

Nonlinear Systems Examples Sheet: Solutions

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Equilibrium points

1. (a). Solving $\dot{x} = \sin^4 x - x^3 = 0$ for x gives $x = 0$ as an equilibrium point. This is the only equilibrium because there is only one point ($x = 0$) where $\sin x = x$ since

$$\begin{aligned} |\sin x| < |x| < 1 &\implies |\sin x|^4 < |x|^3 \text{ for all } |x| \leq 1, x \neq 0 \\ |\sin x| \leq 1 &\implies |\sin x|^4 < |x|^3 \text{ for all } |x| > 1 \end{aligned}$$

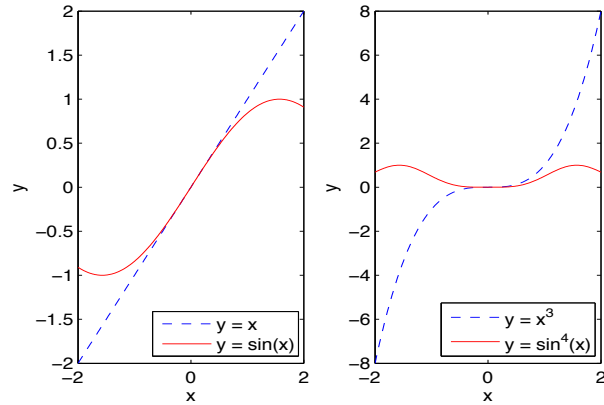


Figure 1: solution of $x^3 = \sin^4 x$ for question 1

- (b). In terms of state variables $(x_1, x_2) = (x, \dot{x})$:

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -(x_1 - 1)^2 x_2^5 - x_1^3 + \sin(\pi x_1/2) \end{aligned}$$

At an equilibrium point $\dot{x}_1 = \dot{x}_2 = 0$. But $\dot{x}_1 = 0$ implies $x_2 = 0$, so

$$\dot{x}_2 = 0 \implies x_1^2 - \sin(\pi x_1/2) = 0 \implies x_1 = 0 \text{ or } 1$$

Therefore equilibrium points are $(x_1, x_2) = (x, \dot{x}) = (0, 0)$ and $(1, 0)$.

Lyapunov's direct method, invariant sets and linearization

2. To explain the significance of constants a, b, c , we first give a derivation of the dynamics (this is not asked for in the question). The angular momentum of the craft in xyz -coordinates (Fig. 2) is given by

$$H = I\omega, \quad I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where I_x, I_y, I_z are the moments of inertia about x, y , and z -axes (assumed to be aligned with the spacecraft's principle axes). Since there is no torque acting on the craft:

$$\frac{d}{dt}(I\omega) = I\dot{\omega} + \omega \times I\omega = 0$$

(where the $\omega \times I\omega$ term is needed because xyz -coordinates are fixed to and hence rotate with the spacecraft). So the full dynamics are given by

$$\begin{aligned} \dot{\omega}_x &= a\omega_y\omega_z & \dot{\omega}_y &= -b\omega_x\omega_z & \dot{\omega}_z &= c\omega_x\omega_y \\ a &= (I_y - I_z)/I_x, & b &= (I_x - I_z)/I_y, & c &= (I_x - I_y)/I_z \end{aligned}$$

and the constants a, b, c are all positive if $I_x > I_y > I_z$.

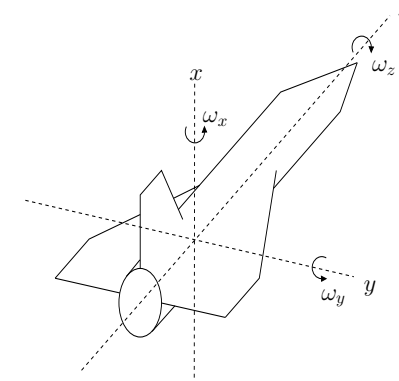


Figure 2: Rotating spacecraft.

- (a). Equilibrium points: $\dot{\omega}_x = 0 \iff \omega_y = 0$ or $\omega_z = 0$, i.e. at least two of ω_x, ω_y and ω_z must be zero for $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Therefore

every point in state space lying on the ω_x -axis, the ω_y -axis, or the ω_z -axis is an equilibrium point.

- (b). To show stability of the equilibrium at $\omega = 0$, try $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$ as a Lyapunov function. Clearly V is positive definite if p, q, r are all positive. Also

$$\begin{aligned}\dot{V} &= 2(p\omega_x\dot{\omega}_x + q\omega_y\dot{\omega}_y + r\omega_z\dot{\omega}_z) \\ &= 2(pa - qb + rc)\omega_x\omega_y\omega_z\end{aligned}$$

Hence choosing p, q, r so that

$$p > 0, \quad q > 0, \quad r > 0, \quad \text{and} \quad pa - qb + rc = 0,$$

(which is always possible since $q = (pa + rc)/b$ is positive for any chosen positive p, r), results in $\dot{V} = 0$, implying that $\omega = 0$ is a stable equilibrium point by Lyapunov's direct method.

- (c). Differentiating the function

$$V = c\omega_y^2 + b\omega_z^2 + [2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)]^2$$

(for constant ω_0) with respect to t along system trajectories yields

$$\begin{aligned}\dot{V} &= \underbrace{2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z}_{=0} \\ &\quad + 2[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)] \underbrace{(4ac\omega_y\dot{\omega}_y + 2ab\omega_z\dot{\omega}_z + 2bc\omega_x\dot{\omega}_x)}_{=0}\end{aligned}$$

i.e. $\dot{V} = 0$. Also $V = 0$ only if $\omega = (\pm\omega_0, 0, 0)$, and $V > 0$ whenever $\omega_x \neq \pm\omega_0$, $\omega_y \neq 0$ or $\omega_z \neq 0$, so that V is a locally positive definite function centered at the equilibrium $(\pm\omega_0, 0, 0)$. Therefore $\dot{V} = 0$ implies that every point on the ω_x -axis in state space is a stable equilibrium, and hence that rotation at any constant velocity about the x -axis alone is stable.

[Note that rotational motion about the z -axis is likewise stable since a, c and ω_x, ω_z can be swapped in the dynamics and in the definition of V . However rotation about the y -axis is unstable, as shown by the

linearized system at $\omega = (0, \omega_0, 0)$:

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & a\omega_0 \\ 0 & 0 & 0 \\ c\omega_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

which has eigenvalues $\pm\sqrt{ac}\omega_0$ and 0, and is therefore unstable.]

3. (a). The positive definite function $V = \frac{1}{2}x^2$ has derivative:

$$\dot{V} = x\dot{x} = -xb(x)$$

which is negative definite due to $xb(x) > 0$ whenever $x \neq 0$. Therefore $x = 0$ is asymptotically stable, and since $V \rightarrow \infty$ as $x \rightarrow \infty$ it follows that $x = 0$ is globally asymptotically stable by Lyapunov's direct method.

- (b). At an equilibrium point $\dot{x} = 0$. Hence $\ddot{x} = -c(x) = 0$ implies $x = 0$ since the condition $xc(x) > 0$ whenever $x \neq 0$ implies that $c(x)$ can only be equal to zero if $x = 0$. Therefore the only equilibrium point is the origin of state space: $(x, \dot{x}) = (0, 0)$.

The function $V(x, \dot{x})$ is positive definite and has derivative

$$\dot{V} = \dot{x}\ddot{x} + c(x)\dot{x} = -\dot{x}b(\dot{x}) \leq 0$$

and hence $(x, \dot{x}) = (0, 0)$ is stable by Lyapunov's direct method.

To apply the local invariant set theorem, we need to show that: (i) the level sets $\{(x, \dot{x}) : V(x, \dot{x}) \leq V_0\}$ are bounded for some V_0 ; (ii) $\dot{V} \leq 0$; (iii) the system dynamics are continuous and V is continuously differentiable in x and \dot{x} . Here (i) is satisfied because V is increasing in both x (since $\text{sign}(c(x)) = \text{sign}(x)$) and \dot{x} ; (ii) is demonstrated above; and (iii) holds since $b(\dot{x})$, $c(x)$, $\partial V/\partial \dot{x} = \dot{x}$, and $\partial V/\partial x = c(x)$ are all continuous functions of x and \dot{x} . Let $\mathcal{R} = \{(x, \dot{x}) : \dot{V} = 0\}$ and let \mathcal{M} be the largest invariant set contained in \mathcal{R} , then

$$\mathcal{R} = \{(x, \dot{x}) : \dot{x} = 0\}$$

and since $\ddot{x} = 0$ is necessary in order that the state remains in \mathcal{R} , we have

$$\mathcal{M} = \mathcal{R} \cap \{(x, \dot{x}) : \ddot{x} = 0\} = \{(x, \dot{x}) : c(x) = 0\} = \{(0, 0)\}.$$

From the local invariant set theorem, (x, \dot{x}) therefore converges asymptotically to \mathcal{R} from all initial conditions within any bounded level set of V , implying that $(0, 0)$ is asymptotically stable.

To show global asymptotic stability we need V to be radially unbounded (in order to apply the global invariant set theorem) or equivalently the level sets of V must cover the entire state space as $V_0 \rightarrow \infty$. This condition requires

$$\int^x c(s) ds \rightarrow \infty \text{ as } x \rightarrow \infty.$$

4. (a). The equilibrium points can be found by solving $\dot{x}_1 = \dot{x}_2 = 0$ for x_1 and x_2 :

$$\begin{aligned} \dot{x}_1 = 0 &\implies x_2 = 0 \\ \dot{x}_1 = \dot{x}_2 = 0 &\implies x_1(x_1^2 - 1) = 0 \implies x_1 = 0, 1, -1. \end{aligned}$$

Hence the equilibrium points are $(x_1, x_2) = \{(0, 0), (1, 0), (-1, 0)\}$.

- (b). The system and function V have the following properties.

- (i). V , \dot{x}_1 and \dot{x}_2 are continuous functions of x_1 and x_2 .
- (ii). The level sets: $\{(x_1, x_2) : V \leq V_0\}$ are finite and V is radially unbounded since $V \rightarrow \infty$ as $|x_1| \rightarrow \infty$ and/or $|x_2| \rightarrow \infty$.
- (iii). Along system trajectories, V has derivative

$$\begin{aligned} \dot{V}(x_1, x_2) &= x_2 \dot{x}_2 + x_1(x_1^2 - 1)\dot{x}_1 \\ &= -x_2^2(x_1 - 1)^2 - x_1 x_2(x_1^2 - 1) + x_1 x_2(x_1^2 - 1) \\ &= -x_2^2(x_1 - 1)^2 \\ &\leq 0. \end{aligned}$$

Using the global invariant set theorem, (i)-(iii) imply that every state trajectory tends to an invariant set on which $\dot{V} = 0$. (The same conclusion can be reached using the local invariant set theorem, since the level sets of V can be made arbitrarily large by choosing V_0 sufficiently large.)

From (iii), $\dot{V}(x_1, x_2) = 0$ is satisfied on the lines $x_2 = 0$ and $x_1 = 1$. The invariant sets within these lines are defined by $\dot{x}_2 = 0$ (on $x_2 = 0$) and $\dot{x}_1 = 0$ (on $x_1 = 1$). But

$$\left. \begin{matrix} x_2 = 0 \\ \dot{x}_2 = 0 \end{matrix} \right\} \implies x_1 = 0, 1, -1, \quad \left. \begin{matrix} x_1 = 1 \\ \dot{x}_1 = 0 \end{matrix} \right\} \implies x_2 = 0$$

and every state trajectory therefore tends asymptotically to one of the three equilibrium points identified in (a).

- (c). Writing the system dynamics in the form $\dot{x} = f(x)$, $x = [x_1 \ x_2]^T$ where the Jacobian matrix of f is

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ -2x_2(x_1 - 1) - (3x_1^2 - 1) & -(x_1 - 1)^2 \end{bmatrix},$$

the linearization of the system at $x_1 = x_2 = 0$ is given by

$$\dot{x} = Ax, \quad A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

A has eigenvalues $-1/2 \pm \sqrt{5}/2$, and it follows that the origin is an unstable equilibrium of the nonlinear system, by Lyapunov's linearization method.

- (d). V has local minimum points at $(x_1, x_2) = (-1, 0)$ and $(1, 0)$ (since

$$\nabla V = \begin{bmatrix} x_1^3 - x_1 \\ x_2 \end{bmatrix} = 0 \quad \frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} 3x_1^2 - 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

at $(x_1, x_2) = (-1, 0)$ and $(1, 0)$). Hence $V + \frac{1}{4}$ is locally positive definite at $(x_1, x_2) = (-1, 0)$ and $(1, 0)$, and from Lyapunov's direct method these equilibrium points are therefore stable because $\dot{V} \leq 0$.

Other approaches for (d): The equilibrium at $(-1, 0)$ can be shown to be stable using the linearization method, since the linearization at this point is stable. However the linearization about $(1, 0)$ has eigenvalues $\pm i\sqrt{2}$, and therefore does not allow any conclusions to be made about the stability of this equilibrium for the nonlinear system.

5. (a). Using matrices A, B, K and the given matrix P we get (2 marks):

$$Q = -(A - BK)^T P - P(A - BK) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where

$$\text{eig}(P) = \lambda : \lambda^2 - 3\lambda + 1 = 0 \implies \lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$$\text{eig}(Q) = \lambda : \lambda^2 - 4\lambda + 3 = 0 \implies \lambda = 1, 3$$

The equilibrium $x = 0$ is locally asymptotically stable since:

- the linearized closed loop system about $x = 0$ is $\dot{x} = (A - BK)x$
- $(A - BK)^T P + P(A - BK) = -Q$ for positive definite P, Q implies $\dot{x} = (A - BK)x$ is stable, i.e. $\text{Re}[\text{eig}(A - BK)] < 0$
- so the nonlinear closed loop system is locally a.s.

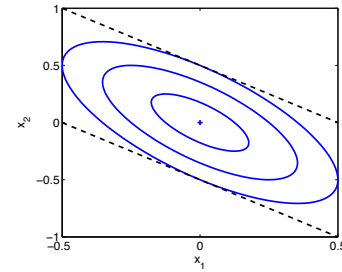
- (b). From $V = x^T P x$ and $\dot{x} = (A - BK)x - x(Kx)$ we get

$$\begin{aligned} \dot{V} &= x^T [(A - BK)^T P + P(A - BK)] x - (Kx)x^T (P + P)x \\ &= -x^T Q x - 2(Kx)x^T P x \\ &\leq -x^T Q x + 2|Kx|x^T P x \end{aligned}$$

But $x^T P x - x^T Q x = x^T (P - Q)x = -x_2^2 \leq 0$,

so $\dot{V} \leq -x^T Q x + 2|Kx|x^T P x$.

- (c). $\dot{V} \leq -x^T Q x(1 - 2|Kx|)$, so \dot{V} is negative definite in the region where $|Kx| < \frac{1}{2}$, which is the strip between the dashed lines in the figure below.



Any level set of V contained entirely within this strip is invariant and hence is a region of attraction for $x = 0$.

The level sets Ω are ellipsoidal, centred on the origin, and decrease in size as α is reduced. Hence Ω must be invariant for small enough α .

Linear and passive systems

6. Let $\Phi = A + \mu I$, then $A^T P + PA + 2\mu P = -Q$ implies

$$\Phi^T P + P\Phi = A^T P + PA + 2\mu P = -Q,$$

so $P, Q > 0$ imply that $\text{Re}\{\text{eig}(\Phi)\} < 0$, so that $\text{Re}\{\text{eig}(A + \mu I)\} < 0$, and therefore $\text{Re}\{\text{eig}(A)\} < -\mu$

(since $A = V\Lambda V^{-1} \implies \Phi = V(\Lambda - \mu I)V^{-1}$).

7. (a). Differentiating V_1 with respect to t gives:

$$\dot{V}_1 = \frac{x_2 e}{L(x_2)} - \frac{R_1}{L(x_2)} x_2^2 = \dot{x}_1 e - \frac{R_1}{L(x_2)} x_2^2$$

and since $V \geq 0$, this implies that the dynamic system with e as input and \dot{x}_1 as output is passive (in fact it is dissipative).

- (b). Let x_3 and x_4 be respectively the charge on the capacitor and flux in the inductor in the right-hand branch of the circuit, and define

$$V_2(x_3, x_4) = \int_0^{x_4} \frac{x}{L(x)} dx + \int_0^{x_3} \frac{x}{C(x)} dx.$$

Differentiating w.r.t. t gives $\dot{V}_2 = \dot{x}_3 e - R_1 x_4^2 / L(x_4)$. Therefore, defining $V = V_1 + V_2$ and using the fact that $\dot{x}_1 + \dot{x}_3 = i$ (since the

currents in the two branches of the circuit must sum to i), we get

$$V = \int_0^{x_2} \frac{x}{L(x)} dx + \int_0^{x_4} \frac{x}{L(x)} dx + \int_0^{x_1} \frac{x}{C(x)} dx + \int_0^{x_3} \frac{x}{C(x)} dx$$

$$\dot{V} = ie - \frac{R_1}{L(x_2)} x_2^2 - \frac{R_2}{L(x_4)} x_4^2.$$

and $V \geq 0$ since $V_1, V_2 \geq 0$.

Opening the switch forces $i = 0$, so

$$\dot{V} = -\frac{R_1}{L(x_2)} x_2^2 - \frac{R_2}{L(x_4)} x_4^2$$

and since the level sets $\{(x_1, x_2, x_3, x_4) : V \leq \bar{V}\}$ are bounded (when \bar{V} is sufficiently small), it follows from the local invariant set theorem that the system is (locally) asymptotically stable.

Specifically, $x = (x_1, x_2, x_3, x_4)$ must converge to the largest invariant set within the set of states such that $\dot{V} = 0$, i.e. $x_2 = x_4 = 0$ and $\dot{x}_2 = \dot{x}_4 = 0$, implying that x converges asymptotically to a steady state such that $x_1/C(x_1) = x_3/C(x_3) = 0$ and $x_2, x_4 = 0$. This asymptotic stability property is global if V_1, V_2 are radially unbounded. Note also that the same analysis can be applied to any number of LCR branches connected in parallel.

8. (a). The rectangular region containing $G(j\omega)$ lies within $D(a, b)$ if $a = -\frac{1}{3}$ and $b = \frac{1}{2}$, since $D(a, b)$ is then just touching its corners (Fig. 3). The open-loop system is stable, and the circle criterion therefore implies that the closed-loop system with $u = -\phi(y)$ will be asymptotically stable if ϕ lies in the sector $[-\frac{1}{3}, \frac{1}{2}]$.

Clearly this is not the only sector bound for ϕ for which the closed-loop system is guaranteed to be stable by the circle criterion. In fact a family of discs $D(a, b)$ containing $G(j\omega)$ is generated as a is increased from $-1/3$, and to allow for the largest possible value of b we need to set $a = 0$ and $b = -1$, corresponding to sector bounds $\phi \in [0, 1]$.

- (b). Closed-loop stability does not apply to nonlinearities ϕ bounded by the union of the two sectors defined in part (a), i.e. $[-\frac{1}{3}, 1]$, since this includes nonlinearities not belonging to either of the sectors $[-\frac{1}{3}, \frac{1}{2}]$ and $[0, 1]$. In particular, the disc centred on the real axis and intersecting the real axis at -1 and 3 does not entirely contain the box in which $G(j\omega)$ is known to lie, so it cannot be concluded from the circle criterion that the closed loop system will be stable.

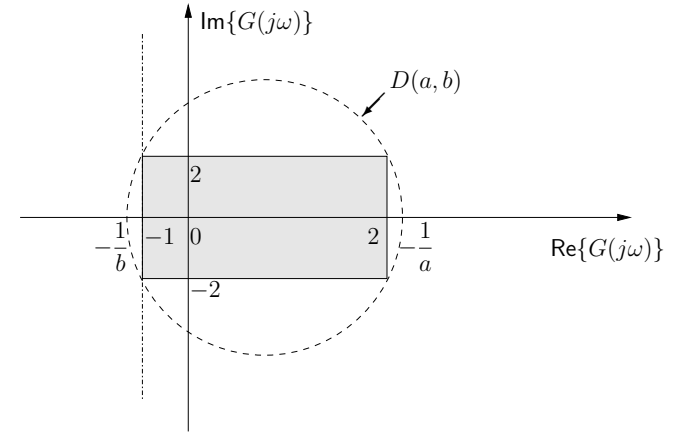


Figure 3: Bounds on the Nyquist plot of $G(j\omega)$.