15.094J: Robust Modeling, Optimization and Computation

Lecture 11: Power of Robust Policies in Adaptive Optimization

Motivation

- RO is tractable
- But how much do we lose in performance?
- Is it worse for multistage optimization?

Stochastic Model

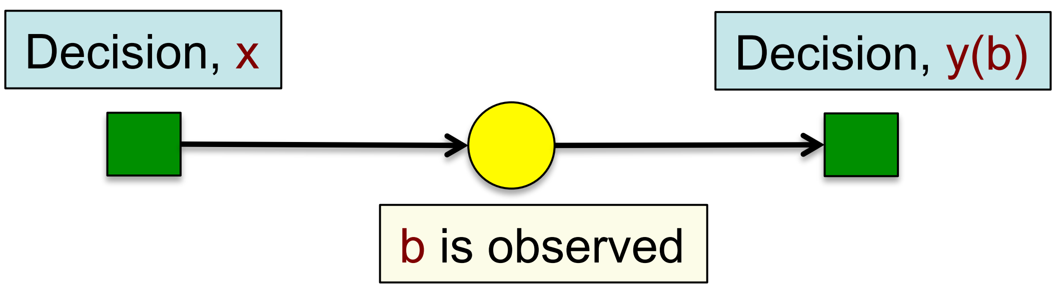
- **Two-stage Stochastic Optimization Model**

$$
\begin{aligned}
\textbf{zStoch} = \min c^T x + \mathbb{E}[d^T y(b)] & \\
Ax + By(b) &\geq b, \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

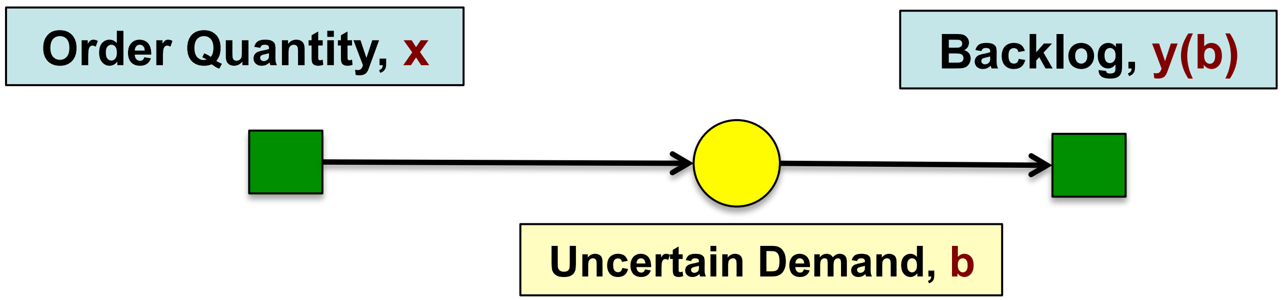
(Minimize **expected cost**)

(Uncertainty Set)

(Uncertain Right Hand Side)



Inventory Management



Minimize $\mathbb{E}_b\left[h(x-b)^+ \; + \; p(b-x)^+\right]$

$$
\begin{aligned}
\min \;\; \mathbb{E}_b\Big[\overbrace{h\big(x+y(b)-b\big)}^{\textbf{Holding Cost}} \; + \; \overbrace{p\cdot y(b)}^{\textbf{Backorder Penalty}}\Big] & \\
x+y(b) \;\; \geq \;\; & b, \;\forall\, b \\
x, y(b) \;\; \geq \;\; & 0
\end{aligned}
$$

Stochastic Model

- **Two-stage Stochastic Optimization Model**

$$
\begin{aligned}
\textsf{zStoch} = \min c^T x + \mathbb{E}[d^T y(b)] & & \\
Ax + By(b) & \geq & b, \ \forall b \in \mathcal{U} \\
x & \in & \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) & \in & \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

- **Computationally intractable in general**
- **Two-stage problem is #P-hard [Dyer and Stougie (2001)]**
- **Multi-stage problem is PSPACE-hard [Dyer and Stougie (2001)]**

Adaptive Optimization Model

- **Two-stage Adaptive Optimization Model**

$$
\begin{aligned}
\textbf{zAdapt} = \min c^T x + \boxed{\max_{b \in \mathcal{U}} \; d^T y(b)} & \\
Ax + By(b) \;&\geq\; b, \; \forall b \in \mathcal{U} \\
x \;&\in\; \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) \;&\in\; \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

(Minimize worst-case cost)

(Uncertainty Set)

(Uncertain Right Hand Side)

- **Still computationally intractable in general**
- **Even approximating LO within an factor of O(log m) is NP-hard [Feige et al.'07]**

Robust Optimization Model

$$
\begin{aligned}
\mathbf{zRob} = \min c^T x + d^T y & \\
Ax + By &\geq b, \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
\boxed{y} &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

(Minimize Cost of a **static solution**)

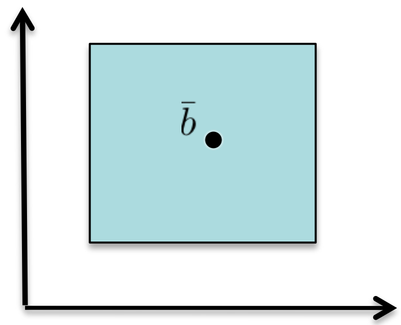
(Uncertainty Set)

(Uncertain Right Hand Side)

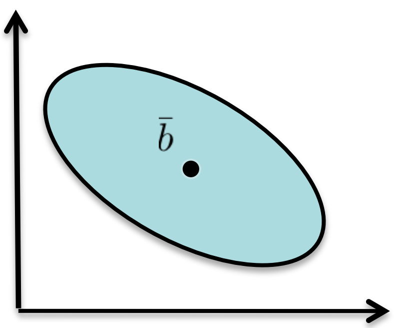
Solution **y** does **not depend on b**

- **Computationally tractable**
- **But does it give a highly conservative solution?**

Uncertainty Sets



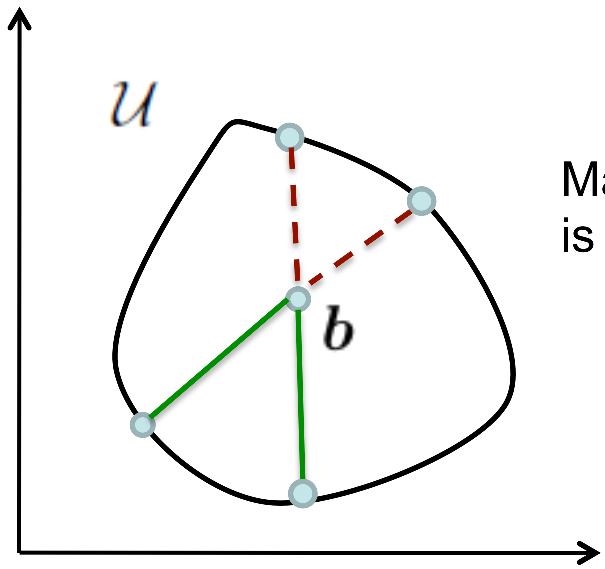
Hypercube : $||b - \bar{b}||_\infty \leq \beta$



Ellipsoid : $||\Sigma(b - \bar{b})||_2 \leq \beta$

Norm-Ball : $||\Sigma(b - \bar{b})||_p \leq \beta$

Symmetry of $\mathcal{U}$



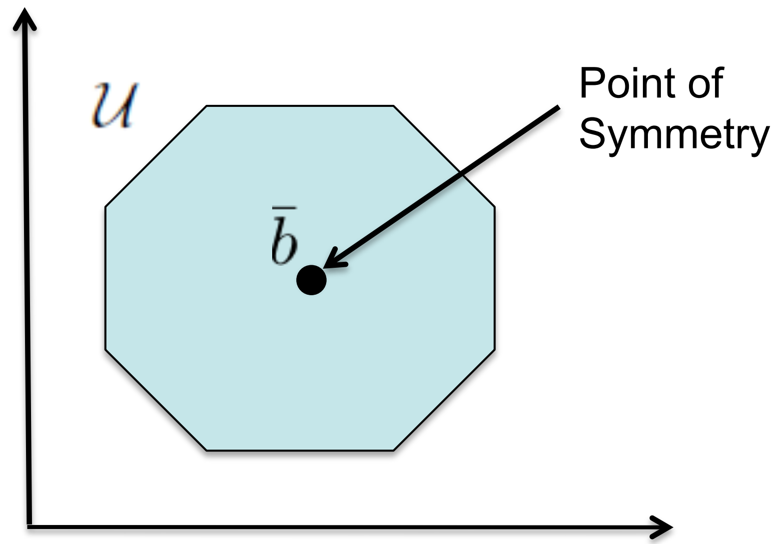
Maximum symmetry point
is **the point of symmetry of $\mathcal{U}$**

$\textbf{sym}(b, \mathcal{U})$: minimum ratio of red and green segments

$$\textbf{sym}(b, \mathcal{U}) = \max\{\alpha \mid b + \alpha \cdot (b - b') \in \mathcal{U}, \ \forall b' \in \mathcal{U}\}$$

$$\textbf{sym}(\mathcal{U}) = \max_{b \in \mathcal{U}} \ \textbf{sym}(b, \mathcal{U})$$

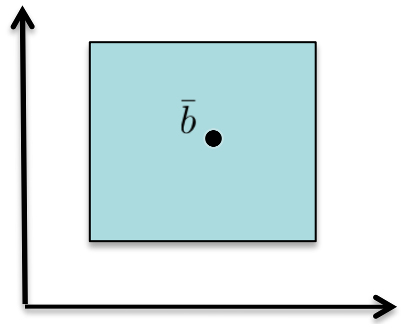
Example (s=1)



$$(\bar{b} - \delta) \in \mathcal{U} \Leftrightarrow (\bar{b} + \delta) \in \mathcal{U}, \ \forall \delta$$
$$\textsf{sym}(\mathcal{U}) = 1$$

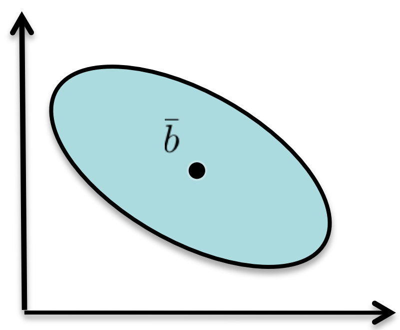
More Examples (s=1)

$\textbf{sym}(\mathcal{U}) = 1$



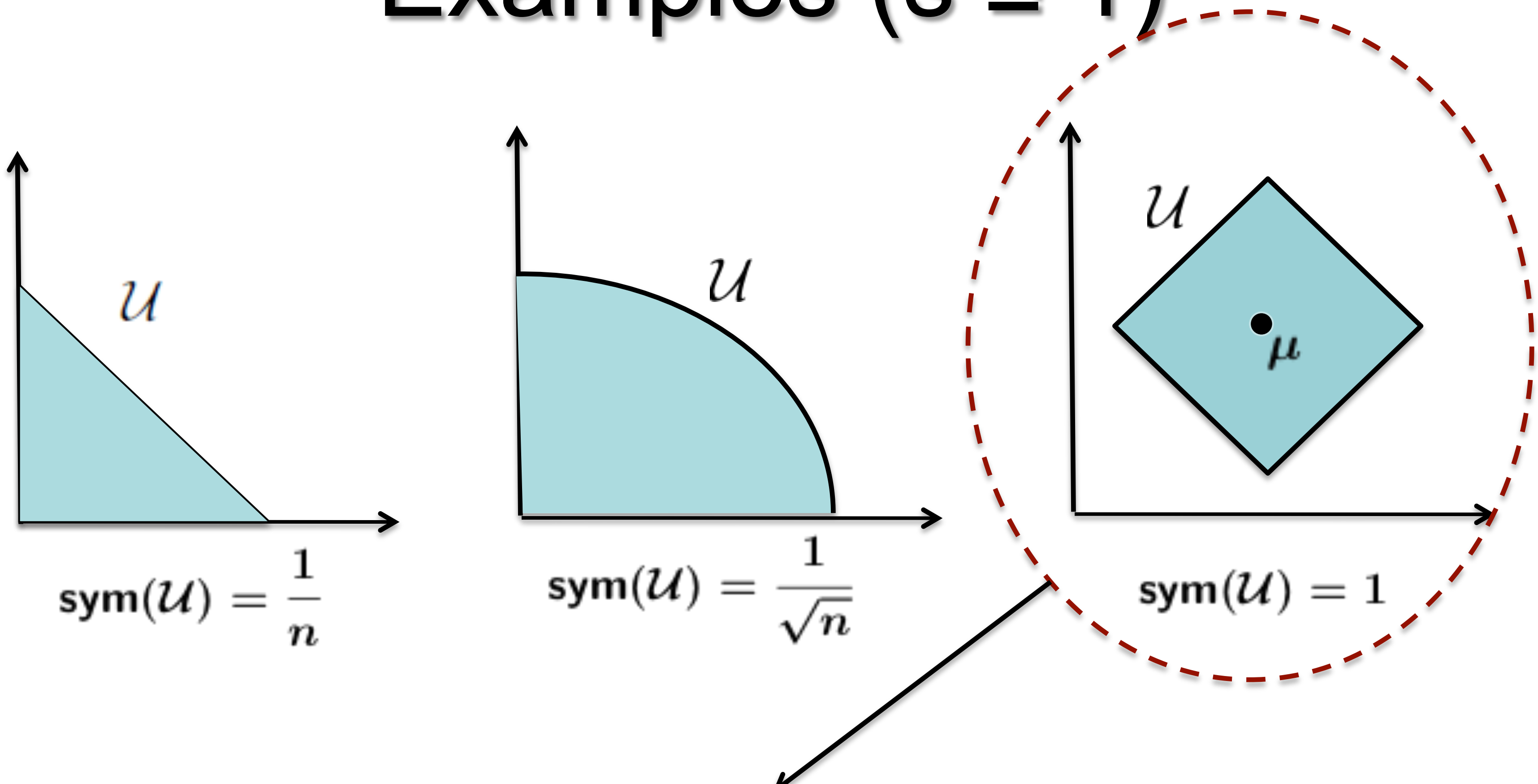
Hypercube : $||b - \bar{b}||_\infty \leq \beta$

$\textbf{sym}(\mathcal{U}) = 1$



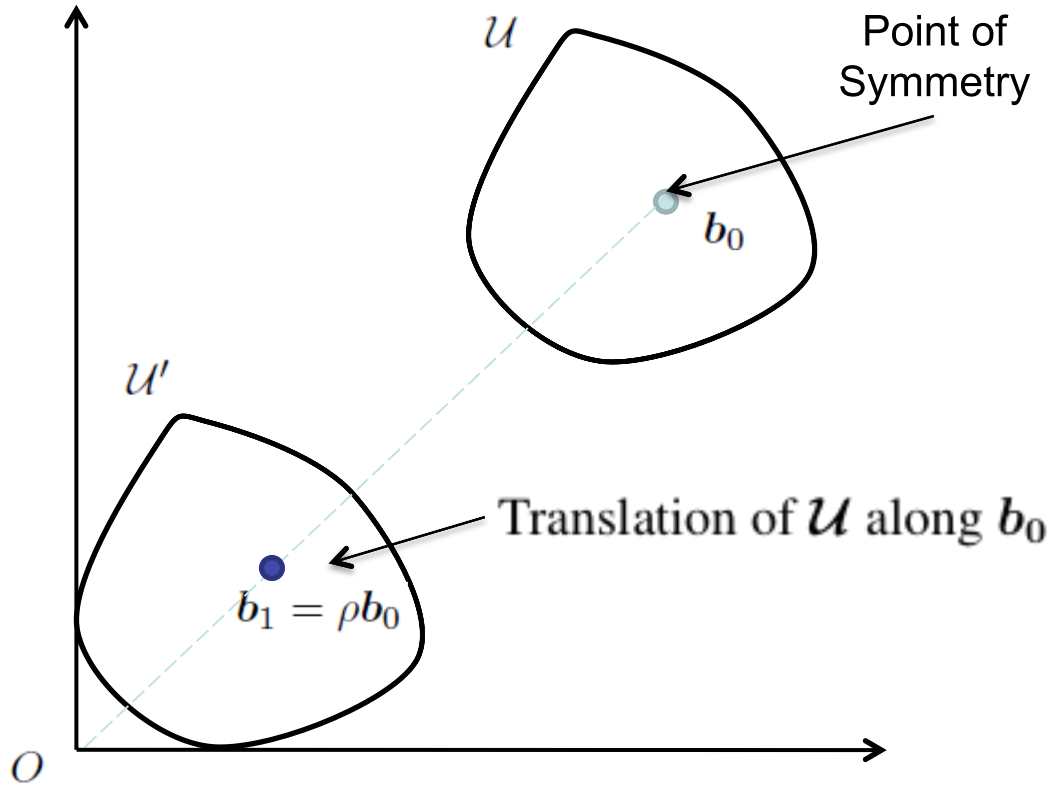
Ellipsoid : $||\Sigma(b - \bar{b})||_2 \leq \beta$

Examples (s ≤ 1)



$$\boxed{\mathcal{U} = \left\{ b \in \mathbb{R}^n_+ : \left| \frac{\sum_{i \in S} b_i - |S|\mu}{\sqrt{|S|}} \right| \leq 2,\ \forall S \subseteq N := \{1, \ldots, n\} \right\}}$$

Translation Factor of $\mathcal{U}$



$$\text{Translation factor of } \mathcal{U},\ \rho(\mathcal{U}) = \frac{||\boldsymbol{b_1}||}{||\boldsymbol{b_0}||}$$

Results: Robust Solutions

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}[d^T y(b)] & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_b d^T y(b) & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Theorem 1 *Let* $\rho = \rho(\mathcal{U})$ *and* $s = \mathsf{sym}(\mathcal{U})$. *Then,*

$$\frac{z_{\mathsf{Rob}}}{z_{\mathsf{Stoch}}} \leq \left(1 + \frac{\rho}{s}\right)$$

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Our Results: Implications

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}[d^T y(b)] & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

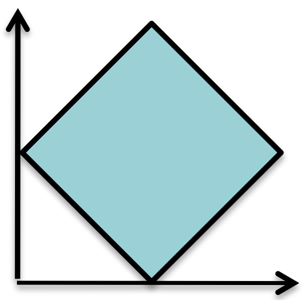
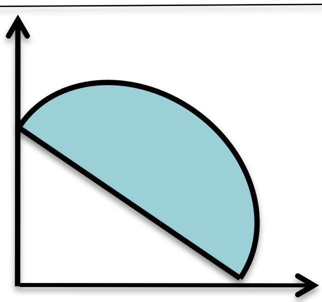
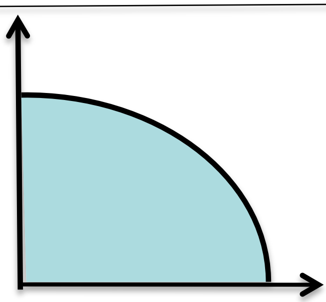
Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_b d^T y(b) & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	(1+ρ) ≤ 2	(1+ρ) ≤ 2
General (1/n < s ≤ 1)	(1+ρ/s)	(1+ρ/s)

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Bounds for different Sets

$\mathcal{U}(\rho = 1)$	sym($\mathcal{U}$)	**Stochasticity Gap**
	$\mathbf{1}$	$\mathbf{2}$
	$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$	$(\mathbf{1} + \sqrt{\mathbf{2}})$
	$\frac{\mathbf{1}}{\sqrt{\boldsymbol{n}}}$	$(\mathbf{1} + \sqrt{\boldsymbol{n}})$

Integer Variables

Stochastic (zStoch)

$$\begin{aligned} \min\ & c^T x + \mathbb{E}[d^T y(b)] \\ & Ax + By(b) \geq b, \ \forall b \in \mathcal{U} \\ & x \in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\ & y(b) \in \mathbb{R}^n_+ \end{aligned}$$

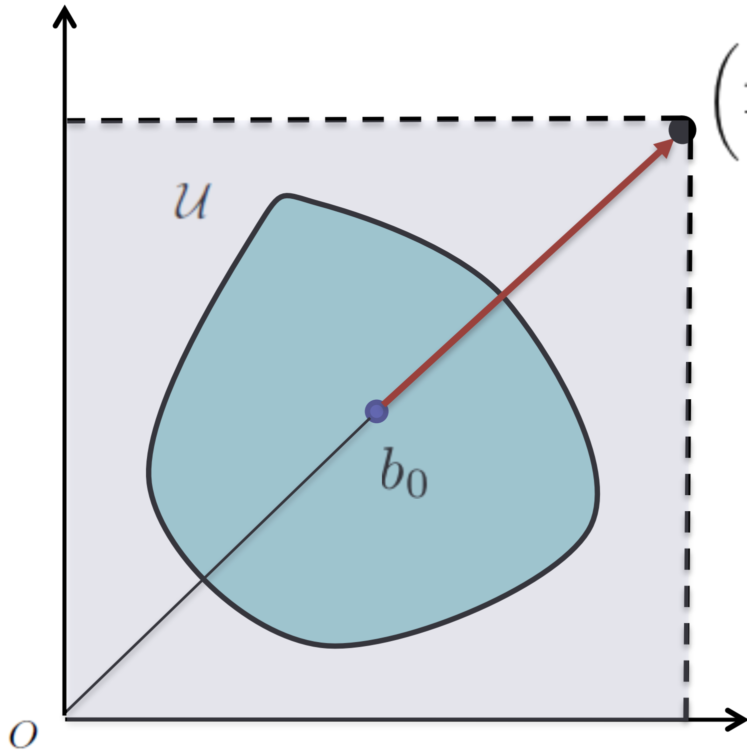
Adaptive (zAdapt)

$$\begin{aligned} \min\ & c^T x + \max_b d^T y(b) \\ & Ax + By(b) \geq b, \ \forall b \in \mathcal{U} \\ & x \in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\ & y(b) \in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \end{aligned}$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$\lceil (1+\rho) \rceil = 2$	$\lceil (1+\rho) \rceil = 2$
General ($1/n < s \leq 1$)	$\lceil (1+\rho/s) \rceil$	$\lceil (1+\rho/s) \rceil$

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Proof



$$\boxed{b \leq \tau \cdot b_0, \ \forall b \in \mathcal{U}}$$

Optimal Stochastic Solution: $x^*, y^*(b), \ \forall b \in \mathcal{U}$

Feasible Static Solution: $(\tau x^*, \tau y^*(b_0))$

$$A(\tau x^*) + B(\tau y^*(b_0)) \geq \tau b_0 \geq b, \forall b \in \mathcal{U}$$

Cost Analysis

$$z_{\textsf{Rob}} \leq \boldsymbol{\tau}(\boldsymbol{c}^T\boldsymbol{x}^* + \boldsymbol{d}^T\boldsymbol{y}^*(\boldsymbol{b}_0))$$

$$z_{\textsf{Stoch}} = \boldsymbol{c}^T\boldsymbol{x}^* + \mathbb{E}_b[\boldsymbol{d}^T\boldsymbol{y}^*(\boldsymbol{b})]$$

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{B}\boldsymbol{y}^*(\boldsymbol{b}) &\geq \boldsymbol{b} \\ \mathbb{E}_b\big[\boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{B}\boldsymbol{y}^*(\boldsymbol{b})\big] &\geq \mathbb{E}_b[\boldsymbol{b}] \\ \boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{B}\mathbb{E}_b[\boldsymbol{y}^*(\boldsymbol{b})] &\geq \boldsymbol{b}_0 \end{aligned}$$

$\mathbb{E}_b[\boldsymbol{y}^*(\boldsymbol{b})]$ is a feasible solution for $\boldsymbol{b}_0 \Rightarrow \boldsymbol{d}^T\boldsymbol{y}^*(\boldsymbol{b}_0) \leq \boldsymbol{d}^T\mathbb{E}_b[\boldsymbol{y}^*(\boldsymbol{b})]$

$$z_{\textsf{Rob}} \leq \boldsymbol{\tau} \cdot z_{\textsf{Stoch}} = \left(1 + \frac{\boldsymbol{\rho}}{\boldsymbol{s}}\right) z_{\textsf{Stoch}}$$

Our Results: Cost, RHS uncertainty

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}_{(b,d)}[d^T y(b,d)] & \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_{(b,d)} \ d^T y(b,d) & \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)		
General ($1/n \leq s \leq 1$)		

Assume: $\mathbf{E}_{b,d}[(b,d)] = (\bar{b},\bar{d})$ where $(\bar{b},\bar{d})$ is the **point of symmetry**

Our Results: Cost, RHS uncertainty

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}_{(b,d)}[d^T y(b,d)] & \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Adaptive (zAdapt)

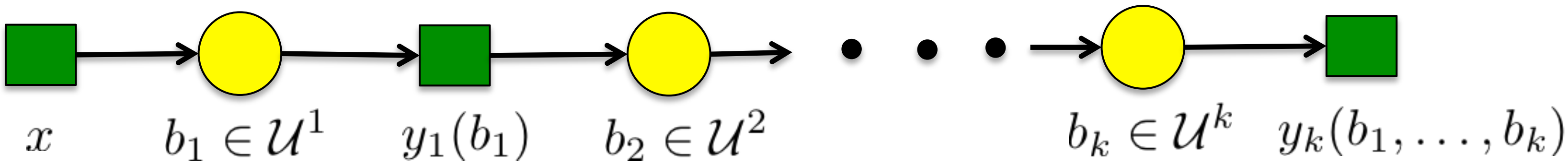
$$\begin{aligned} \min c^T x + \max_{(b,d)} d^T y(b,d) & \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

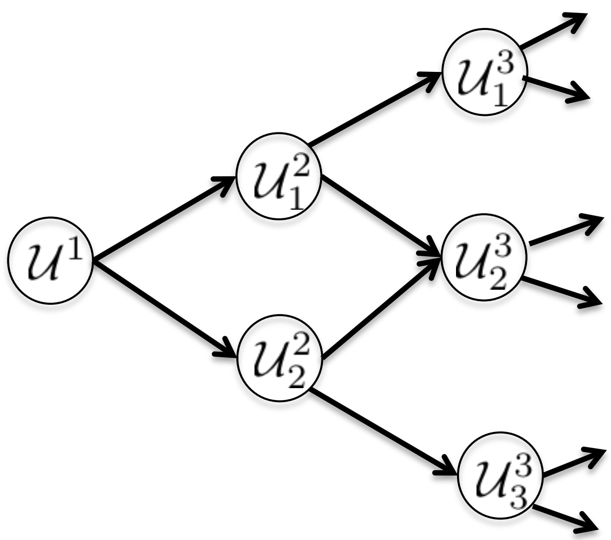
Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$\Omega(m)$	$(1+\rho)^2 \leq 4$
General ($1/n \leq s \leq 1$)	$\Omega(m)$	$(1+\rho/s)^2$

Assume: $E_{b,d}[(b,d)] = (\bar{b},\bar{d})$ where $(\bar{b},\bar{d})$ is the **point of symmetry**

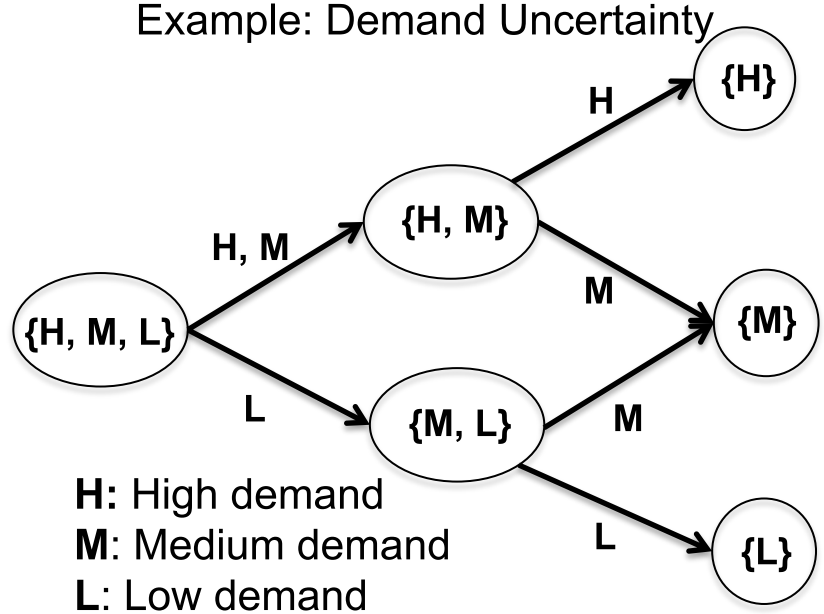
Multi-Stage Problems

Multi-Stage Stochastic Model





Example: Demand Uncertainty



Static solution is not a good approximation

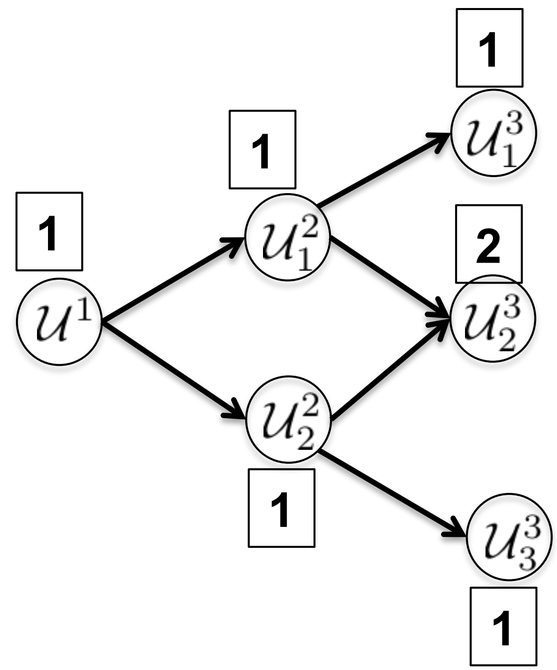
Fully-Adaptive Solution

- Requires optimal decision for each possible scenario
- Uncountable set of scenarios (typically)
- Suffers from the curse of dimensionality
- PSPACE hard to compute in general

Finitely Adaptive Solution

- Partition the scenarios into a small number of sets
- Compute a static solution for each set of scenarios in the partition
- Finite (small) number of solutions in each stage

- partition the scenarios according to the realized paths in the uncertainty network
- Number of paths is finite (small)
- In each stage k, compute a solution for each path from stage 1 to stage k
- For any path P, the solution is feasible for all possible parameter realizations on P



Performance of Finitely Adaptive Solution

Theorem 2 *Let* $\rho = \max\ \rho(\mathcal{U})$ *and* $s = \min\ \mathsf{sym}(\mathcal{U})$ *over all* $\mathcal{U}$*. Also, for all* $\mathcal{U}$*, let,*

$$\mathbb{E}[b] \geq \mathbf{b_0},$$

where $\mathbf{b_0}$ *is the point of symmetry of* $\mathcal{U}$*. Then,*

$$\textit{Cost of an optimal finitely adaptable solution} \leq \left(1 + \frac{\rho}{s}\right) z_{\mathsf{Stoch}}\ .$$

- **Finitely Adaptive solution is a good approximation of the multi-stage stochastic problem**

- **Performance bound = 2 for uncertainty sets with symmetry = 1**

Bounds for Multi-stage Problems

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Symmetric (s=1)	$(1+\rho) \leq 2$	$(1+\rho) \leq 2$
General ($1/n < s \leq 1$)	$(1+\rho/s)$	$(1+\rho/s)$

Finitely Adaptive Solution: Multi-stage Adaptive Optimization Problem

Theorem 3 *Consider the adaptive problem with both rhs and cost uncertainty, i.e., both* $\mathbf{b}, \mathbf{d}$ *are uncertain. Let* $\rho = \max\ \rho(\mathcal{U})$ *and* $s = \min\ \mathsf{sym}(\mathcal{U})$ *over all* $\mathcal{U}$*. Then,*

$$\textit{Cost of an optimal finitely adaptable solution} \leq \left(1 + \frac{\rho}{s}\right)^2 z_{\mathsf{Adapt}} .$$

- **Finitely Adaptive solution is a good approximation of the multi-stage adaptive problem with both rhs and cost uncertainty**
- **Performance bound ≤ 4 for uncertainty sets with symmetry = 1**
- **Finitely adaptive solution is not a good approximation for the corresponding stochastic problem**

Conclusions

- Choose uncertainty sets carefully
- Criteria: Tractability and Symmetry
- Finite Adaptability, which humans heuristically use is **near optimal** if the uncertainty set is symmetric, that is reasonable known unknowns