C21 Nonlinear Systems

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Contents

1 Introduction

1.1 Scope and objectives

The course aims to provide an overview of techniques for analysis and control design for nonlinear systems. Whereas linear system control theory is largely based on linear algebra (eg. when determining the behaviour of solutions to linear differential equations) and complex analysis (eg. when predicting system behaviour from transfer functions), the broader range of behaviour exhibited by nonlinear systems requires a wider variety of techniques. The course gives an introduction to some of the most useful and commonly used tools for determining system behaviour from a description in terms of differential equations. Two main topics are covered:

- **Lyapunov stability** — An intuitive approach to analyzing stability and convergence of dynamic systems without explicitly computing the solutions of their differential equations. This method forms the basis of much of modern nonlinear control theory and also provides a theoretical justification for using local linear control techniques.
- **Passivity and linearity** — These are properties of two important classes of dynamic system which can simplify the application of Lyapunov stability theory to interconnected systems. The stability properties of linear and passive systems are used to derive the circle criterion, which provides an extension of the Nyquist criterion to nonlinear systems consisting of linear and nonlinear subsystems.

The course concentrates on analysis rather than control design, but the techniques form the basis of modern nonlinear control design methods, some of which are covered in other C-paper courses.

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Web pages: http://weblearn.ox.ac.uk
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At the end of the course you should be able to:

- understand the basic Lyapunov stability definitions lecture 1
- analyse stability using the linearization method lecture 2
- analyse stability by Lyapunov's direct method lecture 2
- determine convergence using Barbalat's Lemma lecture 3
- use invariant sets to determine regions of attraction lecture 3
- construct Lyapunov functions for linear systems and passive systems lecture 4
- use the circle criterion to design controllers for systems with static nonlinearities lecture 4

1.2 Books

These notes are self-contained, and they also contain some non-examinable background material (in sections indicated in the text). But for a full understanding of the course it will helpful to read more detailed treatments given by textbooks on nonlinear control. Throughout these notes there are references to additional reading material from the following books.

1. J.-J. Slotine and W. Li, *Applied Nonlinear Control*, Prentice-Hall, 1991.

   The main reference for the course, gives a good overview of Lyapunov stability and convergence methods. Most of the material covered by the course is in chapters 3 and 4.

2. M. Vidyasagar, *Nonlinear Systems Analysis (2nd ed.)*, Prentice-Hall, 1993.

   Chapter 5 gives a comprehensive (fairly technical) treatment of Lyapunov stability analysis. Chapter 2 contains useful background material on nonlinear differential equations.

3. H.K. Khalil, *Nonlinear Systems (2nd ed.)*, Prentice-Hall, 1996.

   A classic textbook. Chapters 1, 3, 4, 10 and 11 are relevant to this course.

1.3 Motivation [1]

There already exists a large amount of theory concerning control of linear systems, including robust and optimal control for multivariable linear systems of arbitrarily high order. So why bother designing controllers explicitly for nonlinear systems?

The main justification for nonlinear control is based on the observations:

- *All physical systems are nonlinear.* Apart from limitations on standard linear modeling assumptions such as Hooke's law, linearity of resistors and capacitors etc., nonlinearity invariably appears due to friction and heat dissipation effects.
- *Linearization is approximate.* Linear descriptions of physical processes are necessarily local (ie. accurate only within a restricted region of operation), and may be of limited use for control purposes.

Nonlinear control techniques are therefore useful when: (i) the required range of operation is large; (ii) the linearized model is inadequate (for example, linearizing the first order system $\dot{x} = xu$ about $x = 0$, $u = 0$ results in the uncontrollable system $\dot{x} = 0$). Further reasons for considering nonlinear controllers are: (iii) the ability of robust nonlinear controllers to tolerate large variations in uncertain system parameters; (iv) the simplicity of nonlinear control designs, which are often based on the physics of the process.

Linear vs. nonlinear system behaviour. The analysis and control of nonlinear systems requires a different set of tools than can be used in the case of linear systems. In particular, for the linear system:

$$\dot{x} = Ax + Bu$$

where x is the state and u the control input, for the case of $u = 0$ we have:

- $x = 0$ is the unique equilibrium point (unless A is singular)
- stability is unaffected by initial conditions

and for $u \neq 0$:

[1] See *Slotine and Li* §1.2 pp4–12, for a more detailed discussion.

- x remains bounded whenever u is bounded if $\dot{x} = Ax$ is stable
- if $u(t)$ is sinusoidal, then in steady state $x(t)$ contains sinusoids of the same frequency as u
- if $u = u_1 + u_2$, then $x = x_1 + x_2$.

However **none** of these properties is in general true for nonlinear systems.

Example 1.1. The system:

$$\ddot{x} + \dot{x} + k(x) = u, \qquad k(x) = x(x^2 + 1) \tag{1.1}$$

is a model of displacement $x(t)$ in a mass-spring-damper system subject to an externally applied force $u(t)$. The nonlinearity appears in the spring term $k(x)$, which stiffens with increasing $|x|$ (figure 1), and might therefore represent the effect of large elastic deflections.

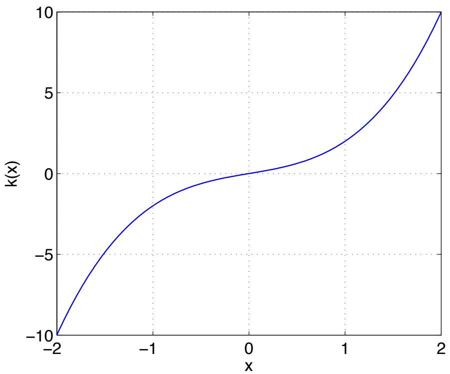


Figure 1: Spring characteristics $k(x)$ in (1.1).

Figure 2 shows the variation of x with time t in response to step changes in input u. Clearly the responses to positive steps are more oscillatory than those for negative step changes in u. This reduction in apparent damping ratio is due simply to the increase in stiffness of the nonlinear spring for large $|x|$. Note also that the steady state value of x for $u = 5$ is not 10 times that for $u = 50$, as would be expected in the linear case. $\Diamond$

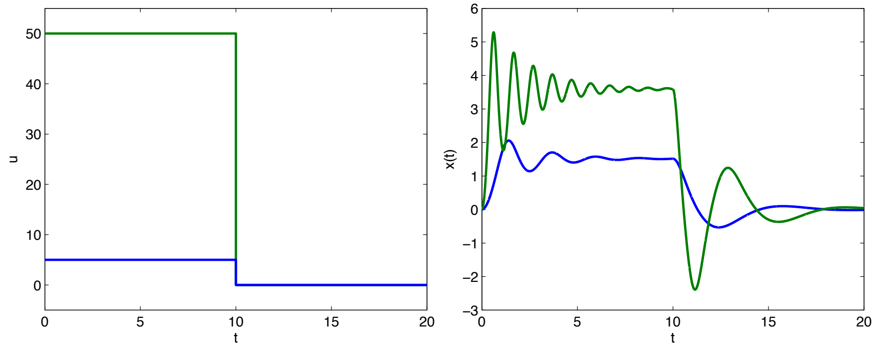
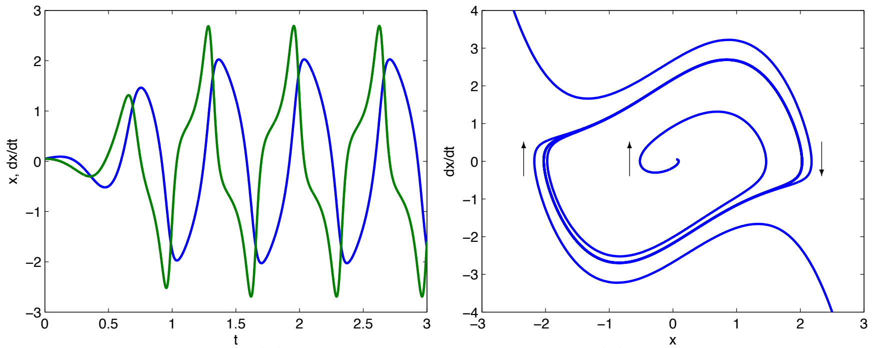


Figure 2: Response to steps in u of 5 (solid line), and 50 (dotted line).

Example 1.2. Van der Pol's equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu > 0, \tag{1.2}$$

is a well-known example of a nonlinear system exhibiting limit cycle behaviour. In contrast to the limit cycles that occur in marginally stable linear systems, the amplitudes of limit cycle oscillations in nonlinear systems are independent of initial conditions. As a result the state trajectories of (1.2) all tend towards a single closed curve (the limit cycle), which can be seen in figure 3b. $\Diamond$



(a) (b)

Figure 3: Limit cycle of Van der Pol's equation (1.2) with $\mu = 0.5$. (a) Response for initial condition $x(0) = dx/dt(0) = 0.05$. (b) State trajectories.

2 Lyapunov stability

Throughout these notes we use state equations of the form

$$\dot{x} = f(x, u, t) \qquad \begin{cases} x: & \text{state variable} \\ u: & \text{control input} \end{cases} \tag{2.1}$$

to represent systems of coupled ordinary differential equations. You should be familiar with the concept of state space from core course lectures, but to refresh your memory, suppose an nth-order system is given by

$$y^{(n)} = h\big(y, \dot{y}, \ldots y^{(n-1)}, u, t\big) \tag{2.2}$$

(where $y^{(i)} = d^i y/dt^i$, $i = 1, 2, \ldots$), for some possibly nonlinear function h. Then an n-dimensional state vector x can be defined for example via

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}; \qquad \begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\vdots \\ x_{n-1} &= y^{(n-2)} \\ x_n &= y^{(n-1)} \end{aligned}$$

and (2.2) is equivalent to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= h(x_1, x_2, \ldots x_n, u, t). \end{aligned}$$

This now has the form of (2.1), with

$$f(x, u, t) = \begin{bmatrix} x_2 \\ x_3 \\ \cdots \\ x_n \\ h(x_1, \ldots x_n, u, t) \end{bmatrix}.$$

We make a distinction between system dynamics which are invariant with time and those which depend explicitly on time.

Definition 2.1 (Autonomous/non-autonomous dynamics)**.** The system (2.1) is **autonomous** if f does not depend explicitly on time t, ie. if (2.1) can be re-written $\dot{x} = f_1(x)$ for some function f_1. Otherwise the system (2.1) is **non-autonomous**.

For example, the closed-loop system formed by $\dot{x} = f(x, u)$ under time-invariant feedback control $u = u(x)$ is given by $\dot{x} = f\big(x, u(x)\big)$, and is therefore autonomous. However a system $\dot{x} = f(x, u)$ under time-varying feedback $u = u(x, t)$ (which would be needed if $x(t)$ were required to follow a time-varying target trajectory) has closed-loop dynamics $\dot{x} = f\big(x, u(x, t)\big)$ which are therefore non-autonomous.

As stated earlier, a nonlinear system may have different stability properties for different initial conditions. It is therefore usual to consider stability and convergence with respect to an equilibrium point, defined as follows.

Definition 2.2 (Equilibrium)**.** A state x^* is an equilibrium point if $x(t_0) = x^*$ implies that $x(t) = x^*$ for all $t \geq t_0$.

Clearly the equilibrium points of the forced system (2.1) are dependent on the control input $u(t)$. In fact the problem of controlling (2.1) so that $x(t)$ converges to a given target state x_d is equivalent to finding a control law $u(t)$ which forces x_d to be a stable equilibrium (in some sense) of the closed-loop system. Thus it is convenient consider the equilibrium points of an **unforced** system:

$$\dot{x} = f(x, t), \tag{2.3}$$

(which could of course be obtained from a forced system under a specified control input $u(t)$). By definition, an equilibrium point x^* of (2.3) is a solution of

$$f(x^*, t) = 0.$$

Note that solving for x^* may not be trivial for general f. The remainder of section 2 considers the stability of equilibrium points of (2.3). Following the usual convention, we define f in (2.3) so that the origin $x = 0$ is an equilibrium point, ie. so that $f(0, t) = 0$ for all t.[2]

The origins of modern stability theory date back to Lagrange (1788), who showed that, in the absence of external forces, an equilibrium of a conservative mechanical system is stable if it corresponds to a local minimum of the potential energy stored in the system. Stability theory remained restricted to conservative dynamics described by Lagrangian equations of motion until 1892, when the Russian mathematician A. M. Lyapunov developed methods applicable to arbitrary differential equations. Lyapunov's work was largely unknown outside Russia until about 1960, when it received widespread attention through the work of La Salle and Lefschetz. With several refinements and modifications, Lyapunov's methods have become indispensable tools in nonlinear system control theory.

2.1 Stability definitions [3]

As might be expected, the most basic form of stability is simply a guarantee that the state trajectories starting from points in the vicinity of an equilibrium point remain close to that equilibrium point at all future times. In addition to this we consider the stronger properties of asymptotic and exponential stability, which ensure convergence of trajectories to an equilibrium point. Although the definitions discussed in this section are mostly intuitively obvious, they are often useful in their own right, particularly in cases where the stability theorems described in section 2.3 cannot be applied directly.

Definition 2.3 (Stability)**.** The equilibrium $x = 0$ of (2.3) is **stable** if, for each time t_0, and for every constant $R > 0$, there exists some $r(R, t_0) > 0$ such that

$$\|x(t_0)\| < r \implies \|x(t)\| < R, \ \forall t \geq t_0.$$

(Here $\|\cdot\|$ can be any vector norm.) It is **uniformly stable** if r is independent of t_0. The equilibrium is **unstable** if it is not stable.

[2] Any equilibrium x^* can be translated to the origin by redefining the state x as $x' = x - x^*$.

[3] See *Slotine and Li* §3.2 pp47–52 and §4.1 pp101–105, or *Vidyasagar* §5.1 pp135–147.

An equilibrium is therefore stable if $x(t)$ can be contained within an arbitrarily small region of state space for all $t \geq t_0$ provided $x(t_0)$ is sufficiently close to the equilibrium point (see figure 4).

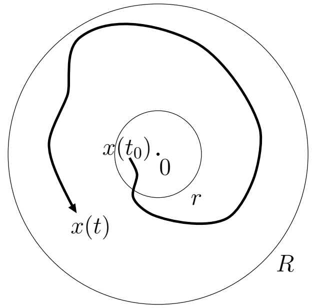


Figure 4: Stable equilibrium.

Note that:

1. Equilibrium points of stable autonomous systems are necessarily uniformly stable. This is because the state trajectories, $x(t)$, $t \geq t_0$, of an autonomous system depend only on initial condition $x(t_0)$ and not on initial time t_0.

2. An equilibrium point $x = 0$ may be unstable even though trajectories starting from points close to $x = 0$ do not tend to infinity. This is the case for Van der Pol's equation (example 1.2), which has an unstable equilibrium at the origin. (All trajectories starting from points within the limit cycle of figure 3b eventually join the limit cycle, and therefore it is not possible to find $r > 0$ in definition 2.3 whenever R is small enough that some points on the closed curve of the limit cycle lie outside the set of points x satisfying $\|x\| < R$.)

It is also worth noting that stability (as opposed to uniform stability) is a very weak condition which implies that an equilibrium point actually tends towards instability as $t \to \infty$. This is because it is only necessary to specify r in definition 2.3 as a function of t_0 if, for fixed R, $r(R, t_0)$ tends to zero as a function of t_0. Otherwise $r(R)$ could be specified independently of t_0 as the minimum value of $r(R, t_0)$ over all t_0.

Definition 2.4 (Asymptotic stability)**.** The equilibrium $x = 0$ of (2.3) is **asymptotically stable** if: (a) it is stable, and (b) for each time t_0 there exists some $r(t_0) > 0$ such that

$$\|x(t_0)\| < r \implies \|x(t)\| \to 0 \text{ as } t \to \infty.$$

It is **uniformly asymptotically stable** if it is asymptotically stable and both r and the rate of convergence in (b) are independent of t_0.

Asymptotic stability therefore implies that the trajectories starting from any point within some region of state space containing the equilibrium point remain bounded and converge asymptotically to the equilibrium (see figure 5).

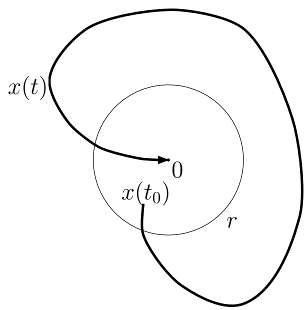


Figure 5: Asymptotically stable equilibrium.

Note that:

1. An asymptotically stable equilibrium of an autonomous system is necessarily uniformly asymptotically stable.

2. The convergence condition (b) of definition 2.4 is equivalent to requiring that, for every constant $R > 0$ there exists a $T(R, r, t_0)$ such that

$$\|x(t_0)\| < r \implies \|x(t)\| < R, \ \forall t \geq t_0 + T,$$

   and the convergence rate is independent of t_0 if T is independent of t_0.

Example 2.5. The first order system

$$\dot{x} = -\frac{x}{1+t}$$

has general solution

$$x(t) = \frac{1+t_0}{1+t}x(t_0), \quad t \geq t_0.$$

Here $x = 0$ is uniformly stable (check the condition of definition 2.3 for the choice $r = R$), and all trajectories converge asymptotically to 0; therefore the origin is an asymptotically stable equilibrium. However the rate of convergence is dependent on initial time t_0 — if $\|x(t_0)\| < r$ then the condition $\|x(t)\| < R, \ \forall t \geq t_0 + T$ requires that $T > (1 + t_0)(r - R)/R$, which, for fixed R, cannot be bounded by any finite constant for all $t_0 \geq 0$. Hence the origin is not uniformly asymptotically stable. ◊

Definition 2.6 (Exponential stability)**.** The equilibrium $x = 0$ of (2.3) is exponentially stable if there exist constants $r, R, \alpha > 0$ such that

$$\|x(t_0)\| < r \implies \|x(t)\| \leq Re^{-\alpha t}, \ \forall t \geq t_0.$$

Note that:

1. Asymptotic and exponential stability are **local** properties of a dynamic system since they only require that the state converges to zero from a finite set of initial conditions (known as a **region of attraction**): x where $\|x\| < r$.

2. If r can be taken to be infinite in definition 2.4 or definition 2.6, then the system is respectively **globally asymptotically stable** or **globally exponentially stable**.[4]

3. A strictly stable linear system is necessarily globally exponentially stable.

[4]There is no ambiguity in talking about global stability of the overall system rather than global stability of a particular equilibrium point since a globally asymptotically or globally exponentially stable system can only have a single equilibrium point.

2.2 Lyapunov's linearization method [5]

In many cases it is possible to determine whether an equilibrium of a nonlinear system is locally stable simply by examining the stability of the linear approximation to the nonlinear dynamics about the equilibrium point. This approach is known as Lyapunov's linearization method since its proof is based on the more general stability theory of Lyapunov's direct method. However the idea behind the approach is intuitively obvious: within a region of state space close to the equilibrium point, the difference between the behaviour of the nonlinear system and that of its linearized dynamics is small since the error in the linear approximation is small for states close to the equilibrium.

The **linearization** of a system

$$\dot{x} = f(x), \qquad f(0) = 0, \tag{2.4}$$

about the equilibrium $x = 0$ is derived from the Taylor's series expansion of f about $x = 0$. Provided $f(x)$ is continuously differentiable[6] we have

$$\dot{x} = Ax + \overline{f}(x),$$

where A is the Jacobian matrix of f evaluated at $x = 0$:

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=0}, \tag{2.5}$$

and by Taylor's theorem the sum of higher order terms, $\overline{f}$, satisfies

$$\lim_{x\to 0} \frac{\|\overline{f}(x)\|_2}{\|x\|_2} = 0, \ \forall t \geq 0. \tag{2.6}$$

Neglecting higher order terms gives the linearization of (2.4) about $x = 0$:

$$\dot{x} = Ax. \tag{2.7}$$

[5]See *Slotine and Li* §3.3 pp53–57 or *Vidyasagar* §5.5 pp209–219.

[6]A function is continuously differentiable if it is continuous and has continuous first derivatives.

The stability of the linear approximation can easily be determined from the eigenvalues of A. This is because the general solution of a linear system can be computed explicitly:

$$\dot{x} = Ax \implies x(t) = e^{A(t-t_0)}x(t_0),$$

and it follows that the linear system is strictly stable if and only if all eigenvalues of A have negative real parts (or marginally stable if the real parts of some eigenvalues of A are equal to zero, and the rest are negative).

Theorem 2.7 (Lyapunov's linearization method)**.** *For the nonlinear system (2.4), suppose that f is continuously differentiable and define A as in (2.5). Then:*

- $x = 0$ *is an* **exponentially stable** *equilibrium of (2.4) if all eigenvalues of A have negative real parts.*
- $x = 0$ *is an* **unstable** *equilibrium of (2.4) if A has at least one eigenvalue with positive real part.*

The proof of the theorem makes use of (2.6), which implies that the error $\overline{f}$ in the linear approximation to f converges to zero faster than any linear function of x as x tends to zero. Consequently the stability (or instability) of the linearized dynamics implies local stability (or instability) of the equilibrium point of the original nonlinear dynamics.

Note that:

1. The linearization approach concerns local (rather than global) stability.
2. If the linearized dynamics are marginally stable then the equilibrium of the original nonlinear system could be either stable or unstable (see example 2.8 below). It is not possible to draw any conclusions about the stability of the nonlinear system from the linearization in this case since the local stability of the equilibrium could be determined by higher order terms that are neglected in the linear approximation.
3. The above analysis can be extended to non-autonomous systems of the form (2.3). However the linearized system is then time-varying (ie. of

the form $\dot{x} = A(t)x$), and its stability is therefore more difficult to determine in general. In this case the equilibrium of the nonlinear system is asymptotically stable if the linearized dynamics are asymptotically stable.

Example 2.8. Consider the following first order system

$$\dot{x} = -\alpha x|x|$$

where α is a constant. For $\alpha > 0$, the derivative $\dot{x}(t)$ is of opposite sign to $x(t)$ at all times t, and the system is therefore globally asymptotically stable. If $\alpha < 0$ on the other hand, then $\dot{x}(t)$ has the same sign as $x(t)$ for all t, and in this case the system is unstable (in fact the general solution shows that $|x(t)| \to \infty$ as $t \to \alpha^{-1}(t_0 + 1/|x(t_0)|)$). However the linearized system is given by

$$\dot{x} = 0$$

which is marginally stable irrespective of the value of α, and therefore contains no information on the stability of the nonlinear system. ◊

The linearization of a forced system $\dot{x} = f(x, u)$ under a given feedback control law $u = u(x)$ is most easily computed by directly neglecting higher order terms in the Taylor's series expansions of f and u about $x = 0, u = 0$. Thus

$$\dot{x} \approx Ax + Bu, \qquad A = \left[\frac{\partial f}{\partial x}\right]_{x,u=0} \quad B = \left[\frac{\partial f}{\partial u}\right]_{x,u=0} \tag{2.8a}$$

$$u \approx Fx, \qquad F = \left[\frac{\partial u}{\partial x}\right]_{x,u=0} \tag{2.8b}$$

and the linearized dynamics are therefore

$$\dot{x} = (A + BF)x. \tag{2.9}$$

The implication of Lyapunov's linearization method is that linear control techniques can be used to design locally stabilizing control laws for nonlinear systems via linearization about the equilibrium of interest. All that is required is a linear control law $u = Fx$, computed using the linearized dynamics (2.8a), which forces the linearized closed-loop system (2.9) to be strictly stable. However this approach is limited by the local nature of the linearization process — there is no guarantee that the resulting linear control law stabilizes the nonlinear system everywhere within the desired operating region of state space, which may be much larger than the region on which the linear approximation to the nonlinear system dynamics is accurate.

2.3 Lyapunov's direct method [7] [8]

The aim of Lyapunov's direct method is to determine the stability properties of an equilibrium point of an unforced nonlinear system without solving the differential equations describing the system. The basic approach involves constructing a scalar function V of the system state x, and considering the derivative $\dot{V}$ of V with respect to time. If V is positive everywhere except at the equilibrium $x = 0$, and if furthermore $\dot{V} \leq 0$ for all x (so that V cannot increase along the system trajectories), then it is possible to show that $x = 0$ is a stable equilibrium. This is the main result of Lyapunov's direct method, and a function V with these properties is known as a **Lyapunov function**. By imposing additional conditions on V and its derivative $\dot{V}$, this result can be extended to provide criteria for determining whether an equilibrium is asymptotically or exponentially stable both locally and globally.

Motivation

The function V can be thought of as a generalization of the idea of the stored energy in a system. Consider for example the second order system:

$$m\ddot{y} + c(\dot{y}) + k(y) = 0 \qquad \begin{cases} m > 0 \\ c(0) = 0, \ \operatorname{sign}\big(c(\dot{y})\big) = \operatorname{sign}(\dot{y}) \\ k(0) = 0, \ \operatorname{sign}\big(k(y)\big) = \operatorname{sign}(y) \end{cases} \tag{2.10}$$

where y is the displacement of a mass m, and $c(\dot{y})$, $k(y)$ are respectively nonlinear damping and spring forces acting on the mass (figure 6).

[7] See *Slotine and Li* §3.4 pp57–68 and §4.2 pp105–113 or *Vidyasagar* §5.3 pp157–176.

[8] The proofs of the various results given in this section are intended to be conceptual rather than rigorous. For a more technical treatment see eg. *Vidyasagar*, though this is not necessary for the level of understanding required in this course.

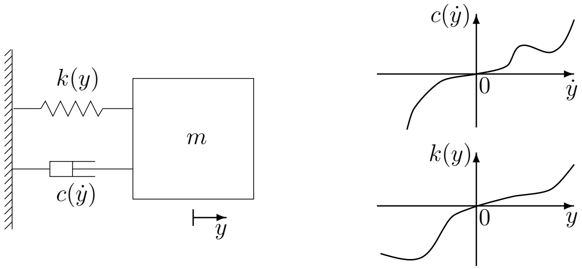


Figure 6: Mass, spring and damper.

From the rate of change of stored energy it is easy to deduce that the equilibrium $y = \dot{y} = 0$ is stable without explicit knowledge of the functions k and c. When released from non-zero initial conditions energy is transfered between the spring and the mass (since the spring force opposes the displacement of the mass), but the total stored energy decreases monotonically over time due to dissipation in the damper (since the damping force opposes the velocity of the mass). It is also intuitively obvious that a small value for the stored energy at time t corresponds to small values of $y(t)$ and $\dot{y}(t)$, and therefore $y = \dot{y} = 0$ must be a stable equilibrium.

To make this argument more precise, let V be the energy stored in the system (ie. the sum of the kinetic energy of the mass and the potential energy of the spring):

$$V(y, \dot{y}) = \frac{1}{2}m\dot{y}^2 + \int_0^y k(y)\,dy. \tag{2.11}$$

Here $k(y)$ has the same sign as y, and V is therefore strictly positive unless $y = \dot{y} = 0$ (in which case $V = 0$). It follows that $(y^2 + \dot{y}^2)^{1/2}$ can only exceed a given bound R if V increases beyond some corresponding bound $\overline{V}$. However, using the chain rule and the system dynamics (2.10), the derivative of V with respect to time is given by

$$\begin{aligned}\dot{V}(y, \dot{y}) &= \frac{1}{2}m\frac{d}{d\dot{y}}\Big[\dot{y}^2\Big]\ddot{y} + \frac{d}{dy}\Big[\int_0^y k(y)\,dy\Big]\dot{y} \\ &= m\dot{y}\ddot{y} + k(y)\dot{y} \\ &= -c(\dot{y})\dot{y}\end{aligned}$$

which implies that $\dot{V} \le 0$ due to the condition on the sign of $c(\dot{y})$ in (2.10). Hence V decreases monotonically along system trajectories. It is therefore clear that if the initial conditions $y(t_0), \dot{y}(t_0)$ are sufficiently close to the equilibrium that $\big(y^2(t_0) + \dot{y}^2(t_0)\big)^{1/2} < r$, where r satisfies

$$(y^2 + \dot{y}^2)^{1/2} < r \implies V(y, \dot{y}) < \overline{V},$$

then $(y^2 + \dot{y}^2)^{1/2}$ cannot exceed the bound R at any future time $t \ge t_0$, which implies that the equilibrium is uniformly stable in the sense of definition 2.3. Figure 7 illustrates the relationship between the constants $\overline{V}$, r and R. To simplify the figure we have assumed that V has radial symmetry (ie. $V(y, \dot{y}) = V(y + \dot{y}^2)$).

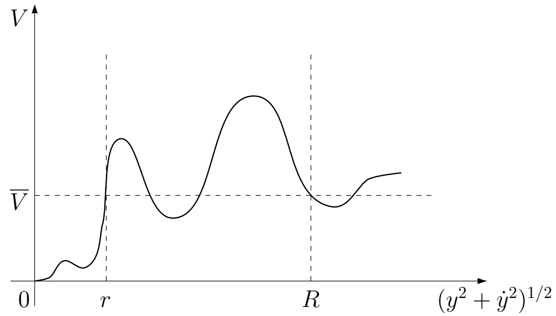


Figure 7: The energy storage function $V(y, \dot{y})$.

Autonomous systems

Positive definite functions. The argument used above to show that the equilibrium of (2.10) is stable relies on the fact that a bound on V implies a corresponding bound on the norm of the state vector and vice-versa. This is the case for a general continuous function V of the state x provided V has the property that

$$\begin{aligned} x \ne 0 &\iff V(x) > 0, \\ x = 0 &\iff V(x) = 0. \end{aligned} \tag{2.12}$$

A continuous function with this property is said to be **positive definite**.[9] Alternatively, if $V(x)$ is continuous and (2.12) holds for all x such that $\|x\| < R_0$ for some $R_0 > 0$, then V is **locally positive definite** (or positive definite for $\|x\| < R_0$).

Derivative of $V(x)$ along system trajectories. If x satisfies the differential equation $\dot{x} = f(x)$, then the time-derivative of a continuously differentiable scalar function $V(x)$ is given by

$$\dot{V}(x) = \nabla V(x)\dot{x} = \nabla V(x) f(x), \tag{2.13}$$

where $\nabla V(x)$ is the gradient of V (expressed as a row vector) with respect to x evaluated at x. The expression (2.13) gives the rate of change of V as x moves along a trajectory of the system state, and $\dot{V}$ is therefore known as the derivative of V along system trajectories.

With these definitions we can give the general statement of Lyapunov's direct method for autonomous systems.

Theorem 2.9 (Stability/asymptotic stability for autonomous systems)**.** *If there exists a continuously differentiable scalar function $V(x)$ such that:*

(a). $V(x)$ is positive definite

(b). $\dot{V}(x) \leq 0$

for all x satisfying $\|x\| < R_0$ for some constant $R_0 > 0$, then the equilibrium $x = 0$ is **stable**. *If, in addition,*

(c). $-\dot{V}(x)$ is positive definite

whenever $\|x\| < R_0$, then $x = 0$ is **asymptotically stable**.

The first part of this theorem can be proved by showing that it is always possible to find a positive scalar r which ensures that, for any given $R > 0$, $x(t)$ is bounded by $\|x(t)\| < R$ for all $t \geq t_0$ whenever the initial condition satisfies $\|x(t_0)\| < r$. To do this, first choose $R < R_0$ and define $\overline{V}$ as the minimum value of $V(x)$ over all x such that $\|x\| = R$ (figure 8a). Then a trajectory $x(t)$ can escape the region on which $\|x\| < R$ only if $V\big(x(t)\big) \geq \overline{V}$

[9] Similarly, V is negative definite if $-V$ is positive definite.

for some $t \geq t_0$. But condition (a) implies that there exists a positive $r < R$ for which $V(x) < \overline{V}$ whenever $\|x\| < r$, whereas condition (b) ensures that $V\big(x(t)\big)$ decreases over time if $\|x(t)\| < R$. It follows that $V\big(x(t)\big)$ cannot exceed $\overline{V}$ for all $t \geq t_0$ if the initial state satisfies $\|x(t_0)\| < r$, and the state $x(t)$ cannot therefore leave the region on which $\|x\| < R$. For the case of $R \geq R_0$, simply repeat this argument with R replaced by R_0.

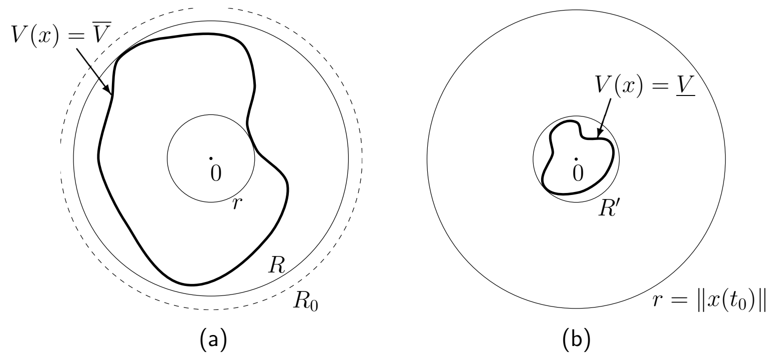


Figure 8: Level sets of $V(x)$ in state space.

The second part concerning asymptotic stability can be proved by contradiction. Suppose that the origin is not asymptotically stable, then for any given initial condition $x(t_0)$ the value of $\|x(t)\|$ must remain larger than some positive number $R' < \|x(t_0)\|$. Conditions (a) and (c) therefore imply that $V\big(x(t)\big) > \underline{V}$ and $\dot{V}\big(x(t)\big) < -\underline{W}$ at all times $t \geq t_0$, for some constants $\underline{V}, \underline{W} > 0$ (figure 8b). But this is a contradiction since $V\big(x(t)\big)$ must decrease to a value less than $\underline{V}$ in a finite time smaller than $[V\big(x(t_0)\big) - \underline{V}]/\underline{W}$ if $\dot{V}\big(x(t)\big) < -\underline{W}$ for all $t \geq t_0$.

Non-autonomous systems [10]

The discussion so far has concerned only autonomous systems (systems with dynamics of the form $\dot{x} = f(x)$). Extensions to non-autonomous systems ($\dot{x} = f(x,t)$) are straightforward, but the situation is complicated by the fact that a Lyapunov function for non-autonomous dynamics may need to be time-varying, ie. of the form $V(x,t)$. For completeness this section gives some results on non-autonomous system stability.

Positive definite time-varying functions. To extend the definition of a positive definite function to the case of time-varying functions, we simply state that $V(x,t)$ is positive definite if there exists a time-invariant positive definite function $V_0(x)$ satisfying

$$V(x,t) \geq V_0(x), \ \forall t \geq t_0, \ \forall x. \tag{2.14}$$

Similarly, V is locally positive definite if $V(x,t)$ is bounded below by a locally positive definite time-invariant function $V_0(x)$ for all $t \geq t_0$.

Decrescent functions. Non-autonomous systems differ from autonomous systems in that the stability of an equilibrium may be uniform or non-uniform (recall that the stability properties of autonomous systems are necessarily uniform). In order to ensure uniform stability, a Lyapunov function V for a non-autonomous system must also be **decrescent**, which requires that

$$V(x,t) \leq V_0(x), \ \forall t \geq t_0, \ \forall x \tag{2.15}$$

for some time-invariant positive definite function $V_0(x)$.

Derivative of $V(x,t)$ along system trajectories. If $x(t)$ satisfies $\dot{x} = f(x,t)$, then the derivative with respect to t of a continuously differentiable function $V(x,t)$ can be expressed

$$\dot{V}(x,t) = \frac{\partial V}{\partial t}(x,t) + \nabla V(x)\dot{x} = \frac{\partial V}{\partial t}(x,t) + \nabla V(x) f(x,t). \tag{2.16}$$

The appearance here of the partial derivative of V with respect to t is due to the explicit dependence of V on time.

[10] The discussion in this section of Lyapunov's direct method for non-autonomous systems is non-examinable and is provided as background material

The main result of Lyapunov's direct method for non-autonomous systems can be stated in terms of these definitions as follows.

Theorem 2.10 (Stability/asymptotic stability for non-autonomous systems)**.**
If there exists a continuously differentiable scalar function $V(x,t)$ such that:

(a). $V(x,t)$ is positive definite

(b). $\dot{V}(x,t) \leq 0$

for all x satisfying $\|x\| < R_0$ for some constant $R_0 > 0$, then the equilibrium $x = 0$ is **stable**. *If, furthermore,*

(c). $V(x,t)$ is decrescent

then $x = 0$ is **uniformly stable**. *If, in addition to (a), (b), and (c),*

(d). $-\dot{V}(x,t)$ is positive definite

whenever $\|x\| < R_0$, then $x = 0$ is **uniformly asymptotically stable**.

This theorem can be proved in a similar way to theorem 2.9. The only difference in the first part concerning stability is that here the definitions of the constants $\overline{V}$ and r must hold for all $t \geq t_0$ (ie. so that $\overline{V}$ is the minimum of $V(x,t)$ for all x such that $\|x\| = R$ *and* all $t \geq t_0$, and likewise $V(x,t) < \overline{V}$ for all x such that $\|x\| < r$ *and* all $t \geq t_0$). Similarly, in the second part concerning asymptotic stability, the constants $\underline{V}$ and $\underline{W}$ must be defined as lower bounds on $V(x,t)$ and $-\dot{V}(x,t)$ for all x such that $\|x\| < R'$ *and* all $t \geq t_0$. In order to show uniform stability, the requirement that V is decrescent in (c) avoids the possibility that $\overline{V}$ tends to infinity as $t \to \infty$ (which would imply that r becomes arbitrarily small as $t \to \infty$) by preventing $V(x,t)$ from becoming infinite within the region on which $\|x\| < R$ at any time $t \geq t_0$. Uniform asymptotic stability is implied by $V\big(x(t_0), t_0\big)$ necessarily being finite (so that the time taken for the value of V to fall below $\underline{V}$ is finite), due to the assumption that V is decrescent.

Global Stability

Lyapunov's direct method also provides a means of determining whether a system is globally asymptotically stable via simple extensions of the theorems for asymptotic stability already discussed. Before giving the details of this approach, we first need to introduce the concept of a radially unbounded function.

Radially unbounded functions. As might be expected, the conditions of theorems 2.9 or 2.10 must hold at all points in state space in order to assert global asymptotic stability for autonomous and non-autonomous systems respectively. However one extra condition on V, which for time-invariant functions can be expressed:

$$V(x) \to \infty \text{ as } \|x\| \to \infty, \tag{2.17}$$

is also required. A function with this property is said to be **radially unbounded**. For time-varying functions this condition becomes

$$V(x,t) \geq V_0(x), \ \forall t \geq t_0, \ \forall x, \tag{2.18}$$

where V_0 is any radially unbounded time-invariant function.

Below we give the global version of theorems 2.9 and 2.10.

Theorem 2.11 (Global uniform asymptotic stability). *If V is a Lyapunov function for an autonomous system (or non-autonomous system) which satisfies conditions (a)–(c) of theorem 2.9 (or conditions (a)–(d) of theorem 2.10 respectively) for all x (ie. in the limit as $R_0 \to \infty$), and V is radially unbounded, then the system is* **globally uniformly asymptotically stable**.

The conditions on V in this theorem ensure that the origin is uniformly asymptotically stable, by theorem 2.9 (or theorem 2.10). To prove the theorem it therefore suffices to show that the conditions on V imply that every state trajectory $x(t)$ tends to the origin as $t \to \infty$. But since $x(t)$ is continuous, this requires that $x(t)$ remains bounded for all $t \geq t_0$ for arbitrary initial conditions $x(t_0)$. Hence the purpose of the radial unboundedness condition on V is to ensure that $x(t)$ remains at all times within the *bounded* region defined by $V(x,t) \leq V(x(t_0), t_0)$. If V were not radially unbounded, then not all contours of constant V in state-space would be closed curves, and it would be possible for $x(t)$ to drift away from the equilibrium even though $\dot{V}$ is negative.

The remainder of the proof involves constructing a finite bound on the time taken for a trajectory starting from arbitrary $x(t_0)$ to enter the region on which $\|x\| < R$, for any $R > 0$. This can be done by finding bounds on V and $-\dot{V}$ using the positive definite properties of V and $-\dot{V}$.

Note that:

1. If the requirement that $\dot{V}$ is negative definite in theorem 2.11 is replaced by the condition that $\dot{V}$ is simply non-positive, then $x(t)$ is guaranteed to be **globally bounded**. This means that, for every initial condition $x(t_0)$, there exists a finite constant $R\big(x(t_0)\big)$ such that $\|x(t)\| < R$ for all $t \geq t_0$.

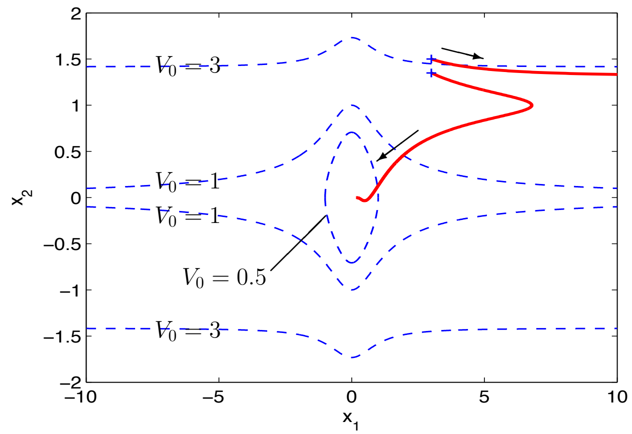


Figure 9: Contours of $V(x_1, x_2) = V_0$ for the function V defined in (2.20) (dashed lines), and state trajectories of (2.19) for two different initial conditions (solid lines).

Example 2.12. To determine whether $x_1 = x_2 = 0$ is a stable equilibrium of the system

$$\begin{aligned} \dot{x}_1 &= (x_2 - 1)x_1^3 \\ \dot{x}_2 &= -\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2} \end{aligned} \tag{2.19}$$

using Lyapunov's direct method, we need to find a scalar function $V(x_1, x_2)$

satisfying some or all of the conditions of theorems 2.9 and 2.11. As a starting point, try the positive definite function

$$V(x_1, x_2) = x_1^2 + x_2^2.$$

Differentiating this function along system trajectories, we have

$$\begin{aligned}\dot V(x_1, x_2) &= 2x_1\dot{x_1} + 2x_2\dot{x_2}\\ &= -2x_1^4 + 2x_2x_1^4 - 2\frac{x_2x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2},\end{aligned}$$

which is not of the required form since $\dot V \not\leq 0$ for some x_1, x_2. However it is not far off since the 2nd and 3rd terms in the above expression nearly cancel each other. After some experimentation with the first term in V we find that, with V redefined as

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2, \tag{2.20}$$

(which is again positive definite) the derivative becomes

$$\begin{aligned}\dot V(x_1, x_2) &= 2\Big[\frac{x_1}{1+x_1^2} - \frac{x_1^3}{(1+x_1^2)^2}\Big]\dot{x_1} + 2x_2\dot{x_2}\\ &= -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2},\end{aligned}$$

so that $\dot V$ is now negative definite. Thus (2.20) satisfies all of the conditions of theorem 2.9, and the equilibrium is therefore asymptotically stable. However V is not radially unbounded since the contours of $V(x_1, x_2) = V_0$ are not closed curves for $V_0 \geq 1$ (figure 9), and it is therefore not possible to conclude that the system (2.19) is globally asymptotically stable. Neither is it possible to conclude that (2.19) is *not* globally asymptotically stable without further analysis, since some other function V might be found which both satisfies the conditions of theorem 2.9 and is radially unbounded. In fact (2.19) is not globally asymptotically stable since it is possible to find initial conditions for which $x_1(t), x_2(t)$ do not converge to zero (for example the trajectory starting from $(x_1, x_2) = (3, 1.5)$ shown in figure 9 escapes to infinity). $\Diamond$

Exponential stability

Exponential stability is a special case of asymptotic stability in which the convergence of the system state to the equilibrium is bounded by an exponentially decaying function of time. To show that an equilibrium is exponentially stable using Lyapunov's direct method, it is therefore necessary to impose further conditions on the rate of change of a Lyapunov function which demonstrates asymptotic stability. The basic approach involves finding a scalar function $V(x, t)$ whose derivative along system trajectories satisfies

$$\dot V(x, t) \leq -aV(x, t) \tag{2.21}$$

for some constant $a > 0$, in a region of state space containing the equilibrium. Suppose therefore that (2.21) holds for all x such that $\|x\| \leq R_0$. If V is also positive definite and decrescent for $\|x\| \leq R_0$, then by theorem 2.10 there exists an $r > 0$ such that all trajectories $x(t)$ with initial conditions $x(t_0)$ satisfying $\|x(t_0)\| \leq r$ remain within the region on which $\|x\| \leq R_0$ for all $t \geq t_0$. On these trajectories the bound (2.21) holds at all times, which implies that

$$V\big(x(t), t\big) \leq V\big(x(t_0), t_0\big)e^{-at}, \ \forall t \geq t_0. \tag{2.22}$$

whenever $\|x(t_0)\| \leq r$. In order to conclude that $\|x(t)\|$ satisfies a similar bound, we need to find a lower bound on $V(x, t)$ in terms of a positive power of $\|x\|$ of the form

$$V(x, t) \geq b\|x\|^p, \tag{2.23}$$

for some constants $b > 0$ and $p > 0$. Note that the positive definiteness of V ensures that it is always possible to construct such a bound for all x satisfying $\|x\| \leq R_0$. Combining inequalities (2.22) and (2.23) leads to the required result:

$$\|x(t_0)\| \leq r \implies \|x(t)\| \leq Re^{-\alpha t}, \ \forall t \geq t_0. \qquad \begin{cases} R = \Big[\dfrac{V\big(x(t_0), t_0\big)}{b}\Big]^{1/p} \\ \alpha = a/p \end{cases}$$

Clearly this argument can be used to show global exponential stability if R_0 can be taken to be infinite and V is radially unbounded. A variation on the above approach assumes that $-\dot{V}$ is greater than some positive power of $\|x\|$, and then imposes both upper and lower bounds on V in terms of positive powers of $\|x\|$ in order to assert exponential stability.

Concluding remarks

- Lyapunov's direct method only gives sufficient conditions for stability. This means for example that the existence of a positive definite V for which $\dot{V} \not\leq 0$ along system trajectories does not necessarily imply that $x = 0$ is unstable.
- Lyapunov analysis can be used to find conditions for instability of an equilibrium point, for example $x = 0$ must be unstable if V and $\dot{V}$ are both positive definite. These results are known as **instability theorems**.
- It is also possible to show that the premises and conclusions of each of theorems 2.9–2.11 are interchangeable, so for example there must be some positive definite function V with $\dot{V}$ negative definite if $x = 0$ is asymptotically stable. These results are known as **converse theorems**.
- There are no general guidelines for constructing Lyapunov functions. In situations where physical insights into the system dynamics do not suggest an obvious choice for V, it is often necessary to resort to a Lyapunov-like analysis based on a function satisfying only some of the conditions of theorems 2.9–2.11. This approach will be discussed in the next section.