Course notes for EE394V
Restructured Electricity Markets: Locational Marginal Pricing

Ross Baldick

9

Locational marginal pricing

(i) Optimal power flow,
(ii) DC optimal power flow,
(iii) Offer-based optimal power flow,
(iv) Examples,
(v) Properties of locational marginal prices,
(vi) Congestion rent (merchandising surplus) and congestion cost,
(vii) Contingency constraints,

(viii) Reactive power,
(ix) Losses,
(x) Decomposition and linearization,
(xi) Homework exercises.

9.1 Optimal power flow

9.1.1 Generalization of economic dispatch

9.1.1.1 Constraints on operation

- Besides generator constraints, capacities of transmission lines between generation and demand can also limit the feasible choices of generation:
  - constraints are typically due to maximum temperature limits from thermal heating of elements due to electrical losses.
  - we will think of these limits as being fixed and given, however,
  - rating depends on how long the flow of power is to be sustained and on ambient conditions.
- Other issues such as voltage constraints and constraints due to need to maintain stability of the dynamics of the generation–transmission system can also constrain operation:
  - we will tacitly assume that these can be translated into thermal **proxy constraints**.

9.1.1.2 Power flow equations

- To check whether or not the line flow and voltage constraints are satisfied, we must expand the detail of representation of the network by explicitly incorporating Kirchhoff's laws, as described in the formulation of the power flow equations in Section 3.2.7.

9.1.1.3 Losses

- As mentioned, flow of power on transmission lines will incur losses.
- Power flowing from remote generators to load may incur greater losses than from generation nearby to load:
  - effectively changes the relative cost of generation depending on location.

9.1.1.4 Other controllable elements

- Besides real power generations, we can also consider adjusting any controllable elements in the system so as to minimize costs and meet constraints.

9.1.2 Formulation

9.1.2.1 Variables

- In the decision vector, we represent:
  - real and reactive power generations at the generators, which are represented in the vectors P and Q (any buses without generators can be represented by a generator with capacity zero),
  - (in the case of demand bids) real and (potentially) reactive power demands at the loads, which are represented in the vectors D and E:
    - in economic dispatch, D was the total demand, but now we must specify demand locationaly.
  - any other controllable quantities in the system, such as the settings of **phase-shifting transformers** and capacitors,
  - the voltage magnitudes at every bus in the system, which are represented in the vector u, and
  - the voltage angles at every bus in the system except for the reference bus, which are represented in the vector $\theta_{-\rho}$, with ρ the reference bus:
    - the voltage angle at the reference bus is constant since, as previously, it represents an arbitrary time reference.

- We collect all the variables into the decision vector $x = \begin{bmatrix} P \\ Q \\ \theta_{-\rho} \\ u \end{bmatrix} \in \mathbb{R}^n$, or

  $x = \begin{bmatrix} D \\ E \\ P \\ Q \\ \theta_{-\rho} \\ u \end{bmatrix} \in \mathbb{R}^n$ in the case of demand bids.
- Recall from Section 8.12.3.1 that we considered the voltage angles and magnitudes to be collected together in a vector $x_{n_P+1} = \begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix} \in \mathbb{R}^{N_{n_P+1}}$.

9.1.2.2 Objective

- As previously, let $f : \mathbb{R}^n \to \mathbb{R}$ represent the total cost of generation.
- Typically:

  f depends only on the entries of x corresponding to real power generations (and, in the case of bid demand, on the demand level); however, in some formulations f also depends somewhat on the entries of x corresponding to reactive power generations (and reactive demands), and

  f is separable since the decisions at one generator do not usually affect the costs at any other generators.
- In this case, we can write the objective as:

$$\forall x \in \mathbb{R}^n, f(x) = \sum_{k=1}^{n_P} f_k(P_k).$$

- In the base of bid demand, this becomes:

$$\forall x \in \mathbb{R}^n, f(x) = -\text{benefit}(D) + \sum_{k=1}^{n_P} f_k(P_k).$$

9.1.2.3 Equality constraints

- Since we are now including the voltage magnitude at the reference bus as a decision variable, we must slightly redefine the power flow equality constraints compared to Section 3.2.7 to be equations in the form:

$$\begin{aligned} \forall \ell, p_\ell\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) - P_\ell + D_\ell &= 0, \\ \forall \ell, q_\ell\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) - Q_\ell + E_\ell &= 0, \end{aligned}$$

- where $p_\ell : \mathbb{R}^{N_{nP}+1} \to \mathbb{R}$ and $q_\ell : \mathbb{R}^{N_{nP}+1} \to \mathbb{R}$ are defined similarly to (3.6)–(3.7):

$$\begin{aligned} \forall \begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix} \in \mathbb{R}^{N_{nP}+1}, p_\ell\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) &= \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], \\ \forall \begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix} \in \mathbb{R}^{N_{nP}+1}, q_\ell\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) &= \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} u_\ell u_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)]. \end{aligned}$$

Equality constraints, continued

- Note that u_{ρ} is now a decision variable, generalizing the case in Section 3.2.6.2 and the subsequent development.
- Recall that $\mathbb{J}(\ell)$ is the set of buses joined by a line to bus ℓ.
- We collect the equations together into a vector equation similar to the form of (3.8):

$$g(x) = \mathbf{0},$$

- where a typical entry of g is of the form:

$$p_{\ell}\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) - P_{\ell} + D_{\ell},$$

- or:

$$q_{\ell}\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) - Q_{\ell} + E_{\ell},$$

- and the decision vector x includes the real and reactive generations (and possibly the demands) as well as the voltage magnitudes and angles.

9.1.2.4 *Inequality constraints*

- Limits on the entries in x:

$$\underline{x} \leq x \leq \overline{x}.$$

- A voltage magnitude limit at bus ℓ could be $0.95 = \underline{u}_\ell \leq u_\ell \leq \overline{u}_\ell = 1.05$.
- A generator real power limit could be $0.15 = \underline{P}_\ell \leq P_\ell \leq \overline{P}_\ell = 0.7$.
- There are also constraints involving functions of x.
- For example, there are typically angle difference constraints of the form:

$$\forall \ell, \forall k \in \mathbb{J}(\ell), -\pi/4 \leq \theta_\ell - \theta_k \leq \pi/4, \qquad (9.1)$$

- and there might be limits on angle differences between buses that are not joined directly by a line.
- What happens if the angle difference between the two ends of a line exceeds $\pi/2$?
- In addition, transmission line flow constraints can be expressed as functional constraints via the power flow equations in terms of x.
- That is, we will also have functional constraints of the form:

$$h(x) \leq \overline{h}.$$

- A typical functional inequality constraint might limit the real power flow on a line that joins bus ℓ to bus k.
- Neglecting shunt elements in the line models, the line flow real power flow function $p_{\ell k} : \mathbb{R}^{N_{nP+1}} \to \mathbb{R}$ is defined by:

$$\forall \begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix} \in \mathbb{R}^{N_{nP+1}}$$

$$p_{\ell k}\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) = u_\ell u_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)] - (u_\ell)^2 G_{\ell k}, \tag{9.2}$$

- If there is a real power flow limit of $\overline{p}_{\ell k}$ on the line joining bus ℓ and k then we represent this limit as an inequality constraint of the form $p_{\ell k}\left(\begin{bmatrix} \theta_{-\rho} \\ u \end{bmatrix}\right) \leq \overline{p}_{\ell k}$ in the inequality constraints $h(x) \leq \overline{h}$.
- Recall that in Section 3.7.1 we derived linearized versions of real power line flow constraints where we linearized voltage angles about a flat start.

Inequality constraints, continued

- Other constraints, such as on complex power flow, and due to stability and voltage issues, can also be represented.

9.1.2.5 Problem

$$\min_{x \in \mathbb{R}^n} \{ f(x) | g(x) = \mathbf{0}, \underline{x} \leq x \leq \overline{x}, h(x) \leq \overline{h} \}. \tag{9.3}$$

- This problem is, in general, non-linear and non-convex:
  - recent work has made progress on solving such general formulations directly,
  - however, current electricity markets typically use approximations based on linearization, including DC OPF.

9.2 DC optimal power flow

9.2.1 Motivation

- Optimal power flow presents several difficulties:
  - solving a non-linear, non-convex optimization problem, both in context of day-ahead and in real-time, and
  - specifying the data, particularly the reactive power and voltage magnitude requirements.
- One simplification involves:
  - replacing the representation of the power flow equations with the DC power flow model, and
  - replacing the functional inequality constraints with a linearized version.
- The simplification neglects losses and reactive power issues and creates a linearly constrained problem:
  - the simplification is used in several day-ahead markets, requiring that the cost of losses and compensation (if any) for reactive power be charged, for example, as uplift.
- Some markets such as New York, New England, Midwest, and PJM include losses.

9.2.2 Formulation

- We will assume that the objective depends only on the real power injections and is additively separable.
- Initially assume specified values $\overline{D}$ and $\overline{E}$ of real and reactive demand:
  - demand bids will be included in particular contexts.
- For simplicity, will assume that the only limits on the entries in x are generator constraints of the form $\underline{P} \leq P \leq \overline{P}$.

  More general generator constraints can also be accommodated,
  For example, we could consider reserves and other ancillary services,
  We could also consider limits on voltage magnitudes.
- We will assume that the functional inequality constraints $h(x) \leq \overline{h}$ represent real power line flow limits only.
- In this case, the the optimal power flow problem is:

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^{n_P} f_k(P_k) \,\middle|\, g(x) = \mathbf{0}, \underline{P} \leq P \leq \overline{P}, h(x) \leq \overline{h}, \right\},$$

- The system constraints are $g(x) = \mathbf{0}, h(x) \leq \overline{h}$.

- Making real and reactive power explicit and separating net generation into generation and demand, we obtain:

$$\min_{x\in\mathbb{R}^n}\left\{\sum_{k=1}^{n_P} f_k(P_k) \;\middle|\; \begin{array}{l} p\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)-P=-\overline{D}, q\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)-Q=-\overline{E}, \\ \\ \underline{P}\leq P\leq\overline{P}, \forall(\ell k)\in\mathbb{K}, p_{\ell k}\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)\leq\overline{p}_{\ell k}\end{array}\right\},$$

- where $\mathbb{K}$ is the set of lines with real power line flow limits and we have assumed that there are specified vectors of real and reactive power demand $\overline{D}$ and $\overline{E}$, respectively:
  - recall that previously in economic dispatch, $\overline{D}$ was the total demand, but now we must specify demand at each location in the system.
- If demand response is considered, then D and, in principle E, should also be part of the decision vector and the equality constraints become:

$$p\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)-P+D=\mathbf{0}, q\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)-Q+E=\mathbf{0}.$$

9.2.3 Further simplifications

- We will further simplify the optimal power flow formulation by:
  - omitting the reactive power flow equations, effectively assuming that we can satisfy them independently of other decisions,
  - deleting the reactive power and voltage magnitude variables from the decision vector,
  - fixing the voltage magnitude schedule at $u^{(0)} = \mathbf{1}$,
  - linearizing the real power flow equations, and
  - linearizing the real power line flow limit equations.
- That is, our decision vector will be re-defined to be $x = \begin{bmatrix} P \\ \theta_{-\rho} \end{bmatrix}$, with $\theta_{-\rho} = x_{n_P+1}$.

9.2.3.1 Omitting reactive power and voltage magnitude

- Omitting the reactive power flow equations and fixing the voltage schedule leaves us with the real power flow equations:

$$p\left(\begin{bmatrix}\theta_{-\rho}\\ u^{(0)}\end{bmatrix}\right)-P=-\overline{D}.$$

9.2.3.2 Linearization of power flow

- We linearize the real power flow equations about $\theta^{(0)}=\mathbf{0}$ to obtain the DC power flow approximation:

$$\frac{\partial p}{\partial\theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta_{-\rho}-P=-\overline{D},$$

- where we assume that $\theta^{(0)}=\mathbf{0}$ and $u^{(0)}=\mathbf{1}$ satisfy the real power flow equations for injections $P^{(0)}=\mathbf{0}$.
- As in Section 3.6.2, we define $J=\frac{\partial p}{\partial\theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$, so that the power flow equations become:

$$J\theta_{-\rho}-P=-\overline{D}.$$

9.2.3.3 Linearization of real power line flow limit constraints

- The real power line flow limit constraints are:

$$\forall(\ell k) \in \mathbb{K}, p_{\ell k}\left(\begin{bmatrix}\theta_{-\rho} \\ u\end{bmatrix}\right) \leq \overline{p}_{\ell k}.$$

- Linearizing these about $\theta^{(0)} = \mathbf{0}$ and maintaining $u^{(0)} = \mathbf{1}$ we obtain:

$$\forall(\ell k) \in \mathbb{K}, \frac{\partial p_{\ell k}}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0} \\ \mathbf{1}\end{bmatrix}\right)\theta_{-\rho} \leq \overline{p}_{\ell k} - p_{\ell k}\left(\begin{bmatrix}\mathbf{0} \\ \mathbf{1}\end{bmatrix}\right) = \overline{p}_{\ell k},$$

- since $p_{\ell k}\left(\begin{bmatrix}\mathbf{0} \\ \mathbf{1}\end{bmatrix}\right) = 0$.
- As in Section 3.7.1, we define a matrix K to have rows $K_{(\ell k)} = \frac{\partial p_{\ell k}}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0} \\ \mathbf{1}\end{bmatrix}\right)$ and also define a vector d to have entries $d_{(\ell k)} = \overline{p}_{\ell k}$, so that the line flow inequality constraints become:

$$K\theta_{-\rho} \leq d.$$

9.2.3.4 Other constraints

- We can also add linearized versions of other constraints such as stability and voltage constraints to the formulation.

9.2.4 *Explicit representation of angles*

9.2.4.1 *Formulation*

- The DC optimal power flow problem is therefore:

$$\min_{P,\theta_{-\rho}} \left\{ \sum_{k=1}^{n_P} f_k(P_k) \,\middle|\, J\theta_{-\rho} - P = -\overline{D}, K\theta_{-\rho} \leq d, \underline{P} \leq P \leq \overline{P} \right\}.$$

- This problem is in the form of our generalized economic dispatch problem:

$$\min_{x\in\mathbb{R}^n} \{ f(x) | Ax = b, Cx \leq d, \forall k = 1,\ldots,n, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \},$$

- where:

$$\begin{aligned} x &= \begin{bmatrix} P \\ \theta_{-\rho} \end{bmatrix} \in \mathbb{R}^n, \\ A &= \begin{bmatrix} -\mathbf{I} & J \end{bmatrix}, \\ b &= -\overline{D}, \\ C &= \begin{bmatrix} \mathbf{0} & K \end{bmatrix}, \end{aligned}$$

Formulation, continued

- and where:

$$
\begin{aligned}
x_k &= [P_k], k = 1, \ldots, n_P, \\
x_{n_P+1} &= \theta_{-\rho}, \\
\forall x \in \mathbb{R}^n, f(x) &= \sum_{k=1}^{n_P} f_k(P_k), \\
\underline{\delta}_k &= [\underline{P}_k], \\
\overline{\delta}_k &= [\overline{P}_k], \\
\Gamma_k &= [1].
\end{aligned}
$$

9.3 Offer-based optimal power flow, angles represented explicitly

- We consider the solution of the optimal power flow problem and write down the pricing rule for offer-based optimal power flow where each generator $k = 1, \ldots, n_P$ offers ∇f_k and specifies its limits $\underline{P}_k$ and $\overline{P}_k$.
- Let $x^\star = \begin{bmatrix} P^\star \\ \theta^\star_{-\rho} \end{bmatrix}$ be the minimizer of the offer-based optimal power flow problem.
- Let $\lambda^\star$ and $\mu^\star$ be the Lagrange multipliers associated with the system constraints $Ax = b$ and $Cx \leq d$, respectively.
- Let A_k and C_k be the columns of A and C, respectively, associated with the decision variables x_k representing generator k.
- That is:

$$\begin{aligned} A_k &= -\mathbf{I}_k, \\ C_k &= \mathbf{0}, \end{aligned}$$

- where $\mathbf{I}_k$ is a vector with all zeros except for a one in the k-th place.
- Note that the corresponding columns for the variables $x_{n_P+1} = \theta_{-\rho}$ are the matrices J and K, respectively.

9.3.1 First-order necessary conditions

$\exists \lambda^\star \in \mathbb{R}^m, \exists \mu^\star \in \mathbb{R}^r, \forall k = 1,\ldots,n_P+1, \exists \underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^{r_k}$ such that:

$$
\begin{aligned}
\forall k = 1,\ldots,n_P+1, \nabla f_k(x_k^\star) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star &= \mathbf{0}; \\
[J]^\dagger \lambda^\star + K^\dagger \mu^\star &= \mathbf{0}; \\
M^\star (Cx^\star - d) &= \mathbf{0}; \\
\forall k = 1,\ldots,n, \underline{M}_k^\star (\underline{\delta}_k - \Gamma_k x^\star) &= \mathbf{0}; \\
\forall k = 1,\ldots,n, \overline{M}_k^\star (\Gamma_k x^\star - \overline{\delta}_k) &= \mathbf{0}; \\
Ax^\star &= b; \\
Cx^\star &\leq d; \\
\forall k = 1,\ldots,n, \Gamma_k x_k^\star &\geq \underline{\delta}_k; \\
\forall k = 1,\ldots,n, \Gamma_k x^\star &\leq \overline{\delta}_k; \\
\mu^\star &\geq \mathbf{0}; \\
\underline{\mu}_k^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^\star &\geq \mathbf{0}.
\end{aligned}
$$

9.3.2 Pricing rule

- From Theorem 8.1, we can write down the price π_{P_k} that induces profit-maximizing generator k to dispatch according to $P_k^\star$:

$$\begin{aligned}\pi_{P_k} &= -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star, \\ &= \lambda_k^\star,\end{aligned}$$

- where $\lambda_k^\star$ is the Lagrange multiplier associated with the system constraint $\frac{\partial p_k}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta_{-\rho} - P_k = -\overline{D}_k$ and where $\frac{\partial p_k}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ is the k-th row of J.
- The price $\lambda_k^\star$ is called the **locational marginal price** or **LMP** at bus k.
- That is, the payment to generator k for generation P_k is:

$$[\pi_{P_k}]^\dagger P_k = \lambda_k^\star \times P_k.$$

- Generator k is paid based on the Lagrange multiplier on the power balance constraint associated with its bus.
- Similarly, demand pays based on the Lagrange multiplier associated with its bus.

Pricing rule, continued

- If the formulation were expanded to include reserves and other ancillary services then the LMPs would also include additional terms related to these services as we derived for the case of reserves without transmission constraints.
- However, current market formulations with ancillary services typically do not represent locational issues in detail:
  - for example, deliverability of spinning reserves may not be considered, or only considered approximately in terms of deliverability to “zones,”
  - this results in some inconsistences in such models.

9.3.3 Sensitivity interpretation

- Typically, the LMP at a bus can be interpreted as the minimum cost per unit energy of delivering an additional infinitesimal amount of power to that bus or the value per unit energy of producing an additional infinitesimal amount of power at that bus.

  "Marginal" means a derivative or infinitesimal change in this context.
- In particular, if the conditions hold to apply Theorem 4.14, then the Lagrange multiplier on a constraint equals the sensitivity of the objective to a change in the right-hand side of the constraint.
- The LMP for bus k is the Lagrange multiplier on the power balance constraint for bus k.
- The LMP therefore represents the sensitivity of cost to changes in production (or demand) at bus k:

  Minimizing the cost (minus benefits) is equivalent to maximizing the benefits minus the cost, or the **surplus**.

  Recall from Section 6.3 that the sensitivity of surplus to changes in production is called the **marginal surplus**.

Sensitivity interpretation, continued

- The sensitivity interpretation is not always valid when constraints are "just" binding.

9.4 Offer-based optimal power flow, angles eliminated

- To understand the relationship between the constraints and the LMPs, we will further re-formulate the optimal power flow problem to eliminate the angles.
- This will also lead to a decomposition approach that can be utilized to represent the AC power flow equations.

9.4.1 Formulation

- Recall the system constraints:

$$\begin{aligned} J\theta_{-\rho} - P &= -\overline{D}, \\ K\theta_{-\rho} &\leq d. \end{aligned}$$

- In Section 3.7.4, we discussed eliminating $\theta_{-\rho}$ by re-writing $J\theta_{-\rho} - P = -\overline{D}$ as:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}\overline{D}, \\ \theta_{-\rho} &= [J_{-\sigma}]^{-1}(P_{-\sigma} - \overline{D}_{-\sigma}), \end{aligned}$$

- where σ is the slack bus and ρ is the angle reference bus.

  For reasons that will become clear, bus σ will also be called the "price reference bus," as distinct from the *angle* reference bus, bus ρ.

  However, typically, we will either choose σ to be the same as the angle reference bus ρ or (in small examples) choose σ to be a bus with demand.

Formulation, continued

- The system equality and inequality constraints then become:

$$
\begin{aligned}
-\mathbf{1}^{\dagger} P &= -\mathbf{1}^{\dagger} \overline{D}, \\
K[J_{-\sigma}]^{-1} P_{-\sigma} &\leq K[J_{-\sigma}]^{-1} \overline{D}_{-\sigma} + d.
\end{aligned}
$$

Formulation, continued

- The DC optimal power flow problem with angles eliminated is therefore:

$$\min_{P}\left\{\sum_{k=1}^{n_P} f_k(P_k)\,\middle|\, -\mathbf{1}^\dagger P = -\mathbf{1}^\dagger \overline{D}, K[J_{-\sigma}]^{-1}P_{-\sigma} \le K[J_{-\sigma}]^{-1}\overline{D}_{-\sigma} + d, \underline{P} \le P \le \overline{P}\right\}.$$

- This is in the form of our generalized economic dispatch problem:

$$\min_{x\in\mathbb{R}^n}\{f(x)|\hat{A}x = \hat{b}, \hat{C}x \le \hat{d}, \forall k = 1,\ldots,n, \underline{\delta}_k \le \Gamma_k x_k \le \overline{\delta}_k\},$$

- where we have used $\hat{}$ to distinguish this formulation from the formulation where angles $\theta_{-\rho}$ were explicit and where:

$$\begin{aligned} x &= P \in \mathbb{R}^{n_P}, \\ \hat{A} &= -\mathbf{1}^\dagger, \\ \hat{b} &= -\mathbf{1}^\dagger \overline{D}, \\ \hat{C} &= \begin{bmatrix} \mathbf{0} & K[J_{-\sigma}]^{-1} \end{bmatrix}, \\ \hat{d} &= K[J_{-\sigma}]^{-1}\overline{D}_{-\sigma} + d, \end{aligned}$$

where we have assumed that the first entry of P corresponds to the price reference bus,

Formulation, continued

- and where:

$$
\begin{aligned}
x_k &= [P_k], \\
\underline{\delta}_k &= [\underline{P}_k], \\
\overline{\delta}_k &= [\overline{P}_k], \\
\Gamma_k &= [1].
\end{aligned}
$$

- Recall that $\hat{C}$ is the **augmented shift factor matrix** that we derived in Section 3.7.5 in the context of linearized power flow.

Formulation, continued

- We again consider the solution of the optimal power flow problem and write down the pricing rule for offer-based optimal power flow where each generator $k = 1, \ldots, n_P$ offers ∇f_k and specifies its limits $\underline{P}_k$ and $\overline{P}_k$.
- Let $P^\star$ be the minimizer of the offer-based optimal optimal power flow problem with angles $\theta_{-\rho}$ eliminated.

  Note that if the offers are the same as in the previous case where we considered angles $\theta_{-\rho}$ explicitly then the minimizer $P^\star$ must be the same as previously in the formulation where we represented angles explicitly!

  Moreover, the angles in the previous solution must also satisfy $\theta^\star_{-\rho} = [J_{-\sigma}]^{-1}(P^\star_{-\sigma} - \overline{D}_{-\sigma})$.
- Let $\hat{\lambda}^\star$ and $\hat{\mu}^\star$ be the Lagrange multipliers associated with the system constraints $\hat{A}x = \hat{b}$ and $\hat{C}x \leq \hat{d}$, respectively.

Formulation, continued

- Let $\hat{A}_k$ and $\hat{C}_k$ be the columns of $\hat{A}$ and $\hat{C}$, respectively, associated with the decision variables x_k representing generator k.
- That is:

$$
\begin{aligned}
\hat{A}_k &= -1, \\
\hat{C}_k &= \begin{cases} K\left[[J_{-\sigma}]^{-1}\right]_k, & \text{if } k \neq \sigma \text{ is not the price reference bus,} \\ \mathbf{0}, & \text{if } k \neq \sigma \text{ is the price reference bus.} \end{cases}
\end{aligned}
$$

- where $\left[[J_{-\sigma}]^{-1}\right]_k$ is the k-th column of $[J_{-\sigma}]^{-1}$.

Formulation, continued

- That is, if k is not the price reference bus then $\hat{C}_k$ is the column of the shift factor matrix corresponding to generator k.
- Each entry of $\hat{C}_k$ represents the fraction of the generation injected by generator k that flows on the corresponding line when withdrawn at the price reference bus.
- The entries of $\hat{C}_\sigma$ are all zero since injecting and withdrawing the same amount of power at the price reference bus has no effect on any line flows.

9.4.2 First-order necessary conditions

$$\exists \hat{\lambda}^\star \in \mathbb{R}^m, \exists \hat{\mu}^\star \in \mathbb{R}^r, \forall k = 1, \ldots, n_P, \exists \underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^{r_k} \text{ such that:}$$

$$\begin{aligned}
\forall k = 1, \ldots, n_P, \nabla f_k(x_k^\star) + [\hat{A}_k]^\dagger \hat{\lambda}^\star + [\hat{C}_k]^\dagger \hat{\mu}^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star &= \mathbf{0}; \\
M^\star(\hat{C}x^\star - \hat{d}) &= \mathbf{0}; \\
\forall k = 1, \ldots, n, \underline{M}_k^\star(\underline{\delta}_k - \Gamma_k x^\star) &= \mathbf{0}; \\
\forall k = 1, \ldots, n, \overline{M}_k^\star(\Gamma_k x^\star - \overline{\delta}_k) &= \mathbf{0}; \\
\hat{A}x^\star &= \hat{b}; \\
\hat{C}x^\star &\leq \hat{d}; \\
\forall k = 1, \ldots, n, \Gamma_k x_k^\star &\geq \underline{\delta}_k; \\
\forall k = 1, \ldots, n, \Gamma_k x^\star &\leq \overline{\delta}_k; \\
\hat{\mu}^\star &\geq 0; \\
\underline{\mu}_k^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^\star &\geq \mathbf{0}.
\end{aligned}$$

9.4.3 Pricing rule

- From Theorem 8.1, we can again write down the price π_{P_k} that induces each profit-maximizing generator k to dispatch according to $P_k^\star$:

$$\begin{aligned}
\pi_{P_k} &= -[\hat{A}_k]^\dagger \hat{\lambda}^\star - [\hat{C}_k]^\dagger \hat{\mu}^\star, \\
&= \begin{cases} \hat{\lambda}^\star - \left[[J_{-\sigma}]^{-1} \right]_k^\dagger K^\dagger \hat{\mu}^\star, & \text{if } k \text{ is not the price reference bus,} \\ \hat{\lambda}^\star, & \text{if } k \text{ is the price reference bus,} \end{cases}
\end{aligned}$$

- where $\hat{\lambda}^\star$ is the Lagrange multiplier associated with the system equality constraint $-\mathbf{1}^\dagger P = -\mathbf{1}^\dagger D$.
- In particular, the payment to generator k for generation P_k is:

$$\begin{aligned}
[\pi_{x_k}]^\dagger P_k &= (\hat{\lambda}^\star - [\hat{C}_k]^\dagger \hat{\mu}^\star) P_k, \\
&= \begin{cases} \left(\hat{\lambda}^\star - \left[[J_{-\sigma}]^{-1} \right]_k^\dagger K^\dagger \hat{\mu}^\star \right) P_k, & \text{if } k \text{ is not the price reference bus,} \\ \hat{\lambda}^\star \times P_k, & \text{if } k \text{ is the price reference bus.} \end{cases}
\end{aligned}$$

Pricing rule, continued

- Generator k is paid based on:

  the Lagrange multiplier on the "overall" power balance constraint associated with the price reference bus, and

  the Lagrange multipliers associated with the line flow limit constraints.

- This is again the **locational marginal price** at bus k.
- In general, $\hat{\lambda}^\star$, the value of the Lagrange multiplier on the "overall" power balance constraint, has a different value to the analogous Lagrange multiplier that would be obtained if the transmission constraints were ignored:

  That is, $\hat{\lambda}^\star$ is not the same as the "unconstrained price" obtained from the offer-based economic dispatch calculation ignoring transmission constraints!

Pricing rule, continued

- The LMP at bus k, $\lambda_k^\star$, is equal to:

  the LMP at the price reference bus,
  minus a weighted sum of the Lagrange multipliers on the line flow limit constraints.

- The weights are "shift factors" to the constraints.
- Since the dispatch $P^\star$ must be the same as in offer-based optimal power flow where we considered angles θ explicitly, it must also be the case that LMPs in each case must provide the same incentives.

Pricing rule, continued

Theorem 9.1

- *Consider the LMPs in the two formulations of offer-based optimal power flow with angles included and with angles eliminated, respectively.*
- *For some choices of Lagrange multipliers $\lambda^\star$ and $\mu^\star$ satisfying the first-order necessary conditions of offer-based optimal power flow with angles included and for some choices of Lagrange multipliers $\hat{\lambda}^\star$ and $\hat{\mu}^\star$ satisfying the first-order necessary conditions of offer-based optimal power flow with angles eliminated we have that:*

$$\begin{aligned} \forall k = 1, \ldots, n, \lambda_k^\star &= \hat{\lambda}^\star - [\hat{C}_k]^\dagger \hat{\mu}^\star, && (9.4) \\ \mu^\star &= \hat{\mu}^\star. && (9.5) \end{aligned}$$

- *If there are unique values of the Lagrange multipliers then (9.4) and (9.5) hold for these values so that the unique LMPs are the same in both formulations.*

Pricing rule, continued

Proof

- Let $P^\star$ be the minimizer of the offer-based optimal optimal power flow problem with angles $\theta_{-\rho}$ eliminated and let $\hat{\lambda}^\star$, $\hat{\mu}^\star$, $\underline{\mu}_k^\star$, and $\overline{\mu}_k^\star$ be the Lagrange multipliers associated with the system constraints $\hat{A}x = \hat{b}$ and $\hat{C}x \leq \hat{d}$ and with the generator constraints, respectively.
- Define $\theta^\star_{-\rho} = [J_{-\sigma}]^{-1}(P^\star_{-\sigma} - \overline{D}_{-\sigma})$, where $P^\star_{-\sigma}$ is the vector obtained from $P^\star$ by deleting the entry corresponding to the row eliminated from J.
- Direct substitution then shows that:

  $$x^\star = \begin{bmatrix} P^\star \\ \theta^\star_{-\rho} \end{bmatrix},$$

  $\lambda^\star$ and $\mu^\star$ defined by and (9.4) and (9.5), and
  $\underline{\mu}_k^\star$, and $\overline{\mu}_k^\star$,
- satisfy the first-order necessary conditions of offer-based power flow with angles included.

□

9.5 Example

- Consider the following one-line two-bus system with MW capacity and per unit impedance (on a 1 MVA base) as shown.
- Let bus $\rho = 1$ be the angle reference bus, so the unknown angle is θ_2.
- Let bus $\sigma = 2$ be the slack/price reference bus.
- There are generators at both buses 1 and 2 and $\overline{D}_2 = 110$ MW at bus 2.
- The offers are specified by:

$$\begin{aligned} \forall P_1 \in [0, 200], \nabla f_1(P_1) &= \$25/\text{MWh}, \\ \forall P_2 \in [0, 50], \nabla f_2(P_2) &= \$35/\text{MWh}. \end{aligned}$$

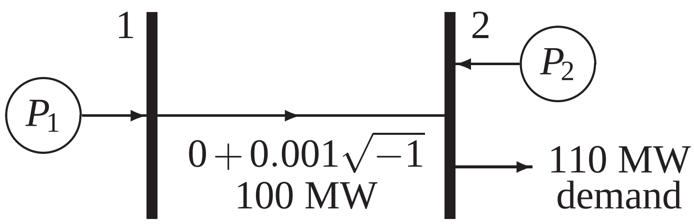

Fig. 9.1. One-line two-bus network.

9.5.1 Admittance matrix

- The line admittance is:

$$\begin{aligned} Y_{12} &= \frac{1}{0+0.001\sqrt{-1}}, \\ &= -1000\sqrt{-1}. \end{aligned}$$

- The bus admittance matrix is:

$$\begin{aligned} \begin{bmatrix} Y_{12} & -Y_{12} \\ -Y_{12} & Y_{12} \end{bmatrix} &= \begin{bmatrix} -1000\sqrt{-1} & 1000\sqrt{-1} \\ 1000\sqrt{-1} & -1000\sqrt{-1} \end{bmatrix}, \\ &= \begin{bmatrix} B_{11}\sqrt{-1} & B_{12}\sqrt{-1} \\ B_{21}\sqrt{-1} & B_{22}\sqrt{-1} \end{bmatrix}. \end{aligned}$$

9.5.2 Jacobian

- Evaluating the sub-matrix of the Jacobian corresponding to real power and angles at the condition of flat start:

$$
\begin{aligned}
J &= \frac{\partial p}{\partial \theta_{-\rho}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right), \\
&= \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right), \\
&= \frac{\partial p}{\partial \theta_{2}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right), \\
&= \begin{bmatrix} -B_{12} \\ B_{12} \end{bmatrix}, \\
&= \begin{bmatrix} -1000 \\ 1000 \end{bmatrix}.
\end{aligned}
$$

9.5.3 DC power flow

- The DC power flow constraints are:

$$J\theta_{-\rho} = \begin{bmatrix} P_1 \\ P_2 - \overline{D}_2 \end{bmatrix}.$$

- Substituting, we obtain:

$$\begin{bmatrix} -1000 \\ 1000 \end{bmatrix} [\theta_2] = \begin{bmatrix} P_1 \\ P_2 - \overline{D}_2 \end{bmatrix}.$$

9.5.4 Eliminating angles

- We eliminate θ_2 to obtain the following form:

$$\begin{aligned} -P_1 - P_2 &= -\overline{D}_2, \\ [\theta_2] &= [J_{-\sigma}]^{-1}[P_1], \end{aligned}$$

- where, to form $J_{-\sigma}$, we have deleted the second row of J corresponding to the price reference/slack bus $\sigma = 2$:

$$\begin{aligned} J_{-\sigma} &= [-1000], \\ [J_{-\sigma}]^{-1} &= [-0.001]. \end{aligned}$$

- Note that the *angle* reference bus is bus $\rho = 1$, whereas the *price* reference bus is bus $\sigma = 2$!

  Example shows that the angle and price reference buses can be different buses!

Eliminating angles, continued

- The power flow equations are then:

$$\begin{aligned} -P_1 - P_2 &= -\overline{D}_2, \\ \theta_2 &= [-0.001][P_1]. \end{aligned}$$

- For positive values of P_1, we have that $\theta_2 < 0 = \theta_1$.
  Power flows from “higher” to “lower” angles.

9.5.5 *Line flow constraints*

- Assume that the real power line flow limit of 100 MW applies only in the direction of the arrow in Figure 9.1.
- Ignore the constraint on flow in the direction opposite to the arrow.
- The line flow constraint is then specified by $K\theta_{-\rho} \leq d$, where:

$$
\begin{aligned}
d &= [\overline{p}_{(12)}], \\
&= [100], \\
K &= [-B_{12}], \\
&= [-1000].
\end{aligned}
$$

- Therefore:

$$
\begin{aligned}
(K[\theta_2] \leq d) &\Leftrightarrow ([-1000][\theta_2] \leq [100]), \\
&\Leftrightarrow (\theta_2 \geq -0.1).
\end{aligned}
$$

- For $|\theta_2| \leq 0.1$ we have that $\sin(\theta_1 - \theta_2) = \sin(-\theta_2) \approx -\theta_2$, so that the DC power flow approximation is reasonable.

9.5.6 Shift factors

- The matrix of shift factors is:

$$\begin{aligned} K[J_{-\sigma}]^{-1} &= [-1000][-0.001], \\ &= [1]. \end{aligned}$$

- That is, if P_1 is injected at bus 1 and withdrawn at bus 2 then $[1][P_1] = P_1$ will flow on the line between bus 1 and bus 2.
- If P_2 is injected at bus 2 and withdrawn at bus 2 then no power will flow on the line between bus 1 and bus 2.

9.5.7 Line flow constraints with angles eliminated

- The system equality and inequality constraints with angles eliminated are:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}\overline{D}, \\ K[J_{-\sigma}]^{-1}[P_1] &\leq K[J_{-\sigma}]^{-1}[0]+d. \end{aligned}$$

- Also, $d = [\,100\,]$, so these constraints become:

$$\begin{aligned} -P_1 - P_2 &= -\overline{D}_2, \\ P_1 &\leq 100. \end{aligned}$$

- We could see this from the picture!!
- What would the constraints be if $\sigma = 1$ were the slack?

9.5.8 Offer-based optimal power flow, angles represented explicitly

- Offer-based optimal power flow involves:

  $P_1^\star = 100$ MW generation from generator 1,
  $P_2^\star = 10$ MW generation from generator 2, and
  flow of 100 MW on the line, so that

  $$\theta_2^\star = [J_{-\sigma}]^{-1}[P_1^\star] = [-0.001][P_1^\star] = [-0.1].$$

- None of the four generator constraints are binding so, by complementary slackness, the Lagrange multipliers on the generator constraints are zero:

$$\begin{aligned} \underline{\mu}_k^\star &= 0, k = 1, 2, \\ \overline{\mu}_k^\star &= 0, k = 1, 2. \end{aligned}$$

- Therefore, both generators are "marginal."
- Except for certain cases where the Lagrange multipliers are not uniquely defined, the number of marginal generators is at least one more than the number of binding transmission constraints.
- See Exercises 9.1, 9.2 and 9.4.

Offer-based optimal power flow, angles represented explicitly, continued

- The first-order necessary conditions include:

$$\begin{aligned}\forall k = 1,2,\ 0 &= \nabla f_k(P_k^\star) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star, \\ &= \nabla f_k(P_k^\star) - \lambda_k^\star.\end{aligned}$$

- Therefore:

$$\begin{aligned}\lambda_1^\star &= \nabla f_1(P_1^\star), \\ &= \$25/\text{MWh}, \\ \lambda_2^\star &= \nabla f_2(P_2^\star), \\ &= \$35/\text{MWh}.\end{aligned}$$

- The LMPs are \$25/MWh and \$35/MWh, respectively.
- These are, respectively, the costs per unit energy of delivering an additional infinitesimal amount of power to buses 1 and 2.
- How would the LMPs change if the slack/price reference bus bus changed to $\sigma = 1$ or if the angle reference bus changed to $\rho = 2$?

9.5.9 Offer-based optimal power flow, angles eliminated

- In this formulation, the first-order necessary conditions include:

$$
\begin{aligned}
\forall k = 1, 2,\ 0 &= \nabla f_k(P_k^\star) + [\hat{A}_k]^\dagger \hat{\lambda}^\star + [\hat{C}_k]^\dagger \hat{\mu}^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star, \\
&= \nabla f_k(P_k^\star) - \hat{\lambda}^\star + [\hat{C}_k]^\dagger \hat{\mu}^\star.
\end{aligned}
$$

- Therefore:

$$
\begin{aligned}
\$35/\mathrm{MWh} &= \nabla f_2(P_2^\star), \\
&= \hat{\lambda}^\star - [0]\hat{\mu}^\star, \\
&= \hat{\lambda}^\star, \\
\$25/\mathrm{MWh} &= \nabla f_1(P_1^\star), \\
&= \hat{\lambda}^\star - [1]\hat{\mu}^\star, \\
&= \$35/\mathrm{MWh} - \hat{\mu}^\star, \\
\hat{\mu}^\star &= \$10/\mathrm{MWh}.
\end{aligned}
$$

- The LMP at the price reference bus, bus $\sigma = 2$, is $\hat{\lambda}^\star = \$35/\mathrm{MWh}$.

9.6 Larger example

- Recall the previous four-line four-bus example from Section 3.8 with MW capacities and per unit impedances (on a 1 MVA base) as shown.
- Let $\rho = 0$ be the angle reference bus, so unknown angles are $\theta_{-\rho} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$.
- Demand is 3000 MW at bus 0 and bus $\sigma = 0$ is the price reference bus.

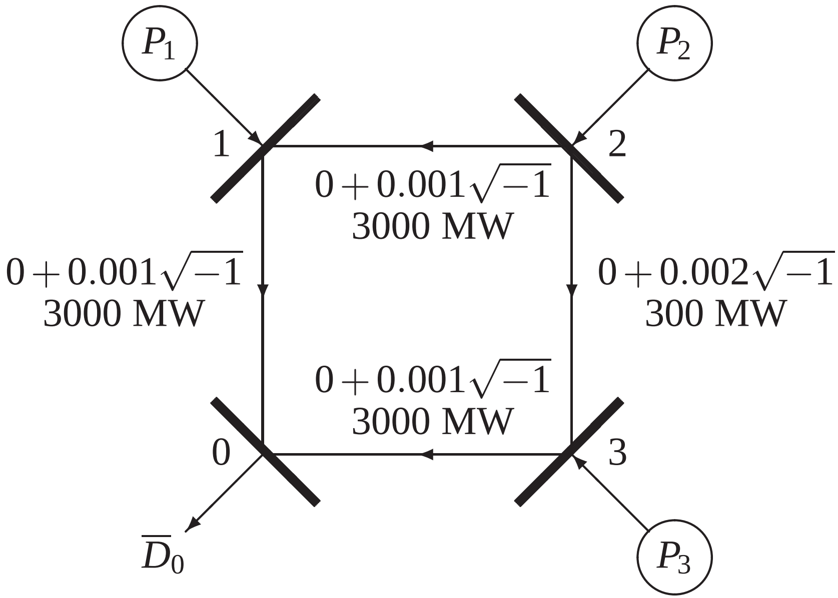

Fig. 9.2. Four-line four-bus network.

Larger example, continued

- Assume that the transmission line capacities are:

$$\begin{aligned}
\overline{p}_{(10)} &= 3000\text{MW}, \\
\overline{p}_{(21)} &= 3000\text{MW}, \\
\overline{p}_{(23)} &= 300\text{MW}, \\
\overline{p}_{(30)} &= 3000\text{MW},
\end{aligned}$$

- in the directions implied by the arrows.
- We ignore limits on these lines in the directions opposite to the arrows.
- The generation offers are:

$$\begin{aligned}
\forall P_1 \in [0,1500], \nabla f_1(P_1) &= \$40/\text{MWh}, \\
\forall P_2 \in [0,1000], \nabla f_2(P_2) &= \$20/\text{MWh}, \\
\forall P_3 \in [0,1500], \nabla f_3(P_3) &= \$50/\text{MWh}.
\end{aligned}$$

- This is the same demand and offers as a previous example, but now we must satisfy the transmission constraints.

9.6.1 DC power flow

- Recall that the DC power flow constraints are:

$$J\theta_{-\rho} = P - \overline{D},$$

- where:

$$J = \begin{bmatrix} -1000 & 0 & -1000 \\ 2000 & -1000 & 0 \\ -1000 & 1500 & -500 \\ 0 & -500 & 1500 \end{bmatrix}.$$

- Since bus 0 is actually a demand bus and there is only generation at buses 1, 2, and 3, the DC power flow constraints are:

$$\begin{bmatrix} -1000 & 0 & -1000 \\ 2000 & -1000 & 0 \\ -1000 & 1500 & -500 \\ 0 & -500 & 1500 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} - \begin{bmatrix} \overline{D}_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

9.6.2 Eliminating angles

- Eliminating the angles yields:

$$
\begin{aligned}
-P_1 - P_2 - P_3 &= -\overline{D}_0, \\
\theta_{-\rho} &= \begin{bmatrix} 0.0008 & 0.0006 & 0.0002 \\ 0.0006 & 0.0012 & 0.0004 \\ 0.0002 & 0.0004 & 0.0008 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}.
\end{aligned}
$$

9.6.3 Line flow constraints

- The line flow constraints are specified by $K\theta_{-\rho} \leq d$, where:

$$\begin{aligned} d &= \begin{bmatrix} \overline{p}_{(10)} \\ \overline{p}_{(21)} \\ \overline{p}_{(23)} \\ \overline{p}_{(30)} \end{bmatrix}, \\ &= \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix}, \\ K &= \begin{bmatrix} 1000 & 0 & 0 \\ -1000 & 1000 & 0 \\ 0 & 500 & -500 \\ 0 & 0 & 1000 \end{bmatrix}. \end{aligned}$$

9.6.4 Shift factors

- The matrix of shift factors is:

$$
\begin{aligned}
K[J_{-\sigma}]^{-1} &= \begin{bmatrix} 1000 & 0 & 0 \\ -1000 & 1000 & 0 \\ 0 & 500 & -500 \\ 0 & 0 & 1000 \end{bmatrix} \begin{bmatrix} 0.0008 & 0.0006 & 0.0002 \\ 0.0006 & 0.0012 & 0.0004 \\ 0.0002 & 0.0004 & 0.0008 \end{bmatrix}, \\
&= \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix}.
\end{aligned}
$$

- The augmented shift factor matrix is:

$$
\begin{aligned}
\hat{C} &= \begin{bmatrix} \mathbf{0} & K[J_{-\sigma}]^{-1} \end{bmatrix}, \\
&= \begin{bmatrix} 0.0 & 0.8 & 0.6 & 0.2 \\ 0.0 & -0.2 & 0.6 & 0.2 \\ 0.0 & 0.2 & 0.4 & -0.2 \\ 0.0 & 0.2 & 0.4 & 0.8 \end{bmatrix}.
\end{aligned}
$$

9.6.5 Line flow constraints with angles eliminated

- The system equality and inequality constraints with angles eliminated are:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}\overline{D}, \\ K[J_{-\sigma}]^{-1}P_{-\sigma} &\leq K[J_{-\sigma}]^{-1}\mathbf{0}+d, \end{aligned}$$

- which yield:

$$\begin{aligned} -P_1-P_2-P_3 &= -\overline{D}_0, \\ \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} &\leq \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix}. \end{aligned}$$

9.6.6 Line flows using solution ignoring transmission constraints

- The solution of offer-based economic dispatch ignoring transmission constraints was $P_1^\star = 1500$ MW, $P_2^\star = 1000$ MW, and $P_3^\star = 500$ MW.
- Substituting, we obtain flows of:

$$\begin{aligned}
\begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} P_1^\star \\ P_2^\star \\ P_3^\star \end{bmatrix} & = \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 1500 \\ 1000 \\ 500 \end{bmatrix}, \\
& = \begin{bmatrix} 1900 \\ 400 \\ 600 \\ 1100 \end{bmatrix}, \\
& \not\leq \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix},
\end{aligned}$$

- since the constraint on flow on the line between buses 2 and 3 would be violated.

Line flows using solution ignoring transmission constraints, continued

- If we dispatched $P_1 = P_3 = 0$ MW and $P_2 = 1000$ MW, then the flow on the line between buses 2 and 3 would be 400 MW, which would still violate the constraint!
- Will we be able to utilize all the low-priced power from bus 2?

Offer-based economic dispatch is sometimes explained by saying that the "offer blocks" are stacked up from lowest to highest offer price until demand is met.

Using this analogy, we might be led to believe that we will not be able to use all of the low-priced power from bus 2 in offer-based optimal power flow, since using the lowest priced "block" alone would violate the transmission constraints.

9.6.7 Offer-based optimal power flow

- To find the offer-based optimal power flow solution, we need to use a formal optimization process.
- Using either the formulation with angles represented or the formulation with angles eliminated, the problem is a linear program, which can be solved, yielding the offer-based optimal power flow solution:

  $P_1^\star = 750$ MW generation from generator 1,
  $P_2^\star = 1000$ MW generation from generator 2,
  $P_3^\star = 1250$ MW generation from generator 3, and
  flow of 300 MW on the line from bus 2 to bus 3.

- In this case, only the generator constraint for generator 2 is binding so, by complementary slackness, the Lagrange multipliers on all the other generator constraints are zero:

$$\begin{aligned} \underline{\mu}_k^\star &= 0, k = 1, 2, 3 \\ \overline{\mu}_k^\star &= 0, k = 1, 3. \end{aligned}$$

- Generators 1 and 3 are marginal.

Offer-based optimal power flow, continued

- Only the line constraint for the line joining bus 2 to bus 3 is binding so, by complementary slackness, the Lagrange multipliers on all the line constraints are zero:

$$\begin{aligned}\hat{\mu}^{\star}_{(10)} &= 0,\\ \hat{\mu}^{\star}_{(21)} &= 0,\\ \hat{\mu}^{\star}_{(30)} &= 0.\end{aligned}$$

9.6.8 Offer-based optimal power flow, angles represented explicitly

- The first-order necessary conditions include:

$$
\begin{aligned}
\forall k = 1,\ldots,4,\ 0 &= \nabla f_k(P_k^\star) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star, \\
&= \nabla f_k(P_k^\star) - \lambda_k^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star, \\
&= \nabla f_k(P_k^\star) - \lambda_k^\star, \text{ for } k = 1,3, \text{ since } \underline{\mu}_k^\star = \overline{\mu}_k^\star = 0 \text{ for } k = 1,3.
\end{aligned}
$$

- Therefore:

$$
\begin{aligned}
\lambda_1^\star &= \nabla f_1(P_1^\star), \\
&= \$40/\text{MWh}, \\
\lambda_3^\star &= \nabla f_3(P_3^\star), \\
&= \$50/\text{MWh}.
\end{aligned}
$$

- The LMPs are \$40/MWh and \$50/MWh, respectively, at buses 1 and 3.
- These are, respectively, the costs per unit energy of delivering an additional infinitesimal amount of power to buses 1 and 3.

  The power is “delivered” to these buses by generating it locally.

9.6.9 Offer-based optimal power flow, angles eliminated

- In this formulation, the first-order necessary conditions include:

$$\begin{aligned}
\forall k = 1,\ldots,4,\ 0 &= \nabla f_k(P_k^\star) + [\hat{A}_k]^\dagger \hat{\lambda}^\star + [\hat{C}_k]^\dagger \hat{\mu}^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star, \\
&= \nabla f_k(P_k^\star) - \hat{\lambda}^\star + [\hat{C}_k]^\dagger \hat{\mu}^\star, \text{ for } k = 1,3, \\
&= \nabla f_k(P_k^\star) - \hat{\lambda}^\star + [\hat{C}_k]^\dagger \begin{bmatrix} 0 \\ 0 \\ \hat{\mu}_{(23)}^\star \\ 0 \end{bmatrix}, \quad \text{since } \hat{\mu}_{(10)}^\star = \hat{\mu}_{(21)}^\star = \hat{\mu}_{(30)}^\star = 0.
\end{aligned}$$

- Therefore, the LMPs at buses 1 and 3 also satisfy:

$$\begin{aligned}
\lambda_1^\star &= \$40/\text{MWh}, \\
&= \nabla f_1(P_1^\star), \\
&= \hat{\lambda}^\star - 0.2 \times \hat{\mu}_{(23)}^\star, \text{ where 0.2 is the shift factor for bus 1}, \\
\lambda_3^\star &= \$50/\text{MWh}, \\
&= \nabla f_3(P_3^\star), \\
&= \hat{\lambda}^\star - (-0.2) \times \hat{\mu}_{(23)}^\star, \text{ where } (-0.2) \text{ is the shift factor for bus 3}.
\end{aligned}$$

Offer-based optimal power flow, angles eliminated, continued

- Solving these equations simultaneously for $\hat{\lambda}^\star$ and $\hat{\mu}^\star_{(23)}$, we obtain:

$$\begin{aligned}\hat{\mu}^\star_{(23)} &= \$25/\text{MWh},\\ \hat{\lambda}^\star &= \$45/\text{MWh}.\end{aligned}$$

- Therefore, the LMPs at buses 0 and 2 are:

$$\begin{aligned}\lambda^\star_0 &= \hat{\lambda}^\star,\\ &= \$45/\text{MWh},\\ &\quad \text{where we note that bus 0 is the price reference bus,}\\ \lambda^\star_2 &= \hat{\lambda}^\star - 0.4 \times \hat{\mu}^\star_{(23)},\\ &\quad \text{where 0.4 is the shift factor for bus 2,}\\ &= \$35/\text{MWh}.\end{aligned}$$

Offer-based optimal power flow, angles eliminated, continued

- Substituting, we obtain flows of:

$$
\begin{aligned}
\begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} P_1^\star \\ P_2^\star \\ P_3^\star \end{bmatrix} & = \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 750 \\ 1000 \\ 1250 \end{bmatrix}, \\
& = \begin{bmatrix} 1450 \\ 700 \\ 300 \\ 1550 \end{bmatrix}, \\
& \leq \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix}.
\end{aligned}
$$

9.6.10 Offer-based optimal power flow solution

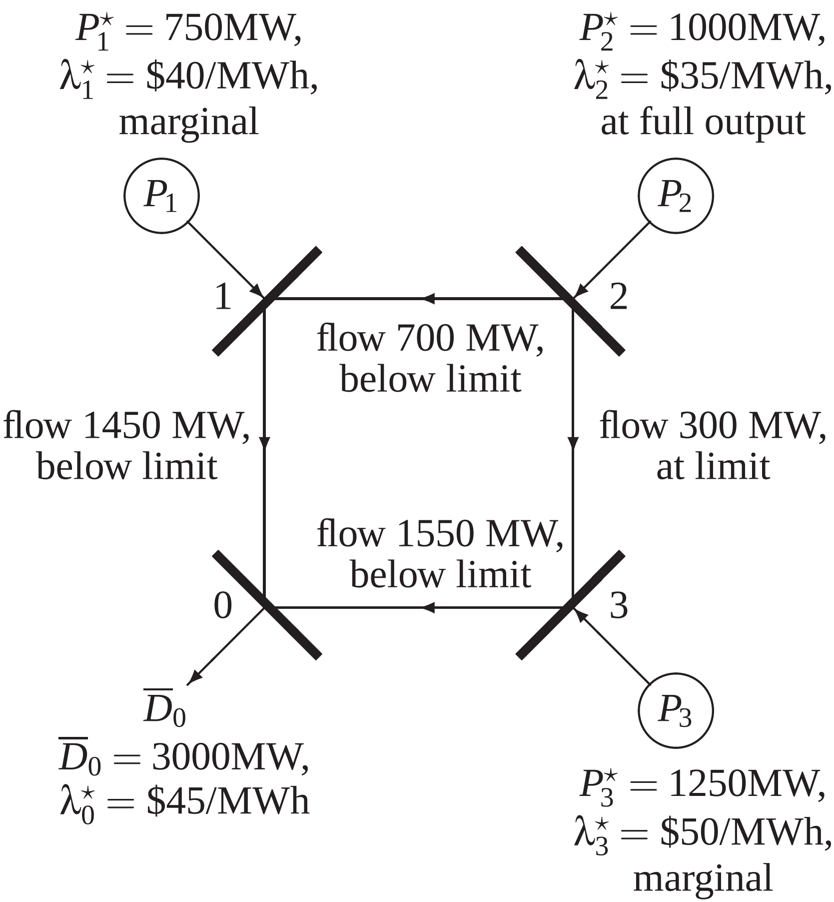

Fig. 9.3. Offer-based optimal power flow for four-line four-bus network.

9.7 Properties of locational marginal prices

- LMPs can be different at every bus:

$$\begin{aligned}\lambda_0^\star &= \$45/\text{MWh},\\ \lambda_1^\star &= \$40/\text{MWh},\\ \lambda_2^\star &= \$35/\text{MWh},\\ \lambda_3^\star &= \$50/\text{MWh}.\end{aligned}$$

- LMPs can be the same as or lower than the offer price at bus:
  - lower than offer price if cheaper imports are feasible,
  - same as offer price if generator is marginal.
- LMPs can be higher than offer price at bus:
  - if no more capacity is available at bus.
- LMPs can be higher or lower at demand than at generation:
  - LMP at demand is higher than LMP at buses 1 and 2,
  - LMP at demand is lower than LMP at bus 3.

Properties of LMPs, continued

- Power can flow from bus with higher LMP to bus with lower LMP:
  - From bus 3, with LMP of \$50/MWh to bus 0 with LMP of \$45/MWh,
  - Injection at bus 3 causes counterflow on line from bus 3 to bus 2, allowing for all the cheap generation at bus 2 to be used.
  - Flow from bus 3 to bus 0 is a *side-effect* of generator 3 injecting at bus 3.
- LMPs can be higher than any generator offer price:
  - if increasing demand necessitates decreasing generation at cheap generator. (See in Homework Exercise 9.2.)
- LMPs can be lower than any generator offer price:
  - if increasing demand by 1 MW allows for more than 1 MW increase at a cheap generator.

Properties of LMPs, continued

- LMPs can be lower than transmission unconstrained price:
  - under offer-based economic dispatch ignoring transmission constraints, unconstrained price was \$50/MWh,
  - under offer-based optimal power flow, LMP is \$45/MWh at bus 0.
  - Note that the transmission unconstrained price is not equal to $\hat{\lambda}^{\star} = \$45/\text{MWh}$.
- LMPs can be higher than transmission unconstrained price:
  - in one line example in Section 9.5, the solution ignoring transmission constraints would result in an LMP of \$25/MWh.

9.8 Congestion rent and congestion cost

9.8.1 Congestion rent

- In the four-line, four-bus example, the payments are:

  Demand pays $\overline{D}_0 \times \lambda_0^\star = 3000\text{MW} \times \$45/\text{MWh} = \$135,000/\text{h}$,
  The generator at bus 1 is paid
  $$P_1^\star \times \lambda_1^\star = 750\text{MW} \times \$40/\text{MWh} = \$30,000/\text{h},$$
  The generator at bus 2 is paid
  $$P_2^\star \times \lambda_2^\star = 1000\text{MW} \times \$35/\text{MWh} = \$35,000/\text{h},$$
  The generator at bus 3 is paid
  $$P_3^\star \times \lambda_3^\star = 1250\text{MW} \times \$50/\text{MWh} = \$62,500/\text{h},$$
- Total payment to the generators is \$127,500/h, which is less than the payment by demand of \$135,000/h.
- The difference between the payment by demand minus the payment to generators is called the **congestion rent**.
- The congestion rent is \$7,500/h for this example.

Congestion rent, continued

- Congestion rent is a revenue stream that accrues to the ISO.
- It is disbursed back to market participants through **financial transmission rights** (known in ERCOT as **congestion revenue rights**):
  - See in Chapter 11.
- Congestion rent is sometimes called **merchandising surplus**.

9.8.2 *Congestion cost*

- A related, but different, concept is the (revealed) **congestion cost**, which is defined as difference between:

  cost of dispatch under offer-based optimal power flow (\$112,500/h), minus
  cost of dispatch under offer-based economic dispatch ignoring transmission constraints (\$105,000/h).

- Congestion cost represents the increased cost of fuel needed due to the finite capability of the transmission network.
- The congestion cost is \$7,500/h in this case.
- The congestion rent is *not* generally equal to the congestion cost.
- In this particular example, the congestion rent and congestion cost happen to be the same!
- More typically, the congestion rent is larger than the congestion cost.
- Congestion rent and congestion cost are often confused:
  - although they are either both zero or both non-zero, there is no direct relationship between them.

9.8.3 Properties of congestion rent

9.8.3.1 Non-negativity

Theorem 9.2 *Congestion rent is always non-negative.*

Proof

- By definition, congestion rent is:

$$\text{payment by demand} - \text{payment to generators} = [\lambda^{\star}]^{\dagger}(\overline{D} - P^{\star}),$$

- where $\lambda^{\star}$ is the vector of LMPs.
- From (9.4), we have that, for some choices of Lagrange multipliers $\hat{\lambda}^{\star}$ and $\hat{\mu}^{\star}$ in the angles eliminated formulation:

$$\forall k = 1, \ldots, n, \lambda_k^{\star} = \hat{\lambda}^{\star} - [\hat{C}_k]^{\dagger} \hat{\mu}^{\star},$$

- where we note that the column of $\hat{C}$ corresponding to the price reference bus was defined to be the zero vector.
- Collect these expressions together into a single vector equation:

$$\lambda^{\star} = \mathbf{1}\hat{\lambda}^{\star} - [\hat{C}]^{\dagger} \hat{\mu}^{\star}. \tag{9.6}$$

- Therefore:

$$\begin{aligned}
\text{congestion rent} &= [\lambda^\star]^\dagger(\overline{D} - P^\star), \\
&= \left[\mathbf{1}\hat{\lambda}^\star - [\hat{C}]^\dagger \hat{\mu}^\star\right]^\dagger (\overline{D} - P^\star), \text{ using the expression for } \lambda^\star, \\
&= [\hat{\lambda}^\star]^\dagger(\mathbf{1}^\dagger\overline{D} - \mathbf{1}^\dagger P^\star) - [\hat{\mu}^\star]^\dagger \hat{C}(\overline{D} - P^\star), \\
&= -[\hat{\mu}^\star]^\dagger \hat{C}(\overline{D} - P^\star),
\end{aligned}$$

- since $\mathbf{1}^\dagger\overline{D} - \mathbf{1}^\dagger P^\star = 0$; that is, $\overline{D}$ and $P^\star$ satisfy the system equality constraint.
- Let $\hat{C}_{(\ell k)}$ be the *row* of $\hat{C}$ corresponding to the line joining buses ℓ and k, let $\overline{p}_{(\ell k)}$ be the corresponding line limit, and let $\hat{\mu}^\star_{(\ell k)}$ be the corresponding Lagrange multiplier on the line limit constraint $\hat{C}(P - \overline{D}) \leq d$.

- Then:

$$\begin{aligned}
[\lambda^\star]^\dagger(\overline{D} - P^\star) &= -[\hat{\mu}^\star]^\dagger \hat{C}(\overline{D} - P^\star), \text{ from the previous page,} \\
&= [\hat{\mu}^\star]^\dagger \hat{C}(P^\star - \overline{D}), \\
&= \sum_{\hat{\mu}^\star_{(\ell k)}=0} \hat{\mu}^\star_{(\ell k)} \hat{C}_{(\ell k)}(P^\star - \overline{D}) + \sum_{\hat{\mu}^\star_{(\ell k)}\neq 0} \hat{\mu}^\star_{(\ell k)} \hat{C}_{(\ell k)}(P^\star - \overline{D}), \\
&= 0 + \sum_{\hat{\mu}^\star_{(\ell k)}\neq 0} \hat{\mu}^\star_{(\ell k)} \hat{C}_{(\ell k)}(P^\star - \overline{D}), \\
&= \sum_{\hat{\mu}^\star_{(\ell k)}\neq 0} \hat{\mu}^\star_{(\ell k)} \overline{p}_{(\ell k)}, \text{ by complementary slackness,} \\
& \text{ since } \hat{C}_{(\ell k)}(P^\star - \overline{D}) \text{ is the flow on the line} \\
& \text{ joining bus } \ell \text{ to } k, \\
&\geq 0,
\end{aligned}$$

- assuming that $\forall \ell, k, \overline{p}_{(\ell k)} \geq 0$, and noting that $\hat{\mu}^\star_{(\ell k)} \geq 0, \forall \ell, k$.

Non-negativity, continued

- That is:

$$\begin{aligned}\text{payment by demand} - \text{payment to generators} &= \sum_{\hat{\mu}^\star_{(\ell k)} \neq 0} \hat{\mu}^\star_{(\ell k)} \overline{p}_{(\ell k)}, \\ &\geq 0.\end{aligned}$$

- We have proved that the congestion rent is non-negative.

$\square$

- Note that we have also proved that the congestion rent is equal to the sum over the binding line constraints of the product of the corresponding Lagrange multiplier and the flow limit.
- In the **flowgate** transmission rights mechanism we associate congestion rent individually to each binding line constraint:
  - the ERCOT zonal market used a flowgate transmission rights mechanism based on inter-zonal flow limits.

9.8.3.2 Changing the dispatch

- Now we will consider a related property that is useful in the discussion of **financial transmission rights**.
- We consider vectors of demand and generation, D' and P', that may differ from the demand $\overline{D}$ and generation $P^{\star}$ in offer-based optimal power flow.
- However, we require that D' and P' satisfy the system constraints, so that:

$$\begin{aligned} -\mathbf{1}^{\dagger}P' &= -\mathbf{1}^{\dagger}D', \\ \hat{C}(P'-D') &\leq d. \end{aligned}$$

- We consider the congestion rent under the following circumstances:

  the prices $\lambda^{\star}$ were determined from the offer-based optimal power flow solution, corresponding to the demand $\overline{D}$ and generation $P^{\star}$, but
  the demand and generation quantities are given by D' and P'.

- That is, we consider $[\lambda^{\star}]^{\dagger}(D'-P')$.

Corollary 9.3

- *Suppose locational marginal prices* $\lambda^\star$ *were determined from the offer-based optimal power flow solution, corresponding to the demand* $\overline{D}$ *and generation* $P^\star$*.*
- *Let* D' *and* P' *be any vectors of demand and generation, respectively, that satisfy the system constraints, so that:*

$$\begin{aligned} -\mathbf{1}^\dagger P' &= -\mathbf{1}^\dagger D', \\ \hat{C}(P'-D') &\leq d. \end{aligned}$$

- *The values* D' *and* P' *may differ from the demand* $\overline{D}$ *and generation* $P^\star$ *in offer-based optimal power flow.*
- *Then:*

$$[\lambda^\star]^\dagger(D'-P') \leq [\lambda^\star]^\dagger(\overline{D}-P^\star). \tag{9.7}$$

Proof We have:

$$
\begin{aligned}
[\lambda^\star]^\dagger(D'-P') &= \left[\mathbf{1}\hat{\lambda}^\star - [\hat{C}]^\dagger \hat{\mu}^\star\right]^\dagger (D'-P'), \text{ by (9.6)},\\
&= [\hat{\lambda}^\star]^\dagger(\mathbf{1}^\dagger D' - \mathbf{1}^\dagger P') - [\hat{\mu}^\star]^\dagger \hat{C}(D'-P'),\\
&= -[\hat{\mu}^\star]^\dagger \hat{C}(D'-P'), \text{ since } \mathbf{1}^\dagger D' - \mathbf{1}^\dagger P' = 0,\\
&= [\hat{\mu}^\star]^\dagger \hat{C}(P'-D'),\\
&= \sum_{\hat{\mu}^\star_{(\ell k)}=0} \hat{\mu}^\star_{(\ell k)} \hat{C}_{(\ell k)}(P'-D') + \sum_{\hat{\mu}^\star_{(\ell k)}\neq 0} \hat{\mu}^\star_{(\ell k)} \hat{C}_{(\ell k)}(P'-D'),\\
&= 0 + \sum_{\hat{\mu}^\star_{(\ell k)}\neq 0} \hat{\mu}^\star_{(\ell k)} \hat{C}_{(\ell k)}(P'-D'),\\
&\leq \sum_{\hat{\mu}^\star_{(\ell k)}\neq 0} \hat{\mu}^\star_{(\ell k)} \overline{p}_{(\ell k)}, \text{ since } \hat{C}(P'-D') \leq d \text{ and } \hat{\mu}^\star_{(\ell k)} \geq 0, \forall \ell, k,\\
&= [\lambda^\star]^\dagger(\overline{D} - P^\star), \text{ from the proof of Theorem 9.2}.
\end{aligned}
$$

□

9.9 Contingency constraints

9.9.1 Pre-contingency versus post-contingency flow

- In the formulation of transmission limits, we have implicitly been considering limits on **pre-contingency flow**.
- However, most transmission systems are **contingency limited**.
- That is, the binding constraint is on a limiting post-contingency flow that would occur on contingency of another line:
  - flows in the post-contingency case result from the generation injections and the post-contingency network.
- These contingency constraints can also be considered in our formulation, but require **outage shift factors**:
  - fraction of post-contingency flow on a line due to injection at generator and withdrawal at price reference bus.

9.9.2 *Example*

- Consider the following eight-line four-bus system.
- To be secure against all single contingencies, we must operate the system so that for any outaged element, the flows on the remaining system are within limits.
- There are eight possible single element outages.

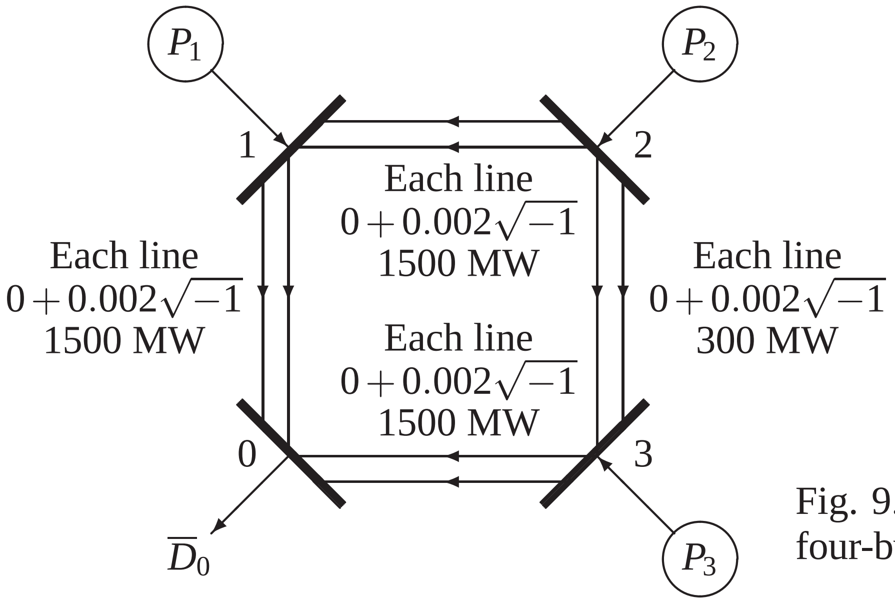

Fig. 9.4. Eight-line four-bus network.

Example, continued

- For example, consider an outage of one of the lines joining bus 2 to bus 3.
- This would yield the system shown.
- We can analyze the contingency constraints by calculating the shift factors for the outage system.
- For example, we would consider the shift factors to the remaining line joining bus 2 to bus 3.

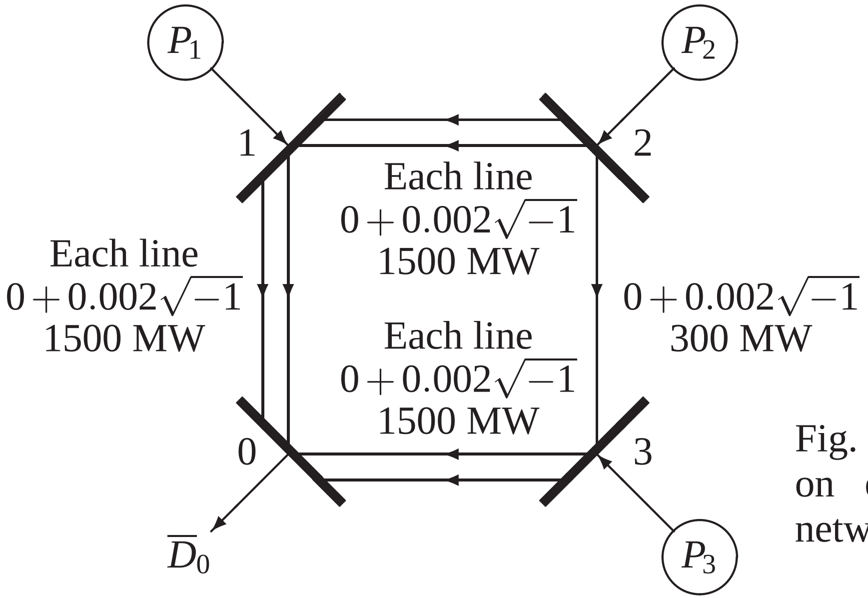

Fig. 9.5. Contingency on eight-line four-bus network.

Example, continued

- For this example, a contingency on one of the lines joining bus 2 and 3 is the most binding contingency:
  - the corresponding post-contingency system happens to have the same admittances and total capacities on each corridor as we considered previously in the pre-contingency limited case.
- To be secure with respect to a contingency on one of the lines joining bus 2 and 3, we must operate so that this contingency would not result in overload of the remaining lines post-contingency.
- Assuming the same offers as previously, the resulting generation dispatch and LMPs are the same as the solution we found for the pre-contingency limited case.

Example, continued

- However, the *pre-contingency* flows resulting from this dispatch are different to the solution we found previously:
  - we must dispatch so that post-contingency flows are within constraints on post-contingency system,
  - but unless the contingency actually occurs, flows will be due to generation injections and the pre-contingency network,
  - pre-contingency flows are typically well below capacities.
- Consider a corridor of two parallel, identical lines each with 100 MW capacity joining two zones. What is the maximum pre-contingency flow on each line to ensure security?

Example, continued

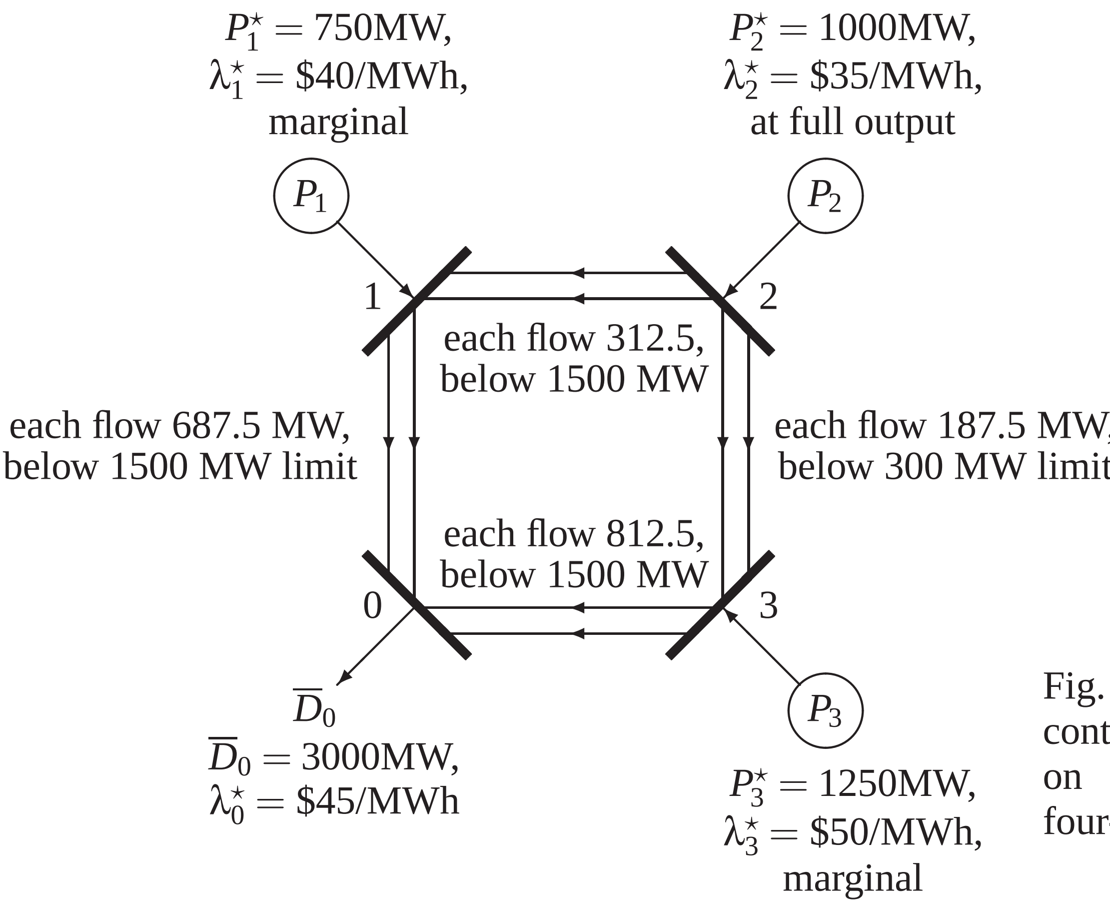

Fig. 9.6. Pre-contingency flows on eight-line four-bus network.

9.9.3 Representation of contingency constraints

- Note that the *post-contingency* flows in the system must be represented in terms of the generation levels *pre-contingency* in the economic dispatch problem:
  - the relevant system constraint is on *post-contingency* flow as a function of generation.

Representation of contingency constraints, continued

- In the ERCOT zonal system, the Commercially Significant Constraints (CSCs) were represented by the effect on *pre-contingency* flow on the CSCs as a function of generation:
  - however, pre- and post-contingency shift factors are different,
  - so the approximation used in the ERCOT zonal system used the *incorrect* derivative of the function representing the post-contingency flows that appear in the system constraints; the wrong shift factors were used.
  - As discussed previously, this distorted the incentives away from inducing the behavior that would be consistent with contingency-constrained economic dispatch.
  - When generators then behaved consistently with their incentives, but inconsistent with actual constraints, ERCOT had to adjust the constraints or take out-of-market actions to maintain feasibility.

Representation of contingency constraints, continued

- In the ERCOT nodal system, contingency constraints are represented in terms of (linearizations of) post-contingency flows.
- Incentives for generators are better aligned with the actual transmission constraints:
  - because more of the constraints are represented, and
  - because the contingency constraints are represented correctly.

9.10 Reactive power prices

9.10.1 Offer-based economic dispatch formulation

- Recall the AC formulation:

$$\min_{x\in\mathbb{R}^n}\left\{\sum_{k=1}^{n_P} f_k(x_k)\;\middle|\; \begin{array}{l} p\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right) - P = -\overline{D}, q\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right) - Q = -\overline{E},\\ \\ \underline{P} \leq P \leq \overline{P}, \forall(\ell k)\in\mathbb{K}, p_{\ell k}\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right) \leq \overline{p}_{\ell k}\end{array}\right\},$$

- where we now explicitly allow the cost (and the offer) for generator k to be a function of both real and reactive power, so that $x_k = \begin{bmatrix}P_k\\ Q_k\end{bmatrix}$.
- We neglect other ancillary services for simplicity.
- In this case, the system equality constraints include both terms for real power and for reactive power.

9.10.2 Pricing rule, angles explicit

- Let the minimizer be $x^\star = \begin{bmatrix} P^\star \\ Q^\star \\ \theta^\star_{-\rho} \\ u^\star \end{bmatrix}$.
- Let $\lambda^\star_{Pk}$ and $\lambda^\star_{Qk}$ be the Lagrange multipliers on real and reactive power balance at generator k.
- From Theorem 8.3, we can write down the pricing rule for generator k:

$$\pi_{x_k} = \begin{bmatrix} \lambda^\star_{Pk} \\ \lambda^\star_{Qk} \end{bmatrix},$$

- so that there are prices for both real and reactive power.
- That is, the payment to generator k is:

$$[\pi_{x_k}]^\dagger x_k = \lambda^\star_{Pk} P_k + \lambda^\star_{Qk} Q_k,$$

- on the basis of both its real and reactive power production.
- How would you expect the values of $\lambda^\star_{Pk}$ and $\lambda^\star_{Qk}$ to compare?

9.10.3 Discussion

- Although the theoretical development of prices for reactive power is straightforward, a difficulty with setting up a market for reactive power is that reactive power does not "travel" far, so that there are serious issues of geographical market power.
- Furthermore, although real power reserves are typically less valuable than energy, reactive reserves may be more valuable than steady-state reactive power:
  - so contingency constraints should be explicitly represented.

9.11 Loss prices

9.11.1 Offer-based economic dispatch formulation

- Again recall the AC formulation:

$$\min_{x\in\mathbb{R}^n}\left\{\sum_{k=1}^{n_P} f_k(P_k)\left|\begin{array}{l} p\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)-P=-\overline{D}, q\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)-Q=-\overline{E}, \\ \\ \underline{P}\leq P\leq\overline{P}, \forall(\ell k)\in\mathbb{K}, p_{\ell k}\left(\begin{bmatrix}\theta_{-\rho}\\ u\end{bmatrix}\right)\leq\overline{p}_{\ell k}\end{array}\right.\right\}.$$

- In this case, we will simplify the optimal power flow formulation by:
  - omitting the reactive power flow equations,
  - omitting other ancillary services,
  - deleting the reactive power and voltage magnitude variables from the decision vector, and
  - fixing the voltage magnitude schedule at $u^{(0)}$.

Offer-based economic dispatch formulation, continued

- However, we will keep the non-linear real power flow equations explicit and our decision vector will be $x = \begin{bmatrix} P \\ \theta_{-\rho} \end{bmatrix}$, to yield:

$$\min_{P,\theta_{-\rho}} \left\{ \sum_{k=1}^{n_P} f_k(P_k) \middle| \begin{array}{l} p\left(\begin{bmatrix} \theta_{-\rho} \\ u^{(0)} \end{bmatrix}\right) - P = -\overline{D}, \underline{P} \leq P \leq \overline{P}, \\ \forall(\ell k) \in \mathbb{K}, p_{\ell k}\left(\begin{bmatrix} \theta_{-\rho} \\ u^{(0)} \end{bmatrix}\right) \leq \overline{p}_{\ell k} \end{array} \right\}.$$

- Let the minimizer be $P^\star$ and $\theta^\star_{-\rho}$.
- We assume that we can find Lagrange multipliers $\lambda^\star$ and $\mu^\star$ on the system equality and inequality constraints.

9.11.2 Pricing rule, angles explicit

- Note that the decision variable for generator k is $x_k = [P_k]$.
- From Theorem 8.3, we can write down the pricing rule:

$$\pi_{P_k} = \lambda_k^\star.$$

9.11.3 Formulation to eliminate angles

9.11.3.1 Transformation

- We consider a similar transformation of the equality constraints to the one we used when we eliminated the angles to formulate the DC power flow equations.
- Define the invertible matrix $\mathcal{M} = \begin{bmatrix} 1 & \mathbf{1}^\dagger \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ and notice that:

$$\left(p\left(\begin{bmatrix}\theta_{-\rho} \\ u^{(0)}\end{bmatrix}\right) - P = -\overline{D}\right)$$
$$\Leftrightarrow \quad \left(\mathcal{M}\, p\left(\begin{bmatrix}\theta_{-\rho} \\ u^{(0)}\end{bmatrix}\right) - \mathcal{M}\, P = -\mathcal{M}\, D\right),$$
$$\Leftrightarrow \quad \left(\mathbf{1}^\dagger p\left(\begin{bmatrix}\theta_{-\rho} \\ u^{(0)}\end{bmatrix}\right) - \mathbf{1}^\dagger P = -\mathbf{1}^\dagger \overline{D}, p_{-\sigma}\left(\begin{bmatrix}\theta_{-\rho} \\ u^{(0)}\end{bmatrix}\right) - P_{-\sigma} = -\overline{D}_{-\sigma}\right),$$

- where $p_{-\sigma}\left(\begin{bmatrix}\theta_{-\rho} \\ u^{(0)}\end{bmatrix}\right)$, $P_{-\sigma}$, and $\overline{D}_{-\sigma}$ are the sub-vectors of $p\left(\begin{bmatrix}\theta_{-\rho} \\ u^{(0)}\end{bmatrix}\right)$, P, and $\overline{D}$, respectively, with the price reference bus deleted.

Transformation, continued

- That is, the AC power flow is equivalent to satisfying:

$$\begin{aligned} \mathbf{1}^\dagger p\left(\begin{bmatrix} \theta_{-\rho} \\ u^{(0)} \end{bmatrix}\right) - \mathbf{1}^\dagger P &= -\mathbf{1}^\dagger \overline{D}, \\ p_{-\sigma}\left(\begin{bmatrix} \theta_{-\rho} \\ u^{(0)} \end{bmatrix}\right) - P_{-\sigma} &= -\overline{D}_{-\sigma}. \end{aligned}$$

9.11.3.2 Losses

- The equality:

$$\mathbf{1}^\dagger p\left(\begin{bmatrix} \theta_{-\rho} \\ u^{(0)} \end{bmatrix}\right) - \mathbf{1}^\dagger P = -\mathbf{1}^\dagger \overline{D},$$

- requires that generation equal demand plus losses.
- That is, losses are $\mathbf{1}^\dagger p\left(\begin{bmatrix} \theta_{-\rho} \\ u^{(0)} \end{bmatrix}\right)$.

9.11.3.3 Inverting the power flow equations

- What does power flow software calculate?
- Given $P_{-\sigma}$ and $\overline{D}_{-\sigma}$, it calculates the corresponding $\theta_{-\rho}$ (and u) that solve the power flow equations.
- That is, it inverts the equations:

$$p_{-\sigma}\left(\begin{bmatrix}\theta_{-\rho}\\ u^{(0)}\end{bmatrix}\right) - P_{-\sigma} = -\overline{D}_{-\sigma},$$

- to solve for the angle $\theta_{-\rho}$ as a function of $P_{-\sigma}$.
- That is, power flow software implicitly defines an inverse function $\hat{\theta}_{-\rho}$ to $p_{-\sigma}$ that satisfies:

$$\forall P_{-\sigma}, p_{-\sigma}\left(\begin{bmatrix}\hat{\theta}_{-\rho}(P_{-\sigma})\\ u^{(0)}\end{bmatrix}\right) - P_{-\sigma} = -\overline{D}_{-\sigma}.$$

- In other words, can use $\hat{\theta}_{-\rho}$ to substitute for angles according to:

$$\theta_{-\rho} = \hat{\theta}_{-\rho}(P_{-\sigma}).$$

9.11.3.4 Loss function

- Define the loss function $L : \mathbb{R}^{n-1} \to \mathbb{R}$ by:

$$\forall P_{-\sigma}, L(P_{-\sigma}) = \mathbf{1}^\dagger p\left(\begin{bmatrix} \hat{\theta}_{-\rho}(P_{-\sigma}) \\ u^{(0)} \end{bmatrix}\right).$$

- Define the function $\hat{g} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\begin{aligned} \forall P, \hat{g}(P) &= \mathbf{1}^\dagger p\left(\begin{bmatrix} \hat{\theta}_{-\rho}(P_{-\sigma}) \\ u^{(0)} \end{bmatrix}\right) - \mathbf{1}^\dagger P + \mathbf{1}^\dagger \overline{D}, \\ &= L(P_{-\sigma}) - \mathbf{1}^\dagger P + \mathbf{1}^\dagger D. \end{aligned}$$

- If we require $\hat{g}(P) = 0$ then we require that generation equal demand plus losses.
- Note that:

$$\frac{\partial \hat{g}}{\partial P_k}(P) = \begin{cases} -1, & \text{if } k \text{ is the price reference bus,} \\ \dfrac{\partial L}{\partial P_k}(P_{-\sigma}) - 1, & \text{if } k \text{ is not the price reference bus.} \end{cases}$$

9.11.3.5 Line flows

- Similarly, define the function $\hat{h}$ and vector $\overline{h}$ by:

$$\forall P, \forall(\ell k) \in \mathbb{K}, \hat{h}_{(\ell k)}(P) = p_{\ell k}\left(\begin{bmatrix} \hat{\theta}_{-\rho}(P_{-\sigma}) \\ u^{(0)} \end{bmatrix}\right), \overline{h}_{(\ell k)} = \overline{p}_{\ell k}.$$

- Then if we require that $\hat{h}(P) \leq \overline{h}$, we have have satisfied the system inequality constraints.

9.11.3.6 Formulation

- We can formulate the offer-based economic dispatch problem as:

$$\min_{P} \left\{ \sum_{k=1}^{n_P} f_k(P_k) \,\middle|\, \hat{g}(P) = 0, \hat{h}(P) \leq \overline{h}, \underline{P} \leq P \leq \overline{P} \right\},$$

- where the functions $\hat{g}$ and $\hat{h}$ are provided by power flow software.
- Let the minimizer be $P^\star$.
- Let $P^\star_{-\sigma}$ be the sub-vector of $P^\star$ with the entry for the price reference bus deleted.
- We assume that we can find Lagrange multipliers $\hat{\lambda}^\star$ and $\hat{\mu}^\star$ on the system equality and inequality constraints.

9.11.4 Pricing rule, angles eliminated

- From Theorem 8.3, we can write down the pricing rule for generator k:

$$\pi_{P_k} = -\left[\frac{\partial \hat{g}}{\partial P_k}(P^\star)\right]^\dagger \hat{\lambda}^\star - \left[\frac{\partial \hat{h}}{\partial P_k}(P^\star)\right]^\dagger \hat{\mu}^\star,$$

$$= \begin{cases} \hat{\lambda}^\star, & \text{if } k \text{ is the price reference bus,} \\ \left(1 - \frac{\partial L}{\partial P_k}(P^\star_{-\sigma})\right)\hat{\lambda}^\star - \left[\frac{\partial \hat{h}}{\partial P_k}(P^\star)\right]^\dagger \hat{\mu}^\star, & \text{if } k \text{ is not the price reference bus,} \end{cases}$$

Pricing rule, angles eliminated, continued

- As previously, these prices must match the corresponding prices from the formulation with angles explicitly represented, so that:

$$\lambda_k^\star = \begin{cases} \hat{\lambda}^\star, & \text{if } k \text{ is the price reference bus,} \\ \left(1 - \frac{\partial L}{\partial P_k}(P_{-\sigma}^\star)\right)\hat{\lambda}^\star - \left[\frac{\partial \hat{h}}{\partial P_k}(P^\star)\right]^\dagger \hat{\mu}^\star, & \text{if } k \text{ is not the price reference bus,} \end{cases}$$

- The LMP at bus k, $\lambda_k^\star$, is equal to:

  the LMP at the price reference bus,
  minus the loss penalty for the effect on marginal losses,
  minus a weighted sum of the Lagrange multipliers on the line flow limit constraints.

- The weights are the “incremental shift factors” to the constraints.
- The LMP at the price reference bus, the marginal losses, and the Lagrange multipliers on the line flow limit constraints will each depend on the location of the price reference bus.
- However, the LMP at each bus is independent of the choice of the location of the price reference bus.

9.11.5 Example

- We modify the one-line two-bus system from Section 9.5 to include losses.
- Bus $\rho = 1$ is the angle reference bus, so the unknown angle is θ_2.
- Bus $\sigma = 2$ is the slack/price reference bus.
- There are generators at both buses 1 and 2.
- There is $\overline{D}_2$ MW of demand at bus 2.
- The offers are specified by:

$$\begin{aligned} \forall P_1 \in [0,200], \nabla f_1(P_1) &= \$25/\text{MWh}, \\ \forall P_2 \in [0,50], \nabla f_2(P_2) &= \$35/\text{MWh}. \end{aligned}$$

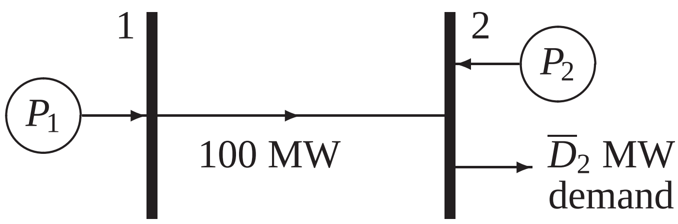

Fig. 9.7. One-line two-bus network.

9.11.5.1 Admittance matrix

- We modify the line admittance to include losses:

$$Y_{12} = 100 - 1000\sqrt{-1}.$$

- The bus admittance matrix is:

$$\begin{aligned}
\begin{bmatrix} Y_{12} & -Y_{12} \\ -Y_{12} & Y_{12} \end{bmatrix} &= \begin{bmatrix} 100 - 1000\sqrt{-1} & -100 + 1000\sqrt{-1} \\ -100 + 1000\sqrt{-1} & 100 - 1000\sqrt{-1} \end{bmatrix}, \\
&= \begin{bmatrix} G_{11} + B_{11}\sqrt{-1} & G_{12} + B_{12}\sqrt{-1} \\ G_{21} + B_{21}\sqrt{-1} & G_{22} + B_{22}\sqrt{-1} \end{bmatrix}.
\end{aligned}$$

- We assume that the voltage magnitudes are maintained equal to one per unit, so that $u^{(0)} = \mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9.11.5.2 Capacity constraint

- We also assume that the thermal capacity constraints are expressed in terms of maximum *current* magnitude:
  - for voltages equal to one per unit, the previous 100 MW constraint becomes a 100 per unit current constraint,
  - there is a constraint on flow at each end of the line.
- Since we have ignored shunt elements, the current is the same at both ends of the line:
  - we can consider current at either end of the line (or at any point on the line.)

- The current I_1 flowing from bus 1 into the line is:

$$
\begin{aligned}
I_1 &= A_{11}V_1 + A_{12}V_2, \\
&\quad \text{where } A \text{ is the bus admittance matrix,} \\
&\quad\quad \text{and } V_k \text{ is the voltage phasor at bus } k = 1,2, \\
&= (100 - 1000\sqrt{-1})(V_1 - V_2). \\
|I_1|^2 &= \left|100 - 1000\sqrt{-1}\right|^2 |V_1 - V_2|^2, \\
&= [(100)^2 + (1000)^2]\left|u_1 - u_2(\cos(\theta_2) + \sin(\theta_2)\sqrt{-1})\right|^2, \\
&= [(100)^2 + (1000)^2]\left|1 - (\cos(\theta_2) + \sin(\theta_2)\sqrt{-1})\right|^2, \\
&\quad\quad \text{since the voltage magnitudes are one per unit,} \\
&= [(100)^2 + (1000)^2][(1 - \cos(\theta_2))^2 + (\sin(\theta_2))^2], \\
&= [(100)^2 + (1000)^2][2 - 2\cos(\theta_2)]
\end{aligned}
$$

- If we require the magnitude of the current to be less than 100 then this requires $\theta_2 \geq -0.0995 = \underline{\theta}_2$ radian:
  – very close to the limit of -0.1 radian we found in the lossless case.

9.11.5.3 Power flow at capacity

- Using the expressions for real power injection, we obtain:

$$
\begin{aligned}
p_1(\theta_2) &= -100\cos(\theta_2) - 1000\sin(\theta_2) + 100, \\
p_1(\underline{\theta}_2) &= -100\cos(\underline{\theta}_2) - 1000\sin(\underline{\theta}_2) + 100, \\
&= 99.87, \\
p_2(\theta_2) &= -100\cos(\theta_2) + 1000\sin(\theta_2) + 100, \\
p_2(\underline{\theta}_2) &= -100\cos(\underline{\theta}_2) + 1000\sin(\underline{\theta}_2) + 100, \\
&= -98.88.
\end{aligned}
$$

- That is, when flow is at capacity, 99.87 MW is injected at bus 1 into the line and 98.88 MW is delivered to bus 2.
- Note that losses are $p_1(\theta_1) + p_2(\theta_2)$, which are 0.99 MW when the line flow is at capacity.
- Injected power at bus 1 is less than 100 MW since some reactive power is injected into the line to maintain voltage equal to 1 per unit.

9.11.5.4 Dispatch and prices for varying demand

- If demand is less than or equal to 98.88 MW then only generator 1 is dispatched to meet demand:
  - the Lagrange multiplier on the line flow constraint is $\hat{\mu}^{\star}_{(12)} = 0$.
  - LMP at bus 1 is $\lambda^{\star}_1 = \$25/\text{MWh}$, reflecting offer at bus 1,
  - generation at bus 1 is slightly more than demand,
  - LMP at bus 2 is slightly more than \$25/MWh, reflecting marginal impact of losses to transmit from bus 1 to bus 2:

$$\begin{aligned}
\lambda^{\star}_2 &= \hat{\lambda}^{\star}, \\
&= \frac{\lambda^{\star}_1}{\left(1 - \dfrac{\partial L}{\partial P_1}(P^{\star}_1)\right)}, \\
&> \lambda^{\star}_1, \text{ since } 0 < \frac{\partial L}{\partial P_1}(P^{\star}_1) < 1.
\end{aligned}$$

Dispatch and prices for varying demand, continued

- If demand is greater than 98.88 MW then both generator 1 and generator 2 are dispatched:
  - the Lagrange multiplier on the line flow constraint is $\hat{\mu}^{\star}_{(12)} > 0$.
  - LMP at bus 1 is $\lambda_1^{\star} = \$25/\text{MWh}$, reflecting offer at bus 1,
  - generation at bus 1 is 99.87 MW,
  - LMP at bus 2 is $\lambda_2^{\star} = \$35/\text{MWh}$, reflecting offer at bus 2,
  - generation at bus 2 is $(\overline{D}_2 - 98.88\text{ MW})$,
  - losses are 0.99 MW.
  - Difference between LMPs at ends of line due to both losses and congestion:

$$\begin{aligned}
\lambda_1^{\star} &= \left(1 - \frac{\partial L}{\partial P_1}(P_1^{\star})\right)\lambda_2^{\star} - \left[\frac{\partial \hat{h}_{(12)}}{\partial P_1}(P^{\star})\right]^{\dagger} \hat{\mu}^{\star}_{(12)}, \\
&= \lambda_2^{\star} - \left[\frac{\partial L}{\partial P_1}(P_1^{\star})\lambda_2^{\star} + \left[\frac{\partial \hat{h}_{(12)}}{\partial P_1}(P^{\star})\right]^{\dagger} \hat{\mu}^{\star}_{(12)}\right], \\
&< \lambda_2^{\star}.
\end{aligned}$$

9.11.6 Surplus

- Losses are well-approximated by a convex quadratic function of injections, so that $\hat{g}$ is convex:
  - the “marginal losses” are approximately double the “average losses.”
- Similarly, thermal line flow limit constraints in $\hat{h}$ are convex for small enough angle differences across the lines.
- From the pricing and uplift Theorem 8.3 for convex non-linear system constraints there will be a surplus.
- That is, assuming thermal constraints are the only binding constraints, pricing that includes the marginal losses will generate a surplus for the ISO:
  - surplus can, in principle, be disbursed back to market participants.

9.12 Decomposition approaches

9.12.1 Inverting the power flow equations

- In general, we cannot explicitly invert the power flow equations to analytically determine the functions $\hat{g}$ and $\hat{h}$.
- As in the discussion of losses, however, for a given choice of generations we can use power flow software to calculate:
  - the power flows,
  - the sensitivity of power flows to generation,
  - the losses, and
  - the sensitivity of losses to generation.
- We can also solve **contingency power flows** for each contingency to evaluate, for given pre-contingency generations:
  - the contingency power flows, and
  - the sensitivity of the contingency power flows to generation.

9.12.2 Successively linearizing constraints

- Using the power flows and sensitivities, we can approximate the losses with a first-order Taylor approximation about the given choice of generations.
- Using the power flows and sensitivities, we can also approximate each pre-contingency line flow constraint and each post-contingency line flow constraint by its first-order Taylor approximation about the given choice of generations.
- We can also linearize other types of constraints in addition to real power flow constraints:
  - constraints on complex power flow,
  - current limits,
  - voltage and reactive power constraints, and
  - transient and dynamic stability constraints.
- We can solve the offer-based optimal power flow by iterating between solving power flow and optimizing the linearized approximation.
- We successively re-linearize the power flow solution at each solution of the optimized linear approximation.

Linearizing constraints, continued

- Voltage and reactive power constraints require solution of AC power flow including reactive power:

  Linearizing voltage constraints in terms of real power yields a proxy thermal limit for the voltage constraints.

  Since the voltage to real power relationship is highly non-linear, the linearization will change significantly from iteration to iteration.

  This is particularly true for voltage-related contingency constraints.

  Moreover, voltage constraints may define a non-convex feasible set.

- Recent work on OPF using a rectangular representation of voltage phasors may allow for more effective modeling of such constraints.
- Transient and dynamic stability constraints require solution of transient behavior.

9.12.3 Iterative re-linearization

(i) Set initial list of indices of binding constraints, $\mathbb{W}$, to be empty.

(ii) Set initial linearization of losses to zero.

(iii) Solve offer-based optimal power flow for generations, given current loss linearization and current set of linearized constraints as specified by indices in $\mathbb{W}$.

(iv) Solve power flow and contingency power flows given generations from solution to step (iii).

(v) Update linearization of losses.

(vi) For each binding or violated pre- or post-contingency constraint (and possibly also some constraints that are close to limits or have been binding at previous iterations):

- form the first-order Taylor approximation to the constraint, and
- include the index of the constraint in $\mathbb{W}$.

(vii) If there are violated constraints or the change from the previous solution of offer-based economic dispatch is too large then go to step (iii).

(viii) Otherwise, end.

Iterative re-linearization, continued

- This decomposition can be used even in the case of DC powerflow to avoid explicitly representing all the line flow constraints into the offer-based optimal power flow calculation.
- In a real-time market, linearization of the line flow constraints can be based on the results of **state estimation**.

9.13 Summary

- In this chapter we have considered transmission constraints.
- We formulated the optimal power flow problem and considered offer-based optimal power flow.
- We applied the previously derived pricing rule to obtain the locational marginal prices.
- We considered properties of the locational marginal prices.
- We discussed several other topics, including congestion rent, prices with AC power flow, losses, and decomposition techniques.

References

- Section 15.6 of *Applied Optimization: Formulation and Algorithms for Engineering Systems*, Cambridge University Press 2006.
- Fred C. Schweppe, Michael C. Caramanis, Richard D. Tabors, and Roger E. Bohn, *Spot Pricing of Electricity*, Kluwer Academic, Boston, MA, 1988.
- William W. Hogan "Contract Networks for Electric Power Transmission," *Journal of Regulatory Economics*, 4(3):211–242, 1992.
- Felix Wu, Pravin Varaiya, Pablo Spiller, and Shmuel Oren, "Folk Theorems on Transmission Access: Proofs and Counterexamples," *Journal of Regulatory Economics*, 10(1):5–23, 1996.
- Tina Orfanogianni and George Gross, "A general formulation for LMP evaluation," *IEEE Transactions on Power Systems*, 22(3):1163–1173, August 2007.
- M. Rivier and J. I. Perez-Arriaga, "Computation and decomposition of spot prices for transmission pricing, *Proceedings of the 11th PSC Conference*, 1993.

Homework exercises

9.1 Consider the example one-line two-bus system as shown in Figure 9.8. Bus $\sigma = 1$ is the slack/price reference bus and bus $\rho = 2$ is the angle reference bus, so the unknown angle is θ_1. There are generators at both buses 1 and 2 with offers again specified by:

$$\begin{aligned} \forall P_1 \in [0,200], \nabla f_1(P_1) &= \$25/\text{MWh}, \\ \forall P_2 \in [0,50], \nabla f_2(P_2) &= \$35/\text{MWh}. \end{aligned}$$

Find the LMPs for the following values of demand $\overline{D}_2$:

(i) $\overline{D}_2 = 90\text{MW}$.
(ii) $\overline{D}_2 = 100\text{MW}$.
(iii) $\overline{D}_2 = 125\text{MW}$.

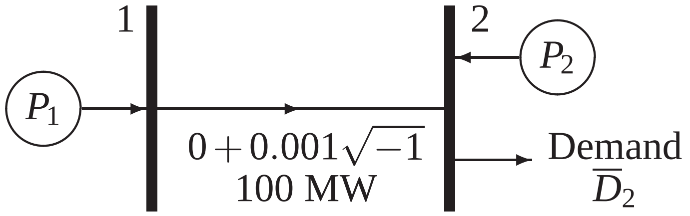

Fig. 9.8. One-line two-bus network for exercise.

9.2 Consider the example four-line four-bus system from Section 9.6 and illustrated in Figure 9.9. Bus $\sigma = 0$ is the slack/price reference bus. Bus $\rho = 0$ is the angle reference bus, so the unknown angles are $\theta_{-\rho} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$. Demand is at bus 0.

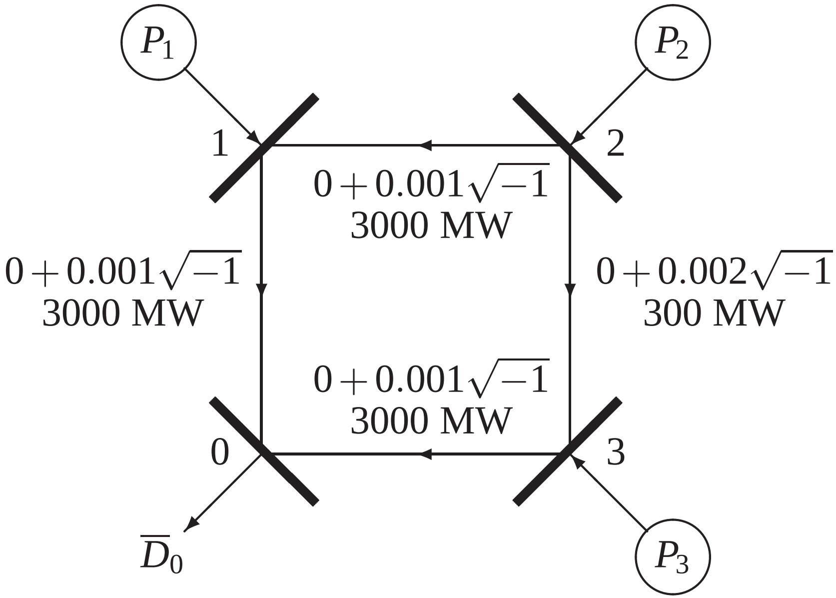

Fig. 9.9. Four-line four-bus network for homework exercise.

Use PowerWorld, the excel solver, or the MATLAB optimization toolbox to solve the following variations on the example in Section 9.6. Continue to the use the DC power flow approximation. In each case, specify the dispatch and the LMPs. You should obtain the Lagrange multipliers from the optimization software in order to facilitate your calculations.

(i) The generation offers are the same as in the example:

$$\begin{aligned}
\forall P_1 \in [0, 1500], \nabla f_1(P_1) &= \$40/\text{MWh}, \\
\forall P_2 \in [0, 1000], \nabla f_2(P_2) &= \$20/\text{MWh}, \\
\forall P_3 \in [0, 1500], \nabla f_3(P_3) &= \$50/\text{MWh}.
\end{aligned}$$

However, the demand changes to $\overline{D}_0 = 1500\text{MW}$.

(ii) The demand is the same as in the example, so that $\overline{D}_0 = 3000$MW. However, the generation offer capacity of generator 3 changes from 1500 MW to 1200 MW. That is, the offers are now:

$$\begin{aligned}
\forall P_1 \in [0,1500], \nabla f_1(P_1) &= \$40/\text{MWh},\\
\forall P_2 \in [0,1000], \nabla f_2(P_2) &= \$20/\text{MWh},\\
\forall P_3 \in [0,1200], \nabla f_3(P_3) &= \$50/\text{MWh}.
\end{aligned}$$

(iii) The generation offers are the same as in the example:

$$\begin{aligned}
\forall P_1 \in [0,1500], \nabla f_1(P_1) &= \$40/\text{MWh},\\
\forall P_2 \in [0,1000], \nabla f_2(P_2) &= \$20/\text{MWh},\\
\forall P_3 \in [0,1500], \nabla f_3(P_3) &= \$50/\text{MWh}.
\end{aligned}$$

The demand is the same as in the example, so that $\overline{D}_0 = 3000$MW. However, the transmission capacity of the line from bus 2 to bus 1 changes from $\overline{p}_{(21)} = 3000$MW to $\overline{p}_{(21)} = 600$MW.

9.3 We again consider the modified one-line two-bus system from Section 9.5 that includes losses, as shown in Figure 9.10. Bus $\rho = 1$ is the angle reference bus, so the unknown angle is θ_2. The power flow injections are:

$$\begin{aligned} p_1(\theta_2) &= -100\cos(\theta_2) - 1000\sin(\theta_2) + 100, \\ p_2(\theta_2) &= -100\cos(\theta_2) + 1000\sin(\theta_2) + 100. \end{aligned}$$

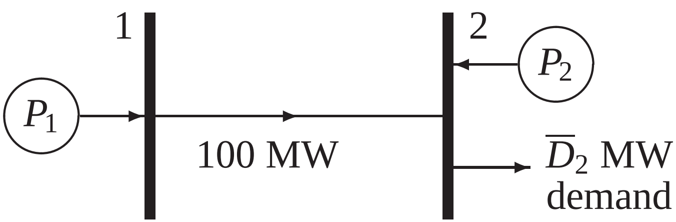

Fig. 9.10. One-line two-bus network.

Adding and subtracting these equations, we obtain:

$$\begin{aligned} p_1(\theta_2)+p_2(\theta_2) &= 200-200\cos(\theta_2), \\ p_2(\theta_2)-p_1(\theta_2) &= p_1(\theta_2)+p_2(\theta_2)-2p_1(\theta_2), \\ &= 2000\sin(\theta_2). \end{aligned}$$

Noting that the injection at bus 1 is $P_1 = p_1(\theta_2)$ and that the losses are $L(P_1) = p_1(\theta_2)+p_2(\theta_2)$, we obtain:

$$\begin{aligned} L(P_1) &= 200-200\cos(\theta_2), \\ L(P_1)-2P_1 &= 2000\sin(\theta_2). \end{aligned}$$

(i) Eliminate θ_2 from the last two equations.

(ii) Use the “quadratic equation” to express the losses as a function of P_1. (There are two solutions. Which operating condition would you prefer: the lower or the higher losses? Use that one.)

(iii) Graph the losses versus P_1.

(iv) Differentiate the expression for losses with respect to P_1.

(v) Find the LMPs and dispatch when demand is $\overline{D}_2 = 98.88$MW, so that the line is just at limit.

(vi) Bonus question: perform several iterations of the iterative linearization algorithm described in Section 9.12.3 to solve for the LMPs and dispatch for the three cases of demand:

  (a) $\overline{D}_2 = 90$MW,
  (b) $\overline{D}_2 = 100$MW, and
  (c) $\overline{D}_2 = 110$MW.

9.4 Use PowerWorld to open the 13 bus system that you downloaded for Exercise 7.1. Select the "Tools" menu and then start the solution by clicking on the "Play" button. Then select the "Add Ons" menu and click on "Primal LP." The system has been set up so that PowerWorld then solves an optimal power flow with all pre-contingency flow limits enforced. In the following parts, each time you modify the system, you need to click on "Primal LP" to re-solve for the optimal power flow.

(i) Click on each generator in turn to obtain the minimum and maximum production capacity for each generator.

(ii) What line is at capacity for the initial configuration of load? How many marginal generators are there?

(iii) Remove the load at bus J by clicking on the associated circuit breaker. Re-solve. What lines are at capacity? How many marginal generators are there?

(iv) Now return the load at bus J to service and remove the line joining bus B to A. Re-solve. What lines are at capacity? How many marginal generators are there?

9.5 Using the DC power flow approximation to linearize the relationship between the real power flows on the lines and the angles, use the MATLAB function `quadprog` to solve the DC optimal power flow that minimizes the cost of production of the generators subject to linearized constraints on the line flows. The system has three buses, buses 1, 2, and 3, and three lines, with the π-equivalent line models specified as follows:

- shunt elements purely capacitive with admittance $0.01\sqrt{-1}$ so that the combined shunt elements at each bus are:

$$Y_1 = Y_2 = Y_3 = 0.02\sqrt{-1},$$

  and
- series elements having admittances:

$$\begin{aligned} Y_{12} &= (0.01 + 0.1\sqrt{-1})^{-1}, \\ Y_{23} &= (0.015 + 0.15\sqrt{-1})^{-1}, \\ Y_{31} &= (0.02 + 0.2\sqrt{-1})^{-1}. \end{aligned}$$

Furthermore, assume the following.

- There are generators at bus 1 and bus 2 and a real power load of 1 at bus 3.

- All lines have real power flow limits of 0.75 in each direction, except for the line joining buses 2 and 3, which has real power flow limits of 0.5 in each direction. That is, there are six transmission constraints in total.
- All voltage magnitudes are set to 1.0 per unit so that u can be ignored in the formulation.
- Zero cost for reactive power production and no constraints on reactive power production nor on reactive power flow so that Q can be ignored in the formulation.
- Costs for real power production at the generators:

$$\begin{aligned} f_1(P_1) &= P_1 \times 1 \frac{\$}{\text{per unit}} + (P_1)^2 \times 0.1 \frac{\$}{(\text{per unit})^2}, \\ f_2(P_2) &= P_2 \times 1.1 \frac{\$}{\text{per unit}} + (P_2)^2 \times 0.05 \frac{\$}{(\text{per unit})^2}, \end{aligned}$$

  where P_k is the real power production at generator $k = 1, 2$, with $0 \leq P_k \leq 1$ for each generator.
- No other constraints on production.
- Reference bus at bus $\rho = 1$ and slack bus at bus $\sigma = 3$.

Use the formulation with explicit representation of angles. Use as initial guess $P_{-3}^{(0)} = \mathbf{0}$ and $\theta_{-1}^{(0)} = \mathbf{0}$.

(i) Specify the decision vector, omitting any known constants.
(ii) Derive the linearized form of the power flow equality constraints.
(iii) Derive the linearized form of the power flow inequality constraints.
(iv) Specify the bound (or box) constraints.
(v) Show the MATLAB code to solve the problem and report the solution of the DC optimal power flow.

9.6 Re-solve the optimal power flow problem in Exercise 9.5 using the angles eliminated formulation. Use as initial guess $P_{-3}^{(0)} = \mathbf{0}$.