Lecture 4

# Linear systems, passivity, and the circle criterion



## Linear systems, passivity, and the circle criterion

- Summary of stability methods
- Lyapunov functions for linear systems
- Passive systems
- Passive linear systems
- The circle criterion
- Example

# Summary of stability methods

- Linearization method

$$\dot{x} = Ax \text{ is strictly stable, } A = \frac{\partial f}{\partial x}\Big|_{x=0}$$
$$\Downarrow$$
$$x = 0 \text{ locally asymptotically stable}$$

- Lyapunov's direct method

| $V(x)$ locally p.d. | $V(x)$ locally p.d. | $V(x)$ p.d. |
|---|---|---|
| $\dot{V}(x) \leq 0$ locally | $\dot{V}(x)$ locally n.d. | $\dot{V}(x)$ n.d. |
| | | $V(x) \to \infty$ as $\lVert x \rVert \to \infty$ |
| $\Downarrow$ | $\Downarrow$ | $\Downarrow$ |
| $x = 0$ stable | $x = 0$ locally asymptotically stable | $x = 0$ globally asymptotically stable |

- Invariant set theorems

| | |
|---|---|
| $V(x)$ p.d. | |
| $\dot{V}(x) \leq 0$ | $\Omega = \{x \ : \ V(x) \leq V_0\}$ bounded |
| $V(x) \to \infty$ as $\lVert x \rVert \to \infty$ | $\dot{V}(x) \leq 0$ for all $x \in \Omega$ |

$$\Downarrow$$

$x(t)$ converges to the union of invariant sets contained in $\{x \ : \ \dot{V}(x) = 0\}$



# Summary of stability methods

- Instability theorems analogous to Lyapunov's direct method, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \text{ p.d.} \end{array}\right\} \implies x = 0 \text{ unstable}$$

- Lyapunov stability criteria are only sufficient, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \not\leq 0 \end{array}\right\} \not\Longrightarrow x = 0 \text{ unstable}$$

(some other $V(x)$ demonstrating stability may exist)

- Converse theorems

$$x = 0 \text{ stable} \implies V(x) \text{ demonstrating stability exists}$$

(can swap premises and conclusions in Lyapunov's direct method)

$$\uparrow$$

But no general method for constructing $V(x)$

# Linear systems

- Systematic method for constructing storage function $V(x) = x^T P x$

  $\dot{x} = Ax$ strictly stable $\quad\Longrightarrow\quad$ can always find constant matrix $P$ so that $\dot{V}(x)$ is negative definite

- Only need consider symmetric $P$

$$x^T P x = \tfrac{1}{2} x^T P x + \tfrac{1}{2} x^T P^T x = \tfrac{1}{2} x^T \underbrace{\left(P - P^T\right)}_{\text{SYMMETRIC}} x$$

- Need $\lambda(P) > 0$ for positive definite $V(x) = x^T P x$

$$\begin{array}{ll} P = U \Lambda U^T & \text{eigenvector/value decomposition} \\ \Downarrow & \\ x^T P x = z^T \Lambda z & z = U^T x \\ \Downarrow & \\ x^T P x \text{ positive definite} & \left\{ \begin{array}{l} \text{notation: } P > 0 \\ \text{or ``}P \text{ is positive definite''} \end{array} \right. \\ \text{iff } \Lambda \text{ strictly positive} & \end{array}$$



# Linear systems

- How is $P$ computed?

$$\left. \begin{aligned} \dot{x} &= Ax \\ V(x) &= x^T P x \end{aligned} \right\} \quad \Longrightarrow \quad \begin{aligned} \dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P x \\ &= x^T (PA + A^T P) x \end{aligned}$$

  $\therefore\ x = 0$ is globally asymptotically stable if, for some $Q$:

$$\boxed{PA + A^T P = -Q \qquad Q = Q^T > 0}$$

Lyapunov matrix equation

- Pick $Q > 0$ and solve $PA + A^T P = -Q$ for $P$, then

$$\boxed{\mathrm{Re}\big[\lambda(A)\big] < 0 \quad \Longleftrightarrow \quad \begin{array}{l} \text{unique solution for } P \\ \text{and } P = P^T > 0 \end{array}}$$

  Proof:

  $\Longleftarrow$ due to $\dot{V}(x) = -x^T Q x$ negative definite

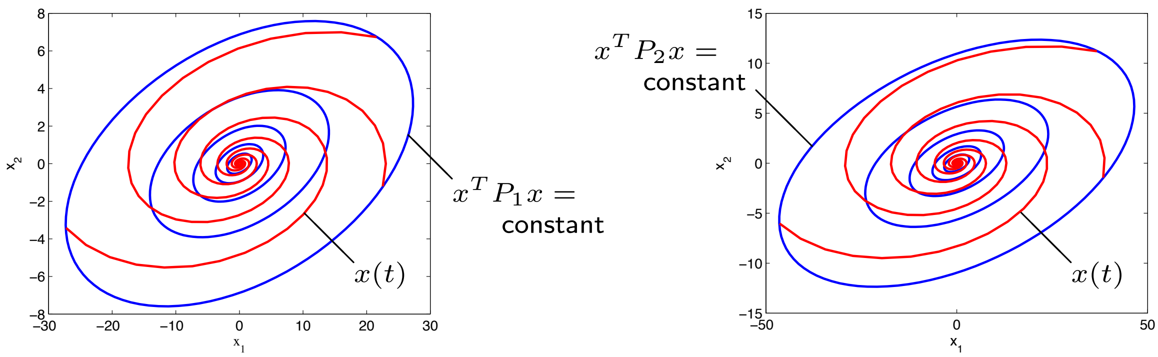
  $\Longrightarrow$ follows from integrating $\dot{V}$ w.r.t. $t$: $P = \displaystyle\int_0^\infty e^{A^T t} Q e^{At}\, dt$

# Example: Lyapunov matrix equation

Stable linear system $\dot{x} = Ax$: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -16\\ 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ $\qquad \lambda(A) = -1 \pm i\sqrt{15}$

Solve $PA + A^TP = -Q$ for $P$:

$Q_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow P_1 = \begin{bmatrix}0.33 & -0.5\\ -0.5 & 4.25\end{bmatrix}$ $\qquad Q_2 = \begin{bmatrix}0.41 & -0.19\\ -0.19 & 0.11\end{bmatrix} \Rightarrow P_2 = \begin{bmatrix}0.12 & -0.21\\ -0.21 & 1.67\end{bmatrix}$



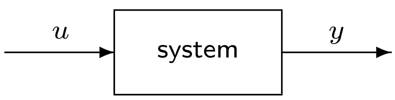

Here:

- ⋆ any choice of $Q > 0$ gives $P > 0$ (since $A$ is strictly stable)
- ⋆ but not every $P > 0$ gives $Q > 0$



# Passive systems

- Systematic method for constructing storage functions
- Input-output representation of system:



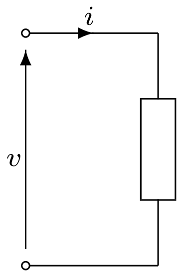

input : $u$
output : $y$

The system is passive if

$$\boxed{\dot{V} = yu - g \quad \text{for some } V(t) \geq 0, \quad g(t) \geq 0}$$

also the system is dissipative if $\displaystyle\int_0^\infty yu\,dt \neq 0 \implies \int_0^\infty g\,dt > 0$

- Motivated by electrical networks with no internal power generation



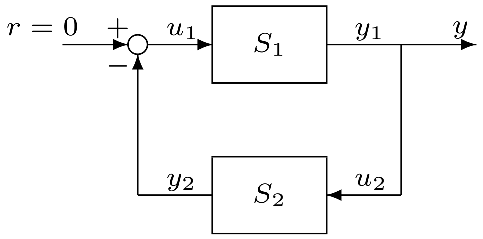

$\left.\begin{array}{l}\text{input: } i\\ \text{output: } v\end{array}\right\}$ stored energy: $\displaystyle V = \int_0^t vi\,dt$

$\dot{V} = iv$

# Passive systems

Passivity is useful for determining storage functions for feedback systems

- Closed-loop system with passive subsystems $S_1$, $S_2$:




$$S_1: \quad V_1 \geq 0 \quad \dot{V}_1 = y_1 u_1 - g_1$$
$$S_2: \quad V_2 \geq 0 \quad \dot{V}_2 = y_2 u_2 - g_2$$

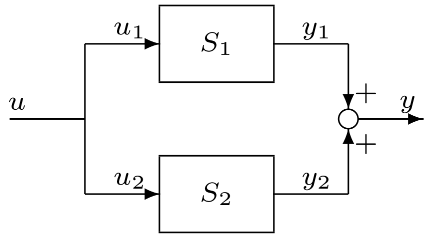
$$\begin{aligned} &V_1 + V_2 \geq 0 \\ &\dot{V}_1 + \dot{V}_2 = y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &\qquad\quad = y_1(-y_2) + y_2 y_1 - g_1 - g_2 \\ &\qquad\quad = -g_1 - g_2 \\ &\qquad\quad \leq 0 \end{aligned}$$

$\implies$ $V = V_1 + V_2$ is a Lyapunov function for the closed-loop system
if $V$ is a p.d. function of the system state



# Interconnected passive systems

- Parallel connection:

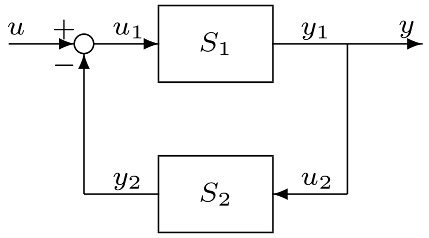



$$\begin{aligned} &V_1 + V_2 \geq 0 \\ &\dot{V}_1 + \dot{V}_2 = y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &\qquad\quad = (y_1 + y_2)u - g_1 - g_2 \\ &\qquad\quad = yu - g_1 - g_2 \end{aligned}$$

$\Downarrow$

Overall system from $u$ to $y$ is passive

- Feedback connection:



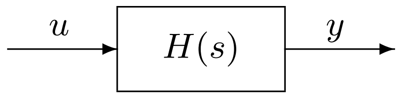

$$\begin{aligned} &V_1 + V_2 \geq 0 \\ &\dot{V}_1 + \dot{V}_2 = y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &\qquad\quad = y(u - y_2) + y_2 y - g_1 - g_2 \\ &\qquad\quad = yu - g_1 - g_2 \end{aligned}$$

$\Downarrow$

Overall system from $u$ to $y$ is passive

# Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$




- $H$ is passive if and only if

$$\boxed{\begin{array}{ll} \text{(i).} & \mathrm{Re}(p_i) \le 0, \text{ where } \{p_i\} \text{ are the poles of } H(s) \\ \text{(ii).} & \mathrm{Re}\big[H(j\omega)\big] \ge 0 \text{ for all } 0 \le \omega \le \infty \end{array}}$$

  - ⋆ $H$ must be stable, otherwise $V(t) = \displaystyle\int_0^t yu\,dt$ is not defined for all $u$
  - ⋆ From Parseval's theorem:

$$\mathrm{Re}\big[H(j\omega)\big] \ge 0 \quad \Longleftrightarrow \quad \int_0^t yu\,dt \ge 0 \text{ for all } u(t) \text{ and } t$$

↑

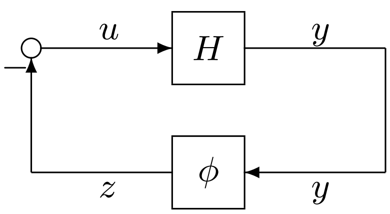
frequency domain criterion for passivity

- $H$ is dissipative if and only if $\mathrm{Re}(p_i) \le 0$ and

$$\mathrm{Re}\big[H(j\omega)\big] > 0 \text{ for all } 0 \le \omega < \infty$$



# Linear system + static nonlinearity




$H$ linear: $\dfrac{Y(s)}{U(s)} = H(s)$

$\phi$ static nonlinearity: $z = \phi(y)$

What are the conditions on $H$ and $\phi$ for closed-loop stability?

- A common problem in practice, due to e.g.
  - ⋆ actuator saturation (valves, dc motors, etc.)
  - ⋆ sensor nonlinearity

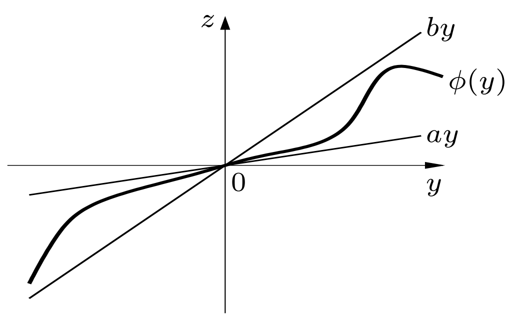
- Determine closed-loop stability given:

$\phi$ belongs to sector $[a, b]$

$\Updownarrow$

$$\boxed{a \le \frac{\phi(y)}{y} \le b}$$

"Absolute stability problem"



# Linear system + static nonlinearity

- Aizerman's conjecture (1949):

  *Closed-loop system is stable if stable for* $\phi(y) = ky$, $a \le k \le b$

  false (necessary but not sufficient)

- Sufficient conditions for closed-loop stability:

  $\left.\begin{array}{l}\text{Popov criterion (1960)}\\ \text{Circle criterion}\end{array}\right\}$ based on passivity

- The passivity approach:

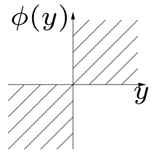
  (1). If $H$ is dissipative (i.e. if $\mathrm{Re}\big[H(j\omega)\big] > 0$ and $H$ is stable), then:

  $$\left.\begin{aligned} V &= x^T P x \\ \dot V &= yu - x^T Q x \end{aligned}\right\} \text{ for some } P > 0,\ \ Q > 0$$

  $$= -y\phi(y) - x^T Q x \qquad \leftarrow x = \text{state of } H$$

  (2). If $\phi$ belongs to sector $[0, \infty)$, then:

  $$y\phi(y) \ge 0 \qquad \leftarrow$$

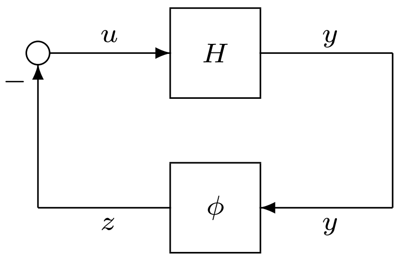



  $$\begin{aligned} (1)\ \&\ (2) \quad &\implies \quad \dot V \le -x^T Q x \\ &\implies \quad x = 0 \text{ is globally asymptotically stable} \end{aligned}$$

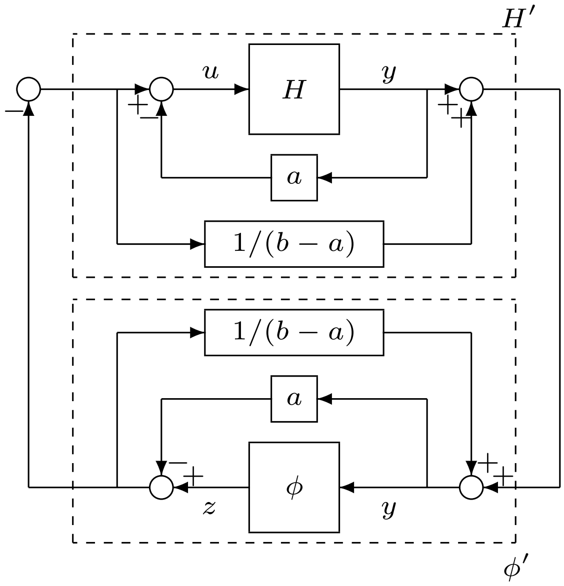


# Circle criterion

Use loop transformations to generalize the approach for $\left\{\begin{array}{l} H \text{ not passive} \\ \phi \notin [0, \infty) \end{array}\right.$




$\longleftrightarrow$ equiv. to




$\phi \in [a, b]$ $\quad a, b$ arbitrary

$$\phi \in [a, b] \implies \phi' \in [0, \infty]$$

$$H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a}$$

# Circle criterion

To make $H'(j\omega) = \dfrac{H(j\omega)}{1 + aH(j\omega)} + \dfrac{1}{b-a}$ dissipative, need:

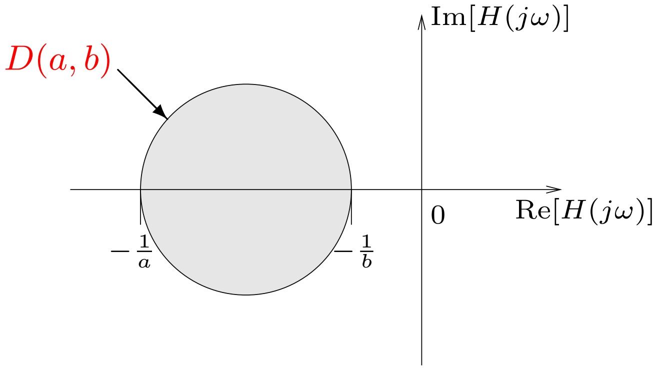
(i). $H'$ stable $\iff$ $\dfrac{H(j\omega)}{1 + aH(j\omega)}$ stable

$\Updownarrow$

Nyquist plot of $H(j\omega)$ goes through $\nu$ anti-clockwise encirclements of $-1/a$ as $\omega$ goes from $-\infty$ to $\infty$

($\nu$ = no. poles of $H(j\omega)$ in RHP)

(ii). $\mathrm{Re}\big[H'(j\omega)\big] > 0$ $\iff$ $\begin{cases} H(j\omega) \text{ lies outside } D(a,b) & \text{if } ab > 0 \\ H(j\omega) \text{ lies inside } D(a,b) & \text{if } ab < 0 \end{cases}$



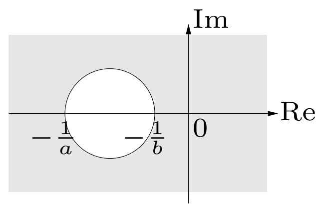



# Graphical interpretation of circle criterion

$x = 0$ is globally asymptotically stable if:

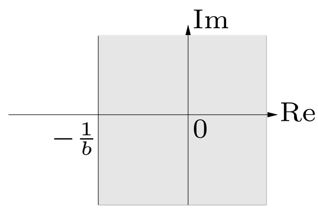
⋆ $0 < a < b$

$H(j\omega)$ lies in shaded region and does $\nu$ anti-clockwise encirclements of $D(a,b)$



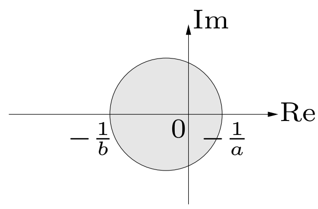

⋆ $b > a = 0$

$H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



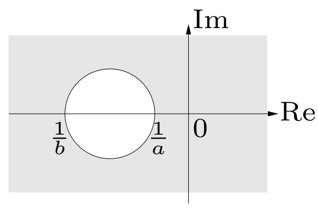

⋆ $a < 0 < b$

$H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)




⋆ $a < b < 0$

$-H(j\omega)$ lies in shaded region and does $\nu$ anti-clockwise encirclements of $D(-b,-a)$



# Circle criterion

- Circle criterion is equivalent to Nyquist criterion for $a = b > 0$

$$\uparrow$$

then $D(a,b) = -\dfrac{1}{a}$ (single point)

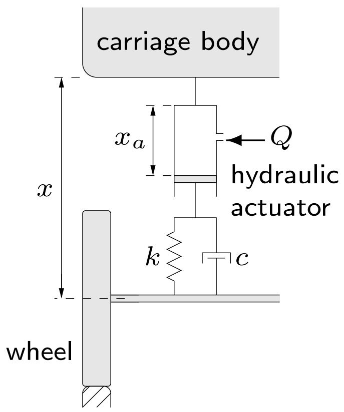
- Circle criterion is only sufficient for closed-loop stability for general $a, b$

- Results apply to time-varying static nonlinearity: $\phi(y,t)$



# Example: Active suspension system

- Active suspension system for high-speed train:




$$\begin{aligned} Q &= \phi(u) \\ \dot{x}_a &= Q/A \end{aligned}$$

$u$ : valve input signal
$Q$ : flow rate
$\phi$ : valve characteristics, $\phi \in [0.005, 0.1]$
$A$ : actuator working area

- Force exerted by suspension system on carriage body: $F_{\text{susp}}$

$$\begin{aligned} F_{\text{susp}} &= k(x_a - x) + c(\dot{x}_a - \dot{x}) \\ &= \big(k \int^t Q\,dt + cQ\big)/A - kx - c\dot{x}, \qquad Q = \phi(u) \end{aligned}$$

- Design controller to compensate for the effects of (constant) unknown load on displacement $x$ despite uncertain valve characteristics $\phi(u)$.

# Active suspension system contd.

- Dynamics:

$$F_{\text{susp}} - F = m\ddot{x}$$

$$\implies \quad m\ddot{x} + c\dot{x} + kx = \big(k\int^t Q\,dt + cQ\big)/A - F, \qquad Q = \phi(u)$$

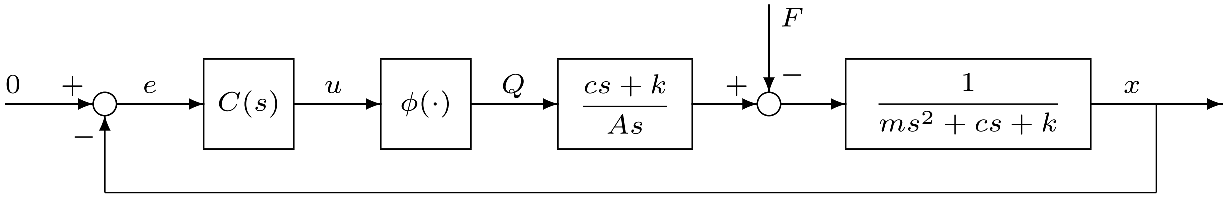
$F$ : unknown load on suspension unit
$m$ : effective carriage mass

- Transfer function model:

$$X(s) = \frac{cs+k}{ms^2+cs+k} \cdot \frac{Q(s)}{As} - \frac{F}{ms^2+cs+k} \qquad Q = \phi(u)$$

- Try linear compensator $C(s)$:

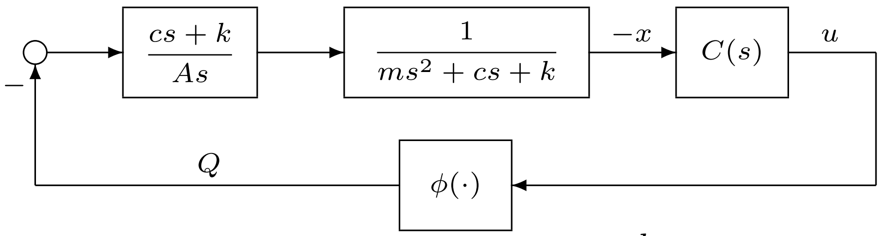
$$U(s) = C(s)E(s) \qquad e = -x, \quad \text{setpoint: } x = 0$$






# Active suspension system contd.

- For constant $F$, we need to stabilize the closed-loop system:




$$\text{linear system:} \quad H(s) = \frac{cs+k}{As(ms^2+cs+k)} \cdot C(s)$$

$$\text{static nonlinearity:} \quad \phi \in [0.005, 0.1]$$

- P+D compensator (no integral term needed):

$$C(s) = K(1+\alpha s) \quad \implies \quad H(s) = \frac{K(1+\alpha s)(cs+k)}{As(ms^2+cs+k)}$$

$H$ open-loop stable ($\nu = 0$)

- From the circle criterion, closed-loop (global asymptotic) stability is ensured if:

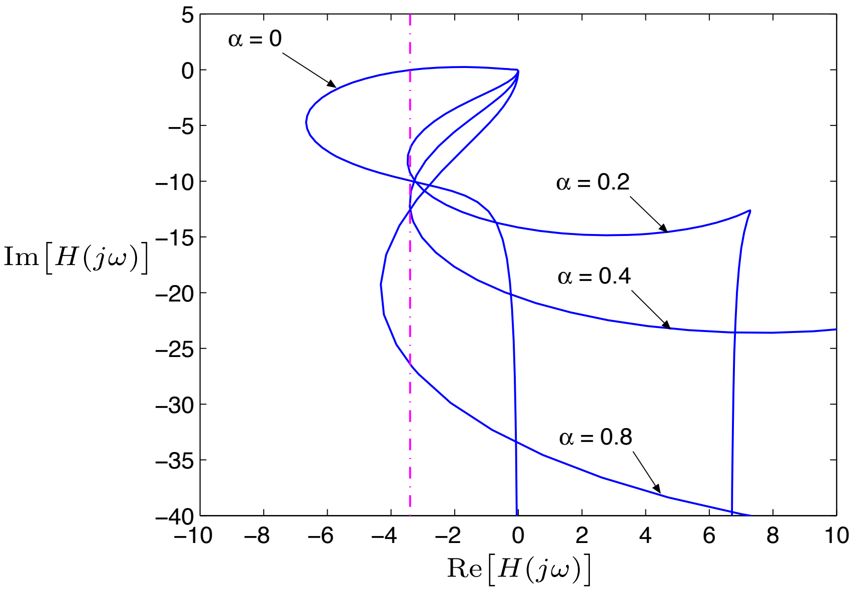
$$H(j\omega) \text{ lies outside } D(0.005, 0.1)$$

$$\uparrow$$

$$\text{sufficient condition: } \mathrm{Re}\big[H(j\omega)\big] > -10$$

# Active suspension system contd.

- Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$:




- To maximize gain margin:

  choose $\alpha = 0.2$ ← allows for largest $K$

  $K \leq 10/3.4 = 2.94$



# Summary

At the end of the course you should be able to do the following:

- Understand the basic Lyapunov stability definitions (lecture 1)
- Analyse stability using the linearization method (lecture 2)
- Analyse stability by Lyapunov's direct method (lecture 2)
- Determine convergence using Barbalat's Lemma (lecture 3)
- Understand how invariant sets can determine regions of attraction (lecture 3)
- Construct Lyapunov functions for linear systems and passive systems (lecture 4)
- Use the circle criterion to design controllers for systems with static nonlinearities (lecture 4)