15.094J: Robust Modeling, Optimization, Computation

Lectures 3: Robust Linear Optimization I: Tractability

February 2015

Outline

Objectives Today

- Tractability of RLO
- Row-wise uncertainty
- General uncertainty

Row-wise Uncertainty

- Primitives: Uncertainty sets $\mathcal{U}_i$, $i = 1, \ldots, m$, $\boldsymbol{b}, c$ (known, WLOG).
- RLO with row-wise uncertainty:

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & a_i'x \leq b_i \qquad \forall a_i \in \mathcal{U}_i, \;\; i = 1, \ldots, m, \\ & x \geq \mathbf{0}. \end{aligned}$$

- Note that the problem has infinitely many constraints.
- Reformulation:

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \max\nolimits_{a_i \in \mathcal{U}_i} a_i'x \leq b_i \\ & x \geq \mathbf{0}. \end{aligned}$$

- Note that the uncertainty for different constraints is independent.

Tractability

- Suppose that $\mathcal{U}_i$, $i = 1, \ldots, m$ are convex sets.
- Given an x, we can solve $i = 1, \ldots, m$:

$$\max_{a_i \in \mathcal{U}_i} a_i' x,$$

  efficiently.
- How should we solve the RLO problem?

Theoretical Tractability

- Solve the nominal problem; find x_0.
- Separation problem: Given an x_0, does there exist an $a_i \in \mathcal{U}_i$ that violates the constraint $a_i'x > b_i$?
- Solution: Solve $\max_{a_i \in \mathcal{U}_i} a_i'x$ and check whether

$$\max_{a_i \in \mathcal{U}_i} a_i'x \leq b_i.$$

- This shows that if $\mathcal{U}_i$ are convex, we can solve the separation problem in polynomial time, thus we can solve the RLO with convex uncertainty sets in polynomial time using the Ellipsoid method (see Chapter 8 of Bertsimas and Tsitsiklis [1997]).
- The key take away from this: Even though RLO has infinitely many constraints it is polynomially solvable.
- Question: How about practically solvable? The Ellipsoid method is not a practical algorithm.

Practical Tractability

- Solve the nominal problem; find x_0.
- Solve $\max_{a_i \in \mathcal{U}_i} a_i' x_0$, solution $\overline{a}_{i,0}$.
- Add the constraint $\overline{a}_{i,0}' x \leq b_i$ to the nominal problem
- Solve (the dual Simplex method is the right choice)

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \overline{a}_{i,0}' x \leq b_i \\ & x \geq \mathbf{0}. \end{aligned}$$

- Find x_1; iterate.

Robust Counterpart-Polyhedral uncertainty

- $$\begin{array}{rl} \max & c'x \\ \text{s.t.} & \max_{a_i \in \mathcal{U}_i} a_i'x \leq b_i. \\ & x \geq \mathbf{0}. \end{array}$$
- $\mathcal{U}_i = \{a_i | \ D_i a_i \leq d_i\}$, $D_i: \ k_i \times n$.
- Consider the problem and its dual:

$$\begin{array}{rl} \max & a_i'x \\ \text{s.t.} & D_i a_i \leq d_i \end{array} \qquad \begin{array}{rl} \min & p_i'd_i \\ \text{s.t.} & p_i'D_i = x' \\ & p_i \geq \mathbf{0}. \end{array}$$

Robust Counterpart continued

- RC becomes

$$\begin{array}{lll} \max\limits_{x,p_i} & c'x & \\ \text{s.t.} & p_i'd_i \leq b_i, & i = 1, \ldots, m, \\ & p_i'D_i = x', & i = 1, \ldots, m, \\ & p_i \geq \mathbf{0}, & i = 1, \ldots, m, \\ & x \geq \mathbf{0}. & \end{array}$$

- Original nominal problem: n variables, m constraints.
- Uncertainty dimension: k_i.
- Size of Robust Counterpart: $n + \sum_{i=1}^{m} k_i$, variables; $m + m \cdot n$ constraints.

Row-wise Ellipsoidal uncertainty

- RO:

$$\begin{array}{ll} \max & c'x \\ \text{s.t.} & \max_{a_i \in \mathcal{U}_i} a_i'x \leq b_i. \\ & x \geq \mathbf{0}. \end{array}$$

- $\mathcal{U}_i = \{a_i |\ a_i = \overline{a}_i + \mathbf{\Delta}_i' u_i,\ ||u_i||_2 \leq \rho\}$, $\mathbf{\Delta}_i :\ k_i \times n,\ u_i : k_i \times 1.$
- RC:

$$\begin{array}{lll} \max & c'x & \\ \text{s.t.} & \overline{a}_i'x + \rho ||\mathbf{\Delta}_i x||_2 \leq b_i, & i = 1, \ldots, m. \\ & x \geq \mathbf{0}. & \end{array}$$

- Second order cone problem, nearly as tractable as linear optimization.

Proof

- $Z^* = \max_{a \in \mathcal{U}} a'x = \overline{a}'x + \max_{||u|| \leq \rho} u'(\boldsymbol{\Delta} x)$
- Lagrangean dual:

$$Z(\lambda) = \overline{a}'x + \max \ u'(\boldsymbol{\Delta} x) - \lambda(u'u/2 - \rho^2/2).$$

- $u^* = \boldsymbol{\Delta} x/\lambda$.
-

$$Z(\lambda) = \overline{a}'x + \frac{1}{2}\left(\frac{||\boldsymbol{\Delta} x||^2}{\lambda} + \lambda\rho^2\right).$$

- For $\lambda \geq 0$, $Z^* \leq Z(\lambda)$ and strong duality: $Z^* = \min_{\lambda \geq 0} Z(\lambda)$.
- $\lambda^* = ||\boldsymbol{\Delta} x||/\rho$.
- $Z^* = \overline{a}'x + \rho||\boldsymbol{\Delta} x||$.

Robust Counterpart-General Norm uncertainty

- RO:

$$\begin{array}{ll} \max & c'x \\ \text{s.t.} & \max_{a_i \in \mathcal{U}_i} a_i'x \le b_i. \\ & x \ge \mathbf{0}. \end{array}$$

- $U_i = \{a_i |\ a_i = \bar{a}_i + \mathbf{\Delta}_i' u_i,\ ||u_i|| \le \rho\}$, $\mathbf{\Delta}_i :\ k_i \times n,\ u_i : k_i \times 1.$
- Dual norm:

$$||s||^* = \max_{\{||x|| \le 1\}} |s'x|.$$

- The dual of the L_p-norm $||x||_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$:
- $||s||^* = ||s||_q$ with $q = 1 + \dfrac{1}{p-1}$.
- The dual norm of the L_2 norm is L_2.
- The dual norm of the L_1 norm is the L_∞ norm.
- RC:

$$\begin{array}{lll} \max & c'x & \\ \text{s.t.} & \bar{a}_i'x + \rho||\mathbf{\Delta}_i x||^* \le b_i, & i = 1, \ldots, m. \\ & x \ge \mathbf{0}. & \end{array}$$

General Polyhedral Uncertainty

- Define the operator $\text{vec}(A) := (a_1, a_2, \ldots, a_m)$ (vector concatenation of the rows of A transposed)
- $$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \tilde{A}x \leq b, \qquad \forall \tilde{A} \in \mathcal{U} \\ & x \in P. \end{aligned}$$
- $\mathcal{U} = \{\tilde{A} \mid G \cdot \text{vec}(\tilde{A}) \leq d\}$,
- $G \in \Re^{l \times (m \cdot n)}$, $d \in \Re^{l \times 1}$, and $\text{vec}(\tilde{A}) \in \Re^{(m \cdot n) \times 1}$.

RC

- The RC is

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & p_i'G = x_i', \quad && i = 1, \ldots, m \\ & p_i'd \le b_i, \quad && i = 1, \ldots, m \\ & p_i \ge \mathbf{0}, \quad && i = 1, \ldots, m \\ & x \in P, \end{aligned}$$

- $p_i \in \Re^{l \times 1}$.
- $x_i \in \Re^{(m \cdot n) \times 1}$, $i = 1, \ldots, m$; x_i contains x in entries $(i-1) \cdot n + 1$ through $i \cdot n$, and zero everywhere else.

Proposition

- Suppose $\mathcal{U} \neq \emptyset$.
- A given $\hat{x}$ satisfies $\tilde{a}_i'\hat{x} \leq b_i$ for all $\tilde{A} \in \mathcal{U}$ if and only if there exists a vector $p_i \in \Re^{l \times 1}$ such that

$$\begin{array}{rcl} p_i'd & \leq & b_i \\ p_i'G & = & \hat{x}_i' \\ p_i & \geq & \mathbf{0} \end{array}$$

- $\hat{x}_i \in \Re^{(m \cdot n) \times 1}$ contains $\hat{x}$ in entries $(i-1) \cdot n + 1$ through $i \cdot n$, and zero everywhere else.

Proof

- Consider the primal-dual pair

$$\begin{array}{cl} \max_A & a_i'\hat{x} \\ \text{s.t.} & G \cdot \text{vec}(A) \leq d \end{array}$$

$$\begin{array}{cl} \min_{p_i} & p_i'd \\ \text{s.t.} & p_i'G = \hat{x}_i' \\ & p_i \geq \mathbf{0}. \end{array}$$

- Suppose that $\hat{x}$ satisfies $\tilde{a}_i'\hat{x} \leq b_i$ for all $\tilde{A} \in \mathcal{U}$.
- Then, $\max_A a_i'\hat{x} \leq b_i$.
- Then primal is feasible and bounded, and so is its dual.
- Thus, there exists a vector $p_i \in \Re^{(m \cdot n) \times 1}$ satisfying the dual constraints.
- By strong duality, the optimal objective function value of the dual equals $\max_A a_i'\hat{x}$ and is less than b_i.

Proof continued

- For the reverse, since $\mathcal{U} \neq \emptyset$, primal is feasible. Suppose there exists a vector $p_i \in \Re^{l \times 1}$ that satisfies the dual constraints.
- Since both problems are feasible, they must be bounded and their optimal objective function values must be equal.
- Then $\min_{p_i} p_i' d \leq p_i' d \leq b_i$.
- By strong duality, $\max_A a_i' \hat{x} = \min_{p_i} p_i' d \leq b_i$, and hence $\hat{x}$ satisfies $a_i' \hat{x} \leq b_i$ for all $\tilde{A} \in \mathcal{U}$.

RC

- RO:

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \tilde{A}x \le b, \qquad \forall \tilde{A} \in \mathcal{U} \\ & x \in P. \end{aligned}$$

- $\mathcal{U} = \{\tilde{A} \mid G \cdot \text{vec}(\tilde{A}) \le d\}$.
- The RC is

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & p_i'G = x_i', \quad & i = 1, \ldots, m \\ & p_i'd \le b_i, \quad & i = 1, \ldots, m \\ & p_i \ge \mathbf{0}, \quad & i = 1, \ldots, m \\ & x \in P. \end{aligned}$$

General uncertainty sets under a general norm

- RO:

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \tilde{A}x \leq b \\ & x \in P \\ & \forall \tilde{A} \in \mathcal{U} = \left\{ \tilde{A} \mid ||M(\text{vec}(\tilde{A}) - \text{vec}(\overline{A}))|| \leq \Delta \right\}. \end{aligned}$$

- RC:

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & \overline{a}_i x + \Delta ||M^{-1} x_i||^* \leq b_i, \quad i = 1, \ldots, m \\ & x \in P, \end{aligned}$$

- M invertible
- $x_i \in \Re^{(m \cdot n) \times 1}$ contains $x \in \Re^{n \times 1}$ in entries $(i-1) \cdot n + 1$ through $i \cdot n$, and 0 everywhere else.

Proof

- $y = \dfrac{M(\text{vec}(\tilde{A}) - \text{vec}(\overline{A}))}{\Delta}$.
- Then, $\mathcal{U} = \{y : \ ||y|| \leq 1\}$.

$$\begin{aligned}
\max_{\{\tilde{A}\in\mathcal{U}\}} \left\{\tilde{a}_i{}'x\right\} &= \max_{\{\tilde{A}\in\mathcal{U}\}} \left\{(\text{vec}(\tilde{A}))'x_i\right\} \\
&= \max_{\{y:\ ||y||\leq 1\}} \left\{(\text{vec}(\overline{A}))'x_i + \Delta(M^{-1}y)'x_i\right\} \\
&= \overline{a}_i'x + \Delta \max_{\{y|||y||\leq 1\}} \left\{y'(M^{-1}x_i)\right\} \\
&= \overline{a}_i x + \Delta||M^{-1}x_i||^*
\end{aligned}$$

References

Dimitris Bertsimas and John Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific, 1997.