15.094J: Robust Modeling, Optimization, Computation

Lecture 21: Robust Optimal Auctions

Outline

Mechanism Design

- Mechanism Design is an area in economics and game theory that has an engineering perspective.
- The goal is to design *economic mechanisms* or *incentives* to implement desired objectives (social or individual) in a strategic setting.
- Mechanism design has important applications in economics (e.g., design of voting procedures, markets, auctions), and more recently finds applications in E-commerce (ebay, ad-auctions).

Auction Theory

- Auction theory is part of Mechanism design theory.
- An auction is one of many ways that a seller can use to sell an object to potential buyers with unknown values.
- Participants: auctioneer, bidders.
- In an auction, the object is sold at a price determined by competition among buyers according to rules set by the seller (auction format), but the seller can use other methods.
- Auction Theory, extensive literature developed in Economics, and Computer Science (more recently).
- Two Nobel Prizes, Vickrey (1996) and Myerson (2007).

Example Auctions

- Open-outcry: ascending, descending
  - Ascending (English): Auctioneer announces ever increasing prices to solicit bids. Continues until only one person left in.
  - Descending (Dutch): Auctioneer announces decreasing prices until someone puts up their hand.
- Sealed-bid: Everyone puts bids in envelopes and gives to seller at the same time.
  - Two types: first-price, second-price
- Internet: EBay.com, Amazon.com, Liquidation.com
- Government: Treasury Bills, mineral rights (e.g. oil fields), assets (e.g. privatization), Electromagnetic spectrum
- Stock Market: IPOs, Opening Bell everyday
- Auctions are everywhere!

Optimal Auction Design

- Design an auction to maximize revenue of the auctioneer.
- Myerson [1981] characterized the optimal auction when
  - Buyers' valuations are sampled from independent probability distributions,
  - Buyers have no budget constraints.
- The optimal auction is a second price auction with a reserve:
  - Bidders submit their bids. If all the bids are less than the reserve, the auction is cancelled.
  - The highest bidder is allocated the item and is charged the second highest bid.

An example

- Inverted Jenny unique Plate Block sold for \$3 million in a NY 2005 auction.
- A reserve was placed.
- The highest bidder won, and paid the second highest price.
- As per Myerson (1981), Nobel prize in Economics 2007.
- But happens if many stamps (part of a collection) are being auctioned?

Myerson Auction

- The reservation price is calculated by solving a non-linear equation

$$\frac{1 - F(r)}{f(r)} = r,$$

  where $F(\cdot)$ is the cdf and $f(\cdot)$ is the pdf of the probability distribution.
- When distributions are not identical, then the reservation price varies with the bidder.
- Myerson auction not optimal for *correlated* valuations and when bidders have *budgets*.

Auctions in the real world

In the real world:

- Typical auctions involve multiple items.
- Bidders have budgets.
- The valuations are correlated.

In these situations, the overall problem is open.

- This is due to the multi-dimensional nature of the problem.
- Modeling using probability distributions leads to this analytical intractability.

Modeling uncertainty in valuations

- For each item $j \in \mathcal{M}$, we model the auctioneer's beliefs on valuations for item j using an uncertainty set $\mathcal{U}_j \in \mathbb{R}^n$.
- Example: Central Limit Theorem states that the normalized sum of random variables

$$\frac{S_n - n\mu}{\sigma \cdot \sqrt{n}}$$

is asymptotically standard normal.

$$\mathcal{U}_j^{\text{CLT}} = \left\{ (v_{1j}, \ldots, v_{nj}) \middle| -\Gamma \leq \frac{\sum_{i=1}^{n} v_{ij} - n \cdot \mu_j}{\sigma_j \cdot \sqrt{n}} \leq \Gamma. \right\}$$

Modeling uncertainty in valuations

- Factor model : $\{\tilde{z}_i\}_{i=1,\ldots,n}$ depend on m factors $\tilde{f} = \left(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m\right)$

  $$\tilde{z}_i = \mathbf{a}_i' \cdot \tilde{f} + \tilde{\epsilon}_i,$$

  $\{\tilde{\epsilon}_i\}$ are i.i.d.

- $$\mathcal{U}^{\text{Corr}} = \left\{ (z_1, \ldots, z_n) \;\middle|\; \begin{array}{ll} z_i = \displaystyle\sum_{j=1}^{m} a_{ij} f_j + \epsilon_i, & \forall i = 1, \ldots, n, \\ -\Gamma_f \leq \dfrac{\displaystyle\sum_{j=1}^{m} f_j - m \cdot \mu_f}{\sigma_f \cdot \sqrt{m}} \leq \Gamma_f, & \\ -\Gamma_\epsilon \leq \dfrac{\displaystyle\sum_{i=1}^{n} \epsilon_i - n \cdot \mu_\epsilon}{\sigma_\epsilon \cdot \sqrt{n}} \leq \Gamma_\epsilon. & \end{array} \right\}.$$

Main Problem

- n buyers, indexed by $i \in \mathcal{N}$, are interested in buying a set of m items, indexed by $j \in \mathcal{M}$ sold by an auctioneer.
- Buyer $i \in \mathcal{N}$ has a valuation v_{ij} for item $j \in \mathcal{M}$, which is not known to the auctioneer, and beliefs modeled by uncertainty sets $\mathcal{U}_j$.
- Buyers are budget constrained with budgets $\{B_1, B_2, \ldots, B_n\}$.

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Problem

Design an auction mechanism that is
(1) individually rational,
(2) budget feasible, and
(3) "worst case" optimal.

Optimization Problem Formulation

- $\mathbf{v} = (\mathbf{v}_j)_{j \in \mathcal{M}}$
- $\mathbf{v} \in \mathcal{U}$ refer to $\mathbf{v}_j \in \mathcal{U}_j$, $j \in \mathcal{M}$.
- $x_{ij}^{\mathbf{v}}$ is the fraction of item j allocated to buyer i
- $p_i^{\mathbf{v}}$ is the total payment charged to Buyer i, when the bid is $\mathbf{v}$.
- Properties :
  (a) *Individual Rationality (IR)* : Buyers do not derive negative utility by participating in the auction.
  (b) *Budget Feasibility (BF)* : Buyers are charged within their budget constraints.
  (c) *Incentive Compatibility (IC)* : The total utility of the i^{th} buyer under truthful bidding is greater or equal to the total utility that Buyer i derives by bidding any other other bid vector $\mathbf{u}_i$.

Optimization Problem Formulation

- Call the optimization problem OPT.

$$
\begin{aligned}
Z^* = \max \quad & W \\
\text{s.t.} \quad & W - \sum_{i \in \mathcal{N}} p_i^{\mathbf{v}} \leq 0, \ \ \forall \mathbf{v} \in \mathcal{U}, \\
& \sum_{i \in \mathcal{N}} x_{ij}^{\mathbf{v}} \leq 1, \ \ \forall j \in \mathcal{M}, \forall \mathbf{v} \in \mathcal{U}, \\
\text{(IC)} \quad & \sum_{j \in \mathcal{M}} v_{ij} \cdot x_{ij}^{(\mathbf{u}_i, \mathbf{v}_{-i})} - p_i^{(\mathbf{u}_i, \mathbf{v}_{-i})} - \sum_{j \in \mathcal{M}} v_{ij} \cdot x_{ij}^{(\mathbf{v}_i, \mathbf{v}_{-i})} \\
& \quad + p_i^{(\mathbf{v}_i, \mathbf{v}_{-i})} \leq 0, \ \ \forall (\mathbf{v}_i, \mathbf{v}_{-i}) \in \mathcal{U}, \forall (\mathbf{u}_i, \mathbf{v}_{-i}) \in \mathcal{U}, \forall i \in \mathcal{N}, \\
\text{(BF)} \quad & p_i^{\mathbf{v}} \leq B_i, \ \ \forall i \in \mathcal{N}, \forall \mathbf{v} \in \mathcal{U}, \\
\text{(IR)} \quad & p_i^{\mathbf{v}} \leq \sum_{j \in \mathcal{M}} v_{ij} \cdot x_{ij}^{\mathbf{v}}, \ \ \forall i \in \mathcal{N}, \forall \mathbf{v} \in \mathcal{U}, \\
& \mathbf{x}^{\mathbf{v}} \geq \mathbf{0}.
\end{aligned}
$$

- Z^* is the "worst case optimal" revenue that we intent to secure.

Robust Optimal Mechanism

- We characterize the mechanism that solves this optimization problem.
- Call it "Robust Optimal Mechanism (ROM)".
- Structure of ROM:
  - Compute Global Reserve R^*.
  - If the total bids result in realized revenue of less than or equal to R^*, then the auctioneer does not allocate the items.
  - Otherwise compute allocations and payments using a linear optimization problem.

Robust Optimal Mechanism

- *ROM* consists of Algorithms *ROM.a* and *ROM.b*.
- In *ROM.a*, which occurs prior to the realization of a specific bid vector $\mathbf{v}$, we compute the quantity R^*, which stands for the global reserve.
  - This involves a bilinear optimization problem.
- In *ROM.b*, when the bid vector $\mathbf{v}$ is realized, we calculate the allocation vector $\left\{a_{ij}^{\mathbf{v}}\right\}_{i\in\mathcal{N},j\in\mathcal{M}}$ and the payments $\{p_i^{\mathbf{v}}\}_{i\in\mathcal{N}}$ by solving linear optimization problems.

ROM.a

- **Input** : Uncertainty set $\mathcal{U}$, and budgets $B_1, \ldots, B_n$,
- **Output** : Global Reserve R^*.
- By solving the bilinear optimization problem, compute

$$\mathbf{R}^* = \min_{\mathbf{v}\in\mathcal{U}} \left\{ \begin{array}{cc} \max\limits_{\left(\{x_{ij}\}_{i\in\mathcal{N}, j\in\mathcal{M}}, \{r_i\}_{i\in\mathcal{N}}\right)} & \sum\limits_{i\in\mathcal{N}} r_i \\ \text{s.t.} & \sum\limits_{j\in\mathcal{M}} x_{ij}\cdot v_{ij} \leq B_i,\ \forall i\in\mathcal{N}, \\ & r_i \leq \sum\limits_{j\in\mathcal{M}} x_{ij}\cdot v_{ij},\ \forall i\in\mathcal{N}, \\ & \sum\limits_{i\in\mathcal{N}} x_{ij} \leq 1,\ \forall j\in\mathcal{M}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \right\}.$$

- Let $\mathbf{z}$ be the argmin of this bilinear optimization problem. Compute $\left(\left\{\xi_j^*\right\}_{j\in\mathcal{M}}, \left\{\eta_i^*\right\}_{i\in\mathcal{N}}, \left\{\theta_i^*\right\}_{i\in\mathcal{N}}\right)$ given by

$$\arg \left\{ \begin{array}{cc} \min\limits_{\{\xi_j,\eta_i,\theta_i\}} & \sum\limits_{j\in\mathcal{M}} \xi_j + \sum\limits_{i\in\mathcal{N}} \eta_i B_i \\ \text{s.t.} & \xi_j + z_{ij}\cdot\eta_i \geq z_{ij}\cdot\theta_i,\ \forall i\in\mathcal{N}, \forall j\in\mathcal{M}, \\ & \theta_i = 1,\ \forall i\in\mathcal{N}, \\ & \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\theta} \geq \mathbf{0}. \end{array} \right\}. \quad (1)$$

ROM.b

- **Input**: Bid vector $\mathbf{v} = \{v_{ij}\}_{i\in\mathcal{N},j\in\mathcal{M}}$, reserve R^*.
- **Output**: Allocation vector $\left\{a_{ij}^{\mathbf{v}}\right\}_{i\in\mathcal{N},j\in\mathcal{M}}$ and the payments $\{p_i^{\mathbf{v}}\}_{i\in\mathcal{N}}$.
- **Algorithm**:
  - If $\mathbf{v} \notin \mathcal{U}$, then do not allocate any item and charge zero, otherwise proceed to next step.
  - Solve the linear optimization problems:

$$\left(\left\{y_{ij}^{\mathbf{v}}\right\}_{i\in\mathcal{N},j\in\mathcal{M}}, \{r_i^{\mathbf{v}}\}_{i\in\mathcal{N}}\right) = \arg\max_{(\mathbf{y},\mathbf{r})\in\mathcal{P}^{\mathbf{v}}} \sum_{i\in\mathcal{N}}\left(\sum_{j\in\mathcal{M}} y_{ij}\cdot v_{ij} - r_i\right), \quad (2)$$

$$\left(\left\{y_{ij,k}^{\mathbf{v}_{-k}}\right\}_{i\in\mathcal{N},j\in\mathcal{M}}, \left\{r_{i,k}^{\mathbf{v}_{-k}}\right\}_{i\in\mathcal{N}}\right) = \arg\max_{(\mathbf{y},\mathbf{r})\in\mathcal{P}^{\mathbf{v}}} \sum_{i\in\mathcal{N}\setminus\{k\}}\left(\sum_{j\in\mathcal{M}} y_{ij}\cdot v_{ij} - r_i\right) \quad (3)$$

ROM.b contd...

- where

$$\mathcal{P}^{\mathbf{v}} = \left\{ \left(\{x_{ij}\}_{i\in\mathcal{N}, j\in\mathcal{M}}, \{r_i\}_{i\in\mathcal{N}} \right) \middle| \begin{array}{ll} \sum_{j\in\mathcal{M}} x_{ij} v_{ij} \leq B_i, & \forall i \in \mathcal{N}, \\ r_i \leq \sum_{j\in\mathcal{M}} x_{ij} v_{ij}, & \forall i \in \mathcal{N}, \\ \sum_{i\in\mathcal{N}} x_{ij} \leq 1, & \forall j \in \mathcal{M}, \\ \sum_{i\in\mathcal{N}} r_i \geq R^*, & \\ \mathbf{x} \geq \mathbf{0}. & \end{array} \right\}. \quad (4)$$

- Compute the allocation vector $\{a_k^{\mathbf{v}}\}_{k\in\mathcal{N}}$ and the payments $\{p_k^{\mathbf{v}}\}_{k\in\mathcal{N}}$ as follows

$$a_k^{\mathbf{v}} = y_k^{\mathbf{v}}, \quad (5)$$

$$p_k^{\mathbf{v}} = r_k^{\mathbf{v}} + \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}_{-k}} \cdot v_{ij} - r_{i,k}^{\mathbf{v}_{-k}} \right) - \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}} \cdot v_{ij} - r_i^{\mathbf{v}} \right). \quad (6)$$

Robust Optimal Mechanism

Theorem

ROM is the worst case optimal auction.

Proof.

Two steps:
(1) Show that the ROM.b leads to allocations and payments that lead to budget feasibility, individual rationality and incentive compatibility. That is, ***show that the allocations and payments are feasible to the primal optimization problem OPT.***
(2) Show that the revenue achieved is optimal by constructing a feasible solution to the dual of OPT that has the same revenue.
(3) By Strong Duality, the result follows. □

Proof (sample)

Budget Feasibility

- Suppose the buyers' response is to bid $\mathbf{v}^{\mathbf{bid}}$.
- The payment charged to each buyer $k \in \mathcal{N}$ is given by

$$
\begin{aligned}
p_k^{\mathbf{v}^{\mathbf{bid}}} &= r_k^{\mathbf{v}^{\mathbf{bid}}} + \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}_{-k}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - r_{i,k}^{\mathbf{v}_{-k}^{\mathbf{bid}}} \right) - \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - r_i^{\mathbf{v}^{\mathbf{bid}}} \right) \\
&= r_k^{\mathbf{v}^{\mathbf{bid}}} + \sum_{j\in\mathcal{M}} y_{kj}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - \sum_{j\in\mathcal{M}} y_{kj}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} + \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}_{-k}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - r_{i,k}^{\mathbf{v}_{-k}^{\mathbf{bid}}} \right) \\
&\quad - \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - r_i^{\mathbf{v}^{\mathbf{bid}}} \right) && (7) \\
&= \sum_{j\in\mathcal{M}} y_{kj}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{kj}^{\mathbf{bid}} + \sum_{i\in\mathcal{N}\setminus\{k\}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}_{-k}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - r_{i,k}^{\mathbf{v}_{-k}^{\mathbf{bid}}} \right) - \sum_{i\in\mathcal{N}} \left(\sum_{j\in\mathcal{M}} y_{ij}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{ij}^{\mathbf{bid}} - r_i^{\mathbf{v}^{\mathbf{bid}}} \right) && (8) \\
&\le \sum_{j\in\mathcal{M}} y_{kj}^{\mathbf{v}^{\mathbf{bid}}} \cdot v_{kj}^{\mathbf{bid}} && (9) \\
&\le B_i, && (10)
\end{aligned}
$$

where (10) follows from (9) because $\left(\left\{ y_{ij}^{\mathbf{v}^{\mathrm{bid}}} \right\}_{i\in\mathcal{N}, j\in\mathcal{M}} \right) \in \mathcal{P}^{\mathbf{v}^{\mathrm{bid}}}$.

Proof (sample)

Worst Case Revenue

- If buyers bid their true valuation, then their utility is non-negative. If, however, buyers bid $\mathbf{v}^{\text{bid}} \notin \mathcal{U}$, then by Step 1 of *ROM.b*, their utility is zero. Therefore, if the buyers bid $\mathbf{v}^{\text{bid}}$, then $\mathbf{v}^{\text{bid}} \in \mathcal{U}$.
- From Step 2 of *ROM.b*, the payments $\left\{ r_i^{\mathbf{v}^{\text{bid}}} \right\}_{i \in \mathcal{N}}$ are feasible to (2) with $\mathbf{v} = \mathbf{v}^{\text{bid}}$. Since $\mathbf{v}^{\text{bid}} \in \mathcal{U}$, then from Step 1 of *ROM.a*,

$$\sum_{i=1}^{n} r_i^{\mathbf{v}^{\text{bid}}} \geq R^*. \tag{11}$$

Furthermore, we have

$$\begin{aligned} p_k^{\mathbf{v}^{\text{bid}}} &= r_k^{\mathbf{v}^{\text{bid}}} + \sum_{i \in \mathcal{N} \setminus \{k\}} \left(\sum_{j \in \mathcal{M}} y_{ij}^{\mathbf{v}_{-k}^{\text{bid}}} \cdot v_{ij}^{\text{bid}} - r_{i,k}^{\mathbf{v}_{-k}^{\text{bid}}} \right) - \sum_{i \in \mathcal{N} \setminus \{k\}} \left(\sum_{j \in \mathcal{M}} y_{ij}^{\mathbf{v}^{\text{bid}}} \cdot v_{ij}^{\text{bid}} - r_i^{\mathbf{v}^{\text{bid}}} \right) \\ &\geq r_k^{\mathbf{v}^{\text{bid}}}, \end{aligned}$$

- This implies that

$$\sum_{i=1}^{n} p_i^{\mathbf{v}^{\text{bid}}} \geq \sum_{i=1}^{n} r_i^{\mathbf{v}^{\text{bid}}} \geq R^*,$$

implying that the worst case revenue is at least R^*.

Proof (sample)

ROM achieves at least Z^*

- Consider the following relaxation of OPT, in which we eliminate the IC constraints:

$$
\begin{aligned}
Z_{\mathbf{1}}^* = \max \quad & W && (12)\\
\text{s.t.} \quad & W - \sum_{i \in \mathcal{N}} p_i^{\mathbf{v}} \le 0, \quad \forall \mathbf{v} \in \mathcal{U}, \\
& \sum_{i \in \mathcal{N}} x_{ij}^{\mathbf{v}} \le 1, \quad \forall j \in \mathcal{M}, \forall \mathbf{v} \in \mathcal{U}, \\
& p_i^{\mathbf{v}} \le B_i, \quad \forall i \in \mathcal{N}, \forall \mathbf{v} \in \mathcal{U}, && (13)\\
& p_i^{\mathbf{v}} \le \sum_{j \in \mathcal{M}} v_{ij} \cdot x_{ij}^{\mathbf{v}}, \quad \forall i \in \mathcal{N}, \forall \mathbf{v} \in \mathcal{U}, && (14)\\
& \mathbf{x}^{\mathbf{v}} \ge \mathbf{0}.
\end{aligned}
$$

The dual of (12) is as follows:

$$
\begin{aligned}
\min \quad & \sum_{\mathbf{v} \in \mathcal{U}} \left(\sum_{j=\mathbf{1}}^{m} \xi_{j,\mathbf{v}} + \sum_{i=\mathbf{1}}^{n} \eta_{i,\mathbf{v}} B_i \right) && (15)\\
\text{s.t.} \quad & \xi_{j,(\mathbf{v}_i,\mathbf{v}_{-i})} - v_{ij} \cdot \theta_{i,(\mathbf{v}_i,\mathbf{v}_{-i})} \ge 0, \quad \forall (\mathbf{v}_i, \mathbf{v}_{-i}) \in \mathcal{U}, \\
& \eta_{i,(\mathbf{v}_i,\mathbf{v}_{-i})} + \theta_{i,(\mathbf{v}_i,\mathbf{v}_{-i})} - \omega_{(\mathbf{v}_i,\mathbf{v}_{-i})} = 0, \quad \forall (\mathbf{v}_i, \mathbf{v}_{-i}) \in \mathcal{U}, \\
& \sum_{\mathbf{v} \in \mathcal{U}} \omega_{\mathbf{v}} = 1, \\
& \omega_{\mathbf{v}} \ge 0, \boldsymbol{\xi}_{\mathbf{v}} \ge \mathbf{0}, \boldsymbol{\eta}_{\mathbf{v}} \ge \mathbf{0}, \boldsymbol{\theta}_{\mathbf{v}} \ge \mathbf{0}.
\end{aligned}
$$

Proof (sample)

ROM achieves at least Z^*

- Since Problem (12) is obtained from OPT by eliminating the IC constraints, we have

$$Z_1^* \geq Z^*. \tag{16}$$

- Let $\mathbf{z}$ be an optimal solution in Step 1 of *ROM.a*. We next construct a feasible solution to the dual problem (15) with objective function equal to R^*. Let

$$\omega_{\mathbf{v}} = \begin{cases} 1, & \text{if } \mathbf{v} = \mathbf{z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_{i,\mathbf{v}} = \begin{cases} \eta_i^*, & \text{if } \mathbf{v} = \mathbf{z}, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in \mathcal{N},$$

$$\xi_{j,\mathbf{v}} = \begin{cases} \xi_j^*, & \text{if } \mathbf{v} = \mathbf{z}, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall j \in \mathcal{M},$$

$$\theta_{i,\mathbf{v}} = \begin{cases} 1 - \eta_i^*, & \text{if } \mathbf{v} = \mathbf{z}, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in \mathcal{N},$$

where $\left(\left\{\xi_j^*\right\}_{j\in\mathcal{M}}, \left\{\eta_i^*\right\}_{i\in\mathcal{N}}, \left\{\theta_i^*\right\}_{i\in\mathcal{N}}\right)$ were computed in **ROM.a**.

Proof (sample)

ROM achieves at least Z^*

- It is easy to verify that this is a dual feasible solution to Problem (15), with objective value given by

$$\sum_{j\in\mathcal{M}} \xi_j^* + \sum_{i\in\mathcal{N}} \eta_i^* B_i,$$

  which is equal to R^*.
- This leads to

$$R^* \geq Z_1^* \geq Z^*.$$

- This concludes the proof.

Summary

- Characterized the worst case optimal auction with budgets, for **any uncertainty set** $\mathcal{U}$.
- Can choose the uncertainty sets carefully, to include distributional information.
  - Capture arbitrary "risk measures" of the auctioneer.
- "Global reserve" structure allows auctioneer to ensure selling even the "bad" items.
- For the case of no budgets, the structure is the same as Myerson auction.
- *ROM* extends to uncertain budgets and indivisible items.

Single Item Auction without Budgets

- *ROM* is a second price auction with reserve price R^*.
- R^* is calculated by a linear optimization problem:

$$\begin{aligned} \min_{r,\mathbf{v}} \quad & r \\ \text{s.t.} \quad & r \geq v_i, \ \forall i \in \mathcal{N}, \\ & (v_1, v_2, \ldots, v_n) \in \mathcal{U}. \end{aligned}$$

Comparision with Myerson Auction

Computational Complexity

- *ROM* and the Myerson auction have the same structure, that of a second price auction with a reservation price.
- In Myerson auction, the reservation price is calculated by solving a non-linear equation

$$\frac{1 - F(r)}{f(r)} = r,$$

  where $F(\cdot)$ is the cdf and $f(\cdot)$ is the pdf of the probability distribution.
- In *ROM*, the reservation price is calculated using a linear optimization problem.

Comparision with Myerson Auction

Robustness to Mis-specification

$$\text{Relative Revenue} = \frac{\text{ROM Revenue} - \text{Myerson Revenue}}{\text{Myerson Revenue}}$$

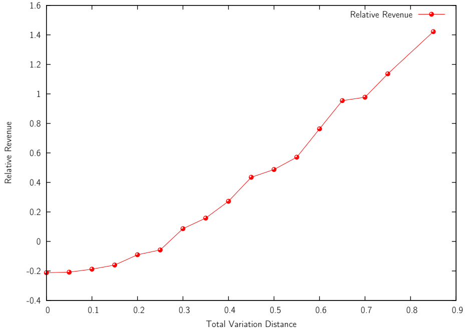

Figure : Robustness of *ROM-Si*.

Comparision with Myerson Auction

Capturing Correlation Information

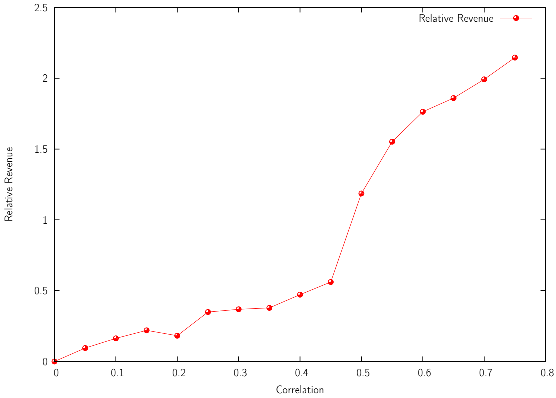

Figure : Effect of Correlations on the Revenue.

Conclusion

- Characterized the "worst case" optimal auction
- with budgeted bidders
  - for any uncertainty set $\mathcal{U}$
- Auction can be interpreted as a VCG auction with a "global reserve".
- Recovered the structure of Myerson Auction for bidders without budgets.
- Has benefits of robustness to mis-specification.
- Extensions: Can naturally model the risk attitudes of the auctioneer by modifying the uncertainty set.