Let's say we want to add three particles with spin 1/2 (for example three electrons). Each of those spins resides in a 2 dimensional Hilbert space. Adding all three will result in the tensor product of these three Hilbert spaces like this:

$$\mathcal{H}^{total} = \mathcal{H}^a \otimes \mathcal{H}^b \otimes \mathcal{H}^c$$

Since each of our Hilbert spaces is 2 dimensional, the total Hilbert space will be 8 dimensional. Thus it must have 8 base vectors (or states) that are orthogonal to each other and normalized. These states can either be the different combinations of "up" and "down" spins made up by the electrons or they can be the eigenvectors of the total spin and the total z axis projection finding this later states can be done as follows:

Take the first two spins and find the "top" state (the state with the highest z-spin component)

$$|\underbrace{j=1}_{\text{total spin}},\quad \overbrace{m_j=1}^{\text{total z component of spin}}\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 = |\uparrow_1\uparrow_2\rangle = \uparrow\uparrow$$

Apply the lowering operator until you get to the "bottom" state (the state with lowest spin projection)

$$\begin{array}{c} \uparrow\uparrow \\ \downarrow \scriptstyle{Lowering} \\ \frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow) \\ \downarrow \scriptstyle{Lowering} \\ \downarrow\downarrow \end{array}$$

Now we have created what I call in the code an "Object": it's a composite state that is made up from smaller spins. However, it has a well defined total spin (in this case the total spin is 1) for all purposes it behaves exactly like a spin-1 particle

$$\uparrow\uparrow \xrightarrow{Equivalent} \uparrow \qquad = |j=1, m_j=1\rangle$$

$$\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow) \xrightarrow{Equivalent} \bullet \qquad = |j=1, m_j=0\rangle$$

$$\downarrow\downarrow \xrightarrow{Equivalent} \downarrow \qquad = |j=1, m_j=-1\rangle$$

Now go to the state with the second highest spin projection and find the state with the same spin projection that is orthogonal to this.

$$\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow) \xrightarrow{Orthogonal} \frac{1}{\sqrt{2}}(\downarrow\uparrow - \uparrow\downarrow)$$

Normally you need to apply the Lowering operator again until you find the lowest state of that Object but this is a singlet Object, it only has one state (if we applied the lowering operator to it, we could see that the resulting state would have a 0 coefficient).Again this here is an object that behaves exactly like a spin-0 particle this time.

Now we need to add these two Objects to our third spin. First we add the first Object. Since it behaves exactly like a spin 1 particle, we can do the algebra with this virtual spin-1 Object and add to it the elctron, and later we can substitute its own states. Remember the first "particle" in these diagrams is spin 1 while the second one is spin 1/2 Again we find the highest state which in this case is [1 , 1/2]

$$\uparrow\uparrow$$

and we apply the lowering operator to complete the object (this new object here behaves like a spin 3/2 particle)

$$\begin{array}{c} \uparrow\uparrow \\ \Big\downarrow {\scriptstyle Lowering} \\ \frac{\sqrt{6}}{3}(\bullet\uparrow) + \frac{\sqrt{3}}{3}(\uparrow\downarrow) \\ \Big\downarrow {\scriptstyle Lowering} \\ \frac{\sqrt{3}}{3}(\downarrow\uparrow) + \frac{\sqrt{6}}{3}(\bullet\downarrow) \\ \Big\downarrow {\scriptstyle Lowering} \\ \downarrow\downarrow \end{array}$$

Now we orthogonalize again from the second highest spin projection state to find the next object.

$$\frac{\sqrt{6}}{3}(\bullet\uparrow) + \frac{\sqrt{3}}{3}(\uparrow\downarrow) \xrightarrow{Orthogonal} \frac{\sqrt{3}}{3}(\bullet\uparrow) - \frac{\sqrt{6}}{3}(\uparrow\downarrow)$$

Again we lower until we complete the object.

$$\begin{array}{c} \frac{\sqrt{3}}{3}(\bullet\uparrow) - \frac{\sqrt{6}}{3}(\uparrow\downarrow) \\ \Big\downarrow {\scriptstyle Lowering} \\ \frac{\sqrt{6}}{3}(\downarrow\uparrow) - \frac{\sqrt{3}}{3}(\bullet\downarrow) \end{array}$$

Then we add to the whole mix the addition of the Singlet Object we had earlier with the electron. Since it's a singlet it can be treated as a spin-0 particle and with the same principle as earlier we find the highest spin state:

$$(\bullet\uparrow)$$

And now by lowering we ge the rest of the states (in this case one more)

$$
\begin{array}{c}
(\bullet\uparrow) \\
\big\downarrow {\scriptstyle Lowering} \\
(\bullet\downarrow)
\end{array}
$$

Again remember that the first spin here is a spin 0 state. Finally we have to substitute the "virtual states" with the actual ones

$$\uparrow\uparrow \xrightarrow{Equivalent} (\uparrow\uparrow)\uparrow = \uparrow\uparrow\uparrow$$

$$\frac{\sqrt{6}}{3}(\bullet\uparrow) + \frac{\sqrt{3}}{3}(\uparrow\downarrow) \xrightarrow{Equivalent} \frac{\sqrt{6}}{3}((\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow)\uparrow) + \frac{\sqrt{3}}{3}((\uparrow\uparrow)\downarrow) = \frac{\sqrt{3}}{3}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$$

$$\frac{\sqrt{3}}{3}(\downarrow\uparrow) + \frac{\sqrt{6}}{3}(\bullet\downarrow) \xrightarrow{Equivalent} \frac{\sqrt{3}}{3}((\downarrow\downarrow)\uparrow) + \frac{\sqrt{6}}{3}((\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow))\downarrow) = \frac{\sqrt{3}}{3}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$$

$$\downarrow\downarrow \xrightarrow{Equivalent} (\downarrow\downarrow)\downarrow = \downarrow\downarrow\downarrow$$

$$\frac{\sqrt{3}}{3}(\bullet\uparrow) - \frac{\sqrt{6}}{3}(\uparrow\downarrow) \xrightarrow{Equivalent} \frac{\sqrt{3}}{3}((\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow))\uparrow) - \frac{\sqrt{6}}{3}((\uparrow\uparrow)\downarrow) = \frac{\sqrt{6}}{6}\downarrow\uparrow\uparrow + \frac{\sqrt{6}}{6}\uparrow\downarrow\uparrow - \frac{\sqrt{6}}{3}\uparrow\uparrow\downarrow$$

$$\frac{\sqrt{6}}{3}(\downarrow\uparrow) - \frac{\sqrt{3}}{3}(\bullet\downarrow) \xrightarrow{Equivalent} \frac{\sqrt{6}}{3}((\downarrow\downarrow)\uparrow) - \frac{\sqrt{3}}{3}((\frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow))\downarrow) = \frac{\sqrt{6}}{3}\downarrow\downarrow\uparrow - \frac{\sqrt{6}}{6}\downarrow\uparrow\downarrow - \frac{\sqrt{6}}{6}\uparrow\downarrow\downarrow$$

$$(\bullet\uparrow) \xrightarrow{Equivalent} (\frac{1}{\sqrt{2}}(\downarrow\uparrow - \uparrow\downarrow))\uparrow = \frac{1}{\sqrt{2}}(\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow)$$

$$(\bullet\downarrow) \xrightarrow{Equivalent} (\frac{1}{\sqrt{2}}(\downarrow\uparrow - \uparrow\downarrow))\downarrow = \frac{1}{\sqrt{2}}(\downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow)$$

So all 8 states are:

$$|j=3/2, m_j=3/2\rangle = \uparrow\uparrow\uparrow$$
$$|j=3/2, m_j=1/2\rangle = \frac{\sqrt{3}}{3}(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow)$$
$$|j=3/2, m_j=-1/2\rangle = \frac{\sqrt{3}}{3}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$$
$$|j=3/2, m_j=-3/2\rangle = \downarrow\downarrow\downarrow$$

$$|j=1/2, m_j=1/2\rangle = \frac{\sqrt{6}}{6}\downarrow\uparrow\uparrow + \frac{\sqrt{6}}{6}\uparrow\downarrow\uparrow - \frac{\sqrt{6}}{3}\uparrow\uparrow\downarrow$$
$$|j=1/2, m_j=-1/2\rangle = \frac{\sqrt{6}}{3}\downarrow\downarrow\uparrow - \frac{\sqrt{6}}{6}\downarrow\uparrow\downarrow - \frac{\sqrt{6}}{6}\uparrow\downarrow\downarrow$$

$$|j=1/2, m_j=1/2\rangle = \frac{1}{\sqrt{2}}(\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow)$$
$$|j=1/2, m_j=-1/2\rangle = \frac{1}{\sqrt{2}}(\downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow)$$

Running the code to add three spin 1/2 particles produces the following result:

```
{3/2: Total Spin 3/2,   z-component of spin 3/2
[' Coefficient: 1 Bases: [1/2, 1/2, 1/2]']

, 1/2: Total Spin 3/2,  z-component of spin 1/2
[' Coefficient: sqrt(3)/3 Bases: [-1/2, 1/2, 1/2]', ' Coefficient: sqrt(3)/3 Bases: [1/2, -1/2, 1/2]',
' Coefficient: sqrt(3)/3 Bases: [1/2, 1/2, -1/2]']

, -1/2: Total Spin 3/2, z-component of spin -1/2
[' Coefficient: sqrt(3)/3 Bases: [-1/2, -1/2, 1/2]', ' Coefficient: sqrt(3)/3 Bases: [-1/2, 1/2, -1/2]'
, ' Coefficient: sqrt(3)/3 Bases: [1/2, -1/2, -1/2]']

, -3/2: Total Spin 3/2, z-component of spin -3/2
[' Coefficient: 1 Bases: [-1/2, -1/2, -1/2]']

}



{1/2: Total Spin 1/2,   z-component of spin 1/2
[' Coefficient: sqrt(6)/6 Bases: [-1/2, 1/2, 1/2]', ' Coefficient: sqrt(6)/6 Bases: [1/2, -1/2, 1/2]',
' Coefficient: -sqrt(6)/3 Bases: [1/2, 1/2, -1/2]']

, -1/2: Total Spin 1/2, z-component of spin -1/2
[' Coefficient: sqrt(6)/3 Bases: [-1/2, -1/2, 1/2]', ' Coefficient: -sqrt(6)/6 Bases: [-1/2, 1/2, -1/2]
', ' Coefficient: -sqrt(6)/6 Bases: [1/2, -1/2, -1/2]']

}



{1/2: Total Spin 1/2,   z-component of spin 1/2
[' Coefficient: sqrt(2)/2 Bases: [-1/2, 1/2, 1/2]', ' Coefficient: -sqrt(2)/2 Bases: [1/2, -1/2, 1/2]']


, -1/2: Total Spin 1/2, z-component of spin -1/2
[' Coefficient: sqrt(2)/2 Bases: [-1/2, 1/2, -1/2]', ' Coefficient: -sqrt(2)/2 Bases: [1/2, -1/2, -1/2]
']

}
```

As you can see, it produced the same states, "+1/2" corresponds to $\uparrow$ and "-1/2" to $\downarrow$ the coefficients are again the same. The algorithm works exactly the same way we derived those states, if we had more particles to add, it would regard the first group of four states as a virtual object with spin 3/2 and the other two pairs a virtual objects of spin 1/2 each. Then it would add these virtual objects to the remaining particle and would produce the states.