3 Lyapunov-like convergence analysis [11]

A common problem with Lyapunov's direct method is that it allows asymptotic stability to be concluded only under extremely restrictive conditions. Thus it may be easy to find a positive definite function V which is non-increasing along system trajectories, but finding a V for which $\dot{V}$ is negative definite along system trajectories is usually a much harder task. However it is often possible to determine whether the system state, or at least some component of the state, converges asymptotically using a Lyapunov-like analysis based on a non-positive (rather than negative definite) derivative $\dot{V}$.

The techniques for Lyapunov-like convergence analyses are slightly different for autonomous and non-autonomous dynamics. This section first considers the case of non-autonomous systems of the form

$$\dot{x} = f(x,t), \qquad f(0,t) = 0, \ \forall t. \tag{3.1}$$

The method for autonomous systems, though more powerful and closer in principle to Lyapunov's direct method, will then be treated as a special case of that for non-autonomous systems.

3.1 Convergence of non-autonomous systems

Suppose that we have found a positive definite function $V(x,t)$ whose derivative $\dot{V}$ along trajectories of (3.1) satisfies

$$\dot{V}(x,t) \leq -W(x) \leq 0 \tag{3.2}$$

for some non-negative function W. The aim of the convergence analysis is to show that $W\big(x(t)\big)$ tends to zero, and therefore that $x(t)$ converges to the set of states on which $W(x) = 0$. Specifically, (3.2) implies that V cannot increase along system trajectories, and since $V \geq 0$ due to the assumption that V is positive definite, it follows that $V\big(x(t),t\big)$ tends to a finite limit as $t \to \infty$:

$$0 \leq \lim_{t\to\infty} V\big(x(t),t\big) \leq V\big(x(t_0),t_0\big),$$

[11] See *Slotine and Li* §3.4.3 pp68–76 and §4.5 pp122–126 or *Vidyasagar* §5.3 pp176–186

where $x(t)$ is a trajectory of (3.1) with initial condition $x(t_0)$. Integration of (3.2) therefore yields a finite bound on the integral of $W(x(t))$ over the interval $t_0 \leq t < \infty$,

$$\int_{t_0}^{\infty} W\big(x(t)\big)\, dt \leq V\big(x(t_0),t_0\big) - \lim_{t\to\infty} V\big(x(t),t\big). \tag{3.3}$$

Under certain conditions on W this bound leads to the conclusion that $W\big(x(t)\big)$ converges to zero as t tends to ∞.

At first sight it may seem obvious that a non-negative function $\phi(t)$, which has finite integral over the infinite interval $0 \leq t < \infty$, necessarily converges to zero. However this is not true in general. For example if ϕ were allowed to be discontinuous, then it would be possible to change the value of ϕ at any individual point t without affecting the integral of ϕ. In fact continuity is not enough to ensure convergence of ϕ, since it is possible to construct functions which are continuous at any finite time t, but which effectively become discontinuous as $t \to \infty$. Figure 10 gives an example of such a function; here $\phi(t)$ does not converge to a limit as $t \to \infty$ even though ϕ is continuous and the integral $\int_0^\infty \phi(t)\, dt$ is finite. A condition which does guarantee that $\phi(t) \to 0$ given that $\int_0^t \phi(s)\, ds$ converges to a finite limit as $t \to \infty$ is provided by a technical result known as Barbalat's lemma, which is stated here in a slightly simplified form.

Barbalat's lemma. For any function $\phi(t)$, if
(a). $\dot{\phi}(t)$ exists and is finite for all t
(b). $\lim_{t\to\infty} \int_0^t \phi(s)\, ds$ exists and is finite
then $\lim_{t\to\infty} \phi(t) = 0$.

From (3.3) it can therefore be concluded that $W(x(t))$ converges to zero asymptotically as $t \to \infty$ provided the derivative $\dot{W}$ of W along trajectories of (3.1) remains finite at all times t. Using the chain rule we have

$$\dot{W}(x) = \nabla W(x) f(x,t),$$

and $\dot{W}\big(x(t)\big)$ must therefore remain finite if W and f are continuous with respect to their arguments and $x(t)$ is bounded for all t.

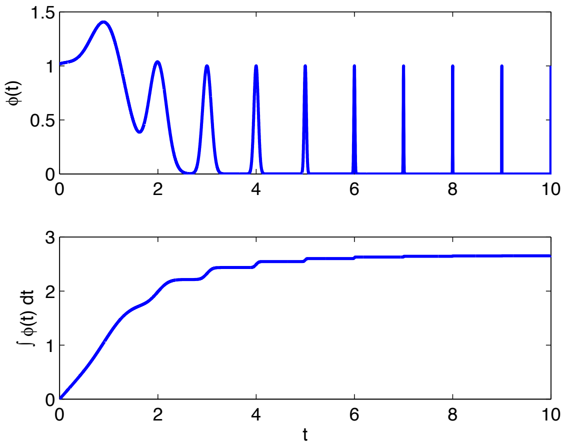

Figure 10: Example of a non-negative continuous function which does not converge to zero even though its integral tends to a finite limit as $t \to \infty$. Upper plot: $\phi(t) = \sum_{k=0}^{\infty} e^{-4^k (t-k)^2}$, lower plot: $\int_0^t \phi(s)\, ds$.

The following theorem summarizes this argument. For convenience the theorem assumes V to be decrescent and radially unbounded in order to ensure that $x(t)$ remains finite for all $t \geq t_0$ given arbitrary initial conditions $x(t_0)$. Clearly these additional assumptions are not needed if the boundedness of $x(t)$ is ascertained via a Lyapunov analysis based on an alternative Lyapunov function.

Theorem 3.1 (Convergence for non-autonomous systems)**.** *Let the function f in (3.1) be continuous with respect to x and t, and assume that there exists a continuously differentiable scalar function $V(x,t)$ such that:*

(a). $V(x,t)$ is positive definite, radially unbounded and decrescent

(b). $\dot{V}(x,t) \leq -W(x) \leq 0$

where W is a continuous function. Then all state trajectories $x(t)$ of (3.1) are globally bounded, and satisfy $W\big(x(t)\big) \to 0$ as $t \to \infty$.

3.2 Convergence of autonomous systems

The preceding analysis can be used to derive stronger convergence properties when the system dynamics are autonomous, ie. of the form

$$\dot{x} = f(x), \qquad f(0) = 0. \tag{3.4}$$

Essentially this is because it is much easier to determine whether the system state remains within a given region of state space if the system is autonomous rather than non-autonomous. Consequently an estimate of a set of points, say R, to which the state $x(t)$ of (3.4) converges can be refined using the criterion that, having entered R, $x(t)$ must remain within R at all future times.

To illustrate the approach, consider again the mass-spring-damper example of section 2.3. We have already shown that the equilibrium $y = \dot{y} = 0$ is stable using the argument of theorem 2.9. But it is not possible to show asymptotic stability by applying Lyapunov's direct method to the function V defined in (2.11), since the derivative $\dot{V} = -c(\dot{y})\dot{y}$ is not a negative definite function of the system state. However the stability of the equilibrium ensures that y and $\dot{y}$ remain finite on any system trajectory with initial conditions sufficiently close to the equilibrium. Application of Barbalat's lemma to $\dot{V}$ therefore shows that such trajectories converge to the set of states on which $\dot{y} = 0$. But if $\dot{y} = 0$, the acceleration $\ddot{y}$ is non-zero whenever the displacement y is non-zero, and consequently the system state cannot remain indefinitely at any point for which $y \neq 0$. This implies that the state must converge to the set on which $\dot{y} = 0$ *and* $y = 0$, ie. the origin. Thus the equilibrium $y = \dot{y} = 0$ is asymptotically stable.

This argument is based on the concept of an invariant set.

Definition 3.2 (Invariant set)**.** A set M is an **invariant set** for a dynamic system if every system trajectory starting in M remains in M at all future times.

Thus an invariant set has the same properties as an equilibrium point but, unlike an equilibrium, can consist of more than just a single point. Useful examples of invariant sets are equilibrium points and limit cycles (ie. system trajectories which form closed curves in state space).

The method used above to show the asymptotic stability of the mass-spring-damper system can be generalized simply by noting that every bounded trajectory of an autonomous system converges to an invariant set. If it can be shown, using Barbalat's lemma for example, that the trajectories $x(t)$ of (3.4) converge to some set R, then it follows that $x(t)$ must converge to an invariant set M contained in R. The following theorem uses this observation to refine theorem 3.1, adapted for the case of autonomous dynamics.

Theorem 3.3 (Invariant set theorem)**.** *Let f in (3.4) be continuous, and assume that there exists a continuously differentiable scalar function $V(x)$ such that:*

(a). $V(x)$ is positive definite and radially unbounded

(b). $\dot{V}(x) \leq 0$

then all solutions $x(t)$ of (3.4) are globally bounded. Furthermore, let R be the set of all x for which $\dot{V}(x) = 0$, and let M be the largest invariant set in R. Then every state trajectory $x(t)$ of (3.4) converges to M as $t \to \infty$.

Note that:

1. The set M is defined in theorem 3.3 as the largest invariant set in R in the sense that M is the union of all invariant sets (eg. equilibrium points or limit cycles) within R.

2. The Lyapunov-like convergence theorems of this section contain the global asymptotic convergence results of Lyapunov's direct method (theorem 2.11) as special cases; namely when $W(x)$ in the convergence theorem for non-autonomous systems is positive definite, and when M in the invariant set theorem consists only of the origin. In fact theorem 3.3 presents a significant generalization of Lyapunov's direct method by providing criteria for convergence to entire state trajectories such as limit cycles.

The region of attraction of the set M in theorem 3.3 is clearly the entire state space, since the conditions on V are required to hold for all x. However, by relaxing slightly the conditions of theorem 3.3, a more general local invariant set theorem can be derived for autonomous systems. This is useful for determining regions of attraction for systems whose stability properties are local rather than global.

Theorem 3.4 (Local invariant set theorem)**.** *Let f in (3.4) be continuous, and assume that there exists a continuously differentiable scalar function $V(x)$ such that:*

(a). for some constant $\overline{V} > 0$, the set Ω defined by $V(x) < \overline{V}$ is bounded

(b). $\dot{V}(x) \leq 0$ for all x in Ω

then Ω is an invariant set for (3.4). Furthermore, let R be the set of all points x in Ω for which $\dot{V}(x) = 0$, and let M be the largest invariant set in R. Then every solution $x(t)$ of (3.4) with initial conditions in Ω converges to M as $t \to \infty$.

To show that Ω is an invariant set, note that condition (b) of the theorem implies that $V\big(x(t)\big)$ cannot exceed the value $\overline{V}$ along any trajectory starting in Ω. Convergence of $x(t)$ to M can be shown by applying Barbalat's lemma to $\dot{V}\big(x(t)\big)$ along trajectories with initial conditions in Ω, and then invoking the condition that $x(t)$ must converge to an invariant set.[12]

Note that:

1. The local invariant set theorem does away with the requirement that V is positive definite by requiring instead that the level set Ω is bounded. This condition performs a similar function to the positive definite condition of Lyapunov's direct method since it ensures that the continuous function V is lower bounded on Ω (ie. $V(x) \geq \underline{V}$ for all x in Ω and some finite $\underline{V}$), and that any trajectory contained in Ω is bounded.

2. The set Ω is a region of attraction of the set M, though not necessarily the largest region of attraction.

[12] Barbalat's lemma is applicable here since V and $\ddot{V}$ are necessarily finite on trajectories in Ω due to the assumptions that V, $\dot{V}$ and f are continuous and Ω is bounded.

Example 3.5. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2 - 1) \end{aligned} \tag{3.5}$$

which has an equilibrium at $x_1 = x_2 = 0$. To determine the stability of this equilibrium point, define V as the function

$$V(x_1, x_2) = x_1^2 + x_2^2.$$

Then the derivative of V along trajectories of (3.5) is given by

$$\dot{V}(x_1, x_2) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1).$$

Let Ω be the set of points (x_1, x_2) satisfying $V(x_1, x_2) < 1$, then Ω is bounded (since Ω is a unit disc centred on the origin) and $\dot{V}(x_1, x_2) \leq 0$ for all x in Ω. Thus all conditions of the local invariant set theorem are satisfied, and since the subset of Ω on which $\dot{V}(x_1, x_2) = 0$ is simply the point $x_1 = x_2 = 0$, it follows that any trajectory starting within Ω converges to the origin. Therefore the origin is an asymptotically stable equilibrium and Ω is contained within its region of attraction.

Even though Ω is the largest region of attraction that can be determined with this choice of V (since $\dot{V}(x_1, x_2) \not\leq 0$ for some (x_1, x_2) in the set $V(x_1, x_2) < \overline{V}$ whenever $\overline{V} > 1$), it cannot be concluded without further analysis that Ω contains every point in the region of attraction of the origin. In this example however, Ω is in fact the largest region of attraction of the origin since the set Ω' defined by $V(x_1, x_2) = 1$ is an unstable limit cycle, ie. all trajectories starting from points arbitrarily close to Ω' either converge asymptotically to the origin or tend to infinity.

To prove this, first note that the first and second derivatives of $x_1^2 + x_2^2 - 1$ along system trajectories are zero at any point satisfying $x_1^2 + x_2^2 = 1$. On every state trajectory with initial conditions $x_1(t_0), x_2(t_0)$ satisfying $x_1^2(t_0) + x_2^2(t_0) = 1$ we therefore have $x_1^2(t) + x_2^2(t) = 1$ for all $t \geq t_0$, which implies that Ω' is a limit cycle of (3.5). Next consider the function

$$V'(x_1, x_2) = (x_1^2 + x_2^2 - 1)^2,$$

which has derivative

$$\dot{V}'(x_1, x_2) = 4(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)^2$$

along trajectories of (3.5). Since V' is minimized on the set Ω' and $\dot{V}'$ is positive whenever $x_1^2 + x_2^2 \neq 0$, it can be concluded that Ω' is an unstable limit cycle. ◊

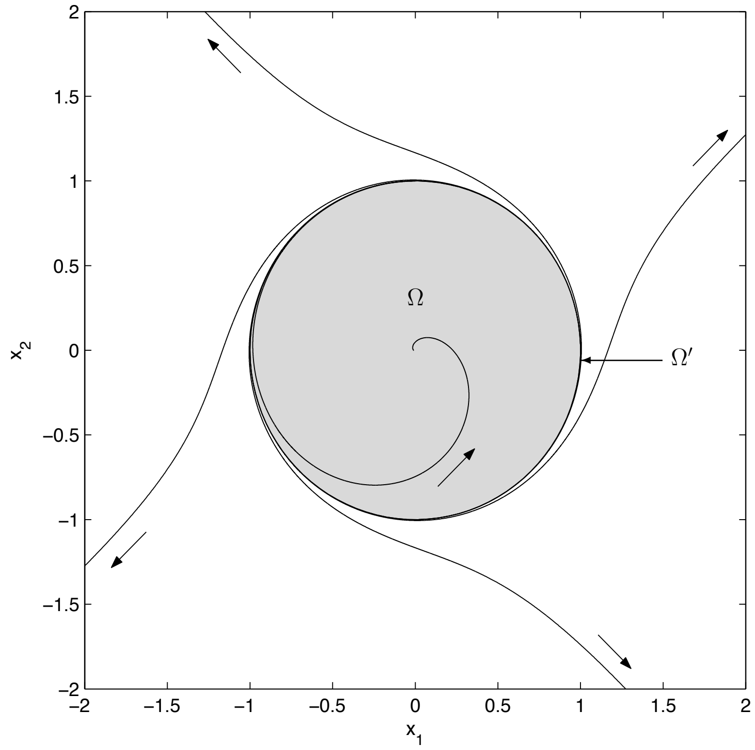

Figure 11: The region of attraction Ω and the unstable limit cycle Ω' in example 3.5.

4 Linear systems and passive systems

This section builds on the discussion of Lyapunov's direct method in the previous section by considering two classes of dynamic system, linear and passive systems. It is possible to find Lyapunov functions for these classes of system systematically, and the problem of determining Lyapunov functions for complex nonlinear systems is therefore simplified if linear or passive subsystems can be identified as components of the original dynamics. This section first considers the stability of linear systems within the Lyapunov stability framework, and then describes passivity and the properties of interconnected passive systems. It concludes with a discussion of the stability of feedback systems in which the forward path contains a linear subsystem and the feedback path contains a memoryless (though possibly time-varying) nonlinearity.

4.1 Linear systems [13]

The Lyapunov stability analysis of linear time-invariant (LTI) systems is based entirely on quadratic forms, i.e. functions of the form

$$V(x) = x^T P x.$$

Before giving the details of the method, we first describe some properties of matrices and quadratic forms.

- Any square matrix P can be expressed as the sum of a symmetric matrix P_1 (satisfying $P_1 = P_1^T$) and a skew-symmetric matrix P_2 (satisfying $P_2 = -P_2^T$):
$$P = P_1 + P_2, \qquad \begin{cases} P_1 = \frac{1}{2}(P + P^T), \\ P_2 = \frac{1}{2}(P - P^T). \end{cases}$$
- If P_2 is skew-symmetric, then $x^T P_2 x = x^T P_2^T x = -x^T P_2 x$, and hence $x^T P_2 x = 0$ for any vector x of conformal dimensions. Therefore a quadratic form $x^T P x$ with non-symmetric P is equivalent to the quadratic form $\frac{1}{2} x^T (P + P^T) x$ involving the symmetric part of P alone.

[13] See *Slotine and Li* §3.5.1 pp77–83 or *Vidyasagar* §5.4.2 pp196–202

- A symmetric matrix P is **positive definite** (denoted $P > 0$) if the quadratic form involving P is a positive definite function, i.e. if
$$x^T P x > 0 \text{ for all } x \neq 0.$$
If $x^T P x \geq 0$ for all $x \neq 0$, then P is **positive semidefinite** (denoted $P \geq 0$). Similarly, P is **negative definite** $(P < 0)$ or **negative semidefinite** $(P \leq 0)$ respectively if $x^T P x < 0$ or $x^T P x \leq 0$ for all non-zero x.
- Any symmetric matrix P can be decomposed as
$$P = U \Lambda U^T,$$
where Λ is a diagonal matrix of eigenvalues of P, and U is an orthogonal matrix $(U^T U = \mathrm{I})$ containing the eigenvectors of P. Thus
$$x^T P x = z^T \Lambda z, \quad z = U^T x,$$
and it follows that P is positive definite if and only if all eigenvalues of P are strictly positive. By the same argument P is positive semidefinite, negative semidefinite, or negative definite if all eigenvalues of P are greater than or equal to zero, less than or equal to zero, or strictly negative, respectively.

To find a Lyapunov function for a given stable linear system

$$\dot{x} = Ax, \tag{4.1}$$

consider the positive definite function

$$V(x) = x^T P x,$$

where P is some positive definite symmetric matrix (i.e. $P = P^T > 0$). The derivative of V along trajectories of (4.1) is

$$\dot{V}(x) = x^T (A^T P + PA) x,$$

and the system is therefore globally asymptotically stable by Lyapunov's direct method if there exists a positive definite matrix Q satisfying the condition

$$A^T P + PA = -Q. \tag{4.2}$$

This is known as a Lyapunov matrix equation.

For arbitrary positive definite P, the matrix Q in (4.2) will not necessarily be positive definite. However it is always possible to choose a matrix $Q = Q^T > 0$ and solve (4.2) for $P = P^T > 0$ whenever (4.1) is stable. To see this, suppose that a particular Q has been chosen. Then along a trajectory of the system (4.1) with initial condition $x(t_0)$, the derivative of the quadratic form V defined in terms of a P satisfying the Lyapunov equation (4.2) is given by

$$\dot{V}\big(x(t)\big) = -x^T(t_0)e^{A^T(t-t_0)}Qe^{A(t-t_0)}x(t_0).$$

Since (4.1) is stable, $V\big(x(t)\big)$ must converge to zero as $t \to \infty$. The integral of $\dot{V}$ with respect to t therefore gives

$$x(t_0)^T Px(t_0) = x^T(t_0)\left[\int_0^\infty e^{A^Tt}Qe^{At}dt\right]x(t_0).$$

This must be true for any initial condition $x(t_0)$, so that

$$P = \int_0^\infty e^{A^Tt}Qe^{At}dt. \tag{4.3}$$

The integral on the RHS is well-defined due to the assumption that (4.1) is stable, and thus defines a positive definite matrix P satisfying the Lyapunov equation (4.2) for given positive definite Q.[14]

This discussion can be summarized as follows.

Theorem 4.1 (Lyapunov stability of LTI systems)**.** *A necessary and sufficient condition for a LTI system $\dot{x} = Ax$ to be stable is that, for any positive definite symmetric matrix Q, the Lyapunov matrix equation (4.2) has a unique solution P which is a positive definite symmetric matrix.*

The stability of the linear system (4.1) can therefore be determined by first choosing Q (e.g. $Q = \mathrm{I}$, the identity matrix, is a simple choice), solving (4.3) for P, and then checking whether P is positive definite. In addition to providing both necessary and sufficient conditions for stability, this approach is constructive in that it provides a systematic method of determining Lyapunov functions for the system (4.1).

[14] The matrix P in (4.3) must be positive definite since $x^TPx = 0$ implies that $x^Te^{A^Tt}Qe^{At}x = 0$ for all $t \geq 0$, and hence $x^TQx = 0$, which implies that $x = 0$ since $Q > 0$.

4.2 Passive systems [15]

Just as a Lyapunov function can be thought of as a generalization of the energy stored within a system, passivity generalizes the property of energy dissipation. Passive systems have no internal sources of power, or, more generally, the rate of dissipation of energy in a passive system exceeds the rate of internal energy generation. Motivated by the principle of energy conservation in physical systems, this leads to the following definition of passivity for nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x,u,t) \\ y &= h(x,t) \end{aligned} \qquad \begin{cases} u: \text{ system input,} \\ y: \text{ system output.} \end{cases} \tag{4.4}$$

Definition 4.2 (Passivity & dissipativity)**.** The system (4.4) is **passive** if there exists a continuous function $V(x,t)$ such that $V \geq 0$, $V(0,t) = 0$, and along system trajectories

$$\dot{V}(x,t) \leq y^T(t)u(t) - g(t) \tag{4.5}$$

for all t, for some function $g \geq 0$.
If $\int_0^\infty g(t)dt > 0$ whenever $\int_0^\infty y^T(t)u(t)dt \neq 0$, then (4.4) is **dissipative**.

The term y^Tu in (4.5) corresponds to the net power input to the system, while g represents the rate of energy dissipation within the system.

Example 4.3. The system:

$$m\ddot{x} + x^2\dot{x}^3 + x^7 = F$$

is a dissipative mapping (from F to $\dot{x}$) because

$$\frac{d}{dt}\Big(\frac{1}{2}m\dot{x}^2 + \frac{1}{8}x^8\Big) = \dot{x}F - x^2\dot{x}^4. \qquad \Diamond$$

There are usually many different possible ways to define the output of a system. However it makes sense to choose as the output a signal which forms the input to another connected subsystem. This highlights the usefulness of the passivity property: it allows a Lyapunov function for a system of interconnected passive

[15] See *Slotine and Li* §4.7 pp132–142

systems to be constructed from the sum of the functions V for each passive subsystem.

Consider for example the feedback connection of a pair of passive systems S_1 and S_2 shown in figure 12. From the definition of passivity, there exist non-negative functions V_1 and V_2 satisfying

$$\begin{aligned} \dot{V}_1 &= y_1^T u_1 - g_1, \qquad & g_1 \geq 0, \\ \dot{V}_2 &= y_2^T u_2 - g_2, \qquad & g_2 \geq 0. \end{aligned}$$

Therefore the function $V_1 + V_2$ has derivative

$$\begin{aligned} \dot{V}_1 + \dot{V}_2 &\leq y_1^T u_1 + y_2^T u_2 - g_1 - g_2 \\ &= y_1^T (u_1 + y_2) - g_1 - g_2 = -g_1 - g_2, \end{aligned}$$

and applying the convergence analysis of section 3, it follows that $g_1(t)$ and $g_2(t)$ converge to zero as $t \to \infty$. Furthermore, if V_1 and V_2 are positive definite decrescent functions of the state of the systems S_1 and S_2 respectively, then $V_1 + V_2$ is a Lyapunov function which shows uniform stability of the equilibrium of the overall system.

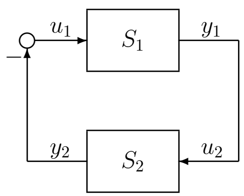

Figure 12: Feedback connection of passive systems.

Passive systems can form the building blocks of larger passive systems. In particular, both the feedback connection of figure 13(a) and the parallel connection of figure 13(b) result in passive dynamics mapping the overall input u to the overall output y whenever the subsystems S_1 and S_2 are both passive. Again this can be shown by considering the sum of the storage functions V_1 and V_2 associated with the individual subsystems S_1 and S_2.

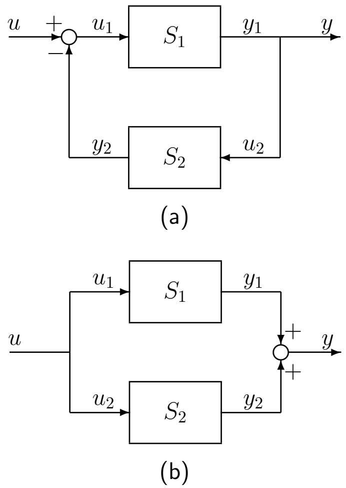

Figure 13: Interconnected passive subsystems. (a) Feedback connection. (b) Parallel connection.

Example 4.4 (Nonlinear adaptive control)**.** Suppose that we want to control the first order plant

$$\dot{x} = u + \theta x^2 \tag{4.6}$$

so that the state x converges to zero, where θ is an unknown but constant parameter. Let $\hat{\theta}$ be an estimate of θ which is allowed to vary over time according to the information on θ contained in measurements of x. The construction of the estimate $\hat{\theta}$ and the design of a control law based on this estimate is an adaptive control problem.

To solve this problem using a Lyapunov-based design method, consider the derivative of the function $V_1 = \frac{1}{2}x^2$ along system trajectories:

$$\dot{V}_1 = x(u + \theta x^2).$$

If the estimate $\hat{\theta}$ were exact (i.e. if $\hat{\theta} = \theta$), then the control law

$$u = -kx - \hat{\theta}x^2, \qquad k > 0, \tag{4.7}$$

would give $\dot{V}_1 = -kx^2$, and would therefore render $x = 0$ an asymptotically stable equilibrium point of the resulting closed-loop system.[16] However, for non-zero parameter estimate errors $\theta - \hat{\theta}$ the control law (4.7) may or may not be stabilizing, depending on the variation of $\hat{\theta}$ over time. With u defined as in (4.7), the derivative of V_1 is given by

$$\dot{V}_1 = (\theta - \hat{\theta})x^3 - kx^2.$$

Thus V_1 satisfies condition (4.5) of the passivity definition (with input $\theta - \hat{\theta}$, output x^3, and $g = kx^2$), which shows that this control law results in passive dynamics from the error $\theta - \hat{\theta}$ to x^3. It follows that both the estimation error and the equilibrium $x = 0$ of the closed-loop system can be stabilized by updating $\hat{\theta}$ so that the dynamics between x^3 and $-(\theta - \hat{\theta})$ are passive. One such update law is

$$\dot{\hat{\theta}} = x^3 \tag{4.8}$$

since this ensures that the function $V_2 = \frac{1}{2}(\theta - \hat{\theta})^2$ has derivative

$$\dot{V}_2 = -(\theta - \hat{\theta})\dot{\hat{\theta}} = -(\theta - \hat{\theta})x^3$$

(see figure 14).

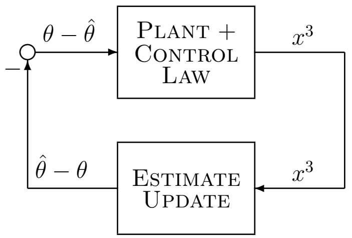

Figure 14: Feedback connection of the subsystems formed by the plant + control law and the estimate update law. Both are passive when the signals $\pm(\theta - \hat{\theta})$ and x^3 are considered as inputs and outputs.

To check that the combination of the control law (4.7) and parameter update law (4.8) meet the control objective, consider the function

$$V = V_1 + V_2 = \tfrac{1}{2}x^2 + \tfrac{1}{2}(\theta - \hat{\theta})^2.$$

[16] For this reason (4.7) is known as a *certainty equivalent* control law.

Along trajectories of the closed-loop system (4.6), (4.7), (4.8), we have

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -kx^2.$$

Therefore the estimate $\hat{\theta}(t)$ remains finite for all t and $x = 0$ is a globally asymptotically stable equilibrium of the closed-loop system. ◊

Linear passive systems

For linear systems passivity has a convenient interpretation in terms of the system frequency response. Consider the transfer function model

$$\frac{Y(s)}{U(s)} = H(s). \tag{4.9}$$

Theorem 4.5. *The system (4.9) is passive if and only if it is stable and the real part of its frequency response function is non-negative:*

$$\mathrm{Re}\big[H(j\omega)\big] \geq 0, \text{ for all } \omega \geq 0. \tag{4.10}$$

This is easy to show using Parseval's theorem, which implies that the integral of the product $y(t)u(t)$ is positive if and only if (4.10) is satisfied. Similarly (4.9) is dissipative if and only if it is stable and[17]

$$\mathrm{Re}\big[H(j\omega)\big] > 0, \text{ for all } \omega \geq 0. \tag{4.11}$$

The **Kalman-Yakubovich lemma** relates the frequency response condition (4.11) to the state space of (4.9). A simplified version of this is given next.

Lemma 4.6 (Kalman-Yakubovich)**.** *If (4.9) is dissipative, then there exist positive definite P and Q such that* [18]

$$V(x) = \tfrac{1}{2}x^T P x, \qquad \dot{V}(x) = yu - \tfrac{1}{2}x^T Q x$$

where x is the state of (4.9).

[17] Passive and dissipative linear systems are sometimes referred to as positive real systems and strictly positive real systems respectively.

[18] Q is only positive semidefinite if the system is passive (positive real) but not dissipative (strictly positive real).

Despite the useful properties of passive systems, it must be recognized that passivity and dissipativity are restrictive conditions for the following reasons.

- A passive system for which the function V in (4.5) is positive definite must be open-loop stable (i.e. stable for $u(t) = 0$).
- The relative degree of a passive system must be 0 or 1.

For linear systems, the relative degree is simply the number of poles of the transfer function $H(s)$ minus the number of zeros of $H(s)$, and the second condition above is a direct consequence of the passivity condition (4.10). More generally, the relative degree of a nonlinear system is defined as the number of times the output must be differentiated before an expression containing the input is obtained.

4.3 Linear systems and nonlinear feedback[19]

This section considers the stability of the class of feedback systems with the structure shown in figure 15. Here the forward path contains a linear time-invariant system H, and the feedback is via a nonlinear function ϕ. The dynamics of the linear system H are given by

$$\dot{x} = Ax + bu \qquad y = c^T x \tag{4.12}$$

where (A, b, c) is a state-space realization of the transfer function $H(s) = c^T(s\mathrm{I} - A)^{-1}b$, and the feedback path is specified by

$$u = -z \qquad z = \phi(y). \tag{4.13}$$

The nonlinearity ϕ is memoryless (i.e. the mapping $z = \phi(y)$ contains no dynamics), but is allowed to be time-varying. It is assumed that ϕ satisfies a sector condition, defined as follows.

Definition 4.7. A continuous function ϕ belongs to the sector $[a, b]$ if there exist two numbers a and b such that

$$a \le \frac{\phi(y)}{y} \le b \tag{4.14}$$

[19]See *Slotine and Li* §4.8 pp142–147, or *Vidyasagar* §5.5 pp219–235.

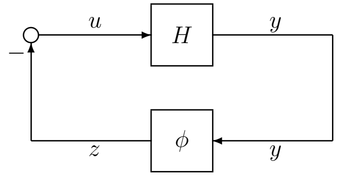

Figure 15: System structure.

whenever $y \neq 0$, and $\phi(0) = 0$.

The graphical interpretation of the sector condition (4.14) is simply that $\phi(y)$ lies between the two lines ay and by, as shown in figure 16.

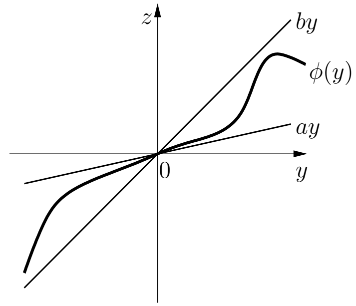

Figure 16: A function ϕ belonging to the sector $[a, b]$.

Systems of this form are of considerable practical interest and a number of different criteria for their stability have been determined. We will derive one of these, known as the circle criterion. This criterion has a graphical interpretation in terms of the frequency response $H(j\omega)$ which generalizes the Nyquist criterion for stability of linear feedback systems.

The circle criterion is based on the concept of passivity introduced in section 4.2. If the linear system H is dissipative, i.e. if all poles of H have strictly negative real part and the frequency response $H(j\omega)$ satisfies condition (4.11), then the closed-loop system is guaranteed to be asymptotically stable whenever ϕ belongs to the sector $[0, \infty)$. This follows from the Kalman-Yakubovich lemma and the fact that $\phi(y)$ then has the same sign as y.

More specifically, if H is dissipative then the Kalman-Yakubovich lemma ensures that there exists a positive definite symmetric matrix P such that the function

$$V(x) = \tfrac{1}{2}x^T P x$$

has derivative

$$\dot{V}(x) = yu - \tfrac{1}{2}x^T Q x,$$

for some positive definite Q, along trajectories $x(t)$ of (4.12). The feedback control law (4.13) therefore gives

$$\dot{V}(x) = -y\phi(y) - \tfrac{1}{2}x^T Q x$$

where $y\phi(y) \geq 0$ for all y due to the assumption that ϕ belongs to the sector $[0, \infty)$. Therefore

$$\dot{V}(x) \leq -\tfrac{1}{2}x^T Q x$$

which implies that the equilibrium $x = 0$ is globally asymptotically stable by Lyapunov's direct method.

This argument can be extended to cover cases in which H is not passive and ϕ belongs to a general sector $[a, b]$ using a technique known as loop transformation. The idea is to construct an equivalent closed-loop system in which the feedback path contains a nonlinearity belonging to the sector $[0, \infty)$ by adding feedforward and feedback loops to the subsystems of figure 15. This approach allows the preceding argument to be used when ϕ lies outside the sector $[0, \infty)$ by placing more severe restrictions than (4.10) on the frequency response $H(j\omega)$. Alternatively it enables the argument to be applied when H is open-loop unstable by exploiting more precise information on the sector to which ϕ belongs.

../Figures 17 and 18 show the two basic types of loop transformation. The signals u, y and z in each of these closed-loop systems are identical to those in the original system in figure 15, given the same initial conditions for the system H. For example the input to H in figure 17 is given by

$$u = u' - ky = -z' - ky = -(z - ky) + ky = -\phi(y),$$

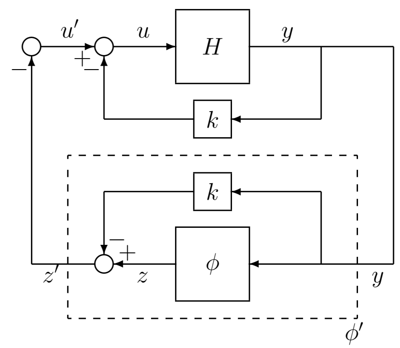

Figure 17: Loop transformation of the system in figure 15. If ϕ belongs to the sector $[a, b]$ then ϕ' belongs to the sector $[a - k, b - k]$.

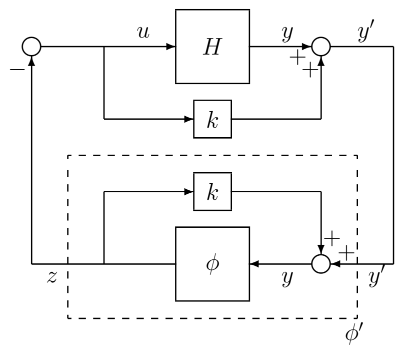

Figure 18: Loop transformation of the system in figure 15. If ϕ belongs to the sector $[a, b]$ then ϕ' belongs to the sector $[a/(1 - ak), b/(1 - bk)]$.

and similarly the input to H in figure 18 is

$$u = -z = -\phi(y' + kz) = -\phi\big((y + ku) + kz\big) = -\phi(y).$$

However the mapping ϕ' between y and z' in figure 17 is given by

$$\begin{aligned} z' &= \phi'(y) \\ &= \phi(y) - ky \end{aligned}$$

so ϕ' belongs to the sector $[a-k, b-k]$ whenever ϕ belongs to $[a,b]$. On the other hand the mapping ϕ' between y' and z in figure 18 satisfies

$$\begin{aligned} z = \phi'(y') \iff y' &= \phi'^{-1}(z) \\ &= \phi^{-1}(z) - kz. \end{aligned}$$

If ϕ belongs to the sector $[a,b]$ (so that the inverse mapping ϕ^{-1} belongs to $[1/b, 1/a]$), then the inverse of ϕ' therefore lies in the sector $[1/b-k, 1/a-k]$, which implies that ϕ' belongs to $[a/(1-ak), b/(1-bk)]$.

Both types of loop transformation are employed in figure 19 in order to construct a nonlinearity ϕ' which belongs to the sector $[0,\infty)$ whenever the original nonlinear function ϕ belongs to $[a,b]$. The Kalman-Yakubovich lemma can therefore be used to show that the closed-loop system in figure 15 is asymptotically stable provided the transformed linear subsystem H' in figure 19 is (i) strictly stable and (ii) satisfies $\mathrm{Re}[H'(j\omega)] > 0$ for all $\omega \geq 0$.

The subsystem H' contains a negative feedback loop around H with gain a. Consequently the stability of H' can be determined via the familiar Nyquist criterion for closed-loop stability. Thus if $H(s)$ has ν poles with positive real part, then all poles of $H'(s)$ have strictly negative real part if and only if the plot of $H(j\omega)$ makes ν anti-clockwise encirclements of the point $-1/a$ as ω goes from $-\infty$ to $+\infty$.

The condition on the real part of $H'(j\omega)$ has the following graphical interpretation. The frequency response of H' is given by

$$H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b-a}$$

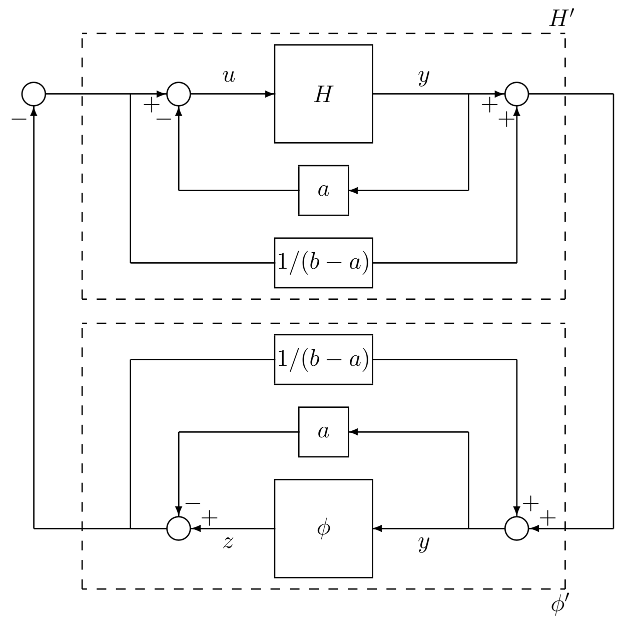

Figure 19: Loop transformation of the system in figure 15. If ϕ belongs to the sector $[a,b]$ then ϕ' belongs to the sector $[0,\infty)$.

and routine algebra shows that $\mathrm{Re}[H'(j\omega)] > 0$ for all $\omega \geq 0$ if and only if

$$\left|H(j\omega) + \tfrac{1}{2}\big(1/a + 1/b\big)\right|^2 > \tfrac{1}{4}\big(1/a - 1/b\big)^2, \quad \text{if } ab > 0 \tag{4.15a}$$

$$\left|H(j\omega) + \tfrac{1}{2}\big(1/a + 1/b\big)\right|^2 < \tfrac{1}{4}\big(1/a - 1/b\big)^2, \quad \text{if } ab < 0 \tag{4.15b}$$

for all $\omega \geq 0$. Let $D(a, b)$ denote the disc in the complex plane with centre $-\frac{1}{2}\big(1/a+1/b\big)$ and radius $\frac{1}{2}\big(1/a-1/b\big)$ (see figure 20). Then condition (4.15a) is satisfied for all $\omega \geq 0$ if the plot of $H(j\omega)$ does not enter the disc $D(a, b)$, while condition (4.15b) is satisfied if the plot of $H(j\omega)$ remains inside the disc $D(a, b)$ for all $\omega \geq 0$.

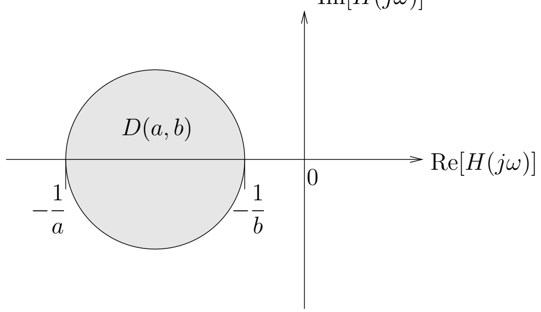

Figure 20: The disc $D(a, b)$ in the circle criterion.

These conditions are summarized in the following statement of the circle criterion.

Theorem 4.8 (Circle Criterion)**.** *If the system (4.12-4.13) satisfies the conditions:*

(a). $H(s)$ has ν poles with positive real part

(b). the nonlinearity ϕ belongs to the sector $[a, b]$

(c). one of the following is true

- *$0 < a < b$, and the Nyquist plot of $H(j\omega)$ does not enter the disc $D(a, b)$ and encircles it ν times anti-clockwise*
- *$0 = a < b$, $\nu = 0$, and the Nyquist plot of $H(j\omega)$ stays in the half-plane $\mathrm{Re}(s) > -1/a$*
- *$a < 0 < b$, $\nu = 0$, and the Nyquist plot of $H(j\omega)$ stays in the interior of the disc $D(a, b)$*
- *$a < b < 0$, and the Nyquist plot of $-H(j\omega)$ does not enter the disc $D(-a, -b)$ and encircles it ν times anti-clockwise*

then the equilibrium point $x = 0$ of the closed-loop system is globally asymptotically stable.

Note that:

1. The circle criterion allows the stability of the nonlinear system to be determined from the frequency response of the linear subsystem, which is relatively easy to compute experimentally.

2. The disc $D(a, b)$ performs roughly the same function in the circle criterion as the critical point $-1/a$ in the Nyquist criterion. As $b \to a$, the sector bound on ϕ shrinks, and the disc $D(a, b)$ tends towards the critical point $-1/a$.

3. Unlike the Nyquist criterion, the circle criterion gives sufficient (and not necessary) conditions for stability.

Example 4.9. A nonlinear system has the structure shown in figure 15, where ϕ is a nonlinear function belonging to some sector $[a, b]$, and H is a linear time-invariant system with transfer function

$$H(s) = \frac{10}{(s-1)(s+3)^2}.$$

Since $H(s)$ has one pole in the right half complex plane (at $s = 1$), the Nyquist plot of $H(j\omega)$ must encircle the disc $D(a, b)$ anti-clockwise once in order to ensure stability by the circle criterion. From the plot of $H(j\omega)$ in figure 21, this condition requires that a and b are both positive and is satisfied for example with $a = 1$, $b = 1.3$. ◊

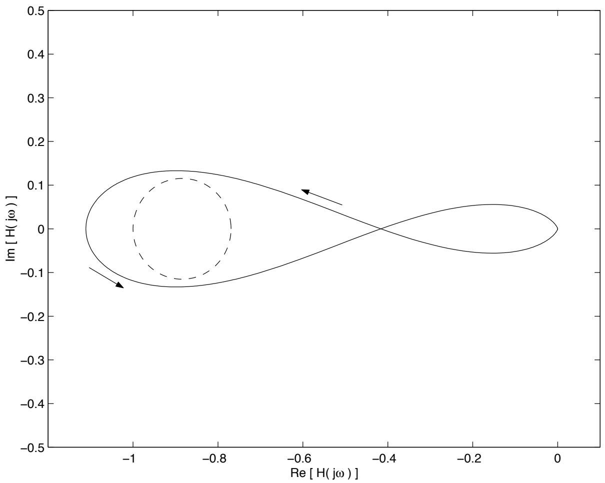

Figure 21: Nyquist plot of $H(j\omega)$ in example 4.9 (solid line) and the disc $D(1, 1.3)$ (dashed line).

4.4 Design example: Hydraulic active suspension system

An active suspension system for a train carriage uses a hydraulic actuator in series with mechanical springs and oil-filled dampers, arranged as shown in

Figure 22 for a single carriage wheel. The purpose of the hydraulic actuator is to reduce tilting caused by slowly varying loads acting on the carriage body (e.g. while cornering). The inclusion of the hydraulic actuator therefore allows for the use of a softer spring-damper assembly, resulting in better vibration isolation and a smoother ride.

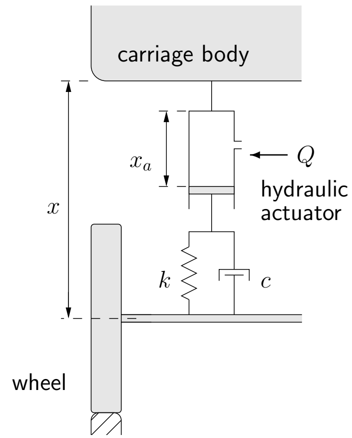

Figure 22: Active suspension for a train carriage.

The flow Q of fluid into the hydraulic actuator and its extension x_a (assuming incompressible fluid) are determined from

$$Q = \phi(u)$$
$$\dot{x}_a = Q/A$$

where u is the valve control signal, ϕ is a nonlinear function giving the valve characteristics, and A is the working area of the actuator. The function ϕ is not known exactly because of fluctuations in the pressure of the hydraulic fluid, but sector bounds on ϕ are available:

$$\phi \in [0.005, 0.1].$$

A controller for u is to be designed in order to compensate for the effects of unknown constant (or slowly varying) disturbance loads on the displacement x of the carriage body despite the uncertain valve characteristics $\phi(u)$.

The force exerted on the carriage by the suspension unit is F_{susp}:

$$\begin{aligned} F_{\mathsf{susp}} &= k(x_a - x) + c(\dot{x}_a - \dot{x}) \\ &= \big(k \int^t Q\,dt + cQ\big)/A - kx - c\dot{x}, \qquad Q = \phi(u) \end{aligned}$$

If F and m are respectively the disturbance load that is to be rejected and the effective carriage mass acting on the suspension, then the dynamic model of carriage plus suspension is given by

$$\begin{aligned} & F_{\mathsf{susp}} - F = m\ddot{x} \\ \implies & m\ddot{x} + c\dot{x} + kx = \big(k \int^t Q\,dt + cQ\big)/A - F, \qquad Q = \phi(u), \end{aligned}$$

or, in terms of transfer functions

$$X(s) = \frac{cs + k}{ms^2 + cs + k} \cdot \frac{Q(s)}{As} - \frac{F}{ms^2 + cs + k}, \qquad Q = \phi(u).$$

Therefore, defining the control signal u feedback via a compensator with transfer fn. $C(s)$:

$$U(s) = C(s)E(s), \quad e = -x, \quad \text{setpoint: } x = 0$$

leads to a closed-loop system with the block diagram of Figure 23.

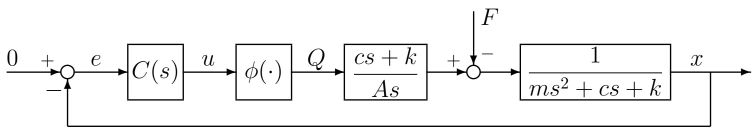

Figure 23: Closed-loop system with compensator $C(s)$.

Using $Q = \phi(u)$, $u = -C(s)X(s)$, this can be rearranged so that the nonlinearity is in the feedback path as shown in Fig. 24, which (for $F = 0$) now has the form of Fig. 15, with

$$\begin{aligned} H(s) &= \frac{cs + k}{As(ms^2 + cs + k)}.C(s) \\ \phi &\in [0.005, 0.1]. \end{aligned}$$

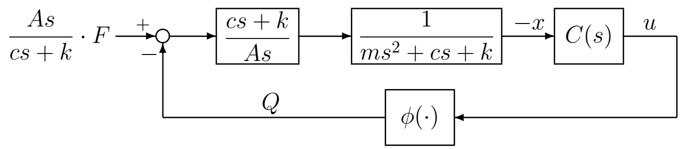

Figure 24: Closed-loop system.

To decide on the structure of $C(s)$, note that the hydraulics introduce an integrator into the forward path transfer function, so no integral term is needed in the controller in order to reject the constant disturbance F. Therefore consider a proportional + derivative compensator:

$$C(s) = K(1 + \alpha s) \qquad \implies H(s) = \frac{K(1 + \alpha s)(cs + k)}{As(ms^2 + cs + k)}$$

From the circle criterion, closed-loop (global asymptotic) stability is ensured if $H(j\omega)$ lies outside the disc $D(0.005, 0.1)$, for which a sufficient condition is that

$$\mathsf{Re}\big[H(j\omega)\big] > -10.$$

From the Nyquist plots of Figure 25, the value of α that maximizes the critical gain K and the corresponding value of K are given by

$$\begin{aligned} \alpha &= 0.4 \\ K &\le 10/3.4 = 2.94 \end{aligned}$$

Note that the maximum value for K is of interest here since the response of the closed-loop system is fastest when K is maximized.

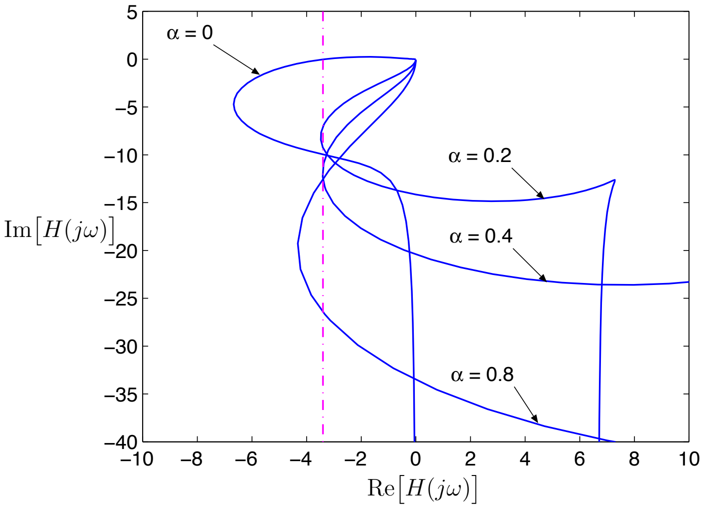

Figure 25: Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$