$ is not connected.

$$\mathbb{Q} = (-\infty, \sqrt{2}) \cap \mathbb{Q} \cup (\sqrt{2}, \infty) \cap \mathbb{Q}$$

"Totally disconnected"