Lecture 2

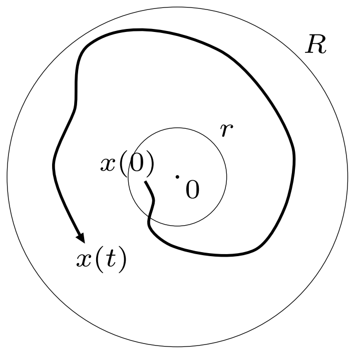
Linearization and Lyapunov's direct method

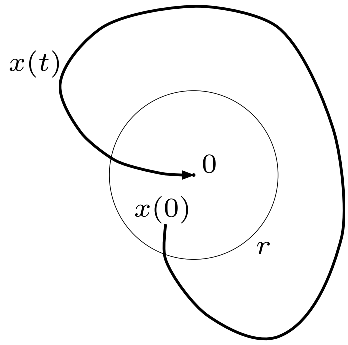
Linearization and Lyapunov's direct method

- Review of stability definitions
- Linearization method
- Direct method for stability
- Direct method for asymptotic stability
- Linearization method revisited

Review of stability definitions

System: $\dot{x} = f(x)$ ⋆ unforced system (i.e. closed-loop)
⋆ consider stability of individual equilibrium points





0 is a stable equilibrium if:

$$\|x(0)\| \le r \implies \|x(t)\| \le R$$
for any $R > 0$

0 is asymptotically stable if:

$$\|x(0)\| \le r \implies \|x(t)\| \to 0$$
as $t \to \infty$

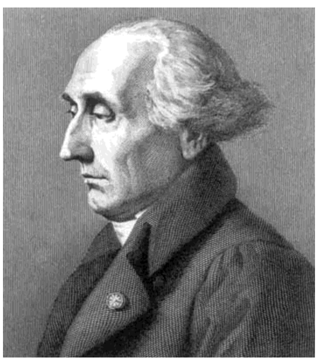
Stability	$\to$	local property
Asymptotic stability	$\to$	global if $r = \infty$ allowed

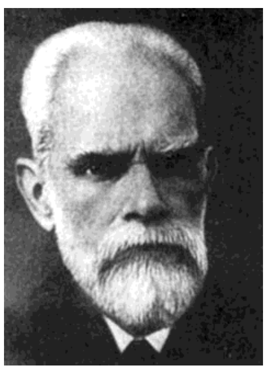
Historical development of Stability Theory

- Potential energy in conservative mechanics (Lagrange 1788):

  An equilibrium point of a conservative system is stable if it corresponds to a minimum of the potential energy stored in the system

- Energy storage analogy for general ODEs (Lyapunov 1892)
- Invariant sets (Lefschetz, La Salle 1960s)


J-L. Lagrange 1736-1813


A. M. Lyapunov 1857-1918


S. Lefschetz 1884-1972

Lyapunov's linearization method

- Determine stability of equilibrium at $x = 0$ by analyzing the stability of the linearized system at $x = 0$.
- Jacobian linearization:

$$\begin{aligned} \dot{x} &= f(x) && \text{original nonlinear dynamics} \\ &= f(0) + \frac{\partial f}{\partial x}\Big|_{x=0} x + R_1 && \text{Taylor's series expansion, } R_1 = O(\|x\|^2) \\ &\approx Ax && \text{since } f(0) = 0 \end{aligned}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \qquad \frac{\partial f}{\partial x} \text{ assumed continuous at } x = 0$$

Lyapunov's linearization method

Conditions on A for stability of original nonlinear system at $x = 0$:

stability of linearization	stability of nonlinear system at $x = 0$
$\mathrm{Re}\big(\lambda(A)\big) < 0$	asymptotically stable (locally)
$\max \mathrm{Re}\big(\lambda(A)\big) = 0$	stable or unstable
$\max \mathrm{Re}\big(\lambda(A)\big) > 0$	unstable

Lyapunov's linearization method

- Some examples

(stable)	$\dot{x} = -x^3$	$\xrightarrow{\text{linearize}}$	$\dot{x} = 0$	(indeterminate)
(unstable)	$\dot{x} = x^3$	$\xrightarrow{\text{linearize}}$	$\dot{x} = 0$	(indeterminate)

$\uparrow$ higher order terms determine stability

- Why does linear control work?
  1. Linearize the model:

$$\begin{aligned}\dot{x} &= f(x,u)\\ &\approx Ax + Bu, \qquad A = \frac{\partial f}{\partial x}(0,0),\ B = \frac{\partial f}{\partial u}(0,0)\end{aligned}$$

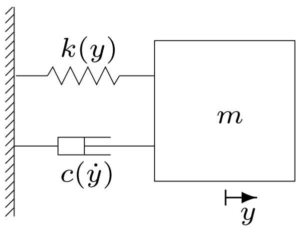
  2. Design a linear feedback controller using the linearized model:

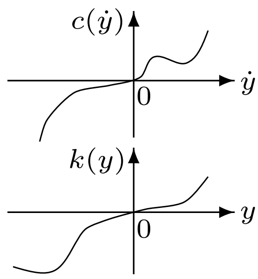
$$u = -Kx, \quad \underbrace{\max \mathrm{Re}\big(\lambda(A - BK)\big) < 0}_{\text{closed-loop linear model strictly stable}}$$

$\Downarrow$

nonlinear system $\dot{x} = f(x, -Kx)$ is locally asymptotically stable at $x = 0$

Lyapunov's direct method: mass-spring-damper example





Equation of motion: $\quad m\ddot{y} + c(\dot{y}) + k(y) = 0$

Stored energy: $\quad V = \text{K.E.} + \text{P.E.} \quad \begin{cases} \text{K.E.} = \frac{1}{2}m\dot{y}^2 \\ \text{P.E.} = \displaystyle\int_0^y k(y)\,dy \end{cases}$

Rate of energy dissipation

$$\begin{aligned}\dot{V} &= \tfrac{1}{2}m\dot{y}\frac{d}{d\dot{y}}\dot{y}^2 + \dot{y}\frac{d}{dy}\left[\int_0^y k(y)\,dy\right]\\ &= m\ddot{y}\dot{y} + \dot{y}k(y)\end{aligned}$$

but $m\ddot{y} + k(y) = -c(\dot{y})$, so

$$\begin{aligned}\dot{V} &= -c(\dot{y})\dot{y}\\ &\le 0 \qquad \leftarrow \text{since } \mathrm{sign}\big(c(\dot{y})\big) = \mathrm{sign}(\dot{y})\end{aligned}$$

Mass-spring-damper example contd.

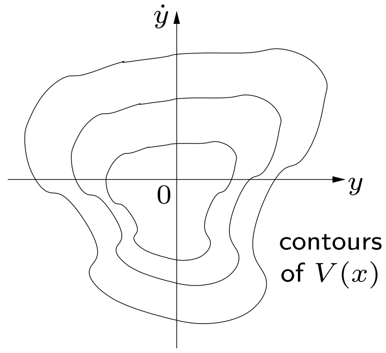
- System state: e.g. $x = [y \;\; \dot{y}]^T$
- $\dot{V}(x) \le 0$ implies that $x = 0$ is stable

  $\uparrow$

  $V(x(t))$ must decrease over time

  but

  $V(x)$ increases with increasing $\|x\|$



- Formal argument:

  for any given $R > 0$:

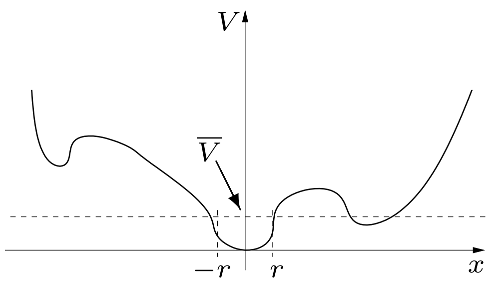
$$
\begin{array}{rll}
\|x\| < R & \text{whenever} & V(x) < \overline{V} \;\; \text{for some } \overline{V} \\
\text{and } V(x) < \overline{V} & \text{whenever} & \|x\| < r \;\; \text{for some } r
\end{array}
$$

$$
\begin{aligned}
\therefore \|x(0)\| < r \quad &\Longrightarrow \quad V\big(x(0)\big) < \overline{V} \\
&\Longrightarrow \quad V\big(x(t)\big) < \overline{V} \qquad \text{for all } t > 0 \\
&\Longrightarrow \quad \|x(t)\| < R \qquad \text{for all } t > 0
\end{aligned}
$$

Positive definite functions

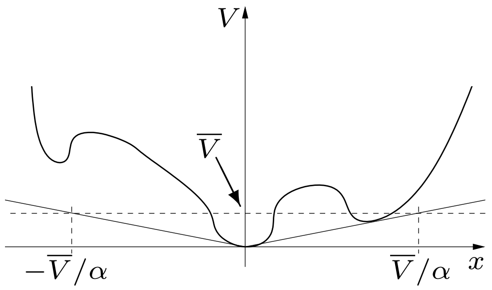
- What if $V(x)$ is not monotonically increasing in $\|x\|$?
- Same arguments apply if $V(x)$ is continuous and positive definite, i.e.

$$
\boxed{
\begin{array}{lll}
\text{(i).} & V(0) = 0 & \\
\text{(ii).} & V(x) > 0 & \text{for all } x \neq 0
\end{array}
}
$$



for any given $\overline{V} > 0$,
can always find r so that

$V(x) < \overline{V}$ whenever $\|x\| < r$



$V(x) \ge \alpha \|x\|$
for some constant α, so

$\|x\| < \overline{V}/\alpha$ whenever $V(x) < \overline{V}$

Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

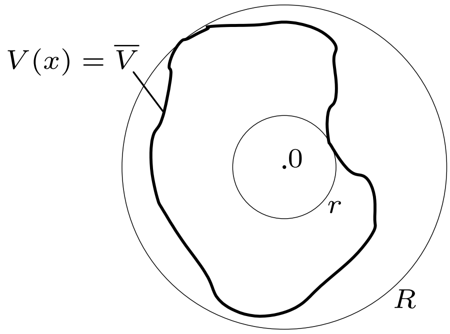
$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot V(x) \le 0 \end{array}}$$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any R and some r:

1. choose $\overline{V}$ as the minimum of $V(x)$ for $\|x\| = R$
2. find r so that $V(x) < \overline{V}$ whenever $\|x\| < r$
3. then $\dot V(x) \le 0$ ensures that

$$V\big(x(t)\big) < \overline{V} \quad \forall t > 0 \quad \text{if } \|x(0)\| < r$$
$$\therefore \|x(t)\| < R \quad \forall t > 0$$



Lyapunov stability theorem

- Lyapunov's direct method also applies if $V(x)$ is locally positive definite, i.e. if

  (i). $V(0) = 0$
  (ii). $V(x) > 0$ for $x \neq 0$ and $\|x\| < R_0$

  then $x = 0$ is stable if $\dot V(x) \le 0$ whenever $\|x\| < R_0$.

- Apply the theorem without determining R, r
  – only need to find p.d. $V(x)$ satisfying $\dot V(x) \le 0$.

- Examples

  (i). $\dot x = -a(t)x, \quad a(t) > 0$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \dot V = x\dot x = -a(t)x^2 \le 0$$

  (ii). $\dot x = -a(x), \quad \mathrm{sign}\big(a(x)\big) = \mathrm{sign}(x)$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \dot V = x\dot x = -a(x)x \le 0$$

Lyapunov stability theorem

- More examples

(iii). $\dot{x} = -a(x)$, $\displaystyle\int_0^x a(x)\,dx > 0$

$$V = \int_0^x a(x)\,dx \quad\Longrightarrow\quad \dot{V} = a(x)\dot{x} = -a^2(x) \le 0$$

(iv). $\ddot{\theta} + \sin\theta = 0$

$$V = \tfrac{1}{2}\dot{\theta}^2 + \int_0^\theta \sin\theta\,d\theta \quad\Longrightarrow\quad \dot{V} = \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta = 0$$

Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$V(x)$	is positive definite
$\dot{V}(x)$	is negative definite

then $x = 0$ is locally asymptotically stable.

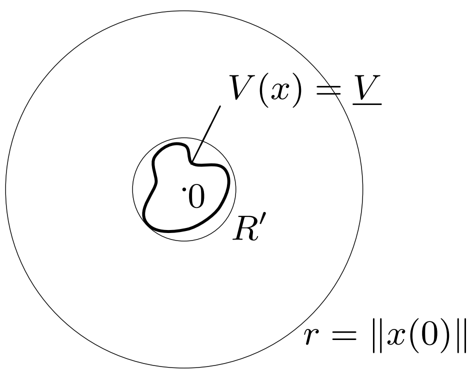
($\dot{V}$ negative definite $\iff$ $-\dot{V}$ positive definite)

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

if $\|x(t)\| > R'$ for all $t \ge 0$, then

$$\left.\begin{aligned} &\dot{V}(x) < -W \\ &V(x) \ge \underline{V} \end{aligned}\right\} \quad \text{for all } t \ge 0$$

contradiction



Linearization method and asymptotic stability

- Asymptotic stability result also applies if $\dot{V}(x)$ is only locally negative definite.

- Why does the linearization method work?

  ⋆ consider 1st order system: $\dot{x} = f(x)$
  linearize about $x = 0$: $= -ax + R \qquad R = O(x^2)$

  ⋆ assume $a > 0$ and try Lyapunov function V:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^2 \\ \dot{V}(x) &= x\dot{x} = -ax^2 + Rx = -x^2(a - R/x) \\ &\leq -x^2(a - |R/x|) \end{aligned}$$

  ⋆ but $R = O(x^2)$ implies $|R| \leq \beta x^2$ for some constant β, so

$$\begin{aligned} \dot{V} &\leq -x^2(a - \beta|x|) \\ &\leq -\gamma x^2 \quad \text{if } |x| \leq (a-\gamma)/\beta \end{aligned}$$

$\Longrightarrow$ $\dot{V}$ negative definite for $|x|$ small enough
$\Longrightarrow$ $x = 0$ locally asymptotically stable

Generalization to nth order systems is straightforward

Global asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{l} \left.\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}\right\} \text{for all } x \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then $x = 0$ is globally asymptotically stable

- If $V(x) \to \infty$ as $\|x\| \to \infty$, then $V(x)$ is radially unbounded

- Test whether $V(x)$ is radially unbounded by checking if $V(x) \to \infty$ as each individual element of x tends to infinity (necessary).

Global asymptotic stability theorem

- Global asymptotic stability requires:

$$\|x(t)\| \text{ finite } \begin{cases} \text{for all } t > 0 \\ \text{for all } x(0) \end{cases}$$

$\uparrow$

not guaranteed by $\dot{V}$ negative definite

in addition to asymptotic stability of $x = 0$

- Hence add extra condition: $V(x) \to \infty$ as $\|x\| \to \infty$

$\updownarrow$ equiv. to

level sets $\{x \ : \ V(x) \leq \overline{V}\}$ are finite

$\updownarrow$ equiv. to

$\|x\|$ is finite whenever $V(x)$ is finite

$\uparrow$

prevents $x(t)$ drifting away from 0 despite $\dot{V} < 0$

Asymptotic stability example

System:

$$\begin{aligned} \dot{x}_1 &= (x_2 - 1)x_1^3 \\ \dot{x}_2 &= -\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2} \end{aligned}$$

- Trial Lyapunov function $V(x) = x_1^2 + x_2^2$:

$$\begin{aligned} \dot{V}(x) &= 2x_1\dot{x_1} + 2x_2\dot{x_2} \\ &= -2x_1^4 + \underbrace{2x_2x_1^4 - 2\frac{x_2x_1^4}{(1+x_1^2)^2}} - 2\frac{x_2^2}{1+x_2^2} \quad \not\leq 0 \end{aligned}$$

$\uparrow$

change V to make
these terms cancel

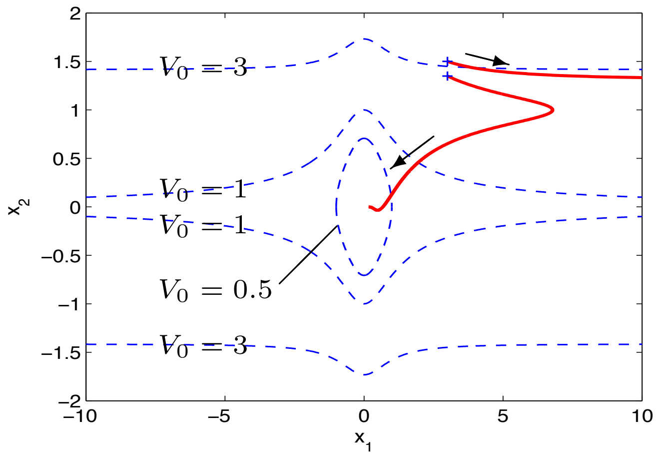
Asymptotic stability example

- New trial Lyapunov function $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$:

$$\dot V(x) = 2\Big[\frac{x_1}{1+x_1^2} - \frac{x_1^3}{(1+x_1^2)^2}\Big]\dot{x_1} + 2x_2\dot{x_2}$$
$$= -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2} \le 0$$

$V(x)$ positive definite, $\dot V(x)$ negative definite $\implies$ $x = 0$ a.s.
But $V(x)$ not radially unbounded, so cannot conclude global a.s.

State trajectories:



Summary

- Positive definite functions
- Derivative of $V(x)$ along trajectories of $\dot x = f(x)$
- Lyapunov's direct method for: stability
asymptotic stability
global stability
- Lyapunov's linearization method