15.094, Spring 2015

Robust Modeling, Optimization and Computation

Midterm

April 1, 2015

- This is a 3 hour exam. The exam is designed so you do not have time pressure. We expect that most of you will finish in 2 hours.
- Please submit all your answers. You must show clear work to receive credit.
- You can use all the material from Stellar (lecture notes, readings, homework solutions, recitations, helper material, etc.) your own notes, and your own homework solutions in either oral or in electronic form.
- You cannot communicate with others in either oral, written or electronic form.
- There are 3 problems with a total of 100 points.
- Please answer the questions robustly. Good luck!

Problem 1 (Short questions)**.** (40 points)

(a) (10 points) Let $\mathbf{r}_1, \ldots, \mathbf{r}_T \in \mathbb{R}^n$ be fixed vectors, and for $\mathbf{x} \in \mathbb{R}^n$ define

$$R_i(\mathbf{x}) = \mathbf{r}_i'\mathbf{x}, \quad i = 1, \ldots, T.$$

Given any $\mathbf{x} \in \mathbb{R}$, let $R_{(1)}(\mathbf{x}), \ldots, R_{(T)}(\mathbf{x})$ denote the sorted values of $R_1(\mathbf{x}), \ldots, R_T(\mathbf{x})$, with

$$R_{(1)}(\mathbf{x}) \leq R_{(2)}(\mathbf{x}) \leq \cdots \leq R_{(T)}(\mathbf{x}).$$

Let $0 < \alpha < 1$ so that αT is an integer. Given a polyhedral set $\mathcal{X} \subseteq \mathbb{R}^n$, formulate the problem

$$\begin{aligned} \max \quad & \frac{1}{\alpha T} \sum_{i=1}^{\alpha T} R_{(i)}(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \end{aligned}$$

as a linear optimization problem.

(b) (10 points) Consider the two-stage adaptive optimization problem

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{a}'\mathbf{y}(\mathbf{b}) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{b}) \geq \mathbf{b} \ \forall \mathbf{b} \in \mathcal{U} \\ & \mathbf{x}, \mathbf{y}(\mathbf{b}) \geq \mathbf{0} \ \forall \mathbf{b} \in \mathcal{U}. \end{aligned} \tag{1}$$

Illustrate under the affine decision rule $\mathbf{y}(\mathbf{b}) = \mathbf{C}\mathbf{b} + \mathbf{d}$ how to solve (1) via a cutting plane method.

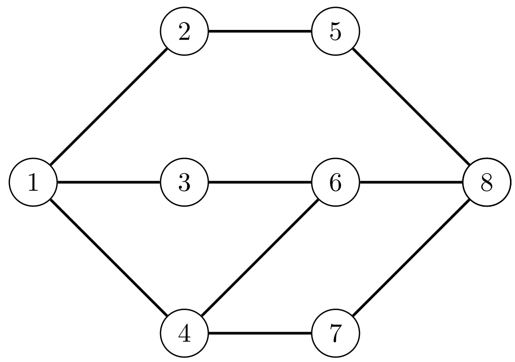
(c) (10 points) Reformulate the problem

$$\min_{\mathbf{x} \geq \mathbf{0}} \max_{\mathbf{c}, \mathbf{d} \in \mathcal{U}} \mathbf{c}'\mathbf{x} + \sum_{j=1}^{n} d_j x_j^2$$

as a quadratic optimization problem when $\mathcal{U} = \{\mathbf{u} : \mathbf{A}\mathbf{u} \leq \mathbf{b},\ \mathbf{u} \geq \boldsymbol{\ell}\} \subseteq \mathbb{R}^n$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b}, \boldsymbol{\ell} \in \mathbb{R}^m$.

(d) (10 points) We consider the problem of optimizing over all discrete 1-dimensional distributions with known support $\{a_1, \ldots, a_n\}$, with fixed and a priori known mean μ, and variance σ^2. Write down the problem of minimizing the mode over all such discrete distributions as a mixed integer optimization problem. (Recall that the mode of a distribution is the a_i such that $p_i = \max_{1 \leq j \leq n} p_j$.)

Problem 2 (Adaptive shortest path)**.** (30 points)



(a) (5 points) Given the graph above, formulate the problem of finding the path with the least cost, starting from node 1 and ending at node 8, as a linear optimization problem. Assume the costs (travel times) for each edge e, denoted by c_e, are known.

(b) (15 points) Now consider an adaptive version of this problem, with the edge costs subject to uncertainty. Observe that we can travel from node 1 to node 8 in three edges. Hence, we model this as a three-stage adaptive optimization problem: at the first stage we decide which of edges $\{1, 2\}$, $\{1, 3\}$, or $\{1, 4\}$ to choose; at the second stage, one among $\{2, 5\}$, $\{3, 6\}$, $\{4, 7\}$, or $\{4, 6\}$; and at the third stage, one edge among $\{5, 8\}$, $\{6, 8\}$, or $\{7, 8\}$.

Consider the following model of uncertainty: Initially, we are at node 1 (stage one), and we know exactly the costs times along edges $\{1, 2\}$, $\{1, 3\}$, and $\{1, 4\}$, while the costs of the other edges are uncertain. When we arrive at stage two, the costs for the edges involved in stage two are known but the edges for stage three are uncertain. Finally when we arrive at stage three, the costs of the remaining edges become known.

Formulate the problem as a three stage adaptive optimization problem.

(c) (10 points) Assuming affine adaptability, reformulate the problem as a linear optimization problem.

Problem 3 (Pareto robust optimization)**.** (30 points)

Consider a robust optimization problem of the form

$$\max_{\mathbf{x}\in\mathcal{X}} \min_{\mathbf{p}\in\mathcal{U}} \mathbf{p}'\mathbf{x}, \tag{2}$$

where $\mathcal{X},\mathcal{U} \subseteq \mathbb{R}^n$. Let $\mathcal{X}^{\mathrm{RO}}$ denote the set of optimal solutions $\mathbf{x} \in \mathcal{X}$ in (2). We say that $\mathbf{x} \in \mathcal{X}^{\mathrm{RO}}$ is *Pareto robust optimal*, or PRO, if there does not exist some $\tilde{\mathbf{x}} \in \mathcal{X}$ so that $\mathbf{p}'\tilde{\mathbf{x}} \geq \mathbf{p}'\mathbf{x}$ for all $\mathbf{p} \in \mathcal{U}$ and $\mathbf{p}'\tilde{\mathbf{x}} > \mathbf{p}'\mathbf{x}$ for some $\mathbf{p} \in \mathcal{U}$. We denote the set of Pareto robust optimal solutions as $\mathcal{X}^{\mathrm{PRO}}$.

(a) (10 points) Consider the problem

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^3} \quad & \min_{\mathbf{p}\in\mathcal{U}} \mathbf{p}'\mathbf{x} \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1 + x_3 = 0 \\ & x_1 \geq 0 \\ & x_1 \leq 1, \end{aligned}$$

where $\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^3 : 1 \leq u_i \leq 2 \ \forall i\}$. Explicitly compute $\mathcal{X}^{\mathrm{RO}}$. Prove for this example that $\mathcal{X}^{\mathrm{PRO}} \subsetneq \mathcal{X}^{\mathrm{RO}}$. (You do *not* need to compute $\mathcal{X}^{\mathrm{PRO}}$, although that approach will be fine.)

(b) (10 points) Suppose that both $\mathcal{X}$ and $\mathcal{U}$ are polyhedral sets and that $\mathcal{U}$ is bounded. If $\mathcal{U}$ has extreme points $\mathbf{u}_1, \ldots, \mathbf{u}_M \in \mathbb{R}^n$, then we define the relative interior of $\mathcal{U}$ to be the set

$$\operatorname{relint}(\mathcal{U}) := \left\{ \sum_{i=1}^{M} \lambda_i \mathbf{u}_i : \sum_{i=1}^{M} \lambda_i = 1 \text{ and } \lambda_i > 0 \ \forall i \right\}.$$

Fix any $\bar{\mathbf{p}} \in \operatorname{relint}(\mathcal{U})$. Prove that any optimal solution $\mathbf{x} \in \mathcal{X}$ to the problem $\max_{\mathbf{x}\in\mathcal{X}^{\mathrm{RO}}} \bar{\mathbf{p}}'\mathbf{x}$ is PRO.

(c) (5 points) Prove or disprove: the result of part (b) remains unchanged if $\mathcal{U}$ is unbounded. For an unbounded polyhedron $\mathcal{U}$ with extreme points $\mathbf{u}_1, \ldots, \mathbf{u}_M \in \mathbb{R}^n$ and a complete set of extreme rays $\mathbf{r}_1, \ldots, \mathbf{r}_k \in \mathbb{R}^n$, we define the relative interior of $\mathcal{U}$ as

$$\operatorname{relint}(\mathcal{U}) := \left\{ \sum_{i=1}^{M} \lambda_i \mathbf{u}_i + \sum_{j=1}^{k} \theta_j \mathbf{r}_j : \sum_{i=1}^{M} \lambda_i = 1,\ \lambda_i > 0 \ \forall i, \text{ and } \theta_j > 0 \ \forall j \right\}.$$

(d) (5 points) Prove or disprove: if $\mathcal{X}$ and $\mathcal{U}$ are non-empty, closed, bounded, convex sets, then $\mathcal{X}^{\mathrm{PRO}}$ is always non-empty.