15.094, Spring 2015

Robust Modeling, Optimization and Computation

Midterm Solutions

April 1, 2015

- This is a 3 hour exam. The exam is designed so you do not have time pressure. We expect that most of you will finish in 2 hours.
- Please submit all your answers. You must show clear work to receive credit.
- You can use all the material from Stellar (lecture notes, readings, homework solutions, recitations, helper material, etc.) your own notes, and your own homework solutions in either oral or in electronic form.
- You cannot communicate with others in either oral, written or electronic form.
- There are 3 problems with a total of 100 points.
- Please answer the questions robustly. Good luck!

Problem 1 (Short questions)**.** (40 points)

(a) (10 points) Let $\mathbf{r}_1, \ldots, \mathbf{r}_T \in \mathbb{R}^n$ be fixed vectors, and for $\mathbf{x} \in \mathbb{R}^n$ define

$$R_i(\mathbf{x}) = \mathbf{r}_i'\mathbf{x}, \quad i = 1, \ldots, T.$$

Given any $\mathbf{x} \in \mathbb{R}$, let $R_{(1)}(\mathbf{x}), \ldots, R_{(T)}(\mathbf{x})$ denote the sorted values of $R_1(\mathbf{x}), \ldots, R_T(\mathbf{x})$, with

$$R_{(1)}(\mathbf{x}) \leq R_{(2)}(\mathbf{x}) \leq \cdots \leq R_{(T)}(\mathbf{x}).$$

Let $0 < \alpha < 1$ so that αT is an integer. Given a polyhedral set $\mathcal{X} \subseteq \mathbb{R}^n$, formulate the problem

$$\begin{aligned} \max \quad & \frac{1}{\alpha T} \sum_{i=1}^{\alpha T} R_{(i)}(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \end{aligned}$$

as a linear optimization problem.

(b) (10 points) Consider the two-stage adaptive optimization problem

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{a}'\mathbf{y}(\mathbf{b}) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{b}) \geq \mathbf{b} \ \forall \mathbf{b} \in \mathcal{U} \\ & \mathbf{x}, \mathbf{y}(\mathbf{b}) \geq \mathbf{0} \ \forall \mathbf{b} \in \mathcal{U}. \end{aligned} \tag{1}$$

Illustrate under the affine decision rule $\mathbf{y}(\mathbf{b}) = \mathbf{C}\mathbf{b} + \mathbf{d}$ how to solve (1) via a cutting plane method.

(c) (10 points) Reformulate the problem

$$\min_{\mathbf{x} \geq \mathbf{0}} \max_{\mathbf{c}, \mathbf{d} \in \mathcal{U}} \mathbf{c}'\mathbf{x} + \sum_{j=1}^{n} d_j x_j^2$$

as a quadratic optimization problem when $\mathcal{U} = \{\mathbf{u} : \mathbf{A}\mathbf{u} \leq \mathbf{b}, \ \mathbf{u} \geq \boldsymbol{\ell}\} \subseteq \mathbb{R}^n$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b}, \boldsymbol{\ell} \in \mathbb{R}^m$.

(d) (10 points) We consider the problem of optimizing over all discrete 1-dimensional distributions with known support $\{a_1, \ldots, a_n\}$, with fixed and a priori known mean μ, and variance σ^2. Write down the problem of minimizing the mode over all such discrete distributions as a mixed integer optimization problem. (Recall that the mode of a distribution is the a_i such that $p_i = \max_{1 \leq j \leq n} p_j$.)

Solution:

(a) Let $\alpha T = s$. Observe that

$$\sum_{i=1}^{s} R_{(i)}(\mathbf{x}) = \min_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=s}} \sum_{i \in S} \mathbf{r}_i' \mathbf{x}.$$

We can rewrite this as

$$\begin{aligned} \min_{\mathbf{z} \in \mathbb{R}^n} \quad & \textstyle\sum_{i=1}^n (\mathbf{r}_i'\mathbf{x}) z_i \\ \text{s.t.} \quad & \textstyle\sum_i z_i = s \\ & 0 \le z_i \le 1 \ \forall i. \end{aligned}$$

Hence, the entire problem can be written as

$$\begin{aligned} \max_{\mathbf{x}} \quad & \left(\begin{aligned} \min_{\mathbf{z}} \quad & \textstyle\sum_{i=1}^n (\mathbf{r}_i'\mathbf{x}) z_i \\ \text{s.t.} \quad & \textstyle\sum_i z_i = s \\ & 0 \le z_i \le 1 \ \forall i \end{aligned} \right) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned}$$

In the usual way, the objective can be dualized to result in a usual linear program.

(b) This can be solved in multiple ways. We use an epigraph approach. We start by rewriting the problem as

$$\begin{aligned} \min_{\mathbf{x}, t, \mathbf{C}, \mathbf{d}} \quad & \mathbf{c}'\mathbf{x} + t \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{C}\mathbf{b} + \mathbf{d}) \ge \mathbf{b} \ \forall \mathbf{b} \in \mathcal{U} \\ & \mathbf{C}\mathbf{b} + \mathbf{d} \ge \mathbf{0} \ \forall \mathbf{b} \in \mathcal{U} \\ & t \ge \mathbf{a}'(\mathbf{C}\mathbf{b} + \mathbf{d}) \ \forall \mathbf{b} \in \mathcal{U} \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

Fix any $\mathbf{b}^{(0)} \in \mathcal{U}$. Let $\mathcal{U}^{(0)} = \{\mathbf{b}^{(0)}\}$. Solve the problem

$$\begin{aligned} \min_{\mathbf{x}, t, \mathbf{C}, \mathbf{d}} \quad & \mathbf{c}'\mathbf{x} + t \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{C}\mathbf{b} + \mathbf{d}) \ge \mathbf{b} \ \forall \mathbf{b} \in \mathcal{U}^{(0)} \\ & \mathbf{C}\mathbf{b} + \mathbf{d} \ge \mathbf{0} \ \forall \mathbf{b} \in \mathcal{U}^{(0)} \\ & t \ge \mathbf{a}'(\mathbf{C}\mathbf{b} + \mathbf{d}) \ \forall \mathbf{b} \in \mathcal{U}^{(0)} \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

Note that this is a usual linear program. There are a few possiblities:

(i) If the problem is infeasible, the entire problem is infeasible. Stop.

(i) If the problem is feasible and has finite objective value, let $(\mathbf{x}^{(0)}, t^*, \mathbf{C}^*, \mathbf{d}^*)$ be an optimal solution.

Check whether the following inequalities hold by solving a linear programs:

$$\begin{aligned}
&\mathbf{A}\mathbf{x}^{(0)} + \mathbf{B}(\mathbf{C}^*\mathbf{b} + \mathbf{d}^*) \geq \mathbf{b} \ \forall \in \mathcal{U} \\
&\mathbf{C}^*\mathbf{b} + \mathbf{d}^* \geq \mathbf{0} \ \forall \mathbf{b} \in \mathcal{U} \\
&t^* \geq \mathbf{a}'(\mathbf{C}^*\mathbf{b} + \mathbf{d}^*) \ \forall \mathbf{b} \in \mathcal{U}
\end{aligned}$$

If any inequalities are violated, add a corresponding $\mathbf{b} \in \mathcal{U}$ so that set $\mathcal{U}^{(0)}$ to create a new (super)set $\mathcal{U}^{(1)}$. If no inequalities are violated, we have reached optimality and can terminate (why?).

If the problem is feasible and unbounded from below, identify an extreme ray and perform a similar process to the one given when the optimal value is finite.

In either case, we have either fully solved the problem or will continue by solving

$$\begin{aligned}
\min_{\mathbf{x}, t, \mathbf{C}, \mathbf{d}} \quad & \mathbf{c}'\mathbf{x} + t \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{C}\mathbf{b} + \mathbf{d}) \geq \mathbf{b} \ \forall \mathbf{b} \in \mathcal{U}^{(1)} \\
& \mathbf{C}\mathbf{b} + \mathbf{d} \geq \mathbf{0} \ \forall \mathbf{b} \in \mathcal{U}^{(1)} \\
& t \geq \mathbf{a}'(\mathbf{C}\mathbf{b} + \mathbf{d}) \ \forall \mathbf{b} \in \mathcal{U}^{(1)} \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned}$$

This process then continues iteratively.

(c) For a given $\mathbf{x}$, the inner problem is

$$\max_{\mathbf{c}, \mathbf{d} \in \mathcal{U}} \mathbf{c}'\mathbf{x} + \sum_{j=1}^{n} d_j x_j^2$$

which is equivalent to the following

$$\max_{\mathbf{c} \in \mathcal{U}} \mathbf{c}'\mathbf{x} + \max_{\mathbf{d} \in \mathcal{U}} \sum_{j=1}^{n} d_j x_j^2$$

Using the usual linear programming duality trick, we can reformulate this using dual variables p_1, q_1, p_2, q_2.

Doing so, the outer problem becomes

$$
\begin{aligned}
\min_{\mathbf{p}_1,\mathbf{q}_1,\mathbf{p}_2,\mathbf{q}_2,\mathbf{y},\mathbf{x}} \quad & (\mathbf{p}_1+\mathbf{p}_2)'\mathbf{b} + (\mathbf{q}_1+\mathbf{q}_2)'\mathbf{l} \\
\text{such that} \quad & \mathbf{p}_1' A + \mathbf{q}_1' = \mathbf{x}' \\
& \mathbf{p}_2' A + \mathbf{q}_2' = \mathbf{y}' \\
& y_i = x_i^2 \quad \forall i \\
& \mathbf{p}_1 \geq \mathbf{0}, \mathbf{p}_2 \geq \mathbf{0} \\
& \mathbf{q}_1 \leq \mathbf{0}, \mathbf{q}_2 \leq \mathbf{0} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

Note that it isn't convex (why?), but it still is a quadratic optimization problem (which was what the question asked).

(d) For a fixed support $a_1, \ldots, a_n$, the set of all probability distributions with given mean, and variance is represented by the following polyhedral set $\mathcal{P}$ in $\mathbf{p} \in \mathbb{R}^n$:

$$
\begin{aligned}
\sum_{i=1}^n a_i p_i &= \mu \\
\sum_{i=1}^n a_i^2 p_i &= \sigma^2 + \mu^2 \\
\sum_{i=1}^n p_i &= 1 \\
\mathbf{p} &\geq \mathbf{0}
\end{aligned}
$$

The mode of this distribution is the a_j whose corresponding p_j is the largest. We define binary variables

$$
\delta_i = \begin{cases} 1 & \text{if } p_i \text{ is the largest} \\ 0 & \text{else} \end{cases} \tag{2}
$$

The mode is then given by $\sum_i \delta_i p_i$. But we need to enforce the condition in 2 mathematically. We do this by the constraints

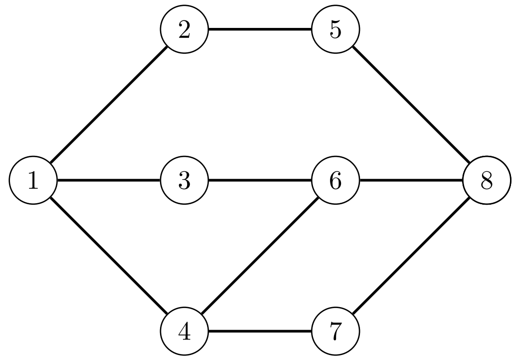
$$
\begin{aligned}
& p_i + 1 - \delta_i \geq p_j \quad \forall i, j \\
& \sum_i \delta_i = 1 \\
& \delta_i \in \{0, 1\} \quad \forall i
\end{aligned}
$$

The first constraint says that if $\delta_i = 1$, then p_i is the largest. If $\delta_j = 0$, then the inequality is satisfied vacuously, as probabilities are upper bounded by 1. The second and third constraints ensure that only one of the δ_i is 1, and the rest are 0.

Thus, the problem of minimizing the mode is given by the following mixed integer (binary) optimization problem:

$$\begin{aligned} \min_{\delta,\mathbf{p}} \quad & \delta'\mathbf{a} \\ \text{s.t.} \quad & p_i + 1 - \delta_i \geq p_j \quad \forall i, j \\ & \sum_i \delta_i = 1 \\ & \delta_i \in \{0, 1\} \quad \forall i \\ & \mathbf{p} \in \mathcal{P} \end{aligned}$$

Problem 2 (Adaptive shortest path)**.** (30 points)



(a) (5 points) Given the graph above, formulate the problem of finding the path with the least cost, starting from node 1 and ending at node 8, as a linear optimization problem. Assume the costs (travel times) for each edge e, denoted by c_e, are known.

(b) (15 points) Now consider an adaptive version of this problem, with the edge costs subject to uncertainty. Observe that we can travel from node 1 to node 8 in three edges. Hence, we model this as a three-stage adaptive optimization problem: at the first stage we decide which of edges $\{1,2\}$, $\{1,3\}$, or $\{1,4\}$ to choose; at the second stage, one among $\{2,5\}$, $\{3,6\}$, $\{4,7\}$, or $\{4,6\}$; and at the third stage, one edge among $\{5,8\}$, $\{6,8\}$, or $\{7,8\}$.

Consider the following model of uncertainty: Initially, we are at node 1 (stage one), and we know exactly the costs times along edges $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$, while the costs of the other edges are uncertain. When we arrive at stage two, the costs for the edges involved in stage two are known but the edges for stage three are uncertain. Finally when we arrive at stage three, the costs of the remaining edges become known.

Formulate the problem as a three stage adaptive optimization problem.

(c) (10 points) Assuming affine adaptability, reformulate the problem as a linear optimization problem.

Solution:

(a) The shortest path can be found by solving the following LP (we don't need x_e to be binary, as this is a network flow problem, or you can also argue that the constraint matrix is totally unimodular)

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \sum_e c_e x_e \\
\text{s.t.} \quad & x_{12} + x_{13} + x_{14} = 1 \\
& x_{12} - x_{25} = 0 \\
& x_{13} - x_{36} = 0 \\
& x_{14} - x_{46} - x_{47} = 0 \\
& x_{36} + x_{46} - x_{68} = 0 \\
& x_{25} - x_{58} = 0 \\
& x_{47} - x_{78} = 0 \\
& x_{58} + x_{68} + x_{78} = 1 \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

(b) Assume the uncertainty is expressed in terms of the second and third stage uncertainty variables $\mathbf{y}_2 \in \mathcal{U}_2, \mathbf{y}_3 \in \mathcal{U}_3$. Let us split the edge set E into the first, second and third stage edges E_1, E_2 and E_3 respectively. The adaptive problem we formulate is:

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \sum_{e \in E_1} c_e x_e^1 + \max_{\mathbf{y}_2 \in \mathcal{U}_2} \min_{\mathbf{x}(\mathbf{y}_2)} \sum_{e \in E_2} c_e(\mathbf{y}_2) x_e + \max_{\mathbf{y}_3 \in \mathcal{U}_3} \min_{\mathbf{x}(\mathbf{y}_3)} \sum_{e \in E_3} c_e(\mathbf{y}_3) x_e \\
\text{s.t.} \quad & x_{12} + x_{13} + x_{14} = 1 \\
& x_{12} - x_{25} = 0 \\
& x_{13} - x_{36} = 0 \\
& x_{14} - x_{46} - x_{47} = 0 \\
& x_{36} + x_{46} - x_{68} = 0 \\
& x_{25} - x_{58} = 0 \\
& x_{47} - x_{78} = 0 \\
& x_{58} + x_{68} + x_{78} = 1 \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

Technically, we would also need to replace the equalities with inequalities in the constraints, as we are robustifying over the uncertainty $\mathbf{y}_2, \mathbf{y}_3$.

(c) Using affine adaptability, this cannot be reformulated as a linear optimization problem. In fact, even assuming $c_e(\mathbf{y}) = \mathbf{y}$, we get a nonlinearity. I gave 10 points to everyone for this part (irrespective of whether there was a genuine attempt or not).

Problem 3 (Pareto robust optimization)**.** (30 points)

Consider a robust optimization problem of the form

$$\max_{\mathbf{x}\in\mathcal{X}} \min_{\mathbf{p}\in\mathcal{U}} \mathbf{p}'\mathbf{x}, \tag{3}$$

where $\mathcal{X},\mathcal{U} \subseteq \mathbb{R}^n$. Let $\mathcal{X}^{\mathrm{RO}}$ denote the set of optimal solutions $\mathbf{x} \in \mathcal{X}$ in (3). We say that $\mathbf{x} \in \mathcal{X}^{\mathrm{RO}}$ is *Pareto robust optimal*, or PRO, if there does not exist some $\tilde{\mathbf{x}} \in \mathcal{X}$ so that $\mathbf{p}'\tilde{\mathbf{x}} \geq \mathbf{p}'\mathbf{x}$ for all $\mathbf{p} \in \mathcal{U}$ and $\mathbf{p}'\tilde{\mathbf{x}} > \mathbf{p}'\mathbf{x}$ for some $\mathbf{p} \in \mathcal{U}$. We denote the set of Pareto robust optimal solutions as $\mathcal{X}^{\mathrm{PRO}}$.

(a) (10 points) Consider the problem

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^3} \quad & \min_{\mathbf{p}\in\mathcal{U}} \mathbf{p}'\mathbf{x} \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1 + x_3 = 0 \\ & x_1 \geq 0 \\ & x_1 \leq 1, \end{aligned}$$

where $\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^3 : 1 \leq u_i \leq 2 \ \forall i\}$. Explicitly compute $\mathcal{X}^{\mathrm{RO}}$. Prove for this example that $\mathcal{X}^{\mathrm{PRO}} \subsetneq \mathcal{X}^{\mathrm{RO}}$. (You do *not* need to compute $\mathcal{X}^{\mathrm{PRO}}$, although that approach will be fine.)

(b) (10 points) Suppose that both $\mathcal{X}$ and $\mathcal{U}$ are polyhedral sets and that $\mathcal{U}$ is bounded. If $\mathcal{U}$ has extreme points $\mathbf{u}_1, \ldots, \mathbf{u}_M \in \mathbb{R}^n$, then we define the relative interior of $\mathcal{U}$ to be the set

$$\mathrm{relint}(\mathcal{U}) := \left\{ \sum_{i=1}^{M} \lambda_i \mathbf{u}_i : \sum_{i=1}^{M} \lambda_i = 1 \text{ and } \lambda_i > 0 \ \forall i \right\}.$$

Fix any $\bar{\mathbf{p}} \in \mathrm{relint}(\mathcal{U})$. Prove that any optimal solution $\mathbf{x} \in \mathcal{X}$ to the problem $\max_{\mathbf{x}\in\mathcal{X}^{\mathrm{RO}}} \bar{\mathbf{p}}'\mathbf{x}$ is PRO.

(c) (5 points) Prove or disprove: the result of part (b) remains unchanged if $\mathcal{U}$ is unbounded. For an unbounded polyhedron $\mathcal{U}$ with extreme points $\mathbf{u}_1, \ldots, \mathbf{u}_M \in \mathbb{R}^n$ and a complete set of extreme rays $\mathbf{r}_1, \ldots, \mathbf{r}_k \in \mathbb{R}^n$, we define the relative interior of $\mathcal{U}$ as

$$\mathrm{relint}(\mathcal{U}) := \left\{ \sum_{i=1}^{M} \lambda_i \mathbf{u}_i + \sum_{j=1}^{k} \theta_j \mathbf{r}_j : \sum_{i=1}^{M} \lambda_i = 1, \ \lambda_i > 0 \ \forall i, \ \text{and } \theta_j > 0 \ \forall j \right\}.$$

(d) (5 points) Prove or disprove: if $\mathcal{X}$ and $\mathcal{U}$ are non-empty, closed, bounded, convex sets, then $\mathcal{X}^{\mathrm{PRO}}$ is always non-empty.

Solution:

(a) To compute $\mathcal{X}^{\text{RO}}$, first note that for any $\mathbf{x} \in \mathcal{X}$ we have that

$$\min_{\mathbf{p}\in\mathcal{U}} \mathbf{p}'\mathbf{x}$$

is equal to $\hat{\mathbf{p}}'\mathbf{x}$, where $\hat{\mathbf{p}} = (1, 1, 2) \in \mathcal{U}$ (this is easy to see because of the box structure of $\mathcal{U}$). Because the set $\mathcal{X}$ is precisely equal to

$$\mathcal{X} = \{(x, x, -x) : x \in [0, 1]\},$$

and for any $\mathbf{x} \in \mathcal{X}$ we have $\hat{\mathbf{p}}'\mathbf{x} = 0$, we conclude that

$$\mathcal{X}^{\text{RO}} = \mathcal{X}.$$

To show that $\mathcal{X}^{\text{PRO}} \subsetneq \mathcal{X}^{\text{RO}}$ we prove that $(0, 0, 0) \notin \mathcal{X}^{\text{PRO}}$. This is easy. Take $\tilde{\mathbf{x}} = (1, 1, -1)$. Then

$$\mathbf{1}'\tilde{\mathbf{x}} = 1 > 0 = \mathbf{1}'(0, 0, 0).$$

Since $\mathbf{1} \in \mathcal{U}$ and $(0, 0, 0), \tilde{\mathbf{x}} \in \mathcal{X}^{\text{RO}}$, this proves that $(0, 0, 0) \notin \mathcal{X}^{\text{PRO}}$.

(b) There are two ways to prove this. We give one proof here and a more general proof in part (d) below. Fix $\bar{\mathbf{p}} = \sum_i \lambda_i \mathbf{u}_i \in \text{relint}(\mathcal{U})$, where $\lambda_i > 0\ \forall i$ and $\sum_i \lambda_i = 1$. Let $S = \text{argmax}_{\mathbf{x}\in\mathcal{X}^{\text{RO}}} \bar{\mathbf{p}}'\mathbf{x}$. Note that if $S = \emptyset$, the claim is vacuously true, so we assume $S \neq \emptyset$. Let $\mathbf{x} \in S$. We must show that $\mathbf{x}$ is PRO. If not, then there exists some $\tilde{\mathbf{x}} \in \mathcal{X}^{\text{RO}}$ so that

$$\hat{\mathbf{p}}'\tilde{\mathbf{x}} > \hat{\mathbf{p}}'\mathbf{x}$$

for some $\hat{\mathbf{p}} \in \mathcal{U}$. Without loss of generality, we may assume that $\hat{\mathbf{p}}$ is an extreme point of $\mathcal{U}$ (this can be found by solving $\max_{\mathbf{p}\in\mathcal{U}} \mathbf{p}'(\tilde{\mathbf{x}} - \mathbf{x})$), say, $\hat{\mathbf{p}} = \mathbf{u}_\ell$. Therefore,

$$\mathbf{u}_\ell'\tilde{\mathbf{x}} > \mathbf{u}_\ell'\mathbf{x}.$$

Since $\mathbf{p}'\tilde{\mathbf{x}} \geq \mathbf{p}'\mathbf{x}\ \forall \mathbf{p} \in \mathcal{U}$, this implies that

$$\begin{aligned}
\bar{\mathbf{p}}'\tilde{\mathbf{x}} &= \left(\sum_i \lambda_i \mathbf{u}_i\right)' \tilde{\mathbf{x}} \\
&= \mathbf{u}_\ell'\tilde{\mathbf{x}} + \sum_{i \neq \ell} \lambda_i \mathbf{u}_i'\tilde{\mathbf{x}} \\
&> \mathbf{u}_\ell'\mathbf{x} + \sum_{i \neq \ell} \lambda_i \mathbf{u} + i'\mathbf{x} \\
&= \left(\sum_i \lambda_i \mathbf{u}_i\right)' \mathbf{x} \\
&= \bar{\mathbf{p}}'\mathbf{x}.
\end{aligned}$$

This contradicts the fact that $\mathbf{x}$ is optimal to

$$\max_{\mathbf{x} \in \mathcal{X}^{\mathrm{RO}}} \bar{\mathbf{p}}'\mathbf{x}.$$

We conclude that our assumption that $\mathbf{x} \in S$ was not PRO is false, and therefore any optimal solution to

$$\max_{\mathbf{x} \in \mathcal{X}^{\mathrm{RO}}} \bar{\mathbf{p}}'\mathbf{x}$$

must be in $\mathcal{X}^{\mathrm{PRO}}$.

(c) The statement is true. We detail two cases: first suppose that $\mathcal{U}$ has at least one extreme. We may assume that $\mathcal{X}^{\mathrm{RO}}$ is non-empty, for otherwise the statement is vacuously true. The proof is essentially the same as part (b). If $\tilde{\mathbf{x}}$ Pareto dominates $\mathbf{x}$, then one can show that

$$\mathbf{r}_j'\tilde{\mathbf{x}} \geq \mathbf{r}_j'\mathbf{x}\ \forall j.$$

Then the proof carries through similarly.

In the case when $\mathcal{U}$ has no extreme points, then we automatically have $\mathbf{r}_j'\tilde{\mathbf{x}} \geq \mathbf{r}_j'\mathbf{x}\ \forall j$ if $\tilde{\mathbf{x}}$ Pareto dominates $\mathbf{x}$. Therefore, the proof is the same.

(d) This statement is true. One can give an argument like part (c), but this would require a more sophisticated argument. Using the notation of part (b), suppose $\mathbf{x} \in\ S$ and $\tilde{\mathbf{x}}$ Pareto dominates $\mathbf{x}$. Let $\hat{\mathbf{p}}$ be an extreme point of $\mathcal{U}$ with

$$\hat{\mathbf{p}}'\tilde{\mathbf{x}} > \hat{\mathbf{p}}'\mathbf{x}.$$

We define a new point $\mathbf{p}_* \in \mathcal{U}$ as follows: let $\mathbf{p}_*$ be the unique point in the affine hull of $\{\bar{\mathbf{p}}, \hat{\mathbf{p}}\}$, other than $\hat{\mathbf{p}}$, which intersects $\mathcal{U}$ (there are other geometric and algebraic ways of defining this point). Then it is easy to argue that $\mathbf{p}_*'\tilde{\mathbf{x}} < \mathbf{p}_*'\mathbf{x}$, which contradicts the fact that $\mathbf{p}'\tilde{\mathbf{x}} \geq \mathbf{p}'\mathbf{x}\ \forall \mathbf{p} \in \mathcal{U}$.

N.B. We have implicitly used an alternative characterization of relative interior to invoke that $\mathbf{p}_*$ is well-defined, but we accepted a variety of arguments for this problem that demonstrated some understanding of things.

N.B. This actually provides an alternative proof of part (b).