15.094J: Robust Modeling, Optimization, Computation

Lecture 13: RO in Inventory Theory

Outline

Single station

- State x_k: stock available at the beginning of the kth period
- Control u_k: stock ordered at the beginning of the kth period
- Randomness w_k: demand during the kth period
- Dynamics: $x_{k+1} = x_k + u_k - w_k$
- Inventory Costs: $\max(hx_{k+1}, -px_{k+1})$
- Fixed costs: $cu_k + K1_{\{u_k>0\}}$.

Modeling Randomness

- $z_k = (w_k - \bar{w}_k)/\hat{w}_k \in [-1, 1]$.
- Uncertainty budget $\sum_{i=0}^{k} |z_i| \leq \Gamma_k$.
- Γ_k: budget of uncertainty controlling tradeoff between robustness and optimality.

The nominal model

- Goal is to solve:

$$
\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} \left(c\, u_t + K\, 1_{\{u_t>0\}} + \max(h\,\overline{x}_{t+1}, -p\,\overline{x}_{t+1}) \right) \\
\text{s.t.} \quad & u_t \geq 0\ \forall t.
\end{aligned}
$$

- Can be formulated as a LO or MIO by replacing $\max(h\,\overline{x}_{t+1}, -p\,\overline{x}_{t+1})$ by new variable y_t
- Use closed-form expression $\overline{x}_{t+1} = x_0 + \sum_{s=0}^{t}(u_s - \overline{w}_s)$
- Model $1_{\{u_t>0\}}$ by $v_t \in \{0,1\}$ with $0 \leq u_t \leq M\, v_t$.

The nominal model continued

$$
\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} (c\, u_t + K\, v_t + y_t) \\
\text{s.t.} \quad & y_t \geq h \left(x_0 + \sum_{s=0}^{t} (u_s - \overline{w}_s) \right), && \forall t, \\
& y_t \geq -p \left(x_0 + \sum_{s=0}^{t} (u_s - \overline{w}_s) \right), && \forall t, \\
& 0 \leq u_t \leq M\, v_t, \;\; v_t \in \{0, 1\}, && \forall t.
\end{aligned}
$$

LO if no fixed costs, MIO if fixed costs.

The robust formulation

- Add uncertainty to the nominal model.
- Example: holding constraint $y_t \geq h\left(x_0 + \sum_{s=0}^{t}(u_s - \overline{w}_s)\right)$.
- Robust approach: at y_t and $u_0, \ldots, u_t$ given, constraint must be feasible for any demand in the uncertainty set:

$$y_t \geq h\left(x_0 + \sum_{s=0}^{t}(u_s - \overline{w}_s - \widehat{w}_s\, z_s)\right)$$

$$\forall z \in Z = \left\{ |z_s| \leq 1\ \forall s, \sum_{s=0}^{t} |z_s| \leq \Gamma_t \right\}.$$

The robust formulation, continued

In particular, it must be feasible for the demand yielding the greatest value of the right-hand side:

$$y_t \geq h\left(x_0 + \sum_{s=0}^{t}(u_s - \overline{w}_s)\right) + \max_{z \in Z}(-h)\sum_{s=0}^{t}\widehat{w}_s\, z_s$$

Auxiliary problem:

$$\begin{array}{llcll} \max & \displaystyle\sum_{s=0}^{t}\widehat{w}_s \cdot (-z_s) & \Rightarrow & \max & \displaystyle\sum_{s=0}^{t}\widehat{w}_s z'_s \\ \text{s.t.} & \displaystyle\sum_{s=0}^{t}|z_s| \leq \Gamma_t, & & \text{s.t.} & \displaystyle\sum_{s=0}^{t} z'_s \leq \Gamma_t, \\ & |z_s| \leq 1,\ \forall s \leq t, & & & 0 \leq z'_s \leq 1,\ \forall s \leq t. \end{array}$$

The robust formulation, continued

By strong duality:

$$\begin{aligned} \max \quad & \sum_{s=0}^{t} \widehat{w}_s z'_s & \Rightarrow \quad \min \quad & q_t \Gamma_t + \sum_{s=0}^{t} r_{st} \\ \text{s.t.} \quad & \sum_{s=0}^{t} z'_s \leq \Gamma_t, & \text{s.t.} \quad & q_t + r_{st} \geq \widehat{w}_s, \ \forall s, \\ & 0 \leq z'_s \leq 1, \ \forall s \leq t, & & q_t \geq 0, \ r_{st} \geq 0, \ \forall s \leq t. \end{aligned}$$

The holding constraint becomes:

$$y_t \geq h \left(x_0 + \sum_{s=0}^{t} (u_s - \overline{w}_s \right) + h \cdot \min_{(q,r) \in Q} \left(q_t \Gamma_t + \sum_{s=0}^{t} r_{st} \right)$$

Enough to find (q, r) feasible: constraint is linear.

LO or MIO

$$
\begin{array}{lll}
\min & \displaystyle\sum_{t=0}^{T-1} (c\, u_t + K\, v_t + y_t) & \\
\text{s.t.} & y_t \geq h \left(x_0 + \displaystyle\sum_{s=0}^{t} (u_s - \overline{w}_s) \right) + h\, A_t, & \forall t, \\
& y_t \geq -p \left(x_0 + \displaystyle\sum_{s=0}^{t} (u_s - \overline{w}_s) \right) + p\, A_t, & \forall t, \\
& A_t = q_t\, \Gamma_t + \displaystyle\sum_{s=0}^{t} r_{st}, & \forall t, \\
& q_t + r_{st} \geq \widehat{w}_s, \;\; q_t \geq 0, r_{st} \geq 0, & \forall t, \;\; s \leq t, \\
& 0 \leq u_t \leq M v_t, \;\; v_t \in \{0, 1\}, & \forall t.
\end{array}
$$

Properties

- No fixed costs: Robust problem optimal ordering policy is also (S, S), or basestock, i.e., there exists a threshold sequence (S_k) such that, at each time period k, it is optimal to order $S_k - x_k$ if $x_k < S_k$ and 0, otherwise. S_k given in closed form.
- Fixed costs, optimal policy for robust problem is (s, S), i.e., there exists a threshold sequence (s_k, S_k) such that, at each time period k, it is optimal to order $S_k - x_k$ if $x_k < s_k$ and 0 otherwise, with $s_k \leq S_k$.
- Constrast to the stochastic case.

Budget of uncertainty

- Expected cost if distribution is known:

$$c \sum_{t=0}^{T-1} u_t + K \sum_{t=0}^{T-1} v_t + \sum_{t=0}^{T-1} E \max(h\, x_{t+1}, -p\, x_{t+1}).$$

- Assume that only first two moments are known. We want an upper bound on

$$E \max(h\, x_{t+1}, -p\, x_{t+1}) = h\, E(x_{t+1}) + (h+p)\, E \max(0, -x_{t+1})$$

Budget of uncertainty, continued

- Bertsimas and Popescu, 2001: $E\max(0, X - a)$

$$\leq \begin{cases} \frac{1}{2}\left(\mu - a + \sqrt{\sigma^2 + (\mu - a)^2}\right), & \text{if } a \geq \frac{\mu}{2} + \frac{\sigma^2}{2\,\mu}, \\ \frac{\mu}{\mu^2 + \sigma^2}\left(\sigma^2 - \mu(a - \mu)\right), & \text{otherwise.} \end{cases}$$

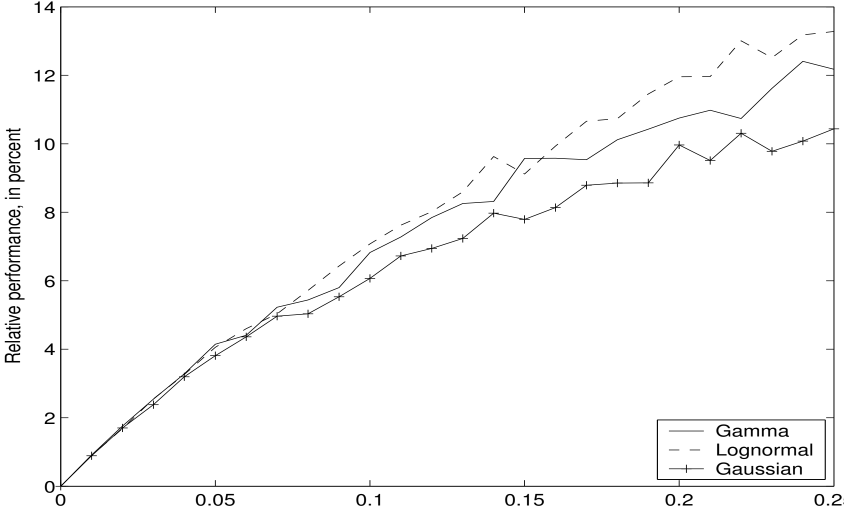
- Bound is convex in a.
- Find the budgets of uncertainty that minimize the upper bound.

Example

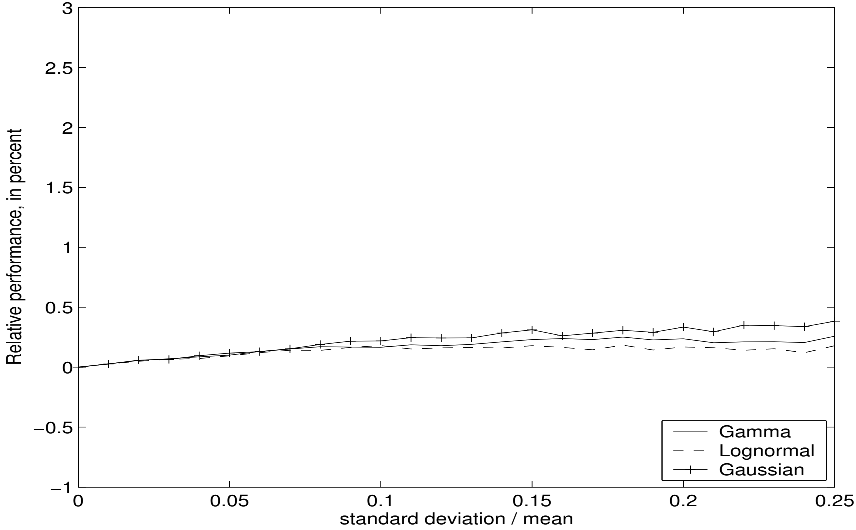
- Goal: compare robust approach with dynamic programming when first two moments of distribution are known.
- Performance measure: $100 \cdot \dfrac{E(DP) - E(ROB)}{E(DP)}$.
- Questions:
  - Does the actual distribution (beyond first two moments) significantly affect performance?
  - What is the impact of the cost parameters?
  - What is the impact of DP assuming a wrong distribution?

Impact of standard deviation

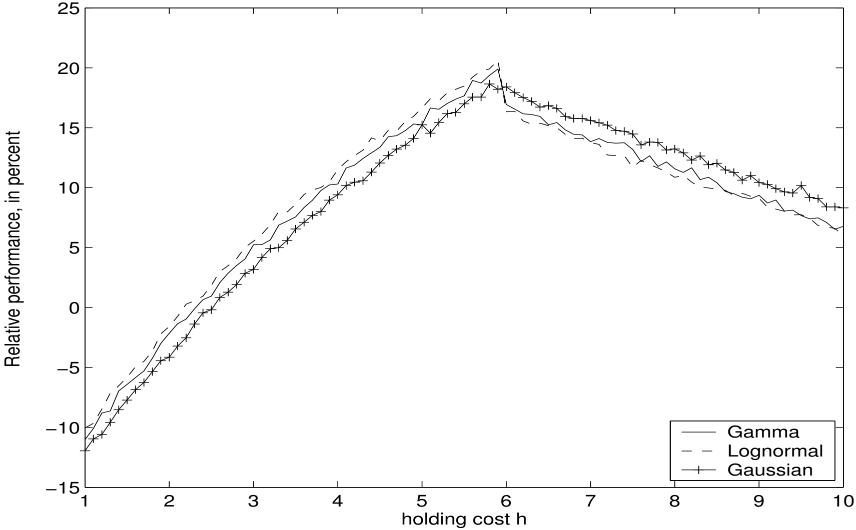
DP assumes binomial; actual distribution is different
$(c = 1, h = 4, p = 6, \overline{w} = 100, \sigma = 20)$.



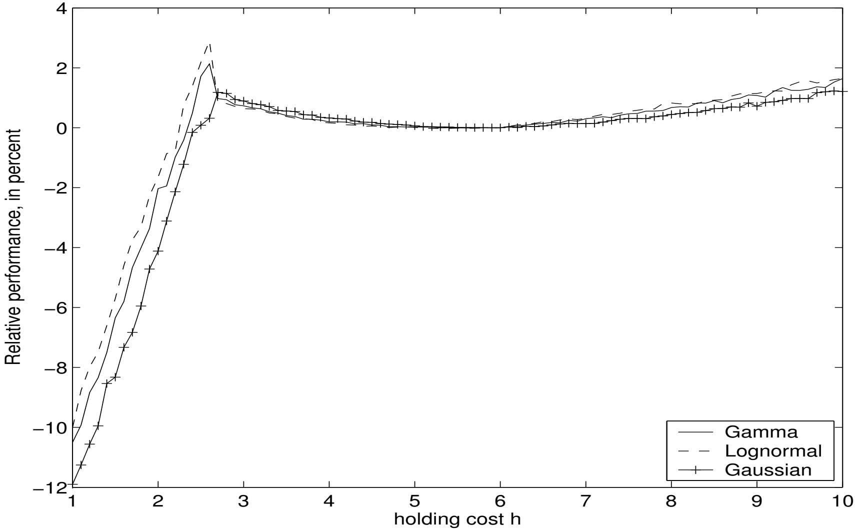
DP assumes almost right distribution (Gaussian)



DP assumes binomial; actual distribution is different



DP assumes right distribution (Gaussian)



Comments

- Robust approach leads to high-quality solutions,
- Outperforms dynamic programming for a wide range of parameters, in particular if assumption on distribution in DP is not accurate,
- It is robust to the actual demand distributions (beyond their first two moments).

Series systems

- Station $k+1$ supplies station k, and the demand at station k is the order made by station $k-1$. Station N is supplied by the outside world and the demand at station 1 is exogenous, subject to randomness. Stock at station k at time t is $I_k(t)$.
- Echelon k is stations 1 to $k \rightarrow X_k(t) = \sum_{j=1}^{k} I_j(t)$.
- Clark and Scarf, 1960: Optimal policy when costs are computed at the echelon level is basestock, when there are no fixed ordering costs except maybe for station N.
- Can the robust approach yield similar theoretical results?

The nominal model

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{N} \sum_{t=0}^{T-1} \left(c_k\, U_k(t) + K_k\, 1_{\{U_k(t)>0\}} \right) \\
& \qquad + \sum_{k=1}^{N} \sum_{t=0}^{T-1} \max\left(h\, \overline{X}_k(t+1), -p\, \overline{X}_k(t+1) \right) \\
\text{s.t.} \quad & \overline{X}_k(t+1) = X_k(0) + \sum_{s=0}^{t} (U_k(s) - \overline{W}_k(s)), \ \forall k, t, \\
& U_k(t) \le I_{k+1}(t), \ \forall k, t, \\
& U_k(t) \ge 0, \ \forall k, t.
\end{aligned}
$$

Again, LO or MIO.

The robust model

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{N} \sum_{t=0}^{T-1} \left(c_k\, U_k(t) + K_k\, V_k(t) + Y_k(t)\right) \\
\text{s.t.} \quad & \overline{X}_k(t+1) = X_k(0) + \sum_{s=0}^{t} (U_k(s) - \overline{W}_k(s)), \ \forall k, t, \\
& A(t) = q(t)\,\Gamma(t) + \sum_{s=0}^{t} r(s,t), \ \forall t, \\
& Y_k(t) \geq h\,(\overline{X}_k(t+1) + A(t)), \ \forall k, t, \\
& Y_k(t) \geq p\,(-\overline{X}_k(t+1) + A(t)), \ \forall k, t, \\
& U_k(t) \leq \overline{X}_k(t+1) - \overline{X}_k(t), \ \forall k, t, \\
& q(t) + r(s,t) \geq \widehat{W}(t), \ q(t) \geq 0, \ r(s,t) \geq 0, \ \forall t, s \leq t, \\
& 0 \leq U_k(t) \leq M\, V_k(t), \ V_k(t) \in \{0,1\}, \ \forall k, t.
\end{aligned}
$$

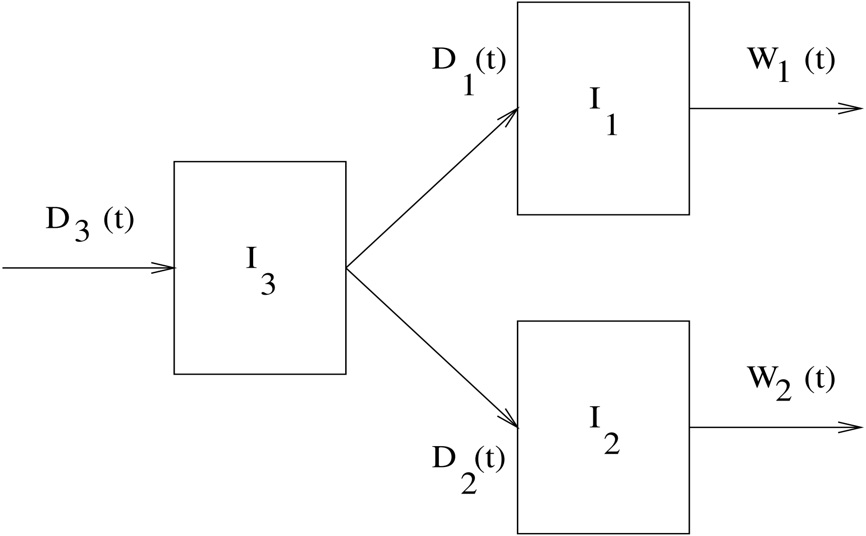
Properties

- The robust model of a series system remains of the same class as the nominal problem:
  - LO if no fixed costs,
  - MIO if fixed costs.
- Optimal robust policy is same as optimal nominal policy for modified demand: $$W'_k(t) = \overline{W}(t) + \frac{p_k - h_k}{p_k + h_k}(A(t) - A(t-1)).$$
- Optimal policy is basestock, parameters are computed by optimization, not DP.

General Supply chains

- **A new theoretical result for tree supply chains:**
  If no fixed costs, the optimal policy in the robust model is still basestock for modified cost parameters. [And we don't know what the optimal stochastic policy is.]
- **Tractability:**
  This approach is tractable for arbitrary supply chains as the robust problem remains an LOP if no fixed costs and a MIOP if fixed costs. [Complexity of the formulation does not increase with complexity of the network. Contrast with DP.]

Example

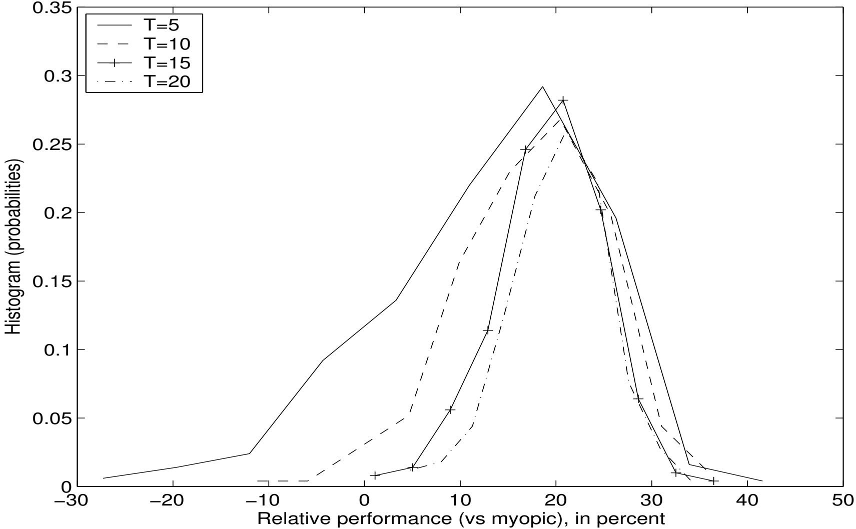


Example, continued

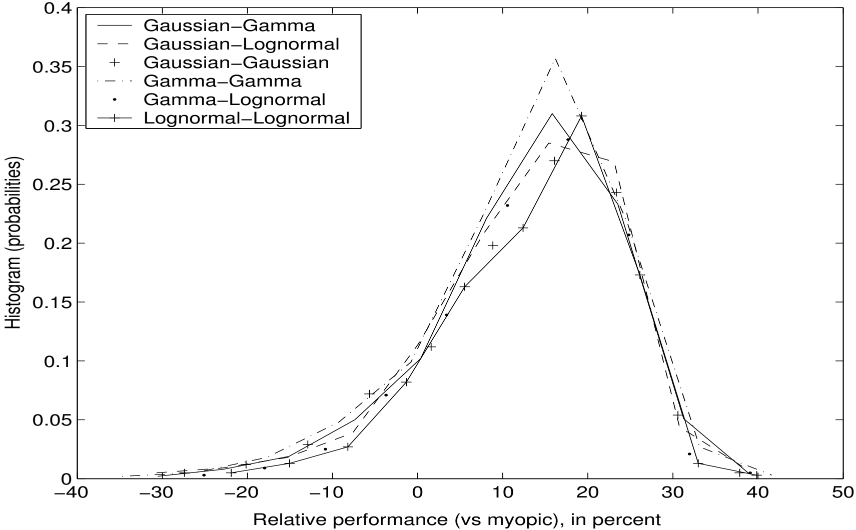
- Parameter selection is similar to single station.
- We will compare the robust approach to the myopic policy.
- Performance measure: histogram of $100 \cdot \dfrac{MYO - ROB}{MYO}$, with MYO (ROB) cost of myopic (robust) policy.
- Questions:
  - Role of time horizon in performance?
  - Role of distributions?

Impact of time horizon

Actual: Gamma, assumed: Gaussian distribution



Gaussian with $T = 5$



Comments

- Robust approach leads to high-quality solutions,
- Performs significantly better than myopic policies, in particular over many time periods, even when actual and assumed distributions are close,
- Is indeed robust to uncertainty on the distributions.

Summary and Conclusions

- Robust approach is numerically tractable even for large dimensions, without the curse of dimensionality.
- It offers qualitatively the same solutions as DP, when DP policies are known.
- Outperforms DP in computational experiments.
- Successfully applied approach to other problems.