Course notes for EE394V Restructured Electricity Markets: Locational Marginal Pricing

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5

Economic dispatch

(i) Formulation,
(ii) Problem characteristics,
(iii) Optimality conditions,
(iv) Examples,
(v) Linear programming approximation,
(vi) Homework exercises.

5.1 Formulation

5.1.1 Variables

- Suppose there are n_P generators.
- We consider their electric energy production over a period of time.
- The length T of this period of time is assumed to be short enough so that the power from each generator can be well approximated by a constant over the period of time.
- We will deal separately with variations of power production and demand that occur over shorter time scales through:
  - economic dispatch defined over shorter time scales, and
  - other **ancillary services**.
- Define $P_k \in \mathbb{R}$ to be the (average) power level of generator k during the time period.
- We collect the production decisions of generators $k = 1, \ldots, n_P$, into a vector $P \in \mathbb{R}^{n_P}$, so that $P = \begin{bmatrix} P_1 \\ \vdots \\ P_{n_P} \end{bmatrix}$ is the decision vector.

5.1.2 Generator constraints

- We assume that generator k has:
  a maximum production capacity, say $\overline{P}_k$, and
  a minimum production capacity, $\underline{P}_k \geq 0$.
- That is, P_k must satisfy:

$$\underline{P}_k \leq P_k \leq \overline{P}_k. \tag{5.1}$$

- Equivalently, the feasible operating set for generator k is:

$$\mathbb{S}_k = [\underline{P}_k, \overline{P}_k].$$

- Some generators have a maximum amount of energy that they can deliver over a time horizon:
  – hydro generators with seasonal inflow to a reservoir,
- Maximum capacity may vary over time:
  – wind generation is limited by wind conditions,
  – equipment failures can affect capacity.
- We will not treat energy-limited resources, intermittent resources, nor failures in detail, except in Sections 8.12.6.5 and 10.11.3.

5.1.3 Production costs

- We suppose that for $k = 1, \dots, n_P$ there are functions $f_k : \mathbb{S}_k \rightarrow \mathbb{R}$ such that $f_k(P_k)$ is the cost for generator k to produce at power level P_k for the time period T.
- We will consider the properties of f_k by first considering the **average cost per unit of production** $f_k(P_k)/P_k$.

Average production costs

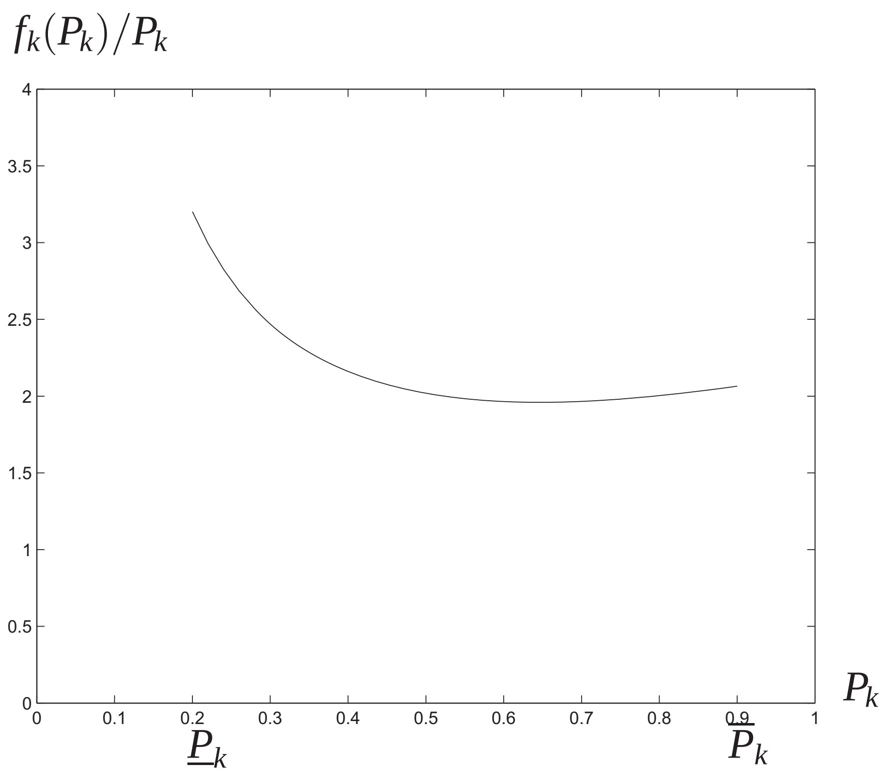

Fig. 5.1. The average production cost $f_k(P_k)/P_k$ versus production P_k for a typical generator for $\underline{P}_k \leq P_k \leq \overline{P}_k$.

Average production costs, continued

- At low levels of production, we would expect the average production cost to be relatively high.
- This is because there are usually "auxiliary" costs that must be incurred whenever the plant is in-service and producing non-zero levels of output.
- As P_k increases from low levels, the average production costs typically decrease because the costs of operating the auxiliary equipment are averaged over a greater amount of production.
- For some P_k, the average costs $f_k(P_k)/P_k$ reach a minimum and then may begin to increase again for larger values of P_k.
- The point where $f_k(P_k)/P_k$ is at a minimum is the point of maximum efficiency of the generator.

Production costs

- If we multiply the values of $f_k(P_k)/P_k$ in Figure 5.1 by P_k, we obtain the production costs $f_k(P_k)$ as illustrated in Figure 5.2.

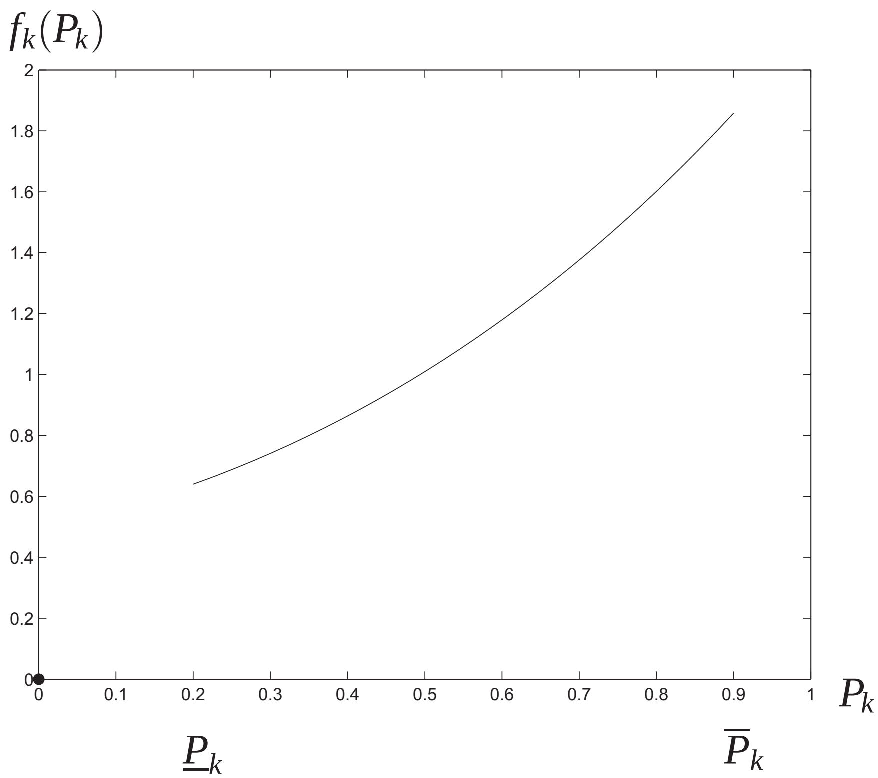

Fig. 5.2. Production cost $f_k(P_k)$ versus production P_k for a typical generator.

Production costs, continued

- Extrapolating the shape of f_k from $\underline{P}_k$ to values $P_k < \underline{P}_k$ we find that at $P_k = 0$ the extrapolated value of the production cost function would be greater than zero due to the auxiliary operating costs.

Convexity

- Figure 5.2 suggests that f_k is convex on $\mathbb{S}_k$.
- It is often reasonable to assume that $f_k : \mathbb{S}_k \to \mathbb{R}$ is quadratic:

$$\forall P_k \in \mathbb{S}_k, f_k(P_k) = \frac{1}{2}Q_{kk}(P_k)^2 + c_k P_k + d_k. \tag{5.2}$$

- We will assume that the cost function of each generator has been extrapolated to a function that is convex on the *whole* of $\mathbb{R}$.
- For convex costs, $Q_{kk} \geq 0$.
- With non-zero auxiliary costs, $d_k > 0$.
- We also usually expect that $c_k > 0$.
- Note that the **marginal costs**, $\nabla f_k(P_k) = Q_{kk}P_k + c_k$, increase with P_k.

5.1.4 Objective

- We must consider the production costs of all generators combined.
- We want to minimize the objective $f : \mathbb{R}^{n_P} \to \mathbb{R}$ defined by:

$$\forall P \in \mathbb{R}^{n_P}, f(P) = \sum_{k=1}^{n_P} f_k(P_k). \tag{5.3}$$

- Adding together the cost functions for all generators, we obtain:

$$\forall P \in \mathbb{R}^{n_P}, f(P) = \frac{1}{2} P^\dagger Q P + c^\dagger P + d,$$

- where $Q \in \mathbb{R}^{n_P \times n_P}$ is a diagonal matrix with k-th diagonal entry equal to Q_{kk},
- $c \in \mathbb{R}^{n_P}$ has k-th entry equal to c_k, and
- $d = \sum_{k=1}^{n_P} d_k \in \mathbb{R}$.

$$\begin{aligned} \forall P \in \mathbb{R}^{n_P}, \nabla f(P) &= \begin{bmatrix} \nabla f_1(P_1) \\ \vdots \\ \nabla f_{n_P}(P_{n_P}) \end{bmatrix}, \\ &= QP + c. \end{aligned}$$

5.1.5 Supply–demand power balance constraint

- Let us assume that during the time period of interest we face (an average) power demand of $\overline{D}$.
- To meet demand, we must satisfy the constraint:

$$\overline{D} = \sum_{k=1}^{n_P} P_k. \tag{5.4}$$

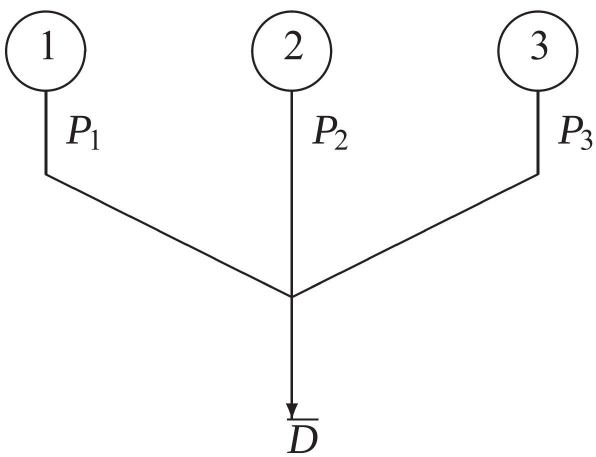

Fig. 5.3. Production from three generators.

5.1.6 Supply–demand power balance constraint, continued

- We can write the power balance constraint in the form $AP = b$ with either of the following two choices for $A \in \mathbb{R}^{1 \times n_P}$ and $b \in \mathbb{R}$:

  $A = \mathbf{1}^\dagger, b = \left[\overline{D}\right]$, or
  $A = -\mathbf{1}^\dagger, b = \left[-\overline{D}\right]$.
- Note that A here is different to the admittance matrix introduced in Section 3.2.4.
- For reasons that will be made clear when we discuss the economic interpretation of the problem, we prefer to use the second choice for A and b:
  - we already used the second choice in the development of linearized power flow in Section 3.6.8.

5.1.7 Generator and power balance constraints combined

- The feasible operating set for all the generators is: $(\prod_{k=1}^{n_P} \mathbb{S}_k) \subset \mathbb{R}^{n_P}$, where the symbol $\prod$ means the **Cartesian product**, so that the feasible set for the problem is:

$$\begin{aligned}\mathbb{S} &= \left(\prod_{k=1}^{n_P} \mathbb{S}_k\right) \cap \{P \in \mathbb{R}^{n_P} | AP = b\}, \\ &= \{P \in \mathbb{R}^{n_P} | AP = b, \underline{P} \leq P \leq \overline{P}\},\end{aligned}$$

- where $\underline{P} \in \mathbb{R}^{n_P}$ and $\overline{P} \in \mathbb{R}^{n_P}$ are constant vectors with k-th entries $\underline{P}_k$ and $\overline{P}_k$, respectively.

5.1.8 Problem

- The **economic dispatch** problem is:

$$\min_{P \in \mathbb{R}^{n_P}} \{f(P) | AP = b, \underline{P} \leq P \leq \overline{P}\} = \min_{\forall k, P_k \in \mathbb{S}_k} \{f(P) | AP = b\}. \tag{5.5}$$

5.2 Problem characteristics

5.2.1 Objective

- For typical cost functions, f_k is convex on $[\underline{P}_k, \overline{P}_k]$.
- Therefore, f is convex.

5.2.2 Equality constraints

- The equality constraint $\overline{D} = \sum_{k=1}^{n_P} P_k$ is linear.

5.2.3 Inequality constraints and the feasible region

- The intersection of the box with the equality constraint restricts the feasible region to being a planar slice through the box.
- This is illustrated in Figure 5.4 for $n_P = 3$, $\overline{D} = 10$, and:

$$\underline{P} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \overline{P} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Inequality constraints and the feasible region, continued

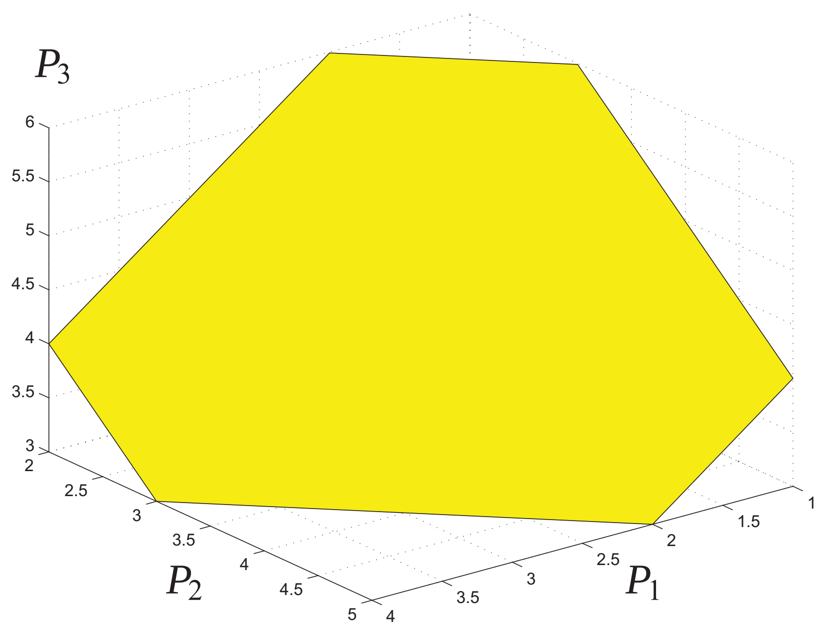

Fig. 5.4. Feasible set for economic dispatch example.

5.2.4 Solvability

- Problem (5.5) is convex.
- It is possible for there to be no feasible points for economic dispatch Problem (5.5).
- Give an example with $n_P = 3$ and $\overline{D} = 10$ of a specification of the economic dispatch problem that is not feasible.
- Give an example with $n_P = 3$, $\underline{P} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\overline{P} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ of a specification of the economic dispatch problem that is not feasible.

5.3 Optimality conditions

5.3.1 First-order necessary conditions

- Assuming that there is a minimizer $P^\star \in \mathbb{R}^{n_P}$, then by Theorem 4.12, the first-order necessary conditions are that (see homework):

$$\begin{aligned}
\exists \lambda^\star \in \mathbb{R}, \exists \underline{\mu}^\star, \overline{\mu}^\star \in \mathbb{R}^{n_P} \text{ such that: } \nabla f(P^\star) - \mathbf{1}\lambda^\star - \underline{\mu}^\star + \overline{\mu}^\star &= \mathbf{0}; \\
\underline{M}^\star(\underline{P} - P^\star) &= \mathbf{0}; \\
\overline{M}^\star(P^\star - \overline{P}) &= \mathbf{0}; \\
-\mathbf{1}^\dagger P^\star &= \left[-\overline{D}\right]; \\
P^\star &\geq \underline{P}; \\
P^\star &\leq \overline{P}; \\
\underline{\mu}^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}^\star &\geq \mathbf{0},
\end{aligned}$$

- where $\underline{M}^\star = \text{diag}\{\underline{\mu}^\star\} \in \mathbb{R}^{n_P \times n_P}$ and $\overline{M}^\star = \text{diag}\{\overline{\mu}^\star\} \in \mathbb{R}^{n_P \times n_P}$ are diagonal matrices with entries specified by the entries of $\underline{\mu}^\star$ and $\overline{\mu}^\star$, respectively, which correspond to the constraints $P \geq \underline{P}$ and $P \leq \overline{P}$.
- These first-order necessary conditions involve the marginal costs ∇f_k.

First-order necessary conditions, continued

- If the generator capacity constraints are not binding then:
  $\underline{\mu}^\star = \overline{\mu}^\star = \mathbf{0}$ and the first and fourth lines of the first-order necessary conditions become:

$$\exists \lambda^\star \in \mathbb{R}, \text{ such that: } \begin{aligned} \nabla f(P^\star) - \mathbf{1}\lambda^\star &= \mathbf{0}; \\ -\mathbf{1}^\dagger P^\star &= \left[-\overline{D}\right]. \end{aligned}$$

  That is, under economic dispatch, the marginal costs for each generator are equalized (and all equal to $\lambda^\star$) and total generation equals demand.

  To interpret, note that if $\nabla f_k(P_k) \neq \nabla f_\ell(P_\ell)$, we could improve dispatch by "backing off" the generator with higher marginal cost and increasing generation at the generator with lower marginal cost.
- If a generator maximum production capacity constraint is binding then its marginal cost is less than or equal to $\lambda^\star$: $\nabla f_k(P_k^\star) = \nabla f_k(\overline{P}_k) = \lambda^\star - \overline{\mu}_k^\star$,
- If a generator minimum production capacity constraint is binding then its marginal cost is greater than or equal to $\lambda^\star$: $\nabla f_k(P_k^\star) = \nabla f_k(\underline{P}_k) = \lambda^\star + \underline{\mu}_k^\star$.

5.3.2 Sensitivity

- By the sensitivity Theorem 4.14, the Lagrange multiplier $\lambda^\star$ equals the sensitivity of the total costs to changes in demand:
  - increasing demand would involve increasing production at the generators,
  - sensitivity of costs to demand is $\lambda^\star$.
- Each Lagrange multiplier $\underline{\mu}_k^\star$ equals the sensitivity of the total costs to changes in the corresponding minimum capacity of generator k.
- Each Lagrange multiplier $\overline{\mu}_k^\star$ equals the sensitivity of the total costs to changes in the corresponding maximum capacity of generator k.

5.3.3 Solving the optimality conditions

5.3.3.1 Capacity constraints not binding

- Assuming that the upper and lower bound constraints are not binding, the first-order necessary conditions are:

$$\begin{aligned}\forall k = 1,\ldots,n_P, \nabla f_k(P_k^\star) - \lambda^\star &= 0,\\ \overline{D} - \sum_{k=1}^{n_P} P_k^\star &= 0.\end{aligned}$$

- If f_k is quadratic then the marginal costs are linear and these equations are linear.
- If the marginal costs are non-linear then these equations are non-linear and can be solved using the Newton–Raphson update.
- If each f_k is strictly convex then there will be a unique minimizer.

5.3.3.2 Dual maximization

- First ignoring the capacity constraints, we can use the analysis from Section 4.4.5.3 for separable objectives.
- We seek the value of Lagrange multiplier $\lambda^\star$ that maximizes the dual.
- For a given λ, the dual function can be evaluated as the minimum of the Lagrangian, which separates into n_P problems that are equivalent to profit maximization for each firm:

$$\forall k = 1, \ldots, n_P, P_k^{(\lambda)} \in \operatorname*{arg\,min}_{P_k \in \mathbb{R}} \{ f_k(P_k) - \lambda P_k \}, \tag{5.6}$$

- This can be generalized to the case of upper and lower bound constraints:

$$\forall k = 1, \ldots, n_P, P_k^{(\lambda)} \in \operatorname*{arg\,min}_{P_k \in \mathbb{R}} \{ f_k(P_k) - \lambda P_k | \underline{P}_k \leq P_k \leq \overline{P}_k \}. \tag{5.7}$$

- Note that we have "dualized" the equality constraint but not the inequality constraints.

5.3.4 Discussion

- Recall the economic interpretation from Sections 4.2.5, 4.4.3, and 4.4.5.3, involving dual maximization and prices.
- We consider the price π paid for producing the energy.
- The Independent System Operator (ISO) solves the economic dispatch problem:
  - the ISO also chooses prices π for production of energy by the generators.
  - the goal of the ISO is to pick prices such that the resulting supply matches demand and total production costs are minimized,
  - as previously, we will assume that the generators choose generation levels that maximize their operating profits given the prices.
- Each generator sells a quantity of production P_k to maximize its operating profits, which is equivalent to minimizing the difference between:

  the production costs $f_k(P_k)$ for the quantity P_k, minus
  the revenue $\pi \times P_k$.
- We will first consider the case without capacity constraints, which repeats previous analysis, and then generalize to the case of capacity constraints.

Discussion, continued

- Ignoring capacity constraints, Problem (5.6) is equivalent to generator profit maximization, given a price $\pi = \lambda$ specified by the ISO and given that the price cannot be influenced by the generator.
- If we solve (5.6) for various possible values of λ, we can construct a function that specifies the profit-maximizing quantity produced, $P_k^{(\lambda)}$, versus given values of λ:
  - By Theorem 4.7, $\nabla f_k(P_k^{(\lambda)}) = \lambda$.
  - Conversely, note that if a generator is choosing production to maximize profits, and it is producing at a level P_k, then the price required by the generator would be $\pi = \nabla f_k(P_k)$.
  - The function specifying the price required by a generator for a given level of generation is called the **offer function**.
- To summarize, if the price will be specified by the ISO and cannot be directly influenced by the generator, then a generator maximizes its profits for each possible price by choosing its offer function equal to its marginal costs.

Discussion, continued

- In the case of capacity constraints, the solution of Problem (5.7) maximizes the generator's profit over its feasible production range given the price $\pi = \lambda$.
- If we solve (5.7) for various values of λ, we can again construct a function that specifies the quantity produced, $P_k^{(\lambda)}$, versus values of λ:
  - By Theorem 4.12, and similarly to the discussion in Section 5.3.1,
$$\nabla f_k(P_k^{(\lambda)}) = \lambda + \underline{\mu}_k^{\star\star} - \overline{\mu}_k^{\star\star},$$
    where $\underline{\mu}_k^{\star\star} \geq 0$ and $\overline{\mu}_k^{\star\star} \geq 0$ are the Lagrange multipliers from Problem (5.7) and are non-zero only if the corresponding production constraints $P_k \geq \underline{P}_k$ or $P_k \leq \overline{P}_k$ are binding.
  - That is, if a generator is choosing production to maximize profits, and it is producing at a level:

    $P_k = \underline{P}_k$, then the price required by the generator would be $\pi \leq \nabla f_k(\underline{P}_k)$,
    $\underline{P}_k < P_k < \overline{P}_k$, then the price required by the generator would be
    $\pi = \nabla f_k(P_k)$,
    $P_k = \overline{P}_k$, then the price required by the generator would be $\pi \geq \nabla f_k(\overline{P}_k)$.

Discussion, continued

- To summarize, within the range of its production constraints, a generator maximizes its operating profits by choosing its offer function equal to marginal costs.
- Setting the price π equal to the Lagrange multiplier $\lambda^{\star}$ in the economic dispatch problem results in profit-maximizing generators collectively meeting demand at the lowest overall production cost.
- In principle, we could imagine that the ISO seeks this price by announcing a sequence of tentative prices:
  - at each iteration, price is raised or lowered to encourage or discourage production depending on whether the total generation is less or more than the demand.
- In practice, there is a more explicit transfer of information from generators to the ISO:
  - the generators provide offer functions to the ISO,
  - we will return to the issue of the correspondence between the offer function and marginal costs in Section 8.11.3.

Discussion, continued

- Recall that the derivative of the the cost function, $\nabla f_k(P_k)$, is called "the marginal cost of production."
- At the optimum, generators not at maximum or minimum capacity limits all have marginal cost of production equal to the Lagrange multiplier.
- The value of the Lagrange multiplier is sometimes called the **shadow price**.

5.4 Examples

5.4.1 Capacity constraints not binding

- Suppose that $n = 3$, with quadratic costs:

$$\begin{aligned}
\forall P_1 \in [0,10],\ f_1(P_1) &= (P_1)^2 \times 0.5\ \$/(\text{MW})^2\text{h}, \\
\forall P_2 \in [0,10],\ f_2(P_2) &= (P_2)^2 \times 1\ \$/(\text{MW})^2\text{h}, \\
\forall P_3 \in [0,10],\ f_3(P_3) &= (P_3)^2 \times 1.5\ \$/(\text{MW})^2\text{h}.
\end{aligned}$$

- That is, the marginal costs are assumed to be linear:

$$\begin{aligned}
\forall P_1 \in [0,10], \nabla f_1(P_1) &= P_1 \times 1\ \$/(\text{MW})^2\text{h}, \\
\forall P_2 \in [0,10], \nabla f_2(P_2) &= P_2 \times 2\ \$/(\text{MW})^2\text{h}, \\
\forall P_3 \in [0,10], \nabla f_3(P_3) &= P_3 \times 3\ \$/(\text{MW})^2\text{h}.
\end{aligned}$$

- Let $\overline{D} = 11$ MW.
- We claim that the minimizer of this economic dispatch problem is $P_1^\star = 6$, $P_2^\star = 3$, and $P_3^\star = 2$.

Capacity constraints not binding, continued

- The optimality conditions are:

$$\begin{array}{rcl}
\exists \lambda^\star \in \mathbb{R}, \exists \underline{\mu}^\star, \overline{\mu}^\star \in \mathbb{R}^{n_P} \text{ such that: } \nabla f(P^\star) - \mathbf{1}\lambda^\star - \underline{\mu}^\star + \overline{\mu}^\star & = & \mathbf{0}; \\
\underline{M}^\star(\underline{P} - P^\star) & = & \mathbf{0}; \\
\overline{M}^\star(P^\star - \overline{P}) & = & \mathbf{0}; \\
-\mathbf{1}^\dagger P^\star & = & \left[-\overline{D}\right]; \\
P^\star & \geq & \underline{P}; \\
P^\star & \leq & \overline{P}; \\
\underline{\mu}^\star & \geq & \mathbf{0}; \text{ and} \\
\overline{\mu}^\star & \geq & \mathbf{0},
\end{array}$$

- We can find the Lagrange multipliers by observing that none of the generators are at their minimum or maximum capacity limits at the claimed solution.

Capacity constraints not binding, continued

- We claim that:

$$\lambda^\star = 6\$/\text{MWh},$$

- and all other Lagrange multipliers have value zero.
- Substituting into the first line of the first-order necessary conditions:

$$\begin{aligned}
\nabla f_1(P_1^\star) - \lambda^\star &= 6 \times 1 - 6, \\
&= 0, \\
\nabla f_2(P_2^\star) - \lambda^\star &= 3 \times 2 - 6, \\
&= 0, \\
\nabla f_3(P_3^\star) - \lambda^\star &= 2 \times 3 - 6, \\
&= 0.
\end{aligned}$$

- The other lines of the first-order necessary conditions are also satisfied.

Capacity constraints not binding, continued

- The sensitivity of total costs to changes in demand is $\lambda^\star = 6$\$/MWh:
  - this is the common value of marginal cost of production for all the generators.
- The sensitivity of total costs to changes in the capacities is zero.
- Estimate how much the total production costs would change if the demand changed by 1 MW.
- Estimate how much the total production costs would change if the capacity of any generator increased by 1 MW.

5.4.2 Capacity constraints binding

- Suppose that $n = 3$, with:

$$\begin{aligned}
\forall P_1 \in [0,1500],\ f_1(P_1) &= P_1 \times 40\$/\text{MWh},\\
\forall P_2 \in [0,1000],\ f_2(P_2) &= P_2 \times 20\$/\text{MWh},\\
\forall P_3 \in [0,1500],\ f_3(P_3) &= P_3 \times 50\$/\text{MWh}.
\end{aligned}$$

- That is, the marginal costs are assumed constant for each machine over their feasible production sets:

$$\begin{aligned}
\forall P_1 \in [0,1500], \nabla f_1(P_1) &= 40\$/\text{MWh},\\
\forall P_2 \in [0,1000], \nabla f_2(P_2) &= 20\$/\text{MWh},\\
\forall P_3 \in [0,1500], \nabla f_3(P_3) &= 50\$/\text{MWh}.
\end{aligned}$$

- Let $\overline{D} = 3000$ MW.
- We claim that the minimizer of this economic dispatch problem is $P_1^\star = 1500$, $P_2^\star = 1000$, and $P_3^\star = 500$.

Capacity constraints binding, continued

- The optimality conditions are:

$$
\begin{aligned}
\exists \lambda^\star \in \mathbb{R}, \exists \underline{\mu}^\star, \overline{\mu}^\star \in \mathbb{R}^{n_P} \text{ such that: } \nabla f(P^\star) - \mathbf{1}\lambda^\star - \underline{\mu}^\star + \overline{\mu}^\star &= \mathbf{0}; \\
\underline{M}^\star(\underline{P} - P^\star) &= \mathbf{0}; \\
\overline{M}^\star(P^\star - \overline{P}) &= \mathbf{0}; \\
-\mathbf{1}^\dagger P^\star &= \left[-\overline{D}\right]; \\
P^\star &\geq \underline{P}; \\
P^\star &\leq \overline{P}; \\
\underline{\mu}^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}^\star &\geq \mathbf{0},
\end{aligned}
$$

- We can find the Lagrange multipliers by observing that only generator 3 is not at its minimum or maximum capacity limits at the claimed solution.

Capacity constraints binding, continued

- We claim that:

$$\begin{aligned}
\lambda^\star &= 50\$/\text{MWh}, \\
\overline{\mu}_1^\star &= 50 - 40 = 10\$/\text{MWh}, \\
\overline{\mu}_2^\star &= 50 - 20 = 30\$/\text{MWh},
\end{aligned}$$

- and all other Lagrange multipliers have value zero.
- Substituting into the first line of the first-order necessary conditions:

$$\begin{aligned}
\nabla f_1(P_1^\star) - \lambda^\star - \underline{\mu}_1^\star + \overline{\mu}_1^\star &= 40 - 50 - 0 + 10, \\
&= 0, \\
\nabla f_2(P_2^\star) - \lambda^\star - \underline{\mu}_2^\star + \overline{\mu}_2^\star &= 20 - 50 - 0 + 30, \\
&= 0, \\
\nabla f_3(P_3^\star) - \lambda^\star - \underline{\mu}_3^\star + \overline{\mu}_3^\star &= 50 - 50 - 0 + 0, \\
&= 0.
\end{aligned}$$

- The other lines of the first-order necessary conditions are also satisfied.

Capacity constraints binding, continued

- The sensitivity of total costs to changes in demand is $\lambda^\star = 50\$/\text{MWh}$:
  - this is the marginal cost of production for generator 3, which is not at its maximum nor minimum capacity limits.
- The sensitivity of total costs to changes in the maximum capacity of generator 1 is $\overline{\mu}_1^\star = 10(\$/\text{h})/\text{MW}$.
- The sensitivity of total costs to changes in the maximum capacity of generator 2 is $\overline{\mu}_2^\star = 30(\$/\text{h})/\text{MW}$.
- The sensitivity of total costs to changes in other capacities is zero.
- Estimate how much the total production costs would change if the demand changed by 1 MW.
- Estimate how much the total production costs would change if the capacity of any generator increased by 1 MW.

Capacity constraints binding, continued

- Note that for levels of demand other than $\overline{D} = 3000$ MW, optimal dispatch would correspond to:
  - for $0 \leq \overline{D} \leq 1000$ MW, lowest marginal cost generator 2 would be dispatched to meet all the demand,
  - for $1000 < \overline{D} \leq 2500$ MW, lowest marginal cost generator 2 would generate at maximum capacity and generator 1 would be dispatched to meet the rest of the demand, and
  - for $2500 < \overline{D} \leq 4000$ MW, lowest marginal cost generator 2 would generate at maximum capacity, generator 1 would generate at maximum capacity, and generator 3 would be dispatched to meet the rest of the demand.
- That is, we use generators in order of their marginal costs, from lowest to highest.
- This is called **merit order**.

Capacity constraints binding, continued

- What is $\lambda^\star$ for $0 \leq \overline{D} \leq 1000$ MW?
- What is $\lambda^\star$ for $1000 < \overline{D} \leq 2500$ MW?
- What is $\lambda^\star$ for $2500 < \overline{D} \leq 4000$ MW?
- What prices would provide the right compensation so that profit-maximizing firms generate consistently with economic dispatch?
- What happens for $\overline{D} > 4000$ MW?

5.5 Linear programming approximation

- Typical generator cost functions are non-linear, for example, quadratic.
- To use linear programming software to solve economic dispatch, we need to approximate the generator costs function.
- A typical approximation is to **piece-wise linearize** the function.
- This approximates the marginal costs as being piece-wise constant.

5.5.1 Piece-wise linearization

- For a function $f : [0,1] \to \mathbb{R}$ we might:
  - define subsidiary variables $\xi_1, \ldots, \xi_5$,
  - include constraints:

$$\begin{aligned} P &= \sum_{j=1}^{5} \xi_j, \\ 0 &\leq \xi_j \leq 0.2, \end{aligned}$$

  - define parameters:

$$\begin{aligned} d &= f(0), \\ c_j &= \frac{1}{0.2}\left[f(0.2\times j) - f(0.2\times(j-1))\right], j = 1,\ldots,5, \end{aligned}$$

    and
  - replace the objective f by the piece-wise linearized objective $\phi : \mathbb{R}^5 \to \mathbb{R}$ defined by:

$$\forall \xi \in \mathbb{R}^5, \phi(\xi) = c^{\dagger}\xi + d.$$

5.5.2 Quadratic example function

$$\forall P \in [0,1], f(P) = (P)^2.$$

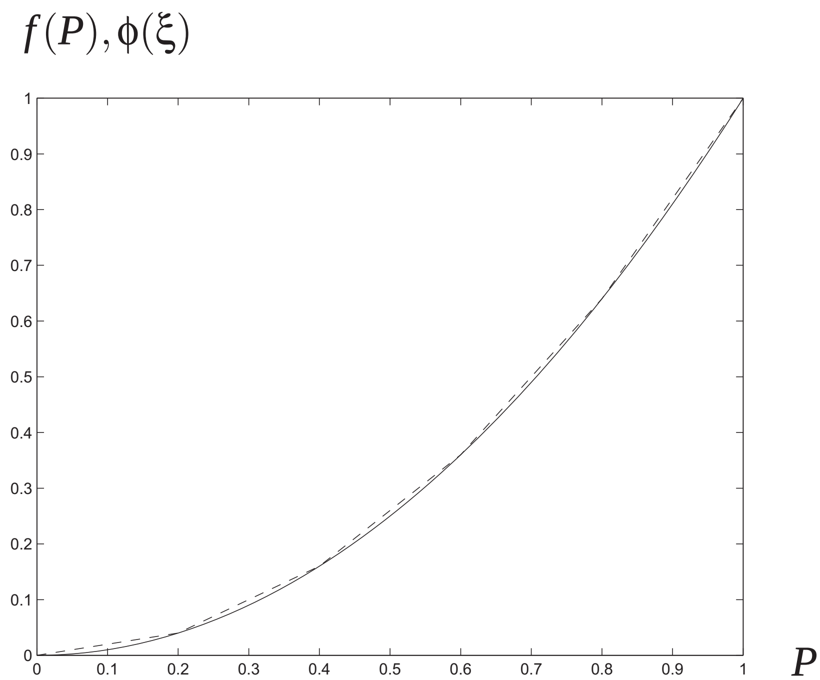

Fig. 5.5. Piece-wise linearization (shown dashed) of a function (shown solid).

Quadratic example function, continued

- For the function f illustrated in Figure 5.5:

$$\begin{aligned} d &= f(0), \\ &= 0, \\ c_j &= \frac{1}{0.2}\left(f(0.2 \times j) - f(0.2 \times (j-1))\right), \\ &= (0.4 \times j) - 0.2, j = 1, \ldots, 5. \end{aligned}$$

- To piece-wise linearize f in an optimization problem, we use ϕ as the objective instead of f, augment the decision vector to include ξ, include the constraints that link ξ and P, and include the bound constraints on ξ.
- Similarly, non-linear constraints can also be piece-wise linearized.

5.6 Summary

(i) Formulation,
(ii) Problem characteristics,
(iii) Optimality conditions,
(iv) Examples,
(v) Linear programming approximation.

This chapter is based on:

- Sections 12.1, 13.5, and 15.1 of *Applied Optimization: Formulation and Algorithms for Engineering Systems*, Cambridge University Press 2006,
- Daniel S. Kirschen and Goran Strbac, *Power System Economics*, Wiley, 2004.

Homework exercises

5.1 In this exercise, we consider the optimality conditions for the economic dispatch Problem (5.5) and for the individual generator profit maximization problem (5.7).

(i) Use Theorem 4.12 to verify the first-order necessary conditions presented in Section 5.3.1 for the economic dispatch problem. (Hint: In Theorem 4.12, let $A = -\mathbf{1}^{\dagger}$, $b = \left[-\overline{D}\right]$, $C = \begin{bmatrix} -\mathbf{I} \\ \mathbf{I} \end{bmatrix}$, and $d = \begin{bmatrix} -\underline{P} \\ \overline{P} \end{bmatrix}$. Define $\underline{\mu}^{\star}$ and $\overline{\mu}^{\star}$ to be suitable sub-vectors of the Lagrange multiplier $\mu^{\star}$ in Theorem 4.12.)

(ii) Given a value of λ, write down the first-order necessary conditions for the individual generator profit maximization problem (5.7). Use the symbols $\underline{\mu}_k^{\star\star}$ and $\overline{\mu}_k^{\star\star}$ for the Lagrange multipliers on the lower and upper bound constraints.

5.2 Consider the economic dispatch Problem (5.5) in the particular case that $n_P = 3$, $\overline{D} = 5$, $\underline{P} = \mathbf{0}$, $\overline{P} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$, and the f_k are of the form:

$$\begin{aligned}
\forall P_1 \in [\underline{P}_1, \overline{P}_1],\ f_1(P_1) &= \frac{1}{2}(P_1)^2 + P_1, \\
\forall P_2 \in [\underline{P}_2, \overline{P}_2],\ f_2(P_2) &= \frac{1}{2} \times 1.1(P_2)^2 + 0.9P_2, \\
\forall P_3 \in [\underline{P}_3, \overline{P}_3],\ f_3(P_3) &= \frac{1}{2} \times 1.2(P_3)^2 + 0.8P_3.
\end{aligned}$$

Solve the economic dispatch problem by solving the first-order necessary conditions in terms of the minimizer $P^\star$ and the Lagrange multipliers $\lambda^\star$, $\underline{\mu}^\star$, and $\overline{\mu}^\star$. (Hint: Because the minimum capacities are low enough and because the maximum capacities are large enough, none of the minimum and maximum capacity constraints will be binding. By complementary slackness, what can you say about $\underline{\mu}^\star$ and $\overline{\mu}^\star$?)

5.3 Consider the economic dispatch Problem (5.5) in the particular case that $n_P = 3$, $\underline{P} = \mathbf{0}$, $\overline{P} = \begin{bmatrix} 150 \\ 1000 \\ 1000 \end{bmatrix}$, and the ∇f_k are of the form:

$$\begin{aligned}
\forall P_1 \in [\underline{P}_1, \overline{P}_1], \nabla f_1(P_1) &= 20, \\
\forall P_2 \in [\underline{P}_2, \overline{P}_2], \nabla f_2(P_2) &= 50, \\
\forall P_3 \in [\underline{P}_3, \overline{P}_3], \nabla f_3(P_3) &= 100.
\end{aligned}$$

Solve the economic dispatch problem and find the minimizer $P^\star$ and the Lagrange multipliers $\lambda^\star$, $\underline{\mu}^\star$, and $\overline{\mu}^\star$ for demand:

(i) $\overline{D} = 500$, and

(ii) $\overline{D} = 1500$.

(Hint: What is the lowest marginal cost generation? How much of that can be used?)