C21 Nonlinear Systems

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4 lectures

Hilary Term 2016

Lecture 1

Introduction and Concepts of Stability

Course outline

1. Types of stability
2. Linearization
3. Lyapunov's direct method
4. Regions of attraction
5. Linear systems and passive systems

Books

- J.-J. Slotine & W. Li *Applied Nonlinear Control*, Prentice-Hall 1991.
  - ⋆ Stability
  - ⋆ Interconnected systems and passive systems
- H.K. Khalil *Nonlinear Systems*, Prentice-Hall 1996.
  - ⋆ Stability
  - ⋆ Passive systems
- M. Vidyasagar *Nonlinear Systems Analysis*, Prentice-Hall 1993.
  - ⋆ Stability & passivity (more technical detail)

Why use nonlinear control?

- Real systems are nonlinear
  - ⋆ friction, non-ideal components
  - ⋆ actuator saturation
  - ⋆ sensor nonlinearity
- Analysis via linearization
  - ⋆ accuracy of approximation?
  - ⋆ conservative?
- Account for nonlinearities in high performance applications
  - ⋆ Robotics, Aerospace, Petrochemical industries, Process control, Power generation ...
- Account for nonlinearities if linear models inadequate
  - ⋆ large operating region
  - ⋆ model properties change at linearization point

Linear vs nonlinear system properties

Free response

Linear system	Nonlinear system
$\boxed{\dot{x} = Ax}$	$\boxed{\dot{x} = f(x)}$
• Unique equilibrium point: $Ax = 0 \iff x = 0$	• Multiple equilibrium points $f(x) = 0$
• Stability independent of initial conditions	• Stability dependent on initial conditions

Linear vs nonlinear system properties

Forced response

Linear system

$$\boxed{\dot{x} = Ax + Bu}$$

- $\|u\|$ finite $\Rightarrow$ $\|x\|$ finite
  if open-loop stable
- Frequency response:
  $u = U \sin \omega t \;\; \Rightarrow \;\; x = X \sin(\omega t + \phi)$
- Superposition:
  $u = u_1 + u_2 \;\; \Rightarrow x = x_1 + x_2$

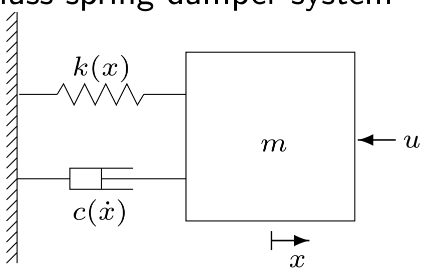
Nonlinear system

$$\boxed{\dot{x} = f(x, u)}$$

- $\|u\|$ finite $\not\Rightarrow$ $\|x\|$ finite
- No frequency response
  $u = U \sin \omega t \;\; \not\Rightarrow \;\; x$ sinusoidal
- No linear superposition
  $u = u_1 + u_2 \;\; \not\Rightarrow x = x_1 + x_2$

Example: step response

Mass-spring-damper system

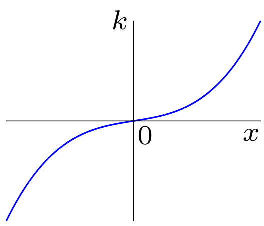


Equation of motion:

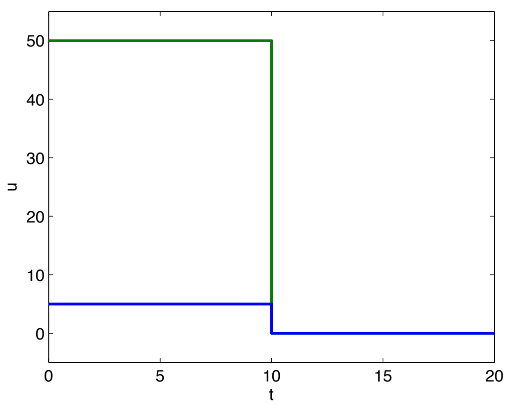
$$\ddot{x} + c(\dot{x}) + k(x) = u$$

$c(\dot{x}) = \dot{x}$

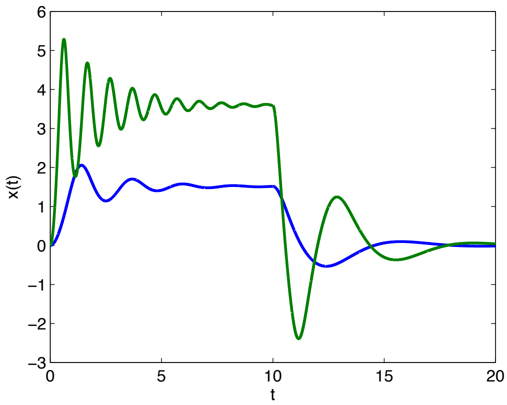
$k(x)$ nonlinear:



Input $u(t)$



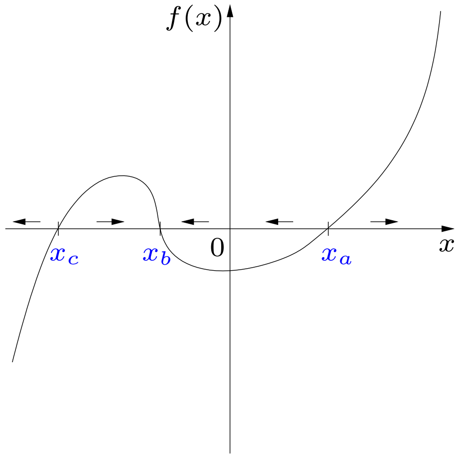
Response $x(t)$



apparent damping ratio depends on size of input step

Example: multiple equilibria

First order system: $\dot{x} = f(x)$



$x > x_a$	$\Longrightarrow$	$f(x) > 0$	$\Longrightarrow$	$x(t)$ increases
$x_b < x < x_a$	$\Longrightarrow$	$f(x) < 0$	$\Longrightarrow$	$x(t)$ decreases
$x_c < x < x_b$	$\Longrightarrow$	$f(x) > 0$	$\Longrightarrow$	$x(t)$ increases
$x < x_c$	$\Longrightarrow$	$f(x) < 0$	$\Longrightarrow$	$x(t)$ decreases

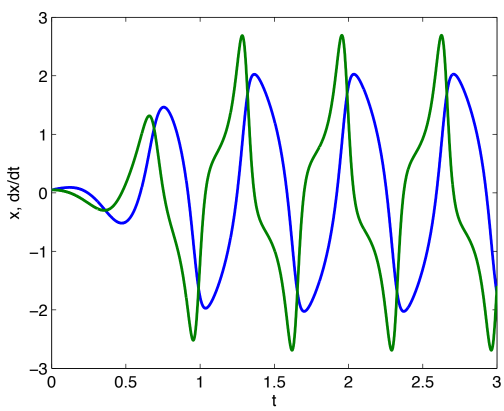
- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

Example: limit cycle

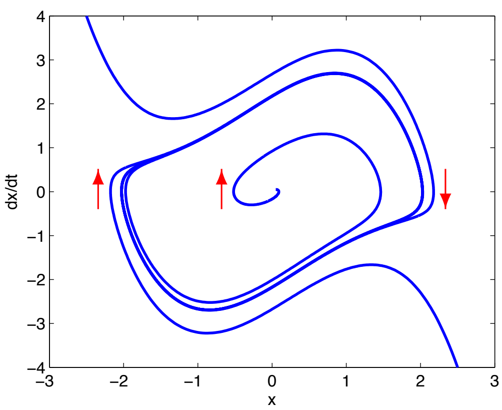
Van der Pol oscillator:

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

- Response $x(t)$ tends to a limit cycle (= trajectory forming a closed curve)
- Amplitude independent of initial conditions



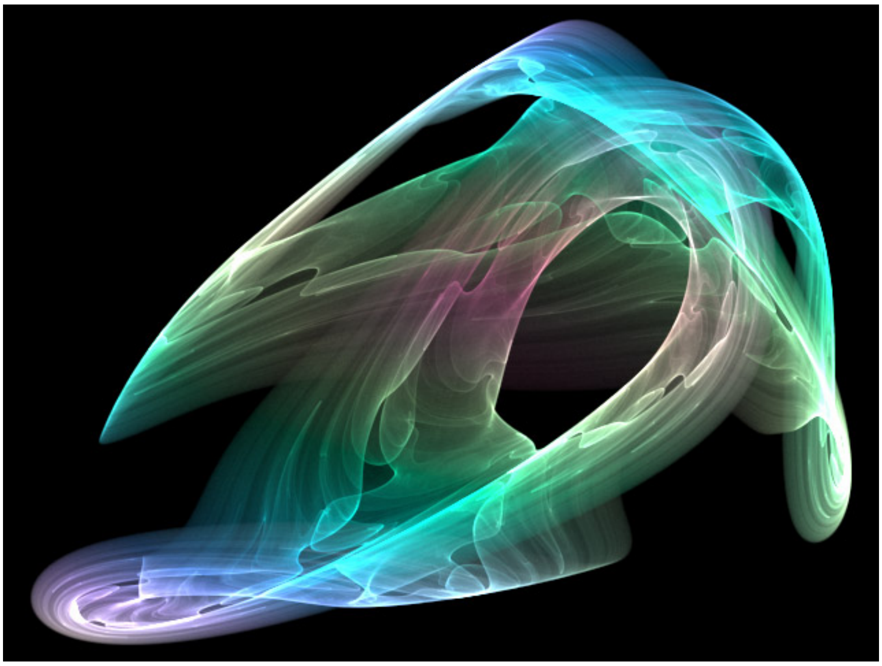
Response with $x(0) = 0.05$, $dx/dt = 0.05$



State trajectories $(x(t), \dot{x}(t))$

Example: chaotic behaviour

Strange attractor



Example: chaotic behaviour

Lorenz attractor

- Simplified model of atmospheric convection:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

- State variables

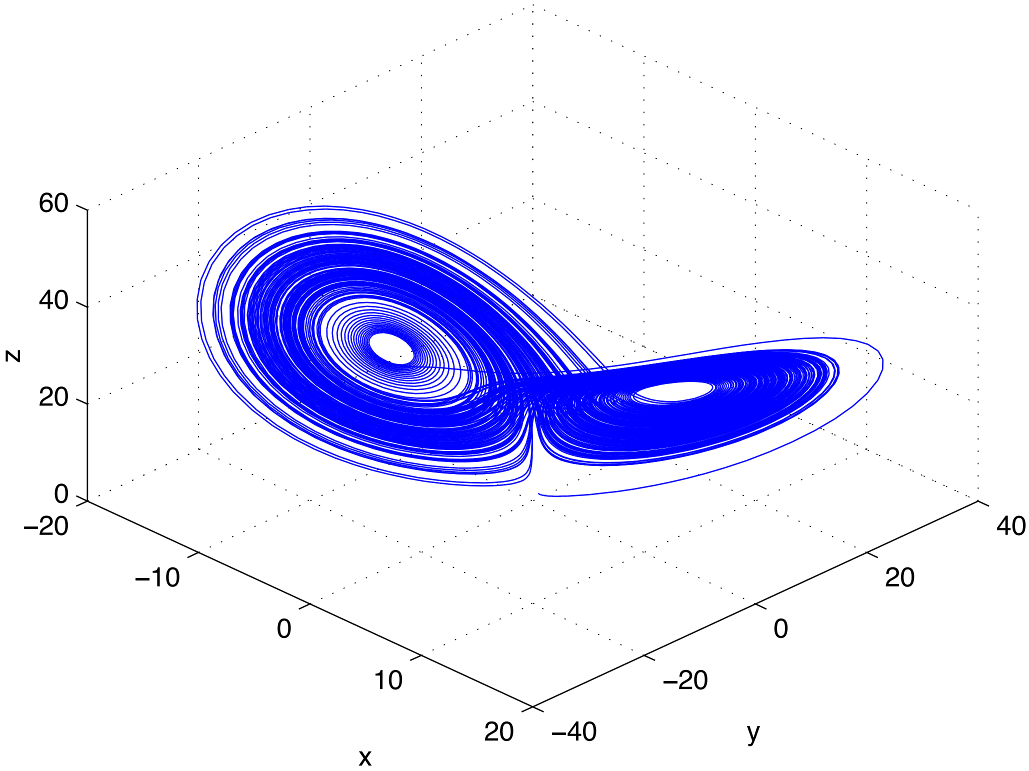
$x(t)$:	fluid velocity
$y(t)$:	difference in temperature of acsending and descending fluid
$z(t)$:	characterizes distortion of vertical temperature profile

- Parameters $\sigma = 10$, $\beta = 8/3$, $\rho =$ variable

Example: chaotic behaviour

Lorenz attractor

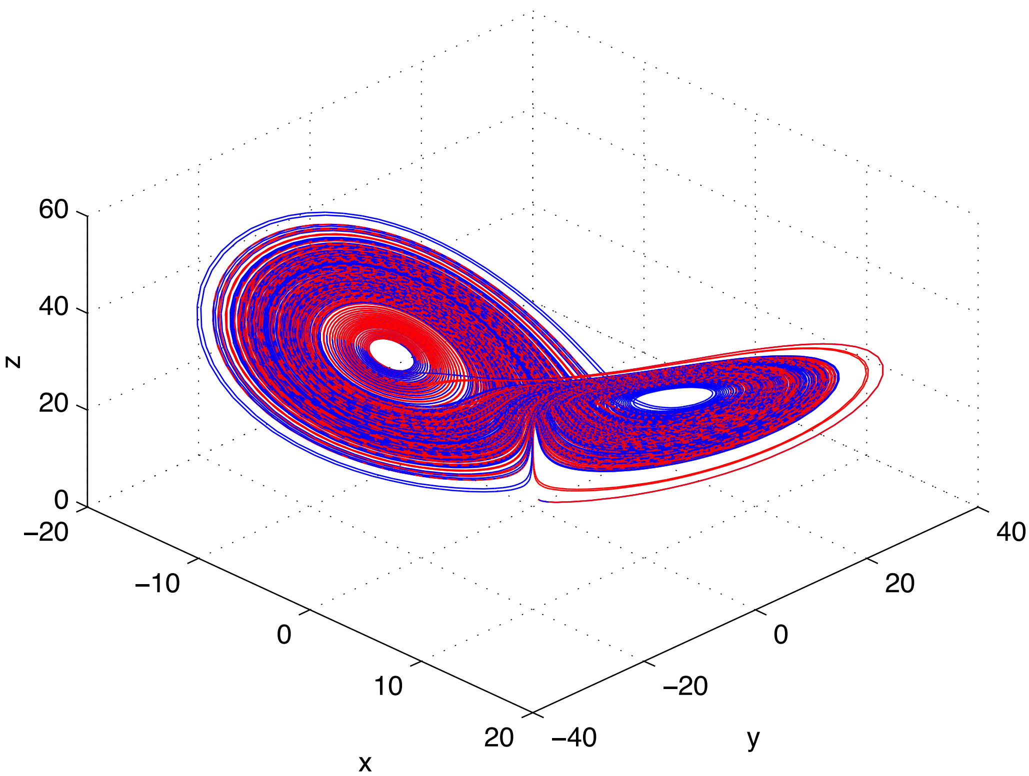
$\rho = 28 \implies$ "strange attractor":



Example: chaotic behaviour

Lorenz attractor

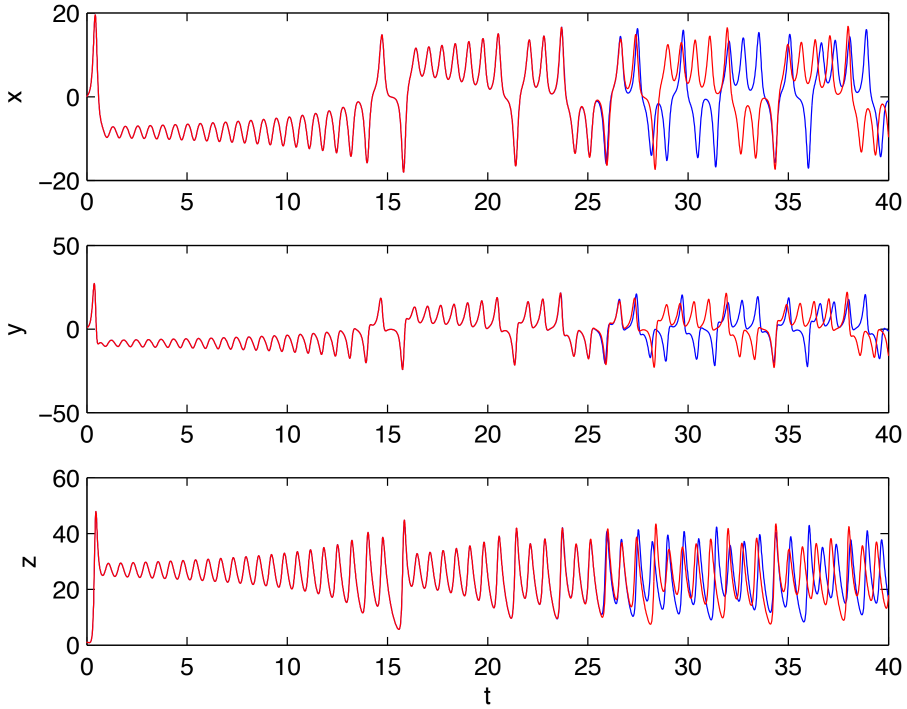
sensitivity to initial conditions



Example: chaotic behaviour

Lorenz attractor

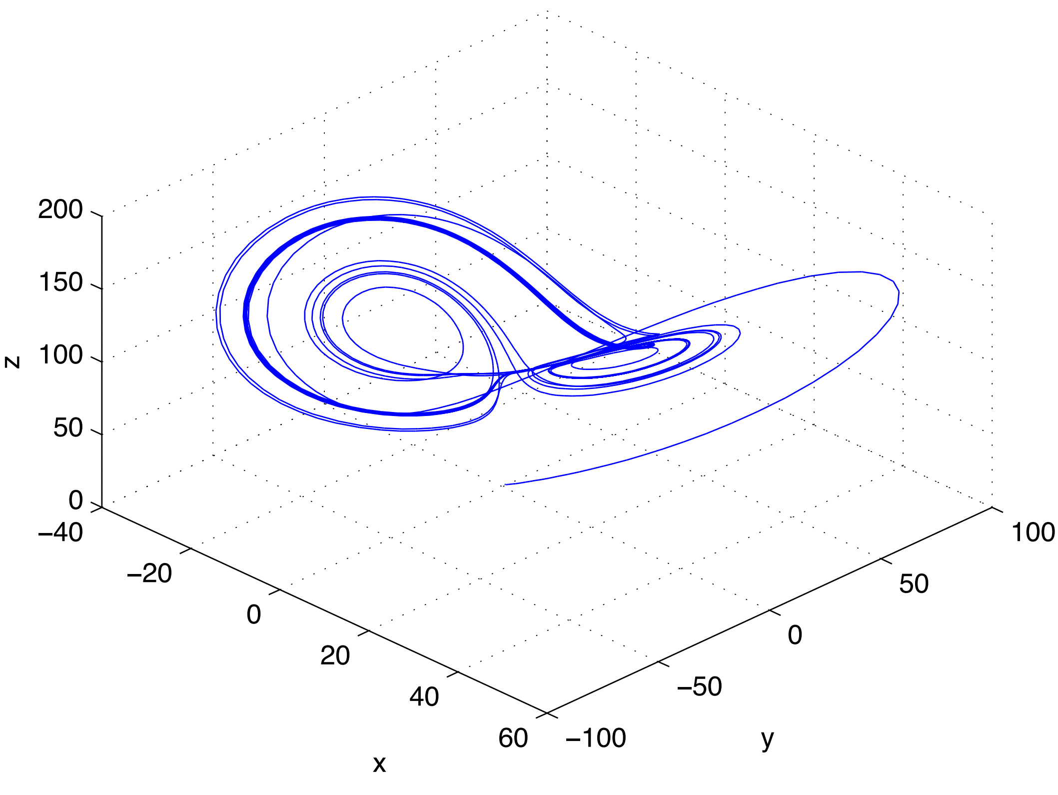
sensitivity to initial conditions blue: $(x, y, z) = (0, 1, 1.05)$
red: $(x, y, z) = (0, 1, 1.050001)$



Example: chaotic behaviour

Lorenz attractor

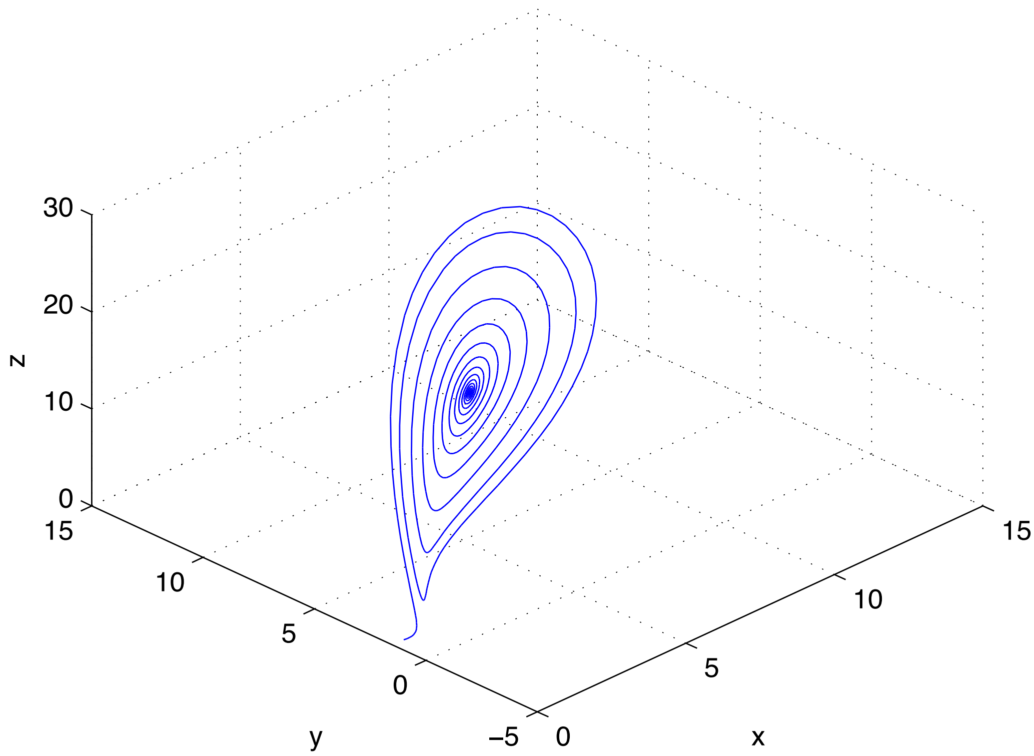
$\rho = 99.96 \implies$ limit cycle:



Example: chaotic behaviour

Lorenz attractor

$\rho = 14 \implies$ convergence to a stable equilibrium:



State space equations

$$\dot{x} = f(x, u, t) \qquad \begin{array}{rcl} x & : & \text{state} \\ u & : & \text{input} \end{array}$$

e.g. nth order differential equation:

$$\frac{d^n y}{dt^n} = h\Big(y, \ldots, \frac{d^{n-1} y}{dt^{n-1}}, u, t\Big)$$

has state vector (one possible choice)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ d^{n-1}y/dt^{n-1} \end{bmatrix}$$

and state space dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ h\big(x_1, x_2, \ldots, x_n, u, t\big) \end{bmatrix} = f(x, u, t)$$

Equilibrium points

x^* is an equilibrium point of system $\dot{x} = f(x)$ iff:

$$x(0) = x^* \quad \text{implies} \quad x(t) = x^* \quad \forall t > 0$$
$$\text{i.e. } f(x^*) = 0$$

Examples:

(a) $\ddot{y} + \alpha \dot{y}^2 + \beta \sin y = 0$ (damped pendulum)

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \ n = 0, \pm 1$$

(b) $\ddot{y} + (y-1)^2 \dot{y} + y - \sin(\pi y/2) = 0$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- ⋆ Consider local stability of individual equilibrium points
- ⋆ Convention: define f so that $x = 0$ is equilibrium point of interest
- ⋆ Autonomous system: $\dot{x} = f(x) \implies x^* =$ constant

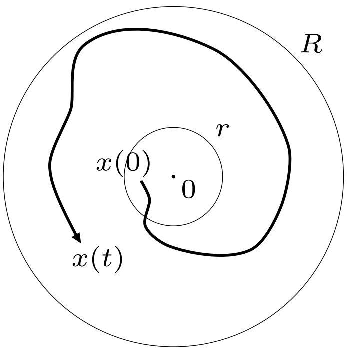
Stability definition

An equilibrium point $x = 0$ is stable iff:

$$\max_t \|x(t)\| \text{ can be made arbitrarily small}$$
$$\text{by making } \|x(0)\| \text{ small enough}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c}\text{for any } R > 0, \text{ there exists } r > 0 \text{ so that} \\ \|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0\end{array}}$$



- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point

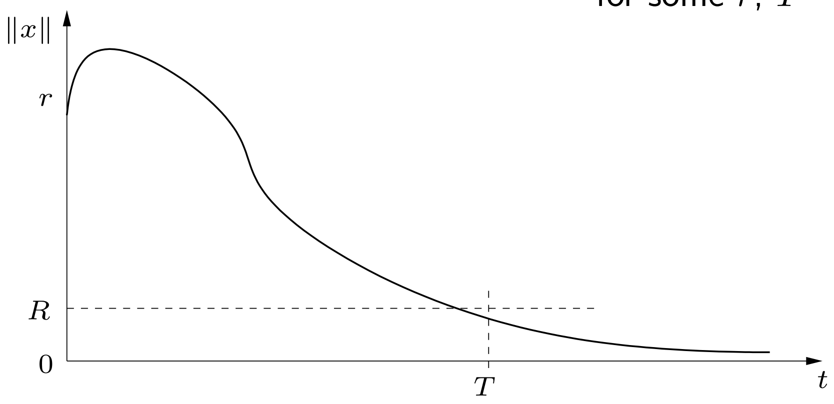
Asymptotic stability definition

An equilibrium point $x = 0$ is asymptotically stable iff:

$$\boxed{\begin{array}{rl} \text{(i).} & x = 0 \text{ is stable} \\ \text{(ii).} & \|x(0)\| < r \implies \|x(t)\| \to 0 \text{ as } t \to \infty \end{array}}$$

(ii) is equivalent to:

$$\begin{array}{l} \text{for any } R > 0, \\ \qquad \|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > T \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } r,\, T \end{array}$$

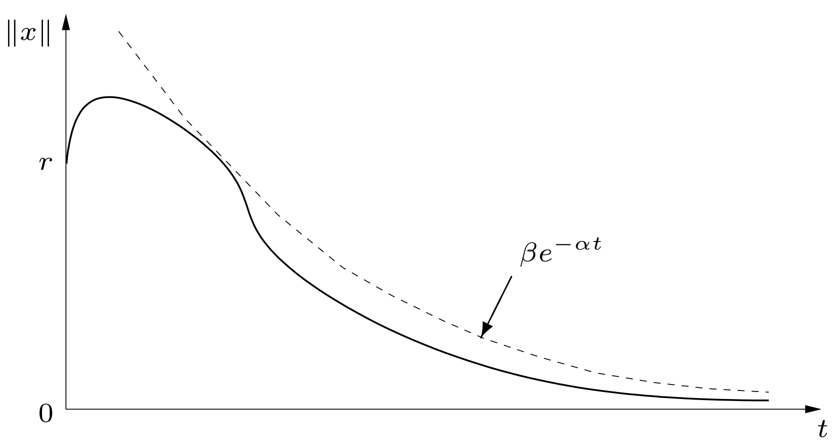


Exponential stability definition

An equilibrium point $x = 0$ is exponentially stable iff:

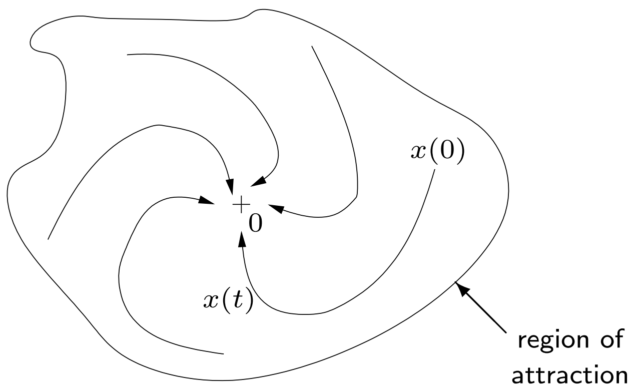
$$\boxed{\|x(0)\| < r \implies \|x(t)\| \le \beta e^{-\alpha t} \quad \forall t > 0}$$

exponential stability is a special case of asymptotic stability



Region of attraction

The region of attraction of $x = 0$ is the set of all initial conditions $x(0)$ for which $x(t) \to 0$ as $t \to \infty$



- Every asymptotically stable equilibrium point has a region of attraction
- $r = \infty \implies$ entire state space is a region of attraction
  $\implies$ $x = 0$ is globally asymptotically stable
- Are stable linear systems asymptotically stable?

Summary

- Nonlinear state space equations: $\dot{x} = f(x, u)$
  $x =$ state vector, $u =$ control input
- Equilibrium points: x^* is an equilibrium point of system $\dot{x} = f(x)$ if $f(x^*) = 0$
- Stable equilibrium point: x^* is stable if state trajectories starting close to x^* remain near x^* at all times
- Asymptotically stable equilibrium point: x^* must be stable and state trajectories starting near x^* must tend to x^* asympotically
- Region of attraction: initial conditions from which state trajectories converge to an asymptotically stable equilibrium x^*