Course notes for EE394V Restructured Electricity Markets: Locational Marginal Pricing

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8

Offer-based Economic dispatch

(i) Overview,
(ii) Surplus,
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(iv) Need for centralized coordination,
(v) Optimization formulation,
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(xii) Generalizations:

- ancillary services (reserves and regulation),
- non-linear system constraints, and
- representation of constraints.

(xiii) Homework exercises.

8.1 Overview

- We will now begin to synthesize the background material in the context of offer-based economic dispatch:

  (i) combine optimization, economic dispatch, and markets,

  (ii) (in Chapter 9) include transmission constraints.

- Offer-based economic dispatch will involve:
  - submission of offer functions by generators (or representatives of generators),
  - specification of demand or demand willingness-to-pay, and
  - the **Independent System Operator** (ISO) using the offer functions and demand information to choose the dispatch of the generators to meet the demand, and set prices.

Overview, continued

- Offer-based economic dispatch is a type of **auction**:
  - set of rules, or **mechanism**, that:
    - takes offers and bids, and
    - calculates quantities sold and prices,
- Auctions have various forms and properties in various contexts:
  - by design, the resulting dispatch and prices from offer-based economic dispatch are intended to be consistent with what would have occurred in the equilibrium of idealized bilateral trading.
- We will discuss the criterion for choosing the dispatch, which will involve maximizing the (revealed) surplus over the feasible production decisions of the generators and (in the case of flexible demand) over the possible levels of demand.

Overview, continued

- Is the ISO a central planner?
  - Yes, for short-term operations.
  - But the ISO applies a well-defined algorithm:
    - takes offers and demand as input, and
    - provides dispatch and prices as output,
- Some initial proposals for restructured markets involved an even more limited role for ISOs:
  - However, as we will discuss in Section 8.4, the need to centrally coordinate the matching of supply to demand in real-time necessitates that ISO performs at least some central operational planning and has some operational authority.

8.2 Surplus

8.2.1 Definition

- What are we trying to achieve with electricity market design?
- As discussed in Chapter 7, one public policy goal is to maximize:

  the *benefits* of electricity consumption, minus
  the *costs* of electricity production,

- We formalize this in:

Definition 8.1 The **surplus** or **welfare** is the value or benefits of consumption minus the costs of production over a particular time horizon.
□

- This is analogous to the definition we used in the apartment example in Chapter 6 and re-states the definition in Chapter 7.
- We assume that benefits and costs can usefully be compared using monetary units.
- In the context of electric power, surplus is the value of the benefits of electricity consumption minus the costs of electricity production, both measured over a particular time horizon.

8.2.2 Discussion

- In our definition, surplus is denominated in monetary units (or monetary units for the duration of the time horizon, or monetary units per unit time) and depends on:
  - the amount of demand power consumed by the load, and
  - the production of the generators.
- In the context of short-term operations, where the time horizon might be an hour or a day, we will think of the costs as being the operating costs associated with fuel and variable maintenance.
- In shorter-term contexts, we sometimes consider the rate of change of surplus with respect to time, in which case we are actually considering the surplus per unit time.
- We typically use the term "surplus" to refer to interchangeably to either surplus or surplus per unit time.
- In some cases we specify the demand power as a fixed desired value, say $\overline{D}$, to be met by supply (if possible):
  - that is, the demand is inelastic.

8.2.3 *Inelastic demand*

- In the case of inelastic demand, the benefits of demand are not explicitly revealed by response to price, but are *implicit*:
  - as in Section 7.8, the derivative of benefits with respect to demand, the **willingness-to-pay**, is implicitly assumed to be "positive infinity" (or to be equal to a very large value w) for demand D in the range from zero to a specified, desired level of demand, $\overline{D}$.
- The lack of an explicit revelation of benefits poses great difficulties!
  - For example, discussions of "reliability" often make the implicit assumption that the derivative of benefits is extremely large.
  - But we may fail to charge for consumption on this basis.
  - This can result in a serious discrepancy between returns on investment for generation and the remuneration from consumers (recall role of demand setting high price during curtailment in description of idealized market in Section 7.8).
  - This is the core of the concerns about capacity adequacy in ERCOT today.
- In some cases, we will posit an explicit form for the benefits of demand.

8.3 Feasible production set

- Also implicit in the definition of surplus is the assumption that production is chosen from a **feasible production set**.
- Constraints that define the feasible production set include:
  - demand-supply power balance,
  - generator capacity constraints, and
  - transmission constraints (treat in Chapter 9).
- As discussed in the context of economic dispatch, we will make demand-supply power balance explicit for the energy produced over each time scale that we consider by requiring that average supply equal average demand over each dispatch interval having duration T:
  - If T is one hour, then we will require that the average supply in the hour is equal to the average demand in the hour.
  - The decision variables represent averages over the dispatch interval.
  - In fact, as discussed in the context of economic dispatch, demand-supply balance must be maintained continuously.
  - The need to match supply and demand continuously is met in the short term by "ancillary services."

Feasible production set, continued

- The demand-supply power balance and transmission constraints are examples of **system constraints**.
- The generator capacity constraints are examples of **generator constraints**.
- Unlike the apartment example where each landlord has a single indivisible apartment to rent, each generator can produce and sell over a continuous range:
  - each generator can sell anything in the range specified by its capacity constraints.

8.4 The need for centralized coordination

8.4.1 Apartment example

- In the apartment example, there was no centralized coordination of leases:
  - individual landlords and renters had enough time between successive months to negotiate price in “bilateral” month-to-month rental agreements,
  - it was assumed implicitly that renters could be evicted when an agreement expired; that is, bilateral contracts are enforced by landlords,
  - either an apartment is rented for a month or it is not rented, and
  - the demand and supply functions for apartments were assumed to be fixed (or very slowly varying).
- A single market clearing price for all apartments arose as a natural outcome of self-interested behavior by landlords and renters:
  - prices might in practice adjust over several months towards the equilibrium.

8.4.2 Characteristics of electricity

- Demand of individual consumers varies continuously and (currently) is mostly price inelastic, in part because of a historical lack of interval metering:
  - Stoft calls the lack of metering and of real-time billing the "first demand-side flaw" (Section 1-1.5 of *Power System Economics*.)
  - residential interval meters have been installed throughout ERCOT and are in place for all large customers.
- The transmission system links all supply and demand collectively.
- Total supply must be controlled to match total demand continuously (or widespread blackouts will result).
- Bilateral contracts in electricity cannot be enforced in real-time since individual customers cannot (currently) be cut off if the demand exceeds their contractual quantity (or if the demand exceeds a contractual maximum or if the customer violates some other contractual condition):
  - Stoft calls the lack of real-time control of power flow to specific customers the "second demand-side flaw."

8.4.3 The role of the system operator

- Because of the characteristics of electricity, we cannot completely avoid central coordination in electricity markets and only rely on bilateral contracts between generators and demand (or between portfolios of generators and aggregated demand).
- Because of the lack of real-time control, a system operator must step in to be the “default supplier” in real time to match supply and demand in order to avoid widespread blackouts:
  - there is no analog of widespread blackouts in the apartment renting example (or in other commodity markets).
- The system operator also must arrange for curtailment of demand and set a price when supply and demand do not intersect:
  - there is no analog of the active need to maintain supply demand balance in the apartment renting example (or in other commodity markets),
  - total apartment supply equals total demand, since landlords enforce each individual bilateral contract,
  - but this is not the case in electricity markets.

The role of the system operator, continued

- To summarize, the system operator is necessary in electricity markets for:
  - matching supply to demand under normal conditions, and
  - curtailing demand to match supply under extreme conditions and setting price.
- To carry out this role, the system operator should be independent of the market participants:
  - the independent system operator (ISO).

8.4.4 Other roles of the system operator

- Demand changes rapidly and varies continuously:
  - hard for individual generators and demand to rapidly adjust prices and establish equilibrium through bilateral contracting when demand changes rapidly, so
  - system operator can facilitate efficient use of generation by explicitly seeking the market clearing price based on offer and bid functions.
- In the ERCOT zonal market (until December 2010):
  - short-term adjustment of supply to demand through ancillary services (AS) procured in day-ahead ancillary services market run by ISO,
  - real-time "balancing" market ($T = 15$ minute) run by ISO, but
  - longer-term decisions taken through bilateral contracting.
- In the ERCOT nodal market:
  - short-term adjustment of supply to demand through ancillary services,
  - real-time market ($T = 5$ minute dispatch intervals) run by ISO,
  - day-ahead market ($T = 1$ hour dispatch intervals) run by ISO including **unit commitment** decisions (Chapter 10) and AS, but
  - even longer-term decisions taken through bilateral contracting.

Other roles of the system operator, continued

- When transmission constraints bind, it is especially difficult for decentralized decision making through bilateral contracts to achieve efficient generation dispatch:
  - role of system operator is particularly important in this case.
- In the ERCOT zonal market:
  - inter-zonal transmission constraints were managed by ERCOT as another function of balancing market that is in addition to maintaining supply-demand balance,
  - intra-zonal transmission constraints were managed by ERCOT **out-of-market**.
- In the ERCOT nodal market:
  - inter-zonal and most intra-zonal transmission constraints are managed by ERCOT in day-ahead and real-time markets,
  - some constraints managed by ERCOT through out-of-market **reliability unit commitment**.

8.5 Optimization formulation

- As we have discussed in Section 8.2, maximizing surplus:
  - in the context of electricity,
  - over the short-term (focusing on operating costs),
  - under the assumption that unit commitment decisions are fixed,
- is the process of **economic dispatch**.
- **Offer-based economic dispatch** is the process by which the ISO:
  - solicits offer functions from generators, as introduced in Section 5.3.4,
  - forecasts demand, or solicits a specification of demand or specification of bids from the representatives of demand, and
  - finds the market clearing prices and quantities, with the goal of maximizing surplus.
- In the next sections, we will describe the offers, the demand, and the formulation of the optimization problem to maximize the surplus.

8.6 Generator offer functions

- Recall from Section 5.3.4 that if the price for energy is specified, and cannot be influenced by a generator, we argued that the generator will maximize its operating profits by specifying its offer function equal to its marginal cost function.
- That is, under suitable assumptions, the offer function will be equal to $\frac{df_k}{dP_k} = \nabla f_k$, where f_k is the generator cost function:
  - in practice, market rules typically restrict the form of the function to being piecewise linear or piecewise constant, so the offer function may only approximate the marginal cost function,
  - since the offer is assumed to reflect a convex cost function, market rules require the offer function to be non-decreasing.
- For now, we will assume that offer functions are specified equal to marginal costs.
- We will re-visit this assumption in Section 8.11.

Generation offer functions, continued

- For now we will also assume that $f_k(0) = f_k(0^+) = 0$, so that we can re-construct f_k from ∇f_k according to:

$$\forall P_k \in \left[0, \overline{P}_k\right], f_k(P_k) = \int_{y=0}^{y=P_k} \nabla f_k(y)\, dy.$$

- In more general cases, where $f_k(0^+) \neq 0$ we would need to add these no-load operating costs $f_k(0^+)$ to evaluate $f_k(P_k)$ for $P_k > 0$.
- Recall from Section 5.3.1 that the optimality conditions for economic dispatch involve only $\nabla f_k, \forall k = 1, \ldots, n_P$, and do not involve f_k, so that the ISO does not have to evaluate f_k to solve the optimality conditions for economic dispatch.
  - In the context of unit commitment and "make-whole" payments, the ISO will have to evaluate $f_k, \forall k = 1, \ldots, n_P$.

8.7 Demand specification

- If the benefit of consumption is implicit, we will specify demand as a quantity such as $\overline{D}$.
- We will also discuss the case where demand cannot be met.

8.8 Demand bid functions

- When demand bids a function representing its willingness-to-pay, we will interpret this function as specifying the derivative of its benefit function with respect to the power level.

8.9 Dispatch calculation by independent system operator

8.9.1 Formulation

- Problem (5.5) defined the economic dispatch problem.
- The **offer-based economic dispatch** problem is the same, and we repeat it here:

$$\min_{P \in \mathbb{R}^{n_P}} \{f(P) | AP = b, \underline{P} \leq P \leq \overline{P}\} = \min_{\forall k, P_k \in \mathbb{S}_k} \{f(P) | AP = b\}.$$

- In Section 5.3.1, we developed optimality conditions for economic dispatch of generators with convex costs and a specified demand.

8.9.2 First-order necessary conditions

- The first-order necessary conditions are:

$$
\begin{array}{rcl}
\exists \lambda^\star \in \mathbb{R}, \exists \underline{\mu}^\star, \overline{\mu}^\star \in \mathbb{R}^{n_P} \text{ such that: } \nabla f(P^\star) - \mathbf{1}\lambda^\star - \underline{\mu}^\star + \overline{\mu}^\star & = & \mathbf{0}; \\
\underline{M}^\star (\underline{P} - P^\star) & = & \mathbf{0}; \\
\overline{M}^\star (P^\star - \overline{P}) & = & \mathbf{0}; \\
-\mathbf{1}^\dagger P^\star & = & \left[-\overline{D} \right]; \\
P^\star & \geq & \underline{P}; \\
P^\star & \leq & \overline{P}; \\
\underline{\mu}^\star & \geq & \mathbf{0}; \text{ and} \\
\overline{\mu}^\star & \geq & \mathbf{0},
\end{array}
$$

- where $\underline{M}^\star = \mathrm{diag}\{\underline{\mu}^\star\} \in \mathbb{R}^{n_P \times n_P}$ and $\overline{M}^\star = \mathrm{diag}\{\overline{\mu}^\star\} \in \mathbb{R}^{n_P \times n_P}$ are diagonal matrices with entries specified by the entries of $\underline{\mu}^\star$ and $\overline{\mu}^\star$, respectively, which correspond to the constraints $P \geq \underline{P}$ and $P \leq \overline{P}$.
- These first-order necessary conditions involve the marginal costs ∇f_k, which we have assumed are given by the offer functions.

8.9.3 Representation of demand bids

- Optimality conditions including demand bids are similar.
- To represent bid demand, we define:
  - an additional entry, say D, in the decision vector to represent the demand, so that the decision vector becomes $x = \begin{bmatrix} D \\ P \end{bmatrix} \in \mathbb{R}^{1+n_P}$,
  - specify a feasible operating set for demand of the form $\mathbb{S}_0 = \left[0, \overline{D}\right]$, and
  - include an additional term, f_0, in the objective that represents *minus* the benefits of consumption.
- We modify the objective (5.3) to:

$$\forall x \in \mathbb{R}^{n_P}, f(x) = f_0(D) + \sum_{k=1}^{n_P} f_k(P_k).$$

- Recall that the power balance constraints (5.4) are:

$$D = \sum_{k=1}^{n_P} P_k.$$

- We can "dispatch" the demand similarly to the case of generators.

8.10 Pricing rule

(i) Lagrange multiplier on power balance constraint,
(ii) Example,
(iii) The case of no feasible solution,
(iv) Re-interpretation of the case where not all specified demand is met.

8.10.1 Lagrange multiplier on power balance constraint

- As discussed in Chapter 7, by Theorem 4.14 the Lagrange multiplier $\lambda^\star$ on the supply-demand balance constraint is the sensitivity of the objective to changes in demand:
  - as mentioned in Section 6.3, this sensitivity is sometimes called the marginal surplus and is the market clearing price.
- Our pricing rule will be to pay (generators) or charge (demands) for all energy uniformly at a price equal $\lambda^\star$:
  - That is, energy is priced at the marginal surplus.
- Generator k is paid $\lambda^\star \times P_k$ for generating P_k.
- If a generator is not at its minimum or maximum production then the first-order necessary conditions of the economic dispatch problem say that generator k's marginal cost will be equal to $\lambda^\star$:
  - such a generator is called **marginal**,
  - the pricing rule is also called **marginal cost pricing**,
  - the marginal generator is sometimes said to "set" the price, although all dispatched offers in fact contribute to determining which generator is marginal and therefore all contribute to "setting" the price.

Lagrange multiplier on power balance constraint, continued

- We will see that if a demand bid is not completely supplied then its (possibly implicit) willingness-to-pay will be equal to $\lambda^\star$.
  - Paralleling the phrasing for generators, we might say that the demand "sets" the price at its willingness-to-pay.
- We will also see that we can generalize the pricing rule to the case where there is supply and demand for multiple commodities.
  - The basic principle will be to price each commodity based on the Lagrange multiplier on the corresponding system constraint.
  - The prices do not depend (directly) on Lagrange multipliers on generator constraints.

8.10.2 Example

- Consider the previous example with $n_P = 3$, $\overline{D} = 3000$ MW, $\lambda^\star = \$50$/MWh, and marginal costs:

$$\begin{aligned}
\forall P_1 \in [0,1500], \nabla f_1(P_1) &= \$40/\text{MWh},\\
\forall P_2 \in [0,1000], \nabla f_2(P_2) &= \$20/\text{MWh},\\
\forall P_3 \in [0,1500], \nabla f_3(P_3) &= \$50/\text{MWh}.
\end{aligned}$$

- Suppose that each generator sets its offer function equal to its marginal cost function.
- All energy is transacted at a price of $\lambda^\star = \$50$/MWh.
- Generator 3 is paid its offer price, which equals its marginal cost.
  - We might say that the marginal cost of generator 3 "sets" the price of $\lambda^\star = \$50$/MWh.

Example, continued

- Generators 1 and 2 are paid more than their offer price; that is, they are paid more than their marginal costs.
  - The marginal costs of generators 1 and 2 do not "set" the price in that their marginal cost differs from the price of $\lambda^\star = \$50/\text{MWh}$ by the Lagrange multipliers on the respective generator constraints.
  - Of course, the marginal cost and capacities of generators 1 and 2 help to determine the economic dispatch that sets the price!
- Demand pays at the price of $\lambda^\star = \$50/\text{MWh}$.

8.10.3 *The case of not meeting all demand*

- If there is enough supply to meet the specified demand then there will be a feasible solution.
- However, if there are no demand bids or insufficient demand bids and supply is insufficient to meet the specified, desired demand then there is no feasible solution:
  - supply does not intersect the desired demand!
- In this case, from a practical perspective, the system operator must curtail some of the desired demand (or violate other constraints) in the economic dispatch problem.
- What should the price be?

The case of not meeting all demand, continued

- Curtailment implies that not all of the specified, desired demand $\overline{D}$ can be served.
- Some demand will be involuntarily limited:
  - we can notionally imagine a marginal *dis-benefit* of involuntary curtailment, the **value of lost load** or **VOLL**, and
  - we re-interpret the specified demand to be a demand that is bid with a willingness-to-pay equal to some value w, which we interpret to be the value of lost load.
  - as mentioned earlier, we define a variable D to represent the demand actually served.

The case of not meeting all demand, continued

- The benefit function is given by:

$$\forall D \in \mathbb{S}_0 = \left[0, \overline{D}\right], \text{benefit}(D) = w \times D,$$

- We require that $0 \leq D \leq \overline{D}$, with corresponding Lagrange multipliers $\underline{\mu}_0^\star$ and $\overline{\mu}_0^\star$.

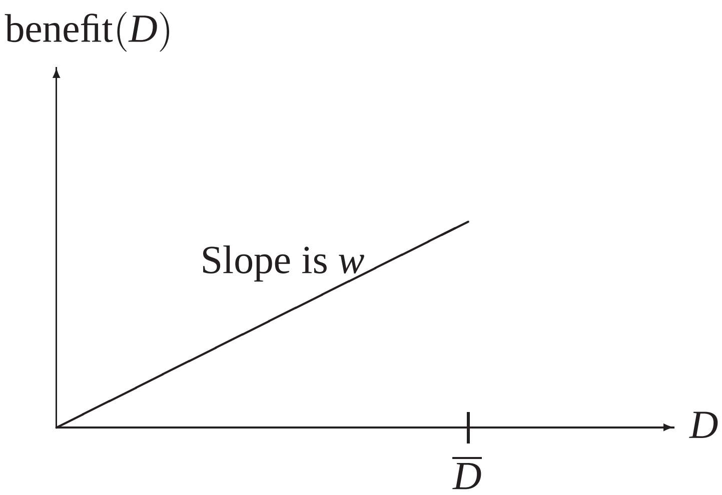

Fig. 8.1. Benefit function for consumption.

The case of not meeting all demand, continued

- The feasible set for consumption is:

$$\mathbb{S}_0 = \{D \in \mathbb{R} | 0 \leq D \leq \overline{D}\}.$$

- We modify the economic dispatch problem to include:
  - an additional term $f_0 = (-\text{benefit})$ in the objective,
  - power balance constraints of the form $D = \sum_{k=1}^{n_P} P_k$.
- The first line in the first-order necessary conditions corresponding to D is then (where $D^\star$ is optimal value):

$$\begin{aligned} 0 = \nabla f_0(D^\star) + \lambda^\star - \underline{\mu}_0^\star + \overline{\mu}_0^\star &= \frac{d(-\text{benefit})}{dD}(D^\star) + \lambda^\star - \underline{\mu}_0^\star + \overline{\mu}_0^\star, \\ &= -w + \lambda^\star - \underline{\mu}_0^\star + \overline{\mu}_0^\star. \end{aligned}$$

- When the desired demand $\overline{D}$ is not completely met:

  $0 < D^\star < \overline{D}$, so by complementary slackness, $\underline{\mu}_0^\star = \overline{\mu}_0^\star = 0$,
  substituting into the first line of the FONC, $\lambda^\star = w$.
  the willingness-to-pay of w "sets" the price in this case,
  generators should be paid and demand should pay at the price w.

8.11 Incentives

(i) Price-taking assumption,
(ii) Profit maximization,
(iii) Offer versus marginal cost of production,
(iv) Infra-marginal revenues,
(v) Investment decisions.

8.11.1 Price-taking assumption

- We assume (for now) that each generator and each consumer of electricity cannot individually influence the price:
  - we say that each market participant is a **price taker in the economics sense**,
  - ("price taker" is also used in the context of electricity markets to mean a market participant who, for example, is at maximum capacity and therefore does not directly "set" the price; however, such a market participant can potentially influence the price and so is not necessarily a price taker in the economics sense.)
- More specifically, we will assume that the Lagrange multipliers $\lambda^\star$, $\underline{\mu}^\star$, and $\overline{\mu}^\star$ that satisfy the optimality conditions for offer-based economic dispatch do not change (significantly) if any particular generator offer or any particular demand bid changes.
- We will show that, under the price-taking assumption, the pricing rule:

  aligns private incentives to maximize profits, with
  the public policy goal of achieving economic dispatch; that is,
  maximizing surplus.

8.11.2 Profit maximization

- Repeating the analysis from Section 5.3.4, again consider a particular generator that has a production cost function $f_k : \mathbb{R} \to \mathbb{R}$ in a particular period of its production.
- If it produces P_k then the cost of production is $f_k(P_k)$.
- It is paid a price $\lambda^\star$ for its production P_k.
- That is, revenue is $\lambda^\star \times P_k$.
- Operating profit is $(\lambda^\star \times P_k) - f_k(P_k)$.
- What should generator k do to maximize profit, given that it cannot affect the Lagrange multipliers $\lambda^\star$, $\underline{\mu}^\star$, and $\overline{\mu}^\star$?
- Given an energy price specified by $\lambda^\star$, and assuming that the generator cannot affect $\lambda^\star$, profit maximization involves finding a value of generation $P_k^{\star\star}$ that solves the following problem:

$$\max_{P_k \in \mathbb{S}_k} \{(\lambda^\star \times P_k) - f_k(P_k)\} = \max_{P_k \in \mathbb{R}} \{(\lambda^\star \times P_k) - f_k(P_k) | \underline{P}_k \leq P_k \leq \overline{P}_k\}.$$

Profit maximization, continued

- Equivalently, the generator could *minimize* the negative of the profit:

$$\min_{P_k \in \mathbb{S}_k} \{ f_k(P_k) - (\lambda^\star \times P_k) \} = \min_{P_k \in \mathbb{R}} \{ f_k(P_k) - (\lambda^\star \times P_k) | \underline{P}_k \leq P_k \leq \overline{P}_k \}.$$

- The optimality conditions for a minimizer $P_k^{\star\star}$ of this problem are:

$$\begin{array}{rcl} \exists \underline{\mu}_k^{\star\star}, \overline{\mu}_k^{\star\star} \in \mathbb{R} \text{ such that: } \nabla f_k(P_k^{\star\star}) - \lambda^\star - \underline{\mu}_k^{\star\star} + \overline{\mu}_k^{\star\star} & = & \mathbf{0}; \\ \underline{\mu}_k^{\star\star}(\underline{P}_k - P_k^{\star\star}) & = & \mathbf{0}; \\ \overline{\mu}_k^{\star\star}(P_k^{\star\star} - \overline{P}_k) & = & \mathbf{0}; \\ P_k^{\star\star} & \geq & \underline{P}_k; \\ P_k^{\star\star} & \leq & \overline{P}_k; \\ \overline{\mu}_k^{\star\star} & \geq & \mathbf{0}; \text{ and} \\ \underline{\mu}_k^{\star\star} & \geq & \mathbf{0}. \end{array}$$

- Generator k seeks $P_k^{\star\star}$, $\underline{\mu}_k^{\star\star}$, and $\overline{\mu}_k^{\star\star}$ satisfying these optimality conditions.
- Generator k enforces its own generator constraints by requiring that $P_k^{\star\star} \geq \underline{P}_k$ and $P_k^{\star\star} \leq \overline{P}_k$.

Profit maximization, continued

- Note that these optimality conditions for generator k are precisely those lines in the first-order necessary conditions for economic dispatch that involve generator k.
- Assuming differentiability and *strict* convexity of f_k, these optimality conditions are *uniquely* satisfied by $P_k^{\star\star} = P_k^\star$, $\underline{\mu}_k^{\star\star} = \underline{\mu}_k^\star$, and $\overline{\mu}_k^{\star\star} = \overline{\mu}_k^\star$.
- When paid at the price $\lambda^\star$ for all of its units of production, the generator making "decentralized" decisions to maximize its own profit will choose to produce at the level $P_k^\star$ that is consistent with economic dispatch.
- The price $\lambda^\star$, together with profit maximizing behavior by the generator, will yield economic dispatch:
  - the price $\lambda^\star$ is a market clearing price, since total supply equals demand.
- In the context of economic dispatch, this market clearing price is said to **strictly support** economic dispatch, meaning that there is a unique profit maximizing production level for generator k and this production level is consistent with economic dispatch.

Profit maximization, continued

- If f_k is convex but not strictly convex then there may be multiple choices that maximize profit.
  - In this case, the choice of generation is not completely decentralized since it requires specification of the value $P_k^\star$ by the ISO.
  - However, $P_k^{\star\star} = P_k^\star$ is still consistent with individual profit maximization.
  - The price is a market clearing price in that supply equals demand for some choice of generation and demand that is consistent with individual profit maximization.
  - To emphasize that the price is insufficient to determine the market clearing quantities, we say that the price does not strictly support economic dispatch.
  - (We say that the price **supports** economic dispatch to include both the strictly supporting and not strictly supporting cases.)
- Several markets allow only piece-wise constant offers:
  - prices will support but will not strictly support economic dispatch.

8.11.3 *Offer versus marginal cost of production*

- We have implicitly assumed that the offer of each generator is the *same* as the derivative of its cost of production:
  - generator is said to have made a **competitive** offer or a **price taking** (in the economics sense) offer.
- Here we will explore the conditions under which it is profit maximizing to make a competitive offer.

Offer versus marginal cost of production, continued

- Let's continue to write $x^{\star}$, $\lambda^{\star}$, $\underline{\mu}^{\star}$, and $\overline{\mu}^{\star}$ for the solution of economic dispatch based on the "true" marginal costs for each generator.
- However, suppose that generator k specifies its offer to be *different* to its true marginal cost:
  - the offer is $\nabla f_k + e$,
  - where ∇f_k is the marginal cost, but
  - where $e : \mathbb{R} \to \mathbb{R}$ is a function representing the mark-up (or mark-down, if negative) of the offer above generator k's marginal cost.
- Since offers are supposed to be derivatives of convex costs, this modified offer must be non-decreasing.
- The ISO uses the modified offer $\nabla f_k + e$ instead of ∇f_k in its economic dispatch calculations, possibly resulting in different dispatch quantities x.
- We continue to assume that the resulting Lagrange multipliers $\lambda^{\star}$, $\underline{\mu}^{\star}$, and $\overline{\mu}^{\star}$ that satisfy the optimality conditions for offer-based economic dispatch do not change due to the modified offer:
  - the conditions under which this assumption is true, or approximately true, will be discussed.

Offer versus marginal cost of production, continued

- Suppose that the ISO's solution to economic dispatch with the offer $\nabla f_k + e$ now involved generator k producing $P_k^{\star\star\star} \neq P_k^\star$.
- But by the discussion in Section 8.11.2, we know that $P_k^\star$ maximizes the profit for k, given the price $\lambda^\star$.
- So, dispatching at $P_k^{\star\star\star}$ *cannot* improve the profit compared to dispatching at $P_k^\star$, although the profit might be no worse than the profit at $P_k^\star$.
- How does generator k guarantee that it is asked by the ISO to generate at its profit maximizing level $P_k^\star$?
  - By setting $e = 0$; that is, offering at its true marginal cost.

Offer versus marginal cost of production, continued

- A similar argument applies for the mis-specification of minimum and maximum capacities for power production, but the corresponding results are somewhat weaker.
  - Suppose that the Lagrange multipliers in the ISO solution of offer-based economic dispatch are not affected.
  - Then, a generator that specifies its “offered capacities” differently to its actual capacities will not experience better profits (in expectation) compared to the case where it specified its limits correctly.
- For example, suppose that a generator “physically withholds” by specifying offered capacity that is less than its actual capacity.
  - If the result of offer-based economic dispatch is for it to operate at its offered capacity then it receives a price at or above its offer price.
  - It would have made at least as much or more profit by generating at a higher level at that price, which it could have achieved by not physically withholding.

Offer versus marginal cost of production, continued

- Conversely, suppose that a generator specifies an offered capacity that is more than its actual capacity.
  - If the result of offer-based economic dispatch is for it to operate at its offered capacity then it will be unable to generate at this level.
  - The implications depend on whether the market is a **forward market** (such as a day-ahead market) or a **real-time** market (see Chapter 11).
  - If the market is a forward market:
    - The generator will have to buy back the energy it is unable to produce from a later market.
    - The generator risks that the price will be higher in the later market.
    - It has effectively made a **virtual offer** for the difference between its offered capacity and its actual capacity,
    - If all else is equal, there will be less supply in the later market, so the buy back price will typically be higher in the later market.
  - If the market is a real-time market then a **deviation penalty** may be assessed if the deviation is large enough:
    - Possibly keyed to economic cost of procuring energy at late notice.

Offer versus marginal cost of production, continued

- We will further discuss the implications of mis-specification of capacity in the context of reserves.
- All of the previous arguments rely on the assumption that the offer of the market participant does not affect the values of the Lagrange multipliers calculated in the ISO offer-based economic dispatch problem.
- If a market participant owns multiple generators or if a single generator is large enough:
  - then Lagrange multipliers in the ISO problem (and hence prices) are affected by the offer of the market participant,
  - so offers that differ from marginal cost can improve profits compared to offering at marginal cost,
  - "economics" definition of market power,
  - discussed in market power course, http://users.ece.utexas.edu/~baldick/classes/394V_market_power/EE394V_market_power.html,
  - we will not treat this case in detail in this course.
- From now on, we will treat offers and marginal costs as synonymous.

Offer versus marginal cost of production, continued

- Note that the examples used throughout the course typically involve "large" generators relative to the size of the market:
  - easier to solve examples with a small number of generators, but
  - firms in such examples have a large amount of market power and would improve profits by not offering competitively!
- Can usually re-cast example by dividing each large generator up into many smaller generators having similar costs:
  - can then typically expect competitive or close-to-competitive behavior.

8.11.4 Infra-marginal revenues

- For simplicity, first suppose that a generator k has constant marginal costs as shown in Figure 8.2.
- Also assume that economic dispatch results in a power level for generator k that is between its minimum and maximum capacity.
- Then, by the first-order necessary conditions, $\nabla f_k(P_k^\star) = \lambda^\star$.
- Ignoring "auxiliary" costs, $f_k(0) = f_k(0^+) = 0$, so that:

$$\begin{aligned} f_k(P_k^\star) &= \int_{y=0}^{y=P_k^\star} \nabla f_k(y)\, dy = \nabla f_k(P_k^\star) \times P_k^\star, \\ &= \lambda^\star \times P_k^\star, \end{aligned}$$

- so that revenues exactly cover operating costs.

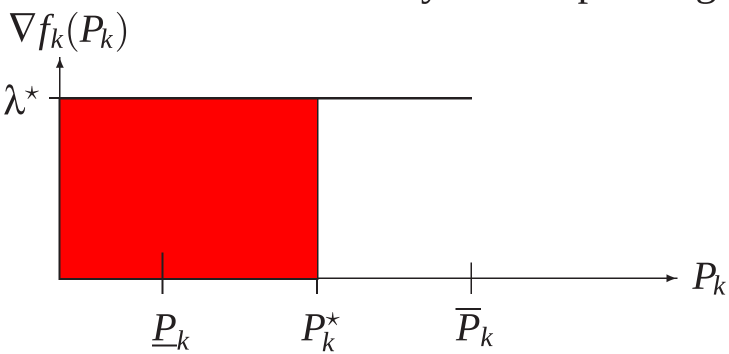

Fig. 8.2. Revenues (shaded region) exactly cover operating costs with constant marginal costs (horizontal thick line).

Infra-marginal revenues, continued

- More typically, marginal costs increase with production as in Figure 8.3, so that, by strict convexity of f_k, if $f_k(0) = f_k(0^+) = 0$ then:

$$\begin{aligned} f_k(P_k^\star) &< \nabla f_k(P_k^\star) \times P_k^\star, \\ &= \lambda^\star \times P_k^\star, \end{aligned}$$

- so that revenues *more* than cover operating costs.

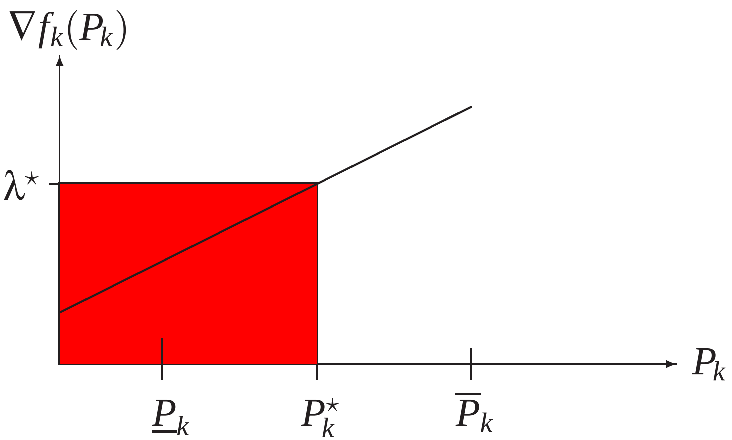

Fig. 8.3. Revenues (shaded region) more than cover operating costs with increasing marginal costs (monotonically increasing thick line).

Infra-marginal revenues, continued

- Moreover, if a generator is at maximum production then:

$$\begin{aligned} \nabla f_k(P_k^\star) &= \lambda^\star - \overline{\mu}_k^\star, \\ &\leq \lambda^\star, \end{aligned}$$

- so that revenues again more than cover operating costs.

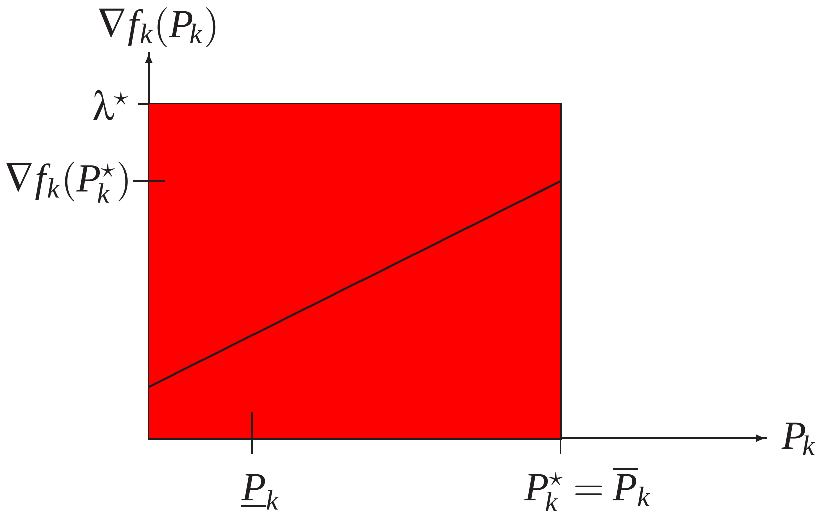

Fig. 8.4. Revenues (shaded region) more than cover operating costs when fully dispatched.

Infra-marginal revenues, continued

- On the other hand, if a generator is at minimum production then:

$$\begin{aligned}\nabla f_k(P_k^\star) &= \lambda^\star + \underline{\mu}_k^\star, \\ &\geq \lambda^\star,\end{aligned}$$

- so that revenues might not cover the operating costs.
- In the context of unit commitment, this situation suggests that the generator should be de-committed.
- We will see that if the ISO needs the generator to stay committed then it will provide a **make-whole** payment to the generator.

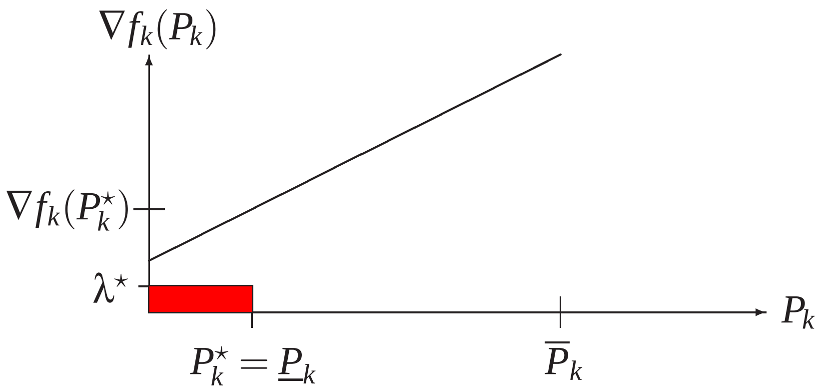

Fig. 8.5. Revenues (shaded region) may not cover operating costs when dispatched at minimum.

Infra-marginal revenues, continued

- When revenues are more than operating costs we say that there are **infra-marginal rents** or **infra-marginal revenues**.
- In practice, there may be a non-zero value of $f_k(0^+)$:
  - as with the case of a generator being operated at minimum, we will deal with non-zero $f_k(0^+)$ with a **make-whole** payment.
- Why allow infra-marginal rents?

(i) Generators have capital and other costs in addition to operating costs.

- If the market price did not cover more than their operating costs then they would all become bankrupt!

Infra-marginal revenues, continued

- Why allow infra-marginal rents?

(ii) Suppose that, in the hope of reducing payments, for example, because of a concern about market power, we changed the pricing rule so that each accepted generator was paid only what it offered:

- Continue to dispatch in order from low price to high price offers.
- Still expect similar highest accepted offer price.
- In such a **pay-as-bid** (or **pay-as-offer**) market, the previous argument about a generator maximizing its profit by offering at its marginal costs is no longer valid:
  - each accepted generator will want to forecast the highest accepted offer price and offer at that price in order to maximize its profit.
- A result in economics called the **revenue equivalence theorem** suggests that changing the pricing rule will not result in changes to the net payments to generators!
- The basic reason is that the offers will change in response to the changed pricing rule so that the payments under the pay-as-bid rule will match the payments under the uniform price rule.

Infra-marginal revenues, continued

- Unfortunately, the restrictive assumptions of the revenue equivalence theorem do not exactly hold in electricity markets.
- However, the result approximately holds and so changing the pricing rule is unlikely to significantly change the revenues.
- Moreover, under a pay-as-bid mechanism, profitability of each generator depends on each generator forecasting the price and offering at that price.
- Due to imperfections in forecasts, these predictions will be wrong and we will get poor dispatch:
  - imagine a nuclear generator who forecasts high prices, and
  - a gas plant that forecasts low prices.
- All electricity markets in North America set prices based on the Lagrange multiplier on supply-demand balance:
  - have varying approaches to pricing under **scarcity**; that is, occasions when not all desired demand can be met.

8.11.5 Investment decisions

- As discussed in Chapter 7, if generators (and generator infrastructure):

  come in small "lumps" and do not exhibit economies of scale in construction,

  can be built quickly, and

  no participant can unilaterally affect prices,

- then investors have incentives to build the "right" amount of generation capacity:
  - if there is too much capacity then prices (and anticipated prices) will typically be low, infra-marginal rents will be small, and there will be little incentive to invest in more generation, while
  - if there is too little capacity then prices (and anticipated prices) will rise until infra-marginal rents are large enough to encourage new investment.
- Investment in "peaking capacity" will only occur if demand sets the price at peak, as discussed in Section 7.8.15, or there is some other mechanism to allow peaking generation to recover more than operating costs.
- If prices are depressed (for example, by market power mitigation rules) then investment in peaking capacity will not occur spontaneously!

8.12 Generalizations

- We will generalize the basic formulation in three ways:
  - including ancillary services such as reserves,
  - including transmission constraints, and
  - including unit commitment decisions.
- We will explicitly consider reserves here and the other generalizations in later chapters.
- We will also consider how generally to set prices on commodities defined by system constraints and discuss the representation of constraints.

(i) Ancillary services,
(ii) Offer-based reserve-constrained economic dispatch,
(iii) More general formulations of economic dispatch,
(iv) Generalized offer-based economic dispatch,
(v) Spinning reserve re-visited,
(vi) Non-linear system constraints,
(vii) Representation of constraints.

8.12.1 Ancillary services

- Because supply must meet demand continuously, our supply-demand constraint on (average) power production over a period of time does not fully specify the requirements for supply-demand balance.
- Moreover, because markets cannot respond instantaneously to changes due to equipment failure, we must explicitly consider **recourse**:
  - we must prepare in advance to be ready to deal with an outage if it occurs.
- As mentioned previously, the additional services to fully satisfy supply-demand balance and satisfy other constraints are called **ancillary services**.

Ancillary services, continued

- We will first focus on **spinning reserve**, which is the capability of a generator to respond to frequency change due to supply-demand imbalance and to further respond to ISO signals to change production:
  - ERCOT uses the term **responsive reserve** to refer both to generation that can provide such reserve, and also to demand that can provide a similar response.
  - for notational simplicity we will ignore the case of demand providing reserves.
- The most critical requirement for spinning reserve is the ability to *increase* production subsequent to a failure of a generator.
  - We will focus on this issue, although being able to decrease production subsequent to the loss of a large load or sudden increase in, for example, wind generation may also be critical.
- We will consider a **co-optimized** market where energy and spinning reserve are considered together in a single market.

8.12.1.1 Variables

- We must explicitly represent the power generation and the spinning reserve of a generator.
- Since spinning reserve is an amount of generation capacity, its units are the same as the units of power.
- Slightly abusing notation, we re-interpret the decision variable associated with generator k to be a vector:

$$\begin{aligned} x_k &= \begin{bmatrix} P_k \\ S_k \end{bmatrix}, \\ &= \begin{bmatrix} \text{amount of average power production by generator } k \text{ during interval} \\ \text{amount of spinning reserve provided by generator } k \text{ during interval} \end{bmatrix}, \\ &\in \mathbb{R}^2. \end{aligned}$$

8.12.1.2 Generator constraints

- We previously considered minimum and maximum power limits.
- Ramp rate limits typically determine the maximum spinning reserve.
- Spinning reserve is also limited by the maximum power limits, since average power production plus spinning reserve is bounded by the minimum and maximum capacity.
- The feasible operating set $\mathbb{S}_k$ for generator k is therefore re-defined to be:

$$\mathbb{S}_k = \{x_k \in \mathbb{R}^2 | \underline{P}_k \leq P_k \leq \overline{P}_k, \underline{S}_k \leq S_k \leq \overline{S}_k, \underline{P}_k \leq P_k + S_k \leq \overline{P}_k\},$$

- where we write $\underline{P}_k$ and $\overline{P}_k$ for the minimum and maximum power production capacities, $\underline{S}_k$ and $\overline{S}_k$ for the lower and upper limits on spinning reserve, with:
  - power produced being required to stay within these limits, and
  - the sum of power produced and spinning reserve being required to stay within these limits (the "power plus reserve constraint"), and
  - where we might use a slightly different formulation if we were separately considering ability to decrease production subsequent to the loss of a large load or increase in wind production.

Generator constraints, continued

- The feasible operating set $\mathbb{S}_k$ for generator k is a region in $\mathbb{R}^2_+$.
- As an example, suppose that:

$$\begin{aligned}
\underline{P}_k &= 0 \text{ MW},\\
\overline{P}_k &= 100 \text{ MW},\\
\underline{S}_k &= 0 \text{ MW},\\
\overline{S}_k &= 20 \text{ MW}.
\end{aligned}$$

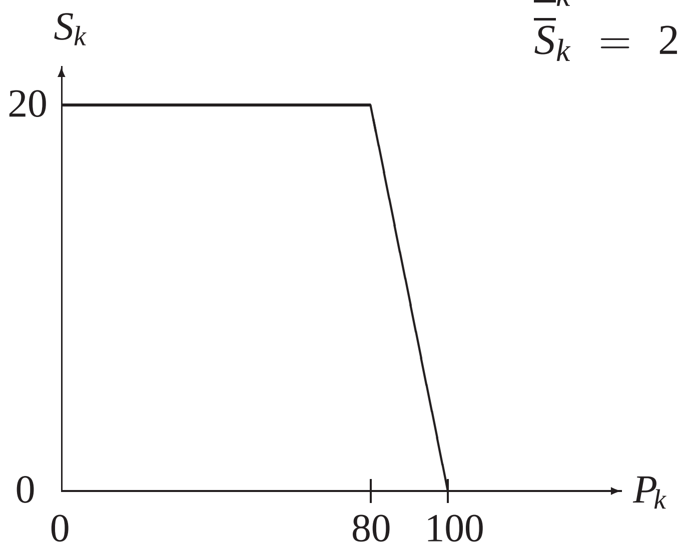

Fig. 8.6. The feasible operating set $\mathbb{S}_k$ for generator k.

Generator constraints, continued

- We can re-write the generator constraints in the form:

$$\mathbb{S}_k = \{x_k \in \mathbb{R}^2 | \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\},$$

- where $\Gamma_k \in \mathbb{R}^{r \times 2}$, $\underline{\delta}_k \in \mathbb{R}^r$, and $\overline{\delta}_k \in \mathbb{R}^r$ are appropriately chosen matrices and vectors with $r = 3$ to represent the generator constraints:

$$\begin{aligned}
\Gamma_k &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \\
\underline{\delta}_k &= \begin{bmatrix} \underline{P}_k \\ \underline{S}_k \\ \underline{P}_k \end{bmatrix}, \\
\overline{\delta}_k &= \begin{bmatrix} \overline{P}_k \\ \overline{S}_k \\ \overline{P}_k \end{bmatrix}.
\end{aligned}$$

- Other formulations of the generator constraints are possible and will again result in constraints such as $\underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k$ but possibly with $r \neq 3$.

8.12.1.3 Objective

- We now consider the cost of generation to be a function of both power and spinning reserve.

$$
\begin{aligned}
x_k &= \begin{bmatrix} P_k \\ S_k \end{bmatrix}, \\
f_k(x_k) &= f_k\left(\begin{bmatrix} P_k \\ S_k \end{bmatrix}\right), \\
\nabla f_k(x_k) &= \begin{bmatrix} \dfrac{\partial f_k}{\partial P_k}\left(\begin{bmatrix} P_k \\ S_k \end{bmatrix}\right) \\[2ex] \dfrac{\partial f_k}{\partial S_k}\left(\begin{bmatrix} P_k \\ S_k \end{bmatrix}\right) \end{bmatrix}.
\end{aligned}
$$

Objective, continued

- It is typically the case that the cost of generation is additively separable into the sum of:

  a cost f_{kP} for producing energy depending only on P_k, and
  a cost f_{kS} for providing spinning reserve depending only on S_k.

$$\forall x_k = \begin{bmatrix} P_k \\ S_k \end{bmatrix} \in \mathbb{S}_k, f_k(x_k) = f_{kP}(P_k) + f_{kS}(S_k),$$

$$\forall x_k = \begin{bmatrix} P_k \\ S_k \end{bmatrix} \in \mathbb{S}_k, \nabla f_k(x_k) = \begin{bmatrix} \frac{\partial f_{kP}}{\partial P_k}(P_k) \\ \frac{\partial f_{kS}}{\partial S_k}(S_k) \end{bmatrix}.$$

- Moreover, reserves typically impose essentially no *direct* operational cost on the generator so that $f_{kS} = 0$.
  - We will see that even if the reserves cost is zero, the payment for reserves can be non-zero if inequality constraints involving reserves are binding.

8.12.1.4 System constraints

Power balance

- As previously, we must satisfy power balance constraints:

$$\overline{D} = \sum_{k=1}^{n_P} P_k,$$

- where we have assumed a fixed demand $\overline{D}$.
- As previously, we can write the constraint in the form $Ax = b$ with:

$$\begin{aligned} x &= \begin{bmatrix} P \\ S \end{bmatrix}, \\ A &= \begin{bmatrix} -\mathbf{1}^\dagger & \mathbf{0} \end{bmatrix}, \\ b &= \begin{bmatrix} -\overline{D} \end{bmatrix}, \end{aligned}$$

- where we have re-ordered the elements of x and partitioned it into two sub-vectors:

  $P \in \mathbb{R}^{n_P}$ consists of all the real power productions, and
  $S \in \mathbb{R}^{n_P}$ consists of all the spinning reserve contributions.

Spinning reserve

- What is the purpose of spinning reserve?
- Formulation 1. To withstand an outage of the generator producing the most in the system:

$$\forall k = 1, \ldots, n_P, \sum_{j \neq k} S_j \geq P_k.$$

- Formulation 2. To withstand outages of the two generators producing the largest and second largest in the system:

$$\forall k = 1, \ldots, n_P, \forall \ell > k, \sum_{j \neq k, \ell} S_j \geq P_k + P_\ell.$$

- One drawback of these two formulations is that they imply a large number of constraints:
  - we will consider formulation 1 in detail in later development, but formulation 2 is similar.

Spinning reserve, continued

- These two formulations also have the *political* drawback of highlighting that large generators contribute to the need for spinning reserve and would, if taken literally, charge the largest generators for their reserve implications:
  - we will nevertheless return to these cases when we consider how to handle multiple system constraints,
  - we will also illustrate that the form of the constraint determines whether the cost of service is **uplifted**,
  - **uplift** means any charges other than payments for commodities.
- Kirschen and Strbac advocate for generators to pay for spinning reserve in *Power System Economics*, (section 5.4.3.1).
- In the Australian market, "frequency control ancillary services" (also known as spinning reserve) for restoring frequency following failure of a generator are paid for by the generators (although not according the pricing rules implied by the above formulations):
  - See, "Guide to Ancillary Services in the National Electricity Market," AEMO, page 11, Available from www.aemo.com.au

Spinning reserve, continued

- Formulation 3. To withstand an outage equal to some "fixed" fraction of the total demand:

$$\sum_k S_k \geq \alpha \overline{D}.$$

  - How is α determined? $(\text{maximum generator capacity})/(\overline{D})$?
  - This formulation has the drawback of not distinguishing the effect of the maximum generation on spinning reserve.
  - A variation on this is to procure spinning reserve in every hour to withstand an outage equal to some fixed fraction of the demand that occurs in the hour of peak demand.
- Formulation 4. To withstand an outage equal to a fixed requirement:

$$\sum_k S_k \geq \overline{F},$$

- where $\overline{F}$ is the "fixed" amount of required spinning reserve.
- We will first think of $\overline{F}$ as a constant, but consider the implications of it actually depending on system conditions.

Spinning reserve, continued

- Whichever formulation we choose, we can write the spinning reserve constraint in the form $Cx \leq d$.
- For example, for the "fixed" requirement formulation, $\sum_k S_k \geq \overline{F}$, formulation we have:

$$\begin{aligned} C &= \begin{bmatrix} \mathbf{0} & -\mathbf{1}^\dagger \end{bmatrix}, \\ d &= [-\overline{F}]. \end{aligned}$$

- We will first consider this formulation in detail.
- Then consider the other formulations to see the implications for uplift.

Formulation

- The reserve-constrained economic dispatch problem is:

$$\min_{\forall k=1,\ldots,n_P, x_k \in \mathbb{S}_k} \{f(x) | Ax = b, Cx \leq d\}$$
$$= \min_{x \in \mathbb{R}^{2n_P}} \{f(x) | Ax = b, Cx \leq d, \forall k = 1, \ldots, n_P, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}.$$

- For concreteness, we will first assume a "fixed" spinning reserve requirement of the form $\sum_k S_k \geq \overline{F}$, so that $C = \begin{bmatrix} \mathbf{0} & -\mathbf{1}^\dagger \end{bmatrix}$ and $d = [-\overline{F}]$.

Formulation, continued

- Recall the definition of x:

$$x = \begin{bmatrix} P_1 \\ \vdots \\ P_k \\ \vdots \\ P_{n_P} \\ S_1 \\ \vdots \\ S_k \\ \vdots \\ S_{n_P} \end{bmatrix}.$$

Formulation, continued

- Recall the specifications of A and C:

$$\begin{aligned} A &= \begin{bmatrix} -\mathbf{1}^\dagger & \mathbf{0} \end{bmatrix}, \\ C &= \begin{bmatrix} \mathbf{0} & -\mathbf{1}^\dagger \end{bmatrix}. \end{aligned}$$

- Define A_k to be the "columns" of A associated with the variables P_k and S_k representing generator k:

$$A_k = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

- Define C_k to be the "columns" of C associated with the variables P_k and S_k representing generator k:

$$C_k = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

- When we generalize to the case of more than one system equality constraint and more than one system inequality constraint, the corresponding matrices A_k and C_k will have columns that are actually vectors!

Minimizer

- Suppose that $x^\star \in \mathbb{R}^{2n_P}$ is the minimizer of the reserve-constrained economic dispatch problem.
- The problem is convex so the first-order necessary conditions are also sufficient.

First-order necessary conditions for economic dispatch

$$\exists \lambda^\star \in \mathbb{R}, \exists \mu^\star \in \mathbb{R}, \forall k = 1,\ldots,n_P, \exists \underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^r \text{ such that:}$$

$$\begin{aligned}
\forall k = 1,\ldots,n_P, \nabla f_k(x_k^\star) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star &= \mathbf{0}; \\
\mu^\star (Cx^\star - d) &= 0; \\
\forall k = 1,\ldots,n_P, \underline{M}_k^\star (\underline{\delta}_k - \Gamma_k x_k^\star) &= \mathbf{0}; \\
\forall k = 1,\ldots,n_P, \overline{M}_k^\star (\Gamma_k x_k^\star - \overline{\delta}_k) &= \mathbf{0}; \\
Ax^\star &= b; \\
Cx^\star &\leq d; \\
\forall k = 1,\ldots,n_P, \Gamma_k x_k^\star &\geq \underline{\delta}_k; \\
\forall k = 1,\ldots,n_P, \Gamma_k x^\star &\leq \overline{\delta}_k; \\
\mu^\star &\geq 0; \\
\underline{\mu}_k^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^\star &\geq \mathbf{0},
\end{aligned}$$

- where $\underline{M}_k^\star = \operatorname{diag}\{\underline{\mu}_k^\star\} \in \mathbb{R}^{r \times r}$ and $\overline{M}_k^\star = \operatorname{diag}\{\overline{\mu}_k^\star\} \in \mathbb{R}^{r \times r}$ are diagonal matrices with entries specified by the entries of $\underline{\mu}_k^\star$ and $\overline{\mu}_k^\star$, respectively.

First-order necessary conditions, continued

- Using the specifications of the matrices A and C:

$$\exists \lambda^\star \in \mathbb{R}, \exists \mu^\star \in \mathbb{R}, \forall k = 1,\ldots,n_P, \exists \underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^r \text{ such that:}$$

$$\begin{array}{rcl}
\forall k = 1,\ldots,n_P, \nabla f_k(x_k^\star) - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star & = & \mathbf{0}; \\
\mu^\star (C x^\star - d) & = & 0; \\
\forall k = 1,\ldots,n_P, \underline{M}_k^\star (\underline{\delta}_k - \Gamma_k x_k^\star) & = & \mathbf{0}; \\
\forall k = 1,\ldots,n_P, \overline{M}_k^\star (\Gamma_k x_k^\star - \overline{\delta}_k) & = & \mathbf{0}; \\
A x^\star & = & b; \\
C x^\star & \leq & d; \\
\forall k = 1,\ldots,n_P, \Gamma_k x_k^\star & \geq & \underline{\delta}_k; \\
\forall k = 1,\ldots,n_P, \Gamma_k x_k^\star & \leq & \overline{\delta}_k; \\
\mu^\star & \geq & 0; \\
\underline{\mu}_k^\star & \geq & \mathbf{0}; \text{ and} \\
\overline{\mu}_k^\star & \geq & \mathbf{0}.
\end{array}$$

First-order necessary conditions, continued

- Note that $\underline{\mu}_k^\star$ and $\overline{\mu}_k^\star$ are now vectors and have different (expanded) interpretations compared to the previous interpretation of $\underline{\mu}_k^\star$ and $\overline{\mu}_k^\star$.
- However, $\underline{\mu}_k^\star$ and $\overline{\mu}_k^\star$ are still Lagrange multipliers on generator constraints for generator $k = 1, \ldots, n_P$.
- In particular, the entries $\underline{\mu}_{k\ell}^\star$ and $\overline{\mu}_{k\ell}^\star$ are the Lagrange multipliers on the constraints $\Gamma_{k\ell} x_k^\star \geq \underline{\delta}_{k\ell}$ and $\Gamma_{k\ell} x_k^\star \leq \overline{\delta}_{k\ell}$, respectively, where:

  $\Gamma_{k\ell}$ is the ℓ-th row of Γ_k, and

  $\underline{\delta}_{k\ell}$ and $\overline{\delta}_{k\ell}$ are the ℓ-th entries of $\underline{\delta}_k$ and $\overline{\delta}_k$, respectively.
- Moreover, $\lambda^\star$ and $\mu^\star$ are Lagrange multipliers on system constraints.
- We will distinguish generator and system constraints and their corresponding Lagrange multipliers in the context of offer-based reserve-constrained economic dispatch.

8.12.1.6 Example

- Example 5.6 from Kirschen and Strbac, *Power System Economics*.
- $\forall k = 1, \ldots, n_P = 4, \underline{P}_k = \underline{S}_k = 0$, and with the other capacities and marginal costs specified by:

$$\begin{aligned}
\overline{P}_1 = 250, \overline{S}_1 = 0, \forall x_1 \in \mathbb{S}_1, \nabla f_1(x_1) &= \begin{bmatrix} \$2/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_2 = 230, \overline{S}_2 = 160, \forall x_2 \in \mathbb{S}_2, \nabla f_2(x_2) &= \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_3 = 240, \overline{S}_3 = 190, \forall x_3 \in \mathbb{S}_3, \nabla f_3(x_3) &= \begin{bmatrix} \$20/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_4 = 250, \overline{S}_4 = 0, \forall x_4 \in \mathbb{S}_4, \nabla f_4(x_4) &= \begin{bmatrix} \$28/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},
\end{aligned}$$

- so that energy costs are non-zero but spinning reserve costs are zero.
- We consider two levels of demand, $\overline{D} = 300$ and $\overline{D} = 500$ MW.
- We require procured spinning reserve to be at least $\overline{F} = 250$ MW in both cases.

Example, continued

- Note that generator 1 cannot provide spinning reserve and has the lowest marginal cost:
  - for any demand equal to or above 250 MW, generator 1 will be fully dispatched, and
  - the optimal values of power and spinning reserve for generator 1 are:

$$\begin{aligned} P_1^\star &= 250 \text{ MW}, \\ S_1^\star &= 0 \text{ MW}. \end{aligned}$$

- Only generators 2 and 3 can provide spinning reserve and neither can provide all the spinning reserve.
- Generator 2 has the lower marginal cost amongst generators 2 and 3:
  - use generator 2 to produce as much energy as possible consistent with meeting the reserve constraint,
  - use generator 3 to provide as much spinning reserve as possible,
  - optimal values:

$$\begin{aligned} S_3^\star &= \overline{S}_3 = 190 \text{ MW}, \\ S_2^\star &= \overline{F} - S_3^\star = 250 - 190 = 60 \text{ MW}. \end{aligned}$$

- For $\overline{D} = 300$ MW:
  - Other optimal values are: $P_2^\star = 50$ MW, $P_3^\star = P_4^\star = S_4^\star = 0$ MW, with:
    - the upper limit on the three generator constraints for generator 1 binding,
    - none of the three sets of generator constraints for generator 2 binding, and
    - only the upper limit on the spinning reserve constraint for generator 3 binding.
  - First line of the first-order necessary conditions yields for generator 2:

$$\begin{aligned} \mathbf{0} &= \nabla f_2(x_2^\star) - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - [\Gamma_2]^\dagger \underline{\mu}_2^\star + [\Gamma_2]^\dagger \overline{\mu}_2^\star, \\ &= \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix} - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix}, \end{aligned}$$

  - by complementary slackness, since none of the three sets of generator constraints for generator 2 are binding.
  - That is, $\lambda^\star = \$17/\text{MWh}$ and $\mu^\star = \$0/\text{MWh}$.

Example, continued

– Recalling the sensitivity results from Theorem 4.14, note that:

  ◦ $\lambda^\star$ is the sensitivity of surplus to changes in the demand, and
  ◦ $\mu^\star$ is the sensitivity of surplus to changes in the requirements for spinning reserve.

– That is:

  ◦ increasing demand would require increasing generator 2 production, costing $\lambda^\star = \$17/\text{MWh}$.
  ◦ increasing spinning reserve requirements would require additional spinning reserve from generator 2, costing $\mu^\star = \$0/\text{MWh}$.

Example, continued

- For $\overline{D} = 500$ MW:
  - The other optimal values are: $P_2^{\star} = \overline{P}_2 - S_2^{\star} = 230 - 60 = 170$ MW, $P_3^{\star} = 50$ MW, $P_4^{\star} = 30$ MW, with:
    - the upper limit on the three generator constraints for generator 1 binding,
    - only the upper limit on the power plus reserve constraint for generator 2 binding,
    - only the upper limit on the spinning reserve constraint and the power plus reserve constraint for generator 3 binding, and
    - only the spinning reserve constraint for generator 4 binding.

– The first line of the first-order necessary conditions yields for generator 4:

$$
\begin{aligned}
\mathbf{0} &= \nabla f_4(x_4^\star) - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - [\Gamma_4]^\dagger \underline{\mu}_4^\star + [\Gamma_4]^\dagger \overline{\mu}_4^\star, \\
&= \begin{bmatrix} \$28/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix} - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underline{\mu}_4^\star + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overline{\mu}_4^\star, \\
&= \begin{bmatrix} \$28/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix} - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - \begin{bmatrix} 0 \\ \underline{\mu}_{42}^\star \end{bmatrix} + \begin{bmatrix} 0 \\ \overline{\mu}_{42}^\star \end{bmatrix},
\end{aligned}
$$

– where, by complementary slackness, $\underline{\mu}_4^\star = \begin{bmatrix} 0 \\ \underline{\mu}_{42}^\star \\ 0 \end{bmatrix}$ and $\overline{\mu}_4^\star = \begin{bmatrix} 0 \\ \overline{\mu}_{42}^\star \\ 0 \end{bmatrix}$ since only the spinning reserve constraint for generator 4 is binding.

– Focusing on the first line of this condition, we obtain $0 = \$28/\text{MWh} - \lambda^\star$.

– The first line of the first-order necessary conditions yields for generator 2:

$$
\begin{aligned}
\mathbf{0} &= \nabla f_2(x_2^\star) - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - [\Gamma_2]^\dagger \underline{\mu}_2^\star + [\Gamma_2]^\dagger \overline{\mu}_2^\star, \\
&= \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix} - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underline{\mu}_2^\star + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overline{\mu}_2^\star, \\
&= \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix} - \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix} - \mathbf{0} + \begin{bmatrix} \overline{\mu}_{23}^\star \\ \overline{\mu}_{23}^\star \end{bmatrix},
\end{aligned}
$$

– where, by complementary slackness, $\underline{\mu}_2^\star = \mathbf{0}$ and $\overline{\mu}_2^\star = \begin{bmatrix} 0 \\ 0 \\ \overline{\mu}_{23}^\star \end{bmatrix}$ since only the upper limit on the power plus reserve constraint for generator 2 is binding.

Example, continued

- Focusing on the first line of the condition, we obtain $0 = \$17/\text{MWh} - \lambda^\star + \overline{\mu}^\star_{23}$, so that $\overline{\mu}^\star_{23} = \$28/\text{MWh} - \$17/\text{MWh} = \$11/\text{MWh}$.
- Focusing on the second line of the condition, we obtain $0 = \$0/\text{MWh} - \mu^\star + \overline{\mu}^\star_{23}$.
- That is, $\lambda^\star = \$28/\text{MWh}$ and $\mu^\star = \$11/\text{MWh}$.
- That is:
  - increasing demand would require increasing generator 4 production, costing $\lambda^\star = \$28/\text{MWh}$.
  - increasing reserve requirements would require additional spinning reserve from generator 2, which would involve generating *less* power from generator 2 and more power from generator 4, which would cost on net $\mu^\star = \$28/\text{MWh} - \$17/\text{MWh} = \$11/\text{MWh}$.
- The Lagrange multiplier on the spinning reserve constraint is non-zero even though the "direct" cost of supplying spinning reserve is zero.

8.12.2 Offer-based reserve-constrained economic dispatch

8.12.2.1 Implementing the results of economic dispatch

- As before, the ISO must ask each generator k to produce at level $x_k^\star$ resulting from reserve-constrained economic dispatch.
- Again we will define a pricing rule that aligns private incentives with the public goal of economic dispatch.
- For energy, we proposed a pricing rule and then verified that it aligned private profit maximization by each generator k with achieving economic dispatch.
- We will see how to derive pricing rules more generally to align incentives when there are multiple commodities.

8.12.2.2 Profit maximization by a generator

- Consider generator k that is paid some price π_P for its power production and some price π_S for its reserve contribution.
- Revenue to generator k is:

$$\pi_P P_k + \pi_S S_k = \begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix}^\dagger x_k.$$

- We continue to assume that generator k cannot directly affect π_P and π_S and that it desires to maximize its profit.
- Equivalently, generator k wants to minimize the difference between costs and revenues.
- The problem faced by generator k is:

$$\begin{aligned} &\min_{x_k \in \mathbb{S}_k} \left\{ f_k(x_k) - \begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix}^\dagger x_k \right\} \\ = &\min_{x_k \in \mathbb{R}^2} \left\{ f_k(x_k) - \begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix}^\dagger x_k \middle| \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \right\}. \end{aligned}$$

8.12.2.3 First-order necessary conditions for profit maximization

- Suppose that $x_k^{\star\star} \in \mathbb{R}^2$ maximizes profit.

$$\exists \underline{\mu}_k^{\star\star}, \overline{\mu}_k^{\star\star} \in \mathbb{R}^r \text{ such that:}$$

$$\begin{aligned}
\nabla f_k(x_k^{\star\star}) - \begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix} - [\Gamma_k]^\dagger \underline{\mu}_k^{\star\star} + [\Gamma_k]^\dagger \overline{\mu}_k^{\star\star} &= \mathbf{0}; \\
\underline{M}_k^{\star\star}(\underline{\delta}_k - \Gamma_k x_k^{\star\star}) &= \mathbf{0}; \\
\overline{M}_k^{\star\star}(\Gamma_k x_k^{\star\star} - \overline{\delta}_k) &= \mathbf{0}; \\
\Gamma_k x_k^{\star\star} &\geq \underline{\delta}_k; \\
\Gamma_k x_k^{\star\star} &\leq \overline{\delta}_k; \\
\underline{\mu}_k^{\star\star} &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^{\star\star} &\geq \mathbf{0},
\end{aligned}$$

- where $\underline{M}_k^{\star\star} = \operatorname{diag}\{\underline{\mu}_k^{\star\star}\} \in \mathbb{R}^{r \times r}$ and $\overline{M}_k^{\star\star} = \operatorname{diag}\{\overline{\mu}_k^{\star\star}\} \in \mathbb{R}^{r \times r}$ are diagonal matrices with entries specified by the entries of $\underline{\mu}_k^{\star\star}$ and $\overline{\mu}_k^{\star\star}$, respectively.
- As previously, generator k enforces its own generator constraints by requiring that $\Gamma_k x_k^{\star\star} \geq \underline{\delta}_k$ and $\Gamma_k x^{\star\star} \leq \overline{\delta}_k$.

8.12.2.4 Aligning the incentives

- How do we make the solution of these first-order necessary conditions for maximizing generator k's profit consistent with the results of economic dispatch?
  - Consider the corresponding lines in the two sets of first-order necessary conditions for generator k's profit maximization and for reserve-constrained economic dispatch, respectively:

$$\begin{aligned}
\nabla f_k(x_k^{\star\star}) - \begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix} - [\Gamma_k]^\dagger \underline{\mu}_k^{\star\star} + [\Gamma_k]^\dagger \overline{\mu}_k^{\star\star} &= \mathbf{0}, \\
\nabla f_k(x_k^\star) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star &= \mathbf{0}.
\end{aligned}$$

  - Define the prices π_P and π_S so that $x_k^{\star\star} = x_k^\star$ is a solution to the first-order necessary conditions for generator k's profit maximization:
    - there may be other solutions if f_k is not strictly convex.

- Comparing the first lines of the two sets of first-order necessary conditions for generator k's profit maximization and for economic dispatch, respectively, we see that $x_k^{\star\star} = x_k^\star$ is a solution if:

$$\begin{aligned}
\begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix} &= -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star, \\
&= -\begin{bmatrix} -1 \\ 0 \end{bmatrix} \lambda^\star - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \mu^\star, \\
&= \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix}, \\
\underline{\mu}_k^{\star\star} &= \underline{\mu}_k^\star, \\
\overline{\mu}_k^{\star\star} &= \overline{\mu}_k^\star.
\end{aligned}$$

Aligning the incentives, continued

- Generalizing the case of energy only:

  if the ISO sets $\begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix} = \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix}$

  then $x_k^{\star\star} = x_k^\star$, $\underline{\mu}_k^{\star\star} = \underline{\mu}_k^\star$, and $\overline{\mu}_k^{\star\star} = \overline{\mu}_k^\star$ satisfy the first-order necessary conditions for profit maximization by k.

- If f_k is strictly convex then:

  $x_k^{\star\star} = x_k^\star$, $\underline{\mu}_k^{\star\star} = \underline{\mu}_k^\star$, and $\overline{\mu}_k^{\star\star} = \overline{\mu}_k^\star$ are the unique profit maximizing solutions, and

  the market clearing prices $\begin{bmatrix} \pi_P \\ \pi_S \end{bmatrix} = \begin{bmatrix} \lambda^\star \\ \mu^\star \end{bmatrix}$ strictly support economic dispatch,

- If f_k is convex but not strictly convex then:

  there may be multiple maximizers,

  $x_k^{\star\star} = x_k^\star$ is consistent with individual profit maximization, and

  the prices non-strictly support economic dispatch.

- If the spinning reserve offer price is zero then f_k is convex but not strictly convex.

Aligning the incentives, continued

- We typically write the units for the price of energy as \$/MWh.
- Since spinning reserve is typically procured on an hourly basis, we typically write units for the price of spinning reserve as \$/MW per hour.
- Note that the spinning reserve price in such a co-optimized energy and reserves market can be strictly positive even with zero offers prices for reserves:
  - a generator providing spinning reserve is paid a non-zero price for providing capacity for spinning reserve that reflects the **opportunity cost** of the foregone profit that it did not earn because it did not use that capacity to make energy,
  - the opportunity cost equals the difference between the energy price and the energy offer.
- For low levels of required spinning reserve, or if there is significant available capacity, the spinning reserve price may be zero or low.
- For high levels of required spinning reserve, or if supply is tight, the spinning reserve price may be high because of the opportunity costs.
- A qualitatively similar pattern will occur for all types of reserves.

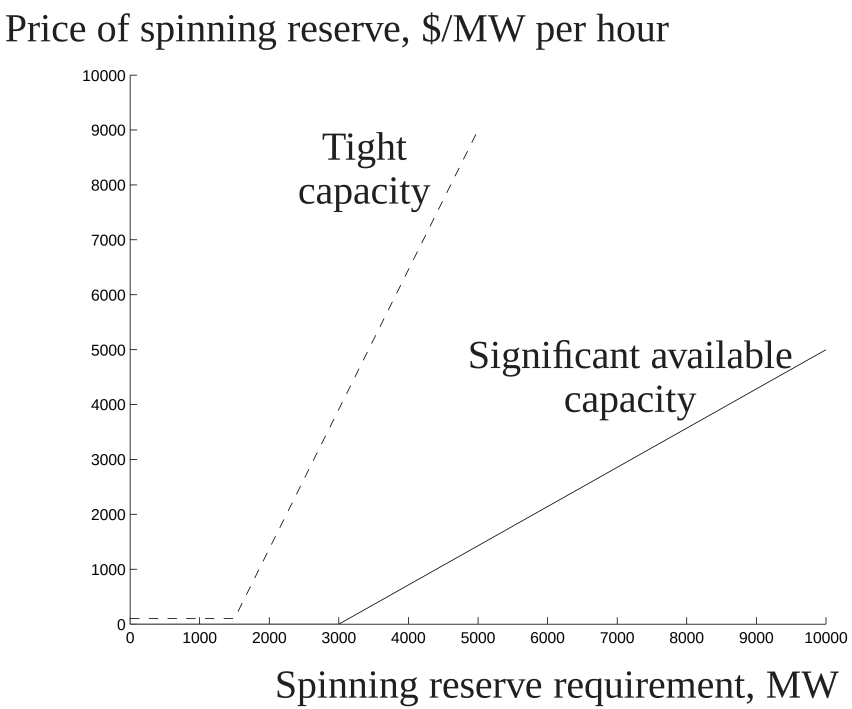

Fig. 8.7. Price of spinning reserve versus the spinning reserve requirement in case of significant available capacity (shown solid) and in case of tight available capacity (shown dashed).

8.12.2.5 Offer versus marginal cost of production

- In the discussion of offers for power there were clear downsides to mis-specifying marginal cost or capacity, given the price-taking assumption:
  - (i) specifying the offer to be different to marginal costs reduced profits;
  - (ii) specifying the capacity to be less than the actual capacity reduced profits;
  - (iii) specifying the capacity to be greater than the actual capacity in a forward market involved a speculative offer;
  - (iv) specifying the capacity to be greater than the actual capacity in a real-time market would result in deviation penalties.

Offer versus marginal cost of production, continued

- Given the price-taking assumption, specifying the energy or spinning reserve offer to be different to the corresponding marginal cost or specifying the capacity to be less than the actual capacity still result in profit reduction.
- What about specifying the upper limit on spinning reserve to be greater than the actual upper limit on spinning reserve?
- Unless the capacity is called on, there is nothing actually "done" in providing spinning reserve.
  - Implication is that non-compliance penalties or operational tests are necessary to ensure that reserves capacity is not exaggerated.

8.12.2.6 Example

- Example 5.6 from Kirschen and Strbac, *Power System Economics*.
- $\forall k = 1, \ldots, 4, \underline{P}_k = \underline{S}_k = 0$, and with the other capacities and marginal costs specified by:

$$\begin{aligned}
\overline{P}_1 = 250, \overline{S}_1 = 0, \forall x_1 \in \mathbb{S}_1, \nabla f_1(x_1) &= \begin{bmatrix} \$2/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_2 = 230, \overline{S}_2 = 160, \forall x_2 \in \mathbb{S}_2, \nabla f_2(x_2) &= \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_3 = 240, \overline{S}_3 = 190, \forall x_3 \in \mathbb{S}_3, \nabla f_3(x_3) &= \begin{bmatrix} \$20/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_4 = 250, \overline{S}_4 = 0, \forall x_4 \in \mathbb{S}_4, \nabla f_4(x_4) &= \begin{bmatrix} \$28/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},
\end{aligned}$$

- so that energy costs are non-zero but spinning reserve costs are zero.
- We consider two levels of demand, $\overline{D} = 300$ and $\overline{D} = 500$ MW.
- We require spinning reserves to be at least $\overline{F} = 250$ MW in both cases.

Example, continued

- Assume that offers reflect marginal costs and actual capacities:
  - price-taking assumption together with additional assumption that offered spinning reserve capacity equals actual spinning reserve capacity.
- For both demand levels, generator 1 will be fully dispatched, with optimal values of power $P_1^\star = 250$ MW and spinning reserve $S_1^\star = 0$ MW.

- For $\overline{D} = 300$ MW:
  - Optimal dispatch is $P_1^\star = 250$ MW, $S_1^\star = 0$ MW, $P_2^\star = 50$ MW, $S_2^\star = 60$ MW, $P_3^\star = 0$ MW, $S_3^\star = 190$ MW, $P_4^\star = 0$ MW, $S_4^\star = 0$ MW.
  - Note that generator 2 is not fully dispatched and has marginal cost \$17/MWh:
    - the energy price is $\lambda^\star = \$17$/MWh, reflecting the offer price of generator 2 of \$17/MWh to meet an additional MW of demand.
  - The capacity constraint of generator 2 is not binding, so $\mu^\star = \$0$/MWh:
    - the spinning reserve price is zero, reflecting the offer price of generator 2 of \$0/MWh to provide an additional MW of spinning reserve,
    - if reserve offers were non-zero then the prices for reserves would be non-zero.

Example, continued

- For $\overline{D} = 500$ MW:
  - Optimal dispatch is $P_1^\star = 250$ MW, $S_1^\star = 0$ MW, $P_2^\star = 170$ MW, $S_2^\star = 60$ MW, $P_3^\star = 50$ MW, $S_3^\star = 190$ MW, $P_4^\star = 30$ MW, $S_4^\star = 0$ MW.
  - Note that generator 4 is not fully dispatched and has marginal cost \$28/MWh:
    - the energy price is $\lambda^\star =$ \$28/MWh, reflecting the offer price of generator 4 of \$28/MWh to meet an additional MW of demand.
  - The capacity constraints of generators 2 and 3 are binding and $\mu^\star =$ \$11/MWh:
    - the spinning reserve price is \$11/MWh, reflecting the difference between the offer prices of generators 4 and 2 for energy,
    - the spinning reserve price is also equal to the opportunity cost for generator 2 to not sell more energy at the price $\lambda^\star =$ \$28/MWh.

Example, continued

– Note that the power plus reserve of generator 2 is equal to its capacity and that generator 2 is *indifferent* to the sharing of its capacity between power and spinning reserve since either it:
  - ○ generates, costing \$17/MWh and being paid \$28/MWh, receiving operating profit \$11/MWh, or
  - ○ provides spinning reserve, costing \$0/MWh, and being paid \$11/MWh, receiving operating profit \$11/MWh.

– The power plus reserve of generator 3 is also at capacity; however, its reserve constraint is binding.

– Generator 3 would prefer to provide even more spinning reserve if it had more spinning reserve capacity since:
  - ○ for generation, it costs \$20/MWh and is paid \$28/MWh, receiving operating profit \$8/MWh, but
  - ○ for spinning reserve, it costs \$0/MWh and is paid \$11/MWh, receiving operating profit \$11/MWh.

– However, generator 3 cannot provide more spinning reserve since its spinning reserve constraint is binding.

Example, continued

- Note that the price for spinning reserve is non-zero when $\overline{D} = 500$ MW, even though the offer prices for spinning reserve were zero:
  - prices would also be non-zero if the offer prices for spinning reserve were non-zero!
- As mentioned above, the price for spinning reserve represents the *opportunity cost* to generator 2 of foregoing the infra-marginal rent from selling energy.
- This opportunity cost is automatically represented in the price for spinning reserve because the energy and spinning reserve are considered together in a single problem:
  - the energy and reserves are **co-optimized** as in the day-ahead ERCOT nodal market.

8.12.2.7 Separated versus co-optimized markets

- In contrast, in the previous ERCOT zonal market, there was a day-ahead market for ancillary services, including reserves, but there was no associated day-ahead energy market:
  - the market for reserves (in the day-ahead AS market) and the market for energy (in the balancing market) were separated,
  - in the separated AS market, prices could only be non-zero if offers were non-zero.
- A profit-maximizing generator would not willingly forego the operating profit from selling energy into the balancing market.
- Therefore, offers into the previous AS market were made at a price that reflected *expectation* of the operating profit that would have been received for selling energy in the balancing market:
  - offers would reflect an estimate of the difference between the energy price and marginal cost.
- Since forecasting the balancing market energy prices is difficult, it was difficult for market participants to find an appropriate offer price into the day-ahead ancillary services market.

8.12.2.8 Price for demand

- When we discussed the pricing rule for energy, we observed that the demand would pay for energy at a price equal to the Lagrange multiplier on the power balance constraint:
  - Demand consumed $\overline{D}$ and it would pay a total of $\lambda^\star \times \overline{D}$,
  - Total power generation was $\overline{D} = \sum_k P_k^\star$ and generators were paid a total of $\lambda^\star \times \sum_k P_k^\star = \lambda^\star \times \overline{D}$.
  - Payment from demand for energy equals payment to generators for energy.

Price for demand, continued

- How about spinning reserve?
  - Total provision of spinning reserve by the generators was $\overline{F} = \sum_k S_k^\star$, which would be paid a total of $\mu^\star \times \sum_k S_k^\star = \mu^\star \times \overline{F}$.
  - We need to pay the generators for providing spinning reserve.
  - However, there was no explicit dependence of the spinning reserve requirement on the demand $\overline{D}$ in the $\sum_k S_k \geq \overline{F}$ formulation of the spinning reserve constraint since $\overline{F}$ is (apparently) fixed independent of the power demand $\overline{D}$.
- In the $\sum_k S_k \geq \overline{F}$ formulation of the spinning reserve constraint, we must "uplift" (charge) the spinning reserve payment of $\mu^\star \times \overline{F}$ to demand:
  - for example, the payment could be charged to the demand by adding a pro rata share of the spinning reserve cost to the energy price paid by demand,
  - this effectively increases the price of energy by $(\mu^\star \times \overline{F}/\overline{D})$, so that total payment by the demand for spinning reserve would be:

$$(\mu^\star \times \overline{F}/\overline{D}) \times \overline{D} = \mu^\star \times \overline{F}.$$

Price for demand, continued

- For another interpretation of the payment for spinning reserve by demand, again assume that the demand is bid with a willingness-to-pay.
- So, D is a scalar representing the consumed power and we include the constraint $0 \leq D \leq \overline{D}$ and a term f_0 in the objective representing minus the benefits of consumption.
- We will consider the formulation where the spinning reserve must cover a "fixed" fraction of demand: $F = \alpha D$.
- Note that in this formulation the demand appears twice in the system constraints:

$$\begin{aligned} D - \sum_{k=1}^{n_P} P_k &= 0, \\ \alpha D - \sum_{k=1}^{n_P} S_k &\leq 0. \end{aligned}$$

- In this formulation, there is an explicit dependence of the reserve requirement in terms of the demand for power:
  - this will change the interpretation of the spinning reserve payment.

Price for demand, continued

- To represent demand into the reserve-constrained economic dispatch formulation, we add columns A_0 and C_0 to A and C of the form:

$$\begin{aligned} A_0 &= [1], \\ C_0 &= [\alpha]. \end{aligned}$$

- The first-order necessary conditions for economic dispatch include:

$$\nabla f_0(D^\star) + [1]\lambda^\star + [\alpha]\mu^\star - \underline{\mu}_k^\star + \overline{\mu}_k^\star = 0.$$

- This is one line of the first-order necessary conditions for the problem:

$$\min_{D \in \mathbb{S}_0} \{ f_0(D) + (\lambda^\star + \alpha\mu^\star)D \}.$$

Price for demand, continued

- Paralleling the previous discussion, the price paid by demand to induce behavior consistent with economic dispatch is $\lambda^\star + \alpha\mu^\star$.
- This is the sum of:

  the marginal energy cost, plus
  a share of the marginal reserve cost.

- With this price, and assuming $D^\star = \overline{D}$, the demand consumes $\overline{D}$ and it would pay:

$$(\lambda^\star + \alpha\mu^\star)\overline{D} = \lambda^\star\overline{D} + \mu^\star\alpha\overline{D}.$$

- Total generation is $\overline{D} = \sum_k P_k^\star$ and is paid a total $\lambda^\star \times \sum_k P_k^\star = \lambda^\star \times \overline{D}$.
- Total spinning reserve is $\alpha\overline{D} = \sum_k S_k^\star$, and is paid a total of $\mu^\star \times \sum_k S_k^\star = \mu^\star \times \alpha\overline{D}$.
- Payment from demand equals payment to generators; there is no uplift:
  - if reserves (or any other ancillary services) are explicitly represented as proportional to demand consumption then there is no uplift since total payment by demand equals total payment to generators,
  - demand is charged based on marginal cost of service.

Price for demand, continued

- The share α is based on the reserve requirement as a fraction of demand:
  - If α is chosen to equal $\overline{F}/\overline{D}$ then the total payment by demand for energy and spinning reserve with the $\sum_k S_k \geq \alpha\overline{D}$ formulation of spinning reserve is the same as the total payment for energy plus pro rata uplift in the $\sum_k S_k \geq \overline{F}$ formulation of spinning reserve.
- That is, the payment under pro rata uplift would provide the right incentives to demand if required spinning reserve was actually proportional to demand:
  - however, this is not a correct model of the need for spinning reserve!
  - pro rata sharing of spinning reserve procurement costs as practiced in ERCOT and other North American markets does not provide efficient incentives for consumption,
  - however, distortion of consumption decisions likely to be small because overall cost of spinning reserve small compared to average energy cost.
- Note that in the context of a day-ahead market, the spinning reserve constraint might be based on peak demand over the day.
  - price should, in principle, be associated with demand in peak hour.

8.12.3 More general formulations of economic dispatch

(i) Decision variables,
(ii) Generator constraints,
(iii) System constraints,
(iv) Commodities.

8.12.3.1 Decision variables

- Previously we considered the case that there were two decision variables associated with each generator:

  power, and
  spinning reserve.

- We generalize the formulation to suppose that each generator k has associated with it multiple decision variables: $x_k \in \mathbb{R}^{N_k}$, for example:

  power,
  regulation,
  spinning reserve, and
  non-spinning reserve.

Decision variables, continued

- If there are variables in the formulation besides the generator decision variables, we will collect them together into a vector: $x_{n_P+1} \in \mathbb{R}^{N_{n_P+1}}$.
  - For example, if we represent voltage angles or magnitudes in an AC power flow formulation,
  - For completeness, we also consider a cost f_{n_P+1} associated with these other variables, but this cost is usually equal to zero.
- We will also sometimes represent the level of demand with a decision variable $x_0 \in \mathbb{R}^{N_0}$.
  - in the simplest case, $N_0 = 1$ and $x_0 = [D]$.
- We collect the decisions variables of all the generators (together with any other variables, x_{n_P+1}, that are necessary to represent the system constraints) into a vector: $x \in \mathbb{R}^n$, where $n = \sum_{k=0}^{n_P+1} N_k$.
- We continue to use n_P for the number of generators.

8.12.3.2 Generator constraints

- There are generator constraints that limit the choices of x_k:

$$\underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k,$$

- where $\Gamma_k \in \mathbb{R}^{r_k \times N_k}$, $\underline{\delta}_k \in \mathbb{R}^{r_k}$, and $\overline{\delta}_k \in \mathbb{R}^{r_k}$ are appropriately chosen matrices and vectors.
- We will again define the feasible operating set for generator k:

$$\mathbb{S}_k = \{x_k \in \mathbb{R}^{N_k} | \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}.$$

- Similarly, for demand, with $x_0 = [D]$, we would have:

$$\begin{aligned} \mathbb{S}_0 &= \{D \in \mathbb{R} | 0 \leq D \leq \overline{D}\}, \\ &= \{x_0 \in \mathbb{R} | \underline{\delta}_0 \leq \Gamma_0 x_0 \leq \overline{\delta}_0\}, \end{aligned}$$

- where $\underline{\delta}_0 = [0], \Gamma_0 = [1], \overline{\delta}_0 = [\overline{D}]$.

8.12.3.3 System equality constraints

- Previously we considered the case that there was one equality constraint in the system constraints associated with supply-demand power balance.
- We generalize the formulation to suppose that there are multiple equality constraints:

$$Ax = b,$$

- where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
- As previously, we define $A_k \in \mathbb{R}^{m \times N_k}$ to be:
  - for $1 \leq k \leq n_P$, the columns of A associated with the decision variables representing generator k,
  - for $k = 0$, the columns of A associated with the decision variables for demand, and
  - for $k = n_P + 1$, the columns of A associated with the other variables.

8.12.3.4 System inequality constraints

- Previously we considered the case that there was one inequality constraint in the system constraints associated with reserve constraints.
- We generalize the formulation to suppose that there are multiple inequality constraints:

$$Cx \leq d,$$

- where $C \in \mathbb{R}^{r \times n}$ and $d \in \mathbb{R}^{r}$.
- As previously, we define $C_k \in \mathbb{R}^{r \times N_k}$ to be:
  - for $1 \leq k \leq n_P$, the columns of C associated with the decision variables representing generator k,
  - for $k = 0$, the columns of C associated with the decision variables for demand, and
  - for $k = n_P + 1$, the columns of C associated with the other variables.

8.12.3.5 Commodities

- In our basic formulation, both generator decision variables, P_k and S_k, of each generator k appeared in the system constraints and there were no other variables besides the generator decision variables.
- In the more general formulation, only some of the generator variables might appear in the system constraints and there may be some other variables besides the generator decision variables.
  - For example, there might be a variable that was necessary to represent the generator constraints, but which did not appear in the system constraints.
  - For example, we may need to represent voltage angles, but these are not generator decision variables.
- Generator decision variables with non-zero coefficients in the system constraints are associated with **commodities**.
- Each row of A and C defines a supply-demand balance or a minimum or maximum requirement for a commodity.
- Each row of A and C will be associated with a Lagrange multiplier that prices the associated commodities.

8.12.3.6 Generalized economic dispatch problem

Formulation

- The generalized economic dispatch problem is:

$$\begin{aligned}
&\min_{\forall k=0,\ldots,n_P+1, x_k\in\mathbb{S}_k}\{f(x)|Ax=b, Cx\leq d\}\\
&= \min_{x\in\mathbb{R}^n}\{f(x)|Ax=b, Cx\leq d, \forall k=0,\ldots,n_P+1, \underline{\delta}_k\leq\Gamma_k x_k\leq\overline{\delta}_k\}.
\end{aligned}$$

- This is a similar formulation to previously except that we have changed the definitions of the matrices and vectors.
- The similar formulation will help us to understand the pricing rule for the generalized economic dispatch problem.

Minimizer

- Suppose that $x^\star\in\mathbb{R}^n$ is the minimizer of the generalized economic dispatch problem.
- The problem is convex so the first-order necessary conditions are also sufficient.

First-order necessary conditions

$\exists \lambda^\star \in \mathbb{R}^m, \exists \mu^\star \in \mathbb{R}^r, \forall k = 0, \ldots, n_P + 1, \exists \underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^{r_k}$ such that:

$$
\begin{aligned}
\forall k = 0, \ldots, n_P + 1, \nabla f_k(x_k^\star) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star &= \mathbf{0}; \\
M^\star (Cx^\star - d) &= \mathbf{0}; \\
\forall k = 0, \ldots, n_P + 1, \underline{M}_k^\star (\underline{\delta}_k - \Gamma_k x_k^\star) &= \mathbf{0}; \\
\forall k = 0, \ldots, n_P + 1, \overline{M}_k^\star (\Gamma_k x_k^\star - \overline{\delta}_k) &= \mathbf{0}; \\
Ax^\star &= b; \\
Cx^\star &\leq d; \\
\forall k = 0, \ldots, n_P + 1, \Gamma_k x_k^\star &\geq \underline{\delta}_k; \\
\forall k = 0, \ldots, n_P + 1, \Gamma_k x_k^\star &\leq \overline{\delta}_k; \\
\mu^\star &\geq \mathbf{0}; \\
\underline{\mu}_k^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^\star &\geq \mathbf{0},
\end{aligned}
$$

First-order necessary conditions, continued

- where $M^\star = \mathrm{diag}\{\mu^\star\} \in \mathbb{R}^{r_k \times r_k}$, $\underline{M}_k^\star = \mathrm{diag}\{\underline{\mu}_k^\star\} \in \mathbb{R}^{r_k \times r_k}$, and $\overline{M}_k^\star = \mathrm{diag}\{\overline{\mu}_k^\star\} \in \mathbb{R}^{r_k \times r_k}$ are diagonal matrices with entries specified by the entries of $\mu^\star$, $\underline{\mu}_k^\star$, and $\overline{\mu}_k^\star$, respectively.

8.12.4 Generalized offer-based dispatch

8.12.4.1 Pricing rule for general linear system constraints

- Recall that each generator $k = 1, \ldots, n_P$, has decision variables x_k and demand has decision variable x_0.
- We will define a price paid for each entry in the decision vector x_k that appears in a system constraint.
- Paralleling the previous development, consider a vector of prices π_{x_k} defined by:

$$\pi_{x_k} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star.$$

- Each generator $k = 1, \ldots, n_P$, is paid $[\pi_{x_k}]^\dagger x_k$.
- Demand is “paid” $[\pi_{x_k}]^\dagger x_k$, or equivalently pays $[-\pi_{x_k}]^\dagger x_k$.

8.12.4.2 Private profit maximization by a generator

- The operating profit maximization problem faced by generator $k = 1, \ldots, n_P$, and the consumer surplus maximization problem faced demand $k = 0$ is equivalent to the following minimization problem:

$$\begin{aligned}
&\min_{x_k \in \mathbb{S}_k} \left\{ f_k(x_k) - [\pi_{x_k}]^\dagger x_k \right\} \\
&\quad = \min_{x_k \in \mathbb{R}^{N_k}} \left\{ f_k(x_k) - [\pi_{x_k}]^\dagger x_k \middle| \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \right\}, \\
&\quad = \min_{x_k \in \mathbb{R}^{N_k}} \left\{ f_k(x_k) + \left[[\lambda^\star]^\dagger A_k + [\mu^\star]^\dagger C_k \right] x_k \middle| \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \right\}.
\end{aligned}$$

- Consider a minimizer, $x_k^{\star\star} \in \mathbb{R}^{N_k}$, of this problem, which is therefore also a maximizer of profit.

8.12.4.3 First-order necessary conditions for profit maximization

- Suppose that $x_k^{\star\star} \in \mathbb{R}^{N_k}$ maximizes profit.

$$
\begin{aligned}
\exists \underline{\mu}_k^{\star\star}, \overline{\mu}_k^{\star\star} \in \mathbb{R}^{r_k} \text{ such that:} \\
\nabla f_k(x_k^{\star\star}) + [A_k]^\dagger \lambda^\star + [C_k]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^{\star\star} + [\Gamma_k]^\dagger \overline{\mu}_k^{\star\star} &= \mathbf{0}; \\
\underline{M}_k^{\star\star}(\underline{\delta}_k - \Gamma_k x_k^{\star\star}) &= \mathbf{0}; \\
\overline{M}_k^{\star\star}(\Gamma_k x_k^{\star\star} - \overline{\delta}_k) &= \mathbf{0}; \\
\Gamma_k x_k^{\star\star} &\geq \underline{\delta}_k; \\
\Gamma_k x_k^{\star\star} &\leq \overline{\delta}_k; \\
\underline{\mu}_k^{\star\star} &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^{\star\star} &\geq \mathbf{0},
\end{aligned}
$$

- where $\underline{M}_k^{\star\star} = \operatorname{diag}\{\underline{\mu}_k^{\star\star}\} \in \mathbb{R}^{r_k \times r_k}$ and $\overline{M}_k^{\star\star} = \operatorname{diag}\{\overline{\mu}_k^{\star\star}\} \in \mathbb{R}^{r_k \times r_k}$ are diagonal matrices with entries specified by the entries of $\underline{\mu}_k^{\star\star}$ and $\overline{\mu}_k^{\star\star}$, respectively.

First-order necessary conditions for profit maximization, continued

- As previously, generator $k = 1, \ldots, n_P$, and demand $k = 0$ enforces its own generator (or demand) constraints by requiring that $\Gamma_k x_k^{\star\star} \geq \underline{\delta}_k$ and $\Gamma_k x^{\star\star} \leq \overline{\delta}_k$.
- Note that the first-order necessary conditions for profit maximization reproduce the corresponding conditions in the first-order necessary conditions for generalized economic dispatch.
- As previously, $x_k^{\star\star} = x_k^{\star}$, $\underline{\mu}_k^{\star\star} = \underline{\mu}_k^{\star}$, and $\overline{\mu}_k^{\star\star} = \overline{\mu}_k^{\star}$ satisfy the first-order necessary conditions for profit maximization by k.

8.12.4.4 Pricing theorem

Theorem 8.1 *Let $\lambda^\star$ and $\mu^\star$ be Lagrange multipliers on the system equality and inequality constraints for the generalized offer-based economic dispatch problem:*

$$\begin{aligned} &\min_{\forall k=0,\ldots,n_P+1, x_k \in \mathbb{S}_k} \{f(x) | Ax = b, Cx \leq d\} \\ &= \min_{x \in \mathbb{R}^n} \{f(x) | Ax = b, Cx \leq d, \forall k = 0, \ldots, n_P + 1, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}. \end{aligned}$$

Define prices paid to generators $k = 1, \ldots, n_P$ to be:

$$\pi_{x_k} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star.$$

Define the prices paid by demand to be:

$$-\pi_{x_0} = [A_0]^\dagger \lambda^\star + [C_0]^\dagger \mu^\star.$$

- *If f_k is strictly convex then:*
  $x_k^{\star\star} = x_k^\star$, $\underline{\mu}_k^{\star\star} = \underline{\mu}_k^\star$, *and* $\overline{\mu}_k^{\star\star} = \overline{\mu}_k^\star$ *are the unique profit maximizing solutions of the profit maximization problem for generator k and demand 0, and*

the prices strictly support economic dispatch,

- *If f_k is convex but not strictly convex then:*

  there may be multiple maximizers,
  $x_k^{\star\star} = x_k^\star$ is consistent with individual profit maximization, and
  the prices non-strictly support economic dispatch.

$\square$

- In some cases, the Lagrange multipliers $\lambda^\star$ and $\mu^\star$ may not be unique:
  - there will be multiple sets of prices that support economic dispatch,
  - we may resort to some criterion such as "fairness" to choose particular values of the Lagrange multipliers.

8.12.5 Spinning reserve re-visited

8.12.5.1 First formulation of spinning reserve constraint

Formulation

- Consider again the first formulation of the spinning reserve constraint from Section 8.12.1.4:

$$\forall k = 1, \ldots, n_P, \sum_{j \neq k} S_j \geq P_k.$$

- We can write the constraint in the form $Cx \leq d$ with:

$$\begin{aligned}
x &= \begin{bmatrix} D \\ P \\ S \end{bmatrix}, \\
C &= \begin{bmatrix}
0 & 1 & 0 & & \cdots & 0 & 0 & -1 & & \cdots & -1 \\
0 & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & -1 & \cdots & -1 & 0 & -1 \\
0 & 0 & \cdots & & 0 & 1 & -1 & \cdots & & -1 & 0
\end{bmatrix}, \\
d &= \mathbf{0}.
\end{aligned}$$

Formulation, continued

- For later use in the pricing rule, we note that for $k = 1, \ldots, n_P$, C_k, the columns of C associated with the variables P_k and S_k, is specified by:

$$C_k = \left[\mathbf{I}_k \quad (\mathbf{I}_k - \mathbf{1}) \right],$$

- where $\mathbf{I}_k$ is column vector with zero everywhere except for a one in the k-th place, and
- $\mathbf{1}$ is a column vector of all ones.
- The column for demand is $C_0 = \mathbf{0}$.

Problem characteristics

- There are as many system inequality constraints as generators.
- We would typically expect than only a few system inequality constraints would be binding at the minimizer.
- If more than one of the spinning reserve constraints is binding then the Lagrange multipliers may be non-unique.

Minimizer and Lagrange multipliers

- Suppose that $x^\star$ is a minimizer of this problem with associated Lagrange multipliers $\lambda^\star \in \mathbb{R}$, $\mu^\star \in \mathbb{R}^{n_P}$, $\underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^{r_k}$.

Pricing rule

- From the previous development, the pricing rule for generator k is:

$$\begin{aligned}
\pi_{x_k} &= \begin{bmatrix} \pi_{P_k} \\ \pi_{S_k} \end{bmatrix}, \\
&= -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star, \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda^\star - \begin{bmatrix} [\mathbf{I}_k]^\dagger \\ (\mathbf{I}_k - \mathbf{1})^\dagger \end{bmatrix} \mu^\star, \\
&= \begin{bmatrix} \lambda^\star - \mu_k^\star \\ \sum_{j \neq k} \mu_j^\star \end{bmatrix}.
\end{aligned}$$

- The pricing rule for demand is:

$$\begin{aligned}
-\pi_{x_0} &= [A_0]^\dagger \lambda^\star + [C_0]^\dagger \mu^\star, \\
&= \lambda^\star, \text{ since } C_0 = \mathbf{0}.
\end{aligned}$$

Interpretation of pricing rule

- To understand, this pricing rule, suppose that generator ℓ generates the most power and that the corresponding constraint $\sum_{j \neq \ell} S_j \geq P_\ell$ is the only binding constraint.
- Therefore, $\mu_\ell^\star$ is the only non-zero Lagrange multiplier on the system inequality constraints.
- All but generator ℓ would be paid for energy at the price $\lambda^\star$:
  - total payment to all generators except generator ℓ for energy would be $\lambda^\star \sum_{j \neq \ell} P_j^\star$.
- All but generator ℓ would be paid for spinning reserves at the price $\mu_\ell^\star$:
  - total payment to all generators except generator ℓ for spinning reserves would be $\mu_\ell^\star \sum_{j \neq \ell} S_j^\star = \mu_\ell^\star P_\ell^\star$.
- Generator ℓ would be paid for energy at the price $(\lambda^\star - \mu_\ell^\star)$:
  - total payment to generator ℓ for energy would be $(\lambda^\star - \mu_\ell^\star) P_\ell^\star$.
- Generation ℓ would receive no payment for spinning reserve.

Interpretation of pricing rule, continued

- Total payment to all generators would be:

$$\begin{aligned}\lambda^\star \sum_{j\neq\ell} P_j^\star + \mu_\ell^\star \sum_{j\neq\ell} S_j^\star + (\lambda^\star - \mu_\ell^\star)P_\ell^\star + 0 &= \lambda^\star \sum_{j\neq\ell} P_j^\star + \mu_\ell^\star P_\ell^\star + (\lambda^\star - \mu_\ell^\star)P_\ell^\star, \\ &= \lambda^\star \sum_{j} P_j^\star, \\ &= \lambda^\star \times D.\end{aligned}$$

- Demand would pay at the price $\lambda^\star$:
  - this price would usually (but not always) be higher than the value obtained in the $\sum_k S_k \geq \alpha D$ formulation of spinning reserve,
  - Total payment by demand would be $\lambda^\star \times D$.
- Payment from demand equals payment to generators and there is no uplift:
  - if spinning reserve (or any other ancillary services) are explicitly represented as depending linearly on generation power $P_1, \ldots, P_{n_P}$ then there is no uplift since total payment by demand equals total payment to generators.

8.12.5.2 Example

- Example 5.6 from Kirschen and Strbac, *Power System Economics*.
- $\forall k = 1, \ldots, n, \underline{P}_k = \underline{S}_k = 0$, and with the other capacities and marginal costs specified by:

$$\overline{P}_1 = 250, \overline{S}_1 = 0, \forall x_1 \in \mathbb{S}_1, \nabla f_1(x_1) = \begin{bmatrix} \$2/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},$$

$$\overline{P}_2 = 230, \overline{S}_2 = 160, \forall x_2 \in \mathbb{S}_2, \nabla f_2(x_2) = \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},$$

$$\overline{P}_3 = 240, \overline{S}_3 = 190, \forall x_3 \in \mathbb{S}_3, \nabla f_3(x_3) = \begin{bmatrix} \$20/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},$$

$$\overline{P}_4 = 250, \overline{S}_4 = 0, \forall x_4 \in \mathbb{S}_4, \nabla f_4(x_4) = \begin{bmatrix} \$28/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},$$

- so that energy costs are non-zero but spinning reserve costs are zero.
- We again consider two levels of demand, $\overline{D} = 300$ and $\overline{D} = 500$ MW.
- We require spinning reserve to cover the largest single contingency, so that the spinning reserve constraints are $\forall k = 1, \ldots, n, \sum_{j \neq k} S_j \geq P_k$.

Example, continued

- It turns out that even with the changed spinning reserve constraints, the dispatch is the same as previously in the two respective cases.
- For $\overline{D} = 300$ MW:
  - Optimal dispatch is $P_1^\star = 250$ MW, $S_1^\star = 0$ MW, $P_2^\star = 50$ MW, $S_2^\star = 60$ MW, $P_3^\star = 0$ MW, $S_3^\star = 190$ MW, $P_4^\star = 0$ MW, $S_4^\star = 0$ MW.
  - Moreover, $\lambda^\star = \$17$/MWh, $\mu_1^\star = \$0$/MWh, and all other Lagrange multipliers on system constraints have value zero.
  - The demand pays and all generation is paid \$17/MWh.
  - Same price for energy as in the $\sum_k S_k \geq \overline{F}$ formulation of spinning reserve.
  - No separate payment by demand for spinning reserve.

- For $\overline{D} = 500$ MW:
  - Optimal dispatch is $P_1^\star = 250$ MW, $S_1^\star = 0$ MW, $P_2^\star = 170$ MW, $S_2^\star = 60$ MW, $P_3^\star = 50$ MW, $S_3^\star = 190$ MW, $P_4^\star = 30$ MW, $S_4^\star = 0$ MW.
  - Moreover, $\lambda^\star = \$28$/MWh, $\mu_1^\star = \$11$/MWh, and all other Lagrange multipliers on system constraints have value zero.
  - The demand pays $\lambda^\star = \$28$/MWh.
  - All generators except generator 1 are paid $\lambda^\star = \$28$/MWh.
  - Generator 1 is paid $(\lambda^\star - \mu_1^\star) = 28 - 11 = \$17$ /MWh.
  - Same price for energy as in the $\sum_k S_k \geq \overline{F}$ formulation of spinning reserve for demand and for all generators except generator 1.
  - No separate payment by demand for spinning reserve.
  - Energy payment to generator 1 discounted by marginal cost of providing spinning reserve.

8.12.5.3 *Pricing rules*

- Each pricing rule derives from the formulation of the corresponding constraint.
- With the $\forall k = 1, \ldots, n_P, \sum_{j \neq k} S_j \geq P_k$ formulation of the spinning reserve constraint:
  - If a particular constraint $\sum_{j \neq \ell} S_j \geq P_\ell$ is binding for some ℓ then each additional unit of power produced by generator ℓ delivers energy to the system, but increases the amount of spinning reserve that must be procured.
  - The price paid to generator ℓ reflects the benefit of the energy minus the cost of increased spinning reserve.
- Note that the different formulations of spinning reserve constraints can involve different requirements for spinning reserve:
  - It is not surprising that changing the formulation of constraints could result in a different dispatch and that this would result in possibly different prices.

8.12.5.4 Pricing rules, continued

- The example shows, however, that even if a change in formulation does *not* result in a change in dispatch, nevertheless the prices can change:
  - the prices reflect the incentives to behave consistently with optimal dispatch.
- Why care about differences in the prices if the dispatch does not change?
  - the prices provide efficient incentives for *both* operation and investment "at the margin,"
  - generator 1 in the example should be paid less than the "base" energy price of $\lambda^\star$ because a marginal increase in its generation capacity would necessitate increased spinning reserve procurement, providing less value to the system than a marginal increase in the capacity of other generators,
  - the large generator 1 should only consider increasing its capacity if the amortized cost of increased capacity is less than \$17/MWh; paying the generator \$28/MWh would fail to reflect the cost of the spinning reserve requirement.

8.12.5.5 Pricing rules, continued

- Formulation of constraints determines the pricing rule!
  - In practice, spinning reserve and other ancillary services are usually charged on (something like) a load-weighted average share of demand:
    - equivalent total payment by demand as in the $\sum_k S_k \geq \alpha D$ formulation of the spinning reserve constraint.
  - Load-weighted average share pricing based on Lagrange multipliers provides the correct incentive for ancillary services that are actually required in proportion to demand.
  - For ancillary services that are not required in proportion to demand, load-weighted average shares are simply an uplift allocation mechanism to achieve revenue neutrality for the ISO:
    - typically motivated by "fairness,"
    - but have no particular claim to "correctness,"
    - typically do not provide incentives for optimal operation and investment, as shown for spinning reserve and the large generator,
    - implicit assumption is that the effect of uplift on energy prices does not distort consumption decisions.

8.12.5.6 *Uplift*

- A system constraint of the form $\sum_k S_k \geq \overline{F}$ necessitates an uplift.
- A system constraint of the form $\sum_k S_k \geq \alpha D$ does not require uplift:
  - may result in the same net payment by demand,
  - in which case the main difference is the interpretation.
- More generally:
  - system constraints of the form $Ax = b$ and $Cx \leq d$, with $b \neq \mathbf{0}$ or $d \neq \mathbf{0}$ will require an uplift (or produce a surplus), whereas,
  - system constraints that can be expressed in the form $Ax = \mathbf{0}$ and $Cx \leq \mathbf{0}$ do not require uplift nor do they produce surplus.

Theorem 8.2 *Let $\lambda^\star$ and $\mu^\star$ be Lagrange multipliers on the system equality and inequality constraints for the generalized offer-based economic dispatch problem with demand represented explicitly as part of the decision vector:*

$$\min_{\forall k=0,\ldots,n_P, x_k \in \mathbb{S}_k} \{ f(x) | Ax = b, Cx \leq d \}$$
$$= \min_{x \in \mathbb{R}^N} \{ f(x) | Ax = b, Cx \leq d, \forall k = 0, \ldots, n_P, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \}.$$

Define prices paid to generators $k = 1, \ldots, n_P$ to be:

$$\pi_{x_k} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star.$$

Define prices paid by demand to be:

$$-\pi_{x_0} = [A_0]^\dagger \lambda^\star + [C_0]^\dagger \mu^\star.$$

- *Then the uplift (or surplus if negative) is equal to $-b^\dagger \lambda^\star - d^\dagger \mu^\star$.*
- *The uplift is zero if $b = \mathbf{0}$ and $d = \mathbf{0}$.*

Proof The uplift is equal to the total payment to the generators minus the payment by demand, which is:

$$
\begin{aligned}
\sum_{k=0}^{n_P} [\pi_{x_k}]^\dagger x_k^\star &= \sum_{k=0}^{n_P} \left[-[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star \right]^\dagger x_k^\star, \\
&= \sum_{k=0}^{n_P} \left[-[\lambda^\star]^\dagger A_k - [\mu^\star]^\dagger C_k \right] x_k^\star, \\
&= \left[-[\lambda^\star]^\dagger A - [\mu^\star]^\dagger C \right] x^\star, \\
&\quad \text{re-assembling the entries of } x \text{ into a single vector,} \\
&= -[\lambda^\star]^\dagger A x^\star - [\mu^\star]^\dagger C x^\star, \\
&= -[\lambda^\star]^\dagger b - [\mu^\star]^\dagger d, \text{ since } Ax^\star = b, \text{ and} \\
&\quad \text{since } [\mu^\star]^\dagger (Cx^\star - d) = 0 \text{ by complementary slackness.}
\end{aligned}
$$

□

8.12.6 Non-linear system constraints

8.12.6.1 Formulation

- We generalize the system constraints to:

$$\begin{aligned} g(x) &= \mathbf{0}, \\ h(x) &\leq \mathbf{0}, \end{aligned}$$

- where $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^r$.
- The non-linear generalized economic dispatch problem is now:

$$\begin{aligned} &\min_{\forall k=0,\ldots,n_P+1, x_k \in \mathbb{S}_k} \{f(x) | g(x) = \mathbf{0}, h(x) \leq \mathbf{0}\} \\ &= \min_{x \in \mathbb{R}^n} \{f(x) | g(x) = \mathbf{0}, h(x) \leq \mathbf{0}, \forall k = 0, \ldots, n_P + 1, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}. \end{aligned}$$

Minimizer

- Suppose that $x^\star \in \mathbb{R}^n$ minimizes the economic dispatch problem and is a regular point of the constraints $g(x) = \mathbf{0}, h(x) \leq \mathbf{0}$.

First-order necessary conditions

$\exists \lambda^\star \in \mathbb{R}^m, \exists \mu^\star \in \mathbb{R}^r, \forall k = 0, \ldots, n_P + 1, \exists \underline{\mu}_k^\star, \overline{\mu}_k^\star \in \mathbb{R}^{r_k}$ such that:

$$
\begin{aligned}
\forall k, \nabla f_k(x_k^\star) + \left[\frac{\partial g}{\partial x_k}(x^\star)\right]^\dagger \lambda^\star + \left[\frac{\partial h}{\partial x_k}(x^\star)\right]^\dagger \mu^\star - [\Gamma_k]^\dagger \underline{\mu}_k^\star + [\Gamma_k]^\dagger \overline{\mu}_k^\star &= \mathbf{0}; \\
M^\star h(x^\star) &= \mathbf{0}; \\
\forall k = 0, \ldots, n_P + 1, \underline{M}_k^\star(\underline{\delta}_k - \Gamma_k x_k^\star) &= \mathbf{0}; \\
\forall k = 0, \ldots, n_P + 1, \overline{M}_k^\star(\Gamma_k x_k^\star - \overline{\delta}_k) &= \mathbf{0}; \\
g(x^\star) &= \mathbf{0}; \\
h(x^\star) &\leq \mathbf{0}; \\
\forall k = 0, \ldots, n_P + 1, \Gamma_k x_k^\star &\geq \underline{\delta}_k; \\
\forall k = 0, \ldots, n_P + 1, \Gamma_k x_k^\star &\leq \overline{\delta}_k; \\
\mu^\star &\geq 0; \\
\underline{\mu}_k^\star &\geq \mathbf{0}; \text{ and} \\
\overline{\mu}_k^\star &\geq \mathbf{0},
\end{aligned}
$$

First-order necessary conditions

- where $M^\star = \mathrm{diag}\{\mu^\star\} \in \mathbb{R}^{r \times r}$, $\underline{M}_k^\star = \mathrm{diag}\{\underline{\mu}_k^\star\} \in \mathbb{R}^{r_k \times r_k}$, and $\overline{M}_k^\star = \mathrm{diag}\{\overline{\mu}_k^\star\} \in \mathbb{R}^{r_k \times r_k}$ are diagonal matrices with entries specified by the entries of $\mu^\star$, $\underline{\mu}_k^\star$, and $\overline{\mu}_k^\star$, respectively.

8.12.6.2 Pricing rule for general non-linear system constraints

- To find the pricing rule, note that in the first-order necessary conditions, $\frac{\partial g}{\partial x_k}(x^\star)$ and $\frac{\partial h}{\partial x_k}(x^\star)$ have the same roles as, respectively, A_k and C_k, in the previous formulation.
- This suggests prices π_{x_k} paid to generators defined by:

$$\pi_{x_k} = -\left[\frac{\partial g}{\partial x_k}(x^\star)\right]^\dagger \lambda^\star - \left[\frac{\partial h}{\partial x_k}(x^\star)\right]^\dagger \mu^\star.$$

- and prices paid by demand defined by:

$$-\pi_{x_0} = \left[\frac{\partial g}{\partial x_0}(x^\star)\right]^\dagger \lambda^\star + \left[\frac{\partial h}{\partial x_0}(x^\star)\right]^\dagger \mu^\star.$$

- Comparison of the first-order conditions for the profit maximization problem for generator $k = 1, \ldots, n_P$, and for demand $k = 0$ confirms that these prices will induce behavior that is consistent with economic dispatch.

Pricing rule for general non-linear system constraints, continued

- Unlike the case of linear constraints, however, non-linear system constraints will generally necessitate an uplift or generate a surplus.

8.12.6.3 Pricing theorem

Theorem 8.3 *Let $\lambda^\star$ and $\mu^\star$ be Lagrange multipliers on the system equality and inequality constraints for the non-linear offer-based economic dispatch problem with demand represented explicitly as part of the decision vector:*

$$\begin{aligned}
&\min_{\forall k=0,\ldots,n_P+1, x_k \in \mathbb{S}_k} \{f(x) | g(x) = \mathbf{0}, h(x) \leq \mathbf{0}\} \\
&= \min_{x \in \mathbb{R}^N} \{f(x) | g(x) = \mathbf{0}, h(x) \leq \mathbf{0}, \forall k = 0, \ldots, n+1, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}.
\end{aligned}$$

Define prices paid to generators $k = 1, \ldots, n_P$ by:

$$\pi_{x_k} = -\left[\frac{\partial g}{\partial x_k}(x^\star)\right]^\dagger \lambda^\star - \left[\frac{\partial h}{\partial x_k}(x^\star)\right]^\dagger \mu^\star.$$

Define prices paid by demand by:

$$-\pi_{x_0} = \left[\frac{\partial g}{\partial x_0}(x^\star)\right]^\dagger \lambda^\star + \left[\frac{\partial h}{\partial x_0}(x^\star)\right]^\dagger \mu^\star.$$

- *If f_k is strictly convex then:*

  $x_k^{\star\star} = x_k^\star$, $\underline{\mu}_k^{\star\star} = \underline{\mu}_k^\star$, *and* $\overline{\mu}_k^{\star\star} = \overline{\mu}_k^\star$ *are the unique profit maximizing solutions of the profit maximization problem for generator* $k = 1, \ldots, n_P$, *and demand* $k = 0$ *and*

  the prices strictly support economic dispatch,

- *If f_k is convex but not strictly convex then:*

  there may be multiple maximizers,

  $x_k^{\star\star} = x_k^\star$ *is consistent with optimal dispatch, and*

  the prices non-strictly support economic dispatch.

- *The uplift (or surplus if negative) is equal to:*

$$
\begin{aligned}
&[\lambda^\star]^\dagger \left[g(x^\star) - \frac{\partial g}{\partial x}(x^\star) x^\star \right] + [\mu^\star]^\dagger \left[h(x^\star) - \frac{\partial h}{\partial x}(x^\star) x^\star \right] \\
&\qquad = \left[-[\lambda^\star]^\dagger \frac{\partial g}{\partial x}(x^\star) - [\mu^\star]^\dagger \frac{\partial h}{\partial x}(x^\star) \right] x^\star .
\end{aligned}
$$

Proof The incentive results follow from the previous discussion. The uplift is equal to the total payment to the generators minus the payment by demand, which is:

$$
\begin{aligned}
\sum_{k=0}^{n_P} [\pi_{x_k}]^\dagger x_k^\star &= \sum_{k=0}^{n_P} \left[-\left[\frac{\partial g}{\partial x_k}(x^\star)\right]^\dagger \lambda^\star - \left[\frac{\partial h}{\partial x_k}(x^\star)\right]^\dagger \mu^\star \right]^\dagger x_k^\star, \\
&= \sum_{k=0}^{n_P} \left[-[\lambda^\star]^\dagger \frac{\partial g}{\partial x_k}(x^\star) - [\mu^\star]^\dagger \frac{\partial h}{\partial x_k}(x^\star) \right] x_k^\star, \\
&= \left[-[\lambda^\star]^\dagger \frac{\partial g}{\partial x}(x^\star) - [\mu^\star]^\dagger \frac{\partial h}{\partial x}(x^\star) \right] x^\star,
\end{aligned}
$$

re-assembling the entries of x into a single vector.

That is:

$$\begin{aligned}
\sum_{k=0}^{n_P} [\pi_{x_k}]^\dagger x_k^\star &= \left[-[\lambda^\star]^\dagger \frac{\partial g}{\partial x}(x^\star) - [\mu^\star]^\dagger \frac{\partial h}{\partial x}(x^\star) \right] x^\star, \\
&= -[\lambda^\star]^\dagger \left[g(x^\star) + \frac{\partial g}{\partial x}(x^\star) x^\star - g(x^\star) \right] \\
&\qquad -[\mu^\star]^\dagger \left[h(x^\star) + \frac{\partial h}{\partial x}(x^\star) x^\star - h(x^\star) \right], \\
&\qquad \text{adding and subtracting terms,} \\
&= -[\lambda^\star]^\dagger \left[\frac{\partial g}{\partial x}(x^\star) x^\star - g(x^\star) \right] - [\mu^\star]^\dagger \left[\frac{\partial h}{\partial x}(x^\star) x^\star - h(x^\star) x^\star \right], \\
&\qquad \text{since } g(x^\star) = \mathbf{0}, \text{ and} \\
&\qquad \text{since } [\mu^\star]^\dagger h(x^\star) = 0 \text{ by complementary slackness,} \\
&= [\lambda^\star]^\dagger \left[g(x^\star) - \frac{\partial g}{\partial x}(x^\star) x^\star \right] + [\mu^\star]^\dagger \left[h(x^\star) - \frac{\partial h}{\partial x}(x^\star) x^\star \right].
\end{aligned}$$

□

- The first page of the proof shows that we could have also evaluated the uplift as:

$$\left[-[\lambda^\star]^\dagger \frac{\partial g}{\partial x}(x^\star) - [\mu^\star]^\dagger \frac{\partial h}{\partial x}(x^\star) \right] x^\star.$$

- However, the form we derived for the uplift:

$$[\lambda^\star]^\dagger \left[g(x^\star) - \frac{\partial g}{\partial x}(x^\star)x^\star \right] + [\mu^\star]^\dagger \left[h(x^\star) - \frac{\partial h}{\partial x}(x^\star)x^\star \right],$$

  highlights that the *non-linearity* of the functions underlies the uplift.

- In particular, if, g and h were linear functions then $\forall x \in \mathbb{R}^n, g(x) = \frac{\partial g}{\partial x}(x)x$ and $\forall x \in \mathbb{R}^n, h(x) = \frac{\partial h}{\partial x}(x)x$, and the uplift would be zero.
- Note that *affine* g and h will result in uplift or surplus as in the previous Theorem 8.2 that considered constraints of the form $Ax = b$ and $Cx \leq d$.

8.12.6.4 Role of linearization

- Note that the derivatives of the functions g and h that represent the system constraints appear in the pricing rule.
- When we approximate the functional form of a constraint function by *neglecting* its dependence on a system variable, we are approximating its derivative by zero and we are neglecting the corresponding term in the pricing rule:
  - in the “fixed” requirements $\sum_k S_k \geq \overline{F}$ form of the spinning reserve constraint, we are pretending that the required amount of reserves is constant independent of the values of generation and demand,
  - but the required amount of reserves actually do depend on the amount of generation or the amount of demand, or both,
  - so the $\sum_k S_k \geq \overline{F}$ form of the spinning reserve constraint provides the wrong incentives (and also necessitates an uplift).

Role of linearization, continued

- Approximations of the derivatives of the functions representing the system constraints will distort the incentives away from inducing behavior that is consistent with economic dispatch, or away from inducing efficient investment, or both.
- We will see this issue in the representation of zonal transmission constraints in the zonal ERCOT market.

8.12.6.5 Example

Regulation

- As another example of ancillary services, consider **regulation**, which is generation capacity available to provide for the deviation of:
  - actual demand from the short-term forecast that is used as input to dispatch, and
  - actual generation from short-term forecast or schedule or dispatch level.
- Providing regulation requires the ability to respond to:
  - short-term frequency variation, and
  - signals from the ISO.
- We write R_k for the regulation provided by generator k.
- A typical requirement for regulation is to have enough regulating capacity to cope with three times the standard deviation of the difference between:
  - the actual demand, minus
  - the actual generation that is not participating in regulation.
- The statistics are calculated for the offer-based economic dispatch market having the finest time resolution (the **real-time market**).

Regulation, continued

- Most thermal and hydro generation is dispatchable so that the actual mechanical power of thermal and hydro generation closely follows the generation level that is dispatched in the offer-based economic dispatch (plus any commands for regulation):
  - dispatchable generation does not contribute to the variation of demand minus generation, so does not contribute to the requirements for regulation,
  - dispatchable generation is able to provide regulation.
- For non-dispatchable generation such as wind, however, there may be differences between the actual generation and the short-term forecast or the short-term schedule that is used in the offer-based economic dispatch.
- There will also be differences between the actual demand and the short-term forecast of demand that is used in the offer-based economic dispatch.
- Consequently, the deviation of actual demand from actual generation is due to non-dispatchable generation and to demand.

Model of short-term wind and demand variability

- The short-term variability of wind at one wind farm is typically independent of the short-term variability of wind at another wind farm.
- Similarly, the short-term variability of demand at one location is typically independent of the short-term variability of demand at another location.
- By **central limit theorem** arguments, it may be reasonable to suppose that the short-term variance of wind and the short-term variance of demand are proportional to the amount of wind production and to the amount of demand, respectively:
  - we will ignore dispatchable wind and demand.
- Moreover, short-term variability of demand and wind are independent, so their variances add.
- Therefore, the standard deviation of short-term actual demand minus short-term actual wind generation can be represented as:

$$\sigma = \sqrt{\beta_D D + \beta_W W},$$

- where β_D and β_W are constants, and
- where W is the total forecast wind generation.

Generator decision variables and constraints

- We will neglect spinning reserve in this formulation, but can be included.
- We will assume that the constraints on generation and regulation are of the form:

$$\mathbb{S}_k = \{x_k \in \mathbb{R}^2 | \underline{P}_k \leq x_k \leq \overline{P}_k, 0 \leq R_k \leq \overline{R}_k, \underline{P}_k \leq P_k - R_k, P_k + R_k \leq \overline{P}_k\},$$

- with wind generators assumed to have $\overline{R}_k = 0$ and where $\overline{P}_k$ should be interpreted as a forecast of the maximum production that is possible from the wind farm given the prevailing wind conditions.
- That is, the generator constraints are analogous to the case for spinning reserve, except that offered regulation R_k is assumed to be available for both increasing and decreasing generation compared to P_k:
  - in ERCOT, regulation offers are separated into “regulation up” and “regulation down,”
  - constraints are therefore slightly different in the ERCOT formulation.
- We will assume that generators 1 to n_W are wind generators, while generators $n_W + 1$ to n_P are dispatchable:
  - so $\overline{R}_k = 0, k = 1, \ldots, n_W$.

Variability of demand and non-dispatchable resources

- The total wind is $W = \sum_{k=1}^{n_W} P_k$.
- The standard deviation of short-term actual demand minus short-term actual wind generation is:

$$\sigma = \sqrt{\beta_D D + \beta_W \sum_{k=1}^{n_W} P_k}.$$

System constraints

- We consider a requirement that we have enough regulation to cover three standard deviations of the variation of demand minus wind generation:
  - requirement is cited in several wind integration studies, but
  - does not fully reflect requirements for regulation in North American Electric Reliability Corporation standards.
- As in the case of spinning reserve, North American markets currently represent regulation as a notionally “fixed” requirement for regulation:

$$\sum_{k=1}^{n_P} R_k \geq \overline{G},$$

- where $\overline{G} = 3\sigma$.
- Analysis of this representation is very similar to the “fixed” requirement formulation for spinning reserve from Section 8.12.1.4:
  - we will not consider this formulation explicitly.

System constraints

- Instead, we will consider a formulation that explicitly considers the dependence of requirements on demand and wind generation:
  - not used in practice in North America, but
  - demonstrates issues with non-linear constraints and uplift.
- We consider system constraints on power balance and regulation:

$$
\begin{aligned}
D - \sum_{k=1}^{n_P} P_k &= 0, \\
\sum_{k=1}^{n_P} R_k &\geq 3 \times \sqrt{\beta_D D + \beta_W \sum_{k=1}^{n_W} P_k},
\end{aligned}
$$

which we can express in the form $g(x) = \mathbf{0}$ and $h(x) \leq \mathbf{0}$ by defining $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ by: $\forall x \in \mathbb{R}^n, g(x) = D - \sum_{k=1}^{n_P} P_k$, and

$$
\forall x \in \mathbb{R}^n, h(x) = \left(3 \times \sqrt{\beta_D D + \beta_W \sum_{k=1}^{n_W} P_k}\right) - \sum_{k=1}^{n_P} R_k.
$$

System constraints, continued

- Note that we could also write the equality constraints in the form $Ax = \mathbf{0}$, with A a row vector with:
  - minus ones in the places corresponding to the generations P_k and a plus one in the place representing the demand, and
  - zeros elsewhere.
- Analogously to the discussion of reserves, we have explicitly considered the dependence of the need for regulation on:
  - the demand variability, and
  - the non-dispatchable generation variability.
- As previously, we should expect this formulation to result in different prices compared to a formulation that assumed a *fixed* regulation requirement that was independent of demand and generation:
  - as mentioned, ERCOT formulation in practice assumes a “fixed” regulation requirement that necessitates uplift of all of the cost of regulation.

Offer-based regulation-constrained economic dispatch

- We consider the following problem:

$$\min_{\forall k=0,\ldots,n_P, x_k \in \mathbb{S}_k} \{f(x) | g(x) = \mathbf{0}, h(x) \leq \mathbf{0}\}$$
$$= \min_{x \in \mathbb{R}^n} \{f(x) | g(x) = \mathbf{0}, h(x) \leq \mathbf{0}, \forall k = 0, \ldots, n_P, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}.$$

- Let $x^\star \in \mathbb{R}^n$ be a minimizer.
- We let $\lambda^\star \in \mathbb{R}$ and $\mu^\star$ be the Lagrange multipliers on the constraints $g(x) = \mathbf{0}$ and $h(x) \leq \mathbf{0}$, respectively.

Pricing rule

- Demand pays in total:

$$
\begin{aligned}
[-\pi_{x_0}]^{\dagger} x_0 &= \left(\left[\frac{\partial g}{\partial x_0}(x^{\star}) \right]^{\dagger} \lambda^{\star} + \left[\frac{\partial h}{\partial x_0}(x^{\star}) \right]^{\dagger} \mu^{\star} \right) D \\
&= \left(\lambda^{\star} + \frac{3\beta_D}{2\sqrt{\beta_D(-x_0^{\star}) + \beta_W \sum_{k=1}^{n_W} P_k^{\star}}} \times \mu^{\star} \right) D.
\end{aligned}
$$

- That is, demand pays for energy and pays for regulation based on its *marginal* contribution to the offered cost of providing energy and regulation, respectively.

Pricing rule

- Dispatchable generation is paid:

$$\begin{aligned}\forall k = n_W + 1, \ldots, n, [\pi_{x_k}]^\dagger x_k &= \left[- \left[\frac{\partial g}{\partial x_k}(x^\star) \right]^\dagger \lambda^\star - \left[\frac{\partial h}{\partial x_k}(x^\star) \right]^\dagger \mu^\star \right]^\dagger x_k, \\ &= \lambda^\star P_k + \mu^\star R_k.\end{aligned}$$

- That is, dispatchable generation is paid for energy and regulation.
- Non-dispatchable generation is paid:

$$\begin{aligned}\forall k = 1, \ldots, n_W, [\pi_{x_k}]^\dagger x_k &= \left[- \left[\frac{\partial g}{\partial x_k}(x^\star) \right]^\dagger \lambda^\star - \left[\frac{\partial h}{\partial x_k}(x^\star) \right]^\dagger \mu^\star \right]^\dagger x_k, \\ &= \left(\lambda^\star - \frac{3\beta_W}{2\sqrt{\beta_D(-x_0^\star) + \beta_W \sum_{k=1}^{n_W} P_k^\star}} \times \mu^\star \right) P_k.\end{aligned}$$

- That is, non-dispatchable generation is paid for energy, but pays for regulation.

Uplift

- Because the power balance constraint is linear and of the form $Ax = \mathbf{0}$, there is no uplift for energy.
- However, the sum of the payments for regulation by demand and by non-dispatchable generation does not equal the payment for regulation to the dispatchable generation;
  - there is still a need for an uplift because of the non-convexity of the non-linear system inequality constraint, but the uplift is smaller than if we had a "fixed" regulation requirement:

$$\overline{G} = 3 \times \sqrt{\beta_D D + \beta_W \sum_{k=1}^{n_W} P_k^{\star}}.$$

  - conversely, *convex* non-linear system inequality constraints result in surplus accruing to the ISO.
- What would the energy and regulation payments to dispatchable generation, to non-dispatchable generation, and from demand be in a formulation where regulation was required to be at least a fixed value $\overline{G}$? What would the uplift be?

8.12.7 Representation of constraints

- In formulating constraints, we should consider what is being represented:
  - a physical law that cannot be broken, such as Kirchhoff's laws, versus
  - a target requirement that could, in principle, be violated.
- The formulations so far have represented all constraints as though the constraints must be satisfied:
  - such a constraint is called a **hard constraint**.
- We will also develop an alternative representation:
  - a **soft constraint** representing a target that can be violated under some circumstances.

8.12.7.1 Reserves

- Market processes can deal with *some* randomness:
  - as will be discussed in Section 10.10, real-time markets deal with deviation of actual load from day-ahead specification by setting a real-time price payable on deviations from day-ahead positions.
  - The greater the participation of price-responsive supply and demand in the real-time market, the more randomness can be accommodated by adjustments in the real-time market by response to prices.
- However, because of the need to match supply and demand continuously and because of random failures, market processes cannot deal (directly) with all randomness in sufficient time to ensure security.
- Reserves and other ancillary services are required for supply and (to a lesser extent) demand fluctuations that occur too fast for the market to respond.
  - Forced outage of generator,
  - Wind die-off.

8.12.7.2 Hard constraints

- The formulations so far have all involved hard constraints:
  - For example, the spinning reserve was required to meet or exceed a specific level.
  - This is an example of a **security constraint**, which is enforced so that at each time we have enough spinning reserve so that we will not get into an operating state that could lead to cascading outages.
- However, reserve above the minimum needed to cope with all single-contingencies (and common-mode double-contingencies) has a role in assuring **adequacy**:
  - hard constraints may not be an appropriate representation for adequacy reserve.

8.12.7.3 Soft constraints

- Soft constraints are appropriate when there is some flexibility in meeting the constraint or where there is some implicit trade-off between:
  - meeting a constraint, and
  - the finite cost (or expected cost) of not meeting the constraint.
- Examples include:
  - constraints that instantiate rules of thumb, and
  - adequacy constraints, where we want to reduce, to some acceptable level, the probability of needing to curtail demand in order to maintain security.

8.12.7.4 Representation of soft constraints

- How to represent soft constraints?
  - Relax a constraint by allowing violation at some assumed **penalty cost**.
  - Penalty cost could vary with the level of violation.
- In practice, all constraints are represented in software implementations as soft constraints using a high penalty cost for violation:
  - notionally “hard constraints” have very high penalty costs,
  - ensures that software will be able to find a “feasible” solution,
  - penalties are different for different types of constraints in order to encourage a particular order of violation of constraints,
  - magnitudes vary widely from ISO to ISO.

8.12.7.5 Representation of adequacy reserve

- Consider reserve for adequacy and assume that we procure an amount of adequacy reserve specified by $\overline{F}^{\text{adequacy}}$
- The corresponding hard constraint would be:

$$\sum_{k=1}^{n_P} S_k^{\text{adequacy}} \geq \overline{F}^{\text{adequacy}},$$

- where S_k^{adequacy} is the contribution to adequacy reserve by the k-th generator:
  - this type of reserve would be in addition to spinning reserve discussed earlier, but we will ignore spinning reserve for now,
  - typical requirements are that the reserve can be deployed within approximately 30 minutes, so could either be in-service capacity (including capacity that could otherwise be used as spinning reserve) or out-of-service capacity that could be committed and dispatched within 30 minutes.
- However, we recognize that violating this requirement for adequacy reserve does not necessarily lead to an immediate violation of security.

Representation of adequacy reserve, continued

- Instead of representing the constraint as a hard constraint, we could:
  - include an extra variable S_0^{adequacy}, which represents the shortfall of adequacy reserve,
  - modify the constraint to:

$$\sum_{k=0}^{n_P} S_k^{\text{adequacy}} \geq \overline{F}^{\text{adequacy}},$$

  - include a non-negativity requirement $S_0^{\text{adequacy}} \geq 0$, and
  - add a penalty term to the objective $f_0^{\text{adequacy}}(S_0^{\text{adequacy}})$, with $f_0^{\text{adequacy}}(0) = 0$.
- The penalty term could be linear in S_0^{adequacy}, implying a fixed marginal penalty for each unit of violation:

$$\forall S_0^{\text{adequacy}}, f_0^{\text{adequacy}}(S_0^{\text{adequacy}}) = c_0^{\text{adequacy}} S_0^{\text{adequacy}}.$$

- Alternatively, it could be non-linear or piece-wise linear, with higher marginal penalty for greater violation.

Representation of adequacy reserve, continued

- Solving the penalty formulation may result in the original hard adequacy reserve constraint being violated.
- That is, the optimal solution will satisfy $S_0^{\text{adequacy}\star} > 0$, and the Lagrange multiplier on the reserve constraint will be equal to:
  - for the linear cost case, c_0^{adequacy},
  - the derivative of the penalty function evaluated at $S_0^{\text{adequacy}\star}$ in the general case.
- The penalty cost c_0^{adequacy} will be reflected in the prices of all commodities, unless a so-called **decontamination** procedure is used to remove the effects of the penalty.

Representation of adequacy reserve, continued

- The actually procured adequacy reserve is given by:

$$F^{\text{adequacy}} = \overline{F}^{\text{adequacy}} - S_0^{\text{adequacy}}.$$

- We can think of F^{adequacy} as the **demand for adequacy reserve**.
- Using this interpretation, we can view the soft adequacy reserve constraint as involving:
  - a supply-demand constraint between procured adequacy reserve and the "demand for adequacy reserve" F^{adequacy}:

$$-\sum_{k=1}^{n_P} S_k^{\text{adequacy}} = -F^{\text{adequacy}},$$

  - together with a "benefit" for adequacy reserve defined by:

$$\text{benefit}(F^{\text{adequacy}}) = -f_0^{\text{adequacy}}(\overline{F}^{\text{adequacy}} - F^{\text{adequacy}}).$$

- As usual, offer-based economic dispatch then involves minimizing costs minus benefits, where the benefits are due to adequacy reserve.

Representation of adequacy reserve, continued

- In case of a linear function f_0^{adequacy}, how should we choose c_0^{adequacy}?
- Compromise:
  - Typically want c_0^{adequacy} larger than the highest offer price for adequacy reserve so that whenever enough adequacy reserve is actually available then it will be procured.
  - However, if the penalty is “too large” then it will produce unreasonably high commodity prices in the market.
- The choice of c_0^{adequacy} should be a proxy to the cost of violating the constraint.
- The use of a penalty for violation of a soft constraint effectively transforms the constraint into a term in the objective, as in dualizing the constraint:
  - as mentioned, this approach is also used in practice even for hard constraints, but with very high penalties.

8.12.7.6 Evaluation of proxy to cost of violating constraint

- What is the cost (reduction in surplus) of violating an adequacy reserve constraint?
- The cost is due to the increase in probability that involuntary curtailment will be necessary to maintain security.
- The change in surplus due to involuntary curtailment is equal to the difference between:

  the value of the lost load, minus
  the savings from not generating.

- Generally, savings from not generating are small compared to the value of lost load.
- So, the cost is approximately VOLL $\times$ expected energy curtailment.

Expected load curtailment

- How to calculate the expected energy curtailed?
  - Involves outage probabilities of each in-service generator,
  - Requires specification of conditions for load curtailment.
- We will sketch an approximation to this analysis.
- It will lead to a re-formulation of adequacy reserve constraint as a **demand bid for adequacy reserve**.

Conditions for curtailment

- What happens if a generator providing energy trips?
  - Spinning reserve is deployed.
- What happens when spinning reserve is deployed or a generator providing spinning reserve trips?
  - Other reserve is deployed to allow restoration of the spinning reserve to be available again for security with respect to the next generator trip.
  - Since the other reserve will take some time to be deployed, the system may be not secure to an additional contingency during this time, but implicit assumption is that multiple outages will not occur in succession.
- The other reserve is providing adequacy:
  - ensuring low probability of curtailing demand to maintain security.
- How much outage capacity $\overline{P}^{\text{outage}}$ can we sustain in a pricing interval of length T before we have to curtail load to preserve security?
  - Until the total outaged capacity exceeds the amount of reserve that was procured for adequacy.
  - Denote the total procured adequacy reserve by F^{adequacy}.

Probability of curtailment

- The probability of curtailment being necessary in any given interval of length T is the probability that the total outage capacity in the interval, $\overline{P}^{\text{outage}}$, exceeds the amount of reserve that was procured for adequacy in that interval, F^{adequacy}.
- That is, the probability of curtailment is $\text{Probability}(\overline{P}^{\text{outage}} \geq F^{\text{adequacy}})$.
- The outage probability distribution depends on the the outage characteristics of generators, total in-service capacity, T, and other system conditions.
- The total outage is a discrete or mixed random variable.
- The resulting amount of curtailed load power is equal to $\min\{D, \max\{0, \overline{P}^{\text{outage}} - F^{\text{adequacy}}\}\}$:
  - cannot curtail more than the demand D,
  - curtailment is at least zero,
  - only curtail an amount of demand that is necessary to maintain security, which occurs only if the amount of outage capacity $\overline{P}^{\text{outage}}$ exceeds the amount of adequacy reserve F^{adequacy}.

Probability of curtailment, continued

- In general, once curtailed, the load cannot be restored to service until some time has elapsed, even if the supply becomes adequate again to support all of the load securely.
- Let the load curtailment time be τT, where τ is the number of dispatch intervals of curtailment:
  - we expect $\tau > 1$,
  - for example, if $T = 5$ minutes and minimum curtailment time is 30 minutes, then $\tau = 6$,
  - effect of curtailment is effectively "magnified" by τ.

Probability of curtailment, continued

- In some cases, the probability distribution of outage capacity $\overline{P}^{\text{outage}}$ can be approximated by an **exponential distribution** with parameters that depend only weakly on system conditions:

$$\text{Probability}(\overline{P}^{\text{outage}} \geq y) \approx a_0 \exp(-y/M), y > 0.$$

- where a_0 and M depend only weakly on system conditions.
- The parameters a_0 and M do depend on T.
- Note that a_0 is the probability of occurrence of an outage, which is typically much smaller than 1:
  - in ERCOT, the forced outage rate is about once per two days, so the probability of an outage in an interval of length $T = 5$ minutes is approximately 0.0017.

Expected energy curtailed

- The probability density function of the amount of outage capacity $\overline{P}^{\text{outage}}$ is $(a_0/M)\exp(-y/M)$.
- So, the expected curtailed power is:

$$
\begin{aligned}
&\text{Expectation}[\min\{D, \max\{0, \overline{P}^{\text{outage}} - F^{\text{adequacy}}\}\}] \\
&= \int_{y=F^{\text{adequacy}}}^{D+F^{\text{adequacy}}} (y - F^{\text{adequacy}})(a_0/M)\exp(-y/M)\,dy, \\
&= \left[-(y - F^{\text{adequacy}})a_0\exp(-y/M)\right]_{y=F^{\text{adequacy}}}^{D+F^{\text{adequacy}}} \\
&\qquad + \int_{y=F^{\text{adequacy}}}^{D+F^{\text{adequacy}}} a_0\exp(-y/M)\,dy, \\
&\qquad \text{integrating by parts,} \\
&= ((D-M)\exp(-D/M)+M)a_0\exp(-F^{\text{adequacy}}/M).
\end{aligned}
$$

- The expected curtailed energy is τT times this curtailed power.

Adequacy reserve cost

- Recall that the expected cost of curtailment associated with a given level of adequacy reserve is:

$$\begin{aligned} &\text{VOLL} \times \text{ expected energy curtailed} \\ &= \text{VOLL} \times ((D-M)\exp(-D/M)+M)a_0\exp(-F^{\text{adequacy}}/M)\tau T. \end{aligned}$$

- Recall that in our dispatch problem, we considered costs per unit time, so dividing by T we obtain the expected cost per unit time of curtailment as:

$$\tau\text{VOLL} \times ((D-M)\exp(-D/M)+M)a_0\exp(-F^{\text{adequacy}}/M),$$

  - note that the cost is "amplified" by the length of curtailment.
- If we include this cost term (or an approximation to it) in the dispatch objective then the ISO will procure enough adequacy reserve to ensure that the probability of curtailment is reduced to an acceptable level as implied by the value of lost load VOLL.
- If there is price-responsive demand then the amount of demand may be modulated to ensure that enough adequacy reserve is available.

8.12.7.7 Re-formulation as demand bid for adequacy reserve

- We can re-interpret the cost of curtailment as being equivalent to minus the *benefits* of demand for adequacy reserve.
- Recall from Section 8.8 that a demand bid is the derivative of the corresponding benefit function.
- Differentiating minus the cost term, we obtain the equivalent demand bid function for adequacy reserve level F^{adequacy}:

$$\tau \text{VOLL} \times ((D/M - 1)\exp(-D/M) + 1) a_0 \exp(-F^{\text{adequacy}}/M).$$

- At a low level of adequacy reserve, $F^{\text{adequacy}} = 0$, the willingness-to-pay is $\tau \text{VOLL} \times ((D/M - 1)\exp(-D/M) + 1) a_0$, which could be a significant fraction of VOLL.
- At a very high level of adequacy reserve, the willingness-to-pay would be close to zero.
- The willingness-to-pay varies depending on the level of adequacy reserve.
- Given offers for adequacy reserve, S_k^{adequacy}, the amount of procured capacity will depend on the intersection of supply and demand.

8.12.7.8 Procurement of spinning and adequacy reserve

- Typically, the ISO will procure both spinning reserve to cope with the immediate effect of a single generator outage (for security) and adequacy reserve to ensure that the probability of curtailment is low.
- Typically, performance requirements for spinning reserve are more restrictive than for adequacy reserve:
  - adequacy reserve can be provided by non-spinning capacity $S_k^{\text{non-spinning}}$ that must be committed, as well as spinning reserve, S_k^{spinning}.
- For a spinning reserve requirement of $\overline{F}^{\text{spinning}}$, as well as adequacy reserve requirement, we require:

$$-\sum_{k=1}^{n_P} S_k^{\text{spinning}} \leq -\overline{F}^{\text{spinning}},$$

$$-\sum_{k=1}^{n_P} (S_k^{\text{spinning}} + S_k^{\text{non-spinning}}) = -\overline{F}^{\text{spinning}} - F^{\text{adequacy}},$$

- and include the demand bid for the adequacy reserve in the objective.

Procurement of spinning and adequacy reserve, continued

- This formulation allows spinning reserve to be used for adequacy:
  - avoids "price reversal" that can occur if spinning reserve cannot be used for adequacy in the case that spinning reserve is available at a lower offer price than non-spinning reserve.
- If we assume that the adequacy reserve is always depleted before there is a shortage of spinning reserve, then we can represent the result as a composite demand bid for spinning plus adequacy reserve:
  - willingness-to-pay for spinning reserve depends on the penalty cost for relaxation of the hard spinning reserve constraint, typically constant at VOLL up to the required spinning reserve $\overline{F}^{\text{spinning}}$, while
  - willingness-to-pay for additional, adequacy reserve depends on level of adequacy reserve, and decreases with increasing adequacy reserve.

Procurement of spinning and adequacy reserve, continued

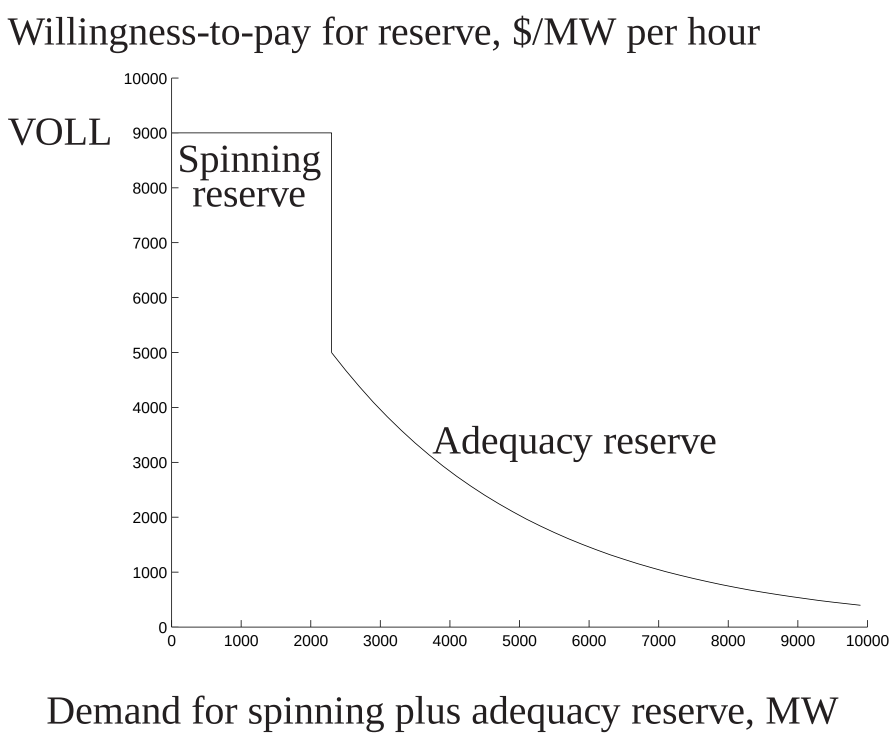

Fig. 8.8. Composite demand curve for spinning plus adequacy reserve.

Procurement of spinning and adequacy reserve, continued

- Actual price for reserve and procured amount of adequacy reserve will depend on intersection of prices for reserves and demand bid:
  - recall that prices for reserves are based on the opportunity cost of using capacity to provide reserves instead of generating energy.
- ERCOT is considering implementing a demand curve for adequacy reserve in order, in part, to encourage new investment:
  - demand bid can set the price for reserves (and energy) high during tight supply conditions,
  - several other ISOs already have such a demand bid for adequacy reserve.
- Other ISOs in North America besides ERCOT also have other mechanisms to provide for new investment to meet forecast of future peak load:
  - installed capacity requirements on load serving entities, or
  - capacity markets that arrange for capacity to be built.

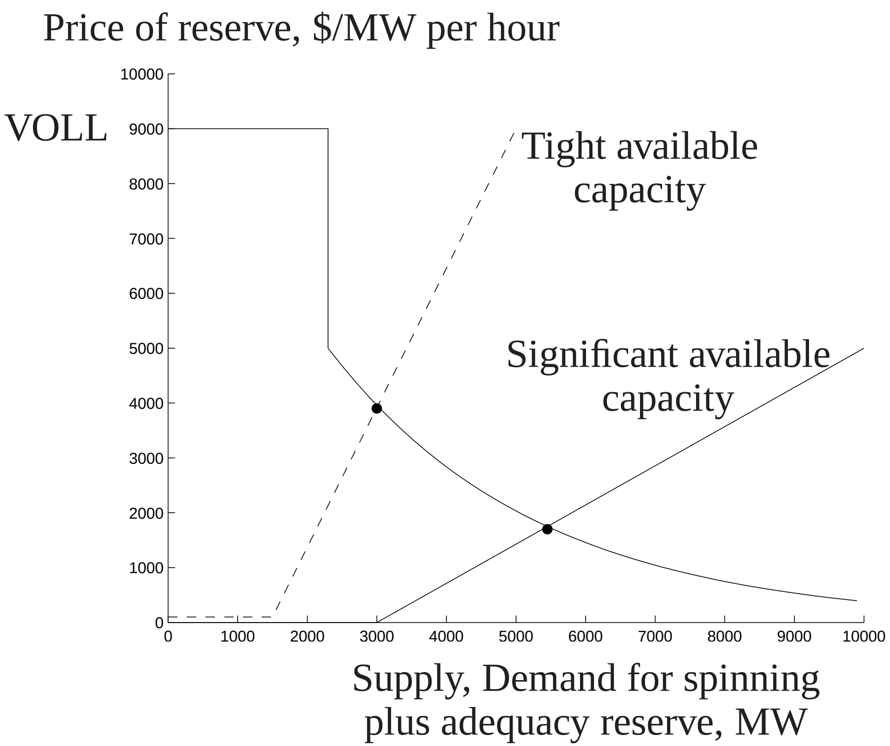

Fig. 8.9. Price and procured reserve shown as bullets for significant available capacity (shown solid) and tight available capacity (shown dotted).

8.13 Summary

- In this chapter we have considered surplus.
- We specialized this to the economic dispatch problem.
- We considered the need for centralized coordination.
- We investigated offer-based economic dispatch and the incentives from the uniform pricing rule.
- We considered the relationship between uplift and the form of the system constraints.
- We considered non-linear system constraints and the representation of constraints.

References

- Sections 15.6 and 18.1 of *Applied Optimization: Formulation and Algorithms for Engineering Systems*, Cambridge University Press 2006.
- M. C. Caramanis, R. E. Bohn, and F. C. Schweppe, “Optimal Spot Pricing: Practice and Theory,” *IEEE Transaction on Power Apparatus and Systems*, PAS-1010(9):3234–3245, September 1982.
- Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford University Press, 1995.
- Steven Stoft, *Power System Economics: Designing Markets for Electricity*, IEEE Press and Wiley Interscience and John Wiley & Sons, Inc., 2002.
- Daniel S. Kirschen and Goran Strbac, *Power System Economics*, Wiley, 2004.
- Ross Baldick, “Single Clearing Price in Electricity Markets,” available from http://www.competecoalition.com/files/Baldick%20study.pdf, February 2009.
- William W. Hogan, “Electricity Scarcity Pricing with an Operating Reserve Demand Curve,” April 2013.

References

- Debabrata Chattopadhyay and Ross Baldick, “Unit Commitment With Probabilistic Reserve,” in Proceedings of *IEEE Power Engineering 2002 Winter Meeting*, 27–31 January 2002.
- Robert Borlick, “Co-optimizing Electric Energy and Contingency Reserve,” April 2013.
- Yen-Yu Lee and Ross Baldick, “A Frequency-Constrained Stochastic Economic Dispatch Model,” to appear in *IEEE Transactions on Power Systems*, 2013.
- Samuel Newell, Kathleen Spees, Johannes Pfeifenberger, *et al.*, “ERCOT Investment Incentive and Resource Adequacy,” The Brattle Group, June 2012.

Homework exercises

8.1 Consider economic dispatch Problem (5.5) in the case that $n_P = 3$, $\overline{D} = 5$, $\underline{P} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\overline{P} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and the f_k are of the form:

$$\begin{aligned}
\forall P_1 \in \mathbb{R}, f_1(P_1) &= \frac{1}{2}(P_1)^2 + P_1, \\
\forall P_2 \in \mathbb{R}, f_2(P_2) &= \frac{1}{2} \times 1.1(P_2)^2 + 0.9P_2, \\
\forall P_3 \in \mathbb{R}, f_3(P_3) &= \frac{1}{2} \times 1.2(P_3)^2 + 0.8P_3.
\end{aligned}$$

A similar problem was solved in a previous homework. Now the generator capacity constraints will be binding.

(i) Solve the economic dispatch problem by solving the first-order necessary conditions in terms of the minimizer $P^\star$ and the Lagrange multipliers $\lambda^\star$, $\underline{\mu}^\star$, and $\overline{\mu}^\star$.

(ii) What price is paid for energy production and consumption assuming that offers are equal to marginal costs?

8.2 Consider Example 5.6 from Kirschen and Strbac, *Power System Economics*, but with one additional generator. Suppose that $\forall k = 1, \ldots, 5, \underline{P}_k = \underline{S}_k = 0$, and with the other capacities and marginal costs specified by:

$$\begin{aligned}
\overline{P}_1 = 250, \overline{S}_1 = 0, \forall x_1 \in \mathbb{S}_1, \nabla f_1(x_1) &= \begin{bmatrix} \$2/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_2 = 230, \overline{S}_2 = 160, \forall x_2 \in \mathbb{S}_2, \nabla f_2(x_2) &= \begin{bmatrix} \$17/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_3 = 240, \overline{S}_3 = 190, \forall x_3 \in \mathbb{S}_3, \nabla f_3(x_3) &= \begin{bmatrix} \$20/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_4 = 250, \overline{S}_4 = 0, \forall x_4 \in \mathbb{S}_4, \nabla f_4(x_4) &= \begin{bmatrix} \$28/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix}, \\
\overline{P}_5 = 250, \overline{S}_5 = 0, \forall x_5 \in \mathbb{S}_5, \nabla f_5(x_5) &= \begin{bmatrix} \$5/\text{MWh} \\ \$0/\text{MWh} \end{bmatrix},
\end{aligned}$$

so that energy costs are non-zero but reserves costs are zero.

Note that generator 5 is very similar to generator 1. We consider one level of demand, $\overline{D} = 750$ MW. Demand in this exercise is 250 MW more than a case considered previously and generator 5 has capacity of 250 MW, so the calculations for this exercise can mostly be deduced from the analogous previous calculations. We assume that offers reflect marginal costs.

(i) Consider the $\sum_k S_k \geq \overline{F}$ formulation of the spinning reserve constraint and suppose that the spinning reserve requirement is fixed at $\overline{F} = 250$ MW. Solve the offer-based reserve-constrained economic dispatch and find the prices for energy and spinning reserve and the uplift.

(ii) Consider the $\sum_k S_k \geq \overline{F}$ formulation of the spinning reserve constraint but now suppose that the reserve requirement is specified as $\overline{F} = \alpha\overline{D}$, with $\alpha = (1/3)$. Solve the offer-based reserve-constrained economic dispatch and find the prices for energy and reserves.

(iii) Suppose that the spinning reserve constraints are $\forall k = 1, \ldots, 5, \sum_{j \neq k} S_j \geq P_k$.

(a) Write out the five spinning reserve constraints explicitly.

(b) Solve the offer-based reserve-constrained economic dispatch for the optimal generations and spinning reserve contributions.

(c) Show that there are two binding spinning reserve constraints.

(d) Find the price for energy.

(e) Show that the Lagrange multipliers on the two binding spinning reserve constraints are not uniquely defined, but that the *sum* of these two Lagrange multipliers has a unique value. Specify the valid values of these two Lagrange multipliers.

(f) Since the values of the Lagrange multipliers are not uniquely defined, we must specify another criterion to determine them. What would be a "fair" specification of the values? Find the resulting price for reserves.

(iv) Consider the following modification of the system:

- Remove generator 5 from the system, so that the generators are as specified in Example 5.6 from Kirschen and Strbac, *Power System Economics*.
- Let demand be $\overline{D} = 500$ MW.
- Use the $\forall k = 1, \ldots, 4, \sum_{j \neq k} S_j \geq P_k$ formulation of the spinning reserve constraints.
- Assume that generators 2 through 4 continue to offer in at marginal cost.
- However, generator 1 offers its 250 MW of capacity at various prices π_1.

Graph the energy price $\lambda^{\star}$, the spinning reserve prices $\mu^{\star}$, and the profit (that is, revenue minus costs) for generator 1 versus its offer price π_1, for prices in the range from \$2/MWh to \$30/MWh. For some offer prices, the Lagrange multipliers may be non-unique: if so, use the same rule that you considered in the previous part to find a "fair" specification of the values in this case.