15.094J: Robust Modeling, Optimization, Computation

Lecture 5: Robust Mixed Integer Optimization

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Outline

Row-wise Polyhedral Uncertainty

- Primitives: Uncertainty sets $\mathcal{U}_i$, $i = 1, \ldots, m$, b, c (known, WLOG).
- RLO with row-wise uncertainty:

$$\begin{array}{rl} \max & c'x \\ \text{s.t.} & a_i'x \le b_i. \quad \forall a_i \in \mathcal{U}_i,\ i = 1, \ldots, m, \\ & x \ge \mathbf{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$

- $\mathcal{U}_i = \{a_i |\ D_i a_i \le d_i\}$, $D_i :\ k_i \times n$.
- RC

$$\begin{array}{rll} \max\limits_{x, p_i} & c'x & \\ \text{s.t.} & p_i' d_i \le b_i, & i = 1, \ldots, m, \\ & p_i' D_i = x', & i = 1, \ldots, m, \\ & p_i \ge \mathbf{0}, & i = 1, \ldots, m, \\ & x \ge \mathbf{0},\ x_i \in \mathcal{Z}, & i = 1, \ldots, k. \end{array}$$

- RMIO reduces to MIO.
- Same even if uncertainty is not row-wise.

Row-wise Ellipsoidal uncertainty

- RO:

$$\begin{array}{rl} \max & c'x \\ \text{s.t.} & \max_{a_i \in \mathcal{U}_i} a_i'x \leq b_i. \\ & x \geq \mathbf{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$

- $\mathcal{U}_i = \{a_i |\ a_i = \overline{a}_i + \mathbf{\Delta}_i' u_i,\ ||u_i||_2 \leq \rho\}$, $\mathbf{\Delta}_i :\ k_i \times n,\ u_i : k_i \times 1.$
- RC:

$$\begin{array}{rl} \max & c'x \\ \text{s.t.} & \overline{a}_i'x + \rho||\mathbf{\Delta}_i x||_2 \leq b_i, \qquad i = 1, \ldots, m. \\ & x \geq \mathbf{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$

- RMIO reduces to Mixed Integer Second order cone problem.
- Current versions of CPLEX and Gurobi support it, but more difficult than MIO.

Row-wise Budget of Uncertainty

- $$\begin{array}{rl} \text{minimize} & \tilde{c}'x \\ \text{subject to} & \tilde{A}x \leq b \\ & x \geq \mathbf{0},\ x_i \in \mathcal{Z},\ i = 1, \ldots, k. \end{array}$$
- **Uncertainty for matrix** A: $a_{ij},\ j \in J_i$ is independent, symmetric and bounded random variable (but with unknown distribution) $\tilde{a}_{ij}, j \in J_i$ that takes values in $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.
- **Uncertainty for cost vector** c: $c_j,\ j \in J_0$ takes values in $[c_j, c_j + d_j]$.

Budget of Uncertainty

- Consider an integer $\Gamma_i \in [0, |J_i|]$, $i = 0, 1, \ldots, m$.
- Γ_i adjusts the robustness of the proposed method against the level of conservativeness of the solution.
- Unlikely that all of the a_{ij}, $j \in J_i$ will change. We want to be protected against all cases that up to Γ_i of the a_{ij}'s are allowed to change.
- Nature will be restricted in its behavior, in that only a subset of the coefficients will change in order to adversely affect the solution.
- We will guarantee that if nature behaves like this then the robust solution will be feasible deterministically. Even if more than Γ_i change, then the robust solution will be feasible with very high probability.

RMIO

$$
\begin{aligned}
RMIO: \quad & \text{minimize} \quad c'x + \max_{\{S_0 |\ S_0 \subseteq J_0, |S_0| \le \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j |x_j| \right\} \\
& \text{subject to} \quad \sum_j a_{ij} x_j + \max_{\{S_i |\ S_i \subseteq J_i, |S_i| \le \Gamma_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij} |x_j| \right\} \le b_i, \quad \forall i \\
& \qquad\qquad\quad x \ge \mathbf{0}, \ x_i \in \mathcal{Z}, \ i = 1, \ldots, k.
\end{aligned}
$$

$$
\begin{aligned}
RC: \quad & \text{minimize} \quad c'x + z_0 \Gamma_0 + \sum_{j \in J_0} p_{0j} \\
& \text{subject to} \quad \sum_j a_{ij} x_j + z_i \Gamma_i + \sum_{j \in J_i} p_{ij} \le b_i \quad && \forall i \\
& \qquad z_0 + p_{0j} \ge d_j y_j && \forall j \in J_0 \\
& \qquad z_i + p_{ij} \ge \hat{a}_{ij} y_j && \forall i \ne 0, j \in J_i \\
& \qquad p_{ij}, y_j, z_i \ge 0 && \forall i, j \in J_i \\
& \qquad -y_j \le x_j \le y_j && \forall j \\
& \qquad x \ge \mathbf{0}, x_i \in \mathcal{Z}, \ i = 1, \ldots, k.
\end{aligned}
$$

Proof

- Given a vector x^*, we define:

$$\beta_i(x^*) = \max_{\{S_i|\ S_i \subseteq J_i, |S_i| = \Gamma_i\}} \left\{ \sum_{j \in S_i} \hat{a}_{ij} |x_j^*| \right\}.$$

- This equals to:

$$\begin{aligned} \beta_i(x^*) = \max \quad & \sum_{j \in J_i} \hat{a}_{ij} |x_j^*| z_{ij} \\ \text{s.t.} \quad & \sum_{j \in J_i} z_{ij} \leq \Gamma_i \\ & 0 \leq z_{ij} \leq 1 \qquad \forall i, j \in J_i. \end{aligned}$$

- Dual:

$$\begin{aligned} \beta_i(x^*) = \min \quad & \sum_{j \in J_i} p_{ij} + \Gamma_i z_i \\ \text{s.t.} \quad & z_i + p_{ij} \geq \hat{a}_{ij} |x_j^*| \quad && \forall j \in J_i \\ & p_{ij} \geq 0 && \forall j \in J_i \\ & z_i \geq 0 && \forall i. \end{aligned}$$

Size

- Original Problem has n variables and m constraints
- RC has $2n + m + l$ variables, where $l = \sum_{i=0}^{m} |J_i|$ is the number of uncertain coefficients, and $2n + m + l$ constraints.
- Sparsity is preserved, attractive!

Probabilistic Guarantees

- x^* an optimal solution of RMIO.
- $\tilde{a}_{ij}, j \in J_i$ independent, symmetric and bounded random variables, support $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.
- $$\Pr\left(\sum_j \tilde{a}_{ij} x_j^* > b_i\right) \le \frac{1}{2^n}\left\{(1-\mu)\sum_{l=\lfloor \nu \rfloor}^{n} \binom{n}{l} + \mu \sum_{l=\lfloor \nu \rfloor + 1}^{n} \binom{n}{l}\right\},$$
  $n = |J_i|$, $\nu = \frac{\Gamma_i + n}{2}$ and $\mu = \nu - \lfloor \nu \rfloor$; bound is tight.
- As $n \to \infty$
  $$\frac{1}{2^n}\left\{(1-\mu)\sum_{l=\lfloor \nu \rfloor}^{n} \binom{n}{l} + \mu \sum_{l=\lfloor \nu \rfloor + 1}^{n} \binom{n}{l}\right\} \sim 1 - \Phi\left(\frac{\Gamma_i - 1}{\sqrt{n}}\right).$$

$\lvert J_i \rvert$	Γ_i
5	5
10	8.3565
100	24.263
200	33.899

Experimental Results

- Knapsack Problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in N} c_i x_i \\ \text{subject to} \quad & \sum_{i \in N} w_i x_i \le b \\ & x \in \{0,1\}^n. \end{aligned}$$

- $\tilde{w}_i$ independently distributed and follow symmetric distributions in $[w_i - \delta_i, w_i + \delta_i]$;
- c is not subject to data uncertainty.
- $|N| = 200$, $b = 4000$,
- w_i randomly chosen from $\{20, 21, \ldots, 29\}$.
- c_i randomly chosen from $\{16, 17, \ldots, 77\}$.
- $\delta_i = 0.1 w_i$.

Experimental Results. continued

Γ	Violation Probability	Optimal Value	Reduction
0	0.5	5592	0%
2.8	0.449	5585	0.13%
36.8	5.71×10^{-3}	5506	1.54%
82.0	5.04×10^{-9}	5408	3.29%
200	0	5283	5.50%

Robust 0-1 Optimization

- Nominal 0-1 optimization:

$$\begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & x \in X \subset \{0,1\}^n. \end{array}$$

- Reformulation:

$$\begin{array}{rll} Z^* = & \text{minimize} & c'x + \max\limits_{\{S|\ S\subseteq J, |S|\leq \Gamma\}} \sum\limits_{j\in S} d_j x_j \\ & \text{subject to} & x \in X, \end{array}$$

Contrast

- Other approaches to robustness are hard. Scenario based uncertainty:

$$\begin{array}{ll} \text{minimize} & \max(c_1'x, c_2'x) \\ \text{subject to} & x \in X. \end{array}$$

  is NP-hard for the shortest path problem.

- $d_1 \geq d_2 \geq \ldots \geq d_n$. Optimal robust solution is

$$Z^* = \min_{l=1,\ldots,n+1} d_l\Gamma + \min_{x\in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l)x_j.$$

- Thus, if nominal problem is polynomially solvable the robust problem is also.

Proof

$$\text{Primal:} \quad Z^* = \min_{x \in X} c'x + \max \sum_j d_j x_j u_j$$
$$\text{s.t.} \quad 0 \leq u_j \leq 1, \qquad \forall\, j$$
$$\sum_j u_j \leq \Gamma$$

$$\text{Dual:} \quad Z^* = \min_{x \in X} c'x + \min \quad \theta\Gamma + \sum_j y_j$$
$$\text{s.t.} \quad y_j + \theta \geq d_j x_j, \qquad \forall\, j$$
$$y_j, \theta \geq 0$$

Proof, continued

- Solution: $y_j = \max(d_j x_j - \theta, 0)$
- $$Z^* = \min_{x \in X, \theta \geq 0} \theta\Gamma + \sum_j (c_j x_j + \max(d_j x_j - \theta, 0))$$
- Since $X \subset \{0,1\}^n$,
$$\max(d_j x_j - \theta, 0) = \max(d_j - \theta, 0)\, x_j$$
- $$Z^* = \min_{x \in X, \theta \geq 0} \theta\Gamma + \sum_j (c_j + \max(d_j - \theta, 0))\, x_j$$

Proof, continued

- $d_1 \geq d_2 \geq \ldots \geq d_n \geq d_{n+1} = 0.$
- For $d_l \geq \theta \geq d_{l+1}$,

$$\min_{x \in X, d_l \geq \theta \geq d_{l+1}} \theta\Gamma + \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - \theta) x_j =$$

$$d_l \Gamma + \min_{x \in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j = Z_l$$

- $$Z^* = \min_{l=1,\ldots,n+1} d_l \Gamma + \min_{x \in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j.$$

Algorithm A

- **Input:** Vectors $c, d \in \Re^n_+$, an integer Γ, and a polynomial time algorithm that solves the problem $Z = \min c'x$ subject to $x \in X \subseteq \{0,1\}^n$ for all $c \geq \mathbf{0}$.
- **Output:** A solution $x^* \in X$ such that
  $x^* = \text{argmin}\left(c'x + \max_{\{S|\ S\subseteq J, |S|=\Gamma\}} \sum_{j\in S} d_j x_j\right)$.

Algorithm A, continued

$$
\begin{array}{l}
x^{\mathbf{1}} \leftarrow \arg\min\{c'x : x \in X\} \\
\textbf{FOR } l \in 2, \ldots, r \\
\quad \textbf{IF } d_l < d_{l-1} \\
\qquad x^l \leftarrow \arg\min\{c'x + \sum_{j=1}^{l}(d_j - d_l)x_j : x \in X\} \\
\qquad Z_l \leftarrow c'x^l + \max_{\{S|\ S \subseteq J, |S| = \Gamma\}} \sum_{j \in S} d_j x_j^l \\
\quad \textbf{ELSE} \\
\qquad x^l \leftarrow x^{l-\mathbf{1}} \\
\qquad Z_l \leftarrow Z_{l-1} \\
\quad \textbf{END IF} \\
\textbf{END FOR}
\end{array}
$$

Algorithm A, continued

$$
\begin{aligned}
11 \quad &: \quad x^{r+1} \leftarrow \arg\min\{c'x + \sum_{j \in J} d_j x_j : x \in X\} \\
12 \quad &: \quad Z_{r+1} \leftarrow c'x^{r+1} + \max_{\{S|\ S \subseteq J, |S| = \Gamma\}} \sum_{j \in S} d_j x_j^{r+1} \\
13 \quad &: \quad \pi \leftarrow \arg\min\{Z_j : j \in J \cup \{r+1\}\} \\
14 \quad &: \quad Z^* = Z_\pi;\ x^* = x^\pi.
\end{aligned}
$$

Theorem

- Algorithm A correctly solves the robust 0-1 optimization problem.
- It requires at most $|J| + 1$ solutions of nominal problems. Thus, if the nominal problem is polynomially time solvable, then the robust 0-1 counterpart is also polynomially solvable.
- Robust minimum spanning tree, minimum assignment, minimum matching, shortest path and matroid intersection, are polynomially solvable.

Robust Approximation Algorithms

- If the nominal problem is α-approximable, is the robust counterpart also α-approximable?
- Use an α-approximate solution to

$$\min_{x \in X} \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j.$$

- Theorem: Overall algorithm is α-approximate.

Ellipsoidal Uncertainty

- $$\min_{x \in X} c'x + \max_{\tilde{s} \in D} \tilde{s}'x$$
- $D = \{s : \|\boldsymbol{\Sigma}^{-1/2} s\|_2 \leq \Omega\}$
- Equivalent to:

$$\min_{x \in X} c'x + \Omega\sqrt{x'\boldsymbol{\Sigma}x}$$

  $\boldsymbol{\Sigma}$ is the covariance matrix of the random cost coefficients: NP-hard
- D a polyhedron: NP-hard.

Uncorrelated uncertainty

- For $\boldsymbol{\Sigma} = diag(d_1^2, \ldots, d_n^2)$,

$$Z^* = \min_{x \in X} c'x + \Omega \sqrt{d'x}$$

Complexity Open.

- Theorem: For $d_1 = \ldots = d_n = \sigma$,

$$Z^* = \min_{w=0,1,\ldots,n} Z(w),$$

$$Z(w) = \begin{cases} \min_{x \in X} \quad \left(c + \dfrac{\Omega\sigma}{2\sqrt{w}} e\right)' x + \dfrac{\Omega\sigma\sqrt{w}}{2} & w = 1, \ldots, n \\ \min_{x \in X} \quad (c + \Omega\sigma e)'x & w = 0. \end{cases}$$

Practical algorithm

- Until $\|x^{k+1} - x^k\| \leq \epsilon$, set $x^{k+1} := \arg\min_{y \in X} (c + \frac{\Omega}{2\sqrt{d' x^k}} d)' y$
- Output x^{k+1}
- Experimented on Shortest Path Problems, Uniform Matroid and Knapsack Problems, under randomly generated cost vectors in dimensions from 200 to 20,000.
- In 998 out of 1000 instances, optimal solution is found in solving less than 6 nominal problems!

Robust Network Flows

- Nominal

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{\{j:(i,j)\in\mathcal{A}\}} x_{ij} - \sum_{\{j:(j,i)\in\mathcal{A}\}} x_{ji} = b_i \quad && \forall i \in \mathcal{N} \\
& 0 \le x_{ij} \le u_{ij} && \forall (i,j) \in \mathcal{A}.
\end{aligned}
$$

- X set of feasible solution flows.
- Robust

$$
\begin{aligned}
Z^* = \quad \min \quad & c'x + \max_{\{S|\ S\subseteq\mathcal{A},|S|\le\Gamma\}} \sum_{(i,j)\in S} d_{ij}x_{ij} \\
\text{s.t.} \quad & x \in X.
\end{aligned}
$$

Theorem

For any fixed $\Gamma \leq |\mathcal{A}|$ and every $\epsilon > 0$, we can find a solution $\hat{x} \in X$:

$$\hat{Z} = c'\hat{x} + \max_{\{S|\ S \subseteq \mathcal{A}, |S| \leq \Gamma\}} \sum_{(i,j) \in S} d_{ij}\hat{x}_{ij}$$

such that

$$Z^* \leq \hat{Z} \leq (1+\epsilon)Z^*$$

by solving $2\lceil \log_2(|\mathcal{A}|\overline{\theta}/\epsilon)\rceil + 3$ network flow problems, where $\overline{\theta} = \max\{u_{ij}d_{ij}\ : (i,j) \in \mathcal{A}\}$.