15.095 Machine Learning Under a Modern Optimization Lens

Lecture 12: Neural Networks and Trees

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What Are Neural Networks?

- Neural networks, a supervised learning technique, have become one of the most widely used machine learning techniques today
- Increased computational power, advances in optimization (stochastic gradient methods), and the massive availability of data sets have made it possible to train large neural networks with many hidden layers
- This methodology, in particular, is known as deep learning
- It has had great successes in the fields of image recognition, natural language processing, and speech recognition

A Sample Neural Network

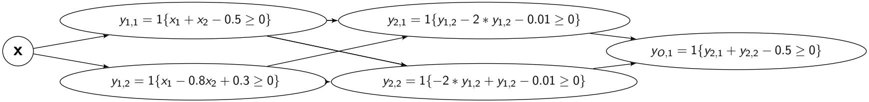

Figure 1: An example of a neural network.

Challenges with Neural Networks

- They rely on heuristics in their training process, like dropout and early stopping
- While neural networks often work well, it is unclear when they work well, why they work well, and if they do not work well how to improve them
- Importantly, given that they have thousands to tens of thousands of parameters, they are not interpretable by humans

Optimal Decision Trees

- However, there has recently been significant progress in finding trees that are near optimal, as discussed in the paper *Optimal Classification Trees*, by Bertsimas and Dunn (2017)
- Using mixed integer optimization and local search methods the authors find optimal classification trees (OCT) that significantly improve upon CART
- Furthermore, their approach allows one to consider hyperplane splits, leading to optimal classification trees with hyperplanes (OCT-Hs), which generalize support vector machines.
- OCT-Hs are less interpretable than trees whose splits rely on only one variable (OCTs), but are still more interpretable than neural networks

Sample Tree with Hyperplane Splits

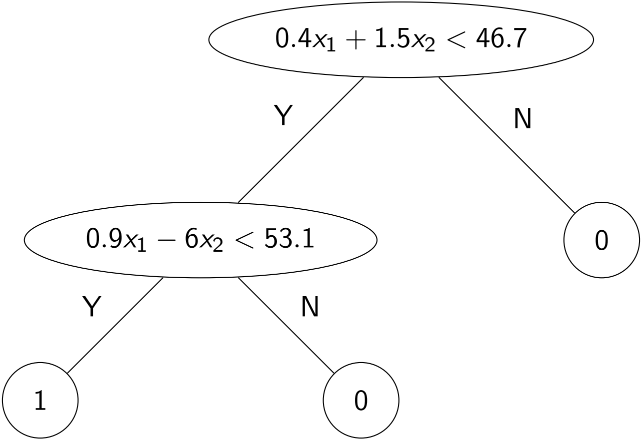

Figure 2: A sample decision tree with hyperplane splits.

Goals

- We investigate the modeling power of neural networks in comparison with OCT-Hs
- We prove that a set of neural networks can be transformed to classification trees with hyperplane splits with the same accuracy in the training set, showing that OCT-Hs are at least as powerful as neural networks
- Conversely, a given classification tree with hyperplane splits can be transformed to a classification neural network with the same accuracy in the training set, showing that these neural networks are at least as powerful as OCT-Hs
- Consequently, we show that OCT-Hs and neural networks are equivalent in terms of modeling power

Implications

- These results link two of the most popular and widely utilized machine learning methods, shedding new light on their strengths and weaknesses
- Given that OCT-Hs have an edge in interpretability compared to neural networks, without loss of modeling power, decision trees might be the method of choice in applications where interpretability matters
- Given the success of stochastic gradient methods in neural networks, it might be worthwhile to investigate their application in the design of optimal trees
- Conversely, given the success of mixed integer optimization and local search methods in optimal trees, it might be worthwhile to investigate their application in neural networks

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Neural Network Structure Overview

- A neural network's architecture is defined by
  - L hidden layers, indexed $\ell = 1, \ldots, L$, and one output layer
  - Hidden layer ℓ consisting of N_ℓ nodes, indexed $i = 1, \ldots, N_\ell$
  - Some non-linear function $\phi(x)$ associated with the hidden layers
  - Some function $\phi_O(x)$ associated with the output layer
- Each node $n_{\ell,i}$ in the neural network has associated weight vector $\mathbf{W}_{\ell,i}$ and bias scalar $b_{\ell,i}$
- After deciding on the values for the architecture parameters, we solve for the weight vectors and bias scalars using stochastic gradient descent
- An example of a neural network can be seen in Figure 3

Sample FNN

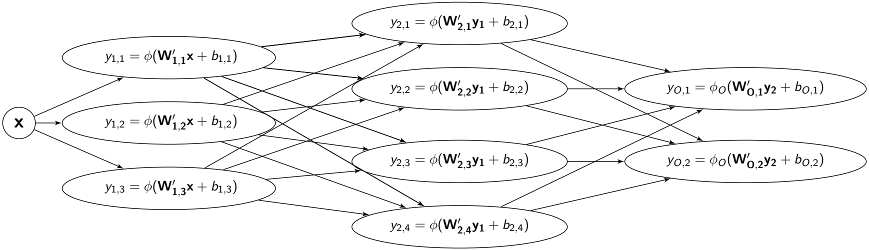

Figure 3: An example of a classification feedforward neural network. It has 2 hidden layers, with 3 nodes in the first hidden layer, 4 nodes in the second, and 2 nodes in the output layer. Each node in this network has its own unique weights $\mathbf{W}_{\ell,i}$ and $b_{\ell,i}$. Also, the nodes in one layer have directed edges leading to all the nodes in the next layer (a trait known as being "fully connected").

Feedforward Neural Network Characteristics (1)

- Node $n_{\ell,i}$ in hidden layer ℓ calculates

$$y_{\ell,i} = \phi(\mathbf{W}_{\ell,i}^T \mathbf{y}_{\ell-1} + b_{\ell,i}), \tag{1}$$

- Here $\phi(x)$ is a nonlinear function, and $\mathbf{y}_{\ell-1}$ is the vector of outputs of the hidden layer $\ell - 1$
- We define $\mathbf{y}_0 \triangleq \mathbf{x}$, the input of the FNN
- Common choices are for $\phi(x)$ are
  1. $1\{x \geq 0\}$, the perceptron function
  2. $\max(x, 0)$, the ReLU function

Feedforward Neural Network Characteristics (2)

- Node $n_{O,i}$ in the output layer calculates

$$y_{O,i} = \phi_O(\mathbf{W}_{O,i}^T \mathbf{y}_L + b_{O,i}) \tag{2}$$

- The only difference between $\phi(x)$ and $\phi_O(x)$ is that $\phi_O(x)$ does not have to be non-linear
- If the perceptron is chosen as the activation function, then typically

$$\phi_O(\mathbf{x}) = 1\{\mathbf{x} \geq 0\}$$

- If the ReLU function is chosen as the activation function, then $\phi_O(\mathbf{x})$ can be defined as

$$(\phi_O(\mathbf{x}))_i = \begin{cases} 1, & \text{where } i = \text{argmax}_{i=1,\ldots,N_0}(x_i), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Feedforward Neural Network Characteristics (3)

- The final prediction of the network is found by calculating $k = \text{arg-lex-max}_{i=1,\ldots,q}(y_{O,i})$
- The lexicographic maximum means that if there is a tie, the smallest index k is our choice
- We then use k as our predicted class value

Tree Parameters (1)

- A tree has depth N_1 if N_1 is the maximum number of split nodes in a tree one visits before reaching a leaf node that contains an output value
- The maximal tree of depth N_1 has $T = 2^{N_1+1} - 1$ nodes
- Nodes 1 through $\lfloor T/2 \rfloor$ of this maximal tree are split nodes, otherwise known as branch nodes, while nodes $\lfloor T/2 \rfloor + 1$ through T are leaf nodes
- Each branch node i, $i = 1, \ldots, \lfloor T/2 \rfloor$ is assigned weight vector and bias scalar $\mathbf{w}_i, b_i$

Tree Parameters (2)

- Given input $\mathbf{x}$, at a given node i we calculate $\mathbf{w}_i^T\mathbf{x} + b_i$
- If $\mathbf{w}_i^T\mathbf{x} + b_i < 0$, we take the left branch of the split to a new tree node; otherwise, we take the right branch
- Once we have passed through at most N_1 different nodes, we arrive at a leaf node.
- Each leaf node is assigned a classification value $k \in \{1, \dots, q\}$ that it uses as the predicted class for all points sorted to it
- Thus, if $\mathbf{x}$ is assigned to leaf node r with classification value k_r, $r \in \{\lfloor T/2 \rfloor + 1, \dots, T\}$ and $k_r \in \{1, \dots, q\}$, the network outputs k_r as the classification value for $\mathbf{x}$

Example Tree

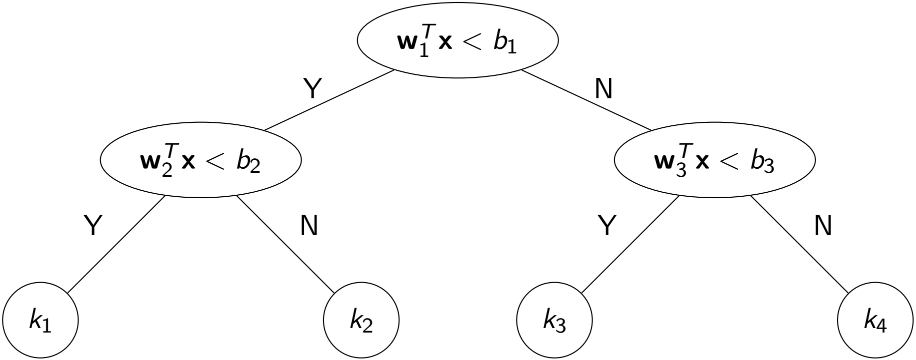

Figure 4: An OCT-H of depth 2. Data in the four leaf nodes are classified as k_1, k_2, k_3, and k_4.

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Transforming a FNN with the Perceptron Activation Function into a DT

- Given a FNN $\mathcal{N}_1$ with the perceptron activation function, we are able to construct a decision tree $\mathcal{T}_1$ with hyperplane splits and maximum depth N_1 that makes the same predictions as $\mathcal{N}_1$
- This construction relies on the fact that a FNN with the perceptron activation function and N_1 nodes in the first hidden layer has at most 2^{N_1} distinct output values
- We can assign these values to the 2^{N_1} leaf nodes of $\mathcal{T}_1$
- Since we know that an OCT-H must classify training data at least as well as $\mathcal{T}_1$, we know that an OCT-H must do at least as well as $\mathcal{N}_1$ too
- This leads to the following theorem

FNN with Perceptron to DT Theorem

Theorem 1

An OCT-H with maximum depth N_1 can classify the data in a training set at least as well as a given classification FNN with the perceptron activation function and N_1 nodes in the first hidden layer.

regression?
using finiteness?

Constructing $\mathcal{T}_1$ (FNN Perceptron)

- We are given a feedforward neural network $\mathcal{N}_1$ with the following characteristics:
  - The perceptron activation function, defined as $\phi(x) = 1\{x \geq 0\}$
  - Output function $\phi_O(\mathbf{x}) : [0,1]^q \to [0,1]^q$
  - L hidden layers and one output layer, indexed $\ell = 1, \ldots, L, O$
  - N_ℓ nodes in each layer, indexed $i = 1, \ldots, N_\ell$
  - Node $n_{\ell,i}$ defined by $\mathbf{W}_{\ell,i}, b_{\ell,i}$
- Given the inequality from the first node in the first hidden layer of $\mathcal{N}_1$ defined by the weight vector $\mathbf{W}_{1,1}$ and bias scalar $b_{1,1}$, we define the first split of $\mathcal{T}_1$ as

$$\mathbf{W}_{1,1}^T \mathbf{x} + b_{1,1} < 0 \tag{4}$$

- This results in the simple split seen in Figure 5

The First Split of $\mathcal{T}_1$

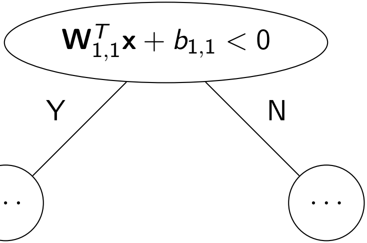


Figure 5: The first split of decision tree $\mathcal{T}_1$.

Continuing building $\mathcal{T}_1$

- Independent of whether inequality (4) is satisfied or not, the second split is given by
$$\mathbf{W}_{1,2}^T\mathbf{x} + b_{1,2} < 0 \tag{5}$$
- We continue this process for all N_1 nodes in the first hidden layer, building a decision tree of depth N_1, with every split at depth N_1 being given by
$$\mathbf{W}_{1,N_1}^T\mathbf{x} + b_{1,N_1} < 0 \tag{6}$$
- This results in the subtree seen in Figure 6

Completed Branches of $\mathcal{T}_1$

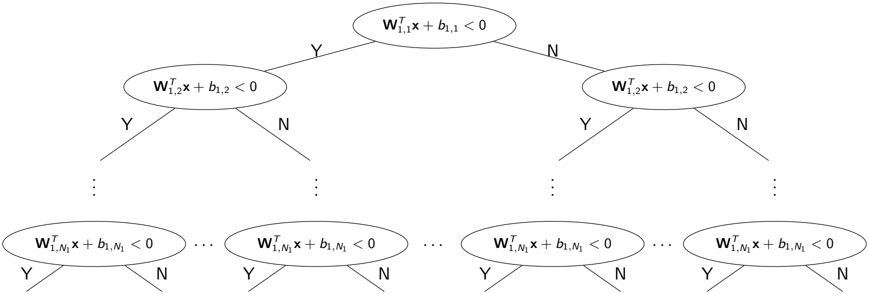


Figure 6: The decision tree $\mathcal{T}_1$. We still need to define the classification values assigned to the leaf nodes.

Defining the Leaf Nodes of $\mathcal{T}_1$

- To complete the construction of $\mathcal{T}_1$, we need to assign a classification value to every leaf node
- Given an input $\mathbf{x}$ of $\mathcal{N}_1$, there are 2^{N_1} possible binary vectors that the first hidden layer of $\mathcal{N}_1$ could output
- These 2^{N_1} vectors, by our construction of $\mathcal{T}_1$, exactly correspond to the 2^{N_1} leaves of $\mathcal{T}_1$
- Given $\mathbf{W}_{\ell,i}$, $b_{\ell,i}$, and $\mathbf{y}_1^r$ (the output of the first hidden layer associated with leaf node r) one can deterministically calculate the the final prediction of $\mathcal{N}_1$, $k(\mathbf{y}_1^r)$, by using the process outlined in the section Overview of Neural Networks
- In every node r of the tree we assign the classification value $k(\mathbf{y}_1^r)$ associated with the corresponding first hidden layer output

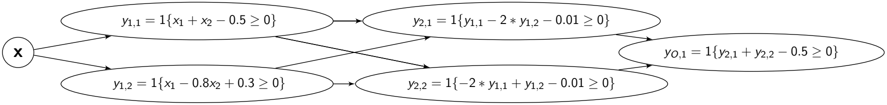
Proving that $\mathcal{T}_1$ and $\mathcal{N}_1$ make the same predictions

- To see that the output of $\mathcal{T}_1$ is the same as the output of $\mathcal{N}_1$, note that if $\mathbf{x}$ is input into $\mathcal{N}_1$, the first hidden layer outputs $\mathbf{y}_1(\mathbf{x})$
- This results in the final network output $k(\mathbf{y}_1)$
- However, in the decision tree, $\mathbf{x}$ is assigned to the leaf node corresponding to $\mathbf{y}_1^r = \mathbf{y}_1(\mathbf{x})$
- There, it is once again assigned output value $k(\mathbf{y}_1^r) = k(\mathbf{y}_1)$ by construction
- Thus, for a given data point $\mathbf{x}$, the network and the tree predict the same classification value
- Since an OCT-H does at least as well as $\mathcal{T}_1$ in classifying the training data, it must do at least as well as $\mathcal{N}_1$ too, completing the proof of the theorem

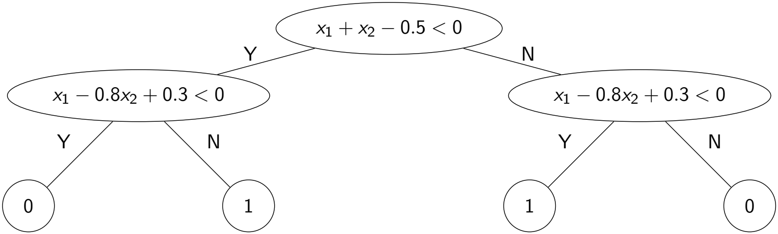
Notes on Theorem 1

- The construction of tree $\mathcal{T}_1$ is independent of L, the number of hidden layers
- While the output function $\phi_O(\mathbf{x})$ affects the values for classification, the construction of $\mathcal{T}_1$ is not affected by $\phi_O(\mathbf{x})$; only the output values of the leaves of $\mathcal{T}_1$ are affected by $\phi_O(\mathbf{x})$

Example: An NN and the Equivalent Tree



(a)



(b)

Figure 7: (a) An FNN with the perceptron activation function performing an XOR operation. (b) The corresponding decision tree.

Transforming a FNN with the ReLU Activation Function into a DT

- Next, we extend Theorem 1 to the case of FNNs with the rectified linear unit activation function (where $\phi(x) = \max(x, 0)$)
- Other than the new activation function, the only other difference in network architecture from the perceptron case is that we assume we use the output function defined in Equation (3)
- We can extend the theorem by constructing a decision tree with maximum depth $q - 1 + \sum_{i=1}^{L} N_\ell$ that makes the same predictions as the given neural network $\mathcal{N}_2$

FNN with ReLU to DT Theorem

Theorem 2

An OCT-H with maximum depth $q - 1 + \sum_{\ell=1}^{L} N_\ell$ can classify data in a training set at least as well as a given classification FNN with the rectified linear unit activation function, L hidden layers, N_ℓ nodes in layer $\ell = 1, \dots, L$, and q nodes in the output layer.

The First Sub-Tree of $\mathcal{T}_2$

- The first subtree is built exactly as it was in the case of an FNN with the perceptron activation function, and can be seen in Figure 8

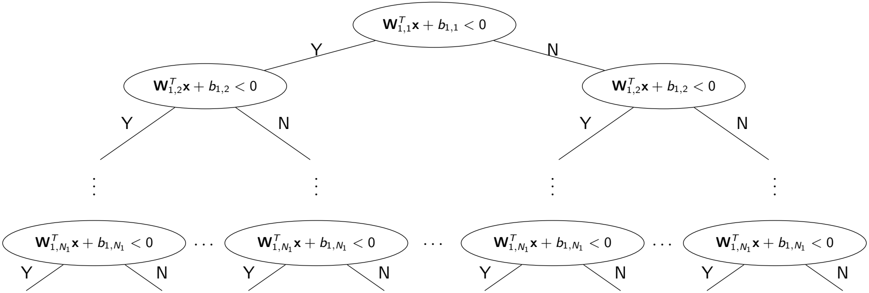


Figure 8: The decision tree $\mathcal{T}_2$ we are building up to depth N_1.

Constructing the Second Sub-Tree of $\mathcal{T}_2$

- After depth N_1, there are 2^{N_1} branches
- These branches correspond to the 2^{N_1} possible output vectors $\mathbf{y}_1$ of the first layer of $\mathcal{N}_2$,

$$(0, \ldots, 0)^T, (\mathbf{W}_{1,1}^T \mathbf{x} + b_{1,1}, \ldots, 0)^T, \ldots,$$
$$(\mathbf{W}_{1,1}^T \mathbf{x} + b_{1,1}, \ldots, \mathbf{W}_{1,N_1}^T \mathbf{x} + b_{1,N_1})^T$$

- We model the second layer of $\mathcal{N}_2$ by constructing after each branch of the first subtree a new subtree of depth N_2 as in Figure 8, but with the corresponding value of $\mathbf{y}_1$ playing the role of $\mathbf{x}$.

The Second Sub-Tree of $\mathcal{T}_2$

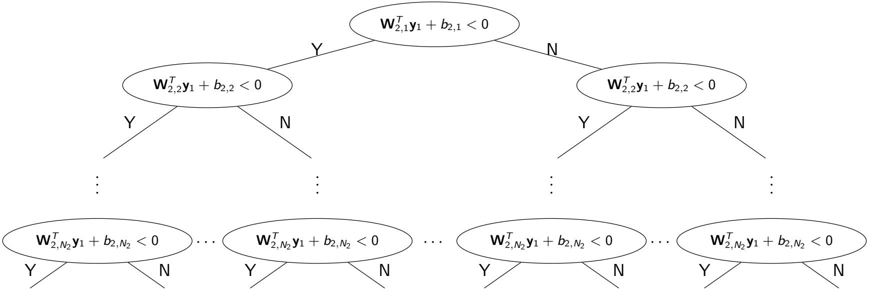


Figure 9: Subtree $\mathcal{T}_{2,2}(\mathbf{y}_1)$ of depth N_2 is concatenated to the corresponding branch of the subtree depicted in the previous slide, resulting in a subtree of depth $N_1 + N_2$.

The Second Sub-Tree of $\mathcal{T}_2$ Example

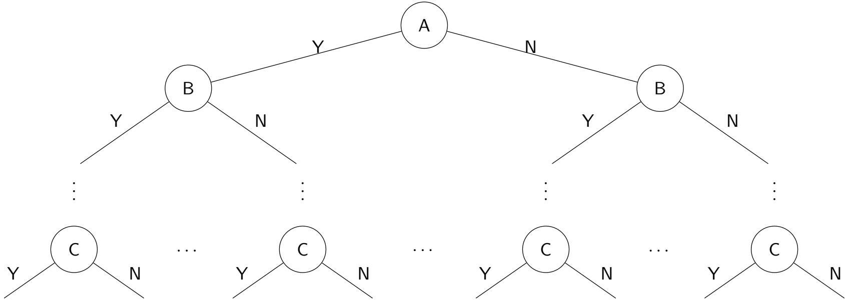


Figure 10: The resulting subtree $\mathcal{T}_{2,2}(\mathbf{y}_1)$ for $\mathbf{y}_1 = (0, \ldots, \mathbf{W}_{1,N_1}^T\mathbf{x} + b_{1,N_1})$. A, B, C are as follows:

- A is $\mathbf{W}_{2,1}^T(0, \ldots, \mathbf{W}_{1,N_1}^T\mathbf{x} + b_{1,N_1})^T + b_{2,1} < 0$.
- B is $\mathbf{W}_{2,2}^T(0, \ldots, \mathbf{W}_{1,N_1}^T\mathbf{x} + b_{1,N_1})^T + b_{2,2} < 0$.
- C is $\mathbf{W}_{2,N_2}^T(0, \ldots, \mathbf{W}_{1,N_1}^T\mathbf{x} + b_{1,N_1})^T + b_{2,N_2} < 0$.

Building the Final Sub-Tree of $\mathcal{T}_2$

- This process continues for the remaining L hidden layers of the network
- We then need to model the output layer, which calculates

$$\text{argmax}_{i=1,\ldots,q}(\mathbf{W}_{O,i}^T \mathbf{y}_L + b_{O,i}) \tag{7}$$

- We simulate this calculation by building a new subtree $\mathcal{T}_{2,O}(\mathbf{y}_L)$, where at each branch we perform some pairwise comparison of the entries of the vector

$$\mathbf{W}_O^T \mathbf{y}_L + \mathbf{b}_O$$

First branches of the Final Sub-Tree of $\mathcal{T}_2$

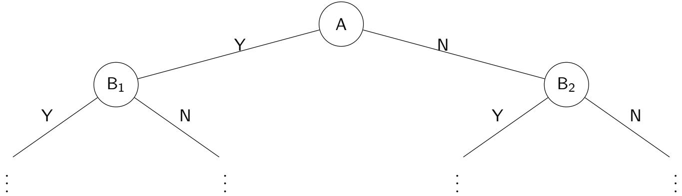


Figure 11: A portion of the subtree $\mathcal{T}_{2,O}(\mathbf{y}_L)$. The labels A, B_1, B_2 are as follows:

- A is $(\mathbf{W}_{O,1} - \mathbf{W}_{O,2})^T \mathbf{y}_L + b_{O,1} - b_{O,2} < 0$
- B_1 is $(\mathbf{W}_{O,2} - \mathbf{W}_{O,3})^T \mathbf{y}_L + b_{O,2} - b_{O,3} < 0$
- B_2 is $(\mathbf{W}_{O,1} - \mathbf{W}_{O,3})^T \mathbf{y}_L + b_{O,1} - b_{O,3} < 0$

Completing $\mathcal{T}_2$

- Each branch of the tree explicitly calculates which output node outputs the highest value (using a lexicographic decision rule in case of ties)
- We can then assign the class associated with that node as the output for the appropriate leaf nodes of $\mathcal{T}_{2,O}(\mathbf{y}_L)$
- Since at each branch we know $\mathbf{y}_L$ as an explicit linear function of $\mathbf{x}$, we also have that $\mathcal{T}_{2,O}(\mathbf{y}_L)$ is a decision tree where all inequalities are explicitly written linear functions of $\mathbf{x}$
- Once we append it to $\mathcal{T}_2$, by construction we have a decision tree that makes the same predictions as $\mathcal{N}_2$
- Since an OCT-H does at least as well as $\mathcal{T}_2$ in classifying the training data, it must do at least as well as $\mathcal{N}_2$ too, completing the proof of the theorem

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Transforming a Classification DT into a Classification FNN

- We have shown that it is possible to take a neural network and find an equivalent decision tree
- Here we show that the converse is also possible, specifically that one can take a decision tree and find an equivalent FNN with the perceptron activation function

Classification DT to Classification FNN Theorem

Theorem 3

A neural network with perceptron activation functions, two hidden layers, N_1 nodes in the first hidden layer, and N_2 nodes in the second can classify training data at least as well as a given classification decision tree with N_1 split nodes and N_2 leaf nodes.

Example Decision Tree

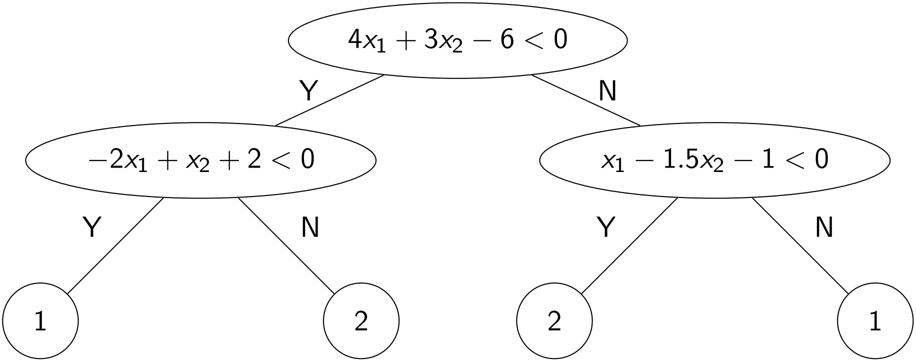

Figure 12: A decision tree that outputs 1 or 2.

The Corresponding Neural Network

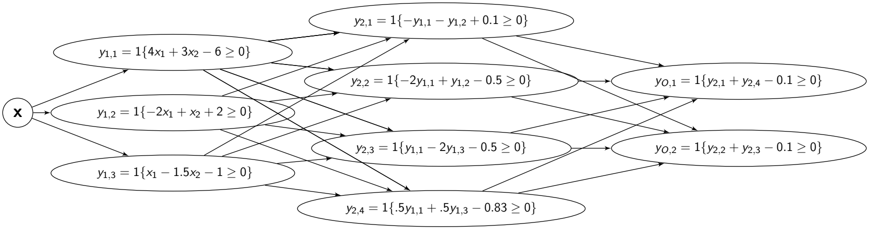

Figure 13: A neural network that performs the same predictions as the decision tree.

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The Data

- We examined the performance of FNNs and OCT-Hs on seven datasets
- We trained several different formulations of each model, and compared their out-of-sample accuracies
- The Data Sets are

Dataset	# Parameters	# Class Values	# Data Points
Bank Marketing	17	2	45,211
Framingham heart study	15	2	3,658
Image Segmentation	18	7	210
Letter Recognition	16	26	20,000
Magic Gamma Telescope	10	2	19,020
Skin Segmentation	3	2	245,057
Thyroid Disease ANN	21	3	3772

Table 1: The datasets used and their parameters.

Computational Result Models – Trees

- To train the optimal trees, we used the Optimal Tree software from Jack Dunn
- For each data set, we trained and cross validated OCTs and OCT-Hs of varying depths, which was automatically done by the software
- Once we had the best OCT and OCT-H based on the validation process, we calculated the models' accuracy in classifying the data in the test set – this out-of-sample accuracy is included as the "Accuracy" value in the table
- The tree depth is listed in the column labeled "Size Parameters"

Computational Result Models – Neural Networks

- To train the neural networks, we used TensorFlow code
- We used sigmoid functions here as the activation functions to make the networks easier to train, as they are a continuous approximation of the perceptron activation functions
- We then trained and validated neural networks with sizes based on the tree depths we used, based on the proof in the Transforming a Decision Tree into a Neural Network section
- For example, a maximal tree with depth 2 has 3 split nodes and 4 leaf nodes, so for a neural network built with a size based on the tree we would have $N_1 = 3$ and $N_2 = 4$

Computational Result Models – Neural Network Training Parameters

- The N_1 and N_2 values are listed in the column labeled "Size Parameters"
- We validated the networks using the training parameters

$$\text{Step sizes} \in \{0.001,\ 0.01,\ 0.1,\ 1\}$$

and

$$\text{Regularization coefficients} \in \{1 \times 10^{-6},\ 1 \times 10^{-5}, \ldots, 0.1,\ 1\}$$

Computational Result Models – Neural Network Optimization

- For each network size, we trained 50 neural networks with different random starts using grid search for the parameters
- We then used the parameters with the best performance on a validation set to obtain the models' accuracy in classifying the data in the test set
- This out-of-sample accuracy is included as the "Accuracy" value in the table

Computational Results – Abridged

Model	Dataset	Parameters	Test Results
FNN	MGT	$N_1 = 15,\ N_2 = 16,\ q = 2$	87.5 %
FNN	MGT	$N_1 = 31,\ N_2 = 32,\ q = 2$	88.4 %
FNN	MGT	$N_1 = 63,\ N_2 = 64,\ q = 2$	88.1%
FNN	MGT	$N_1 = 255,\ N_2 = 256,\ q = 2$	88.3 %
OCT	MGT	maximum depth $= 4$, chosen depth 4	84.1 %
OCT	MGT	maximum depth $= 6$, chosen depth 6	85.3 %
OCT	MGT	maximum depth $= 8$, chosen depth 8	85.7 %
OCT-H	MGT	maximum depth $= 4$, chosen depth 4	86.7 %
OCT-H	MGT	maximum depth $= 6$, chosen depth 5	88.6 %
OCT-H	MGT	maximum depth $= 8$, chosen depth 5	87.0 %
FNN	Letter Recognition	$N_1 = 15,\ N_2 = 16,\ q = 26$	58.6 %
FNN	Letter Recognition	$N_1 = 63,\ N_2 = 64,\ q = 26$	66.8%
FNN	Letter Recognition	$N_1 = 255,\ N_2 = 256,\ q = 26$	67.0 %
OCT	Letter Recognition	maximum depth $= 4$, chosen depth 4	37.9 %
OCT	Letter Recognition	maximum depth $= 6$, chosen depth 6	59.1 %
OCT	Letter Recognition	maximum depth $= 8$, chosen depth 8	68.2 %
OCT-H	Letter Recognition	maximum depth $= 4$, chosen depth 4	44.6 %
OCT-H	Letter Recognition	maximum depth $= 6$, chosen depth 6	72.0 %
OCT-H	Letter Recognition	maximum depth $= 8$, chosen depth 8	80.3 %

Computational Result Conclusions

- In six out of the seven datasets (the exception is Letter Recognition) the FNN and the OCT-H have very similar accuracy
- Moreover, in these datasets OCT and OCT-H also have very similar accuracy
- The accuracy of the FNN was relatively insensitive to the size of the network and similarly the accuracy of the OCT-H was relatively insensitive to the depth of the OCT-H.
- In Letter Recognition OCT-H has a performance edge both with respect to FNN and OCT

Computational Result Implications

- We have proven in the paper that OCT-Hs and FNNs are equivalent in terms of power
- However, the proofs require trees of large depths
- These empirical results provide preliminary evidence that OCT-Hs (and OCTs) have comparable performance even with small depth
- This indicates that there is indeed practical merit to use OCT-Hs in applications
- The fact that OCTs give similar results to FNNs is particularly noteworthy as OCTs are very interpretable

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Concluding Remarks

- We have shown that optimal decision trees are at least as powerful as neural networks in terms of modeling power
- In the case of classification problems OCT-Hs and NNs have the same modeling power
- While our constructions require deep trees that may be impractical to compute, we have also found that in seven data sets the modeling power of OCT-Hs and FNNs is indeed very similar even if the trees have small depth
- While more empirical research is needed, we feel these findings are promising as OCT-Hs and especially OCTs are more interpretable than FNNs
- They bring us closer to a significant objective of machine learning to build interpretable models with state of the art performance

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Other Computational Results (1)

Model	Dataset	Parameters	Test Results
FNN	Bank	$N_1 = 7,\ N_2 = 8,\ q = 2$	89.6 %
FNN	Bank	$N_1 = 15,\ N_2 = 16,\ q = 2$	89.4 %
FNN	Bank	$N_1 = 63,\ N_2 = 64,\ q = 2$	89.6%
FNN	Bank	$N_1 = 255,\ N_2 = 256,\ q = 2$	89.6%
OCT	Bank	maximum depth = 4, chosen depth 4	89.3%
OCT	Bank	maximum depth = 6, chosen depth 6	89.6%
OCT	Bank	maximum depth = 8, chosen depth 6	89.6 %
OCT-H	Bank	maximum depth = 4, chosen depth 3	89.6 %
OCT-H	Bank	maximum depth = 6, chosen depth 3	89.6 %
OCT-H	Bank	maximum depth = 8, chosen depth 3	89.6 %
FNN	Framingham	$N_1 = 3,\ N_2 = 4,\ q = 2$	82.1%
FNN	Framingham	$N_1 = 15,\ N_2 = 16,\ q = 2$	82.1 %
FNN	Framingham	$N_1 = 63,\ N_2 = 64,\ q = 2$	82.1%
FNN	Framingham	$N_1 = 255,\ N_2 = 256,\ q = 2$	81.7%
OCT	Framingham	maximum depth = 6, chosen depth 6	83.1%
OCT	Framingham	maximum depth = 8, chosen depth 8	82.4%
OCT-H	Framingham	maximum depth = 4, chosen depth 2	83.3%
OCT-H	Framingham	maximum depth = 6, chosen depth 2	83.3%
OCT-H	Framingham	maximum depth = 8, chosen depth 2	83.3%

Other Computational Results (2)

Model	Dataset	Parameters	Test Results
FNN	Image Segregation	$N_1 = 15,\ N_2 = 16,\ q = 7$	88.4 %
FNN	Image Segregation	$N_1 = 31,\ N_2 = 32,\ q = 7$	83.7 %
FNN	Image Segregation	$N_1 = 63,\ N_2 = 64,\ q = 7$	83.7%
FNN	Image Segregation	$N_1 = 255,\ N_2 = 256,\ q = 7$	83.7 %
OCT	Image Segregation	maximum depth = 4, chosen depth 4	88.4%
OCT	Image Segregation	maximum depth = 6, chosen depth 6	88.4%
OCT	Image Segregation	maximum depth = 8, chosen depth 6	88.4 %
OCT-H	Image Segregation	maximum depth = 4, chosen depth 4	86.0%
OCT-H	Image Segregation	maximum depth = 6, chosen depth 5	86.0%
OCT-H	Image Segregation	maximum depth = 8, chosen depth 5	86.0 %
FNN	Skin Segmentation	$N_1 = 15,\ N_2 = 16,\ q = 2$	99.9 %
FNN	Skin Segmentation	$N_1 = 63,\ N_2 = 64,\ q = 2$	99.9%
FNN	Skin Segmentation	$N_1 = 127,\ N_2 = 128,\ q = 2$	99.9 %
FNN	Skin Segmentation	$N_1 = 255,\ N_2 = 256,\ q = 2$	99.9 %
OCT	Skin Segmentation	maximum depth = 4, chosen depth 4	98.9%
OCT	Skin Segmentation	maximum depth = 6, chosen depth 6	99.8 %
OCT	Skin Segmentation	maximum depth = 8, chosen depth 8	99.9 %
OCT-H	Skin Segmentation	maximum depth = 4, chosen depth 4	99.9 %
OCT-H	Skin Segmentation	maximum depth = 6, chosen depth 6	99.9 %
OCT-H	Skin Segmentation	maximum depth = 8, chosen depth 7	99.9 %

Other Computational Results (3)

Model	Dataset	Parameters	Test Results
FNN	Thyroid	$N_1 = 7,\ N_2 = 8,\ q = 3$	97.7%
FNN	Thyroid	$N_1 = 15,\ N_2 = 16,\ q = 3$	98.1 %
FNN	Thyroid	$N_1 = 63,\ N_2 = 64,\ q = 3$	97.4%
FNN	Thyroid	$N_1 = 255,\ N_2 = 256,\ q = 3$	98.0%
OCT	Thyroid	maximum depth = 4, chosen depth 4	99.7 %
OCT	Thyroid	maximum depth = 6, chosen depth 4	99.7 %
OCT	Thyroid	maximum depth = 8, chosen depth 4	99.7 %
OCT-H	Thyroid	maximum depth = 4, chosen depth 3	99.9 %
OCT-H	Thyroid	maximum depth = 6, chosen depth 3	99.9%
OCT-H	Thyroid	maximum depth = 8, chosen depth 3	99.9 %

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