15.094J: Robust Modeling, Optimization, Computation

Lecture 12: Affinely Adaptive Optimization

Outline

1. Motivation
2. Preliminaries
3. Optimality of affine policies
4. Suboptimality of affine policies
5. Affine policies in inventory theory
6. Polynomial polices in multi-echelon systems
7. Conclusions

Motivation

- Affine policies have strong empirical performance.
- Under what circumstances affine policies are optimal?
- How suboptimal are they?
- How can we improve them?

Witnesses of robustness

- AO:

$$
\begin{aligned}
z_{Adapt}(\mathcal{U}) = \min \ c^T x + \max_{b \in \mathcal{U}} d^T y(b) \\
Ax + By(b) \geq b, \ \forall b \in \mathcal{U} \\
x, y(b) \geq 0,
\end{aligned}
$$

- Suppose $x^*, y^*(b)$ for all $b \in \mathcal{U}$ is an optimal solution of AO, where the uncertainty set $\mathcal{U}$ is a polytope. Let $b^1, \ldots, b^K$ be the extreme points of $\mathcal{U}$. Then, the worst case cost is achieved at some extreme point, i.e.,

$$
\max_{b \in \mathcal{U}} d^T y^*(b) = \max_{j=1,\ldots,K} d^T y^*(b^j).
$$

Proof

- $\{b^1, \ldots, b^K\} \subseteq \mathcal{U}$:

$$\max_{b \in \mathcal{U}} d^T y^*(b) \geq \max_{j=1,\ldots,K} d^T y^*(b^j).$$

- For the sake of contradiction, suppose

$$\max_{b \in \mathcal{U}} d^T y^*(b) > \max_{j=1,\ldots,K} d^T y^*(b^j).$$

Let $\hat{b} = \text{argmax}\{d^T y^*(b) \mid b \in \mathcal{U}\}$, such that $\hat{b} \notin \{b^1, \ldots, b^K\}$.

- Therefore,

$$d^T y^*(\hat{b}) > \max_{j=1,\ldots,K} d^T y^*(b^j).$$

- Since $\hat{b} \in \mathcal{U}$, $\hat{b} = \sum_{j=1}^K \alpha_j \cdot b^j$, where $\alpha_j \geq 0$ for all $j = 1, \ldots, K$ and $\alpha_1 + \ldots + \alpha_K = 1$.

Proof, continued

- Consider the solution: $\hat{y}(\hat{b}) = \sum_{j=1}^{K} \alpha_j \cdot y^*(b^j)$.
- $\hat{y}(\hat{b})$ is feasible for $\hat{b}$ as,

$$Ax^* + B\hat{y}(\hat{b}) = A\left(\sum_{j=1}^{K} \alpha_j\right) x^* + B\left(\sum_{j=1}^{K} \alpha_j \cdot y^*(b^j)\right) =$$

$$\sum_{j=1}^{K} \alpha_j \cdot Ax^* + \sum_{j=1}^{K} \alpha_j \cdot By^*(b^j) = \sum_{j=1}^{K} \alpha_j \cdot \left(Ax^* + By^*(b^j)\right) \geq \sum_{j=1}^{K} \alpha_j \cdot b^j = \hat{b},$$

- Objective function value:

$$\begin{aligned} d^T\hat{y}(\hat{b}) &= d^T\left(\sum_{j=1}^{K} \alpha_j \cdot y^*(b^j)\right) = \sum_{j=1}^{K} \alpha_j \cdot d^T y^*(b^j) \\ &\leq \sum_{j=1}^{K} \alpha_j \cdot \max\{d^T y^*(b^k) \mid k = 1, \dots, K\} \\ &= \max\{d^T y^*(b^k) \mid k = 1, \dots, K\} \\ &< d^T y^*(\hat{b}). \end{aligned}$$

- This implies that $y^*(\hat{b})$ is not an optimal solution for $\hat{b}$; a contradiction.

Optimality of affine policies over the simplex

- For AO with
$$\mathcal{U} = \mathrm{conv}(b^1, \ldots, b^{m+1}),$$
- $b^j \in \mathbb{R}^m_+$ for all $j = 1, \ldots, m$ such that $b^1, \ldots, b^{m+1}$ are affinely independent.
- Then, there is an optimal two-stage solution $\hat{x}, \hat{y}(b)$ for all $b \in \mathcal{U}$ such that $\hat{y}(b)$ is an affine function of b, i.e., for all $b \in \mathcal{U}$,
$$\hat{y}(b) = Pb + q,$$

Proof

- $x^*, y^*(b)$ optimal for AO.

$$Q = \left[\left(b^1 - b^{m+1}\right), \ldots, \left(b^m - b^{m+1}\right)\right]$$

$$Y = \left[\left(y^*(b^1) - y^*(b^{m+1})\right), \ldots, \left(y^*(b^m) - y^*(b^{m+1})\right)\right]$$

- Since $b^1, \ldots, b^{m+1}$ are affinely independent, $(b^1 - b^{m+1}), \ldots, (b^m - b^{m+1})$ are linearly independent.
- Q is a full-rank matrix and thus, invertible. For any $b \in \mathcal{U}$:

$$\hat{y}(b) = YQ^{-1}\left(b - b^{m+1}\right) + y^*(b^{m+1}).$$

- Since $b \in \mathcal{U}$, $b = \sum_{j=1}^{m+1} \alpha_j b^j$, where $\alpha_j \geq 0$ for all $j = 1, \ldots, m+1$ and $\alpha_1 + \ldots + \alpha_{m+1} = 1$.

Proof, continued

- We have

$$b = \sum_{j=1}^{m} \alpha_j b^j + \left(1 - \sum_{j=1}^{m} \alpha_j\right) b^{m+1} = \sum_{j=1}^{m} \alpha_j \left(b^j - b^{m+1}\right) + b^{m+1}$$

$$= Q \cdot \alpha + b^{m+1},\ \alpha = (\alpha_1, \ldots, \alpha_m)^T$$

- Since Q is invertible, $Q^{-1}\left(b - b^{m+1}\right) = \alpha$, and thus

$$\begin{aligned}
\hat{y}(b) &= Y \cdot \alpha + y^*(b^{m+1}) \\
&= \sum_{j=1}^{m} \alpha_j \left(y^*(b^j) - y^*(b^{m+1})\right) + y^*(b^{m+1}) \\
&= \sum_{j=1}^{m} \alpha_j y^*(b^j) + \left(1 - \sum_{j=1}^{m} \alpha_j\right) y^*(b^{m+1}) \\
&= \sum_{j=1}^{m+1} \alpha_j y^*(b^j)
\end{aligned}$$

Proof, continued

- As before, $\hat{y}(b)$ is a feasible solution for all $b \in \mathcal{U}$.
- Since the worst case occurs at one of the extreme points of $\mathcal{U}$,

$$z_{Adapt}(\mathcal{U}) = \max_{b \in \mathcal{U}} \left(c^T x^* + d^T y^*(b)\right) = \max_{j=1,\ldots,m+1} \left(c^T x^* + d^T y^*(b^j)\right).$$

- Note that $\hat{y}(b^j) = y^*(b^j)$ for all $j = 1, \ldots, m+1$. Therefore,

$$\begin{aligned} \max_{b \in \mathcal{U}} \left(c^T x^* + d^T \hat{y}(b)\right) &= \max_{j=1,\ldots,m+1} \left(c^T x^* + d^T \hat{y}(b^j)\right) \\ &= \max_{j=1,\ldots,m+1} \left(c^T x^* + d^T y^*(b^j)\right) \\ &= z_{Adapt}(\mathcal{U}). \end{aligned}$$

Suboptimality of Affine Policies for Uncertainty Sets with $(m+2)$ Extreme Points

- Data $c = 0,\ d = (1, \dots, 1)',\ A = 0$, and for all $j = 1, \dots, m$

$$B_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{\sqrt{m}} & \text{otherwise} \end{cases}$$

- $\mathcal{U} = \mathrm{conv}\left(\left\{b^0, b^1, \dots, b^{m+2}\right\}\right),\ b^0 = 0,\ b^j = e_j,\ \forall j = 1, \dots, m$

$$b^{m+1} = \left(\underbrace{\frac{1}{\sqrt{m}}, \dots, \frac{1}{\sqrt{m}}}_{m/2}, \underbrace{0, \dots, 0}_{m/2} \right),\quad b^{m+2} = \left(\underbrace{0, \dots, 0}_{m/2}, \underbrace{\frac{1}{\sqrt{m}}, \dots, \frac{1}{\sqrt{m}}}_{m/2} \right)$$

- Given any $\delta > 0$, consider AO with data and uncertainty set $\mathcal{U}$ as above. Then,

$$z_{Aff}(\mathcal{U}) > (2 - \delta) \cdot z_{Adapt}(\mathcal{U}).$$

A Large Gap Example for Affine Policies

- Data $n_1 = n_2 = m$, $m^\delta > 200$, $c = 0$, $d = (1, \ldots, 1)^T$, $A = 0$,

$$B_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \theta_0 & \text{otherwise} \end{cases}$$

- $\mathcal{U} = \text{conv}\left(\left\{b^0, b^1, \ldots, b^N\right\}\right)$, $\theta_0 = \frac{1}{m^{(1-\delta)/2}}$, $r = \lceil m^{1-\delta} \rceil$, $N = \binom{m}{r} + m + 2$ and

$$\begin{aligned} b^0 &= 0 \\ b^j &= e_j, \ \forall j = 1, \ldots, m \\ b^{m+1} &= \frac{1}{\sqrt{m}} \cdot e \\ b^{m+2} &= \theta_0 \cdot \left(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0 \right), \end{aligned}$$

A Large Gap Example for Affine Policies, continued

- Exactly r coordinates are non-zero, each equal to θ_0.
- Extreme points b^j, $j \geq m+3$ are permutations of the non-zero coordinates of b^{m+2}.
- $\mathcal{U}$ has exactly $\binom{m}{r}$ extreme points of the form of b^{m+2}.
- All the non-zero extreme points of $\mathcal{U}$ are roughly on the boundary of the unit hypersphere centered at zero.
- Theorem: For the instance above with uncertainty set $\mathcal{U}$,

$$z_{Aff}(\mathcal{U}) = \Omega\left(m^{1/2-\delta}\right) \cdot z_{Adapt}(\mathcal{U}),$$

for any given $\delta > 0$.

Performance Guarantee for Affine Policies

- Consider AAO with $\mathcal{U} \subseteq \mathbb{R}^m_+$ convex, compact and full-dimensional and $A \geq 0$.
- Then
$$z_{Aff}(\mathcal{U}) \leq 3\sqrt{m} \cdot z_{Adapt}(\mathcal{U}),$$
- Worst case cost of an optimal affine policy is at most $3\sqrt{m}$ times the worst case cost of an optimal fully adaptable solution.
- In general,
$$z_{Aff}(\mathcal{U}) \leq 4\sqrt{m} \cdot z_{Adapt}(\mathcal{U}),$$
- Full characterization of AAO performance: $z_{Aff}(\mathcal{U}) = \Theta(\sqrt{m}) \cdot z_{Adapt}(\mathcal{U})$,
- Contrast with $z_{Rob}(\mathcal{U}) = \Theta(m) \cdot z_{Adapt}(\mathcal{U})$,

Single Echelon Case

- $x_{k+1} = x_k + u_k - w_k$
- x_k : inventory at period k
- w_k : unknown, bounded demands from customers, $w_k \in [\underline{w}_k, \overline{w}_k]$
- u_k : replenishment orders; no lead-time, but capacities, $u_k \in \left[L_k, U_k\right]$
- Linear ordering costs + any convex inventory cost $h_k(x_k)$

$$\mathcal{C}_k(u_k, x_k) = c_k\, u_k + h_k(x_k)$$

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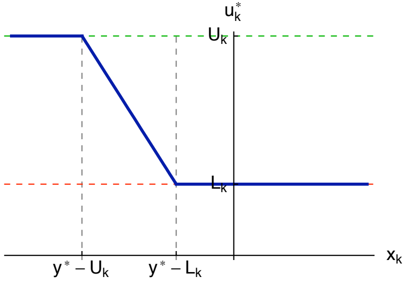
- Typical inventory example: holding and backlogging costs

$$h_k(x_k) = H_k \cdot \max(x_k, 0) + B_k \cdot \max(-x_k, 0)$$

Optimal Policies by Dynamic Programming

- (Modified) Base-stock policies optimal
    - Kasugai Kasegai (1960, 1961)



Optimality of Affine Policies in the Demands.

Theorem (Bertsimas, Iancu, Parrilo 2009a)

Ordering policies that are affine in the history of demands are optimal. In fact, for every time step $k = 1, \ldots, T$, the following quantities exist:

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- *an affine ordering policy,* $u_k(w_{[k]}) \stackrel{def}{=} u_{k,0} + \sum_{t=1}^{k-1} u_{k,t} w_t$,
- *an affine inventory cost,* $z_{k+1}(w_{[k+1]}) \stackrel{def}{=} z_{k+1,0} + \sum_{t=1}^{k} z_{k+1,t} w_t$,

such that the following conditions are obeyed:

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such that the following conditions are obeyed:

- $u_k(w_{[k]}) \in \left[L_k, U_k\right], \quad \forall\, w_{[k]}$
- $z_{k+1}(w_{[k+1]}) \geq h_{k+1}\Big(x_1 + \sum_{t=1}^{k}\big(u_t(w_{[t]}) - w_t\big)\Big), \quad \forall\, w_{[k+1]}$
- $$J_1^\star(x_1) = \max_{w_1,\ldots,w_k} \Big[\sum_{t=1}^{k}\big(c_t \cdot u_t(w_{[t]}) + z_t(w_{[t+1]})\big) + J_{k+1}^\star\Big(x_1 + \sum_{t=1}^{k}\big(u_t(w_{[t]}) - w_t\big)\Big)\Big]$$

Proof Outline. DP, Induction, Geometry.

- Forward induction on k
- Assume true $1, \dots, k$. The problem for uncertainties at k is

$$J_{mM} = \max_{(\theta_1, \theta_2) \in \Theta} \left[\ \theta_1 + J^{\star}_{k+1}(\theta_2) \ \right]$$

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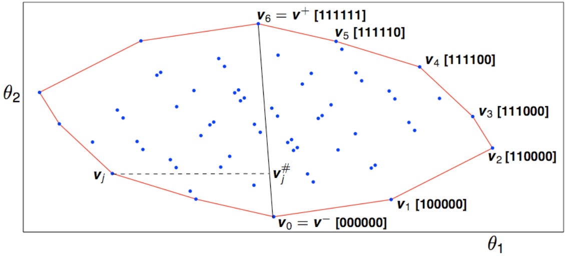

Why Is This Relevant?

1. *Computational result*
   For piecewise affine costs (with m_k pieces), must solve a single LOP with $O\left(T^2 \cdot \max_k\{m_k\}\right)$ variables and constraints

2. *Insight*
   Decomposition of demand satisfaction by means of future orders,

3. *Tight existential result*
   E.g., such policies not optimal for $\sum_{t=1}^{k} u_t \in \left[\hat{L}_k, \hat{U}_k\right]$

Extensions : Supply Contracts, Service Level Constraints

- Supply contracts
  - Order bounds L_k, U_k not *fixed*, but part of contract
  - Retailer pays supplier $f(U) \geq 0$, and receives $g(L) \geq 0$ from supplier
  - Retailer decides L, U beforehand (time $k = 0$), and ordering policies u_k

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If f convex and g concave $\Rightarrow$ solve optimally by sub-gradient methods

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Theorem

If f convex and g concave $\Rightarrow$ solve optimally by sub-gradient methods

If f, g also piecewise affine $\Rightarrow$ solve a single LOP

- Can easily accommodate service-level constraints
  - Satisfy 90% of demand upon arrival
  - Never backlog more than P periods

General Multi-Echelon Problem

$$
\begin{aligned}
&\min_{u_1}\Big[\mathcal{C}_1(x_1,u_1)+\max_{w_1}\min_{u_2}\Big[\mathcal{C}_2(x_2,u_2)+\dots+\max_{w_T}\mathcal{C}_{T+1}(x_{T+1})\Big]\dots\Big]\Big],\\
&x_{k+1}=A_k\,x_k+B_k\,u_k-w_k,\\
&\quad f_k\geq D_k\,x_k+E_k\,u_k, \qquad\qquad k\in\{1,\dots,T\}.
\end{aligned}
$$

- Affine policies *not* optimal

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- Affine policies *not* optimal
- Consider polynomial policies in $w_{[k]} \stackrel{\text{def}}{=} \left[w_1, w_2, \dots, w_{k-1}\right]$

General Multi-Echelon Problem

$$\min_{u_1}\Big[\mathcal{C}_1(x_1,u_1)+\max_{w_1}\min_{u_2}\Big[\mathcal{C}_2(x_2,u_2)+\cdots+\max_{w_T}\mathcal{C}_{T+1}(x_{T+1})\Big]\ldots\Big]\Big],$$
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- Affine policies *not* optimal
- Consider polynomial policies in $w_{[k]}\stackrel{\text{def}}{=}\left[w_1,w_2,\ldots,w_{k-1}\right]$
  - Example: degree $d=2$, $w_{[3]}=(w_1,w_2)$

$$u_3(w_{[3]})=\ell_0+\ell_1 w_1+\ell_2 w_2+\ell_{1,1}w_1^2+\ell_{1,2}w_1\,w_2+\ell_{2,2}w_2^2$$

Why Polynomials? [Bertsimas, Iancu, Parrilo 2009b]

1. Natural extension of affine case
2. Good approximation when optimal policies are continuous
3. Little burden on modeller : only choice of polynomial degree d
4. Can provide semidefinite programming relaxation
   - $T\left(\max_k r_k + \max_k m_k\right)$ SDP constraints, each of size $\binom{n_w T + d}{d}$
   - Solvable by interior-point methods
5. Degree d controls accuracy vs. computation trade-off

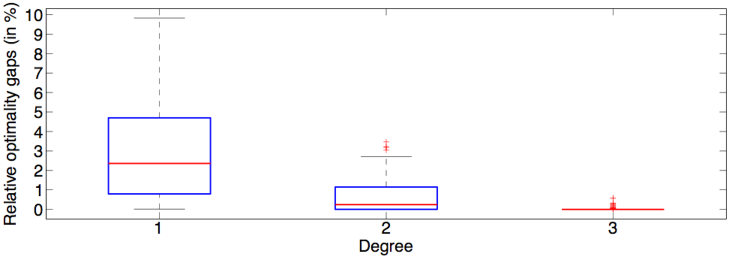
Single-echelon with Cumulative Orders

Single-echelon with Cumulative Orders

Relative optimality gaps (in %) for polynomial policies

	Degree $d = 1$					Degree $d = 2$					Degree $d = 3$				
T	avg	std	mdn	min	max	avg	std	mdn	min	max	avg	std	mdn	min	max
4	2.84	2.41	2.18	0.02	9.76	0.75	0.85	0.47	0.00	3.79	0.03	0.12	0.00	0.00	0.91
5	2.82	2.29	2.52	0.04	11.22	0.62	0.71	0.39	0.00	3.92	0.02	0.09	0.00	0.00	0.56
6	3.09	2.63	2.36	0.01	9.82	0.69	0.89	0.25	0.00	3.47	0.03	0.10	0.00	0.00	0.59
7	3.25	2.95	2.58	0.13	15.00	0.83	0.99	0.43	0.00	4.79	0.06	0.17	0.00	0.00	0.93
8	3.66	3.29	2.69	0.03	18.36	1.06	1.17	0.74	0.00	5.81	0.10	0.17	0.00	0.00	0.99
9	2.93	2.78	2.12	0.05	11.56	0.80	0.86	0.55	0.00	3.39	0.07	0.13	0.00	0.00	0.61
10	3.44	3.60	2.09	0.00	18.20	0.76	1.16	0.26	0.00	5.76	0.05	0.12	0.00	0.00	0.74

Polynomial policies for $T = 6$



Single-echelon with Cumulative Orders

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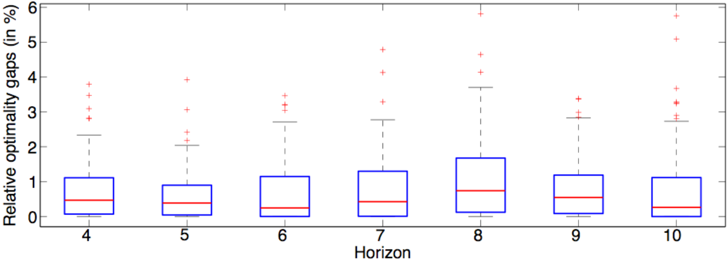
Performance of quadratic policies



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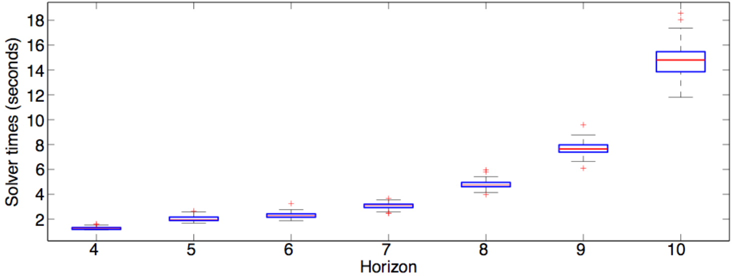
Solver times for quadratic policies



Serial Supply Chain

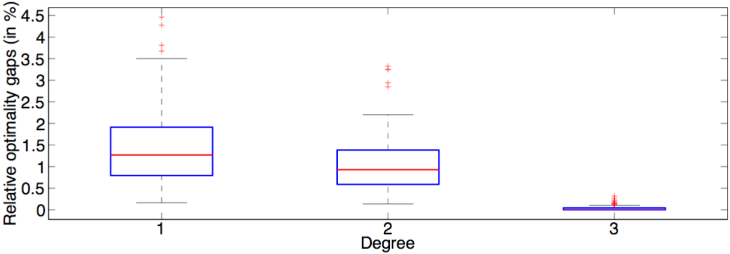
Serial supply chain

Serial Supply Chain

Relative gaps (in %) for the serial supply chain example

	Degree $d = 1$					Degree $d = 2$					Degree $d = 3$				
J	avg	std	mdn	min	max	avg	std	mdn	min	max	avg	std	mdn	min	max
2	1.87	1.48	1.47	0.00	8.27	1.38	1.16	1.11	0.00	6.48	0.06	0.14	0.01	0.00	0.96
3	1.47	0.89	1.27	0.16	4.46	1.08	0.68	0.93	0.14	3.33	0.04	0.06	0.00	0.00	0.32
4	1.14	2.46	0.70	0.05	24.63	0.67	0.53	0.53	0.01	2.10	0.04	0.07	0.00	0.00	0.38
5	0.35	0.37	0.21	0.03	1.85	0.27	0.32	0.15	0.00	1.59	0.02	0.03	0.00	0.00	0.15

Polynomial policies for $J = 3$ echelons.



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    - Affine policies are optimal
    - Newsvendor costs $\Rightarrow$ a single LOP
    - Supply contracts - capacity pre-commitment problem
- Multi-echelon case:
    - Framework to compute polynomial policies - solve a single SDOP
    - Polynomial degree d controls performance-computation trade-off
    - Perform well in several inventory examples