Course notes for EE394V Restructured Electricity Markets: Locational Marginal Pricing

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Power flow

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3.1 Review of power concepts

- **Power** is the rate of doing work, measured in W, kW, MW, or GW.
- **Energy** is the work accomplished over time, measured in Wh, kWh, MWh, or GWh.
- When power varies over time, the energy is the integral over time of the power.

Review of power concepts, continued

- In principle, power could be generated, transmitted, and consumed using either **direct current** (DC) or **alternating current** (AC).
- Cost-effective and low loss transmission of bulk power relies on being able to create high voltages:
  - power capability is proportional to the product of current and voltage, so higher voltages allow for higher power levels,
  - for a given power, higher voltage means lower current, which implies lower losses for a given resistance of conductor.
- Generation and consumption is more convenient at lower voltages:
  - we will mostly focus on generation and transmission, modeling consumption through aggregated loads at distribution substations,
  - probability of generation, transmission, and distribution failures, effects due to presence of other loads, and other issues affect the **quality of supply** to end users,
  - local effects on distribution system typically affect quality of supply more noticeably than generation and transmission failures.

Review of power concepts, continued

- Until the advent of power electronics, only AC power could easily be transformed from one voltage to another:
  - basic reason for ubiquity of AC power systems.
- AC transmission of power also involves the back-and-forth flow of power between electric and magnetic fields:
  - this back-and-forth flow is called **reactive power**,
  - to distinguish reactive power from the power that can actually be consumed by a load, the latter is called **real power**.

Review of power concepts, continued

- The relationship between voltage and current in a circuit is determined by the characteristics of the circuit elements and **Kirchhoff's laws**.
- Kirchhoff's current law:
  - due to conservation of charge passing a bus or node of circuit,
  - implies that supply of *electric* power always equals demand of electric power plus losses,
  - mis-match between mechanical and electrical power is smoothed by inertia of system and results in frequency change,
  - enforcing supply–demand balance between mechanical power and electrical power is different to enforcing supply–demand balance in typical markets, such as a market for apartments to be described in Section 6.
- Kirchhoff's voltage law:
  - sum of voltages around loop is zero,
  - electric transmission network behaves differently to most other transportation networks.

Review of power concepts, continued

- Kirchhoff's laws implicitly determine the voltages and currents due to the real and reactive power injections at the generators and the withdrawals at the loads.
- The problem of using Kirchhoff's laws to solve for the voltages and currents in a circuit consisting of generators, the transmission and/or distribution system, and loads is called the power flow problem:
  - assumes a particular operating condition,
  - **quasi-static** assumption that ignores dynamics and changes.
- The solution provides information about the flow of current and power on the transmission and distribution lines.
- The lines have limited capacities, so calculation of power flow enables us to decide whether or not a particular pattern of generation would result in acceptable flows on lines:
  - constraints on transmission operation implicitly determine limitations on the patterns of injections and withdrawals,
  - the locational marginal pricing market reflects these limitations into prices that vary by bus (or node).

3.2 Formulation of power flow

3.2.1 *Variables*

3.2.1.1 *Phasors*

- We can use complex numbers, called **phasors**, to represent the magnitude and angle of the AC voltages and currents at a fixed frequency.
- The **magnitude** of the complex number represents the root-mean-square magnitude of the voltage or current.
- The **angle** of the complex number represents the angular displacement between the sinusoidal voltage or current and a reference sinusoid.

3.2.1.2 *Reference angle*

- The angles of the voltages and currents in the system would all change if we changed the angle of our reference sinusoid, but this would have no effect on the physical system.
- We can therefore arbitrarily assign the angle at one of the buses to be zero and measure all the other angles with respect to this angle.
- We call this bus the **reference bus** or the **angle reference bus**, bus ρ, and typically number the buses so that $\rho = 0$ or $\rho = 1$.

3.2.1.3 Representation of complex numbers

- To represent a complex number V with real numbers requires two real numbers, either:
  - the **magnitude** $|V|$ and the **angle** $\angle V$, so that $V = |V| \exp(\angle V \sqrt{-1})$, or
  - the **real** $\Re\{V\}$ and **imaginary** $\Im\{V\}$ parts, so that $V = \Re\{V\} + \Im\{V\}\sqrt{-1}$.
- Since we need to compare voltage magnitudes to limits to check satisfaction of voltage limit constraints, we will represent voltages as magnitudes and angles:
  - Some recent developments in power flow have used the real and imaginary parts representation.

3.2.1.4 Scaling and "per unit"

- There are voltage transformers throughout a typical power system:
  - "step-up" voltage at a generator to transmission voltages to enable transfer from generator to transmission system,
  - transform from one transmission voltage to another,
  - "step-down" voltage at a distribution substation and in distribution feeder for convenient use by load.
- The nominal voltage magnitude varies considerably across the system by several orders of magnitude.
- We scale the voltage magnitude so that an actual value of 121 kV in the 110 kV part of the system would be represented by a scaled value of:

$$\frac{121 \text{ kV}}{110 \text{ kV}} = 1.1,$$

- while an actual value of 688.5 kV in the 765 kV part of the system would be represented by a scaled value of:

$$\frac{688.5 \text{ kV}}{765 \text{ kV}} = 0.9.$$

3.2.2 Symmetry

3.2.2.1 Three-phase circuits

- Generation-transmission systems are usually operated as balanced **three-phase systems**, with generators, lines, and (roughly) distribution system loads arranged as symmetric triplets.

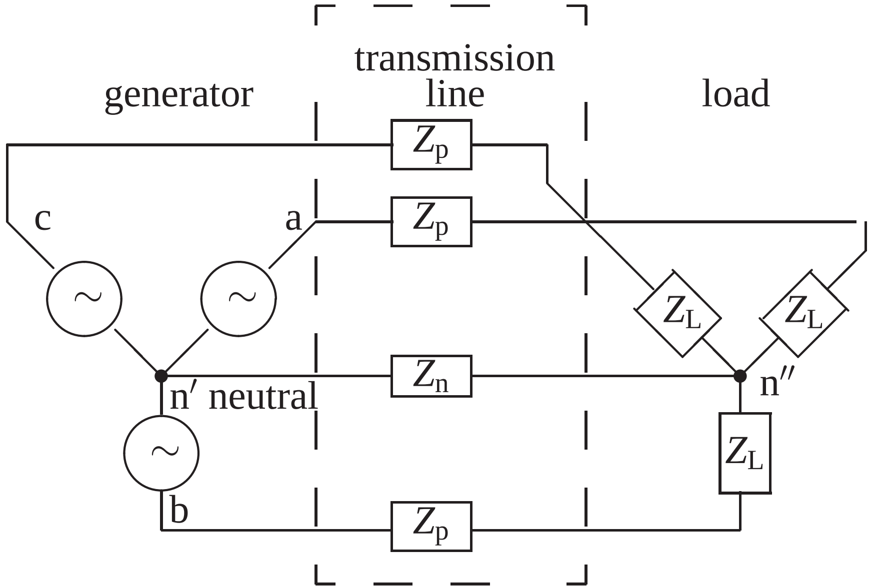

Fig. 3.1. An example balanced three-phase system.

3.2.2.2 Per-phase equivalent

- The behavior of a balanced three-phase circuit can be completely determined from the behavior of a **per-phase equivalent circuit**.
- Figure 3.2 shows the a-phase equivalent circuit of the three-phase circuit of Figure 3.1.

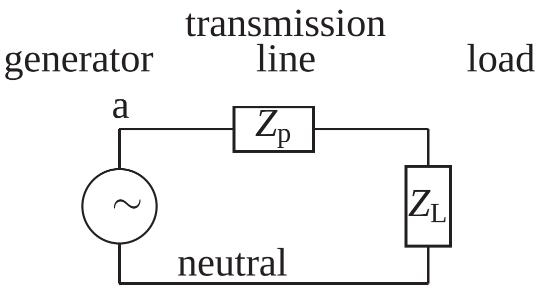

Fig. 3.2. Per-phase equivalent circuit for the three-phase circuit in Figure 3.1.

3.2.3 *Transmission lines*

- Transmission lines are physically extended objects, so the boxes in Figures 3.1 and 3.2 are actually **distributed parameter circuits**,
- We can represent the terminal behavior of such distributed parameter circuits with a **π-equivalent** circuit.
- Each component of the π-equivalent has an impedance (or, equivalently, an **admittance**) determined by the characteristics of the line.

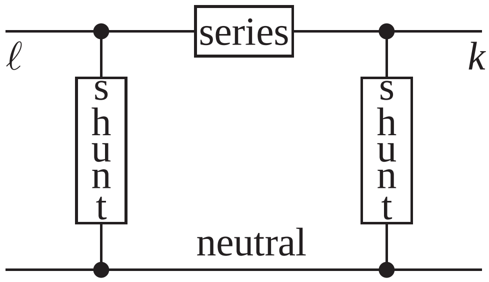

Fig. 3.3. Equivalent π circuit of per-phase equivalent of transmission line.

3.2.4 Bus admittance matrix and power flow equations

- Consider the per-phase equivalent of a three bus, three line transmission system as illustrated in Figure 3.4.
- For each bus $\ell = 1, 2, 3$, the pair of shunt π elements joining node ℓ to neutral can be combined together to form a single shunt element.

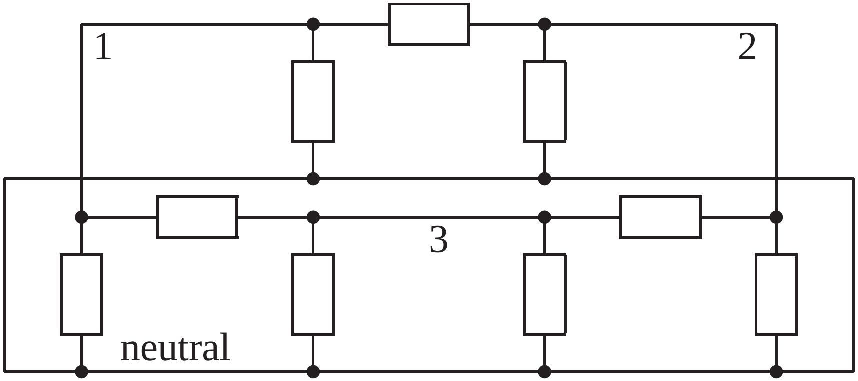

Fig. 3.4. Per-phase equivalent circuit model for three bus, three line system.

Bus admittance matrix and power flow equations, continued

- This yields a circuit with:
  - one element corresponding to each of the buses $\ell = 1, 2, 3$, joining node ℓ to neutral, and
  - one element corresponding to each line,
- as illustrated in Figure 3.5.

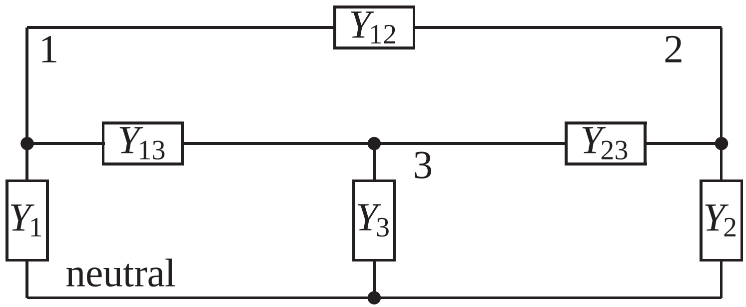

Fig. 3.5. Per-phase equivalent circuit model for three bus, three line system with parallel components combined.

Bus admittance matrix and power flow equations, continued

- Let us write Y_ℓ for the admittance of the element joining node ℓ to neutral, and
- $Y_{\ell k}$ for the admittance of the series element corresponding to a line joining buses ℓ and k.
- The series element is most easily characterized in terms of its impedance.
- For a series impedance $Z_{\ell k} = R_{\ell k} + X_{\ell k}\sqrt{-1}$ between buses ℓ and k, the corresponding admittance $Y_{\ell k}$ is given by:

$$\begin{aligned} Y_{\ell k} &= \frac{1}{Z_{\ell k}}, \\ &= \frac{1}{R_{\ell k} + X_{\ell k}\sqrt{-1}} \\ &= \frac{1}{R_{\ell k} + X_{\ell k}\sqrt{-1}} \times \frac{R_{\ell k} - X_{\ell k}\sqrt{-1}}{R_{\ell k} - X_{\ell k}\sqrt{-1}} \\ &= \frac{R_{\ell k} - X_{\ell k}\sqrt{-1}}{(R_{\ell k})^2 + (X_{\ell k})^2}. \end{aligned} \tag{3.1}$$

- If $Z_{\ell k} = 0.1 + \sqrt{-1}$, what is $Y_{\ell k} = 1/Z_{\ell k}$?

Bus admittance matrix and power flow equations, continued

- Let V be the vector of phasor voltages at all the buses in the system and let I be the vector of phasor current injections into the transmission network at all of the buses in the system.
- Using Kirchhoff's laws, we can obtain a relationship of the form $AV = I$ between current and voltage, where:

$$\forall \ell, k, A_{\ell k} = \begin{cases} Y_\ell + \sum_{k' \in \mathbb{J}(\ell)} Y_{\ell k'}, & \text{if } \ell = k, \\ -Y_{\ell k}, & \text{if } k \in \mathbb{J}(\ell) \text{ or } \ell \in \mathbb{J}(k), \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

- where $\mathbb{J}(\ell)$ is the set of buses joined directly by a transmission line to bus ℓ.
- The linear simultaneous equations $AV = I$ represent conservation of current at each of the buses.
- A is called the **bus admittance matrix**:
  - the ℓ-th diagonal entry is the sum of the admittances connected to bus ℓ,
  - the ℓk-th off-diagonal entry is minus the admittance connecting bus ℓ and k.

3.2.5 Generators and loads

- When electricity is bought and sold, the (real) power and energy are the quantities that are usually priced, not the voltage or current.
- However, real power does not completely describe the interaction between generators or loads and the system.
- We also have to characterize the injected reactive power.
- We can combine the real and reactive powers into the **complex power**, which is the sum of:

  the real power, and
  $\sqrt{-1}$ times the reactive power.

Generators and loads, continued

- The usefulness of this representation is that, for example, the net complex power S_ℓ injected at node ℓ into the network is given by:

$$S_\ell = V_\ell I_\ell^*,$$

- where the superscript $*$ indicates **complex conjugate**:
  - note difference between complex conjugate, denoted superscript $*$, and optimal or desired value, denoted superscript $\star$.
- The current I_ℓ equals the sum of:

  the current flowing into the shunt element Y_ℓ, and
  the sum of the currents flowing into each line connecting ℓ to a bus $k \in \mathbb{J}(\ell)$ through admittance $Y_{\ell k}$.
- We can substitute for the currents to obtain:

$$\begin{aligned} S_\ell &= V_\ell [A_{\ell\ell} V_\ell + \sum_{k \in \mathbb{J}(\ell)} A_{\ell k} V_k]^*, \\ &= |V_\ell|^2 A_{\ell\ell}^* + \sum_{k \in \mathbb{J}(\ell)} A_{\ell k}^* V_\ell V_k^*. \end{aligned} \tag{3.3}$$

Generators and loads, continued

- Let $A_{\ell k} = G_{\ell k} + B_{\ell k}\sqrt{-1}, \forall \ell, k$, where we note that by (3.1) and (3.2):
  - we have that $G_{\ell k} < 0$ and $B_{\ell k} > 0$ for $\ell \neq k$, and
  - we have that $G_{\ell\ell} > 0$ and the sign of $B_{\ell\ell}$ is indeterminate but typically less than zero;
- let $S_\ell = P_\ell + Q_\ell\sqrt{-1}, \forall \ell$, with:
  - for generator buses, $P_\ell > 0$ and Q_ℓ is typically positive,
  - for load buses, $P_\ell < 0$ and $Q_\ell < 0$;
- and let $V_\ell = u_\ell \exp(\theta_\ell\sqrt{-1}), \forall \ell$, with:
  - the voltage magnitude $u_\ell \approx 1$ in scaled units to satisfy voltage limits,
  - the voltage angle θ_ℓ typically between $-\pi/4$ and $\pi/4$ radians.
- Sometimes we will explicitly distinguish the real power injected by a generator from the real power consumed by a load, by writing D_ℓ for the real power load at bus ℓ:
  - the net real power injection at a bus with generation P_ℓ and load D_ℓ is then $P_\ell - D_\ell$, with both P_ℓ and D_ℓ typically positive.
  - Similarly, we will write E_ℓ for the reactive power load at bus ℓ, so that the net reactive power injection is $Q_\ell - E_\ell$.

Generators and loads, continued

- For notational convenience in the following development, we will write P_ℓ and Q_ℓ for the net real and reactive injections:
  - later cases where we explicitly distinguish generation from load will be clear from context.
- We can separate (3.3) into real and imaginary parts:

$$P_\ell = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], \tag{3.4}$$

$$Q_\ell = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)]. \tag{3.5}$$

- The equations (3.4) and (3.5), which are called the **power flow equality constraints**, must be satisfied at each bus ℓ.
- That is, there are two constraints that must be satisfied at each bus.
- For a 5000 bus system, how many power flow equality constraints must be satisfied?

3.2.6 The power flow problem

- The power flow problem is to find values of voltage angles and magnitudes that satisfy the power flow equality constraints.

3.2.6.1 Real and reactive power balance

- For convenience, we will say *PQ* **bus** for a bus where the real and reactive power injection is specified.
- We specify:
  - the real and reactive generations at the *PQ* generator buses according to the generator control settings, and
  - the (typically negative) real and reactive net power injections at the *PQ* load buses according to supplied data.
- At each such bus, we have two specified parameters (the real and reactive power injection) and two unknowns that are entries in the decision vector, the voltage magnitude and angle.
- However, we cannot arbitrarily specify the real and reactive power at all the buses since this would typically violate the first law of thermodynamics!
  - Not all the buses can be *PQ* buses.

3.2.6.2 Slack bus

- A traditional, but *ad hoc* approach to finding a solution to the equations is to single out a **slack bus**, bus σ.
- At this slack bus, instead of specifying injected real and reactive power, there is assumed to be a generator that produces whatever is needed to "balance" the real and reactive power for the rest of the system, assuming that such a solution exists.
- Typically, the slack bus is the same as the reference bus, but this is not necessarily the case, and the slack can even be (conceptually) "distributed" across multiple buses.
- For reasons that will become clear in the context of locational marginal pricing, we also call the slack bus the **price reference bus**.

Slack bus, continued

- We re-interpret P_σ and Q_σ to be decision variables in our power flow formulation and calculate them to satisfy the real and reactive power balance in the system:
  - for reasons that will become clear in the next section, we will not have to represent P_σ and Q_σ explicitly in the decision vector x,
  - in Section 5 in the context of **economic dispatch** where we are considering the choice of generation at all the buses, we will also consider the other real and reactive generations to be decision variables and so P_σ and Q_σ together with all the other real and reactive generations will be explicitly in the decision vector x.

Slack bus, continued

- The generator at the slack bus supplies whatever power is necessary for real and reactive power balance.
- To keep the number of unknowns equal to the number of equations, the voltage magnitude at the slack bus is specified as any particular value:
  - in Section 9 in the context of **optimal power flow** we will re-interpret the voltage magnitude at the slack bus to also be part of the decision vector.
- If the reference bus and the slack bus are the same bus, then we can call it a $V\theta$ bus, since both the voltage magnitude and angle are specified.
- At the $V\theta$ bus, we still have two specified parameters (the voltage magnitude and angle) and two unknowns (the real and reactive power injections).
- For most of the rest of the development of power flow, we will typically assume that the reference bus and the slack bus are the same and typically number the buses so that the reference/slack bus is bus 1:
  - we will sketch how to consider the case where the reference and slack buses are different.

3.2.7 Non-linear equations

- We have n_{PQ} PQ buses, including both the PQ generators and the loads.
- Let $n = 2n_{PQ}$ and define a decision vector $x \in \mathbb{R}^n$ consisting of the voltage magnitudes and angles at the PQ buses:
  - unknown real P_σ and reactive Q_σ generation at the slack bus will be evaluated in terms of x and so not represented explicitly in the decision vector.
- For every bus ℓ (that is, including the slack bus as well as the PQ buses) define functions $p_\ell : \mathbb{R}^n \to \mathbb{R}$ and $q_\ell : \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall x \in \mathbb{R}^n, p_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], \quad (3.6)$$

$$\forall x \in \mathbb{R}^n, q_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)]. \quad (3.7)$$

- The functions p_ℓ and q_ℓ represent the real and reactive power flow, respectively, from bus ℓ into the lines in the rest of the system.
- Kirchhoff's laws require that the net real and reactive flow out of a bus must be zero, so that $p_\ell(x) - P_\ell = 0$ and $q_\ell(x) - Q_\ell = 0$ at every bus ℓ.

Non-linear equations, continued

- For convenience, let $\rho = \sigma = 1$ and define a vector function $g : \mathbb{R}^n \to \mathbb{R}^n$ with entries given by $p_\ell - P_\ell$ and $q_\ell - Q_\ell$ for all the PQ buses:

$$\forall x \in \mathbb{R}^n, g(x) = \begin{bmatrix} p_2(x) - P_2 \\ p_3(x) - P_3 \\ \vdots \\ q_2(x) - Q_2 \\ q_3(x) - Q_3 \\ \vdots \end{bmatrix}.$$

- If we solve the non-linear simultaneous equations:

$$g(x) = \mathbf{0}, \tag{3.8}$$

- for $x^\star$ and then set:

$$\begin{aligned} P_1 &= p_1(x^\star), \\ Q_1 &= q_1(x^\star), \end{aligned}$$

- then we will have satisfied the power flow equality constraints at all buses including the slack bus.

Non-linear equations, continued

- We have formulated the power flow problem as the solution of non-linear simultaneous equations:

$$g(x) = \mathbf{0}.$$

- The vector g includes real and reactive power entries for each bus except the reference/slack bus:
  - we will calculate the real and reactive power injection at the slack bus after we have solved $g(x) = \mathbf{0}$.
- The vector x includes voltage angles and magnitudes for each bus except the reference/slack bus:
  - the voltage angle and magnitude for the reference/slack bus are specified.
- Recall that as we develop other problems, we will re-define x and g as needed for the formulation.

Non-linear equations, continued

- For future notational convenience:
  - Let θ, u, p, q, P, and Q be vectors consisting, respectively, of the entries θ_ℓ, u_ℓ, p_ℓ, q_ℓ, P_ℓ, and Q_ℓ for *all* the buses.
- We will often need to refer to a sub-vector with a particular entry omitted:
  - let subscript $-k$ on a vector denote that vector with the entry k omitted,
  - so θ_{-k}, u_{-k}, p_{-k}, q_{-k}, P_{-k}, and Q_{-k} are, respectively, the sub-vectors of θ, u, p, q, P, and Q with the entries θ_k, u_k, p_k, q_k, P_k, and Q_k omitted.
- We will maintain these definitions of θ, u, p, q, P, and Q throughout the course:
  - recall that we will change the definition of x and g depending on the particular problem being formulated.
- Also, let subscript $-k$ on a matrix denote that matrix with row k omitted:
  - so A_{-k} is the admittance matrix with the k-th row omitted.

Non-linear equations, continued

- With $\rho = \sigma = 1$ the reference/slack bus then, $x = \begin{bmatrix} \theta_{-1} \\ u_{-1} \end{bmatrix}$ and $g = \begin{bmatrix} (p_{-1}) - P_{-1} \\ (q_{-1}) - Q_{-1} \end{bmatrix}$.
- With this notation, $g(x) = \mathbf{0}$ can also be expressed in the equivalent form:

$$p_{-1}(x) = p_{-1}\left(\begin{bmatrix} \theta_{-1} \\ u_{-1} \end{bmatrix}\right) \; = \; P_{-1},$$
$$q_{-1}(x) = q_{-1}\left(\begin{bmatrix} \theta_{-1} \\ u_{-1} \end{bmatrix}\right) \; = \; Q_{-1}.$$

- For a 5000 bus system, how many entries are in θ, u, p and q? How about in θ_{-1}, u_{-1}, p_{-1}, q_{-1}, x, and g?

Non-linear equations, continued

- In summary, to solve Kirchhoff's equations for the electric power network, we:
  (i) solve (3.8), $g(x) = \mathbf{0}$, which is a system of non-linear simultaneous equations, and
  (ii) substitute the solution $x^\star$ into (3.4) and (3.5) for the slack bus $\ell = \sigma$ to find the real and reactive power generated at the slack bus.
- The real power generation at the slack bus is $P_\sigma = p_\sigma(x^\star)$, so $x^\star$ also satisfies $p(x^\star) = P$ and, moreover:

$$\mathbf{1}^\dagger P = \mathbf{1}^\dagger p(x^\star).$$

- This expression evaluates the total losses in the system, since it sums the total net real power injected into the transmission lines.
- Line currents and real and reactive power flows can also be calculated once x is known.

3.2.8 Example

- For example, for a three bus system with buses $\ell = 1,2,3$ and bus 1 the reference/slack bus, the entries of $x \in \mathbb{R}^4$ and $g : \mathbb{R}^4 \to \mathbb{R}^4$ would be:

$$x = \begin{bmatrix} \theta_{-1} \\ u_{-1} \end{bmatrix} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ u_2 \\ u_3 \end{bmatrix},$$

$$\forall x \in \mathbb{R}^4, g(x) = \begin{bmatrix} p_{-1}(x) - P_{-1} \\ q_{-1}(x) - Q_{-1} \end{bmatrix} = \begin{bmatrix} p_2(x) - P_2 \\ p_3(x) - P_3 \\ q_2(x) - Q_2 \\ q_3(x) - Q_3 \end{bmatrix},$$

- where P_2 is the net generation (generation minus demand) at bus 2, and similarly for other buses and for the reactive power at the buses.

Example, continued

- If we solve $g(x) = \mathbf{0}$, we can then use the resulting solution $x^\star$ to evaluate the real power and reactive power that must be produced at the reference/slack bus to satisfy real and reactive power balance at every bus:

$$\begin{aligned} P_1 &= p_1(x^\star), \\ Q_1 &= q_1(x^\star). \end{aligned}$$

- Losses in the system are given by:

$$\mathbf{1}^\dagger p(x^\star) = P_1 + P_2 + P_3.$$

3.3 Problem characteristics and solution

3.3.1 Number of variables and equations

- There are the same number of variables as equations in (3.8).
- For a 5000 bus system, with one $V\theta$ bus and the rest PQ buses, how many variables and equations are there?

3.3.2 Non-existence of direct algorithms

- Because the equations are non-linear, there is no direct algorithm, such as factorization, to solve for the solution $x^\star$ for arbitrary systems.
- The Newton–Raphson algorithm from Section 2.3 can be applied to this problem, requiring:
  - an initial guess $x^{(0)}$,
  - evaluation of partial derivative terms in the Jacobian, $\frac{\partial g}{\partial x}$, and
  - solution of the Newton–Raphson update (2.9)–(2.10) at each iteration.

3.3.3 *Number of solutions*

- There may be no solutions, one solution, or even multiple solutions to (3.8).
- However, power systems are usually designed and operated so that the voltage magnitudes are near to nominal and the voltage angles are relatively close to 0°.
- If we restrict our attention to solutions such that voltage magnitudes are all close to 1 (and make some other assumptions) then we can find conditions for the there to be at most one solution.
- How many solutions are there to $2+\sin(\theta)=0$?
- How many solutions are there to $0.1+\sin(\theta)=0$?
- How many solutions are there to $0.1+\sin(\theta)=0$ with $-\pi/4\leq\theta\leq\pi/4$?

3.3.4 Admittance matrix

3.3.4.1 Symmetry

- The admittance matrix A is symmetric.

3.3.4.2 Sparsity

- The matrix A is only sparsely populated with non-zero entries and each component of g depends on only a few components of x.
- Sparsity is the key to practical solution of problems with large numbers of buses.
- For a 5000 bus system having 5000 lines, how many non-zero entries are there in the admittance matrix A?

3.3.4.3 Values

- A typical line impedance has positive real and imaginary parts.
- The corresponding line admittance $Y_{\ell k}$ therefore has positive real part and negative imaginary part.
- If there is a line between buses ℓ and k then the entries $A_{\ell k} = G_{\ell k} + \sqrt{-1}B_{\ell k}$ in the admittance matrix satisfy $G_{\ell k} < 0, B_{\ell k} > 0$.
- The diagonal entries $A_{\ell\ell} = G_{\ell\ell} + \sqrt{-1}B_{\ell\ell}$ in the admittance satisfy $G_{\ell\ell} > 0$ and, typically, $B_{\ell\ell} < 0$.
- The resistance $R_{\ell k}$ of transmission lines is relatively small compared to the inductive reactance $X_{\ell k}$.
- Furthermore, the shunt elements are often also negligible compared to the inductive reactance.
- This means that:

$$\forall \ell, \forall k \in \mathbb{J}(\ell) \cup \{\ell\}, |G_{\ell k}| \ll |B_{\ell k}|.$$

3.4 Linearized power flow

3.4.1 Base-case

- Suppose that we are given values of real and reactive generation $P^{(0)} \in \mathbb{R}^{n_{PQ}+1}$ and $Q^{(0)} \in \mathbb{R}^{n_{PQ}+1}$ that specify a **base-case**.
  - For example, the base-case real and reactive generations could be the current operating conditions.
  - As another example, $P^{(0)} = \mathbf{0}$ is the (unrealistic) condition of zero net real power injection.
- Also suppose that we have a solution $x^{\star}$ to the base-case equations.
- That is, $g(x^{\star}) = \mathbf{0}$, or equivalently:

$$\begin{aligned} p_{-1}(x^{\star}) &= P^{(0)}_{-1}, \\ q_{-1}(x^{\star}) &= Q^{(0)}_{-1}, \end{aligned}$$

- where $P^{(0)}_{-1}$ and $Q^{(0)}_{-1}$ are the sub-vectors of $P^{(0)}$ and $Q^{(0)}$, respectively, that omit the reference/slack bus.

3.4.2 Change-case

- Now suppose that the real and reactive power generations change:
  - from $P^{(0)}$ and $Q^{(0)}$,
  - to $P = P^{(0)} + \Delta P$ and $Q = Q^{(0)} + \Delta Q$, respectively.
- Similarly, we suppose that the value of x changes from $x^\star$ to $x^\star + \Delta x$ to re-establish satisfaction of the power flow equations $g(x) = \mathbf{0}$.
- That is, the **change-case** power flow equations are given by:

$$\begin{aligned} p_{-1}(x^\star + \Delta x) &= P_{-1}^{(0)} + \Delta P_{-1}, \\ q_{-1}(x^\star + \Delta x) &= Q_{-1}^{(0)} + \Delta Q_{-1}, \end{aligned}$$

- where ΔP_{-1} and ΔQ_{-1} are the sub-vectors of ΔP and ΔQ, respectively, that omit the reference/slack bus.
- The equations are non-linear equations in Δx.

Change-case, continued

- Note the change in net generation at the reference/slack bus is required to be consistent with the change Δx.
- So, we also have that:

$$\begin{aligned} p_1(x^\star + \Delta x) &= P_1^\star + \Delta P_1, \\ q_1(x^\star + \Delta x) &= Q_1^\star + \Delta Q_1. \end{aligned}$$

- That is, the change in generation at the reference/slack bus can be calculated (or estimated) once Δx is known or estimated.

3.4.3 First-order Taylor approximation

- To find an approximate solution to the change-case equations, we form **first-order Taylor approximations** to p_{-1} and q_{-1}:

$$\begin{aligned} p_{-1}(x^\star + \Delta x) &\approx p_{-1}(x^\star) + \frac{\partial p_{-1}}{\partial x}(x^\star)\Delta x, \\ q_{-1}(x^\star + \Delta x) &\approx q_{-1}(x^\star) + \frac{\partial q_{-1}}{\partial x}(x^\star)\Delta x. \end{aligned}$$

- For future reference, note that the matrices $\frac{\partial p_{-1}}{\partial x}(x^\star)$ and $\frac{\partial q_{-1}}{\partial x}(x^\star)$ form the Jacobian of the system of equations $p_{-1}(x) = P_{-1}, q_{-1}(x) = Q_{-1}$:

$$\begin{bmatrix} \dfrac{\partial p_{-1}}{\partial x} \\ \dfrac{\partial q_{-1}}{\partial x} \end{bmatrix}.$$

3.4.4 Linearization of change-case equations

- Substituting the first-order Taylor approximations into the change-case equations, we obtain:

$$
\begin{aligned}
p_{-1}(x^\star) + \frac{\partial p_{-1}}{\partial x}(x^\star)\Delta x &\approx P_{-1}^{(0)} + \Delta P_{-1}, \\
q_{-1}(x^\star) + \frac{\partial q_{-1}}{\partial x}(x^\star)\Delta x &\approx Q_{-1}^{(0)} + \Delta Q_{-1}.
\end{aligned}
$$

- From the base-case solution, we have $p_{-1}(x^\star) = P_{-1}^{(0)}$ and $q_{-1}(x^\star) = Q_{-1}^{(0)}$.
- Ignoring the error in the first-order Taylor approximation, we have:

$$
\begin{aligned}
\frac{\partial p_{-1}}{\partial x}(x^\star)\Delta x &= \Delta P_{-1}, \\
\frac{\partial q_{-1}}{\partial x}(x^\star)\Delta x &= \Delta Q_{-1}.
\end{aligned}
$$

Linearization of change-case equations, continued

- Typically, the Jacobian $\begin{bmatrix} \dfrac{\partial p_{-1}}{\partial x}(x^{\star}) \\ \dfrac{\partial q_{-1}}{\partial x}(x^{\star}) \end{bmatrix}$ is non-singular.
- That is, we can solve:

$$\begin{bmatrix} \dfrac{\partial p_{-1}}{\partial x}(x^{\star}) \\ \dfrac{\partial q_{-1}}{\partial x}(x^{\star}) \end{bmatrix} \Delta x = \begin{bmatrix} \Delta P_{-1} \\ \Delta Q_{-1} \end{bmatrix},$$

- for Δx.
- These are **sparse** linear equations, which can be solved efficiently for Δx.
- This approximation to the solution of the change-case power flow equations is equivalent to performing one iteration of the Newton–Raphson method, starting at the base-case specified by $x^{\star}$.
- For a 5000 bus system, what is the size of the coefficient matrix of the linear equations?

Linearization of change-case equations, continued

- Moreover, the change in real and reactive power at the reference/slack bus will approximately satisfy:

$$\begin{aligned}\Delta P_1 &= \frac{\partial p_1}{\partial x}(x^\star)\Delta x,\\ \Delta Q_1 &= \frac{\partial q_1}{\partial x}(x^\star)\Delta x.\end{aligned}$$

3.4.5 Jacobian

3.4.5.1 Terms

- Recall that the entries in $p : \mathbb{R}^n \to \mathbb{R}^{n_{PQ}+1}$ are defined by:

$$\forall x \in \mathbb{R}^n, p_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)].$$

- The entries in $q : \mathbb{R}^n \to \mathbb{R}^{n_{PQ}+1}$ are defined by: $q_\ell : \mathbb{R}^n \to \mathbb{R}$:

$$\forall x \in \mathbb{R}^n, q_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)].$$

- The entries in the vector x are either of the form θ_k or of the form u_k.
- To examine the terms in the Jacobian, partition x so that all the voltage angles appear first in a sub-vector θ_{-1} followed by all the voltage magnitudes in a sub-vector u_{-1}.

Terms, continued

- There are four qualitative types of partial derivative terms corresponding to each combination:

$$
\forall x \in \mathbb{R}^n, \frac{\partial p_\ell}{\partial \theta_k}(x)
= \begin{cases} \sum_{j \in \mathbb{J}(\ell)} u_\ell u_j [-G_{\ell j} \sin(\theta_\ell - \theta_j) + B_{\ell j} \cos(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise}, \end{cases}
$$

$$
\forall x \in \mathbb{R}^n, \frac{\partial p_\ell}{\partial u_k}(x)
= \begin{cases} 2u_\ell G_{\ell\ell} + \sum_{j \in \mathbb{J}(\ell)} u_j [G_{\ell j} \cos(\theta_\ell - \theta_j) + B_{\ell j} \sin(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise}, \end{cases}
$$

Terms, continued

$$\forall x \in \mathbb{R}^n, \frac{\partial q_\ell}{\partial \theta_k}(x)$$

$$= \begin{cases} \sum_{j \in \mathbb{J}(\ell)} u_\ell u_j [G_{\ell j} \cos(\theta_\ell - \theta_j) + B_{\ell j} \sin(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell u_k [-G_{\ell k} \cos(\theta_\ell - \theta_k) - B_{\ell k} \sin(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

$$\forall x \in \mathbb{R}^n, \frac{\partial q_\ell}{\partial u_k}(x)$$

$$= \begin{cases} -2u_\ell B_{\ell\ell} + \sum_{j \in \mathbb{J}(\ell)} u_j [G_{\ell j} \sin(\theta_\ell - \theta_j) - B_{\ell j} \cos(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ u_\ell [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise.} \end{cases}$$

3.4.5.2 Partitioning by types of terms

- Based on the partitioning of x, we can partition the Jacobian into four blocks:

$$\begin{bmatrix} \frac{\partial p_{-1}}{\partial \theta_{-1}}(x) & \frac{\partial p_{-1}}{\partial u_{-1}}(x) \\ \frac{\partial q_{-1}}{\partial \theta_{-1}}(x) & \frac{\partial q_{-1}}{\partial u_{-1}}(x) \end{bmatrix}.$$

- For a 5000 bus system, with 5000 lines, how many non-zero entries are there in each of these blocks?

3.4.6 Decoupled equations

- Recall that for typical lines $\forall \ell, \forall k \in \mathbb{J}(\ell) \cup \{\ell\}, |G_{\ell k}| \ll |B_{\ell k}|$.
- Also note that for typical lines ℓk, $|\theta_\ell - \theta_k| \ll \pi/2$.
- This implies that the terms in the matrices $\dfrac{\partial p_{-1}}{\partial u_{-1}}$ and $\dfrac{\partial q_{-1}}{\partial \theta_{-1}}$ are small compared to the terms in the matrices $\dfrac{\partial p_{-1}}{\partial \theta_{-1}}$ and $\dfrac{\partial q_{-1}}{\partial u_{-1}}$.
- If we neglect all the terms in $\dfrac{\partial p_{-1}}{\partial u_{-1}}$ and $\dfrac{\partial q_{-1}}{\partial \theta_{-1}}$, then we can then approximate the Jacobian by $\begin{bmatrix} \dfrac{\partial p_{-1}}{\partial \theta_{-1}}(x) & \mathbf{0} \\ \mathbf{0} & \dfrac{\partial q_{-1}}{\partial u_{-1}}(x) \end{bmatrix}$.

Decoupled equations, continued

- Letting $\Delta x = \begin{bmatrix} \Delta\theta_{-1} \\ \Delta u_{-1} \end{bmatrix}$, this allows decoupling of the linearized equations into:

$$\begin{aligned} \frac{\partial p_{-1}}{\partial \theta_{-1}}(x^\star)\Delta\theta_{-1} &= \Delta P_{-1}, \\ \frac{\partial q_{-1}}{\partial u_{-1}}(x^\star)\Delta u_{-1} &= \Delta Q_{-1}, \end{aligned}$$

- The first set of equations relate real power and angles, while the second set of equations relate reactive power and voltage magnitudes.
- These decoupled equations require less computation than solving the full system.
- For a 5000 bus system, what is the size of the coefficient matrix of each of the the decoupled linear equations?

3.5 Fixed voltage schedule

- If **real power** generations and flows are our main concern and there is adequate **voltage support** in the form of controllable reactive sources then we may be justified in assuming that the voltage magnitudes can be held fixed by controlling reactive power:
  – instead of each bus except the reference/slack bus being a PQ bus, we re-interpret them as having a specified real power and voltage magnitude.
  – These are called PV buses.
  – A typical assumption is that all voltage magnitudes are 1 per unit, $u = \mathbf{1}$.
  – More generally, any fixed voltage schedule $u^{(0)}$ can be used.

Fixed voltage schedule, continued

- Assuming all buses, except the reference/slack bus, are *PV* buses:
  - The unknowns are: the voltage angles at all the buses except the reference/slack bus; the real power generation at the reference/slack bus; and the reactive power generations at all buses.
  - We first solve $p_{-1}\left(\begin{bmatrix}\theta^{\star}_{-1}\\ u^{(0)}_{-1}\end{bmatrix}\right) = P_{-1}$ for $\theta^{\star}_{-1}$, given the fixed voltage schedule $u^{(0)}$.
  - To complete the solution:
    - real power at the reference/slack bus, $P^{\star}_{1}$ is chosen to satisfy $P^{\star}_{1} = p_{1}\left(\begin{bmatrix}\theta^{\star}_{-1}\\ u^{(0)}_{-1}\end{bmatrix}\right)$, and
    - reactive generations at all buses, including the reference/slack bus, $Q^{\star}$ are chosen to satisfy $Q^{\star} = q\left(\begin{bmatrix}\theta^{\star}_{-1}\\ u^{(0)}_{-1}\end{bmatrix}\right)$, in order to achieve the voltage schedule $u^{(0)}$.

3.6 DC power flow

- We combine the ideas of fixed voltage profile and linearization.

3.6.1 Fixed voltage schedule

- We again assume that there are controllable voltage sources available to provide a fixed voltage schedule $u^{(0)}$.
- Based on the analysis in the previous section, we could first solve $p_{-1}\left(\begin{bmatrix} \theta^{\star}_{-1} \\ u^{(0)}_{-1} \end{bmatrix}\right) = P_{-1}$ for $\theta^{\star}_{-1}$ and then evaluate $P^{\star}_{1} = p_{1}\left(\begin{bmatrix} \theta^{\star}_{-1} \\ u^{(0)}_{-1} \end{bmatrix}\right)$.
- This again enables us to focus on real power generation and angles.
- However, instead of solving $p_{-1}\left(\begin{bmatrix} \theta^{\star}_{-1} \\ u^{(0)}_{-1} \end{bmatrix}\right) = P_{-1}$ exactly for $\theta^{\star}_{-1}$, we solve a linearized version that is linearized about a base-case in order to estimate a change-case solution.

3.6.2 Linearization

- We linearize about a *fixed* base-case solution, $x^{(0)} = \begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}$.
- The change-case power flow equations are given by:

$$p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ u_{-1}^{(0)} \end{bmatrix}\right) = P_{-1}^{(0)} + \Delta P_{-1}.$$

- To find an approximate solution to the change-case equations, we form a first-order Taylor approximation to p_{-1}:

$$p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ u_{-1}^{(0)} \end{bmatrix}\right) \approx p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) \Delta\theta_{-1}.$$

- Substituting the first-order Taylor approximations into the change-case equations, we obtain:

$$p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) \Delta\theta_{-1} \approx P_{-1}^{(0)} + \Delta P_{-1}.$$

Linearization, continued

- From the base-case solution, we have $p_{-1}\left(\begin{bmatrix}\theta_{-1}^{(0)}\\ u_{-1}^{(0)}\end{bmatrix}\right) = P_{-1}^{(0)}$.
- Ignoring the error in the first-order Taylor approximation, we have:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\theta_{-1}^{(0)}\\ u_{-1}^{(0)}\end{bmatrix}\right)\Delta\theta_{-1} = \Delta P_{-1},$$

- which can be solved for $\Delta\theta_{-1}$.
- The change in the real power generation at the slack bus is then approximately:

$$\Delta P_1 = \frac{\partial p_1}{\partial \theta_{-1}}\left(\begin{bmatrix}\theta_{-1}^{(0)}\\ u_{-1}^{(0)}\end{bmatrix}\right)\Delta\theta_{-1}.$$

Linearization, continued

- The base-case power generations $P^{(0)}$ that determine the base-case solution are chosen to be convenient for calculations.
- A typical base-case involves:
  - zero net real power generation at all buses, so that $P^{(0)} = \mathbf{0}$, and
  - all voltage magnitudes 1 per unit, so that $u^{(0)} = \mathbf{1}$.
- If the transmission lines have zero real values for their shunt elements then $\theta^{(0)} = \mathbf{0}$ solves the base-case.
- $x^{(0)} = \begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is called a **flat start**.
- At the flat start, the linearization yields the following equations:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1} = \Delta P_{-1}.$$

Linearization, continued

- To summarize, we have linearized about the flat start condition to approximate the change-case solution $\theta = \theta^{(0)} + \Delta\theta = \mathbf{0} + \Delta\theta = \Delta\theta$ corresponding to injections $P = P^{(0)} + \Delta P = \mathbf{0} + \Delta P = \Delta P$.
- We now interpret:

  $P^{(0)} + \Delta P = \Delta P = P$ to be the power generation for the change-case we are trying to solve, and

  $\theta^{(0)} + \Delta\theta = \Delta\theta = \theta$ to be the solution for the angles for the change-case we are trying to solve.

- That is, we solve the linearized power flow equations for θ_{-1}:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta_{-1} = P_{-1},$$

- where $\frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ is a constant matrix,
- θ_{-1} is the vector of unknown angles at the change-case solution, and
- P_{-1} is the sub-vector of P that omits the slack bus.

Linearization, continued

- These equations are in the form $J_{-1}\theta_{-1} = P_{-1}$, where the coefficient matrix is:

$$J_{-1} = \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right).$$

- The coefficient matrix J_{-1} relates real power and angles.
- The subscript -1 is referring to bus 1 as the slack bus:
  - for the general case of bus σ as the slack bus, we will consider $J_{-\sigma} = \frac{\partial p_{-\sigma}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$,
  - where the reference bus is still bus 1.
- The equations $J_{-1}\theta_{-1} = P_{-1}$ are sparse linear equations, which can be solved efficiently for θ_{-1}.
- Paralleling the earlier observation, this approximation to the solution of the power flow equations for power generation P_{-1} is equivalent to performing one iteration of the Newton–Raphson method, starting at a flat start.

Linearization, continued

- Moreover, the real power at the slack bus for the change-case can be estimated by:

$$P_1 = \frac{\partial p_1}{\partial \theta_{-1}} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right) \theta_{-1}, \tag{3.9}$$

- or $P_\sigma = \frac{\partial p_\sigma}{\partial \theta_{-1}} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right) \theta_{-1}$ if bus σ is the slack bus.
- We will see that the estimation of the real power generation at the slack bus can be simplified under certain assumptions on the base-case system.
- Letting $J = \frac{\partial p}{\partial \theta_{-1}} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right)$, note that the approximation satisfies $J\theta_{-1} = P$, which are called the **DC power flow equations**.
- Values of θ_{-1} and P that satisfy the DC power flow equations then approximately satisfy the power flow equality constraints (3.4) for all buses ℓ.
- Recall that we have assumed that Q is chosen to satisfy the power flow equality constraints (3.5) for all buses ℓ.

3.6.3 Interpretation

- We have interpreted the DC power flow approximation as equivalent to performing one iteration of the Newton–Raphson method, starting at a base-case specified by a flat start, or equivalently the power flow equations linearized about a flat start.
- This differs from the "traditional" interpretation that emphasizes:
  - the small angle approximations for cos and sin, and
  - the solution of DC power flow being the same as the solution of an analogous DC circuit with current sources specified by the power injections and voltages specified by the angles.
- Our interpretation provides a clearer and more general perspective on the conditions when the DC power flow provides a good approximation:
  - see homework.
- It also provides a connection to **decomposition** algorithms:
  - iteration between solution and linearization of the power flow equations, and calculation of a desired generation operating point.
- The traditional interpretation is useful for solving small systems by hand:
  - See in Section 3.9.

3.6.4 Terms in Jacobian

- The entries for the sub-matrix $J_{-1} = \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ of the Jacobian are:

$$\frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) = \begin{cases} \sum_{j\in\mathbb{J}(\ell)} B_{\ell j}, & \text{if } k=\ell, \\ -B_{\ell k}, & \text{if } k\in\mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases} \tag{3.10}$$

$$= \begin{cases} \sum_{j\in\mathbb{J}(\ell)} \begin{pmatrix} \text{minus the susceptance} \\ \text{joining buses } \ell \text{ and } j \end{pmatrix}, & \text{if } k=\ell, \\ -B_{\ell k}, & \text{if } k\in\mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} -B_{\ell k}, & \text{if } k\in\mathbb{J}(\ell)\cup\{\ell\}, \\ 0, & \text{otherwise,} \end{cases} \left\{ \begin{array}{r} \text{if the shunt susceptances} \\ \text{are all equal to zero.} \end{array} \right.$$

- Note that these entries correspond to the imaginary part of the admittance matrix, B, where $A = G + B\sqrt{-1}$, *not* to the inverse of the line inductive reactances, as is often stated in derivations of the DC power flow:
  - these are different if the resistance is non-zero.

Terms in Jacobian, continued

- In the next slides, we will consider entries of $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$, which includes a row consisting of the derivatives of p_1 corresponding to the slack bus.
- If the shunt admittances are all equal to zero then:

  J is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted,

  J_{-1} is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted and the row corresponding to the slack bus deleted, and

  if the slack bus is bus σ, then $J_{-\sigma}$ is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted and the row corresponding to the slack bus σ deleted.

- If the shunt admittances are non-zero then the entries of J corresponding to diagonal entries $B_{\ell\ell}$ of B will differ from the entries of $-B$ by the shunt admittance connected to bus ℓ.

Terms in Jacobian, continued

- Summing the entries in the ℓ-th column of $\frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$, we obtain:

$$
\begin{aligned}
\forall k, \sum_{\ell} \frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) &= \frac{\partial p_k}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) + \sum_{\ell \neq k} \frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right), \\
& \quad \text{where the summation over } \ell \text{ includes} \\
& \quad \text{the slack bus,} \\
&= \sum_{j \in \mathbb{J}(k)} B_{kj} - \sum_{\ell \in \mathbb{J}(k)} B_{\ell k}, \text{general case from (3.10),} \\
& \quad \text{considering possibly non-zero shunt admittances,} \\
&= \sum_{j \in \mathbb{J}(k)} B_{kj} - \sum_{\ell \in \mathbb{J}(k)} B_{k\ell}, \text{ since } B_{\ell k} = B_{k\ell}, \\
&= 0.
\end{aligned}
$$

- That is, each column of $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ sums to zero.

3.6.5 Slack injection

- Equivalently,

$$\mathbf{1}^{\dagger}\frac{\partial p}{\partial\theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)=\mathbf{0}. \tag{3.11}$$

- From (3.9), we can estimate the net injection at the slack bus as:

$$\begin{aligned}
P_1 &= \frac{\partial p_1}{\partial\theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)\theta_{-1},\\
&= -\mathbf{1}^{\dagger}\frac{\partial p_{-1}}{\partial\theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)\theta_{-1},\\
&\quad \text{since } \mathbf{1}^{\dagger}\frac{\partial p}{\partial\theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)=\mathbf{0}\text{ from (3.11)},\\
&= -\mathbf{1}^{\dagger}P_{-1},
\end{aligned}$$

- given the DC power flow approximation.

3.6.6 Losses

- Moreover, the losses in the system are estimated as:

$$
\begin{aligned}
\mathbf{1}^\dagger P &= \mathbf{1}^\dagger \begin{bmatrix} P_1 \\ P_{-1} \end{bmatrix}, \\
&= \mathbf{1}^\dagger \begin{bmatrix} \dfrac{\partial p_1}{\partial \theta_{-1}} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right) \theta_{-1} \\[2ex] \dfrac{\partial p_{-1}}{\partial \theta_{-1}} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right) \theta_{-1} \end{bmatrix}, \\
&= \mathbf{1}^\dagger \frac{\partial p}{\partial \theta_{-1}} \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right) \theta_{-1}, \\
&= \mathbf{0}\theta_{-1}, \text{ by (3.11)}, \\
&= 0,
\end{aligned}
$$

- so that the linearized representation is lossless, given the DC power flow assumption of the flat start as base-case:
  - note that this applies whether or not the shunt admittances are non-zero.

3.6.7 Solving for the angles

- In the usual case that the matrix $J_{-1} = \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$ is non-singular, we can write $\theta_{-1} = [J_{-1}]^{-1} P_{-1}$, allowing us to solve for the angles θ_{-1}.
- (If J_{-1} is singular, then the system is at a steady-state stability limit.)
- Given the lossless assumption and a specification of P_{-1}, we have already evaluated the net generation at the reference/slack bus:

$$P_1 = -\mathbf{1}^{\dagger} P_{-1},$$

- so that the reference/slack bus exactly compensates for the net demand or withdrawal summed across all other buses, since the approximation is lossless.
- Summarizing, the DC power flow equations $J\theta_{-1} = P$ are equivalent to:

$$\begin{aligned} P_1 &= -\mathbf{1}^{\dagger} P_{-1}, \\ \theta_{-1} &= [J_{-1}]^{-1} P_{-1}. \end{aligned}$$

- See Exercise 3.4 for a formal demonstration of the equivalence of the DC power flow equations to this representation.

3.6.8 Demand

- So far, the vector P has represented the vector of *net* injections at the buses.
- In some formulations, we want to consider demand and generation separately.
- For example, if the net injection is $P - D$, where:

  P is now the vector of generations, and
  D is the vector of demands,
- then the DC power flow equations are equivalent to:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta_{-1} &= [J_{-1}]^{-1}(P_{-1} - D_{-1}), \end{aligned}$$

- where P_{-1} and D_{-1} are the sub-vectors of P and D, respectively, that omit the reference/slack bus.

3.6.9 Slack bus and reference bus choices

- If the slack bus is bus σ then the DC power flow equations are equivalent to:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta_{-1} &= [J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}), \end{aligned}$$

- where $P_{-\sigma}$ and $D_{-\sigma}$ are the sub-vectors of P and D, respectively, that omit the slack bus, and
- where the reference bus is still assumed to be bus 1.
- If the slack bus is bus σ and the reference bus is bus ρ then the DC power flow equations are equivalent to:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta_{-\rho} &= \left[J'_{-\sigma}\right]^{-1}(P_{-\sigma} - D_{-\sigma}), \end{aligned}$$

- where $J' = \dfrac{\partial p}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ is the matrix of partial derivatives with the reference bus assumed to be bus ρ and where the matrix J' differs from J in one column.

3.7 Line flow

- We typically use the results of power flow to evaluate whether the flow along a line is within limits:
  - this is most straightforward for flow limits expressed in terms of real power flow,
  - we can also consider flow limits expressed in terms of current magnitude or the magnitude of complex power.
- There is typically a flow limit in each direction on the line.
- For a line joining bus ℓ to bus k we can consider:
  - real and reactive flows $p_{\ell k}$ and $q_{\ell k}$ along the line from bus ℓ in the direction of bus k, and
  - real and reactive flows $p_{k\ell}$ and $q_{k\ell}$ along the line from bus k in the direction of bus ℓ.
- Without loss of generality, we explicitly consider only $p_{\ell k}$ and $q_{\ell k}$.

Line flow, continued

- Ignoring shunt elements in the models, we have that the real and reactive flows are given by:

$$
\begin{aligned}
\forall x \in \mathbb{R}^n, p_{\ell k}(x) &= u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)] - (u_\ell)^2 G_{\ell k}, \\
\forall x \in \mathbb{R}^n, q_{\ell k}(x) &= u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)] + (u_\ell)^2 B_{\ell k}.
\end{aligned}
$$

- (The linearization analysis including shunt elements has the same result that we will present, but is notationally inconvenient since we need to define parameters for the shunt elements in each line.)
- We will approximate these expressions by again linearizing about a base-case:
  - for convenience, we will again assume $\rho = \sigma = 1$ in the derivation and then sketch the extensions to the general case.
- We linearize the expressions for $p_{\ell k}$ and $q_{\ell k}$ about $\theta_{-1}^{(0)}$.
- We continue to assume that the voltage magnitudes are fixed at $u^{(0)}$.

3.7.1 Linearized line flow

$$
\begin{aligned}
& p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ u_{-1}^{(0)} \end{bmatrix}\right) \\
\approx \;& p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + \frac{\partial p_{\ell k}}{\partial \theta_{-1}}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) \Delta\theta_{-1}, \\
= \;& p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + u_{\ell}^{(0)} u_{k}^{(0)} \begin{bmatrix} -G_{\ell k} \sin(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \\ +B_{\ell k} \cos(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \end{bmatrix} (\Delta\theta_{\ell} - \Delta\theta_{k}), \\
& q_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ u_{-1}^{(0)} \end{bmatrix}\right) \\
\approx \;& q_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + \frac{\partial q_{\ell k}}{\partial \theta_{-1}}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) \Delta\theta_{-1}, \\
= \;& q_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + u_{\ell}^{(0)} u_{k}^{(0)} \begin{bmatrix} G_{\ell k} \cos(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \\ +B_{\ell k} \sin(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \end{bmatrix} (\Delta\theta_{\ell} - \Delta\theta_{k}).
\end{aligned}
$$

Linearized Line flow, continued

- We focus on the real power $p_{\ell k}$ flowing along the line from bus ℓ in the direction of bus k.
- Define the row vector $K_{(\ell k)}$ of partial derivatives by:

$$\forall j \neq 1, K_{(\ell k)j} = \begin{cases} u_\ell^{(0)} u_k^{(0)} [-G_{\ell k} \sin(\theta_\ell^{(0)} - \theta_k^{(0)}) + B_{\ell k} \cos(\theta_\ell^{(0)} - \theta_k^{(0)})], & \text{if } j = \ell, \\ -u_\ell^{(0)} u_k^{(0)} [-G_{\ell k} \sin(\theta_\ell^{(0)} - \theta_k^{(0)}) + B_{\ell k} \cos(\theta_\ell^{(0)} - \theta_k^{(0)})], & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

- That is, $K_{(\ell k)j}$ is the j-th entry in the row vector $K_{(\ell k)}$, which has entries for every bus except the reference bus.
- Then the linear approximation to $p_{\ell k}$ is given by:

$$p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ u_{-1}^{(0)} \end{bmatrix}\right) \approx p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + K_{(\ell k)} \Delta\theta_{-1}.$$

3.7.2 Line flow constraints

- Suppose that we have line flow constraints of the form $p_{\ell k}(x) \leq \overline{p}_{\ell k}$.
- Using the linear approximation, we obtain:

$$p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right) + K_{(\ell k)} \Delta\theta_{-1} \leq \overline{p}_{\ell k}.$$

- We now consider the case that there are line flow constraints on each line (ℓk).
- By defining a matrix K with rows $K_{(\ell k)}$ and a vector d with entries $d_{(\ell k)}$ of the form:

$$d_{(\ell k)} = \overline{p}_{\ell k} - p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ u_{-1}^{(0)} \end{bmatrix}\right),$$

- we can approximate the collection of line flow constraints in the form $K \Delta\theta_{-1} \leq d$.

3.7.3 DC power flow approximation to line flow constraints

- Using a flat start $x^{(0)} = \begin{bmatrix} \theta^{(0)}_{-1} \\ u^{(0)}_{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ as the base-case for the linearization, we find:

$$\begin{aligned}
p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ u^{(0)}_{-1} \end{bmatrix}\right) &= u^{(0)}_{\ell}(u^{(0)}_{k} - u^{(0)}_{\ell})G_{\ell k}, \\
p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) &= 0, \\
K_{(\ell k)j} = \frac{\partial p_{\ell k}}{\partial \theta_j}\left(\begin{bmatrix} \mathbf{0} \\ u^{(0)}_{-1} \end{bmatrix}\right) &= \begin{cases} u_\ell u_k B_{\ell k}, & \text{if } j = \ell, \\ -u_\ell u_k B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \\
K_{(\ell k)j} = \frac{\partial p_{\ell k}}{\partial \theta_j}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) &= \begin{cases} B_{\ell k}, & \text{if } j = \ell, \\ -B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \\
d_{(\ell k)} &= \overline{p}_{\ell k} - p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right), \\
&= \overline{p}_{\ell k}.
\end{aligned}$$

DC power flow approximation to line flow constraints, continued

- Summarizing, we can approximate the flows at the angle $\theta_{-1} = \theta_{-1}^{(0)} + \Delta\theta_{-1} = \Delta\theta_{-1}$ using the linearized equations $K\theta_{-1} \leq d$, where:

$$\begin{aligned}
\forall(\ell k), \forall j \neq 1, K_{(\ell k)j} &= \begin{cases} B_{\ell k}, & \text{if } j = \ell, \\ -B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases} \\
\forall(\ell k), d_{(\ell k)} &= \overline{p}_{\ell k}.
\end{aligned}$$

- If the reference bus changes to bus ρ then the linearized line flow constraints would be $K'\theta_{-\rho} \leq d$, where:

$$\begin{aligned}
\forall(\ell k), \forall j \neq \rho, K'_{(\ell k)j} &= \begin{cases} B_{\ell k}, & \text{if } j = \ell, \\ -B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases} \\
\forall(\ell k), d_{(\ell k)} &= \overline{p}_{\ell k}.
\end{aligned}$$

- The matrix K' differs from the matrix K in one column.

3.7.4 Eliminating the angles

- We previously found that the power flow equations could be expressed as:

$$\begin{aligned} -\mathbf{1}^\dagger P &= -\mathbf{1}^\dagger D, \\ \theta_{-1} &= [J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}), \end{aligned}$$

- where σ is the slack bus and $\rho = 1$ is the reference bus.
- We use the second equation to substitute into $K\theta_{-1} \leq d$ to obtain the equality and inequality constraints with the angles eliminated:

$$-\mathbf{1}^\dagger P = -\mathbf{1}^\dagger D, \tag{3.12}$$

$$K[J_{-\sigma}]^{-1}P_{-\sigma} \leq K[J_{-\sigma}]^{-1}D_{-\sigma} + d. \tag{3.13}$$

- This approximation to the flows is not always good:
  - it is used to represent transmission constraints in most day-ahead electricity markets,
  - will explore accuracy in homework exercise.

3.7.5 Shift factor matrix

- The matrix $K[J_{-\sigma}]^{-1}$ is the matrix of DC **shift factors**.
- That is, entries in the matrix represent the fraction of flow along each line for:

  injection at the buses represented in the vector $P_{-\sigma}$, and
  withdrawal at the slack bus σ.

- We occasionally want to express line flows in terms of the vector P of all net injections.
- For $\sigma = 1$, define the augmented shift factor matrix $\hat{C} = \begin{bmatrix} \mathbf{0} & K[J_{-1}]^{-1} \end{bmatrix}$.
- That is, $\hat{C}$ consists of the columns of $K[J_{-1}]^{-1}$ augmented by an additional zero column corresponding to P_1.
- Each entry of $\hat{C}_k$ represents the fraction of the generation from generator at bus k that flows on the corresponding line.
- The flows are given by $\hat{C}(P - D)$.
- Similarly, if the slack bus is some other bus σ, we can again define a corresponding augmented shift factor matrix $\hat{C}$ such that the flows are given by $\hat{C}(P - D)$.

3.8 Example

- Consider the following system with MW capacities and per unit impedances (on a 1 MVA base) as shown.
- Bus $\sigma = 0$ is the slack bus and there are no shunt admittances.
- Bus $\rho = 1$ is reference bus, so the unknown angles are $\theta_{-1} = \begin{bmatrix} \theta_0 \\ \theta_2 \\ \theta_3 \end{bmatrix}$.

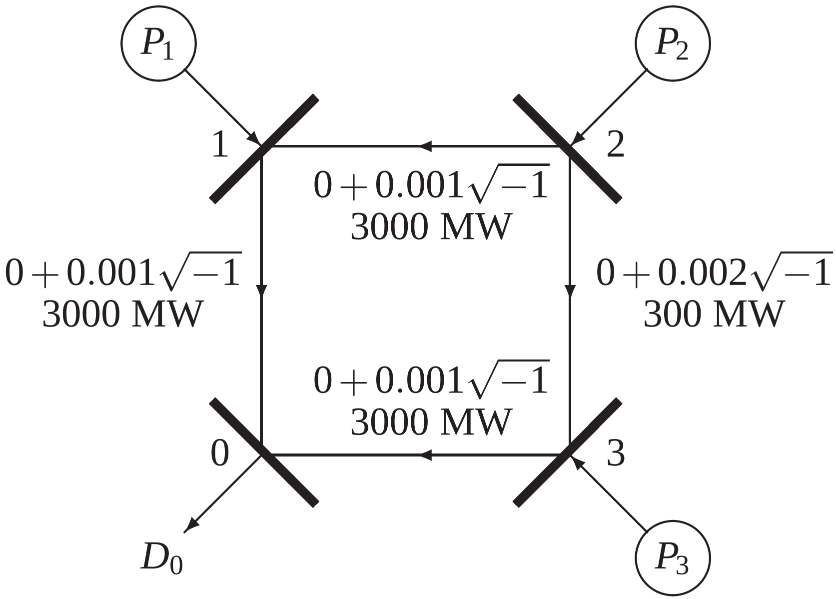

Fig. 3.6. Four-line four-bus network with generators at buses 1, 2, and 3, and demand at bus 0.

3.8.1 Admittance matrix

- The line admittances are:

$$
\begin{aligned}
Y_{01} = Y_{10} = Y_{12} = Y_{21} = Y_{03} = Y_{30} &= \frac{1}{0 + 0.001\sqrt{-1}} = -1000\sqrt{-1}, \\
Y_{23} = Y_{32} &= \frac{1}{0 + 0.002\sqrt{-1}} = -500\sqrt{-1}.
\end{aligned}
$$

Admittance matrix, continued

- The bus admittance matrix is:

$$\begin{bmatrix} Y_{01}+Y_{03} & -Y_{01} & 0 & -Y_{03} \\ -Y_{10} & Y_{10}+Y_{12} & -Y_{12} & 0 \\ 0 & -Y_{21} & Y_{21}+Y_{23} & -Y_{23} \\ -Y_{30} & 0 & -Y_{32} & Y_{30}+Y_{32} \end{bmatrix}$$

$$= \begin{bmatrix} -2000\sqrt{-1} & 1000\sqrt{-1} & 0 & 1000\sqrt{-1} \\ 1000\sqrt{-1} & -2000\sqrt{-1} & 1000\sqrt{-1} & 0 \\ 0 & 1000\sqrt{-1} & -1500\sqrt{-1} & 500\sqrt{-1} \\ 1000\sqrt{-1} & 0 & 500\sqrt{-1} & -1500\sqrt{-1} \end{bmatrix},$$

$$= \begin{bmatrix} B_{00}\sqrt{-1} & B_{01}\sqrt{-1} & 0 & B_{03}\sqrt{-1} \\ B_{10}\sqrt{-1} & B_{11}\sqrt{-1} & B_{12}\sqrt{-1} & 0 \\ 0 & B_{21}\sqrt{-1} & B_{22}\sqrt{-1} & B_{23}\sqrt{-1} \\ B_{30}\sqrt{-1} & 0 & B_{32}\sqrt{-1} & B_{33}\sqrt{-1} \end{bmatrix}.$$

3.8.2 Jacobian

- Since there are no shunts, $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)$ is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted:

$$J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right) = \begin{bmatrix} -B_{00} & 0 & -B_{03} \\ -B_{10} & -B_{12} & 0 \\ 0 & -B_{22} & -B_{23} \\ -B_{30} & -B_{32} & -B_{33} \end{bmatrix},$$

$$= \begin{bmatrix} 2000 & 0 & -1000 \\ -1000 & -1000 & 0 \\ 0 & 1500 & -500 \\ -1000 & -500 & 1500 \end{bmatrix}.$$

- Note that the rows of J are indexed by 0, 1, 2, 3, while the columns are indexed by 0, 2, 3.

3.8.3 DC power flow

- We can solve for θ_{-1} to obtain the following form for the DC power flow equations:

$$\begin{aligned} -\mathbf{1}^{\dagger} P &= -\mathbf{1}^{\dagger} D, \\ \theta_{-1} &= [J_{-0}]^{-1}(P_{-0} - D_{-0}). \end{aligned}$$

- where J_{-0} is J with the row corresponding to $\sigma = 0$ deleted, so that:

$$\begin{aligned} J_{-0} &= \begin{bmatrix} -1000 & -1000 & -0 \\ 0 & 1500 & -500 \\ -1000 & -500 & 1500 \end{bmatrix}, \\ [J_{-0}]^{-1} &= \begin{bmatrix} -0.0008 & -0.0006 & -0.0002 \\ -0.0002 & 0.0006 & 0.0002 \\ -0.0006 & -0.0002 & 0.0006 \end{bmatrix}. \end{aligned}$$

- Note that the subscript -1 on θ_{-1} is referring to $\rho = 1$, the reference bus, with the entry θ_1 omitted, (and columns of J also omit terms for θ_1),
- whereas the subscript -0 on J_{-0}, P_{-0}, and D_{-0} is referring to $\sigma = 0$, the slack bus, with row $\dfrac{\partial p_0}{\partial \theta_{-1}}$, and terms P_0 and D_0, respectively, omitted.

DC power flow, continued

- So far, the development considered generation and demand at all buses.
- The example only has demand at bus 0 and has generation at buses 1, 2, and 3.
- Since there is only demand at bus 0 then the DC power flow equations are:

$$\begin{aligned} -P_1 - P_2 - P_3 &= -D_0, \\ \theta_{-1} &= \begin{bmatrix} -0.0008 & -0.0006 & -0.0002 \\ -0.0002 & 0.0006 & 0.0002 \\ -0.0006 & -0.0002 & 0.0006 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}. \end{aligned}$$

3.8.4 DC power flow approximation to line flow constraints

- In principle, there are limits on flow in both directions on each line.
- We will assume that the only binding limits are in the directions from buses 1 to 0, 2 to 1, 2 to 3, and 3 to 0, respectively, as suggested by the arrows in Figure 3.6.
- These four line flow inequality constraints are then specified by $K\theta_{-1} \leq d$, where:

$$d = \begin{bmatrix} \overline{p}_{10} \\ \overline{p}_{21} \\ \overline{p}_{23} \\ \overline{p}_{30} \end{bmatrix} = \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix},$$

$$K = \begin{bmatrix} -B_{10} & 0 & 0 \\ 0 & B_{21} & 0 \\ 0 & B_{23} & -B_{23} \\ -B_{30} & 0 & B_{30} \end{bmatrix} = \begin{bmatrix} -1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 500 & -500 \\ -1000 & 0 & 1000 \end{bmatrix}.$$

- Note that the rows of K are indexed by $(10), (21), (23), (30)$, while the columns are indexed by 0, 2, 3.

3.8.5 DC shift factors

- The matrix of DC shift factors is:

$$
\begin{aligned}
K[J_{-0}]^{-1} &= \begin{bmatrix} -1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 500 & -500 \\ -1000 & 0 & 1000 \end{bmatrix} \begin{bmatrix} -0.0008 & -0.0006 & -0.0002 \\ -0.0002 & 0.0006 & 0.0002 \\ -0.0006 & -0.0002 & 0.0006 \end{bmatrix}, \\
&= \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix}.
\end{aligned}
$$

- The augmented shift factor matrix is:

$$
\begin{aligned}
\hat{C} &= \begin{bmatrix} \mathbf{0} & K[J_{-0}]^{-1} \end{bmatrix}, \\
&= \begin{bmatrix} 0.0 & 0.8 & 0.6 & 0.2 \\ 0.0 & -0.2 & 0.6 & 0.2 \\ 0.0 & 0.2 & 0.4 & -0.2 \\ 0.0 & 0.2 & 0.4 & 0.8 \end{bmatrix}.
\end{aligned}
$$

DC shift factors, continued

- For example, for power injected at bus 1 and withdrawn at bus $\sigma = 0$, the shift factors to the lines 1 to 0, 2 to 1, 2 to 3, and 3 to 0 are, respectively $0.8, -0.2, 0.2, 0.2$.
- Moreover, the flow on any particular line is the sum of the flows due to individual injections at particular buses.
- For power injected at bus 0 and withdrawn at bus $\sigma = 0$, what are the shift factors to the lines from buses 1 to 0, 2 to 1, 2 to 3, and 3 to 0?
- If 1 MW is injected at bus 1, 10 MW is injected at bus 2, and 100 MW is injected at bus 3, with 111 MW withdrawn at bus $\sigma = 0$, what is the flow on the line from bus 1 to bus 0?

3.8.6 Line flow constraints in terms of shift factors

- The flows on the lines are given by $\hat{C}(P-D)$ or, equivalently, $K[J_{-1}]^{-1}(P_{-0}-D_{-0})$.
- The equality and inequality constraints with angles eliminated are:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ K[J_{-0}]^{-1}P_{-0} &\leq K[J_{-0}]^{-1}D_{-0}+d. \end{aligned}$$

- Again note that $D_{-0}=\mathbf{0}$ for this particular example.
- Also, $d=\begin{bmatrix}3000\\3000\\300\\3000\end{bmatrix}$, so these constraints become:

$$\begin{aligned} -P_1-P_2-P_3 &= -D_0, \\ \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} &\leq \begin{bmatrix}3000\\3000\\300\\3000\end{bmatrix}. \end{aligned}$$

Line flow constraints in terms of shift factors, continued

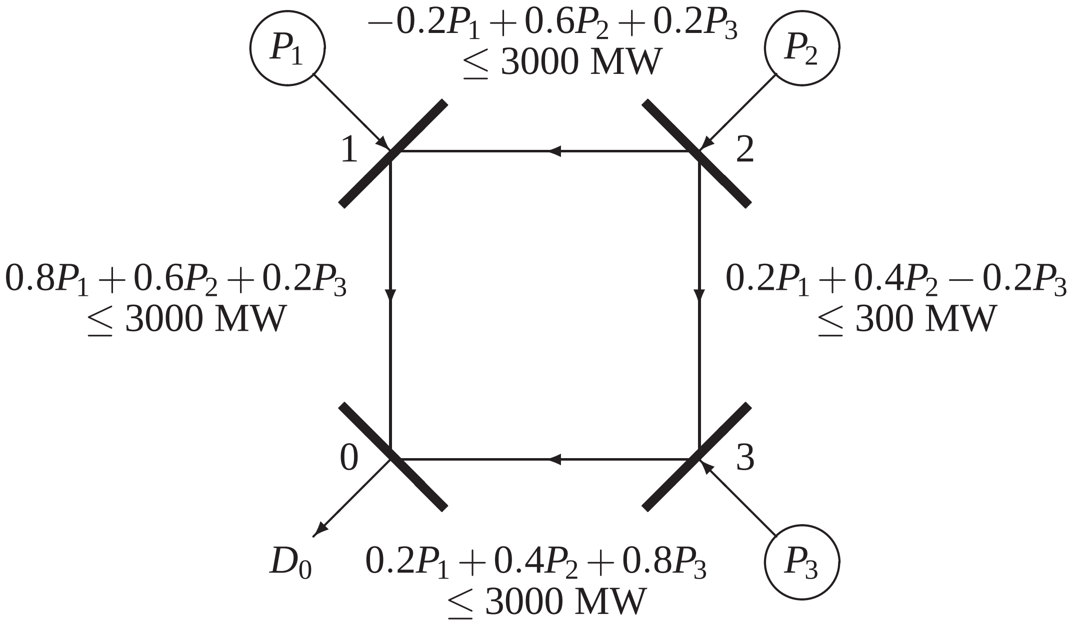

Fig. 3.7. DC power flow approximation to line flow constraints for four-line four-bus network.

3.8.7 Line flow constraints at other operating points

- The derivation so far used the flat start condition as the base-case for evaluating the shift factors and the line flow constraints.
- Other base-cases could be used, such as:
  - another assumed operating point, or
  - a measured or estimated operating point from a **state estimator**.
- The lossless assumption will typically not hold at other base-cases nor for the estimated change-case.

3.9 DC power flow circuit interpretation

- As mentioned in Section 3.6.3, we can interpret the DC power flow approximation in terms of an analogous DC circuit:
  - the DC circuit interpretation is useful to solve small systems by hand.
- Recall that the power flow equations are in the form $J\theta_{-\rho} = P$, where ρ is the reference bus.
- If the shunt admittances are zero then J is given by $-B$ with the column corresponding to the reference bus deleted.
- Consider the following analogy with a DC circuit:
  - Bus ρ is the datum node in the circuit with DC voltage defined to be 0,
  - Real power injections P are analogous to DC current injections i at all buses,
  - Angles $\theta_{-\rho}$ are analogous to DC voltages $v_{-\rho}$ at all nodes except the datum node,
  - Entries in J are analogous to the admittance matrix of a circuit having resistors joining nodes ℓ and k with "conductance" $g_{\ell k} = |B_{\ell k}|$.
- The analogous DC circuit satisfies $Jv_{-\rho} = i$:
  - applies whether or not there are non-zero shunt admittances.

DC power flow circuit interpretation, continued

- *Current* injections and flows in the DC circuit correspond to *power* injections and flows in the power system.
- Recall that currents in a DC circuit can be superposed:
  - the current flowing in a branch due to multiple current injections is equal to the sum of the currents flowing in that branch due to each current injection considered separately.
- Therefore, we can superpose *power* flow in the DC power flow approximation in the same way as we superpose *current* flow in a DC circuit.
- Suppose that the DC circuit has two nodes joined by two conductances, which we view as two "paths" between these nodes:
  - From circuit theory, recall that if current is injected at one node and withdrawn at another node then current is shared on these paths in proportion to their conductances.
- Moreover, if there is current injected at multiple nodes then the resulting total current in any branch is equal to the superposition of the currents in that branch due to the individual current injections.

DC power flow circuit interpretation, continued

- Recall the example from Figure 3.6, repeated in Figure 3.8.
- We will illustrate the DC circuit using this example.

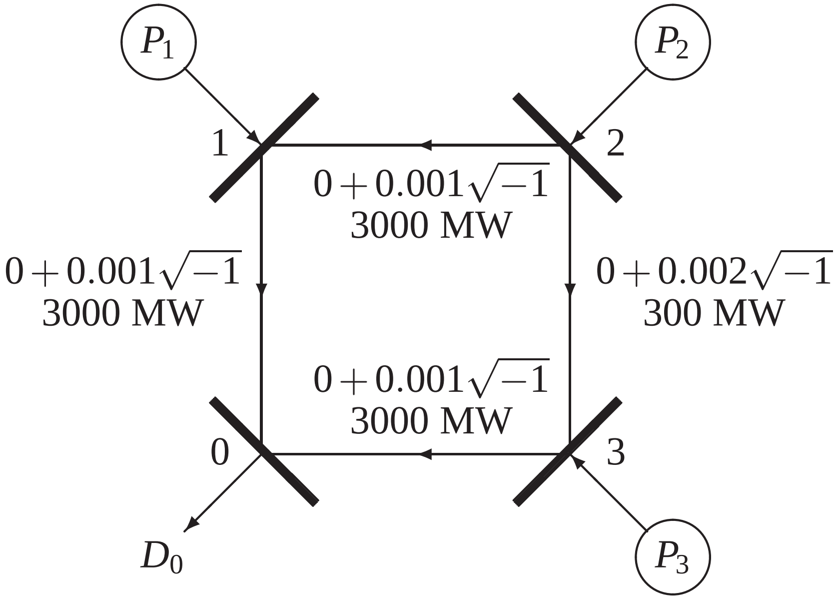

Fig. 3.8. Four-line four-bus network repeated from Figure 3.6.

DC power flow circuit interpretation, continued

- Suppose current is injected at node 1 in the analogous DC circuit and withdrawn at the node $\sigma = 0$.
- Note that the actual impedance directly joining buses 1 and 0 in the power system is $0 + 0.001\sqrt{-1}$:
  - analogous "conductance" of this path is $g_{10} = |B_{10}| = 1000$,
  - current on this analogous conductance is proportional to 1000.
- We can also think of the lines from buses 1 to 2, 2 to 3, and 3 to 0 as another impedance joining buses 1 and 0 in the power system:
  - total impedance in this path is:

$$0 + 0.001\sqrt{-1} + 0 + 0.002\sqrt{-1}0 + 0.001\sqrt{-1} = 0.004\sqrt{-1},$$

  - analogous "conductance" of path is $g_{1230} = 250$,
  - current on this analogous conductance due to the lines 1 to 2, 2 to 3, and 3 to 0 is proportional to 250.
- Current injected at node 1 and withdrawn at node 0 is shared between the analogous conductances in the paths in the proportion $1000 : 250 = 0.8 : 0.2$.

DC power flow circuit interpretation, continued

- In the power system, the DC power flow approximation means that power injected at bus 1 and withdrawn at the slack bus $\sigma = 0$ will be shared in the ratio 0.8:0.2 between:
  - the path consisting of the line directly joining buses 1 and 0, and
  - the lines forming the path from buses 1 to 2, 2 to 3, 3 to 0.
- That is, the shift factors to the lines from buses 1 to 0, 2 to 1, 2 to 3, and 3 to 0 are, respectively 0.8, $-0.2, 0.2, 0.2$, exactly as calculated in the previous section.
- By superposition, we can calculate the total power flow on a line as the sum of the power flows due to individual power injections.

3.10 Summary

- In this chapter we formulated the power flow problem.
- We considered a linearization of power flow.
- We considered fixed voltage profiles.
- We considered the DC power flow.

This chapter is based on:

- Sections 8.2 and 9.2 of *Applied Optimization: Formulation and Algorithms for Engineering Systems*, Cambridge University Press 2006.
- Ross Baldick, "Variation of Distribution Factors with Loading," *IEEE Transactions on Power Systems*, 18(4):1316–1323, November 2003.
- Brian Stott, Jorge Jardim, and Ongun Alsaç, "DC Power Flow Revisited," *IEEE Transactions on Power Systems*, 24(3):1290–1300, August 2009.

Homework exercises

3.1 Consider a power system consisting of two buses and one transmission line:

- bus 1 (the reference/slack bus), where there is a generator, and
- bus 2, where there is load.

Suppose that the reference/slack bus voltage is specified to be $V_1 = 1\angle 0^\circ$ and that real power flow from bus 2 into the line is given by:

$$\forall u_2 \in \mathbb{R}_+, \forall \theta_2 \in \mathbb{R}, p_2(\theta_2, u_2) = u_2 \sin\theta_2.$$

(That is, we assume that $G_{22} = G_{12} = 0$ and $B_{12} = 1$.) Suppose $u_2 = 1.0$.

(i) What is the largest value of demand D_2 at bus 2 for which there is a solution to the equation $p_2(\theta_2, 1.0) + D_2 = 0$? What is the corresponding value of θ_2? We will write $\underline{\theta}_2$ for this value of θ_2.

(ii) What happens if θ_2 is smaller than $\underline{\theta}_2$?

(iii) Show that there are two solutions to the equation $p_2(\theta_2, 1.0) + D_2 = 0$ with $0 \geq \theta_2 > -2\pi$ if $D_2 = 0.5$. What are the corresponding values of θ_2?

(iv) Use the DC power flow to approximate the relationship between θ_2 and D_2.

(v) When do you expect the DC power flow to be a poor approximation to the exact solution?

3.2 Consider the example in Section 3.8, but suppose that bus 0 is both the reference and the slack bus, so that $\rho = \sigma = 0$.

(i) What is vector of unknown angles θ_{-0}?

(ii) Evaluate $J' = \frac{\partial p}{\partial \theta_{-0}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$.

(iii) Evaluate $[J'_{-0}]^{-1}$.

(iv) Write down the DC power flow equations in terms of generation at buses 1, 2, and 3, and demand at bus 0.

(v) Evaluate the matrix K' in the linearized representation of line flow inequality constraints $K'\Delta\theta_{-0} \leq d$.

(vi) Evaluate the shift factor matrix $K'[J'_{-0}]^{-1}$.

(vii) Write down the line flow inequality constraints in terms of the shift factors.

(viii) What do you notice about the line flow inequality constraints? Did the choice of reference bus change the form of the line flow constraints?

(ix) Repeat the previous parts, but with bus 1 both the reference and the slack bus, so that $\rho = \sigma = 1$.

3.3 Consider the three bus, two line system shown in Figure 3.9 and suppose that bus 1 is both the reference and the slack bus, so that $\rho = \sigma = 1$. The line capacities are shown. Assume that the susceptance joining bus 2 to bus 1 and the susceptance joining bus 3 to bus 2 are both non-zero.

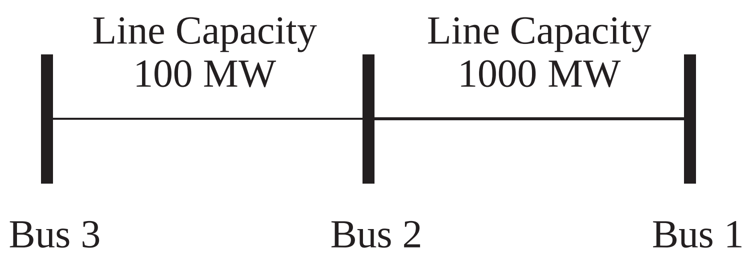

Fig. 3.9. Three bus, two line radial network.

(i) What is vector of unknown angles θ_{-1}?

(ii) Evaluate $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$. (Hint: See (3.10).)

(iii) Write down the DC power flow equations in terms of generation and demand at buses 2 and 3.

(iv) Evaluate $[J_{-1}]^{-1}$.

(v) Assume that the only flow constraints are from bus 2 to bus 1, and from bus 3 to bus 2. Evaluate the matrix K in the linearized representation of real power line flow limit inequality constraints $K\Delta\theta_{-1} \leq d$.

(vi) Evaluate the shift factor matrix $K[J_{-1}]^{-1}$.

(vii) Write down the real power line flow limit inequality constraints in terms of the shift factors as in (3.13).

(viii) Interpret these inequality constraints in terms of the figure.

3.4 Assuming that J_{-1} is invertible, show that the DC power flow equations $J\theta_{-1} = P$ are equivalent to:

$$\begin{aligned} P_1 &= -\mathbf{1}^\dagger P_{-1}, \\ \theta_{-1} &= [J_{-1}]^{-1} P_{-1}. \end{aligned}$$

That is, show that θ_{-1} satisfies $J\theta_{-1} = P$ if and only if θ_{-1} satisfies $\theta_{-1} = [J_{-1}]^{-1} P_{-1}$ and $P_1 = -\mathbf{1}^\dagger P_{-1}$. (Hint: See discussion in Section 3.6.4.)