Lecture 3

Convergence and invariant sets

Convergence and invariant sets

- Review of Lyapunov's direct method
- Convergence analysis using Barbalat's Lemma
- Invariant sets
- Global and local invariant set theorem
- Example

Review of Lyapunov's direct method

Positive definite functions

- If
$$V(0) = 0$$
$$V(x) > 0 \quad \text{for all } x \neq 0$$
then $V(x)$ is positive definite

- If $\mathcal{S}$ is a set containing $x = 0$ and
$$V(0) = 0$$
$$V(x) > 0 \quad \text{for all } x \neq 0,\ x \in \mathcal{S}$$
then $V(x)$ is locally positive definite (within S)

- e.g.
$$V(x) = x^T x \qquad \leftarrow \qquad \text{positive definite}$$
$$V(x) = x^T x(1 - x^T x) \qquad \leftarrow \qquad \text{locally positive definite within } \mathcal{S} = \{x \ : \ x^T x < 1\}$$

Review of Lyapunov's direct method

System: $\dot{x} = f(x), \quad f(0) = 0$

Storage function: $V(x)$

Time-derivative of V: $\dot{V}(x) = \dfrac{\partial V}{\partial x}\dfrac{dx}{dt} = \nabla V(x)^T \dot{x} = \nabla V(x)^T f(x)$

- If
$$\left.\begin{array}{l}\text{(i). } V(x) \text{ is positive definite} \\ \text{(ii). } \dot{V}(x) \leq 0\end{array}\right\} \text{for all } x \in \mathcal{S}$$
then the equilibrium $x = 0$ is stable

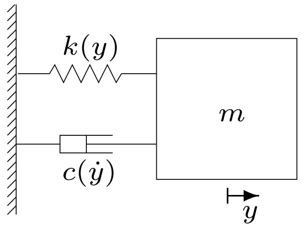
- If
$$\text{(iii). } \dot{V}(x) \text{ is negative definite} \qquad \text{for all } x \in \mathcal{S}$$
then the equilibrium $x = 0$ is asymptotically stable

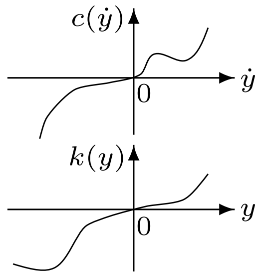
- If
$$\text{(iv). } S = \text{entire state space}$$
$$\text{(v). } V(x) \to \infty \text{ as } \|x\| \to \infty$$
then the equilibrium $x = 0$ is globally asymptotically stable

Convergence analysis

- What can be said about convergence of $x(t)$ to 0
  if $\dot{V}(x) \leq 0$ but $\dot{V}(x)$ is not negative definite?

- Revisit m-s-d example:





Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$

Storage function: $V = \text{K.E.} + \text{P.E.} = \frac{1}{2}m\dot{y}^2 + \int_0^y k(y)\, dy$

$\dot{V} = -c(\dot{y})\dot{y}$

Convergence analysis

- V is p.d. and $\dot{V} \leq 0$ so: $(y, \dot{y}) = (0, 0)$ is stable
  and $V(y, \dot{y})$ tends to a finite limit as $t \to \infty$

- but does $(y, \dot{y})$ converge to $(0, 0)$?

$\updownarrow$ equivalent to

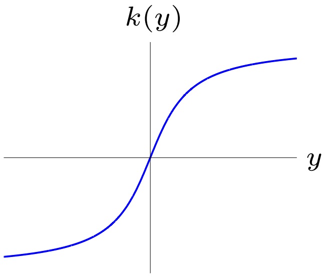
can $V(y, \dot{y})$ "get stuck" at $V = V_0 \neq 0$ as $t \to \infty$?

$\downarrow$

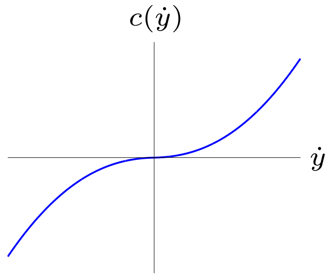
need to consider motion at points $(y, \dot{y})$ for which $\dot{V} = 0$

Example

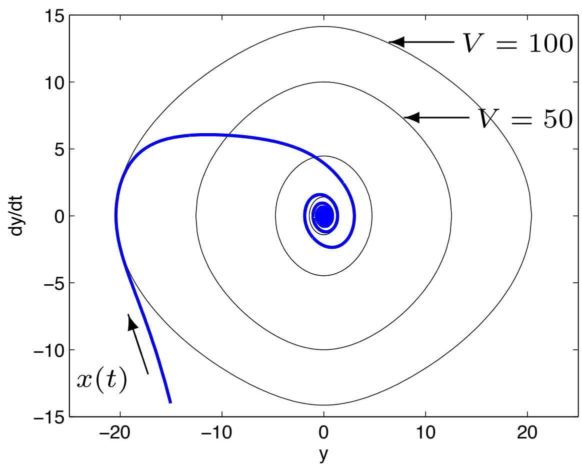
Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



$k(y) = 5\tan^{-1}(y/5)$



$c(\dot{y}) = 0.1\dot{y}(0.5 + |\dot{y}|)(2 - e^{-0.1|\dot{y}|})$



Storage function:

$$V = \tfrac{1}{2}\dot{y}^2 + \int_0^y 5\tan^{-1}(y/5)\,dy$$

$$\dot{V} = -c(\dot{y})\dot{y} \le 0$$

$$\downarrow$$

$$\dot{V} = 0 \text{ when } \dot{y} = 0$$

but $k(y) \neq 0 \implies \ddot{y} \neq 0 \implies \ddot{V} \neq 0$

$$\Downarrow$$

V continues to decrease until $y = \dot{y} = 0$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times

$$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$$

This approach is the basis of the invariant set theorems

Barbalat's Lemma

Barbalat's lemma: For any function $\phi(t)$, if

$$\boxed{\begin{array}{l} \text{(i). } \displaystyle\int_0^t \phi(\tau)\,d\tau \text{ converges to a finite limit as } t \to \infty \\ \text{(ii). } \dot{\phi}(t) \text{ is finite for all } t \end{array}}$$

then $\lim_{t\to\infty} \phi(t) = 0$

- Obvious for the case that $\phi(t) \geq 0$ for all t
- Condition (ii) is needed to ensure that $\phi(t)$ remains continuous for all t

$\uparrow$

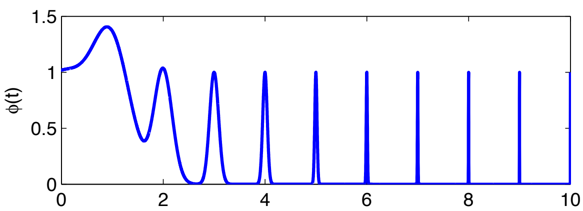
Can construct discontinuous $\phi(t)$ for which $\displaystyle\int_0^t \phi(\tau)\,d\tau$ converges

but $\phi(t) \not\to 0$ as $t \to \infty$

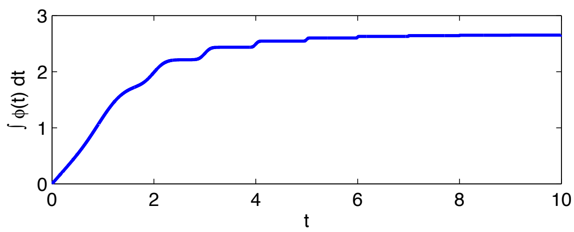
Barbalat's Lemma

Example: pulse train $\phi(t) = \sum_{k=0}^{\infty} e^{-4^k (t-k)^2}$:

$\phi(t)$:



$\displaystyle\int_0^t \phi(\tau)\,d\tau$:



From the plots it is clear that

$$\int_0^t \phi(s)\,ds \text{ tends to a finite limit}$$

but $\quad \phi(t) \not\to 0$ as $t \to \infty \qquad$ because $\quad \dot{\phi}(t) \to \infty$ as $t \to \infty$

Barbalat's Lemma contd.

Apply Barbalat's Lemma to $\dot{V}\big(x(t)\big) = \phi(t) \leq 0$:

- Integrate:
  $$\int_0^t \phi(s)\,ds = V\big(x(t)\big) - V\big(x(0)\big) \qquad \leftarrow \text{finite limit as } t \to \infty$$

- Differentiate:
  $$\dot{\phi}(t) = \ddot{V}\big(x(t)\big) = f^T(x)\frac{\partial^2 V}{\partial x^2}(x)f(x) + \nabla V(x)\frac{\partial f}{\partial x}(x)f(x)$$
  $= \text{finite for all } t \text{ if } f(x) \text{ continuous and } V(x) \text{ continuously differentiable}$

$$\Downarrow$$

$$\boxed{\dot{V}(x) \to 0 \text{ as } t \to \infty}$$

The above arguments rely on $\|x(t)\|$ remaining finite for all t,
which is implied by:

$V(x)$ positive definite
$\dot{V}(x) \leq 0$
$V(x) \to \infty$ as $\|x\| \to \infty$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   $\to$ true whenever $\dot{V} \leq 0$ & V, f are smooth & $\|x(t)\|$ is bounded
   [by Barbalat's Lemma]

2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
   $\to$ algebra!

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times
   $\to$ $\mathcal{M}$ must be invariant

$$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$$

This approach is the basis of the invariant set theorems

Invariant sets

- A set of points $\mathcal{M}$ in state space is invariant if

$$\boxed{x(t_0) \in \mathcal{M} \quad \Longrightarrow \quad x(t) \in \mathcal{M} \quad \text{for all } t > t_0}$$

  Examples:

  - ⋆ Equilibrium points
  - ⋆ Limit cycles
  - ⋆ Level sets of $V(x)$ provided $\dot{V}(x) \le 0$ $\quad\leftarrow\quad$ i.e. $\{x \ : \ V(x) \le V_0\}$ for constant V_0

- If $\dot{V}(x) \to 0$ as $t \to \infty$, then

$$\boxed{\text{$x(t)$ must converge to an invariant set $\mathcal{M}$ contained within the set of points on which $\dot{V}(x) = 0$}}$$

as $t \to \infty$

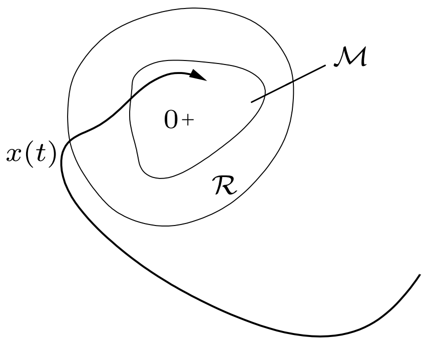
Global invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \le 0 \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then: (i). $\dot{V}(x) \to 0$ as $t \to \infty$
(ii). $x(t) \to \mathcal{M} =$ the largest invariant set contained in $\mathcal{R}$

where $\mathcal{R} = \{x \ : \ \dot{V}(x) = 0\}$



- $\dot{V}(x)$ negative definite $\Longrightarrow \mathcal{M} = 0$ (c.f. Lyapunov's direct method)
- Determine $\mathcal{M}$ by considering system dynamics within $\mathcal{R}$

Global invariant set theorem

Revisit m-s-d example (for the last time)

- $V(x)$ is positive definite, $V(x) \to \infty$ as $\|x\| \to \infty$, and
$$\dot{V}(y, \dot{y}) = -c(\dot{y})\dot{y} \le 0$$
- therefore $\dot{V} \to 0$, implying $\dot{y} \to 0$ as $t \to \infty$
i.e. $\mathcal{R} = \{(y, \dot{y}) \ : \ \dot{y} = 0\}$
- but $\dot{y} = 0$ implies $\ddot{y} = -k(y)/m$
- therefore $\ddot{y} \ne 0$ unless $y = 0$, so $\dot{y}(t) = 0$ for all t only if $y(t) = 0$
i.e. $\mathcal{M} = \{(y, \dot{y}) \ : \ (y, \dot{y}) = (0, 0)\}$

$$\Downarrow$$

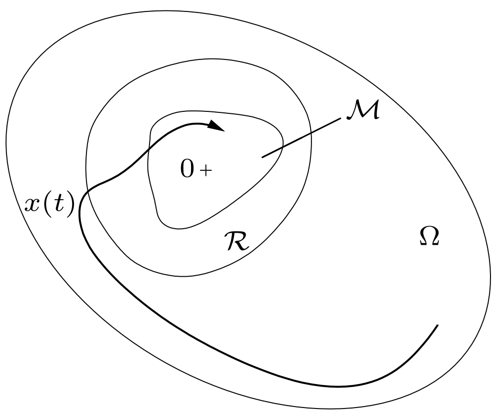
$(y, \dot{y}) = (0, 0)$ is a globally asymptotically stable equilibrium!

Local invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

> the level set $\Omega = \{x \ : \ V(x) \le V_0\}$ is bounded for some V_0
> and $\dot{V}(x) \le 0$ whenever $x \in \Omega$

then:
(i). Ω is an invariant set
(ii). $x(0) \in \Omega \quad \Longrightarrow \quad \dot{V}(x) \to 0$ as $t \to \infty$
(iii). $x(t) \to \mathcal{M} =$ largest invariant set contained in $\mathcal{R}$
where $\mathcal{R} = \{x \ : \ \dot{V}(x) = 0\}$



Local invariant set theorem

- $V(x)$ doesn't have to be positive definite or radially unbounded

- Result is based on Barbalat's Lemma applied to $\dot{V}$

  $\uparrow$

  applies here because finite Ω implies $\|x(t)\|$ finite for all t
  since $x(0) \in \Omega$ and $\dot{V} \leq 0$

- Ω is a region of attraction for $\mathcal{M}$

Example: local invariant set theorem

- Second order system: $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 - 1)^2 x_2^3 - x_1 + \sin(\pi x_1/2) \end{aligned}$$

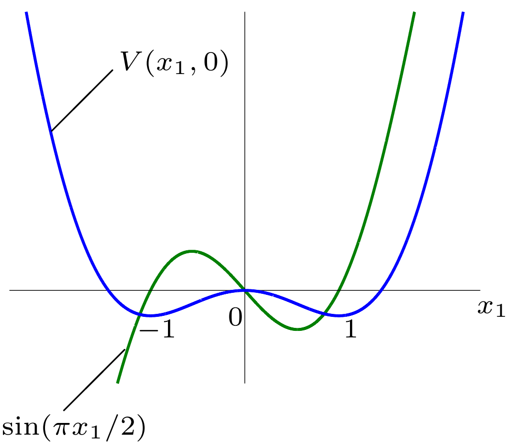
- Equilibrium points: $(x_1, x_2) = (0, 0), (1, 0), (-1, 0)$

- Trial storage function:
  $$V(x) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} \big(y - \sin(\pi y/2)\big)\, dy$$

V is not positive definite
but $V(x) \to \infty$ if $x_1 \to \infty$ or $x_2 \to \infty$

$\Downarrow$

level sets of V are finite



Example: local invariant set theorem contd.

- Differentiate: $\dot V(x) = -(x_1-1)^2 x_2^4 \le 0$

$$\dot V(x) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{R} = \{x \ : \ x_1 = 1 \ \text{ or } \ x_2 = 0\}$$

- From the system model, $x \in \mathcal{R}$ implies:

$$x_1 = 1 \quad \Longrightarrow \quad (\dot x_1, \dot x_2) = (x_2, 0)$$

and

$$x_2 = 0 \quad \Longrightarrow \quad (\dot x_1, \dot x_2) = (0, \sin(\pi x_1/2) - x_1)$$

therefore $\begin{cases} x(t) \text{ remains on line } x_1 = 1 \text{ only if } x_2 = 0 \\ x(t) \text{ remains on line } x_2 = 0 \text{ only if } x_1 = 0,\ 1 \text{ or } -1 \end{cases}$

$$\Longrightarrow \ \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$$

- Apply local invariant set theorem to any level set $\Omega = \{x \ : \ V(x) \le V_0\}$:

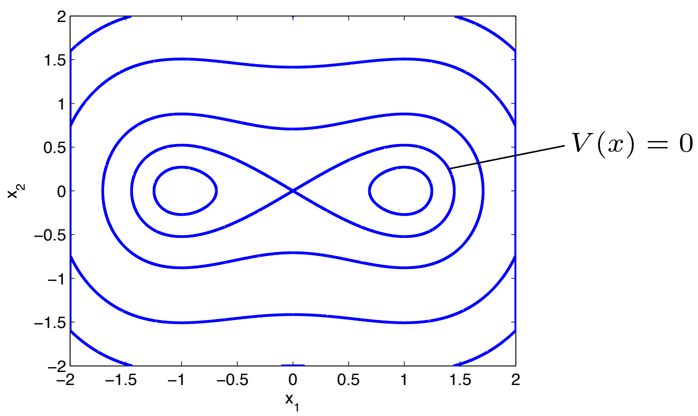
$$\left.\begin{array}{l} \Omega \text{ is finite} \\ \dot V \le 0 \end{array}\right\} \Longrightarrow \quad x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\} \text{ as } t \to \infty$$

Example: local invariant set theorem contd.

- From any initial condition, $x(t)$ converges asymptotically to $(0,0)$, $(1,0)$ or $(-1,0)$

  but $x = (0,0)$ is unstable
  (linearized system at $(0,0)$ has poles $\pm\sqrt{\frac{\pi}{2} - 1}$ so is unstable)

- Contours of $V(x)$:



Use local invariant set theorem on level sets $\Omega = \{x : V(x) \le V_0\}$ for $V_0 < 0$

$$\downarrow$$

$x = (1,0)$, $x = (-1,0)$ are stable equilibrium points

Summary

- Convergence analysis using Barbalat's lemma
- Invariant sets
- Invariant set methods for convergence: local invariant set theorem
  global invariant set theorem