Sparse Optimization
Lecture: Basic Sparse Optimization Models

Instructor: Wotao Yin

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online discussions on piazza.com

Those who complete this lecture will know

- basic ℓ_1, $\ell_{2,1}$, and nuclear-norm models
- some applications of these models
- how to reformulate them into standard conic programs
- which conic programming solvers to use

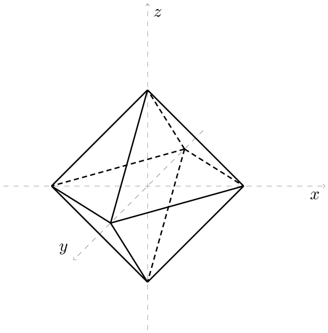
Examples of Sparse Optimization Applications

See online seminar at piazza.com

Basis pursuit

$$\min\{\|\mathbf{x}\|_1 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

- find least ℓ_1-norm point on the affine plane $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$
- tends to return a sparse point (sometimes, the sparsest)



ℓ_1 ball

Basis pursuit

$$\min\{\|\mathbf{x}\|_1 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

- find least ℓ_1-norm point on the affine plane $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$
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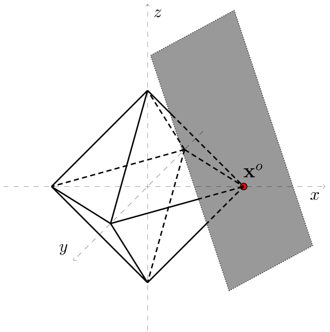


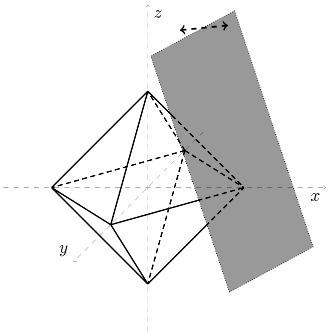
ℓ_1 ball touches the affine plane

Basis pursuit denoising, LASSO

$$\min_{\mathbf{x}} \{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 : \|\mathbf{x}\|_1 \le \tau\}, \tag{1a}$$

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\mu}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \tag{1b}$$

$$\min_{\mathbf{x}} \{\|\mathbf{x}\|_1 : \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \le \sigma\}. \tag{1c}$$



all models allow $\mathbf{A}\mathbf{x}^* \neq \mathbf{b}$

Basis pursuit denoising, LASSO

$$\min_{\mathbf{x}} \{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 : \|\mathbf{x}\|_1 \le \tau\}, \tag{2a}$$

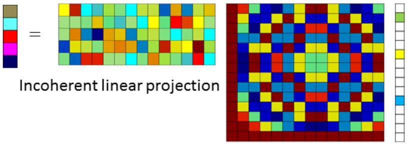
$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\mu}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \tag{2b}$$

$$\min_{\mathbf{x}} \{\|\mathbf{x}\|_1 : \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \le \sigma\}. \tag{2c}$$

- $\|\cdot\|_2$ is most common for error but can be generalized to loss function $\mathcal{L}$
- (2a) seeks for a least-squares solution with "bounded sparsity"
- (2b) is known as LASSO (least absolute shrinkage and selection operator). it seeks for a balance between sparsity and fitting
- (2c) is referred to as BPDN (basis pursuit denoising), seeking for a sparse solution from tube-like set $\{\mathbf{x} : \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \le \sigma\}$
- they are equivalent (see later slides)
- in terms of regression, they select a (sparse) set of features (i.e., columns of $\mathbf{A}$) to linearly express the observation $\mathbf{b}$

Sparse under basis Ψ / ℓ_1-synthesis model

$$\min_{\mathbf{s}}\{\|\mathbf{s}\|_1 : \mathbf{A}\Psi\mathbf{s} = \mathbf{b}\} \tag{3}$$



- signal $\mathbf{x}$ is *sparsely synthesized* by atoms from Ψ, so vector $\mathbf{s}$ is sparse
- Ψ is referred to as the *dictionary*
- commonly used dictionaries include both analytic and trained ones
- analytic examples: Id, DCT, wavelets, curvelets, gabor, etc., also their combinations; they have analytic properties, often easy to compute (for example, multiplying a vector takes $O(n \log n)$ instead of $O(n^2)$)
- Ψ can also be numerically learned from *training data* or *partial signal*
- they can be orthogonal, frame, or general

Sparse under basis Ψ / ℓ_1-synthesis model

If Ψ is **orthogonal**, problem (3) is equivalent to

$$\min_{\mathbf{x}}\{\|\Psi^*\mathbf{x}\|_1 : \mathbf{A}\mathbf{x} = \mathbf{b}\} \tag{4}$$

by change of variable $\mathbf{x} = \Psi\mathbf{s}$, equivalently $\mathbf{s} = \Psi^*\mathbf{x}$.

Related models for noise and approximate sparsity:

$$\begin{aligned}
&\min_{\mathbf{x}}\{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 : \|\Psi^*\mathbf{x}\|_1 \le \tau\},\\
&\min_{\mathbf{x}}\, \|\Psi^*\mathbf{x}\|_1 + \frac{\mu}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2,\\
&\min_{\mathbf{x}}\{\|\Psi^*\mathbf{x}\|_1 : \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \le \sigma\}.
\end{aligned}$$

Sparse after transform / ℓ_1-analysis model

$$\min_{\mathbf{x}}\{\|\Psi^*\mathbf{x}\|_1 : \mathbf{A}\mathbf{x} = \mathbf{b}\} \tag{5}$$

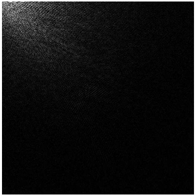
Signal $\mathbf{x}$ becomes sparse under the transform Ψ (may not be orthogonal)

Examples of Ψ:

- DCT, wavelets, curvelets, ridgelets, ...
- tight frames, Gabor, ...
- (weighted) total variation

When Ψ is not orthogonal, the analysis is more difficult

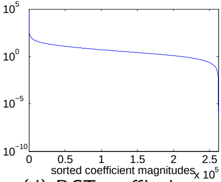
Example: sparsify an image



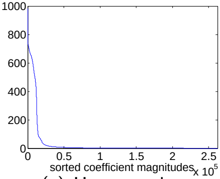
(a) DCT coefficients

(b) Haar wavelets

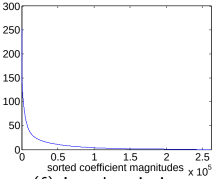
(c) Local variation



(d) DCT coeff's decay



(e) Haar wavelets



(f) Local variation

Figure: the DCT and wavelet coefficients are scaled for better visibility.

Questions

1. Can we trust these models to return intended sparse solutions?
2. When will the solution be unique?
3. Will the solution be robust to noise in $\mathbf{b}$?
4. Are constrained and unconstrained models equivalent? in what sense?

   Questions 1–4 will be addressed in *next lecture*.

5. How to choose parameters?
   - τ (sparsity), μ (weight), and σ (noise level) have different meanings
   - applications determine which one is easier to set
   - generality: use a test data set, then scale parameters for real data
   - cross validation: reserve a subset of data to test the solution

Joint/group sparsity

Joint sparse recovery model:

$$\min_{\mathbf{X}}\{\|\mathbf{X}\|_{2,1} : \mathcal{A}(\mathbf{X}) = \mathbf{b}\} \tag{6}$$

where

$$\|\mathbf{X}\|_{2,1} := \sum_{i=1}^{m} \|[x_{i1}\ x_{i,2} \cdots x_{in}]\|_2 \,.$$

- ℓ_2-norm is applied to each row of $\mathbf{X}$
- $\ell_{2,1}$-norm ball has sharp boundaries "across different rows", which tend to be touched by $\{\mathbf{X} : \mathcal{A}(\mathbf{X}) = \mathbf{b}\}$, so the solution tends to be *row-sparse*
- also $\|\mathbf{X}\|_{p,q}$ for $1 < p \le \infty$, affects magnitudes of entries on the same row
- complex-valued signals are a special case

Joint/group sparsity

Decompose $\{1, \ldots, n\} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_S$.

- non-overlapping groups: $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset,\ \forall i \neq j$.
- otherwise, groups may overlap (modeling many interesting structures).

Group-sparse recovery model:

$$\min_{\mathbf{x}} \{\|\mathbf{x}\|_{\mathcal{G},2,1} : \mathbf{A}\mathbf{x} = \mathbf{b}\} \tag{7}$$

where

$$\|\mathbf{x}\|_{\mathcal{G},2,1} = \sum_{s=1}^{S} w_s \|\mathbf{x}_{\mathcal{G}_s}\|_2.$$

Auxiliary constraints

Auxiliary constraints introduce additional structures of the underlying signal into its recovery, which sometimes *significantly* improve recovery quality

- nonnegativity: $\mathbf{x} \geq \mathbf{0}$
- bound (box) constraints: $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$
- general inequalities: $\mathbf{Q}\mathbf{x} \leq \mathbf{q}$

They can be very effective in practice. They also generate "corners."

Reduce to conic programs

Sparse optimization often has *nonsmooth* objectives.

Classic conic programming solvers do not handle nonsmooth functions.

Basic idea: *model nonsmoothness by inequality constraints*.

Example: for given $\mathbf{x}$, we have

$$\|\mathbf{x}\|_1 = \min_{\mathbf{x}_1,\mathbf{x}_2}\{\mathbf{1}^T(\mathbf{x}_1+\mathbf{x}_2) : \mathbf{x}_1-\mathbf{x}_2=\mathbf{x}, \mathbf{x}_1 \geq \mathbf{0}, \mathbf{x}_2 \geq \mathbf{0}\}. \tag{8}$$

Therefore,

- $\min\{\|\mathbf{x}\|_1 : \mathbf{A}\mathbf{x}=\mathbf{b}\}$ reduces to a linear program (LP)
- $\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\mu}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2$ reduces to a bound constrained quadratic program (QP)
- $\min_{\mathbf{x}}\{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2 : \|\mathbf{x}\|_1 \leq \tau\}$ reduces to a bound constrained QP
- $\min_{\mathbf{x}}\{\|\mathbf{x}\|_1 : \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2 \leq \sigma\}$ reduces to a second-order cone program (SOCP)

Conic programming

Basic form:

$$\min_{\mathbf{x}} \{\mathbf{c}^T \mathbf{x} : \mathbf{F}\mathbf{x} + \mathbf{g} \succeq_{\mathcal{K}} \mathbf{0}, \mathbf{A}\mathbf{x} = \mathbf{b}.\}$$

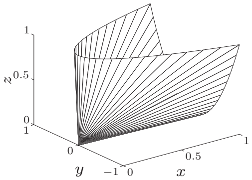
"$\mathbf{a} \succeq_{\mathcal{K}} \mathbf{b}$" stands for $\mathbf{a} - \mathbf{b} \in \mathcal{K}$, which is a convex, closed, pointed cone.

Examples:

- first orthant (cone): $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}$.
- norm cone (2nd order cone): $\mathcal{Q} = \{(\mathbf{x}, t) : \|\mathbf{x}\| \leq t\}$
- polyhedral cone: $\mathcal{P} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{0}\}$
- positive semidefinite cone: $\mathbf{S}_+ = \{\mathbf{X} : \mathbf{X} \succeq \mathbf{0}, \mathbf{X}^T = \mathbf{X}\}$

  Example:

$$\left\{(x, y, z) : \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+ \right\}$$



Linear program

Model

$$\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \succeq_{\mathcal{K}} \mathbf{0}\}$$

where $\mathcal{K}$ is the nonnegative cone (first orthant).

$$\mathbf{x} \succeq_{\mathcal{K}} \mathbf{0} \Longleftrightarrow \mathbf{x} \geq \mathbf{0}.$$

Algorithms

- the Simplex method (move between vertices)
- interior-point methods (IPMs) (move inside the polyhedron)
- decomposition approaches (divide and conquer)

In primal IPM, $\mathbf{x} \geq 0$ is replaced by its logarithmic barrier:

$$\psi(\mathbf{y}) = \sum_i \log(y_i)$$

log-barrier formulation:

$$\min\{\mathbf{c}^T\mathbf{x} - (1/t)\sum_i \log(x_i) : \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

Second-order cone program

Model

$$\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \succeq_{\mathcal{K}} \mathbf{0}\}$$

where $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_K$; each $\mathcal{K}_k$ is the second-order cone

$$\mathcal{K}_k = \left\{\mathbf{y} \in \mathbb{R}^{n_k} : y_{n_k} \geq \sqrt{y_1^2 + \cdots + y_{n_k-1}^2}\right\}.$$

IPM is the standard solver (though other options also exist)

Log-barrier of $\mathcal{K}_k$:

$$\psi(\mathbf{y}) = \log\left(y_{n_k}^2 - (y_1^2 + \cdots + y_{n_k-1})\right)$$

Semi-definite program

Model

$$\min\{\mathbf{C} \bullet \mathbf{X} : \mathcal{A}(\mathbf{X}) = \mathbf{b}, \mathbf{X} \succeq_{\mathcal{K}} \mathbf{0}\}$$

where $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_K$; each $\mathcal{K}_k = \mathbf{S}_+^{n_k}$.

IPM is the standard solver (though other options also exist)

Log-barrier of $\mathbf{S}_+^{n_k}$ (still a concave function):

$$\psi(\mathbf{Y}) = \log\det(\mathbf{Y}).$$

(from Boyd & Vandenberghe, *Convex Optimization*)

properties (without proof): for $y \succ_K 0$,

$$\nabla\psi(y) \succeq_{K^*} 0, \qquad y^T\nabla\psi(y) = \theta$$

- nonnegative orthant $\mathbf{R}^n_+$: $\psi(y) = \sum_{i=1}^n \log y_i$

$$\nabla\psi(y) = (1/y_1, \ldots, 1/y_n), \qquad y^T\nabla\psi(y) = n$$

- positive semidefinite cone $\mathbf{S}^n_+$: $\psi(Y) = \log\det Y$

$$\nabla\psi(Y) = Y^{-1}, \qquad \mathbf{tr}(Y\nabla\psi(Y)) = n$$

- second-order cone $K = \{y \in \mathbf{R}^{n+1} \mid (y_1^2 + \cdots + y_n^2)^{1/2} \leq y_{n+1}\}$:

$$\nabla\psi(y) = \frac{2}{y_{n+1}^2 - y_1^2 - \cdots - y_n^2}\begin{bmatrix} -y_1 \\ \vdots \\ -y_n \\ y_{n+1} \end{bmatrix}, \qquad y^T\nabla\psi(y) = 2$$

(from Boyd & Vandenberghe, *Convex Optimization*)

Central path

- for $t > 0$, define $x^\star(t)$ as the solution of

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

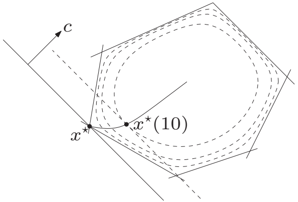
  (for now, assume $x^\star(t)$ exists and is unique for each $t > 0$)

- central path is $\{x^\star(t) \mid t > 0\}$

example: central path for an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, 6 \end{array}$$

hyperplane $c^T x = c^T x^\star(t)$ is tangent to level curve of ϕ through $x^\star(t)$



Log-barrier formulation:

$$\min\{tf_0(\mathbf{x}) + \phi(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

Complexity of log-barrier interior-point method:

$$k \sim \left\lceil \frac{\log((\sum_i \theta_i)/(\varepsilon t^{(0)}))}{\log \mu} \right\rceil$$

Primal-dual interior-point methods

more efficient than barrier method when high accuracy is needed

- update primal and dual variables at each iteration; no distinction between inner and outer iterations
- often exhibit superlinear asymptotic convergence
- search directions can be interpreted as Newton directions for modified KKT conditions
- can start at infeasible points
- cost per iteration same as barrier method

ℓ_1 minimization by interior-point method

Model

$$\min_{\mathbf{x}}\{\|\mathbf{x}\|_1 : \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2 \leq \sigma\}$$

$\Leftrightarrow$

$$\min_{\mathbf{x}} \min_{\mathbf{x}_1,\mathbf{x}_2}\{\mathbf{1}^T(\mathbf{x}_1+\mathbf{x}_2) : \mathbf{x}_1 \geq \mathbf{0}, \mathbf{x}_2 \geq \mathbf{0}, \mathbf{x}_1-\mathbf{x}_2=\mathbf{x}, \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2 \leq \sigma\}$$

$\Leftrightarrow$

$$\min_{\mathbf{x}_1,\mathbf{x}_2}\{\mathbf{1}^T(\mathbf{x}_1+\mathbf{x}_2) : \mathbf{x}_1 \geq \mathbf{0}, \mathbf{x}_2 \geq \mathbf{0}, \|\mathbf{A}(\mathbf{x}_1-\mathbf{x}_2)-\mathbf{b}\|_2 \leq \sigma\}$$

$\Leftrightarrow$

$$\min_{\mathbf{x}_1,\mathbf{x}_2}\{\mathbf{1}^T\mathbf{x}_1+\mathbf{1}^T\mathbf{x}_2 : \mathbf{A}\mathbf{x}_1-\mathbf{A}\mathbf{x}_2+\mathbf{y}=\mathbf{b}, z=\sigma, (\mathbf{x}_1,\mathbf{x}_2,z,\mathbf{y}) \succeq_{\mathcal{K}} \mathbf{0}\}$$

where $(\mathbf{x}_1,\mathbf{x}_2,z,\mathbf{y}) \succeq_{\mathcal{K}} \mathbf{0}$ means

- $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{R}^n_+$,
- $(t,\mathbf{y}) \in \mathcal{Q}^{m+1}$.

Solver: Mosek, SDPT3, Gurobi.

Also, modeling language CVX and YALMIP.

Nuclear-norm minimization by interior-point method

If we can model

$$\min_{\mathbf{X}} \{\|\mathbf{X}\|_* : \mathcal{A}(\mathbf{X}) = \mathbf{b}\} \tag{9}$$

as an SDP ... (how? see next slide) ...

then, we can also model

- $\min_{\mathbf{X}} \{\|\mathbf{X}\|_* : \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_F \leq \sigma\}$
- $\min_{\mathbf{X}} \{\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_F : \|\mathbf{X}\|_* \leq \tau\}$
- $\min_{\mathbf{X}} \mu\|\mathbf{X}\|_* + \frac{1}{2}\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_F^2$

as well as problems involving $\mathrm{tr}(\mathbf{X})$ and spectral norm $\|\mathbf{X}\|$.

$$\|\mathbf{X}\| \leq \alpha \Longleftrightarrow \alpha I - \mathbf{X} \succeq \mathbf{0}.$$

Sparse calculus for ℓ_1

- inspect $|x|$ to get some ideas:
  $y, z \geq 0$ and $\sqrt{yz} \geq |x| \implies \frac{1}{2}(y+z) \geq \sqrt{yz} \geq |x|$.
  moreover, $\frac{1}{2}(y+z) = \sqrt{yz} = |x|$ if $y = z = |x|$.
- observe

$$y, z \geq 0 \text{ and } \sqrt{yz} \geq |x| \iff \begin{bmatrix} y & x \\ x & z \end{bmatrix} \succeq \mathbf{0}.$$

  So,

$$\begin{bmatrix} y & x \\ x & z \end{bmatrix} \succeq \mathbf{0} \implies \frac{1}{2}(y+z) \geq |x|.$$

- we attain $\frac{1}{2}(y+z) = |x|$ if $y = z = |x|$.

Therefore, given x, we have

$$|x| = \min_{\mathbf{M}} \left\{ \frac{1}{2}\operatorname{tr}(\mathbf{M}) : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bullet \mathbf{M} = x, \mathbf{M} = \mathbf{M}^T, \mathbf{M} \succeq \mathbf{0} \right\}.$$

Generalization to nuclear norm

- Consider $\mathbf{X} \in \mathbb{R}^{m \times n}$ (w.o.l.g., assume $m \leq n$) and let's try imposing

$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Z} \end{bmatrix} \succeq \mathbf{0}$$

- Diagonalize $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$, $\|\mathbf{X}\|_* = \sum_i \sigma_i$.

$$\begin{aligned} [\mathbf{U}^T, -\mathbf{V}^T] \begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ -\mathbf{V} \end{bmatrix} &= \mathbf{U}^T\mathbf{Y}\mathbf{U} + \mathbf{V}^T\mathbf{Z}\mathbf{V} - \mathbf{U}^T\mathbf{X}\mathbf{V} - \mathbf{V}^T\mathbf{X}^T\mathbf{U} \\ &= \mathbf{U}^T\mathbf{Y}\mathbf{U} + \mathbf{V}^T\mathbf{Z}\mathbf{V} - 2\Sigma \ \succeq \mathbf{0}. \end{aligned}$$

So, $\mathrm{tr}(\mathbf{U}\mathbf{Y}\mathbf{U}^T + \mathbf{V}\mathbf{Z}\mathbf{V}^T - 2\Sigma) = \mathrm{tr}(\mathbf{Y}) + \mathrm{tr}(\mathbf{Z}) - 2\|\mathbf{X}\|_* \geq 0$.

- To attain "$=$", we can let $\mathbf{Y} = \mathbf{U}\Sigma\mathbf{U}^T$ and $\mathbf{Z} = \mathbf{V}\Sigma_{n\times n}\mathbf{V}^T$.

Therefore,

$$\|\mathbf{X}\|_* = \min_{\mathbf{Y},\mathbf{Z}} \left\{ \frac{1}{2}(\mathrm{tr}(\mathbf{Y}) + \mathrm{tr}(\mathbf{Z})) : \begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Z} \end{bmatrix} \succeq \mathbf{0} \right\} \tag{10}$$

$$= \min_{\mathbf{M}} \left\{ \frac{1}{2}\mathrm{tr}(\mathbf{M}) : \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \bullet \mathbf{M} = \mathbf{X}, \mathbf{M} = \mathbf{M}^T, \mathbf{M} \succeq \mathbf{0} \right\}. \tag{11}$$

Exercise: express the following problems as SDPs

- $\min_{\mathbf{X}} \{\|\mathbf{X}\|_* : \mathcal{A}(\mathbf{X}) = \mathbf{b}\}$
- $\min_{\mathbf{X}} \mu\|\mathbf{X}\|_* + \frac{1}{2}\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_F$
- $\min_{\mathbf{L},\mathbf{S}} \{\|\mathbf{L}\|_* + \lambda\|\mathbf{S}\|_1 : \mathcal{A}(\mathbf{L} + \mathbf{S}) = \mathbf{b}\}$

Practice of interior-point methods (IPMs)

- LP, SOCP, SDP are well known and have reliable (commercial, off-the-shelf) solvers
- Yet, the most reliable solvers cannot handle large-scale problems (e.g., images, video, manifold learning, distributed stuff, ...)
  - Example: to recover a still image, there can be 10M variables and 1M constraints. Even worse, the constraint coefficients are dense. Result: Out of memory.
- Simplex and active-set methods: matrix containing $\mathbf{A}$ must be inverted or factorized to compute the next point (unless $\mathbf{A}$ is very sparse).
- IPMs approximately solve a Newton system and thus also factorize a matrix involving $\mathbf{A}$.
- Even large and dense matrices can be handled, for sparse optimization, one should take advantages of the solution sparsity.
- Some compressive sensing problems have $\mathbf{A}$ with structures friendly for operations like $\mathbf{A}\mathbf{x}$ and $\mathbf{A}^T\mathbf{y}$.

Practice of interior-point methods (IPMs)

- The Simplex, active-set, and IPMs have *reliable* solvers; good to be the benchmark.
- They have nice interfaces (including *CVX* and *YALMIP*, which save you time.)
  CVX and *YALMIP* are not solvers; they translate problems and then call solvers; see `http://goo.gl/zUlMK` and `http://goo.gl/1u0xP`.
- They can return *highly accurate* solutions; some first-order algorithms (coming later in this course) do not always.
- There are other remedies; see next slide.

Papers of large-scale SDPs

- Low-rank factorizations:

  • S. Burer and R. D. C. Monteiro, A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization, Math. Program., 95:329–357, 2003.

  • LMaFit, `http://lmafit.blogs.rice.edu/`

- First-order methods for conic programming:

  • Z. Wen, D. Goldfarb, and W. Yin. Alternating direction augmented Lagrangian methods for semidefinite programming. Math. Program. Comput., 2(3-4):203–230, 2010.

- Matrix-free IPMs:

  • K. Fountoulakis, J. Gondzio, P. Zhlobich. Matrix-free interior point method for compressed sensing problems, 2012. `http://www.maths.ed.ac.uk/~gondzio/reports/mfCS.html`

Subgradient methods

Sparse optimization is typically nonsmooth, so it is natural to consider subgradient methods.

- apply subgradient descent to, say, $\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \frac{\mu}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.
- apply projected subgradient descent to, say, $\min_{\mathbf{x}}\{\|\mathbf{x}\|_1 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

Good: subgradients are easy to derive, methods are simple to implement.

Bad: convergence requires carefully chosen step sizes (classical ones require diminishing step sizes). Convergence rate is weak on paper (and in practice, too?)

Further readings: `http://arxiv.org/pdf/1001.4387.pdf`, `http://goo.gl/qFVA6`, `http://goo.gl/vC21a`.