15.094J: Robust Modeling, Optimization, Computation

Lecture 17: RO and Risk preferences

Outline

1. Risk Measures
2. Coherent Risk Measures
3. From coherent risk measures to convex uncertainty sets
4. From comonotone measures to polyhedral uncertainty sets
5. Summary

Primitives

- $\mathcal{X}$ set of all random variables distributed on $\mathbb{R}$.
- A **risk measure** is a functional $\mu : \mathcal{X} \to \mathbb{R}$.
- if $X_1, X_2 \in \mathcal{X}$, then X_1 is preferable (under μ) to X_2 if and only if $\mu(X_1) \leq \mu(X_2)$.
- Random variables with higher risk measures as less desirable, i.e., associated with *greater losses*.

Standard Deviation as a Measure of Risk

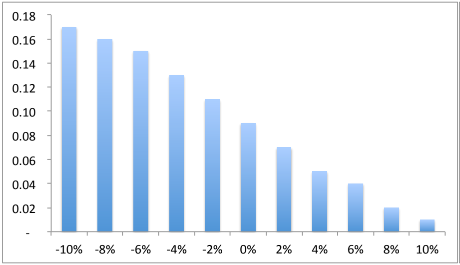
- Let R denote a random variable for the (annual) return on a risky portfolio.

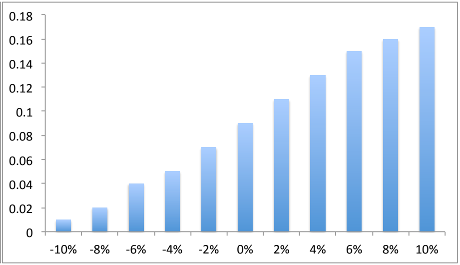
Year	Return
Year 1	12%
Year 2	-3%
Year 3	15%
Year 4	-7%
Year 5	8%

- $\mu = \frac{12-3+15-7+8}{5} = 5\%$.
- $\sigma^2 = \frac{1}{5}(12-5)^2 + \frac{1}{5}(-3-5)^2 + \ldots + \frac{1}{5}(8-5)^2 = 73.2$
- $\sigma = 8.55$.

A Criticism of Standard Deviation

Losses and Gains are symmetrically weighted.





These two distributions have the same standard deviation, but not the same “risk."

Value at Risk (VaR)

Given a time horizon, VaR at a given level α is a threshold value such that the probability that the loss on the portfolio over the time horizon exceeds this value is the given probability level.

Math: The α-quantile $q_\alpha(R) = \min\{r : P(R \leq r) \geq \alpha\} \quad 0 \leq \alpha \leq 1$.
The Value at Risk at level α: $VaR_\alpha(R) = -q_\alpha(R)$.

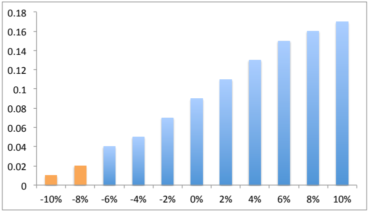


Figure : For $\alpha = 3\%$, $VaR_\alpha(R) = 8\%$.

Widely used by government regulators and banks.

Example

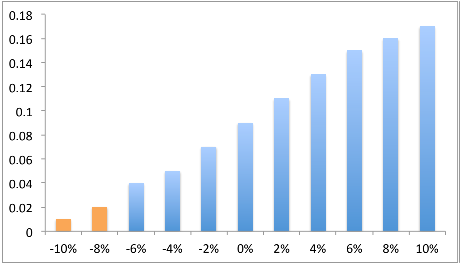
- Historical returns

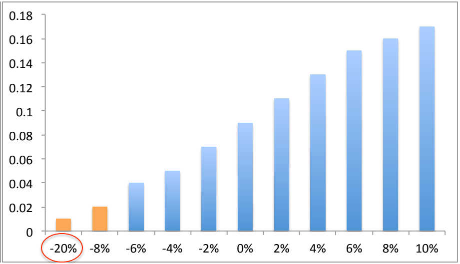
Year	Return
Year 1	12%
Year 2	-3%
Year 3	15%
Year 4	-7%
Year 5	8%

- For $\alpha = 40\%$, $q_\alpha(R) = -3\%$ since $P(R \leq -3) = 0.4$ and $P(R < -3) < .4$.
- $VaR_\alpha(R) = -q_\alpha(R) = 3\%$.
- Note that if year 4 we lost 25%, still $VaR_\alpha = 3\%$.

Criticisms of Value at Risk

VaR is indifferent to severity of losses beyond the "critical" threshold.





Both distributions have same VaR_α for $\alpha = 3\%$, but different "risks."

Coherent Risk Measures

So what is an appropriate definition of risk?

P. Artzner, F. Delbaen, J. Eber, D. Heath (1999). "Coherent Measures of Risk". Mathematical Finance, 9, 3, 203–228.

$\rho(R)$ is a *coherent risk measure* if it satisfies the following four properties:

Monotonicity

If the return of Portfolio 2 is better than the return of Portfolio 1 under all scenarios, then **the risk** of Portfolio 2 is less than **the risk** of Portfolio 1.

Mathematically: If $R_1 \leq R_2$, then $\rho(R_1) \geq \rho(R_2)$.

Diversification

The risk of two portfolios together cannot be worse than the sum of their risks separately.

Mathematically: $\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2)$.

Leverage

If you double your portfolio then you double your risk.

Mathematically: If $\alpha \geq 0$, then $\rho(\alpha R) = \alpha \rho(R)$.

Influence of Cash

If you add cash in your portfolio you reduce your risk by the amount the cash you add.

Mathematically: $\rho(R+\alpha) = \rho(R) - \alpha$.

Is Standard Deviation Coherent?

Consider $R_1 = 10\%$ deterministically,

$$R_2 = \begin{cases} 10\%, & \text{with probability } 1/2, \\ 30\% & \text{with probability } 1/2. \end{cases}$$

- $R_1 \leq R_2$ in all cases.
- $Stdev(R_1) = 0$.
- $Stdev(R_2) = \sqrt{\frac{1}{2}(10-20)^2 + \frac{1}{2}(30-20)^2} = 10\% > 0 = Stdev(R_1)$.
- **Standard Deviation violates monotonicity**; all other properties are satisfied.

Is Value at Risk Coherent?

- Return of first portfolio

$$R_1 = \begin{cases} 10\%, & \text{with probability } 0.6, \\ 3\%, & \text{with probability } 0.4. \end{cases}$$

- For $\alpha = 50\%$, $VaR_\alpha(R_1) = -10\%$, since $P(R_1 \leq 10\%) \geq .5$ and $P(R_1 < 10\%) < .5$.
- Return of second portfolio

$$R_2 = \begin{cases} 2\%, & \text{with probability } 0.6, \\ -10\%, & \text{with probability } 0.4. \end{cases}$$

- For $\alpha = 50\%$, $VaR_\alpha(R_2) = -2\%$, since $P(R_2 \leq 2\%) \geq .5$ and $P(R_2 < 2\%) < .5$.

Is Value at Risk Coherent? (continued)

- Then

$$R_1 + R_2 = \begin{cases} 12\%, & \text{with probability } 0.36, \\ 5\%, & \text{with probability } 0.24, \\ 0\%, & \text{with probability } 0.24, \\ -7\%, & \text{with probability } 0.16. \end{cases}$$

- For $\alpha = 50\%$, $VaR_\alpha(R_1 + R_2) = -5\%$, since $P(R_1 + R_2 \leq 5\%) \geq .5$ and $P(R_1 + R_2 < 5\%) < .5$.
- $VaR_\alpha(R_1 + R_2) > VaR_\alpha(R_1) + VaR_\alpha(R_2)$.
- **VaR does not satisfy the diversification property, i.e., it is not coherent.**
- It turns out that if Returns are normal, then VaR satisfies the diversification property.

Conditional Value at Risk

Natural question: What is the *expected* loss incurred in the α% worst cases in our portfolio? (Contrast to VaR_α)

Conditional Value at Risk:
$cVaR_\alpha$= Expected Loss when Loss is less than or equal to VaR_α.

$$cVaR_\alpha = -E[R \mid R \leq VaR_\alpha(R)].$$

- If R has a continuous distribution, $cVaR_\alpha$ is a coherent risk measure.

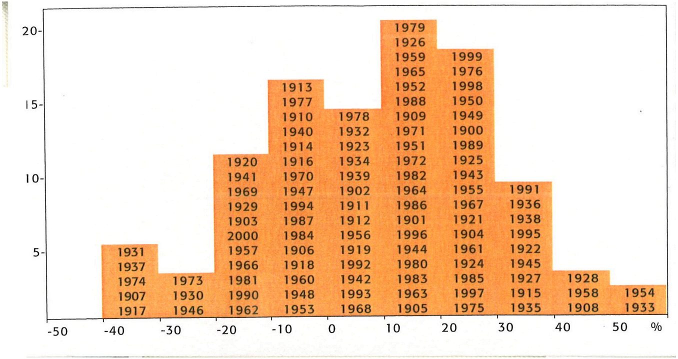
Example (revisited)

Year	Return
Year 1	12%
Year 2	-3%
Year 3	15%
Year 4	-7%
Year 5	8%

- Recall for $\alpha = 40\%$, $VaR_\alpha = 3\%$.
- $cVaR_\alpha = -\frac{1}{2}[-3\% - 7\%] = 5\%$
- $cVaR_\alpha = 5\%$.
- Expected loss in the 40% of the worst case years is 5%.

Real Returns of US Equities in 1900-2000

- Average Real Return: 6.7%
- Maximum annual loss: **Drawdown** $= CVaR = VaR = 40\%$.



Representation theorem for coherent risk measures

- Let $\mathcal{X}$ be the set of all random variables on $\mathbb{R}$ having support with cardinality N. A risk measure $\mu : \mathcal{X} \to \mathbb{R}$ is coherent if and only if there exists a family of probability measures $\mathcal{Q}$ that

$$\mu(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q [X], \qquad \forall\, X \in \mathcal{X},$$

  where $\mathbb{E}_Q [X]$ denotes the expectation of the random variable X under the measure Q (as opposed to the measure of X itself).
- All coherent risk measures may be represented as the worst-case expected value over a family of "generalized scenarios."

Distorted probability measures and the Choquet integral

- Given a random variable $X \in \mathcal{X}$, we define: $S(x) = \mathbb{P}[X \geq x]$.
- A **distortion function** g is any non-decreasing function on $[0,1]$ such that $g(0) = 0$ and $g(1) = 1$. The **distorted probability distribution** for a random variable $X \in \mathcal{X}$ given by $S(x)$ is the unique distribution defined by $S^*(x) = g(S(x))$.
- The **Choquet integral** of a random variable $X \in \mathcal{X}$ with respect to the distortion function g is defined as

$$\pi_g(X) = \int_0^\infty S^*(x)dx + \int_{-\infty}^0 [S^*(x) - 1]\, dx.$$

- The Choquet integral π_g satisfies monotonicity, translation invariance, and positive homogeneity. In addition, π_g satisfies subadditivity if and only if g is concave. Thus, π_g is coherent if and only if g is concave.

Value-at-risk

- For some $\alpha \in [0,1]$ we definte

$$g(u) = \begin{cases} 0, & \text{if } u < \alpha, \\ 1, & \text{otherwise.} \end{cases}$$

Then we have

$$\mu_g(X) = \int_0^{\infty} g(S(x))dx = \int_0^{S^{-1}(\alpha)} dx = S^{-1}(\alpha) = \inf\{x \mid \mathbb{P}[X \geq x] \leq \alpha\}$$

value-at-risk at level α, or $\mathrm{VaR}_\alpha(X)$.

- Note that g is not concave, which implies, that value-at-risk is *not* a coherent risk measure.

Conditional value at Risk

- We define $g(u) = \min(u/\alpha, 1)$ for some $\alpha \in [0, 1]$. Then we have

$$
\begin{aligned}
\mu_g(X) &= \int_0^\infty g(S(x))dx \\
&= \frac{1}{\alpha} \int_{S^{-1}(\alpha)}^{\infty} S(x)dx + \int_0^{S^{-1}(\alpha)} dx \\
&= \frac{x}{\alpha} S(x)\Big|_{S^{-1}(\alpha)}^{\infty} - \frac{1}{\alpha} \int_{S^{-1}(\alpha)}^{\infty} xdS(x) + S^{-1}(\alpha) \\
&= \mathbb{E}[X|X \geq S^{-1}(\alpha)] \\
&= \mathbb{E}[X|X \geq \mathrm{VaR}_\alpha(X)].
\end{aligned}
$$

- Note that this is a coherent risk measure since g is concave. It is **conditional value at risk of X at level α** and denoted $CVAR_\alpha X$.

Comonotone Risk measures

- Is it true that all coherent risk measures can be represented as a Choquet integral under a concave distortion function?
- The set $A \subseteq \mathbb{R}^n$ is a **comonotonic set** if for all $x \in A$, $y \in A$, we have either $x \leq y$ or $y \leq x$.
- Clearly any one-dimensional set is comonotonic.
- A random variable $X = (X_1, \ldots, X_n)$ is **comonotonic** if its support $A \subseteq \mathbb{R}^n$ is a comonotonic set.
- An example of a comonotonic random variable is the joint payoff of a stock and a call option on the stock. Indeed, let S be the stock value at the exercise time, C be the call value, and K be the strike price. Then $C = \max(0, S - K)$. It is easy to see that any pair of payoffs (S_1, C_1), (S_2, C_2) satisfy either $S_1 \geq S_2$ and $C_1 \geq C_2$ or $S_1 \leq S_2$ and $C_1 \leq C_2$, and hence the support of the random variable (S, C) is comonotonic.

Comonotone Risk measures, continued

- A risk measure $\mu : \mathcal{X} \to \mathbb{R}$ is **comonotonically additive** if, for any comonotonic random variables X and Y, we have

$$\mu(X+Y) \quad = \quad \mu(X) + \mu(Y).$$

- If a coherent risk measure is comonotonically additive, we say the risk measure is **comonotone**.
- A risk measure $\mu : \mathcal{X} \to \mathbb{R}$ can be represented as the Choquet integral with a concave distortion function if and only if μ is comonotone.
- The subadditive property says we can do no worse by aggregating risk when dealing with a coherent risk measure;
- We will not benefit from diversifying risk when our risk measure is a Choquet integral and the underlying random variables are comonotonic.
- We can construct a coherent risk measure which violates comonotonic additivity.

Landscape of Risk Measures

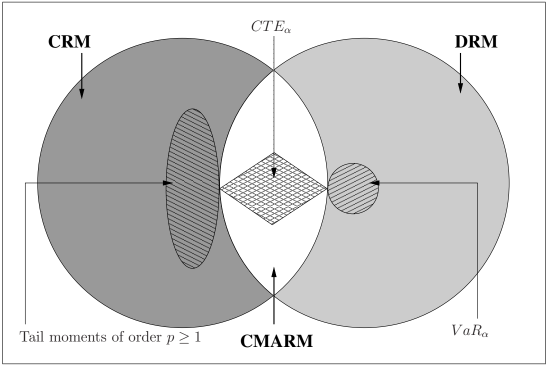


Figure : *Venn diagram of the risk measure universe. The box represents all functions $\mu : \mathcal{X} \to \mathbb{R}$. In bold are the three main classes of risk measures: coherent (CRM), distorted (DRM), and their intersection, comonotone (CMARM). Also illustrated are the specific subclasses CVAR$_\alpha$, VaR$_\alpha$, and tail moments of higher order.*

From coherent risk measures to convex uncertainty sets

- Consider $\tilde{a}'x \geq b$.
- Assumption A: The uncertain vector $\tilde{a}$ has support $\mathcal{A} = \{a_1, \ldots, a_N\}$ and distribution

$$\mathbb{P}[\tilde{a} = a] \quad = \quad \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}(a_j = a),$$

- For a linear optimization problem with uncertain data $\tilde{a}$ and real number b, along with a risk measure μ, we define the **risk averse problem** to be

$$\begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & \mu(\tilde{a}'x) \leq b. \end{array}$$

Key connection

- If the risk measure μ is coherent and $\tilde{a}$ is distributed as in Assumption A, then the risk averse problem is equivalent to the robust optimization problem

$$\begin{array}{rll} \text{minimize} & c'x & \\ \text{subject to} & a'x \le b & \forall\, a \in \mathcal{U}, \end{array}$$

where

$$\mathcal{U} = \mathrm{conv}\left(\left\{ \sum_{i=1}^{N} q_i a_i \;\middle|\; q \in \mathcal{Q} \right\} \right) \subseteq \mathrm{conv}\left(\mathcal{A}\right),$$

and $\mathcal{Q}$ is the set of generating measures for μ

Proof

$$
\begin{aligned}
\mu(\tilde{a}'x) &= \sup_{q \in \mathcal{Q}} \mathbb{E}_q \left[\tilde{a}'x \right] \\
&= \sup_{q \in \mathcal{Q}} \sum_{i=1}^{N} (a_i'x) q_i \\
&= \sup_{q \in \mathcal{Q}} \left(\sum_{i=1}^{N} q_i a_i \right)' x \\
&= \sup_{a \in \tilde{\mathcal{U}}} a'x \\
&= \sup_{a \in \mathcal{U}} a'x,
\end{aligned}
$$

where $\tilde{\mathcal{U}} = \{\sum_{i=1}^{N} q_i a_i \mid q \in \mathcal{Q}\}$ and $\mathcal{U} = \text{conv}(\tilde{\mathcal{U}})$.

Summary

- The decision maker has some primitive risk measure μ
- If it is coherent, then, there is an *explicit* uncertainty set that should be used in the robust optimization framework.
- This uncertainty set is convex and its structure depends on the generating family $\mathcal{Q}$ for μ and the realizations $\mathcal{A}$ of $\tilde{a}$.

From comonotone measures to polyhedral uncertainty sets

- If $\mathcal{Q}$ is a finite set, then we have the following:

$$\sup_{Q\in\mathcal{Q}} \mathbb{E}_Q[X] = \max_{Q\in\mathcal{Q}} \mathbb{E}_Q[X] = \max_{Q\in\mathrm{conv}(\mathcal{Q})} \mathbb{E}_Q[X],$$

  where $\mathrm{conv}(\mathcal{Q})$ denotes the convex hull of $\mathcal{Q}$.

- For a comonotone risk measure with distortion function g on a random variable Y with support $\{y_1,\ldots,y_N\}$ such that $\mathbb{P}[Y=y_i]=1/N$, we have

$$\mu_g(Y) = \sum_{i=1}^{N} q_i y_{(i)},$$

  where $y_{(i)}$ is the ith order statistic of Y, i.e., $y_{(1)}\leq\ldots\leq y_{(N)}$, and

$$q_i = g\left(\frac{N+1-i}{N}\right) - g\left(\frac{N-i}{N}\right).$$

Proof

WLOG $y_i = y_{(i)}$ for all $i \in \{1, \dots, N\}$. Also $y_1 \geq 0$.

$$S_Y(y) = \begin{cases} 1, & \text{if } y < y_1, \\ \frac{N-i}{N}, & \text{if } y_i \leq y < y_{i+1},\ i = 1, \dots, N-1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mu_g(Y) &= \int_0^\infty g\left(S_Y(y)\right) dy = \int_0^{y_1} g(1) dy + \sum_{i=1}^{N-1} \int_{y_i}^{y_{i+1}} g\left(\frac{N-i}{N}\right) dy \\ &= g(1) y_1 + \sum_{i=1}^{N-1} g\left(\frac{N-i}{N}\right)\left(y_{i+1} - y_i\right) \\ &= \sum_{i=1}^{N} \left(g\left(\frac{N-i+1}{N}\right) - g\left(\frac{N-i}{N}\right)\right) y_i \\ &= \sum_{i=1}^{N} q_i y_i = \sum_{i=1}^{N} q_i y_{(i)}. \end{aligned}$$

Connection with LO

In order to compute μ_g for a comonotone risk measure, we need an order statistic on the N possible values of the random variable Y

$$\sum_{i=1}^{N} q_i y_{(i)} = z^*_{q,y},$$

where $z^*_{q,y}$ is the optimal value of the LOP

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{N} q_i \sum_{j=1}^{N} w_{ij} y_j \\ \text{subject to} & w \in W(N), \end{array}$$

$$W(N) = \left\{ w \in \mathbb{R}_+^{N^2} \;\middle|\; \sum_{i=1}^{N} w_{ij} = 1 \;\forall j \in \{1, \ldots, N\},\; \sum_{j=1}^{N} w_{ij} = 1 \;\forall i \in \{1, \ldots, N\} \right\}.$$

The generator $\mathcal{Q}$

$$\mathcal{Q} \quad = \quad \left\{ \sum_{j=1}^{N} w_{ij} q_i \;\middle|\; w \in W(N) \right\},$$

or, alternatively,

$$\mathcal{Q} \quad = \quad \{ p \in \mathbb{R}^N \mid \exists\, \sigma \in \mathrm{S}_N \;:\; p_i = q_{\sigma(i)},\ \forall\, i \in \mathcal{N} \},$$

where S_N is the symmetric group on N elements.

Polyhedral uncertainty set

- For a risk averse problem with comonotone risk measure μ_g generated by a measure $q \in \Delta^N = \{p \in \mathbb{R}_+^N \mid e'p = 1\}$ and uncertain vector $\tilde{a}$ distributed as in Assumption A the risk averse problem is equivalent to

$$\begin{array}{rll} \text{minimize} & c'x & \\ \text{subject to} & a'x \leq b & \forall\, a \in \pi_q(\mathcal{A})\,. \end{array}$$

$$\pi_q(\mathcal{A}) \quad = \quad \operatorname{conv}\left(\left\{ \sum_{i=1}^{N} q_{\sigma(i)} a_i \;\middle|\; \sigma \in \mathrm{S}_N \right\}\right).$$

Comonotone measures and CVAR

- We will show that the set of all comonotone risk measures is finitely generated by the class of conditional value at risk measures.
- Note that as a comonotone risk measure produces generators with $q_i \leq q_{i+1}$ for all $i \in \{1, \ldots, N-1\}$, the space of possible generators q is

$$\tilde{\Delta}^N \quad = \quad \left\{ q \in \Delta^N \mid q_1 \leq \cdots \leq q_N \right\}.$$

Bijection

- There exists a bijection between $\tilde{\Delta}^N$ and the space of comonotone risk measures on random variables with a finite sample space of cardinality N.
- Clearly, any such comonotone risk measure defines a $q \in \tilde{\Delta}^N$ via $q_i = g\left(\frac{N+1-i}{N}\right) - g\left(\frac{N-i}{N}\right)$.
- Conversely, given any $q \in \tilde{\Delta}^N$, we may define a distortion function (on N values) as

$$\begin{aligned} g(0) &= 0, \\ g\left(\frac{i}{N}\right) &= \sum_{j=1}^{i} q_{N-j+1}, \qquad i = 1, \dots, N. \end{aligned}$$

- One can easily verify that such a g satisfies:

$$\begin{aligned} g(1) &= 1, \\ g(i/N) &\geq g((i-1)/N) \qquad \forall\, i \in \mathcal{N}, \\ g(i/N) - g((i-1)/N) &\leq g((i-1)/N) - g((i-2)/N) \qquad \forall\, i \in \{2, \dots, N \end{aligned}$$

- So g is a valid distortion function corresponding to a comonotone risk measure.

Key Connection

- Theorem: The restricted simplex $\tilde{\Delta}^N$ is generated by the N-member family

$$\mathcal{G}_N \quad = \quad \{q \in \tilde{\Delta}^N \mid \exists\ k \in \mathcal{N} :\ q_i = 0\ \forall\ i \leq N - k,\ q_i = 1/k\ \forall\ i > N - k\},$$

  i.e., $\mathrm{conv}\,(\mathcal{G}_N) = \tilde{\Delta}^N$. Moreover, each $\hat{q} \in \mathcal{G}_N$ corresponds to the risk measure $CVAR_{i/N}\cdot$ for some $i \in \mathcal{N}$.
- Thus $CVAR_{\alpha}\cdot$ measures are fundamental.

Summary

- Given a coherent risk measure, we can define a (convex) uncertainty set..
- If further the risk measure is comonotone (related to $CVAR$), the uncertainty set is polyhedral.