15.094J: Robust Modeling, Optimization, Computation

Lecture 20: Robust Queueing Theory - Queueing Networks Analysis

Background

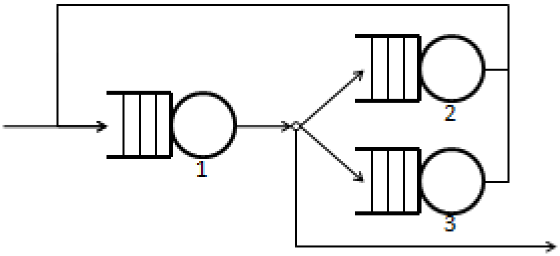
- Under assumption of Poisson arrivals and Exponential service times, performance analysis is tractable.
  - Jackson networks [1957]
  - Kelly networks [1975]

- Departing from exponentiality, *steady-state* performance analysis problems become difficult or intractable.
  - *No tractable theory for networks of $G/G/m$ queues*

    Lack of Burke's theorem

    Approximations exist: QNA (Whitt [1983])

Queueing Network Analysis



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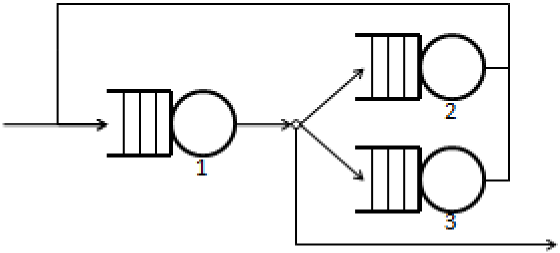
  1. Queueing Node Operator (Output of a Queue)
  2. Superposition Operator
  3. Thinning Operator



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Robust Burke Theorem

- **Main Result**

"When the Arrival Process is a polytope and the Service Process is a polytope, the Departure Process is also a polytope identical to the arrival process polytope."

- **Implications**
  - Generalization of Burke's theorem beyond Markov arrival and service processes
  - Exact analysis of a network of queues can be carried away by regarding the network as a collection of single queues

Robust Burke Theorem

Theorem

If $\{T_i\}_{i\geq 1} \in \mathcal{U}^a$, $\{X_i\}_{i\geq 1} \in \mathcal{U}^s_m$, $\alpha_a = \alpha_s = \alpha$ and $\rho = \lambda/m\mu < 1$, then the interdeparture times $\{D_i\}_{i\geq 1}$ belong to the uncertainty set

$$\mathcal{U}^d = \left\{ (D_1, D_2, \ldots, D_n) \;\middle|\; \frac{\left| \displaystyle\sum_{i=k+1}^{n} D_i - \frac{n-k}{\lambda} \right|}{(n-k)^{1/\alpha}} \leq \Gamma_a + \mathcal{O}\left(\frac{1}{(n-k)^{1/\alpha}} \right), \ \forall k \leq n-1 \right\}.$$

Robust Burke Theorem (Proof)

Proof (single-server case)

- The n^{th} interdeparture time is expressed as

$$D_n = T_n + W_n - W_{n-1} + X_n - X_{n-1} = T_n + S_n - S_{n-1}.$$

- This implies

$$\sum_{i=k+1}^{n} D_i = \sum_{i=k+1}^{n} T_i + S_n - S_k. \tag{1}$$

- Since $S_\ell = W_\ell + X_\ell$, and by the Lindley recursion

$$W_\ell = \max_{1 \le j \le \ell-1} \left(\sum_{i=j}^{\ell-1} X_i - \sum_{i=j+1}^{\ell} T_i \right)$$

the sojourn time can be expressed as

$$S_\ell = \max_{1 \le j \le \ell-1} \left(\sum_{i=j}^{\ell} X_i - \sum_{i=j+1}^{\ell} T_i \right). \tag{2}$$

Robust Burke Theorem (Proof)

- The sum of interdeparture times can be written as

$$\sum_{i=k+1}^{n} T_i - S_k \leq \sum_{i=k+1}^{n} D_i \leq \sum_{i=k+1}^{n} T_i + S_n. \tag{3}$$

- We seek to minimize the left-hand-side and maximize the right-hand side of Eq. (3) over sets $\mathcal{U}^s$ and $\mathcal{U}^a$. Given our assumption on the interarrival times uncertainty set, we can bound

$$\frac{n-k}{\lambda} - \Gamma_a (n-k)^{1/\alpha} \leq \sum_{i=k+1}^{n} T_i \leq \frac{n-k}{\lambda} + \Gamma_a (n-k)^{1/\alpha}$$

Robust Burke Theorem (Proof)

- Bounding the sum of interdepartures, and dividing by $(n-k)^{1/\alpha}$, we obtain

$$-\Gamma_a - \frac{\widehat{S}_k}{(n-k)^{1/\alpha}} \leq \frac{\displaystyle\sum_{i=k+1}^{n} D_i - \frac{n-k}{\lambda}}{(n-k)^{1/\alpha}} \leq \Gamma_a + \frac{\widehat{S}_n}{(n-k)^{1/\alpha}}, \tag{4}$$

where

$$\widehat{S}_\ell = \max_{\mathbf{X}\in\mathcal{U}^s, \mathbf{T}\in\mathcal{U}^a} \left\{ \max_{1\leq j\leq \ell-1} \left(\sum_{i=j}^{\ell} X_i - \sum_{i=j+1}^{\ell} T_i \right) \right\}.$$

- Bounding the sum of service and interarrival times given uncertainty sets $\mathcal{U}^s$ and $\mathcal{U}^a$, $widehatS_\ell$ can therefore be expressed as

$$\widehat{S}_\ell = \max_{1\leq j\leq \ell-1} \left\{ \Gamma_a (\ell-j)^{1/\alpha} + \Gamma_s (\ell-j+1)^{1/\alpha} - (\ell-j)\frac{1-\rho}{\lambda} \right\} + \frac{1}{\mu}. \tag{5}$$

Robust Burke Theorem (Proof)

- The one-dimensional concave maximization problem in Eq. (5) is of the form

$$\max_{1\le x\le \ell-1} \beta\cdot x^{1/\alpha}+\delta\,(x+1)^{1/\alpha}-\gamma\cdot x \;\le\; \max_{1\le x\le \ell-1}(\beta+\delta)(x+1)^{1/\alpha}-\gamma(x+1)+\gamma,$$
$$\le\; \frac{\alpha-1}{\alpha^{\alpha/(\alpha-1)}}\,\frac{(\beta+\delta)^{\alpha/(\alpha-1)}}{\gamma^{1/(\alpha-1)}}+\gamma, \qquad (6)$$

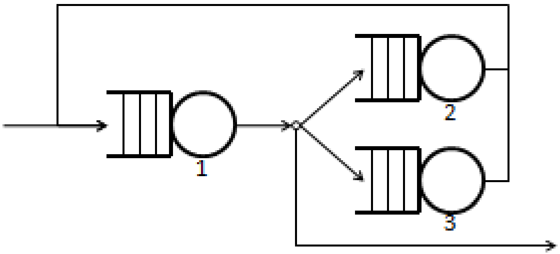
where $\beta=\Gamma_a$, $\delta=\Gamma_s$, $\gamma=(1-\rho)/\lambda>0$, given $\rho<1$. Note that bound (6) is not tight unless $\ell\ge[(\beta+\delta)/\alpha\gamma]^{\alpha/(\alpha-1)}$.

- For $k<n$ where n is large, substituting β, δ and γ by their respective values yields

$$\frac{\widehat{S}_k}{(n-k)^{1/\alpha}}\le\frac{\widehat{S}_n}{(n-k)^{1/\alpha}}\le\frac{1}{(n-k)^{1/\alpha}}\left(\frac{\alpha-1}{\alpha^{\alpha/(\alpha-1)}}\cdot\frac{\lambda^{1/(\alpha-1)}\cdot(\Gamma_a+\Gamma_s)^{\alpha/(\alpha-1)}}{(1-\rho)^{1/(\alpha-1)}}+\frac{1}{\lambda}\right)$$
$$=\mathcal{O}\left(\frac{1}{(n-k)^{1/\alpha}}\right).$$

Applying the above bounds to Eq. (4) completes the proof.

Queueing Network Analysis



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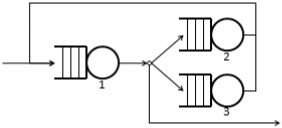
Superposition Operator



- Consider a queue j fed by m arrival processes.
- Let $\mathcal{U}_j^a$ denote the uncertainty set representing the interarrival times $\{T_i^j\}_{i\geq 1}$ from arrival process $j = 1, \ldots, J$.
- Denote the uncertainty set of the combined arrival process by $\mathcal{U}_{sup}^a$.
- Given the primitives $(\lambda_j, \Gamma_{a,j}, \alpha)$, $j = 1, \ldots, J$, we define the *superposition operator*

$$(\lambda_{sup}, \Gamma_{a,sup}, \alpha_{sup}) = Combine\Big\{ (\lambda_j, \Gamma_{a,j}, \alpha), j = 1, \ldots, J \Big\},$$

where $(\lambda_{sup}, \Gamma_{a,sup}, \alpha_{sup})$ characterize the merged arrival process $\{T_i\}_{i\geq 1}$.

Superposition Operator

Theorem (Superposition Operator)

The superposition of arrival processes characterized by the uncertainty sets

$$\mathcal{U}_j^a = \left\{ (T_1, \ldots, T_n) \;\middle|\; \frac{\left| \displaystyle\sum_{i=k+1}^{n} T_i - \frac{n-k}{\lambda_j} \right|}{(n-k)^{1/\alpha}} \leq \Gamma_{a,j}\,,\ \forall k \leq n-1 \right\},\ j = 1, \ldots, J,$$

results in a merged arrival process characterized by the uncertainty set

$$\mathcal{U}_{sup}^a = \left\{ (T_1, \ldots, T_n) \;\middle|\; \frac{\left| \displaystyle\sum_{i=k+1}^{n} T_i - \frac{n-k}{\lambda_{sup}} \right|}{(n-k)^{1/\alpha}} \leq \Gamma_{a,sup}\,,\ \forall k \leq n-1 \right\},$$

where

$$\lambda_{sup} = \sum_{j=1}^{J} \lambda_j, \quad \alpha_{sup} = \alpha, \quad \Gamma_{a,sup} = \frac{\left(\displaystyle\sum_{j=1}^{J} \left(\lambda_j \Gamma_{a,j} \right)^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha}}{\displaystyle\sum_{j=1}^{J} \lambda_j}.$$

Superposition Operator (Proof)

Proof

Consider the case of superposing two arrival processes, and then generalize the result through induction.

(a) Case where $J = 2$

- Let $\{T_i^j\}_{i\geq 1} \in \mathcal{U}_j^a$, $j = 1, 2$ with

$$\lambda_j \sum_{i=k_j+1}^{n_j} T_i^j \leq (n_j - k_j) + \lambda_j \Gamma_{a,j} (n_j - k_j)^{1/\alpha}, \ j = 1, 2.$$

- Summing over index $j = 1, 2$, we obtain

$$\sum_{i=k_1+1}^{n_1} \lambda_1 T_i^1 + \sum_{i=k_2+1}^{n_2} \lambda_2 T_i^2 \leq (n_1 - k_1 + n_2 - k_2) + \lambda_1 \Gamma_{a,1} (n_1 - k_1)^{1/\alpha} + \lambda_2 \Gamma_{a,2} (n_2 - k_2)^{1/\alpha} \quad (7)$$

Superposition Operator (Proof)

- Consider the time window T between the arrival of the k_1^{th} and the n_1^{th} jobs from arrival process 1, and assume that, within period T, the queue sees arrivals of jobs (k_2+1) up to (n_2-1) from arrival process 2

$$T = \sum_{i=k_1+1}^{n_1} T_i^1 \leq \sum_{i=k_2+1}^{n_2} T_i^2. \tag{8}$$

- During time window T, the queue receives a total of $(n_1 - k_1 + n_2 - k_2)$ jobs, with $(n_1 - k_1 + 1)$ arrivals detected from the first arrival process (including job k_1), and $(n_2 - k_2 - 1)$ arrivals from second arrival process.
- Therefore, period T can also be written in terms of the combined interarrival times $\{T_i\}_{i\geq 1}$ as

$$T = \sum_{i=k+1}^{n} T_i, \tag{9}$$

where $k = k_1 + k_2$ and $n = n_1 + n_2$.

Superposition Operator (Proof)

- Combining Eqs. (8) and (9) yields

$$(\lambda_1 + \lambda_2) \sum_{i=k+1}^{n} T_i \leq \lambda_1 \sum_{i=k_1+1}^{n_1} T_i^{1} + \lambda_2 \sum_{i=k_2+1}^{n_2} T_i^{2}$$

which by Eq. (7) can be written as

$$(\lambda_1 + \lambda_2) \sum_{i=k+1}^{n} T_i \leq (n-k) + \lambda_1 \Gamma_{a,1} (n_1 - k_1)^{1/\alpha} + \lambda_2 \Gamma_{a,2} (n_2 - k_2)^{1/\alpha}.$$

- Rearranging and dividing both sides by $(\lambda_1 + \lambda_2)$ and $(n-k)^{1/\alpha}$, we obtain

$$\frac{\displaystyle\sum_{i=k+1}^{n} T_i - \frac{n-k}{\lambda_{sup}}}{(n-k)^{1/\alpha}} \leq \Gamma_{a,sup}(n,k),$$

where $\lambda_{sup} = \lambda_1 + \lambda_2$, $\alpha_{sup} = \alpha$, and

$$\Gamma_{a,sup}(n,k) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \Gamma_{a,1} \left(\frac{n_1 - k_1}{n_1 - k_1 + n_2 - k_2} \right)^{1/\alpha} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \Gamma_{a,2} \left(\frac{n_2 - k_2}{n_1 - k_1 + n_2 - k_2} \right)^{1/\alpha}$$

Superposition Operator (Proof)

- By letting
$$x = \frac{n_1 - k_1}{n_1 - k_1 + n_2 - k_2}, \tag{10}$$
the maximum value that $\Gamma_{a,sup}(n,k)$ can achieve over the range of (n,k) can be determined by optimizing the following one-dimensional concave maximization problem over $x \in (0,1)$
$$\max_{x\in(0,1)} \left\{ \beta x^{1/\alpha} + \delta\left(1-x\right)^{1/\alpha} \right\} = \left(\beta^{\alpha/(\alpha-1)} + \delta^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha}, \tag{11}$$
where
$$\beta = \frac{\lambda_1}{\lambda_1+\lambda_2}\Gamma_{a,1}, \text{ and } \delta = \frac{\lambda_2}{\lambda_1+\lambda_2}\Gamma_{a,2}.$$
- Substituting β and δ by their respective values in Eq. (11) completes the proof for $m = 2$ with
$$\Gamma_{a,sup} = \frac{\left[(\lambda_1 \Gamma_{a,1})^{\alpha/(\alpha-1)} + (\lambda_2 \Gamma_{a,2})^{\alpha/(\alpha-1)} \right]^{(\alpha-1)/\alpha}}{\lambda_1+\lambda_2}.$$
We refer to this procedure of combining two arrival processes by the operator
$$(\lambda_{sup}, \Gamma_{a,sup}, \alpha_{sup}) = Combine\left\{ (\lambda_1, \Gamma_{a,1}, \alpha), (\lambda_2, \Gamma_{a,2}, \alpha) \right\}.$$

Superposition Operator (Proof)

(b) Case for $J > 2$

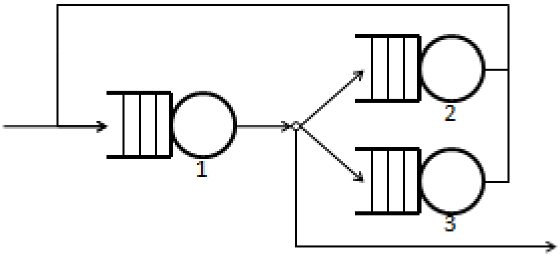
- Suppose that the arrivals to a queue come from arrival processes 1 through $(m-1)$. We assume that the combined arrival process belongs to the proposed uncertainty set, with

$$\overline{\lambda} = \sum_{j=1}^{m-1} \lambda_j \text{ and } \overline{\Gamma}_a = \frac{\left(\sum_{j=1}^{m-1} \left(\lambda_j \Gamma_{a,j}\right)^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha}}{\overline{\lambda}}.$$

- Extending the proof to m sources can be easily done by repeating the procedure shown in part **(a)** through the operator

$$(\lambda_{sup}, \Gamma_{a,sup}, \alpha_{sup}) = Combine \left\{ \left(\overline{\lambda}, \overline{\Gamma}_a, \alpha\right), (\lambda_m, \Gamma_{a,m}, \alpha) \right\}.$$

Queueing Network Analysis



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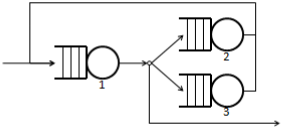


- Consider an arrival process $\{T_i\}_{i\geq 1}$ in which a fraction f of arrivals are classified as type I and the remaining are classified as type II.
- Given the primitives (λ, Γ_a) at the original process and the fraction f, we define the *thinning operator*

$$(\lambda_{split}, \Gamma_{a,split}, \alpha) = Split\Big\{ (\lambda, \Gamma_a, \alpha)\, , f \Big\}$$

where $(\lambda_{split}, \Gamma_{a,split}, \alpha)$ characterizes the thinned arrival process $\{T_i^{split}\}_{i\geq 1}$.

Thinning Operator

Theorem (Thinning Operator)

The thinned arrival process of a fraction f of arrivals belonging to $\mathcal{U}^a$ is described by the uncertainty set

$$\mathcal{U}^a_{split} = \left\{ (T_1^{split}, \ldots, T_n^{split}) \;\middle|\; \frac{\left| \displaystyle\sum_{i=k+1}^{n} T_i^{split} - \frac{n-k}{\lambda_{split}} \right|}{(n-k)^{1/\alpha}} \leq \Gamma_{a,split}, \quad \forall k \leq n-1 \right\}, \quad (12)$$

where $\lambda_{split} = \lambda \cdot f$ *and* $\Gamma_{a,split} = \Gamma_a \cdot \left(\frac{1}{f}\right)^{1/\alpha}$.

Overall Network analysis

Proof

- Consider an arrival process described by $\mathcal{U}^a$ and consider the time window between the k^{th} arrival and the n^{th} arrival. Suppose that a fraction f of these arrivals are type I arrivals, i.e., out of the total of $(n-k)$ arrivals excluding the k^{th} customer, $(n_{split} - k_{split})$ are type I arrivals, such that

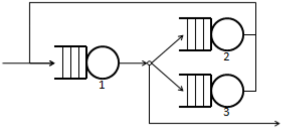
$$f = \frac{n_{split} - k_{split}}{n-k}.$$

- Let $\{T_i^{split}\}_{i \geq 1}$ denote the interarrival times in the thinned arrival process. Note that

$$\sum_{i=k_{split}+1}^{n_{split}} T_i^{split} \leq \sum_{i=k+1}^{n} T_i \leq \frac{n-k}{\lambda} + (n-k)^{1/\alpha}\,\Gamma_a.$$

with the first inequality is tight when the k^{th} and n^{th} customers are both classified as type I. The upper bound in Eq. (12) is obtained by substituting $(n-k)$ by $(n_{split} - k_{split})/f$. The lower bound is derived similarly, hence completing the proof.

Overall Network analysis



- Consider a network of J queues serving a single class of jobs. Each job enters the network through some queue j, and either leaves the network or departs towards another queue right after completion of his service.
- The primitive data in the queueing network are:
  - **(a)** External arrival processes with parameters $(\lambda_j, \Gamma_{a,j}, \alpha_{a,j})$ that arrive to each node $j = 1, \ldots, J$.
  - **(b)** Service processes with parameters $(\mu_j, \Gamma_{s,j}, \alpha_{s,j})$, and the number of servers $m_j, \; j = 1, \ldots, J$.
  - **(c)** Routing matrix $\mathbf{F} = [f_{ij}]$, $i, j = 1, \ldots, J$, where f_{ij} denotes the fraction of jobs passing through queue i routed queue j. The fraction of jobs leaving the network from queue i is $1 - \sum_j f_{ij}$.

Overall Network analysis

Theorem

The behavior of a single class queueing network is equivalent to that of a collection of independent queues, with the arrival process to node j characterized by the uncertainty set

$$\mathcal{U}_j^a = \left\{ (T_1^j, \dots, T_n^j) \;\middle|\; \frac{\left| \displaystyle\sum_{i=k+1}^{n} T_i^j - \frac{n-k}{\overline{\lambda}_j} \right|}{(n-k)^{1/\alpha}} \leq \overline{\Gamma}_{a,j}, \quad \forall k \leq n-1 \right\}, \; j = 1 \dots, J,$$

where $\{\overline{\lambda}_1, \overline{\lambda}_2, \dots, \overline{\lambda}_J\}$ and $\{\overline{\Gamma}_{a,1}, \overline{\Gamma}_{a,2}, \dots, \overline{\Gamma}_{a,j}\}$ *satisfy the set of equations for all* $j = 1, \dots, J$

$$\overline{\lambda}_j = \lambda_j + \sum_{i=1}^{J} \left(\overline{\lambda}_i f_{ij} \right), \tag{13}$$

$$\overline{\Gamma}_{a,j} = \frac{\left[\left(\lambda_j \cdot \Gamma_{a,j}\right)^{\alpha/(\alpha-1)} + \displaystyle\sum_{i=1}^{J} \left(\overline{\lambda}_i \cdot \overline{\Gamma}_{a,i} \right)^{\alpha/(\alpha-1)} \cdot f_{ij} \right]^{(\alpha-1)/\alpha}}{\overline{\lambda}_j}. \tag{14}$$

Overall Network analysis (Proof)

Proof

- Consider a queue j receiving jobs from
  (a) external arrivals described by parameters $(\lambda_j, \Gamma_{a,j}, \alpha)$, and
  (b) internal arrivals routed from queues $i, i = 1, \dots, J$ resulting from splitting the effective departure process from queue i by f_{ij}. By the Robust Burke theorem, the effective departure process from queue i has the same form as the effective arrival process to queue i described by the parameters $\left(\overline{\lambda}_i, \overline{\Gamma}_{a,i}, \alpha\right)$.
- The effective arrival process to queue j can therefore be represented as

$$\left(\overline{\lambda}_j, \overline{\Gamma}_{a,j}, \alpha\right) = \mathit{Combine}\left\{ (\lambda_j, \Gamma_{a,j}, \alpha), \left(\mathit{Split}\left\{ \left(\overline{\lambda}_i, \overline{\Gamma}_{a,i}, \alpha\right), f_{ij} \right\}\right), i = 1, \dots, J \right\} \quad (15)$$

- By the Splitting Operator, we substitute the split processes by their resulting parameters and obtain the superposition of $J + 1$ arrival processes

$$\left(\overline{\lambda}_j, \overline{\Gamma}_{a,j}, \alpha\right) = \mathit{Combine}\left\{ (\lambda_j, \Gamma_{a,j}, \alpha), \left(f_{ij}\overline{\lambda}_i, \overline{\Gamma}_{a,i} \left(\frac{1}{f_{ij}}\right)^{1/\alpha}, \alpha \right), i = 1, \dots, J \right\} \quad (16)$$

- Applying the Combine Operator yields Eqs. (13) and (14).

Overall Network analysis: Solving a Linear System

- Note that finding the overall network parameters $(\overline{\lambda}, \overline{\Gamma})$ amounts to solving a set of linear equations
- This could be achieved by defining

$$x_j = \left(\overline{\lambda}_j \overline{\Gamma}_{a,j}\right)^{\alpha/(\alpha-1)}$$

and rewriting the system as

$$\begin{cases} \overline{\lambda}_j = \lambda_j + \sum_{i=1}^{J} \left(\overline{\lambda}_i P_{ij}\right) & \forall j \\ x_j = \left(\lambda_j \Gamma_{a,j}\right)^{\alpha/(\alpha-1)} + \sum_{i=1}^{J} P_{ij} x_i & \forall j \end{cases}$$

Computational Results

Objectives

- Compare the performance of RQNA to the Queueing Network Analyzer (QNA) proposed by Whitt [1983] and simulations
- Investigate the relative performance of RQNA with respect to
  - system's network size and degree of feedback
  - maximum traffic intensity
  - diversity of external arrival distributions

Primitive Data

- Consider instances of stochastic queueing networks
- External arrivals $(\lambda_j, \sigma_{a,j}, \alpha_{a,j})$ and $c_{a,j}^2 = \lambda_j^2 \sigma_{a,j}^2$
- Service processes $(\mu_j, \sigma_{s,j}, \alpha_{s,j}, m_j)$ and $c_{s,j}^2 = \mu_j^2 \sigma_{s,j}^2$
- Routing matrix $\mathbf{P} = [P_{ij}]$

Precomputing Robust Variability Parameters

Similarly to QNA, we use simulation to construct the parameters (Γ_a, Γ_s)

- Consider a single queue with m servers characterized by $(\rho, \sigma_a, \sigma_s, \alpha_a, \alpha_s)$ and model

$$\Gamma_a = \sigma_a \text{ and } \Gamma_s = f(\rho, \sigma_a, \sigma_s, \alpha_a, \alpha_s).$$

- Motivated by Kingman's bound, we consider the functional form $f(\cdot)$

$$f\left(\rho, \sigma_s, \sigma_a, \alpha_a, \alpha_s\right) = \left(\theta_0 + \theta_1 \cdot \sigma_s^2/m + \theta_2 \cdot \sigma_a^2 \rho^2 m\right)^{(\alpha-1)/\alpha} - \sigma_a m^{(\alpha-1)/\alpha}$$

- For each service distribution, we run simulation over multiple instances of a single queue while varying parameters $(\rho, \sigma_a, \sigma_s, \alpha_a, \alpha_s)$ for different arrival distributions to compute corresponding coefficients $(\theta_0, \theta_1, \theta_2)$.

The RQNA Algorithm

ALGORITHM 1. **Robust Queueing Network Analyzer**

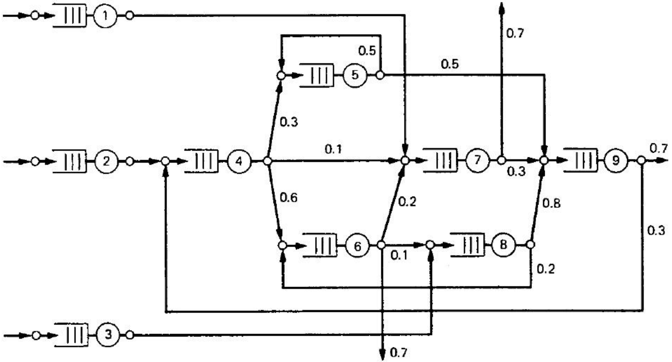
Input: External arrivals $(\lambda_j, \sigma_{a,j}, \alpha_{a,j})$, service parameters $(\mu_j, \sigma_{s,j}, \alpha_{s,j})$, and routing matrix $\mathbf{P} = [P_{ij}]$

Output: Waiting times $\widehat{W}$ at each node j, $j = 1, \ldots, J$.

1. For each external arrival process i in the network, set $\Gamma_{a,i} = \sigma_{a,i}$.
2. For each queue j in the network with parameters $(\mu_j, \sigma_{s,j}, \alpha_{s,j})$, compute
   - **(a)** the effective parameters $\left(\overline{\lambda}_j, \overline{\Gamma}_{a,j}, \overline{\alpha}_{a,j}\right)$ and set $\rho_j = \overline{\lambda}_j / \mu_j$,
   - **(b)** the variability parameter $\Gamma_{s,j} = f\left(\rho_j, \overline{\Gamma}_{a,j}, \sigma_{s,j}, \overline{\alpha}_{a,j}, \alpha_{s,j}\right)$,
   - **(c)** the waiting time $\widehat{W}$ at node j.

Performance of RQNA Compared to QNA and Simulation

- Kuehn Network with nine single-server queues
- Simulations run for normal and Pareto distributed service times



Performance of RQNA Compared to QNA and Simulation

- Kuehn Network with nine single-server queues
- Simulations run for normal and Pareto distributed service times

Case	Pareto Distribution		Normal Distribution	
$(c_{a,j}^2, c_{s,j}^2)$	QNA	RQNA	QNA	RQNA
$(0.25, 0)$	22.78	3.291	15.28	1.389
$(0.25, 1)$	18.48	-3.478	12.08	3.869
$(0.25, 4)$	20.13	-3.052	11.57	-3.882
$(1, 0)$	19.01	1.056	12.68	-3.797
$(1, 1)$	14.06	1.799	5.84	-2.555
$(1, 4)$	10.15	2.893	-10.45	-0.681
$(4, 0)$	21.82	-1.934	10.95	1.290
$(4, 1)$	23.71	-2.139	14.18	-3.508
$(4, 4)$	17.51	-2.974	11.55	1.671

Table : Single-Server Network: Sojourn time percent errors relative to simulation.

Performance of RQNA as a Function of Network Parameters

- Randomly generated networks of queues.
- Queues in the network are randomly assigned 3, 6, or 10 servers independently of each other.

Performance with respect to networks size and degree of feedback

% Loops/Nodes	n=10	n=15	n=20	n=25	n=30
0%	3.594	3.546	3.756	3.432	3.846
20%	3.696	4.014	4.02	4.392	4.452
35%	4.32	4.776	4.956	5.034	4.878
50%	4.95	4.806	5.358	5.67	6.192
70%	5.016	5.556	5.934	5.958	6.03

Table : Multi-Server Networks: Percent error versus network size and degree of feedback

Performance of RQNA as a Function of Network Parameters

- Randomly generated networks of queues.
- Queues in the network are randomly assigned 3, 6, or 10 servers independently of each other.

Performance with respect to traffic intensity and arrival distributions

# Arr. Dist.	$\rho = 0.95$	$\rho = 0.9$	$\rho = 0.8$	$\rho = 0.65$	$\rho = 0.5$
1	4.05	4.092	3.618	3.678	3.228
2	5.082	7.104	6.42	6.108	3.714
3	5.916	6.318	6.9	7.344	5.676
4	7.672	8.644	7.284	6.852	5.37

Table : Multi-Server Networks: Percent error versus traffic intensity and arrival distributions.

Summary of Computational Results

- RQNA produces results that are often significantly closer to simulated values compared to QNA.
- RQNA is somewhat sensitive to the degree of diversity of external arrival disributions
- RQNA is to a large extent insensitive to the
  - number of servers per queue
  - heavy-tailed nature of services
  - network size
  - traffic intensity

Summary and Conclusions

- Explored an alternative approach to model single-class queues by modeling primitive data through uncertainty sets
- Robust approach yields closed-form solutions for the waiting times in multi-server queues for heavy-tailed arrival and service processes operating under both transient and steady-state domains
- Analysis extends to arbitrary networks of queues through the key principle: (a) the departure from a queue, (b) the superposition, and (c) the thinning of arrival processes have the same uncertainty set representation as the original arrival processes

Summary and Conclusions

- Modeling queues via Uncertainty Sets
  - Capture heavy tails
  - Model multi-servers
- We obtain the following benefits
  - Tractability: Closed form expressions and tractable optimization problems.
  - Generalizability: Multi server analysis, Robust Burke theorem, Transient analysis, etc.
  - Accuracy: Computational results - errors within 8%.