15.094J: Robust Modeling, Optimization, Computation

Lecture 15: Robust Multi-period Portfolio Optimization

Outline

1. Multi-period portfolios under certainty
2. Robust Portfolios
3. Insights from Computations
4. Conclusions

Primitives

- M risky assets
- One riskless asset (asset 0)
- N trading periods, $t = 0, \dots, N-1$.
- Goal is to manage the portfolio of assets in a manner that maximizes expected final wealth.
- Decision variables: x_t^m, $m = 0, 1, \dots, M$, $t = 0, 1, \dots, N$ dollar holdings at the beginning of time period t on asset m
- u_t^m dollar sells on asset m at time t
- v_t^m dollar buys on asset m at time t
- $c_{sell} u_t^m$ and $c_{buy} v_t^m$ transaction costs
- $\tilde{r}_t^m$ returns over $(t, t+1]$ for asset m
- r_t^0 for riskless asset known.

Dynamics

- Dynamics under certainty, $t = 1, \ldots, N$

$$
\begin{aligned}
x_t^m &= (1 + r_{t-1}^m)\left(x_{t-1}^m - u_{t-1}^m + v_{t-1}^m\right) \\
x_t^0 &= (1 + r_{t-1}^0)\left(x_{t-1}^0 + \sum_{m=1}^{M}(1 - c_{sell})u_{t-1}^m - \sum_{m=1}^{M}(1 + c_{buy})v_{t-1}^m\right)
\end{aligned}
$$

- Under known returns

$$
\begin{aligned}
\text{Max} \quad & \sum_{m=0}^{M} x_N^m \\
\text{s.t.} \quad & x_t^m = (1 + \tilde{r}_{t-1}^m)(x_{t-1}^m - u_{t-1}^m + v_{t-1}^m) \\
& x_t^0 = (1 + r_{t-1}^0)\left(x_{t-1}^0 + \sum_{m=1}^{M}(1 - c_{sell})u_{t-1}^m - \sum_{m=1}^{M}(1 + c_{buy})v_{t-1}^m\right) \\
& x_t^m \geq 0 \\
& u_t^m \geq 0,\ v_t^m \geq 0
\end{aligned}
$$

Initial Reformulation

$$
\begin{aligned}
R_0^m &= 1 \\
\tilde{R}_t^m &= (1+\tilde{r}_0^m)(1+\tilde{r}_1^m)\ldots(1+\tilde{r}_{t-1}^m) \\
\xi_t^m &= \frac{x_t^m}{\tilde{R}_t^m}, \quad \eta_t^m = \frac{u_t^m}{\tilde{R}_t^m}, \quad \zeta_t^m = \frac{v_t^m}{\tilde{R}_t^m}
\end{aligned}
$$

$$
\begin{aligned}
\text{Max} \quad & w \\
\text{s.t.} \quad & w \le \sum_{m=1}^{M} \tilde{R}_N^m \xi_N^m + R_N^0 \xi_N^0 \\
& \xi_t^m = \xi_{t-1}^m - \eta_{t-1}^m + \zeta_{t-1}^m \\
& \xi_t^0 = \xi_{t-1}^0 + \sum_{m=1}^{M} (1-c_{sell}) \frac{\tilde{R}_{t-1}^m}{R_{t-1}^0} \eta_{t-1}^m - \sum_{m=1}^{M} (1+c_{buy}) \frac{\tilde{R}_{t-1}^m}{R_{t-1}^0} \zeta_{t-1}^m \\
& \xi_t^m, \ \eta_t^m, \ \zeta_t^m \ge 0.
\end{aligned}
$$

Final Reformulation

- Replace equalities by inequalities:

$$
\begin{aligned}
\text{Max} \quad & w \\
\text{s.t.} \quad & w \leq \sum_{m=1}^{M} \tilde{R}_N^m \xi_N^m + R_N^0 \xi_N^0 \\
& \xi_t^m = \xi_{t-1}^m - \eta_{t-1}^m + \zeta_{t-1}^m \\
& \xi_t^0 \leq \xi_{t-1}^0 + \sum_{m=1}^{M} (1 - c_{sell}) \frac{\tilde{R}_{t-1}^m}{R_{t-1}^0} \eta_{t-1}^m - \sum_{m=1}^{M} (1 + c_{buy}) \frac{\tilde{R}_{t-1}^m}{R_{t-1}^0} \zeta_{t-1}^m \\
& \xi_t^m,\ \eta_t^m,\ \zeta_t^m \geq 0.
\end{aligned}
$$

Modeling Uncertainty

- Uncertainty sets:

$$U_1 = \left\{ \begin{array}{l} ||\mathbf{\Sigma}_1^{-\frac{1}{2}} \left(\tilde{R}_1 - \overline{R}_1 \right) || \leq \Delta \\ \underline{\delta}_2^m \tilde{R}_1^m \leq \tilde{R}_2^m \leq \overline{\delta}_2^m \tilde{R}_1^m, \ m = 1, \ldots, M \\ \vdots \\ \underline{\delta}_N^m \tilde{R}_{N-1}^m \leq \tilde{R}_N^m \leq \overline{\delta}_N^m \tilde{R}_{N-1}^m, \ m = 1, \ldots, M \end{array} \right\}$$

  where $\underline{\delta}_t^m$ and $\overline{\delta}_t^m$ are of the form $(1 + \underline{r}_t^m)$ and $(1 + \overline{r}_t^m)$, respectively.

- If data on the covariance matrices of future cumulative returns are available:

$$U_2 = \left\{ \ ||\mathbf{\Sigma}_t^{-\frac{1}{2}} \left(\tilde{R}_t - \overline{R}_t \right) || \leq \Delta_t, \ t = 1, \ldots, N \ \right\},$$

  where Δ_t are budgets of uncertainty.

Robust counterpart for U_1

$$
\begin{aligned}
\max \quad & w \\
\text{s.t.} \quad & w \le R_N^0 \xi_N^0 - \left(p^N - q^N\right)' \boldsymbol{\Sigma}_1^{-\frac{1}{2}} \overline{R}_1 - \frac{\Delta}{d} \cdot (u^N)' e \\
& \xi_t^m = \xi_{t-1}^m - \eta_{t-1}^m + \zeta_{t-1}^m \\
& \xi_{t+1}^0 \le \xi_t^0 - (p^t - q^t)' \boldsymbol{\Sigma}_1^{-\frac{1}{2}} \overline{R}_1 - \frac{\Delta}{d} \cdot (u^t)' e \\
& \xi_1^0 \le \xi_0^0 + \sum_{m=1}^{M} (1 - c_{sell}) \eta_0^m - \sum_{m=1}^{M} (1 + c_{buy}) \zeta_0^m \\
& (p^1 - q^1)' \boldsymbol{\Sigma}_1^{-\frac{1}{2}} + \begin{pmatrix} \underline{\delta}_2^1(1)\alpha_2^1(1) - \overline{\delta}_2^1(1)\beta_2^1(1) \\ \vdots \\ \underline{\delta}_2^M(1)\alpha_2^M(1) - \overline{\delta}_2^M(1)\beta_2^M(1) \end{pmatrix}' = \begin{pmatrix} -\frac{(1-c_{sell})}{R_1^0}\eta_1^1 + \frac{(1+c_{buy})}{R_1^0}\zeta_1^1 \\ \vdots \\ -\frac{(1-c_{sell})}{R_1^0}\eta_1^M + \frac{(1+c_{buy})}{R_1^0}\zeta_1^M \end{pmatrix} \\
& (p^t - q^t)' \boldsymbol{\Sigma}_1^{-\frac{1}{2}} + \begin{pmatrix} \underline{\delta}_2^1(t)\alpha_2^1(t) - \overline{\delta}_2^1(t)\beta_2^1(t) \\ \vdots \\ \underline{\delta}_2^M(t)\alpha_2^M(t) - \overline{\delta}_2^M(t)\beta_2^M(t) \end{pmatrix}' = \mathbf{0}'
\end{aligned}
$$

Robust counterpart for U_1, continued

$$
\begin{aligned}
&-p^t - q^t + u^t = \mathbf{0}\\
&\left((u^t)'e\right) \cdot e \geq d \cdot u^t\\
&-\alpha_\tau^m(t) + \beta_\tau^m(t) + \underline{\delta}_{\tau+1}^m \alpha_{\tau+1}^m(t) - \overline{\delta}_{\tau+1}^m \beta_{\tau+1}^m(t) = 0,\\
&-\alpha_t^m(t) + \beta_t^m(t) + \underline{\delta}_{t+1}^m \alpha_{t+1}^m(t) - \overline{\delta}_{t+1}^m \beta_{t+1}^m(t) = -\frac{(1-c_{sell})}{R_t^0}\eta_t^m + \frac{(1+c_{buy})}{R_t^0}\zeta_t^m,\\
&-\alpha_N^m(t) + \beta_N^m(t) = 0\\
&-\alpha_N^m(N) + \beta_N^m(N) = -\xi_N^m,\\
&\xi_t^m,\ \eta_t^m,\ \zeta_t^m,\ \alpha_\tau^m(t),\ \beta_\tau^m(t),\ p^t,\ q^t,\ u^t \geq 0,
\end{aligned}
$$

Robust counterpart for U_2

$$
\begin{aligned}
\max \quad & w \\
\text{s.t.} \quad & w \leq \overline{R}_N' \boldsymbol{\xi}_N - \frac{\Delta_N}{d} \left(w^N\right)' e \\
& \xi_t^m = \xi_{t-1}^m - \eta_{t-1}^m + \zeta_{t-1}^m \\
& \xi_1^0 \leq \xi_0^0 + \sum_{m=1}^{M} (1 - c_{sell})\, \eta_0^m - \sum_{m=1}^{M} (1 + c_{buy})\, \zeta_0^m \\
& \xi_{t+1}^0 - \xi_t^0 - \begin{pmatrix} -\frac{(1-c_{sell})}{R_t^0}\eta_t^1 + \frac{(1+c_{buy})}{R_t^0}\zeta_t^1 \\ \vdots \\ -\frac{(1-c_{sell})}{R_t^0}\eta_t^M + \frac{(1+c_{buy})}{R_t^0}\zeta_t^M \end{pmatrix}' \overline{R}_t + \frac{\Delta_t}{d} \left(w^t\right)' e \leq 0 \\
& \left(2p^N - w^N\right)' = -\boldsymbol{\xi}_N' \boldsymbol{\Sigma}_N^{\frac{1}{2}} \\
& \left(2p^t - w^t\right)' = - \begin{pmatrix} -\frac{(1-c_{sell})}{R_t^0}\eta_t^1 + \frac{(1+c_{buy})}{R_t^0}\zeta_t^1 \\ \vdots \\ -\frac{(1-c_{sell})}{R_t^0}\eta_t^M + \frac{(1+c_{buy})}{R_t^0}\zeta_t^M \end{pmatrix}' \boldsymbol{\Sigma}_t^{\frac{1}{2}} \\
& \left(w^t\right)' e \geq d \cdot w^t \\
& w^t \geq p^t, \; t = 1, \ldots, N \\
& p^t, \; \xi_t^m, \; \eta_t^m, \; \zeta_t^m \geq \mathbf{0}
\end{aligned}
$$

Size

- The robust counterpart for U_1 has $2MN^2 + MN - 2M + N$ variables and $3MN^2 + 8MN - 3M + N + 2$ constraints
- For a portfolio of 500 stocks optimized over 6 time periods (e.g., rebalanced monthly for half a year): 38,006 variables and 76,508 constraints
- The robust counterpart for U_2 : $5MN - 2M + N$ variables and $8MN - 2M + 2N + 1$ constraints.
- For a portfolio of 500 stocks optimized over 6 time periods 14,006 variables and 23,016 constraints.

Approaches

1. Single Period Mean-Variance (MR1)
2. Single Period Robust using U_1 (SPR)
3. Multiperiod Robust with U_1 (MR1)
4. Multiperiod Robust with U_2 (MR2)
5. Multiperiod Nominal (MPN)

Questions

1. How do the different portfolio optimization approaches perform when single period returns are drawn from the same distribution at each time period, and when there is no noise, i.e., the simulated distributions have the same parameters as the ones used as input to the corresponding optimization problems?
2. How do the approaches perform when returns are drawn from distributions with different expected returns at every time period? Does the ability to "see ahead" help the multi-period approaches perform better?
3. How do the approaches perform when nature does not behave the way we expect probabilistically, e.g., when returns are drawn from different distributions than the ones specified at the beginning, or when the parameters of the distributions are perturbed?

Normal Single Period Returns

- $M = 3$ $\boldsymbol{\sigma} = [0.15, 0.20, 0.22]$, and their correlation matrix, which remains constant over time, is

$$Cor = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 1 & -0.2 \\ 0.7 & -0.2 & 1 \end{bmatrix}.$$

  The single period riskless return is 0.025.
- $N = 5$ and transaction costs are 1% of the amount traded, .
- Experiments 1 and 2: $\underline{\delta}_t^m$ and $\overline{\delta}_t^m$ in U_2 are both set equal to the expected value of the corresponding single period asset return
- In Experiment 1, the returns are simulated from the same multivariate normal distribution at every time period.
- In Experiments 2-4, single period returns are simulated from multivariate normal distributions with different expected values at each time period
- In Experiment 2, the upper and lower bounds in the MR1 formulation are equal, and are set to the expected values of the single period returns.

Normal Single Period Returns, continued

- Experiment 3 is the same as Experiment 2, but the upper and lower bounds in the MR1 formulation are set to expected values $\pm 50\%$ of the standard deviation of the corresponding stock.
- Experiment 4 is the same with Experiment 3, but returns are simulated from a multivariate normal distribution whose expected values are lower than the expected values used by the four algorithms by 50% of the standard deviation for the corresponding single period return.

Asset Returns

Trading Period t	Expected Returns		
	Stock 1	Stock 2	Stock 3
0	0.080	0.090	0.120
1	0.075	0.080	0.110
2	0.080	0.100	0.070
3	0.080	0.110	0.090
4	0.080	0.105	0.110

Results over 1000 simulations

Exp. No.	Method	Mean	StdDev	Min	Max	Ratio	Prob (%)
1	SMV	0.0985	0.0674	-0.0982	0.3204	1.46	7.40
	SPR	0.0978	0.0676	-0.1052	0.3227	1.45	7.50
	MR1	0.0986	0.0506	-0.0081	0.3255	1.95	0.10
	MPN	0.0972	0.0962	-0.1855	0.4105	1.01	15.90
2	SMV	0.0892	0.0667	-0.1241	0.2987	1.34	9.30
	SPR	0.0883	0.0694	-0.1488	0.2989	1.27	9.80
	MR1	0.0905	0.0387	0.0170	0.2657	2.34	0.00
	MPN	0.0831	0.0949	-0.2280	0.3780	0.88	19.10
3	SMV	0.0876	0.0664	-0.1454	0.3113	1.32	8.90
	SPR	0.0876	0.0694	-0.1675	0.2899	1.26	9.40
	MR1	0.0902	0.0370	0.0211	0.2415	2.44	0.00
	MPN	0.0852	0.0971	-0.3109	0.3569	0.88	18.40
4	SMV	0.0781	0.0670	-0.1242	0.3202	1.17	11.80
	SPR	0.0775	0.0693	-0.1299	0.3006	1.12	13.40
	MR1	0.0876	0.0393	0.0176	0.2892	2.23	0.00
	MPN	0.0720	0.0951	-0.2177	0.3496	0.76	22.30

Discussion

- MR1 achieves better average return, probability of loss, and mean-to-standard deviation ratio
- Dominance in the mean-to-standard deviation ratio is particularly important, because it shows that the risk of the portfolio is decreased at no cost to the expected portfolio return.
- Standard deviation is appropriate as a measure of risk in these experiments, because the return distributions are symmetric (normal).
- MR1 also has better worst-case performance.
- In Experiment 3, where the the upper and lower bounds on future stock returns are set to be 50% of the corresponding returns' standard deviations away from the returns' expected values (as opposed to expected values, as in Experiment 2), MR1 becomes more conservative, and its worst case scenario performance improves.

Asymmetric (Lognormal) Single Period Returns

- 25 stocks over 5 time periods
- Returns given by factor model

$$\begin{aligned} \ln(1 + r_t^m) &= \boldsymbol{\Omega}_m' \left[\kappa \cdot e + \sigma \cdot \boldsymbol{\nu}_t\right], \; t = 0, 1, \ldots, N-1, m = 1, \ldots, M \\ \ln(1 + r_t^0) &= \kappa, \; t = 0, 1, \ldots, N-1, \end{aligned}$$

where $\{\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{N-1}\}$ are independent K-dimensional Gaussian random vectors with zero mean and unit covariance matrix; $e \in \Re^K = (1, \ldots, 1)'$; $\boldsymbol{\Omega}_m \in \Re_+^K$ are fixed vectors; and $\kappa, \sigma > 0$ are fixed reals. Single period returns are therefore lognormal.

Experiments

- **Experiment 1**: The single period returns for each asset are assumed to be independent and identically distributed across time periods. The upper and lower bounds in MR1 are set to the expected values of the corresponding single period returns.
- **Experiment 2** is the same as Experiment 1, but the simulated returns are perturbed: after a realization of the vector $\boldsymbol{\nu}_t$ is obtained and the returns are computed, 10% of the value of each realized return is subtracted. The simulated values are therefore lower on average than the optimization problems "expect."
- **Experiment 3** is the same as Experiment 2, but the upper and lower bounds for each asset in the MR1 formulation are set to be 50% of the standard deviation of the corresponding asset.
- **Experiment 4**, the single period expected returns for the first time period are the same as the expected returns in Experiments 1-3, but the expected single period returns in later time periods are different. The MR1 and the MR2 "know" the expected returns more than one time period ahead. The upper and lower bounds for each asset in the MR1 formulation are set to the expected values of the single period returns.

Experiments, continued

- **Experiment 5** is the same as Experiment 4; however, 10% of the realized returns is subtracted in all simulations. The expected values for returns used as inputs in all optimization models, as well as the covariance matrices used in the formulation of MR2, are therefore not correct, so the models are misspecified. The upper and lower bounds for each asset return in the MR1 formulation are set to 50% of the standard deviation of the single period returns. The results of Experiment 5 are particularly important, because the setting of the experiment is the most realistic one.
- **Experiment 6** is the same as Experiment 1, but single period returns are drawn from a multivariate normal distribution (instead of a lognormal distribution) with the same expected value and single period covariance matrix as the single period lognormal distribution for returns.
- T, d-norm equal to $\sqrt{\text{Number of Stocks}}$ and $d = \text{Number of Stocks}$, and the values of λ and Δ change correspondingly.

Results

Exp. No.	Mhd	Mean	Std Dev	Ratio	Min	5th Per	50th Per	95th Per	Max	Prob (%)
1	SMV	0.1933	0.1396	1.39	-0.1983	-0.0255	0.1798	0.4283	0.7441	7.50
	SPR	0.1935	0.1535	1.26	-0.2285	-0.0387	0.1793	0.4637	0.8517	8.40
	MR1	0.2875	0.2338	1.23	-0.1866	-0.0130	0.2458	0.7497	1.4013	6.70
	MR2	0.3468	0.2598	1.34	-0.1135	0.0100	0.2964	0.8543	1.4306	4.70
	MPN	0.1805	0.1793	1.01	-0.2507	-0.0834	0.1660	0.5021	0.9159	15.90
2	SMV	0.0715	0.1252	0.57	-0.2199	-0.1176	0.0632	0.2951	0.5728	30.10
	SPR	0.0727	0.1359	0.53	-0.2689	-0.1289	0.0600	0.3152	0.6669	32.20
	MR1	0.1455	0.1971	0.74	-0.2386	-0.1121	0.1117	0.5098	1.3134	21.90
	MR2	0.1844	0.2058	0.90	-0.2320	-0.0889	0.1496	0.5493	1.0864	18.40
	MPN	0.0617	0.1598	0.39	-0.3128	-0.1739	0.0468	0.3491	0.6539	39.20
3	SMV	0.0709	0.1219	0.58	-0.2915	-0.1201	0.0657	0.2724	0.5364	29.90
	SPR	0.0702	0.1321	0.53	-0.2870	-0.1301	0.0634	0.3001	0.6871	31.20
	MR1	0.1357	0.1763	0.77	-0.2093	-0.0996	0.1102	0.4688	0.9867	22.60
	MR2	0.1774	0.2019	0.88	-0.2021	-0.0905	0.1494	0.5380	1.5102	17.60
	MPN	0.0553	0.1562	0.35	-0.3413	-0.1691	0.0435	0.3371	0.6691	38.40
4	SMV	0.1968	0.1623	1.21	-0.3010	-0.0647	0.1948	0.4738	0.6814	11.20
	SPR	0.1869	0.1839	1.02	-0.3858	-0.0973	0.1791	0.5112	0.7703	15.10
	MR1	0.6941	0.4544	1.53	-0.2022	0.0899	0.6129	1.5331	2.7689	1.90
	MR2	0.2872	0.2014	1.43	-0.1629	0.0009	0.2577	0.6504	1.5458	4.80
	MPN	0.1642	0.2069	0.79	-0.3493	-0.1578	0.1557	0.5184	0.9441	22.40
5	SMV	0.0733	0.1552	0.47	-0.3266	-0.1579	0.0588	0.3403	0.8522	33.20
	SPR	0.0623	0.1703	0.37	-0.3687	-0.1903	0.0460	0.3635	0.8555	38.90
	MR1	0.4212	0.3794	1.11	-0.2355	-0.0158	0.3404	1.0652	2.9457	6.60
	MR2	0.1426	0.1818	0.78	-0.2280	-0.0973	0.1151	0.4712	1.2849	21.20
	MPN	0.0502	0.1950	0.26	-0.4059	-0.2256	0.0306	0.3878	1.0548	43.70
6	SMV	0.2312	0.1376	1.68	-0.2354	-0.0046	0.2369	0.4454	0.5953	5.40
	SPR	0.2278	0.1831	1.24	-0.3578	-0.0721	0.2324	0.5214	0.7698	11.50
	MR1	0.3239	0.2934	1.10	-0.6136	-0.1254	0.3054	0.8267	1.3822	12.50
	MR2	0.4399	0.2956	1.49	-0.3519	-0.0042	0.4140	0.9790	1.5433	5.10
	MPN	0.1743	0.2794	0.62	-0.7380	-0.3052	0.1796	0.6384	1.1475	26.40

Pairwise Comparisons-% row policy beats column policy

	Exp. 1, $d = \sqrt{\text{Number of Stocks}}$					Exp. 1, $d =$ Number of Stocks				
	SMV	SPR	MR1	MR2	MPN	SMV	SPR	MR1	MR2	MPN
SMV		47.4	28.4	12.5	52.1		42.5	25.4	2.7	51.4
SPR	52.6		26.6	6.9	54.3	57.5		27.3	2.6	54.5
MR1	71.6	73.4		37.6	79.4	74.6	72.7		36.5	79.6
MR2	87.5	93.1	62.4		94.1	97.3	97.4	63.5		90.0

	Exp. 2, $d = \sqrt{\text{Number of Stocks}}$					Exp. 2, $d =$ Number of Stocks				
	SMV	SPR	MR1	MR2	MPN	SMV	SPR	MR1	MR2	MPN
SMV		45.0	28.9	12.6	52.5		41.3	29.1	5.5	53.5
SPR	55.0		26.7	6.5	54.1	58.7		29.7	5.5	57.2
MR1	71.1	73.3		37.6	79.7	70.9	70.3		33.5	80.1
MR2	87.4	93.5	62.4		91.2	94.5	94.5	66.5		87.4

	Exp. 3, $d = \sqrt{\text{Number of Stocks}}$					Exp. 3, $d =$ Number of Stocks				
	SMV	SPR	MR1	MR2	MPN	SMV	SPR	MR1	MR2	MPN
SMV		44.4	27.4	13.9	55.9		42.5	9.5	4.5	47.8
SPR	55.6		24.9	7.7	58.0	57.5		10.6	5.0	51.7
MR1	72.6	75.1		31.4	83.1	90.5	89.4		31.8	83.9
MR2	86.1	92.3	68.6		93.5	95.5	95.0	68.2		86.9

Pairwise Comparisons

	Exp. 4, $d = \sqrt{\text{Number of Stocks}}$					Exp. 4, $d = \text{Number of Stocks}$				
	SMV	SPR	MR1	MR2	MPN	SMV	SPR	MR1	MR2	MPN
SMV		58.7	1.2	31.9	56.9		50.1	0.0	31.2	54.9
SPR	41.3		0.5	31.4	53.5	49.9		0.1	31.6	55.7
MR1	98.8	99.5		88.1	96.9	100.0	99.9		88.0	96.0
MR2	68.1	68.6	11.9		73.7	68.8	68.4	12.0		71.7

	Exp. 5, $d = \sqrt{\text{Number of Stocks}}$					Exp. 5, $d = \text{Number of Stocks}$				
	SMV	SPR	MR1	MR2	MPN	SMV	SPR	MR1	MR2	MPN
SMV		59.4	0.9	35.0	56.3		52.9	0.0	33.8	55.6
SPR	40.6		0.0	32.7	53.5	47.1		0.2	35.1	56.0
MR1	99.1	100.0		85.3	93.2	100.0	99.8		82.9	91.2
MR2	65.0	67.3	14.7		71.6	66.2	64.9	17.1		67.7

	Exp. 6, $d = \sqrt{\text{Number of Stocks}}$					Exp. 6, $d = \text{Number of Stocks}$				
	SMV	SPR	MR1	MR2	MPN	SMV	SPR	MR1	MR2	MPN
SMV		48.6	35.6	16.6	59.7		46.9	34.4	7.0	58.2
SPR	51.4		31.8	9.1	58.9	53.1		35.0	6.4	62.2
MR1	64.4	68.2		30.0	81.3	65.6	65.0		25.4	81.4
MR2	83.4	90.9	70.0		93.7	93.0	93.6	74.6		92.6

Discussion

- MR1 and MR2 both have excellent performance in all experiments,
- MR2 tends to do better than MR1 when returns are identically distributed in all time periods (Experiments 1, 2, 3, and 6).
- MR1, on the other hand, performs extremely well when returns are not identically distributed, or when parameters in the model are misspecified, as is the case in Experiments 4 and 5. Moreover, it appears that its worst case performance can be improved by widening the bounds on returns in the formulation. This can be seen by comparing MR1's worst case performance in Experiments 2 and 3.

Conclusions

- Robust modeling: Systematic way for modeling.
- Use of cumulative returns.
- Tractability of RO in practice.