

# Markov Decision Processes

CMPT 729 G100

Jason Peng

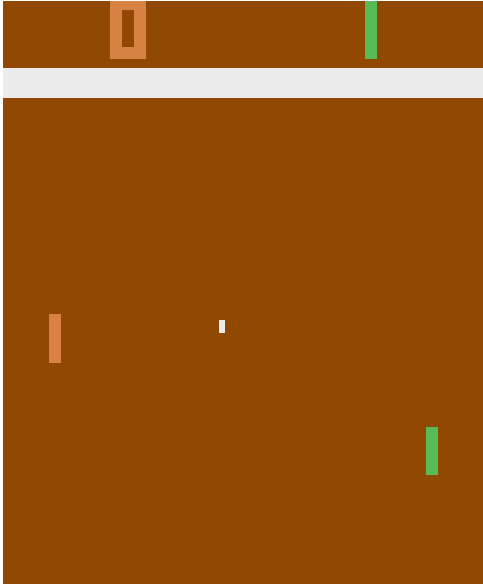
# Overview

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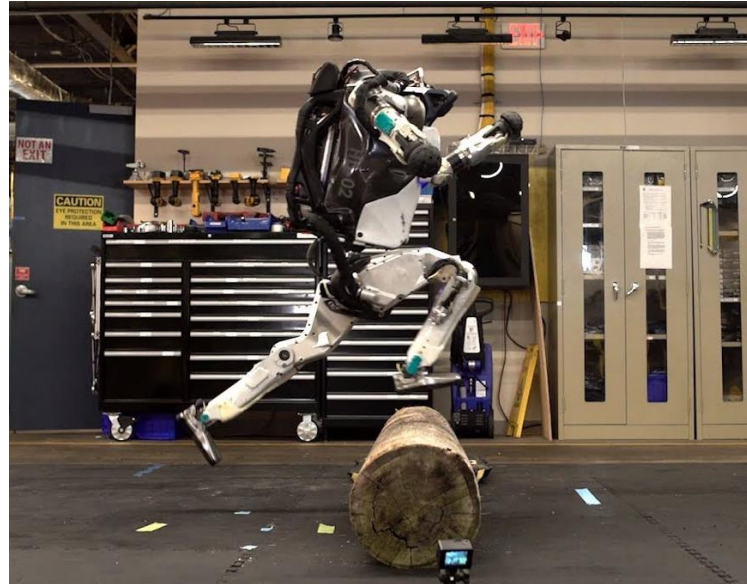
- Agent-Environment Interface
- Markov Decision Processes
- Partially Observable Markov Decision Processes

# Environment Interaction

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Pong [Atari]



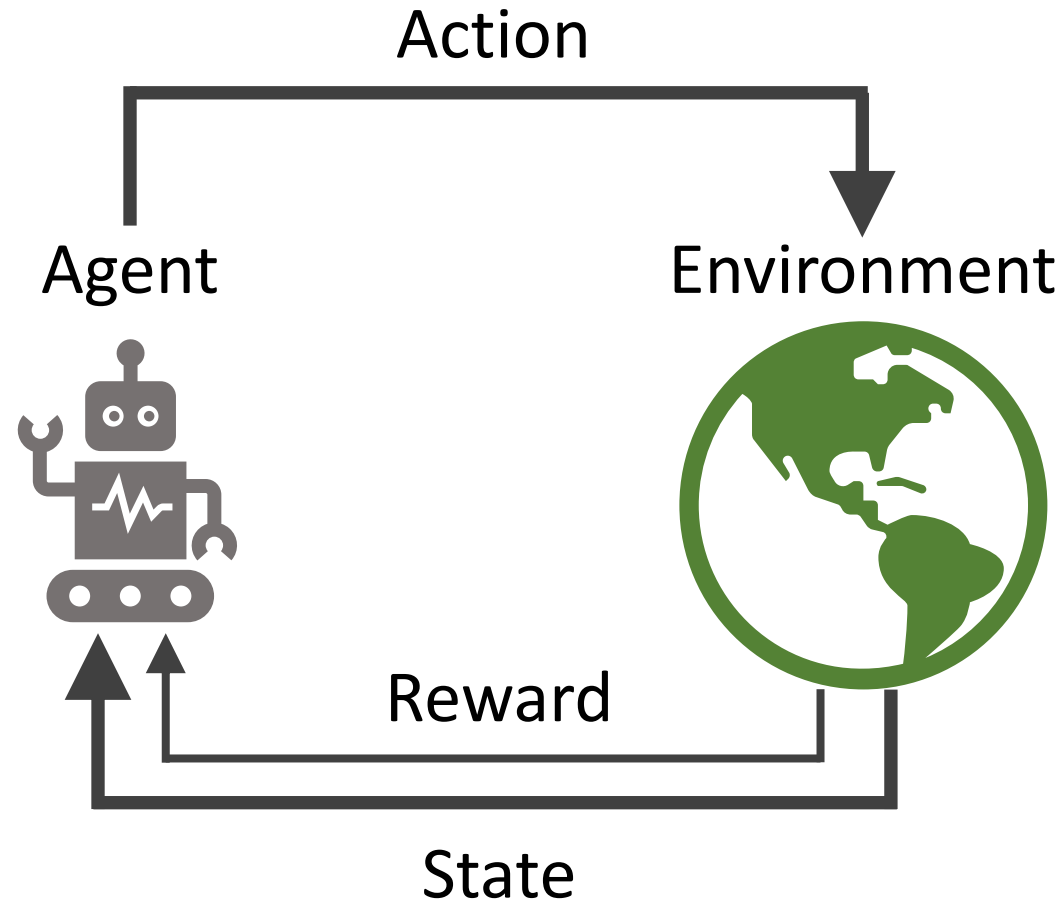
Atlas [Boston Dynamics]



[Smithsonian Magazine]

# Agent-Environment Interface

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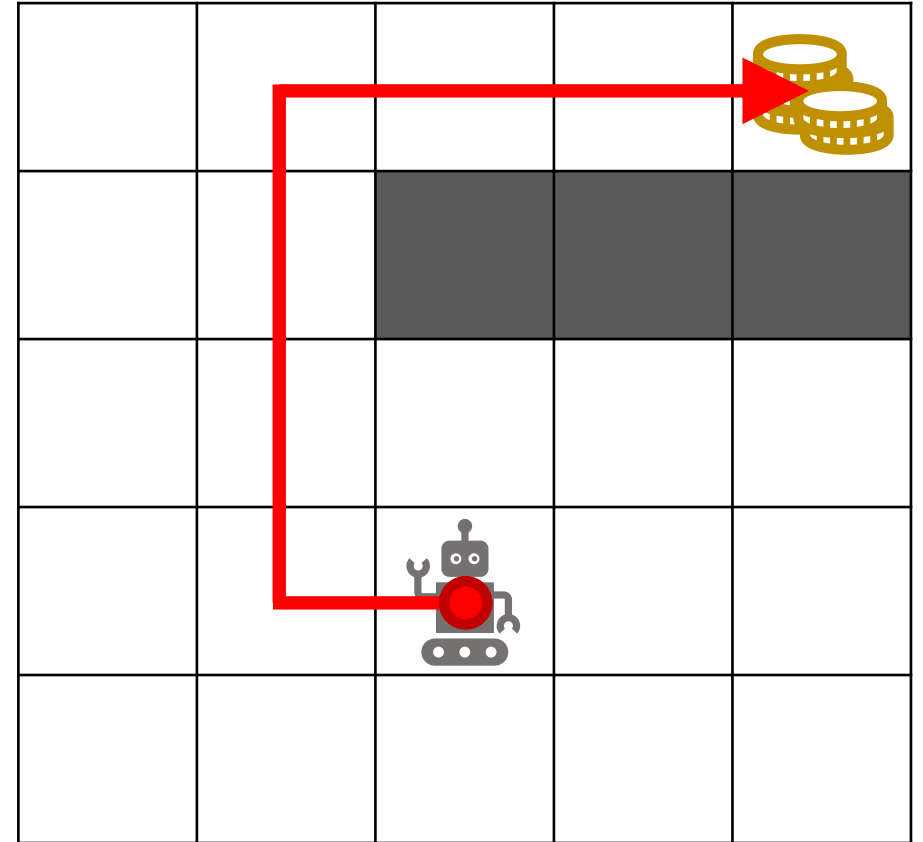


# Maze

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State:

- position



# Maze

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## State:

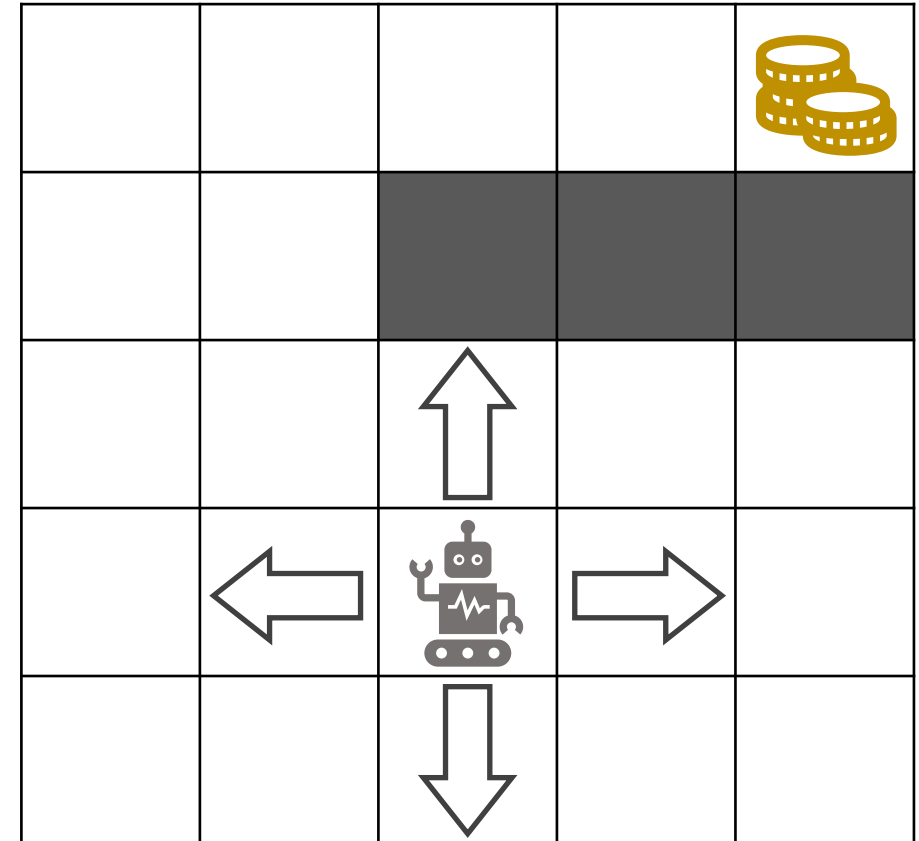
- position

## Action:

- up / down / left / right / stop

## Reward:

- 1 if goal reached
- 0 otherwise



# Pong

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## State:

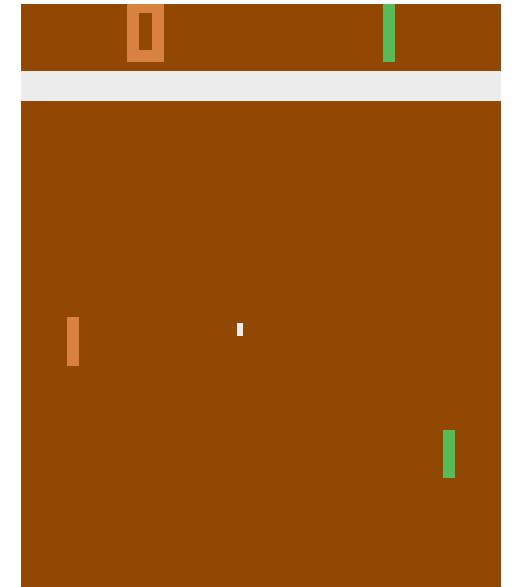
- position + velocity of paddles
- position + velocity of the ball
- scores

## Action:

- up / down / stop

## Reward:

- 1 if agent scores
- -1 if opponent scores



Pong [Atari]

# Humanoid Walking

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State:

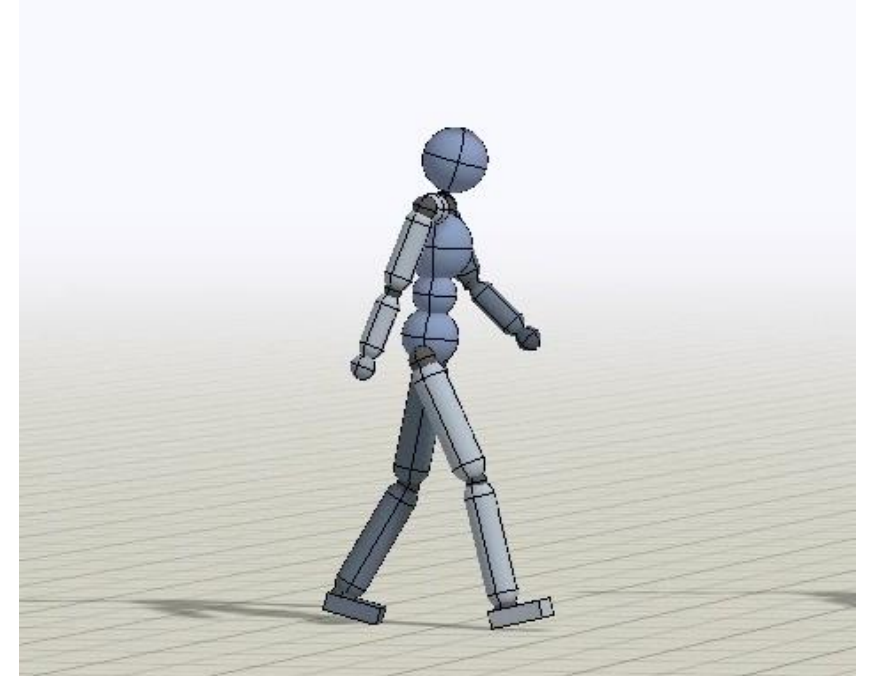
- position + velocity of body parts

Action:

- motor forces

Reward:

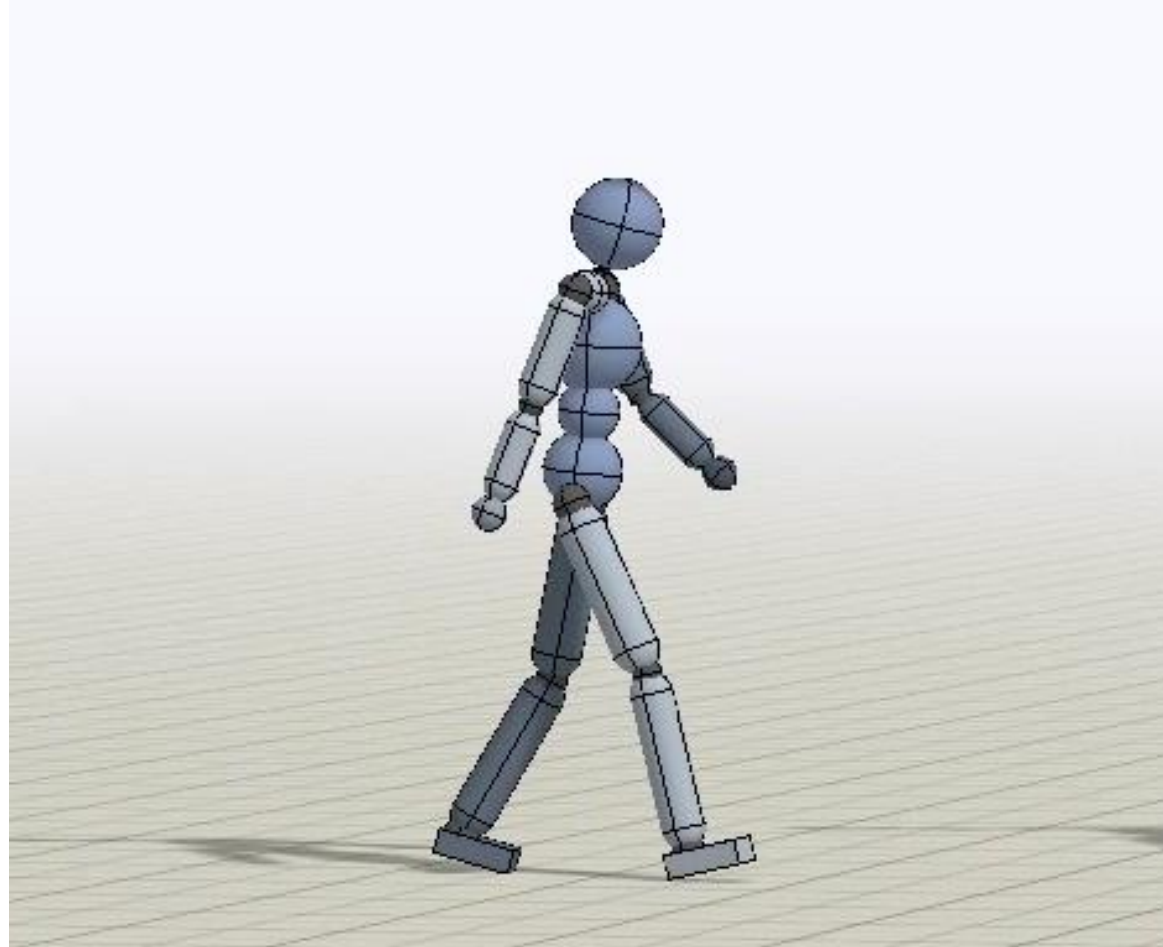
- speed





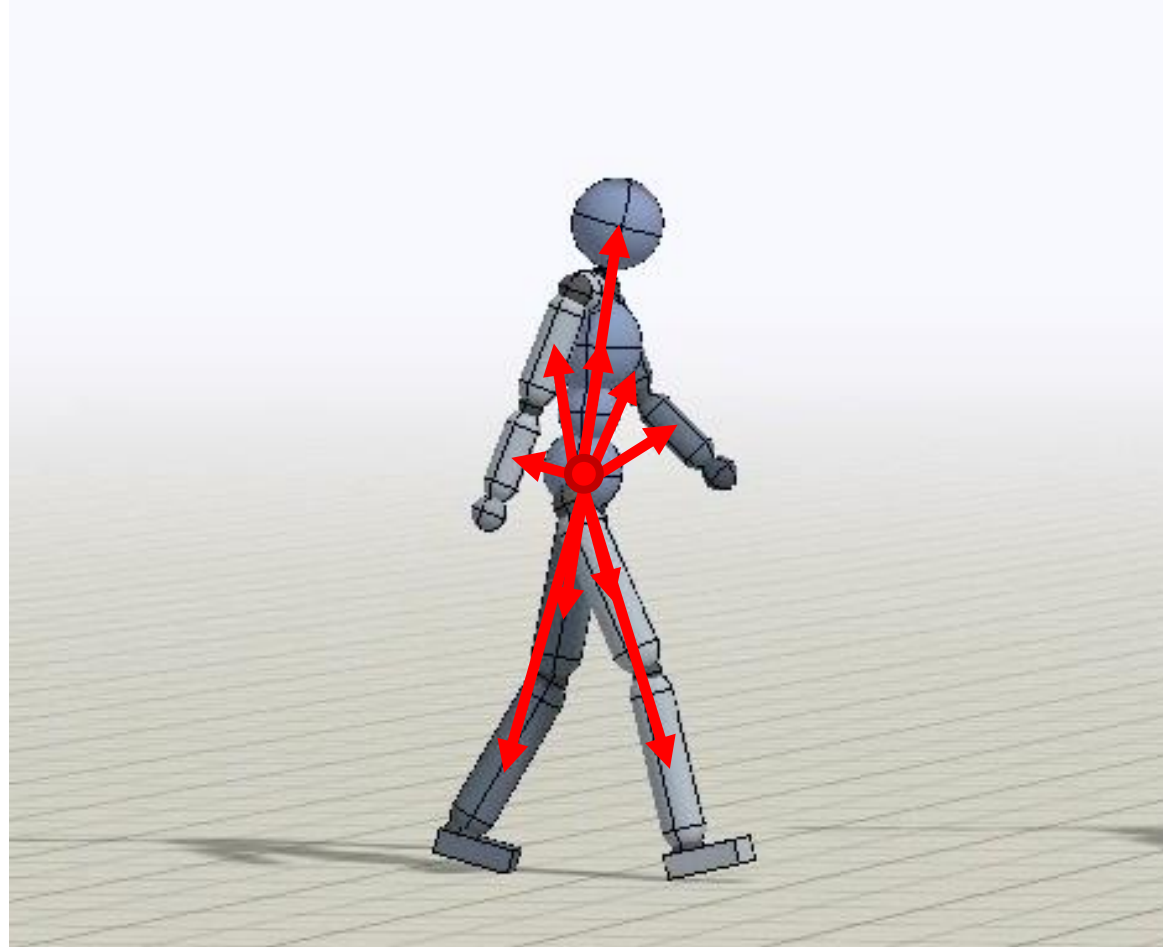
# Humanoid Walking

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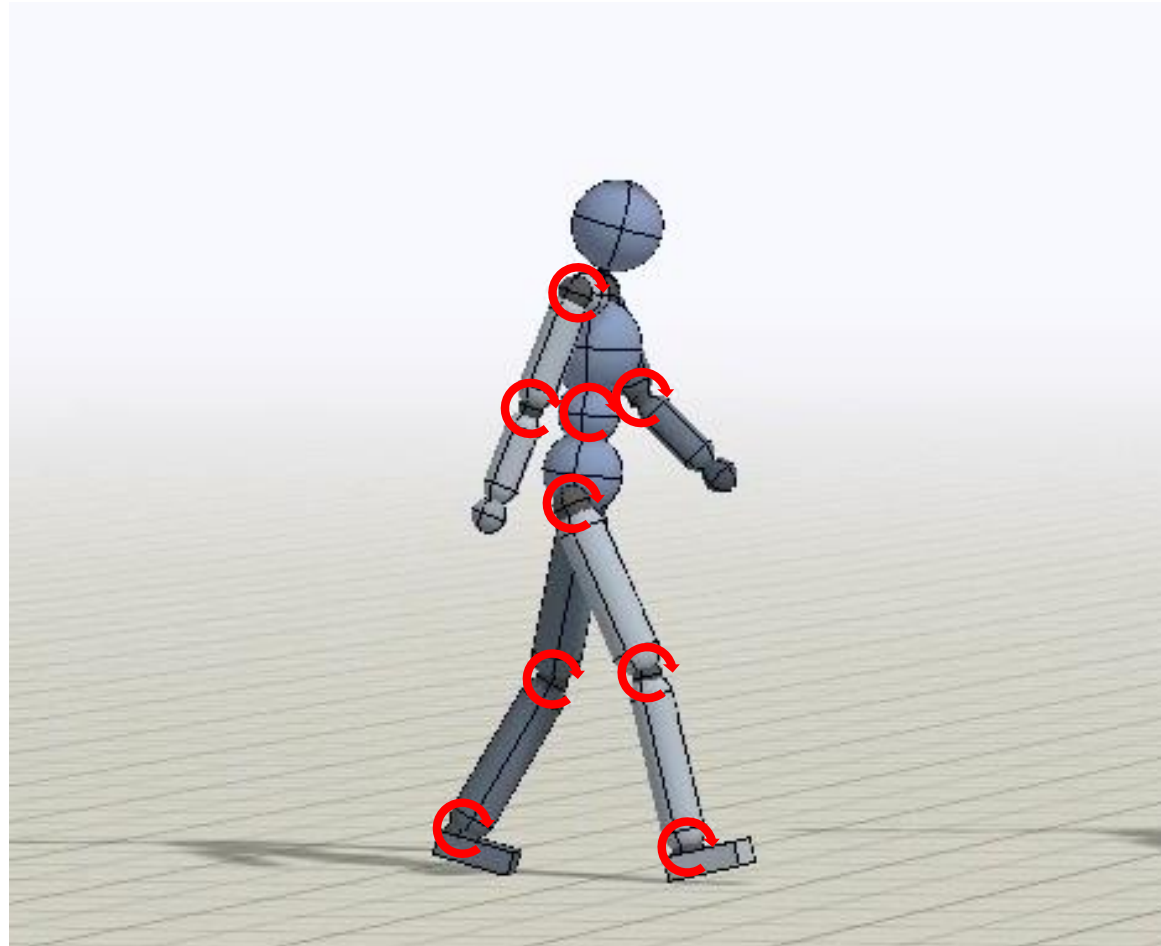
# Humanoid Walking (State)

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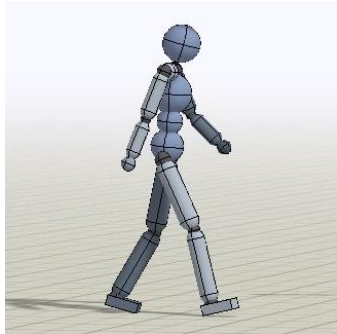
# Humanoid Walking (Action)

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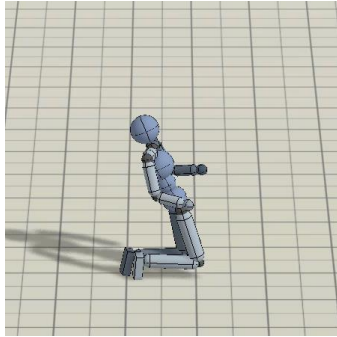
# Humanoid Walking (Reward)

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$$\text{reward} \left( \text{img} \right) = \text{thumbs up}$$
A blue humanoid robot is shown in a walking pose on a light brown, textured floor. The robot has a blue head, torso, and limbs, with a white joint at the hip. It is walking towards the right. The background is a light blue gradient.

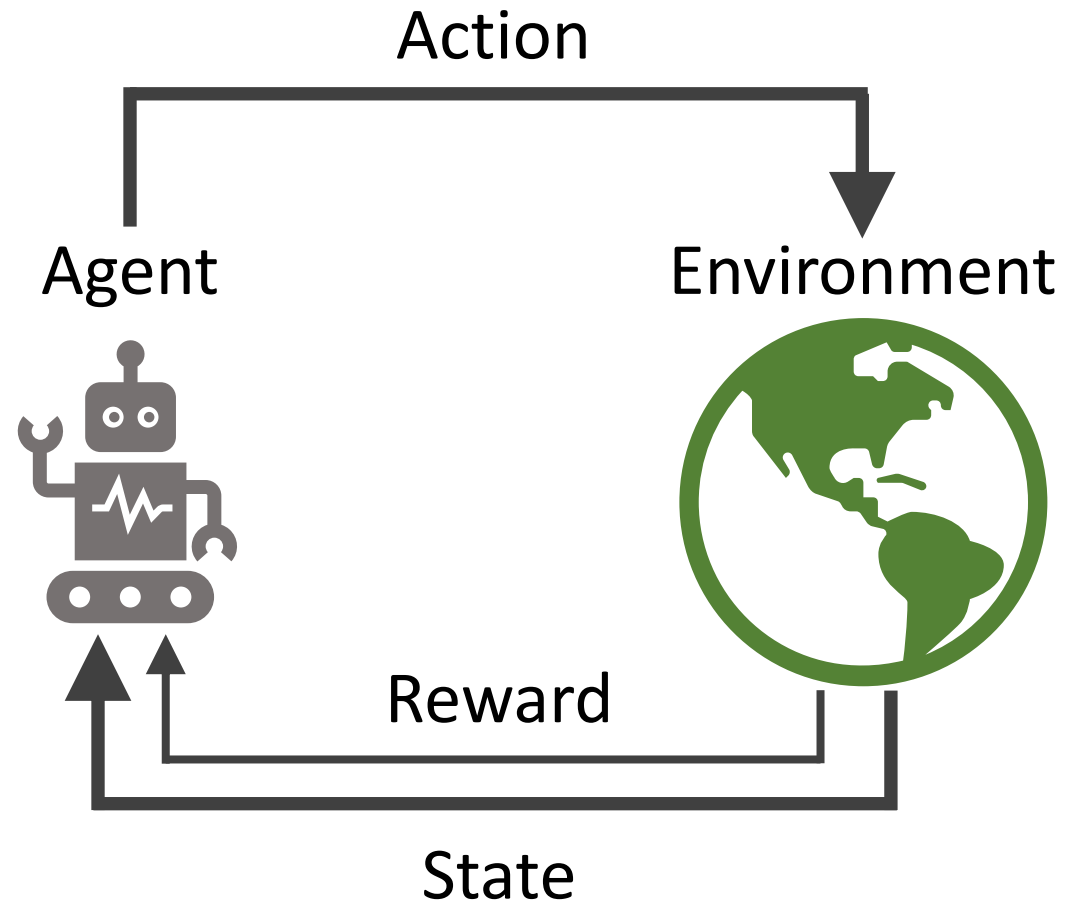
# Humanoid Walking (Reward)

---

$$\text{reward} \left( \text{img} \right) = \text{thumbs down}$$
A small, blue humanoid robot is shown in a falling or stumbling pose on a light-colored tiled floor. The robot's arms are outstretched, and its legs are bent in a way that suggests it has lost its balance. This image is used to represent a negative reward state in a reinforcement learning context for humanoid walking.

# Agent-Environment Interface

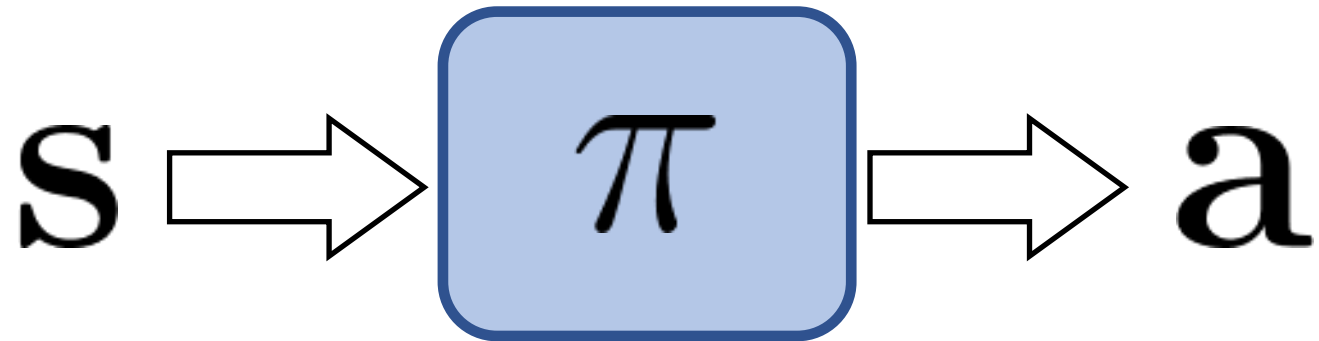
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# Policy

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$$\mathbf{a} = \underline{\pi}(\mathbf{s})$$



# Policy

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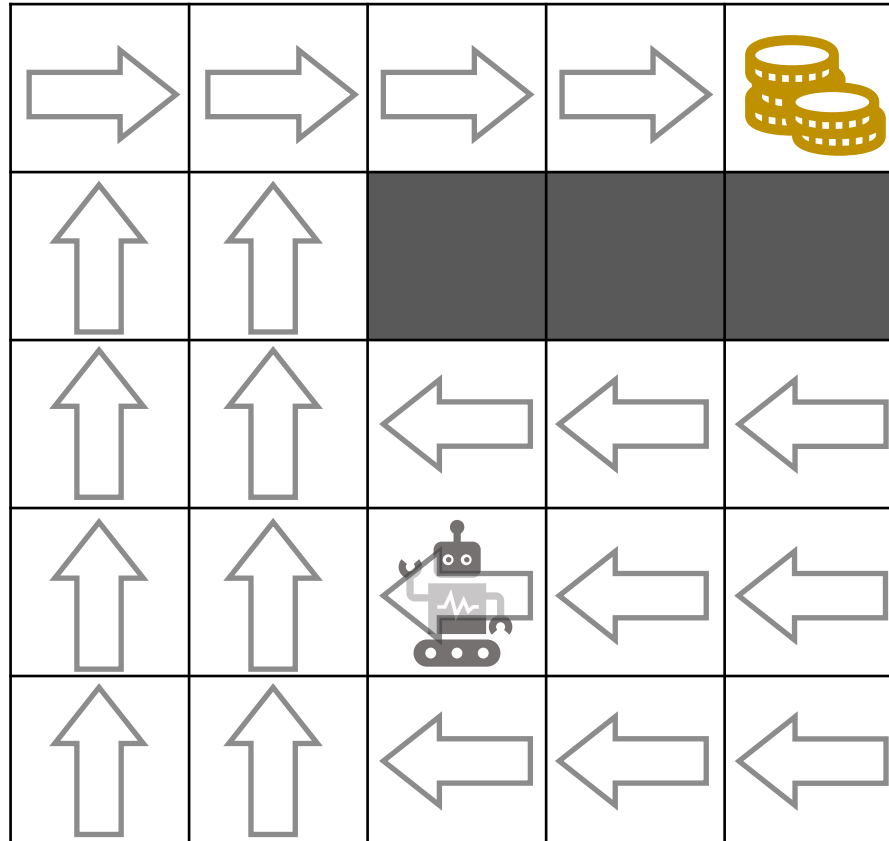
$$\mathbf{a} = \underline{\pi}(\mathbf{s})$$





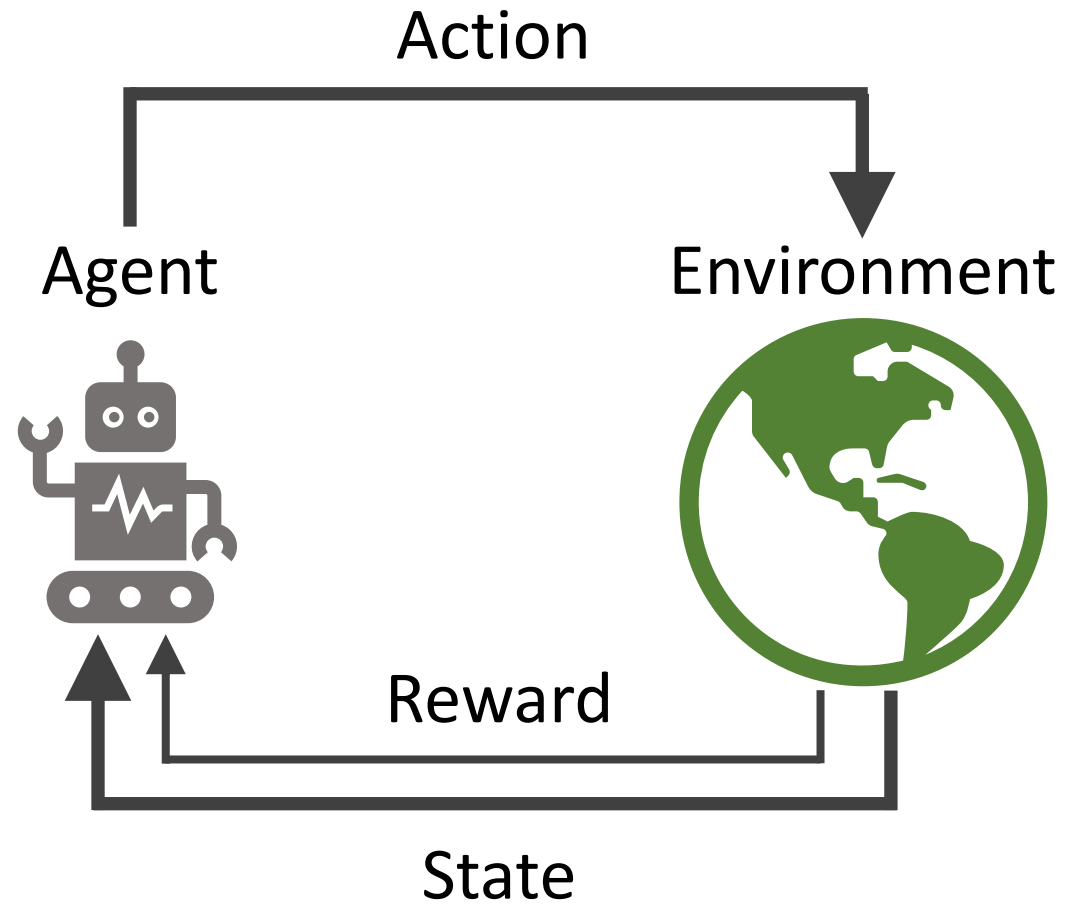
# Maze

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# Agent-Environment Interface

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# Markov Decision Process

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$\mathbf{s} \in \mathcal{S}$  – state space

$\mathbf{a} \in \mathcal{A}$  – action space

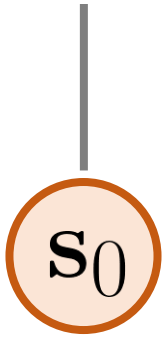
$p(\mathbf{s}'|\mathbf{s}, \mathbf{a})$  – dynamics function

$r(\mathbf{s}, \mathbf{a}, \mathbf{s}')$  – reward function

# MDP

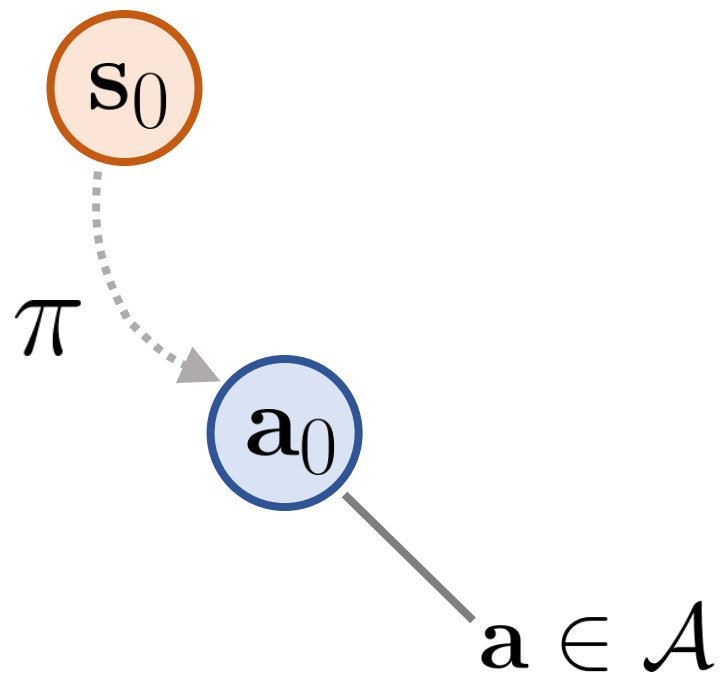
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$s \in \mathcal{S}$



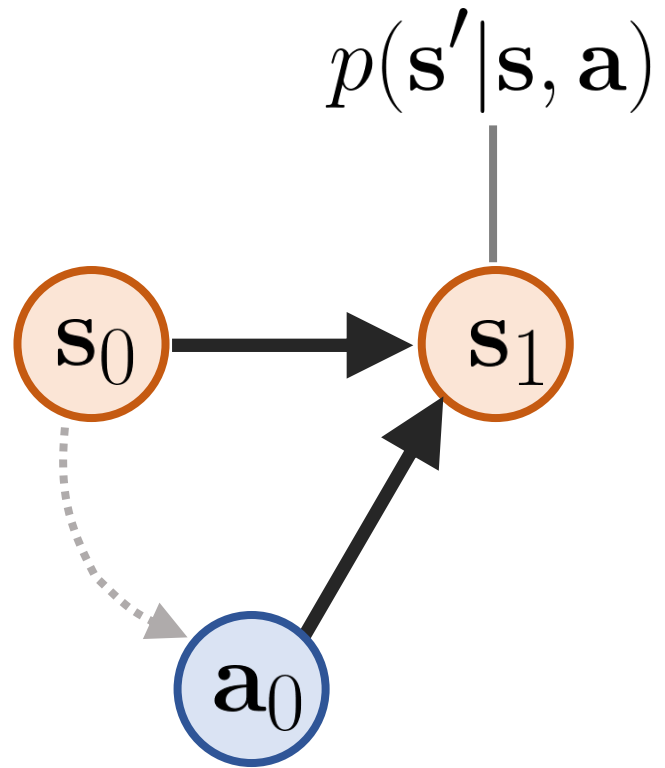
# MDP

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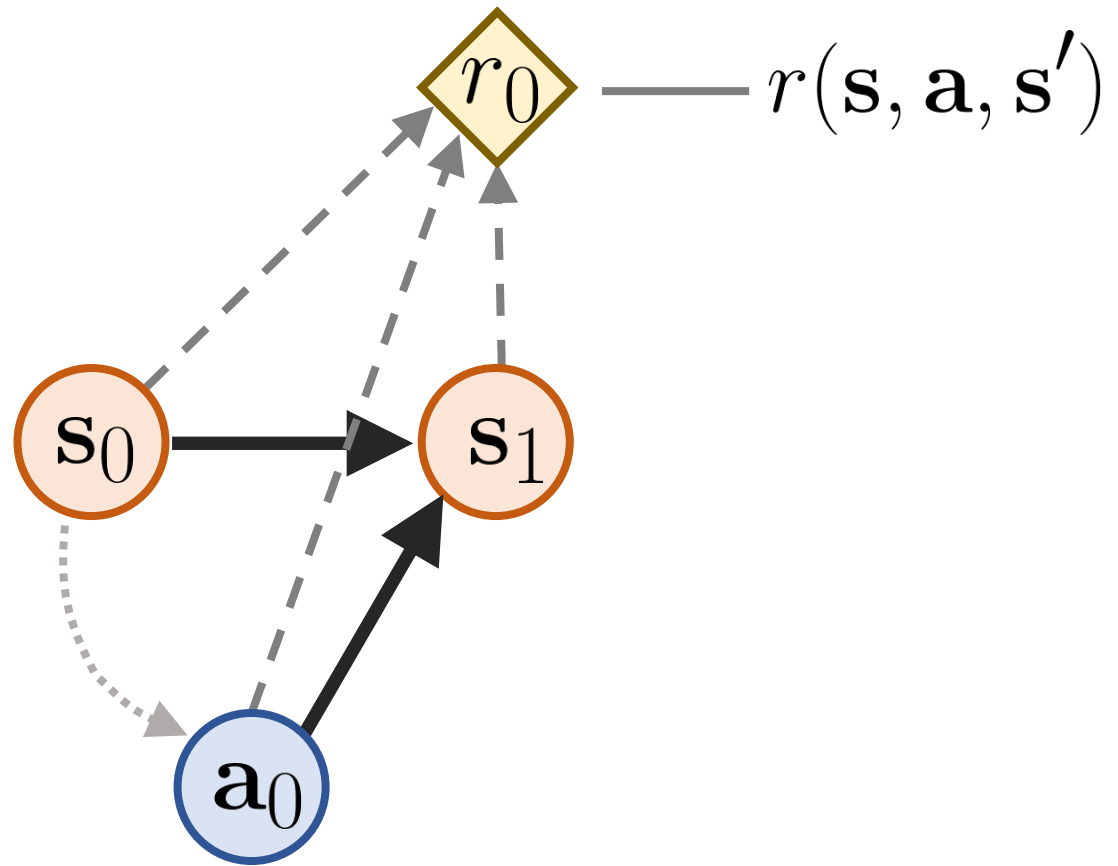
# MDP

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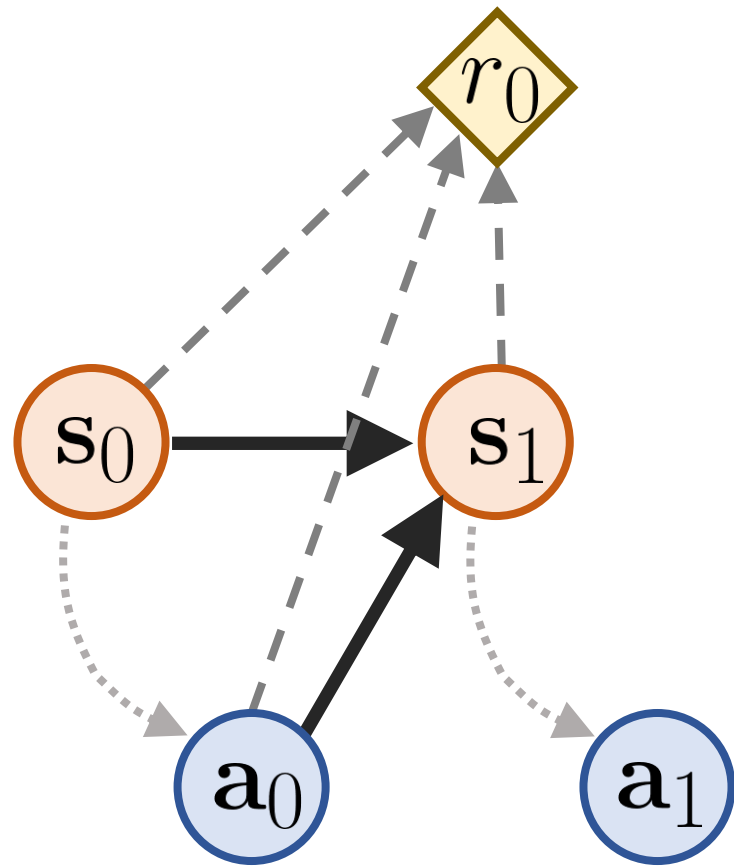
# MDP

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# MDP

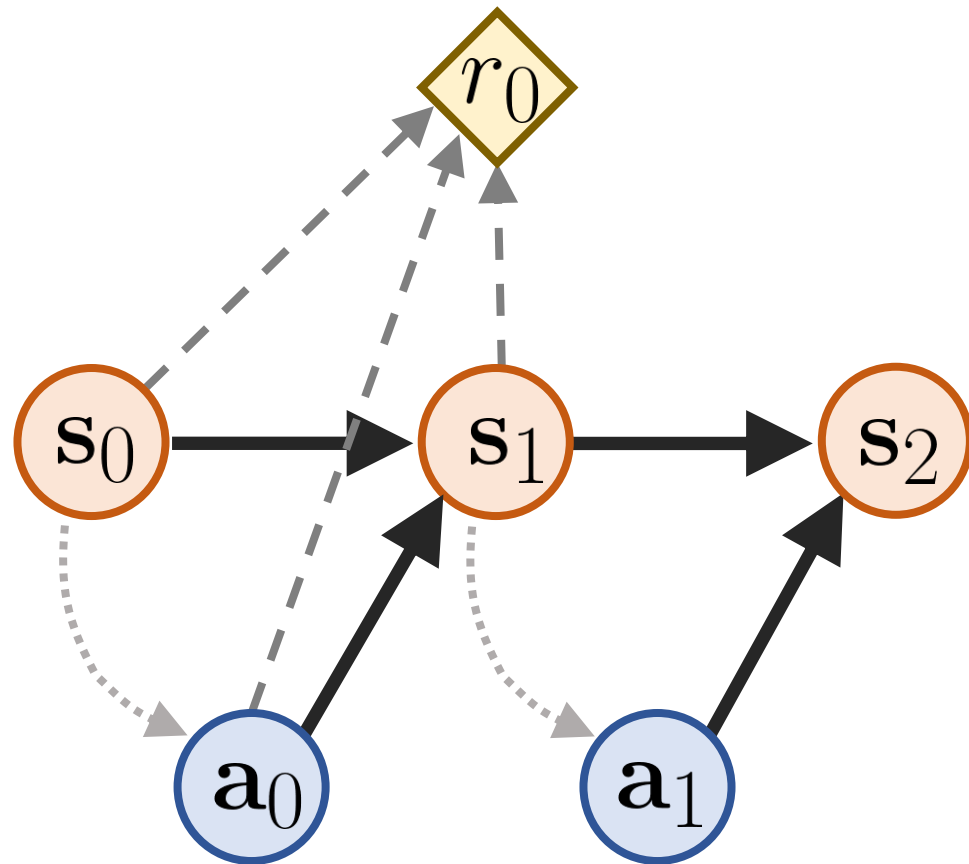
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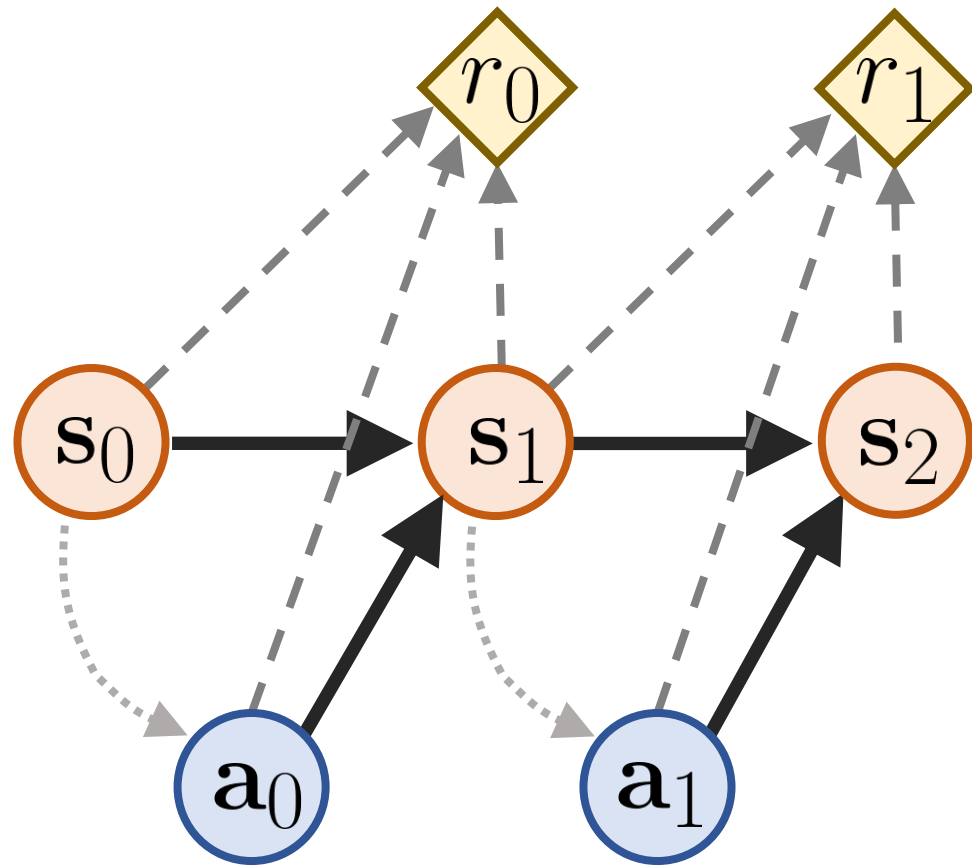
# MDP

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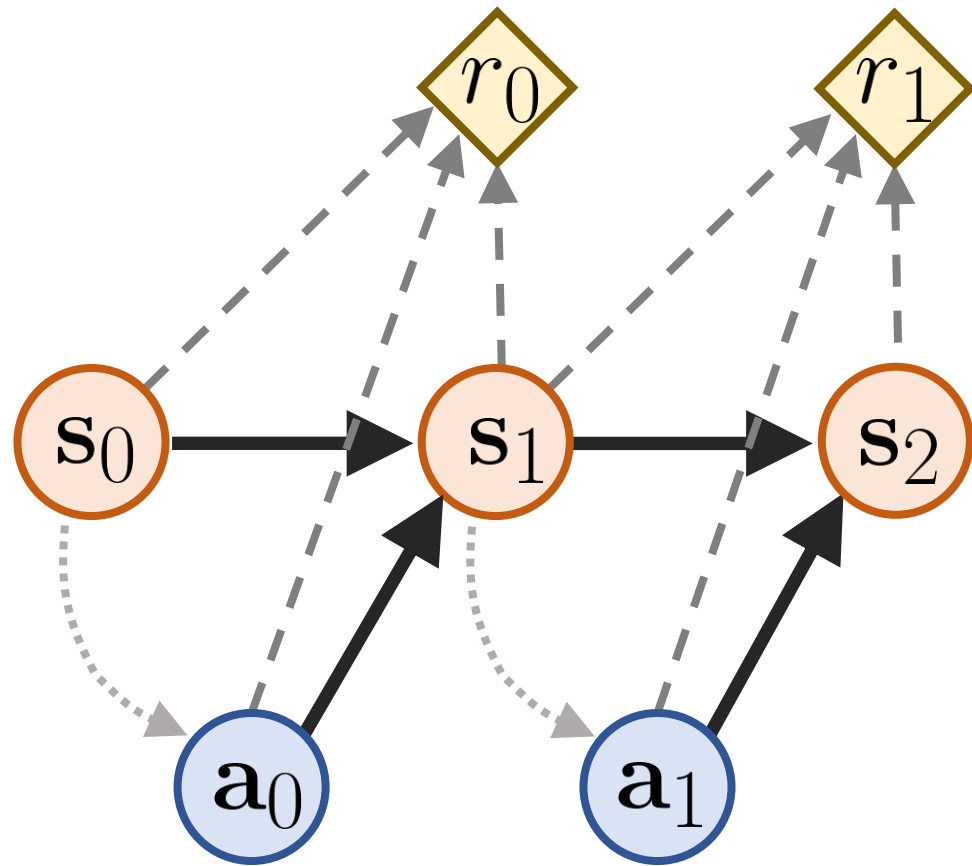
# MDP

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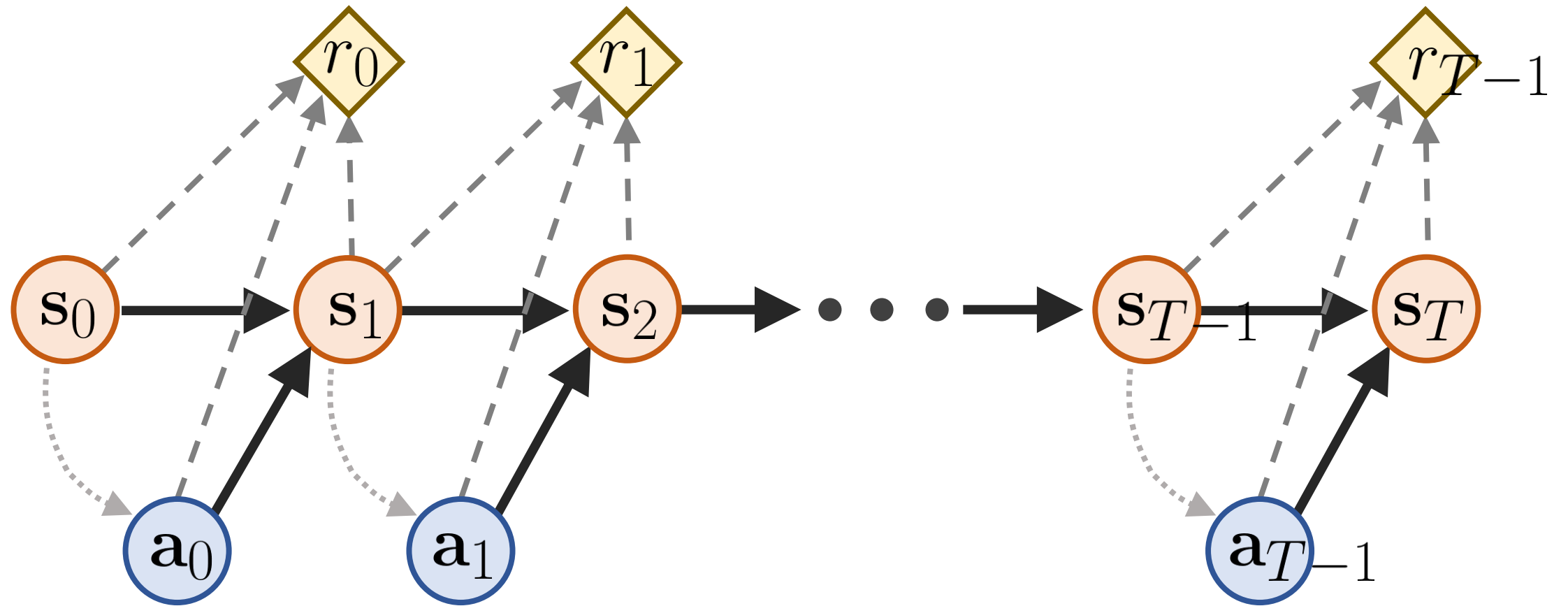
# MDP

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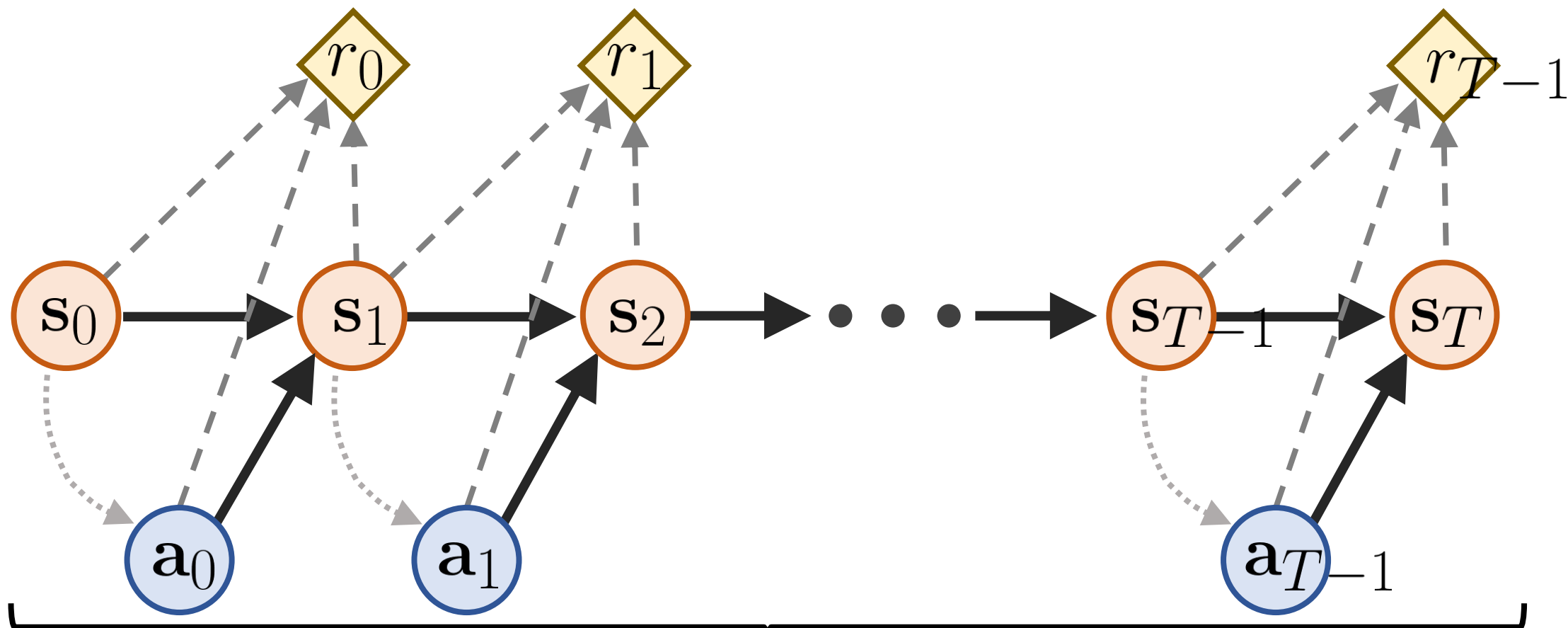


# MDP

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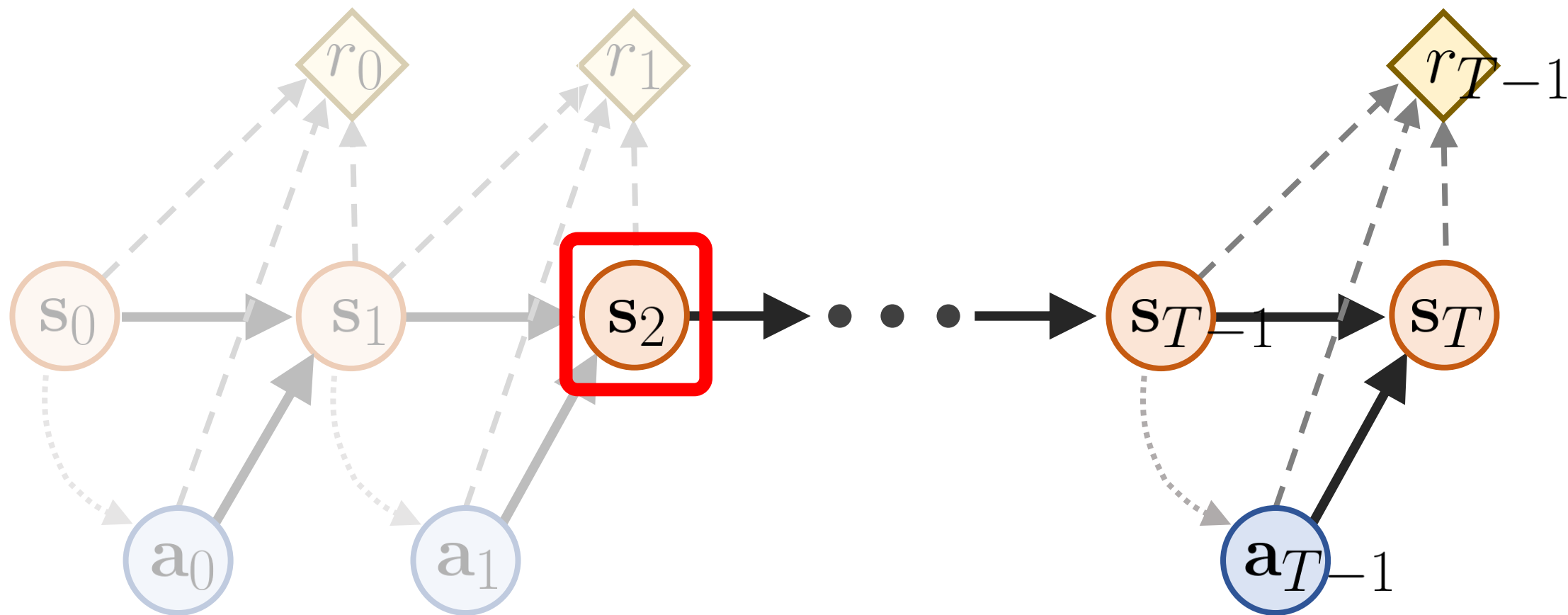
# MDP



$\max \sum_{t=0}^{T-1} \gamma^t r_t$  — return

# MDP

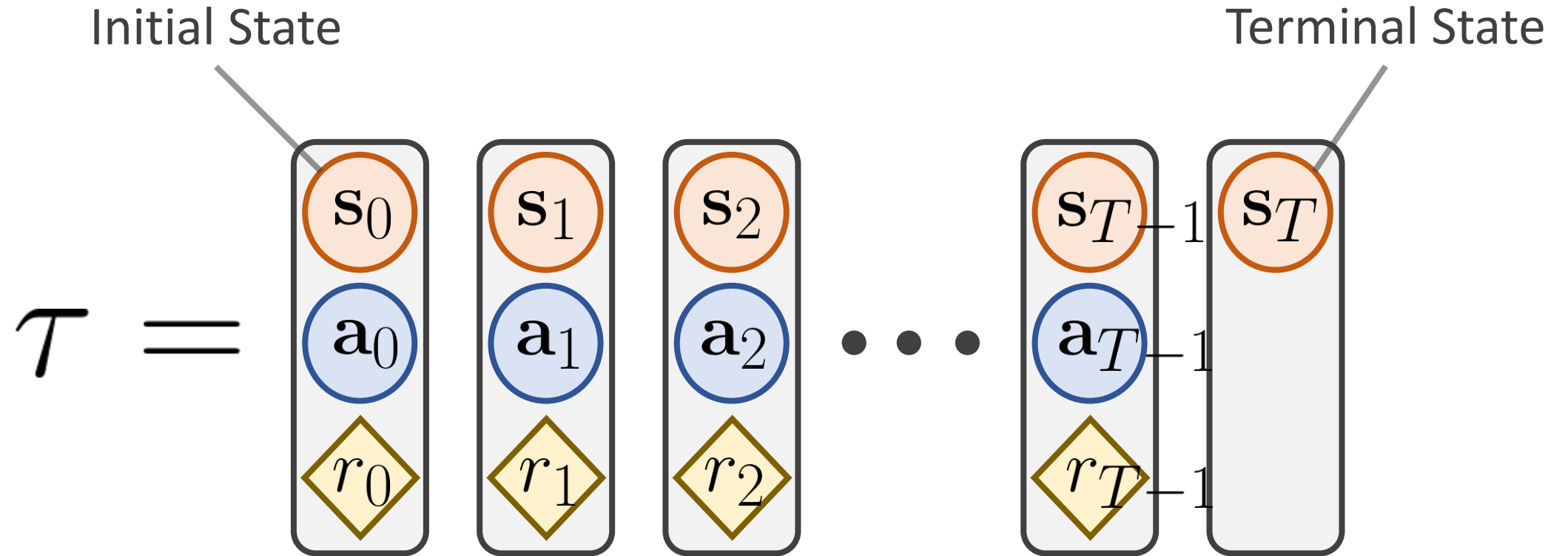
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Markov property:  $p(s'|s, a)$

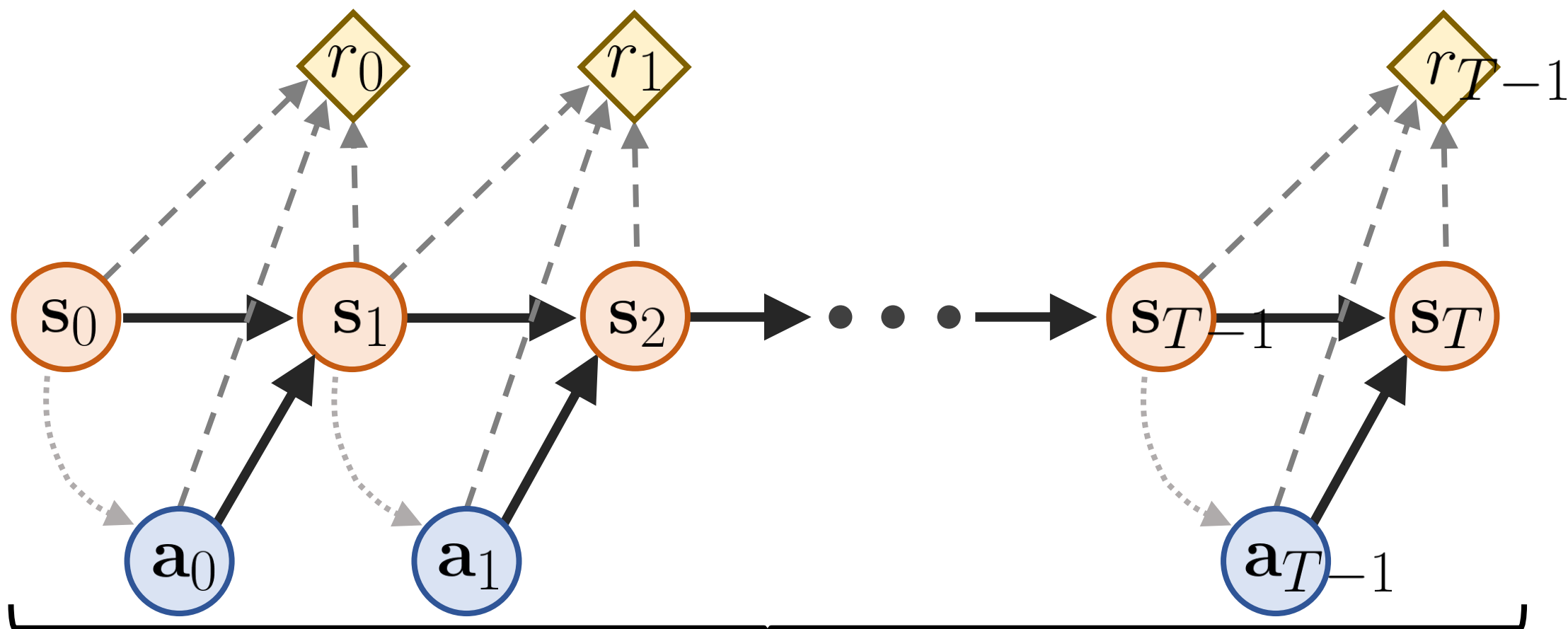
# Trajectory

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**Episode:** One trajectory/rollout.

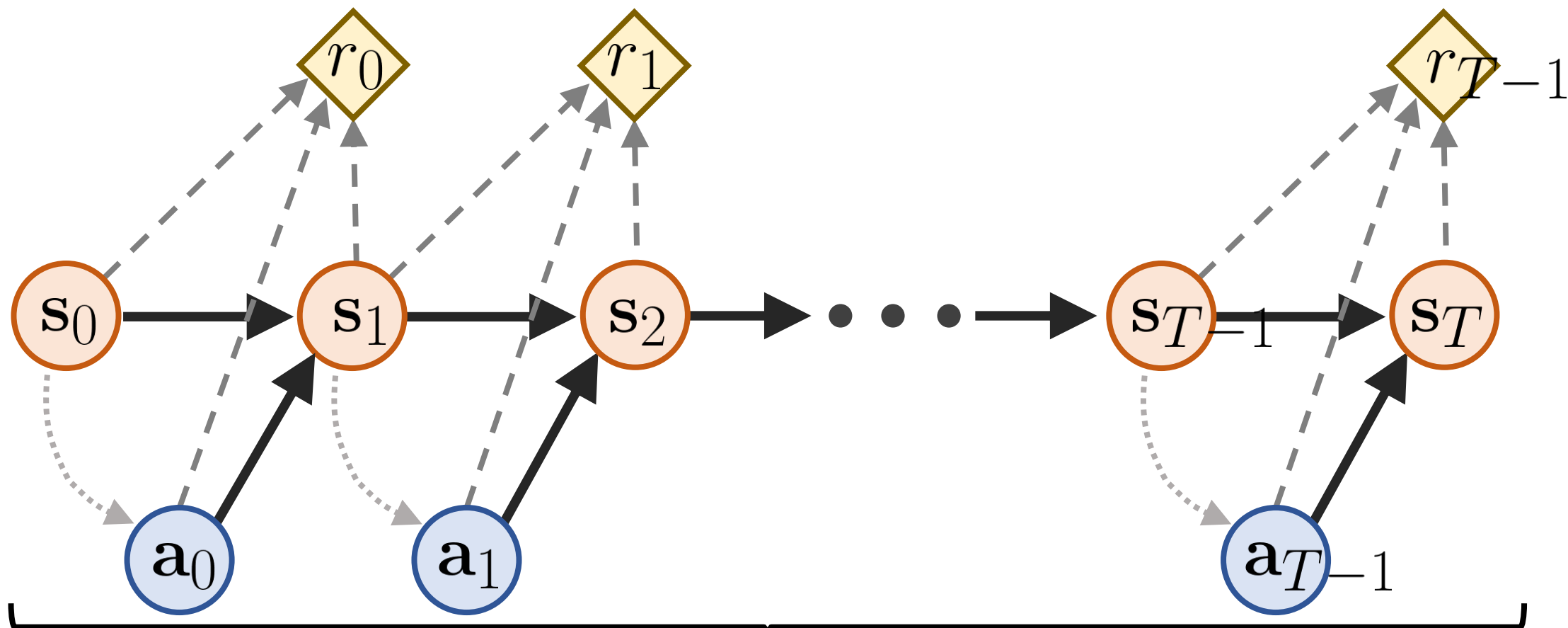
# MDP



$$T \in [1, \infty] \quad \max \sum_{t=0}^{T-1} \gamma^t r_t$$



# MDP



$$\max \sum_{t=0}^{T-1} \gamma^t r_t \quad \gamma \in [0, 1]$$

# Discount Factor

---

$$\underline{J(\pi)} = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

# Discount Factor

---

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# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim \underline{p(\tau|\pi)}} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \underline{\gamma^t r_t} \right]$$

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots + \gamma^{T-1} r_{T-1}$$

$\gamma = 1$ : maximize rewards at all timesteps equally

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + r_1 + r_2 + \dots + r_{T-1}$$

$\gamma = 1$ : maximize rewards at all timesteps equally

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots + \gamma^{T-1} r_{T-1}$$

$\gamma = 0$ : only maximize reward at first timestep



# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0$$

$\gamma = 0$ : only maximize reward at first timestep

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + 0.5r_1 + 0.5^2r_2 + \dots + 0.5^{T-1}r_{T-1}$$

$$\gamma = 0.5$$

# Inflation

---



Now

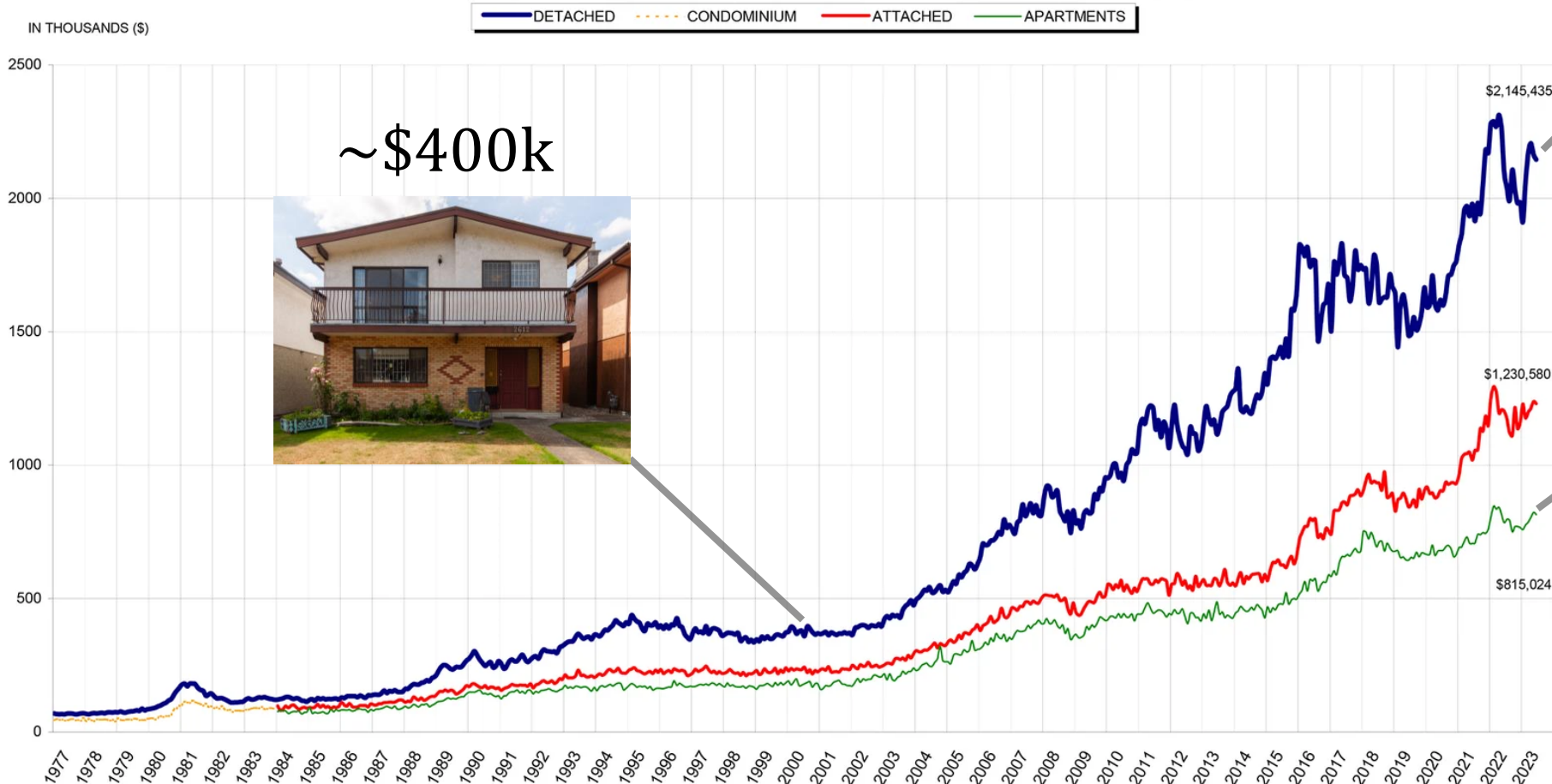


1 Year

# Inflation



Residential Average Sale Prices - January 1977 to June 2023



~\$400k



~\$2 mil

~\$600k



NOTE: From 1977 - 1984 condominium averages were not separated into attached & apartment.



# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots + \gamma^{T-1} r_{T-1}$$

Large  $\gamma$  : maximize long term rewards (far-sighted)

Small  $\gamma$  : maximize short term rewards (short-sighted / greedy)

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots + \gamma^{T-1} r_{T-1}$$

$\gamma = 1$ : maximize rewards at all timesteps equally

$T \neq \infty$

Expectation undefined if  $T = \infty$

# Discount Factor

---

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[ \sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots + \gamma^{T-1} r_{T-1}$$

If  $T = \infty$ ,

Then  $\gamma < 1$  for expectation to be defined.

# Return Bounds

---

- $r \in [r_{\min}, r_{\max}]$ ,  $r \geq 0$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots + \gamma^{T-1} r_{T-1}$$



# Return Bounds

---

- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

# Return Bounds

---

- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t r_t$$

# Return Bounds

---

- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t r_t \leq \sum_{t=0}^{T-1} \gamma^t r_{\max}$$

# Return Bounds

---

- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

$$\begin{aligned} R &= \sum_{t=0}^{T-1} \gamma^t r_t \leq \sum_{t=0}^{T-1} \gamma^t r_{\max} \\ &\leq \sum_{t=0}^{\infty} \gamma^t r_{\max} \end{aligned}$$

# Return Bounds

---

- $r \in [r_{\min}, r_{\max}]$ ,  $r \geq 0$
- Geometric series with ratio  $\gamma < 1$

$$\begin{aligned} R &= \sum_{t=0}^{T-1} \gamma^t r_t \leq \sum_{t=0}^{T-1} \gamma^t r_{\max} \\ &\leq \sum_{t=0}^{\infty} \gamma^t r_{\max} = r_{\max} \sum_{t=0}^{\infty} \gamma^t \end{aligned}$$

# Return Bounds

---

- $r \in [r_{\min}, r_{\max}]$ ,  $r \geq 0$
- Geometric series with ratio  $\gamma < 1$

$$\begin{aligned} R &= \sum_{t=0}^{T-1} \gamma^t r_t \leq \sum_{t=0}^{T-1} \gamma^t r_{\max} \\ &\leq \sum_{t=0}^{\infty} \gamma^t r_{\max} = r_{\max} \sum_{t=0}^{\infty} \gamma^t \\ &\leq r_{\max} \frac{1}{1 - \gamma} \end{aligned}$$

# Return Bounds

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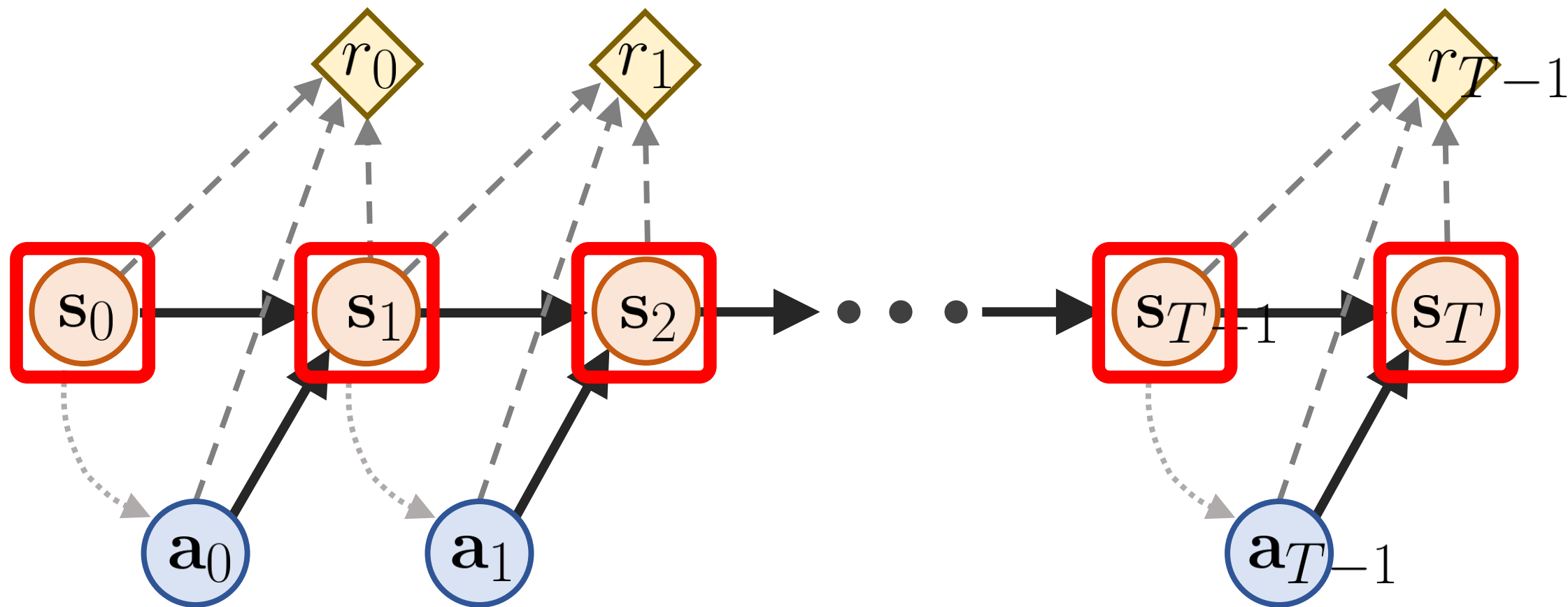
- $r \in [r_{\min}, r_{\max}]$ ,  $r \geq 0$
- Geometric series with ratio  $\gamma < 1$

$$R \leq R_{\max}$$

$$R_{\max} = \frac{1}{1 - \gamma} r_{\max}$$

# MDP

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# State Spaces

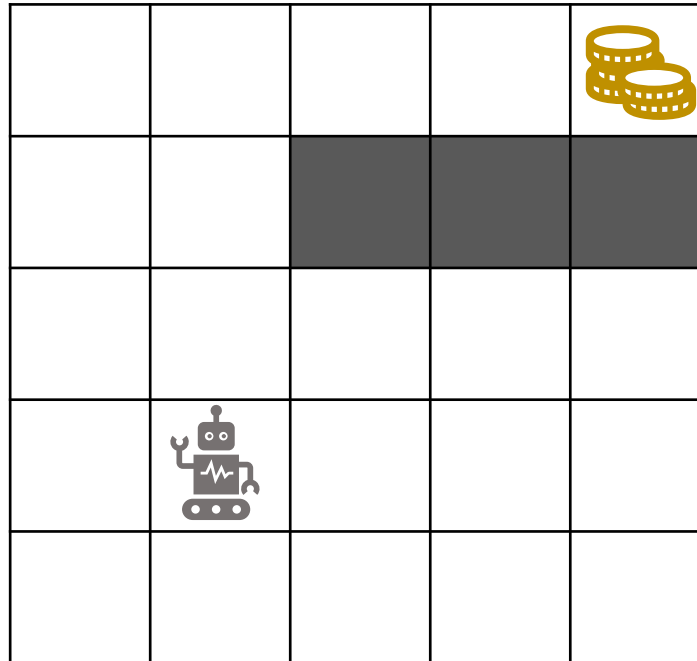
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Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



# State Spaces

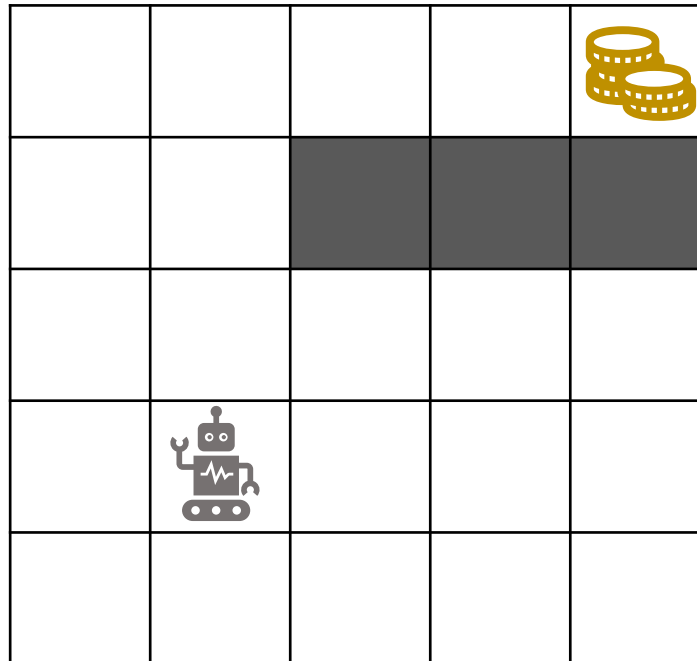
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Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



# State Spaces

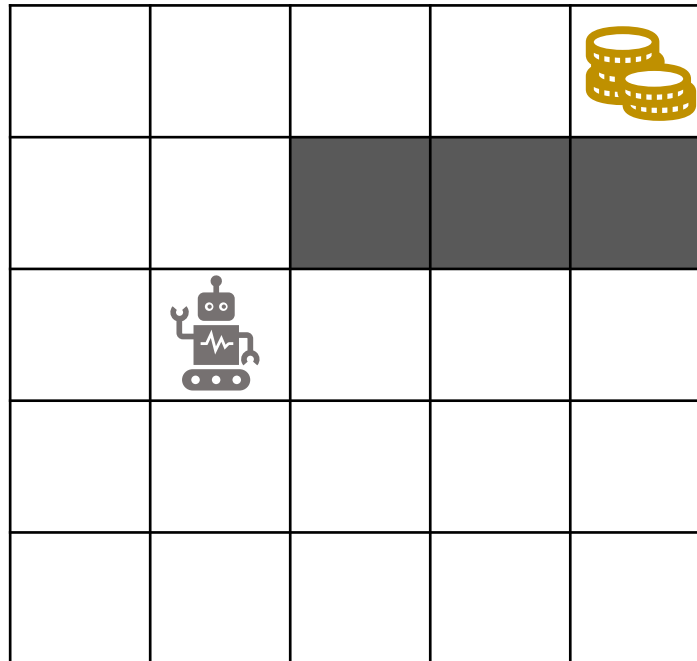
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Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

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# State Spaces

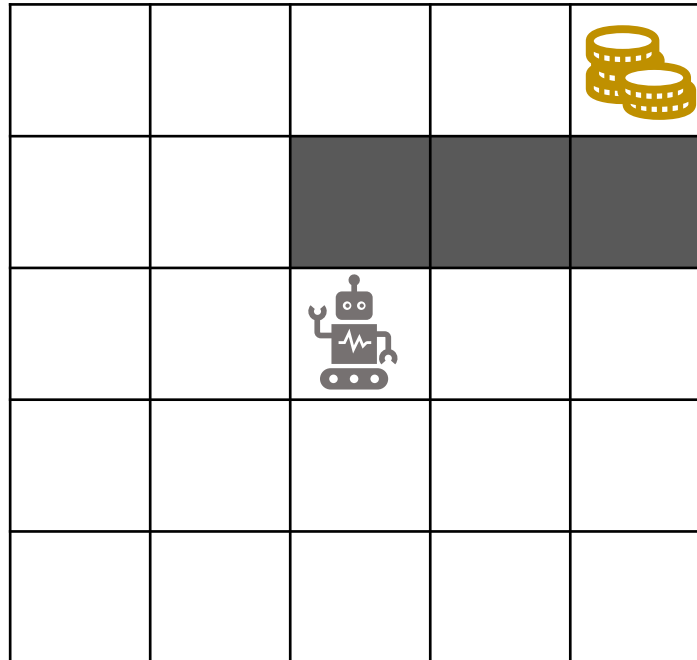
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Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



# State Spaces

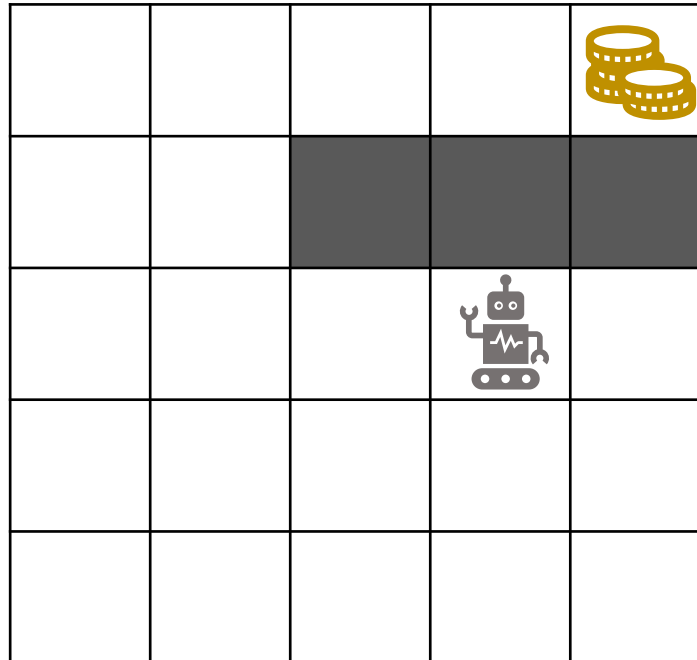
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Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



# State Spaces

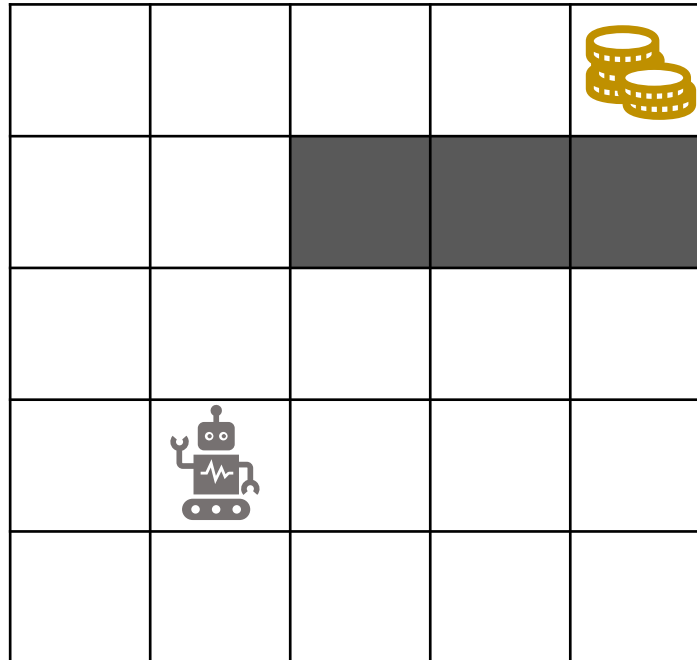
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Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



# State Spaces

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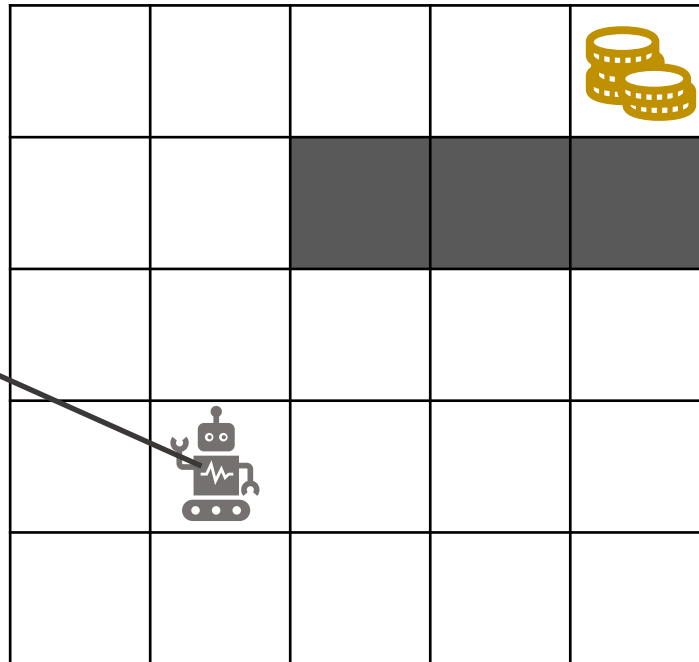
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



# State Spaces

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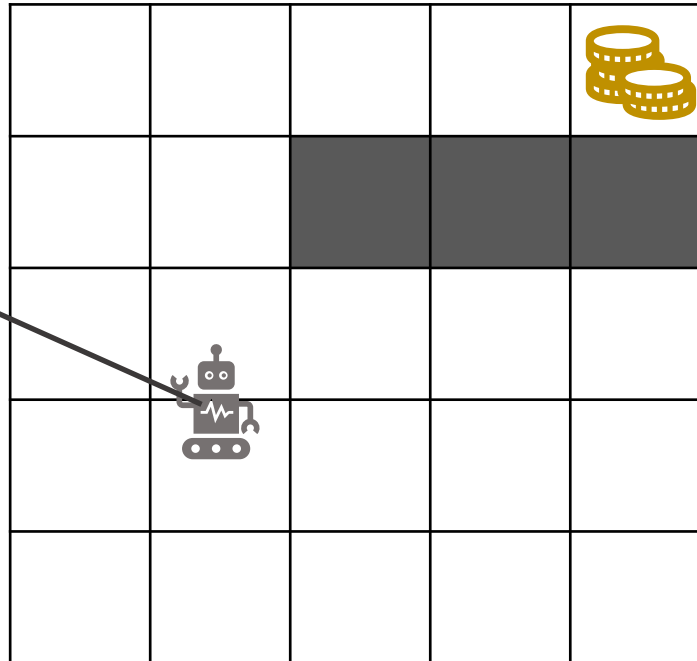
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$





# State Spaces

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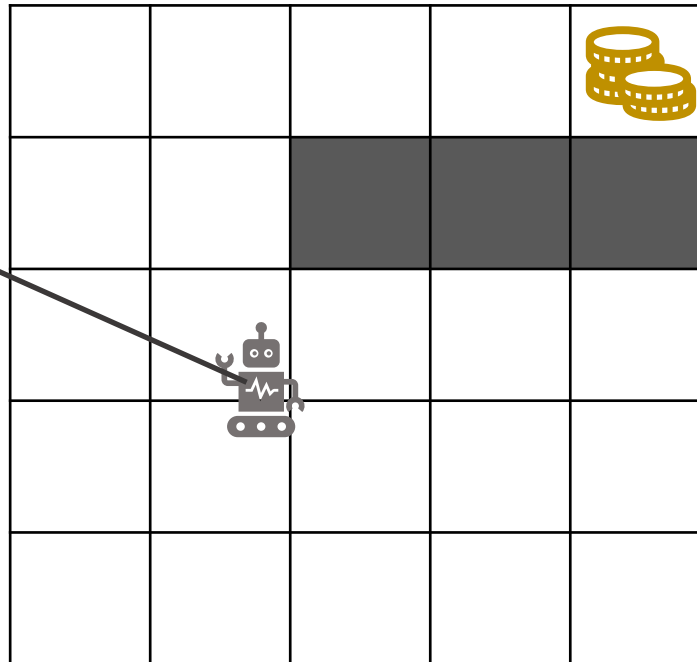
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



# State Spaces

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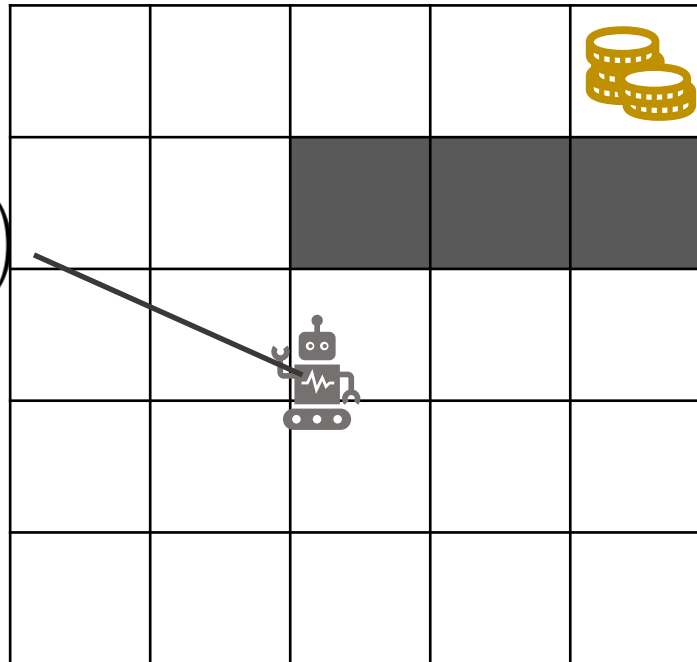
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

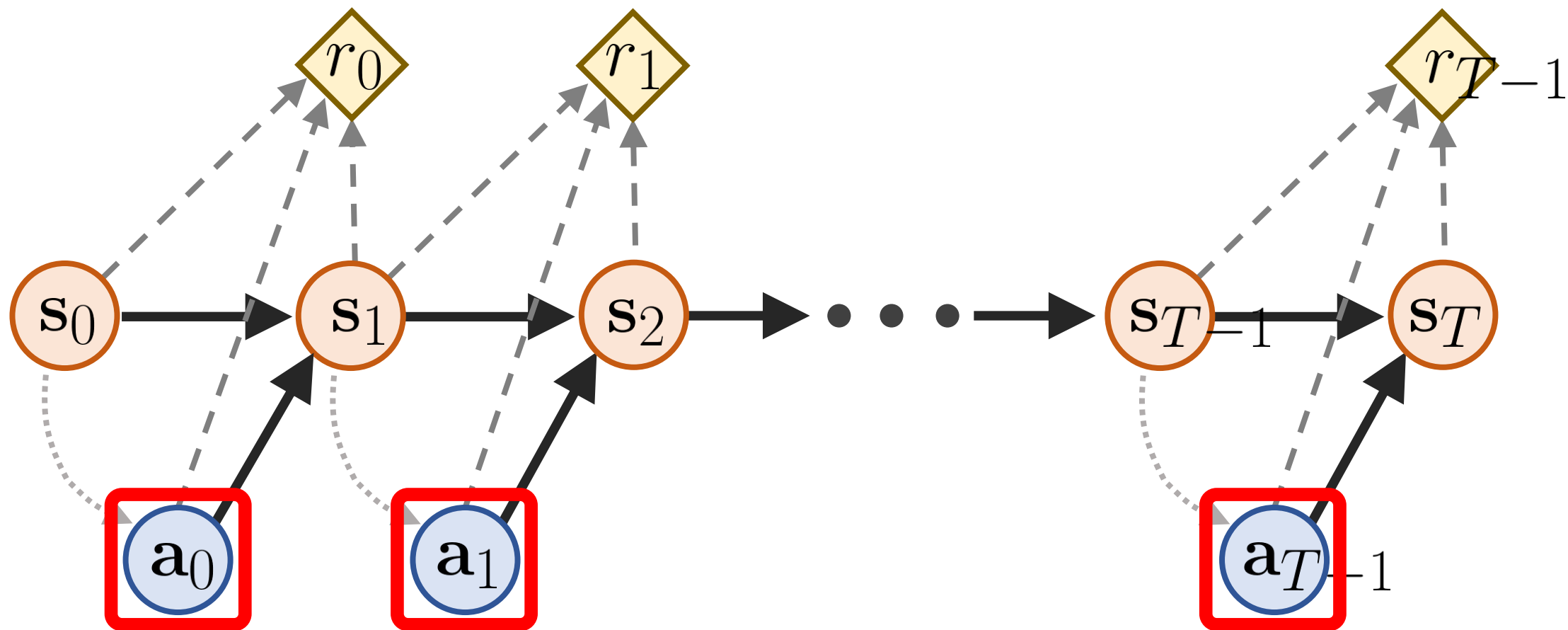
$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



# MDP

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# Action Spaces

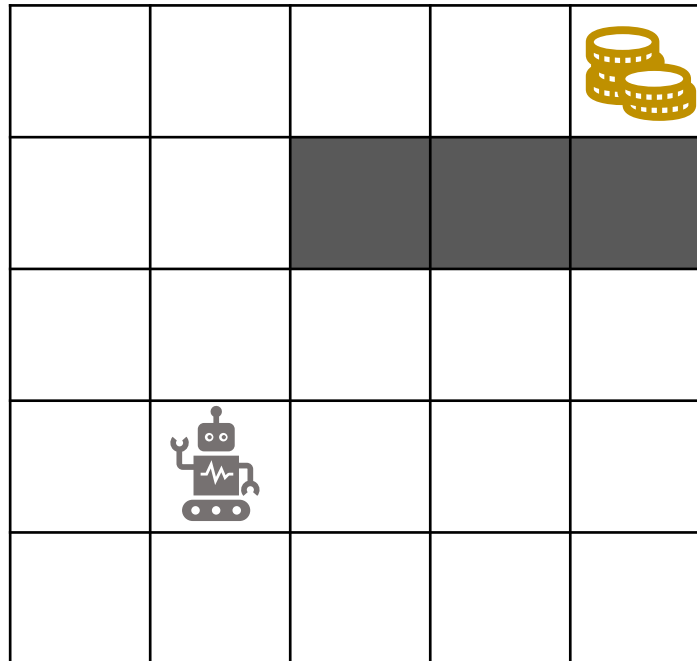
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Discrete

$$\mathbf{a} \in \{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^m\}$$

Continuous

$$\mathbf{a} \in \mathbb{R}^m$$



# Action Spaces

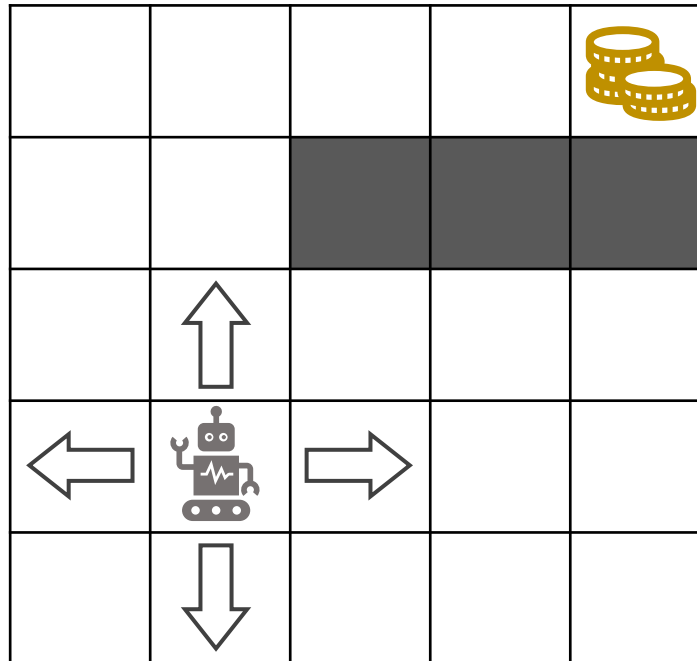
---

Discrete

$$\mathbf{a} \in \{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^m\}$$

Continuous

$$\mathbf{a} \in \mathbb{R}^m$$



# Action Spaces

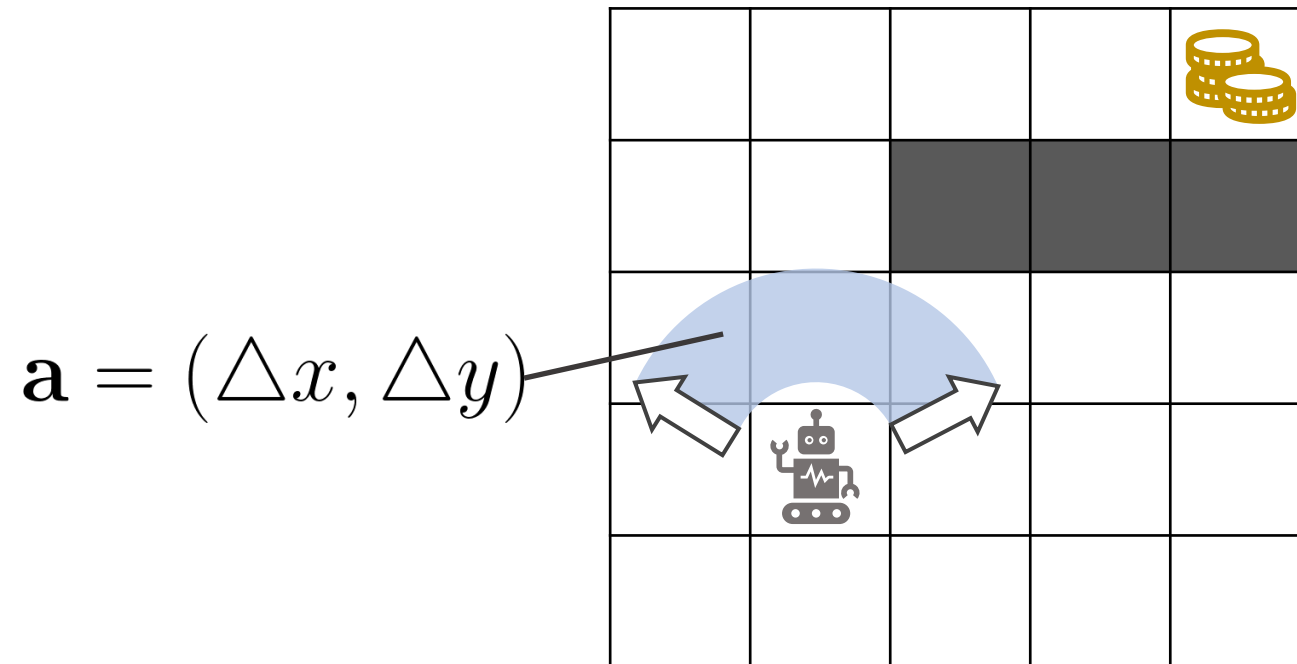
---

Discrete

$$\mathbf{a} \in \{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^m\}$$

Continuous

$$\mathbf{a} \in \mathbb{R}^m$$

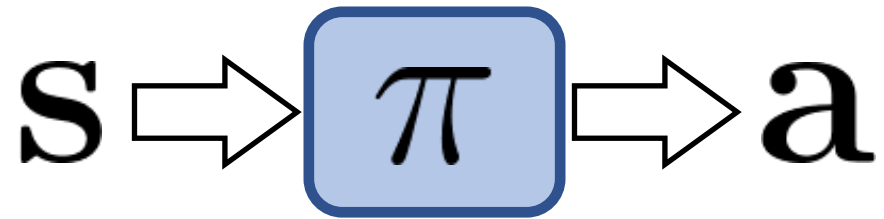


# Policies

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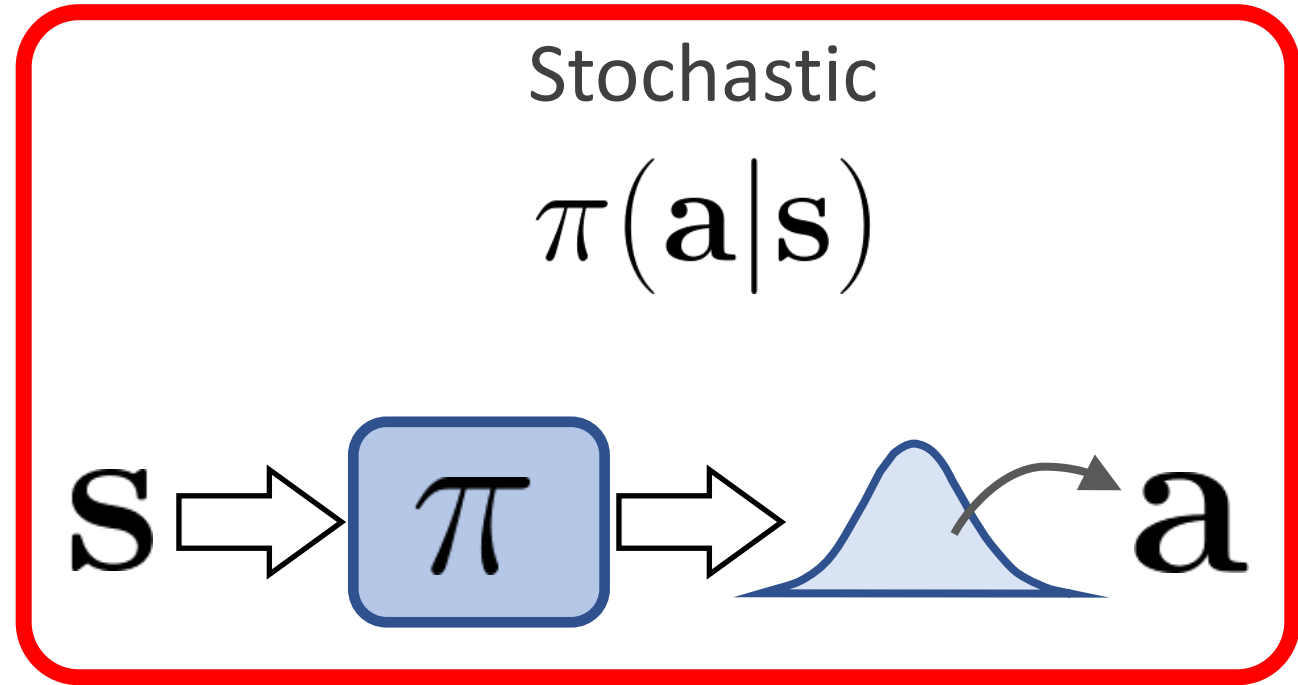
Deterministic

$$\mathbf{a} = \pi(\mathbf{s})$$



Stochastic

$$\pi(\mathbf{a}|\mathbf{s})$$



## Deterministic Policy

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$$\pi^{\text{det}}(\mathbf{s}) = \mathbf{a}^*$$

$$\pi(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \pi^{\text{det}}(\mathbf{s}) \\ 0 & \text{otherwise} \end{cases}$$



# Deterministic Policy

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$$\pi(\mathbf{a}|\mathbf{s})$$

# Deterministic Policy

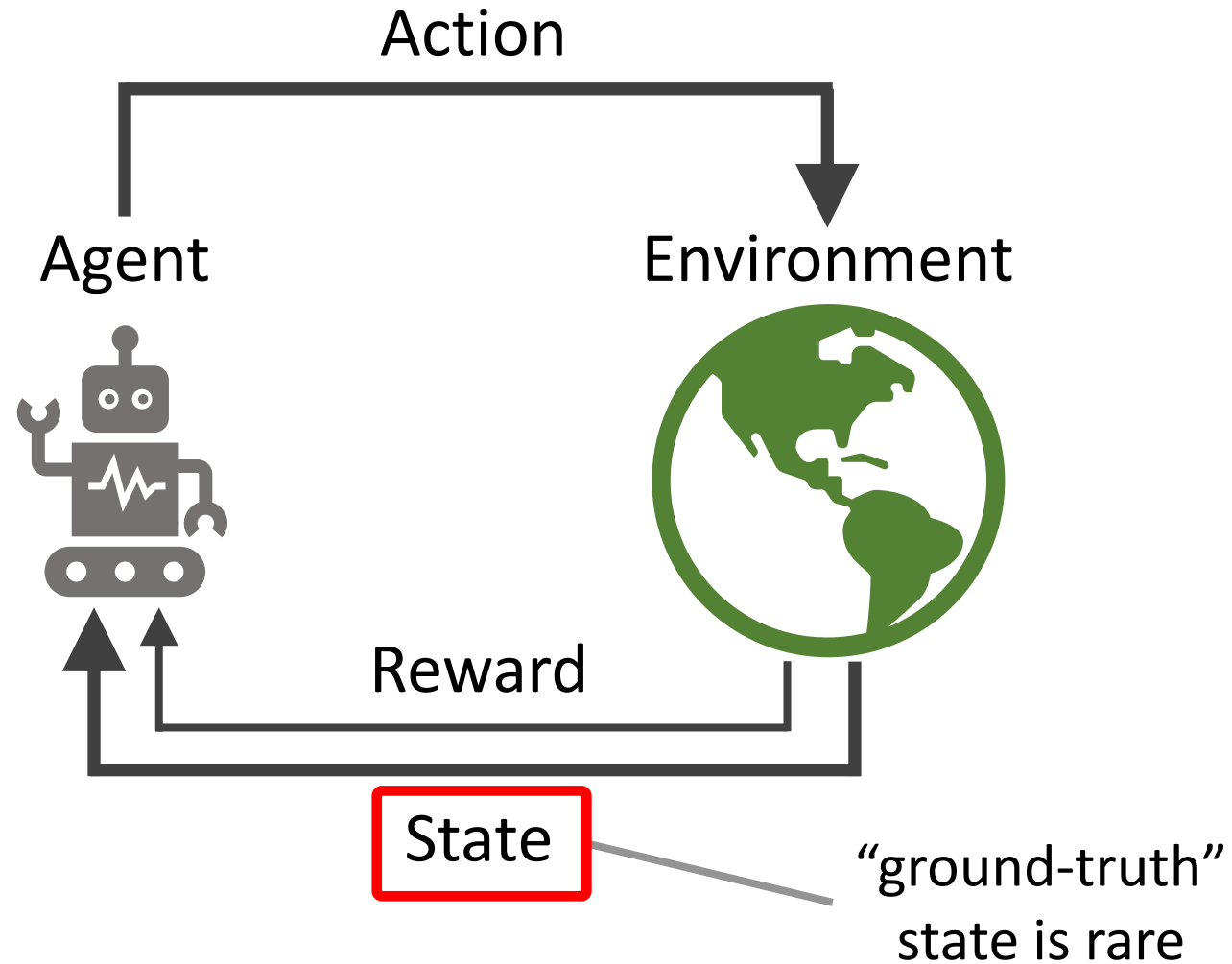
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$$\mathbf{a}^* = \arg \max_{\mathbf{a}} \pi(\mathbf{a}|\mathbf{s})$$

$$\pi^{\text{det}}(\mathbf{s}) = \mathbf{a}^*$$

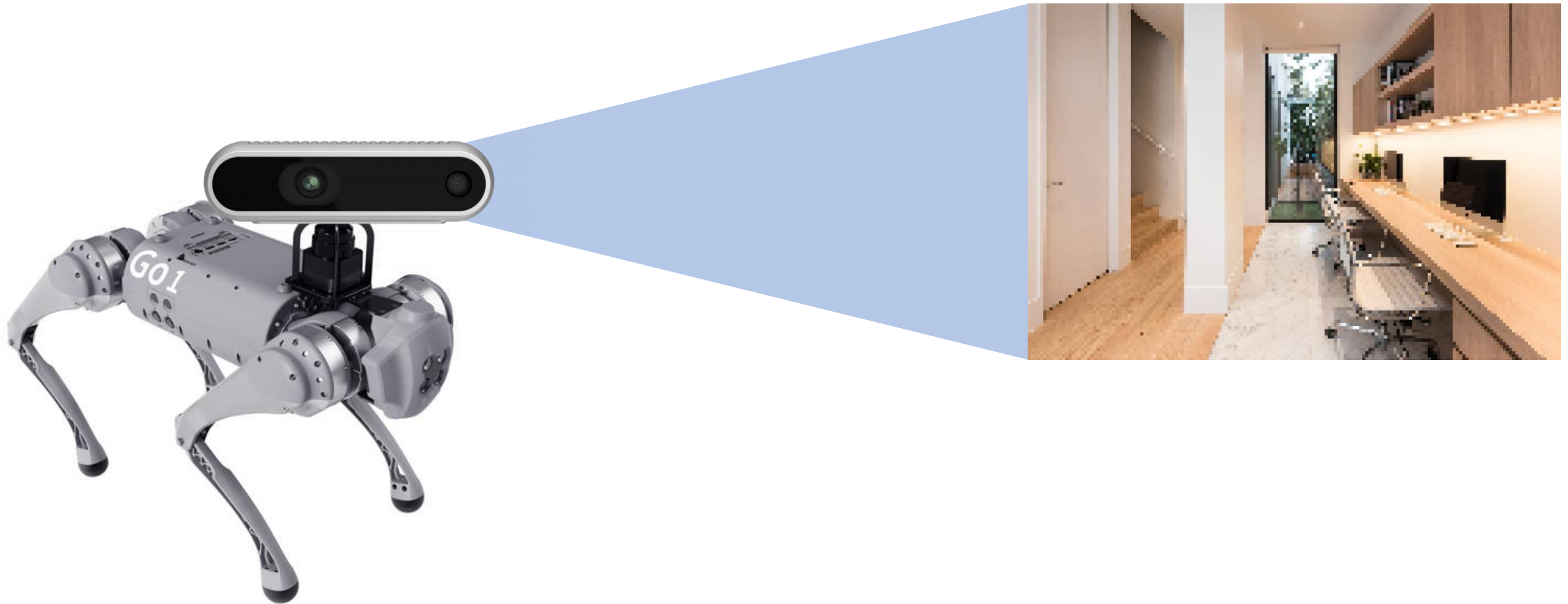
# Agent-Environment Interface

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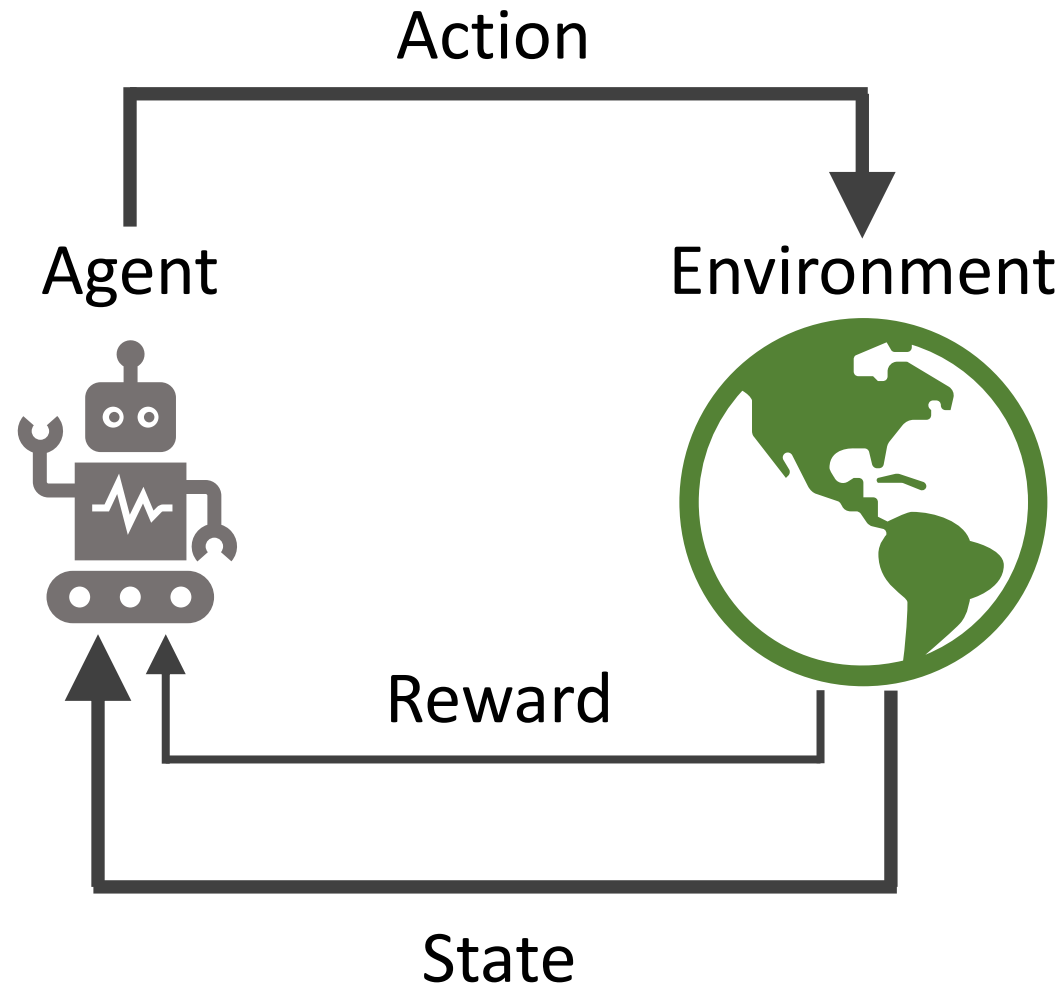
# Observations

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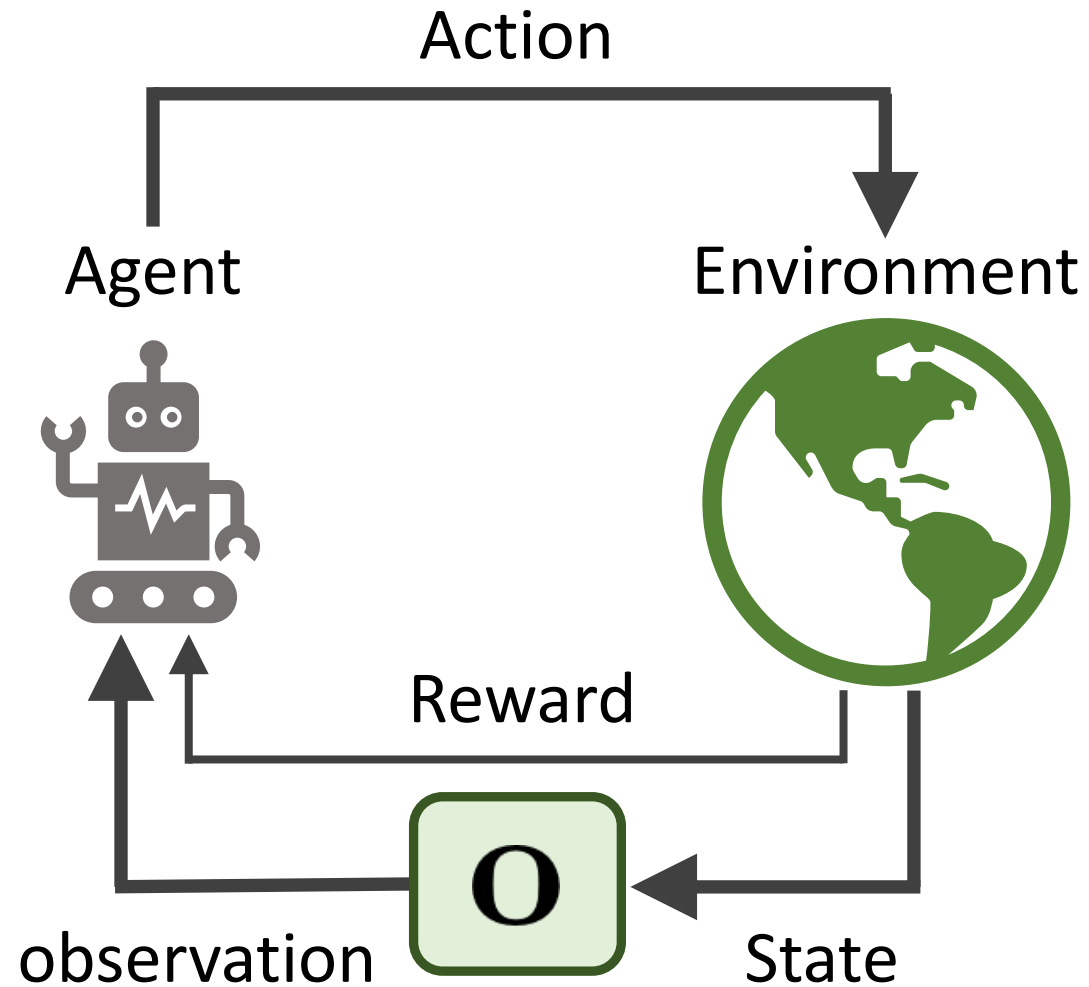
# Agent-Environment Interface

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# Agent-Environment Interface

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# Partially Observable Markov Decision Process

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$\mathbf{s} \in \mathcal{S}$  – state space

$\mathbf{o} \in \mathcal{O}$  – observation space

$\mathbf{a} \in \mathcal{A}$  – action space

$p(\mathbf{o}|\mathbf{s})$  – observation function

$p(\mathbf{s}'|\mathbf{s}, \mathbf{a})$  – dynamics function

$r(\mathbf{s}, \mathbf{a}, \mathbf{s}')$  – reward function

# POMDP

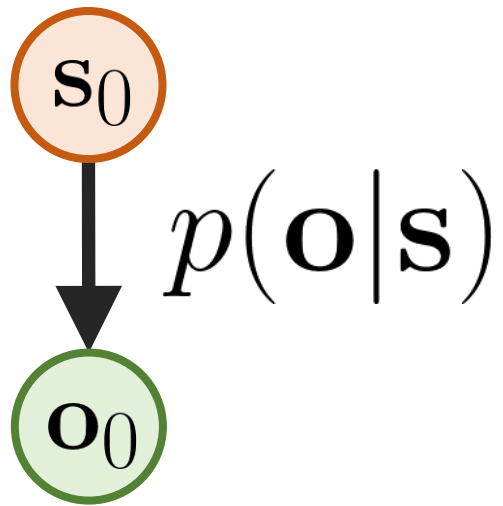
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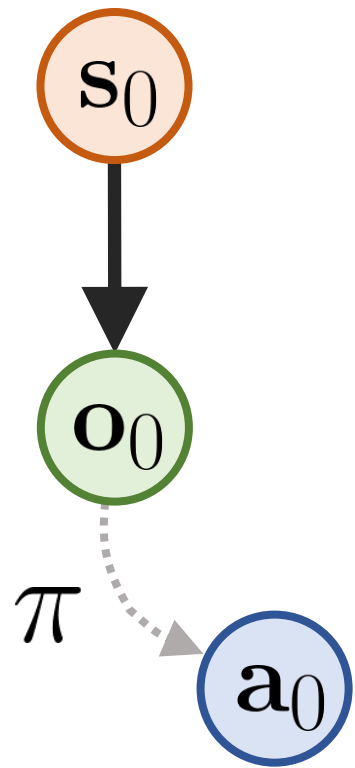
# POMDP

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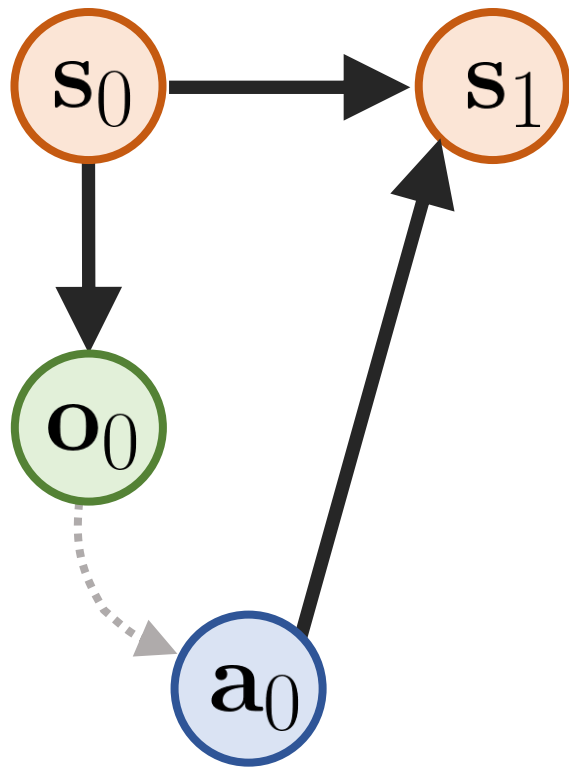
# POMDP

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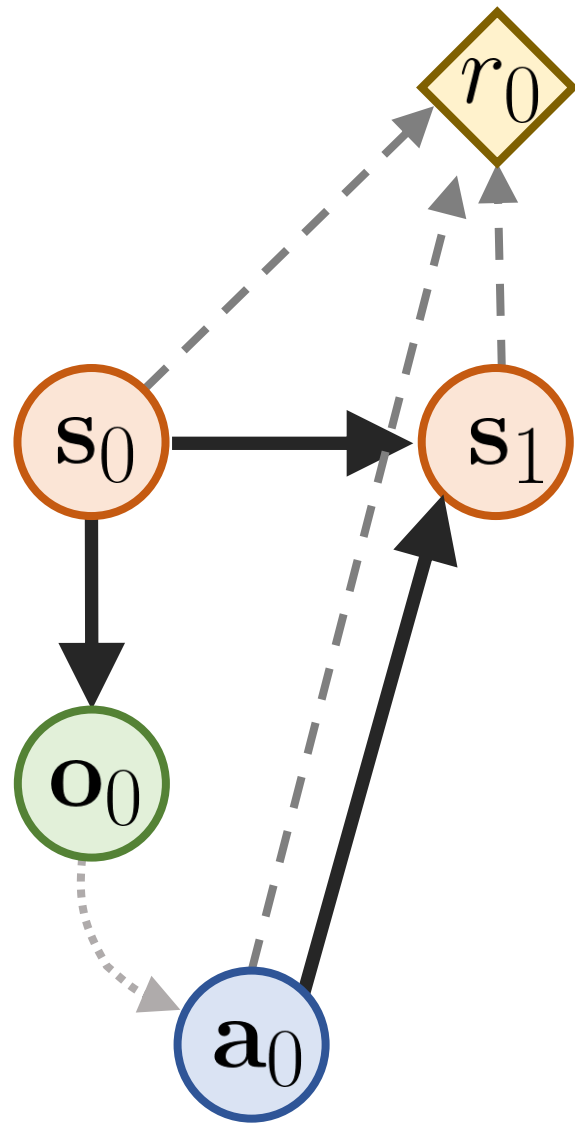
# POMDP

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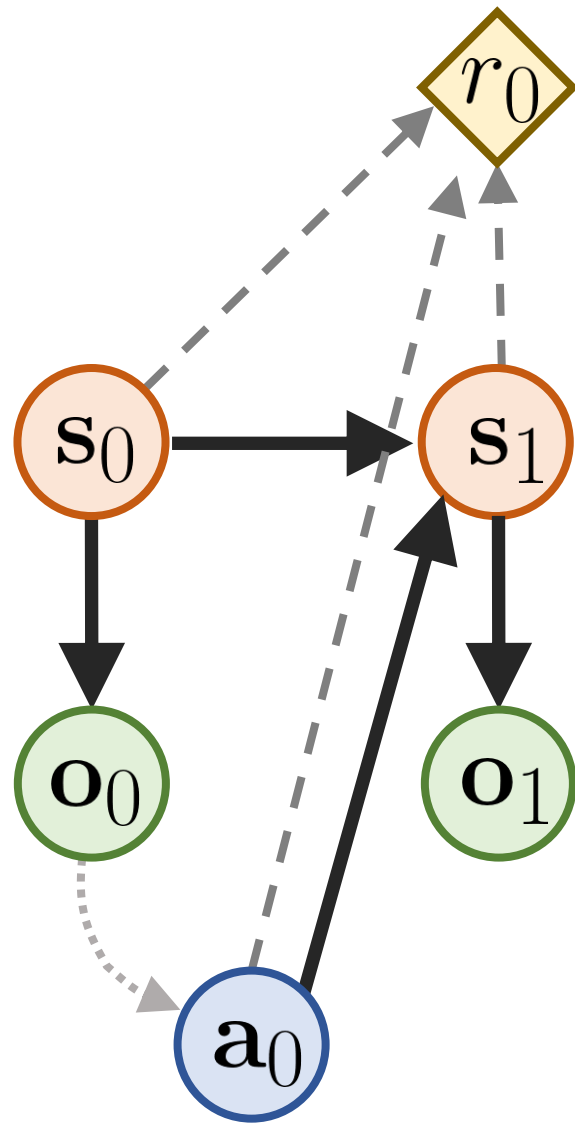
# POMDP

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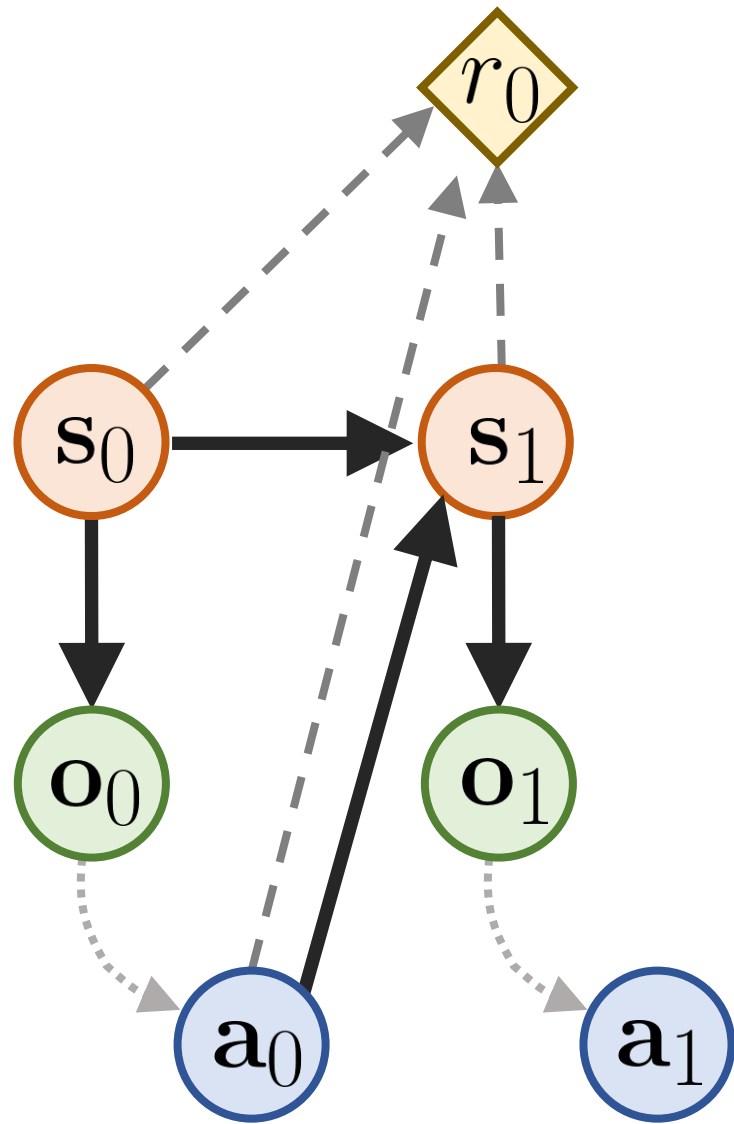
# POMDP

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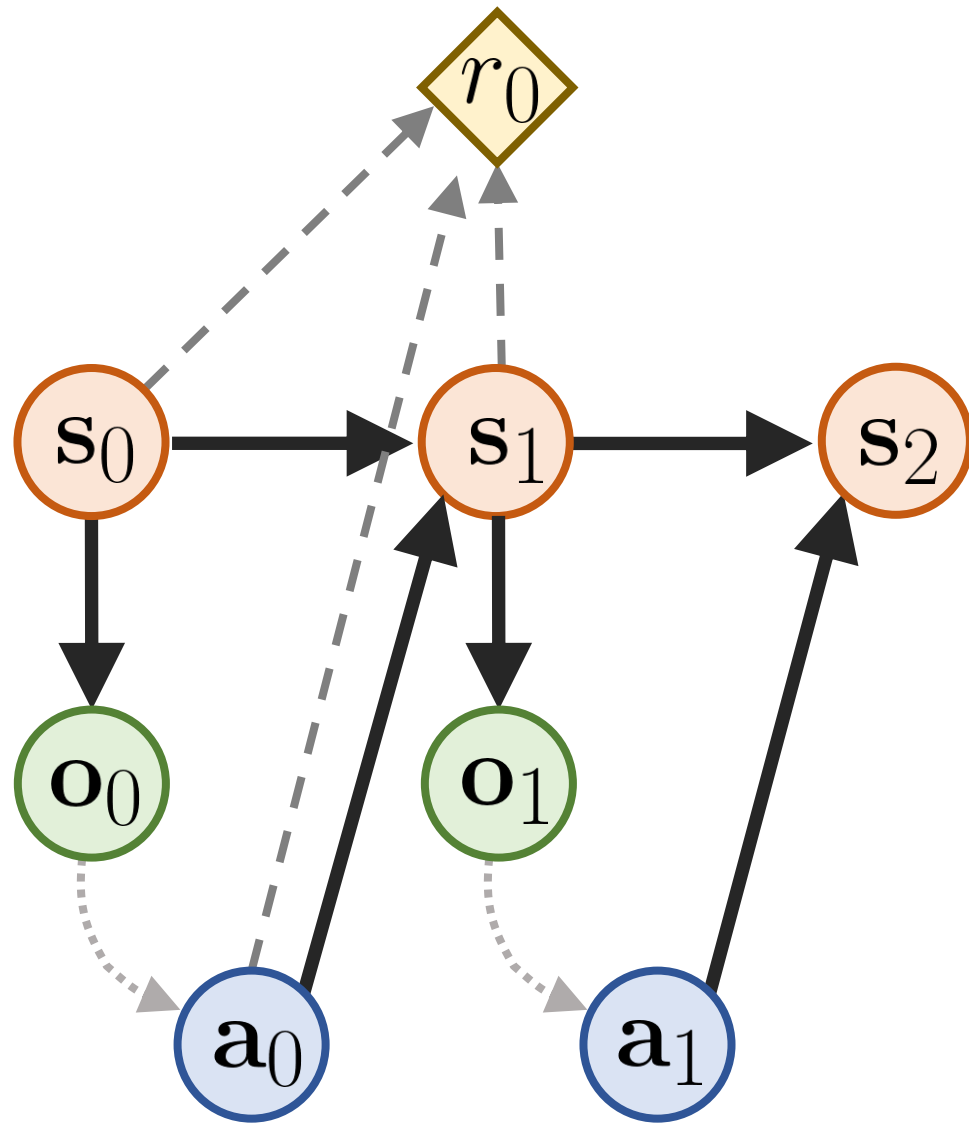
# POMDP

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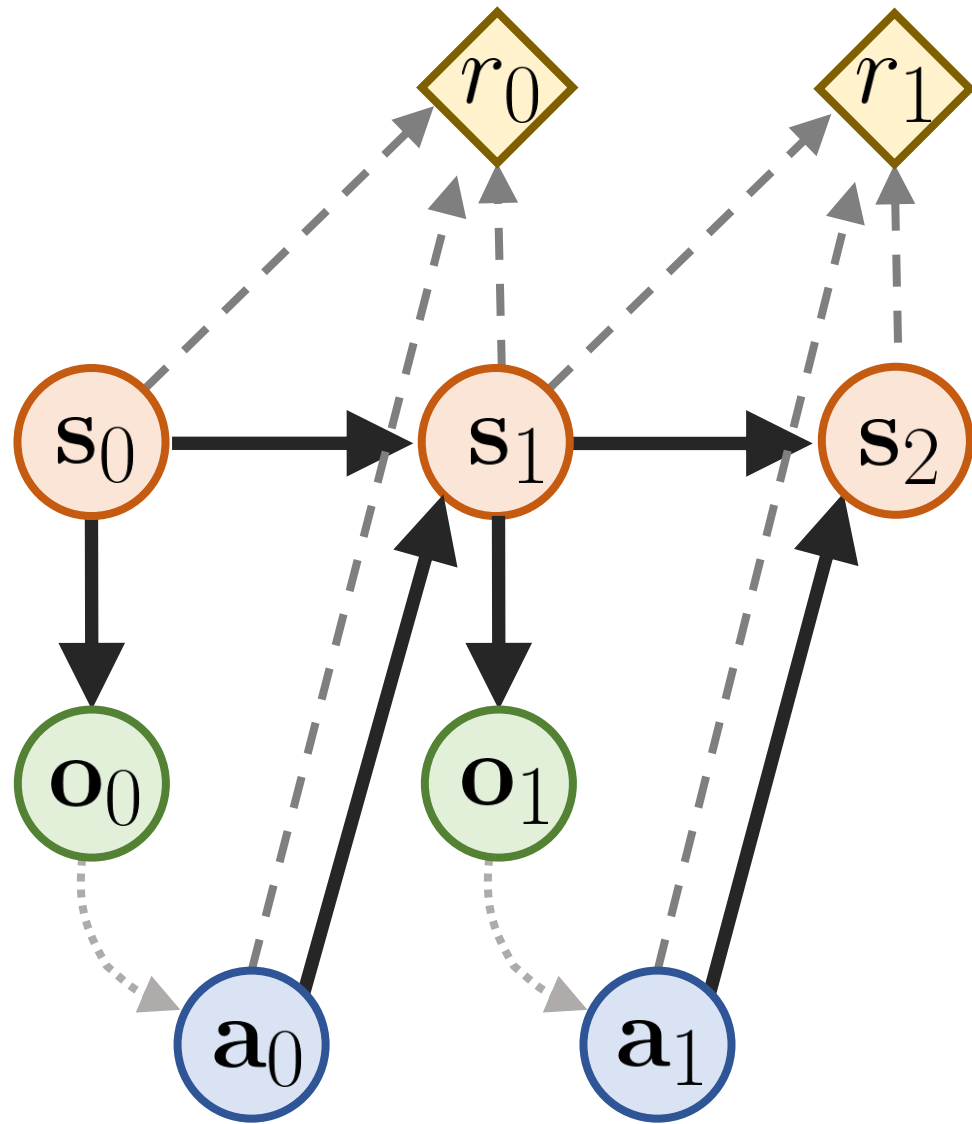
# POMDP

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# POMDP

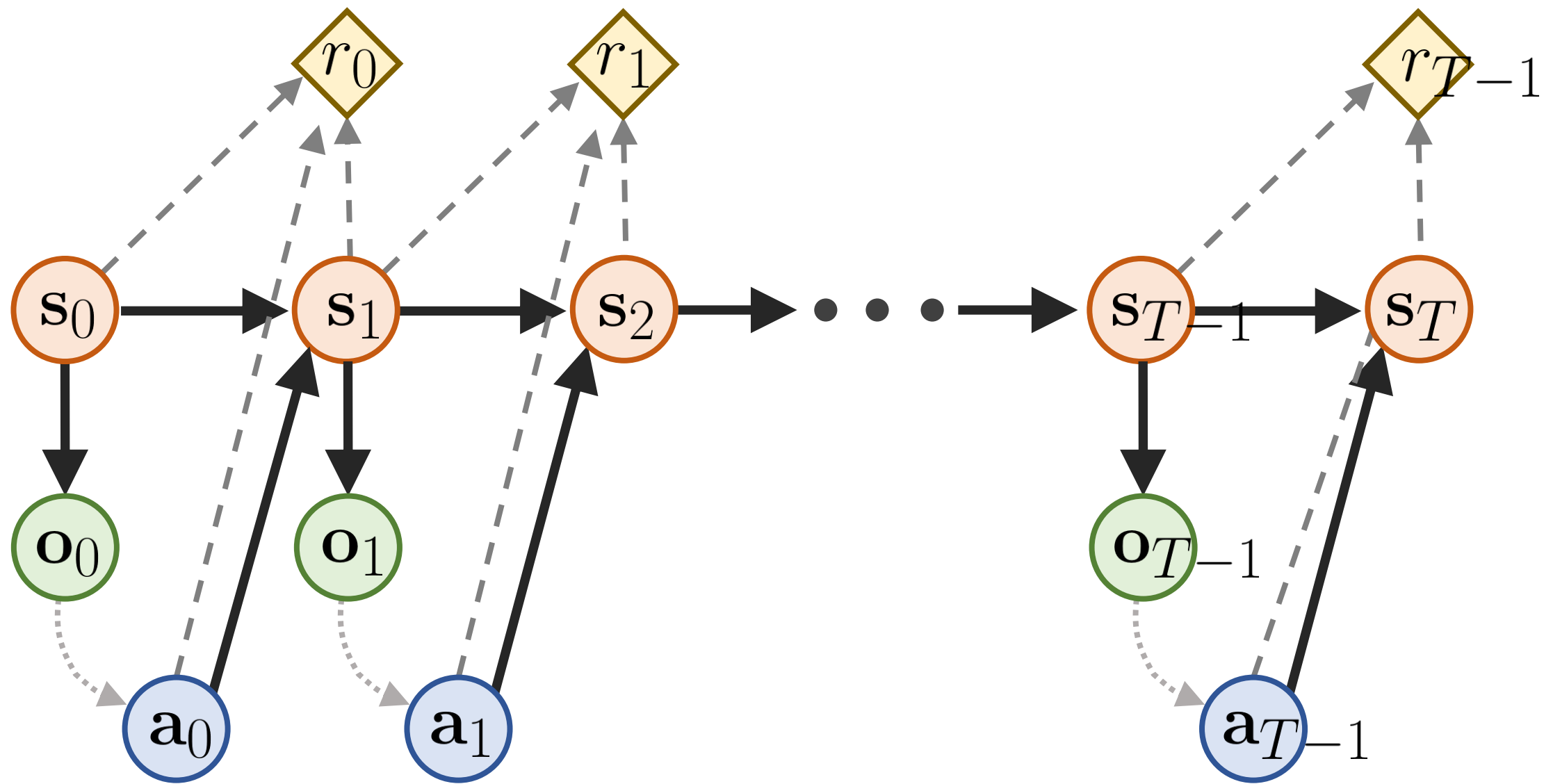
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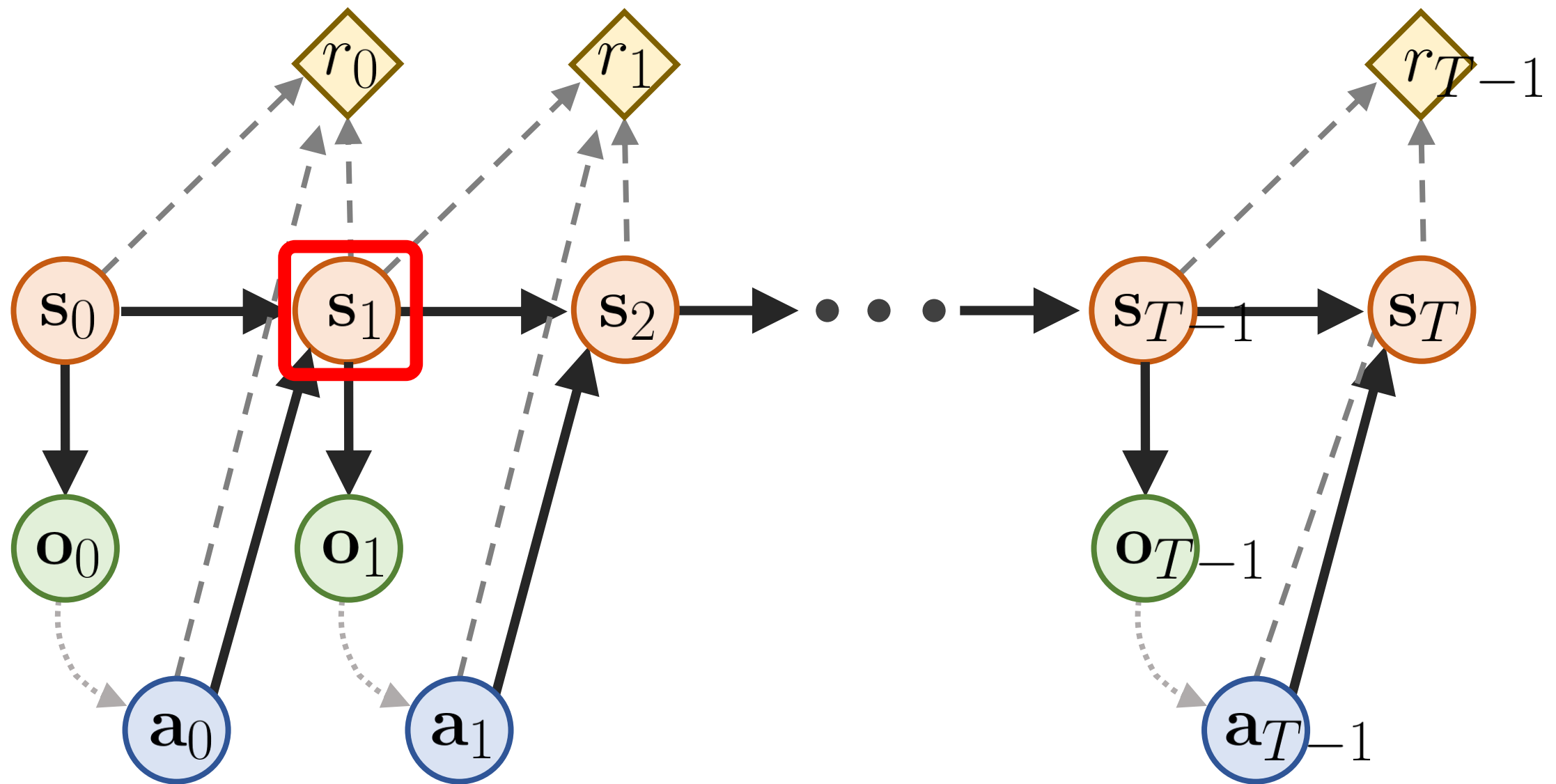
# POMDP

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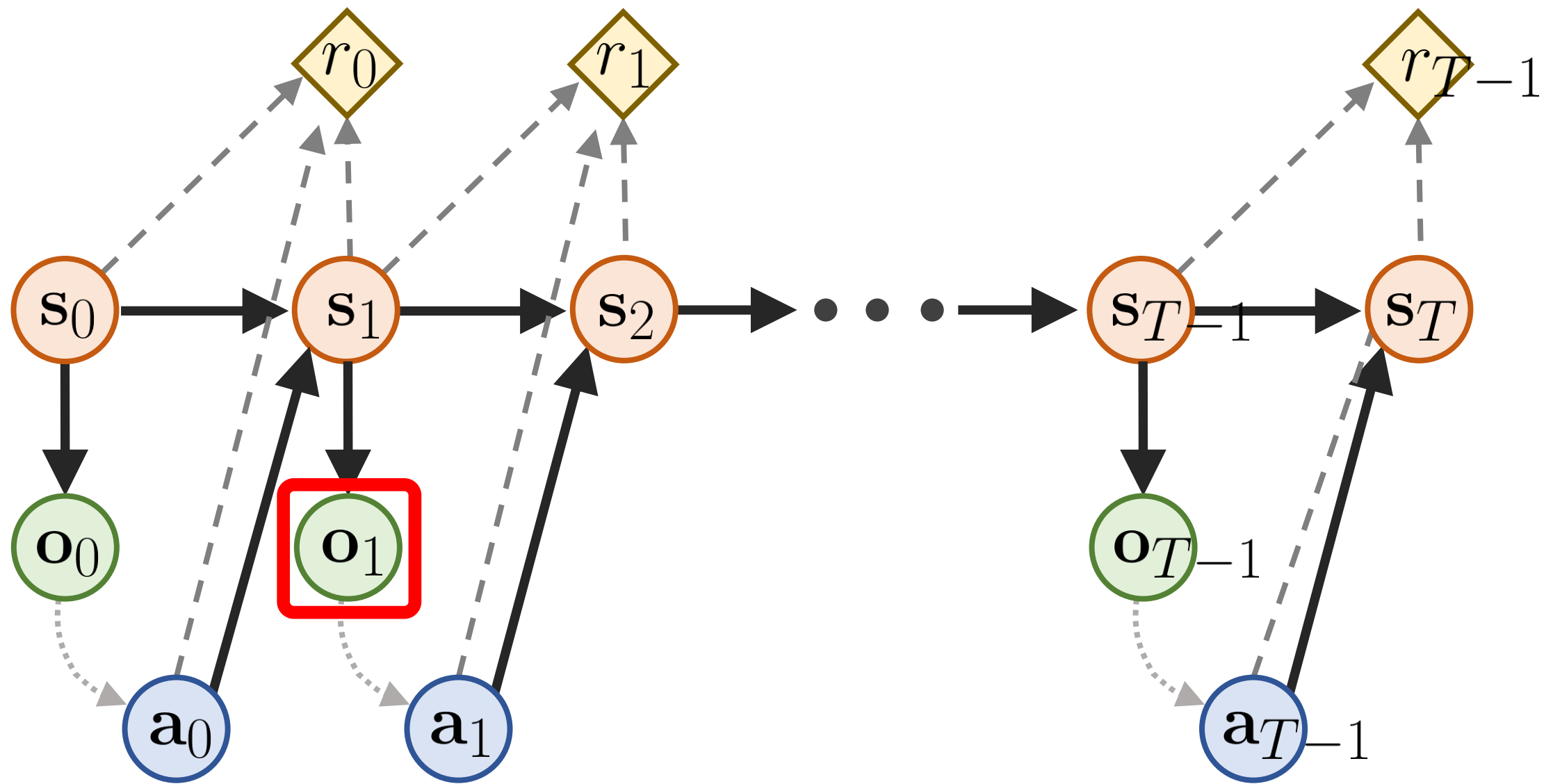
# POMDP

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# POMDP

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# Observation Function

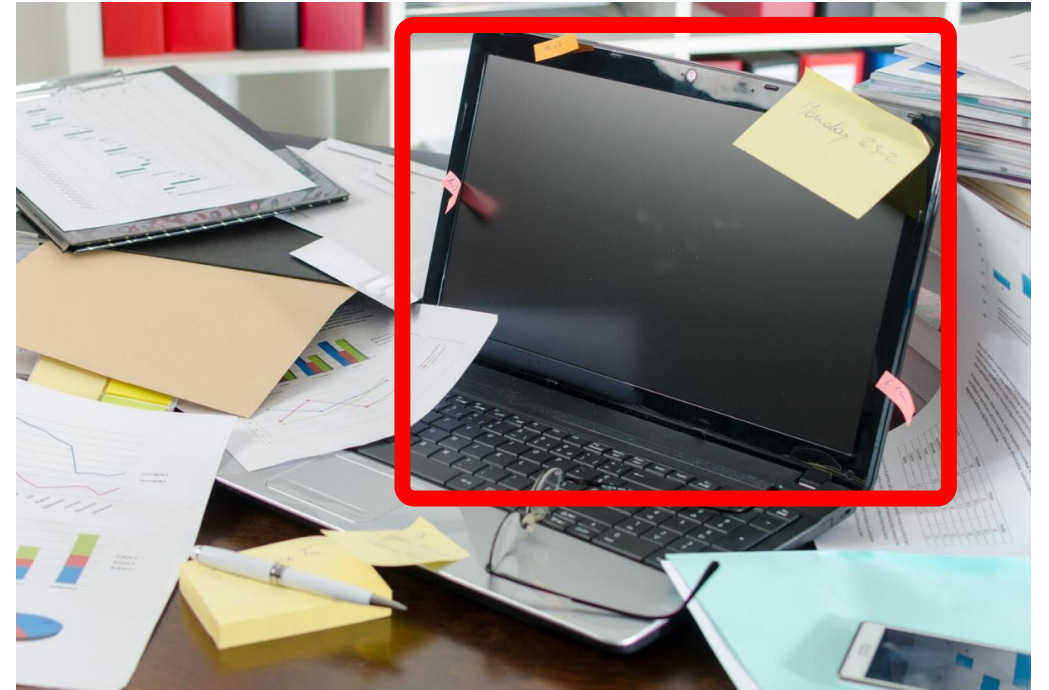
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State:

- Location of every item

Observation

- image of the desk



# Pong

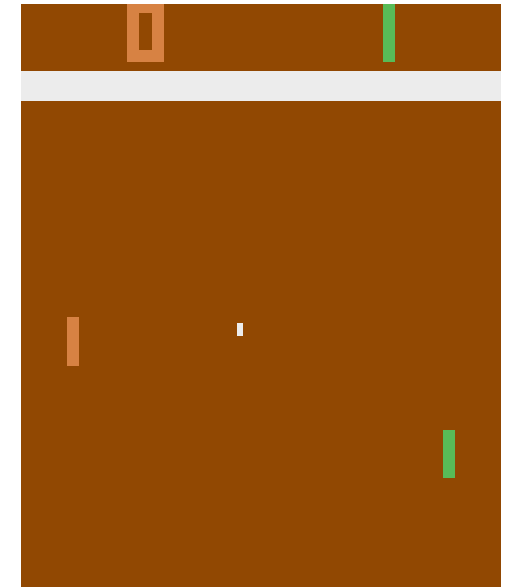
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## State:

- position + velocity of paddles
- position + velocity of the ball
- scores

## Observation

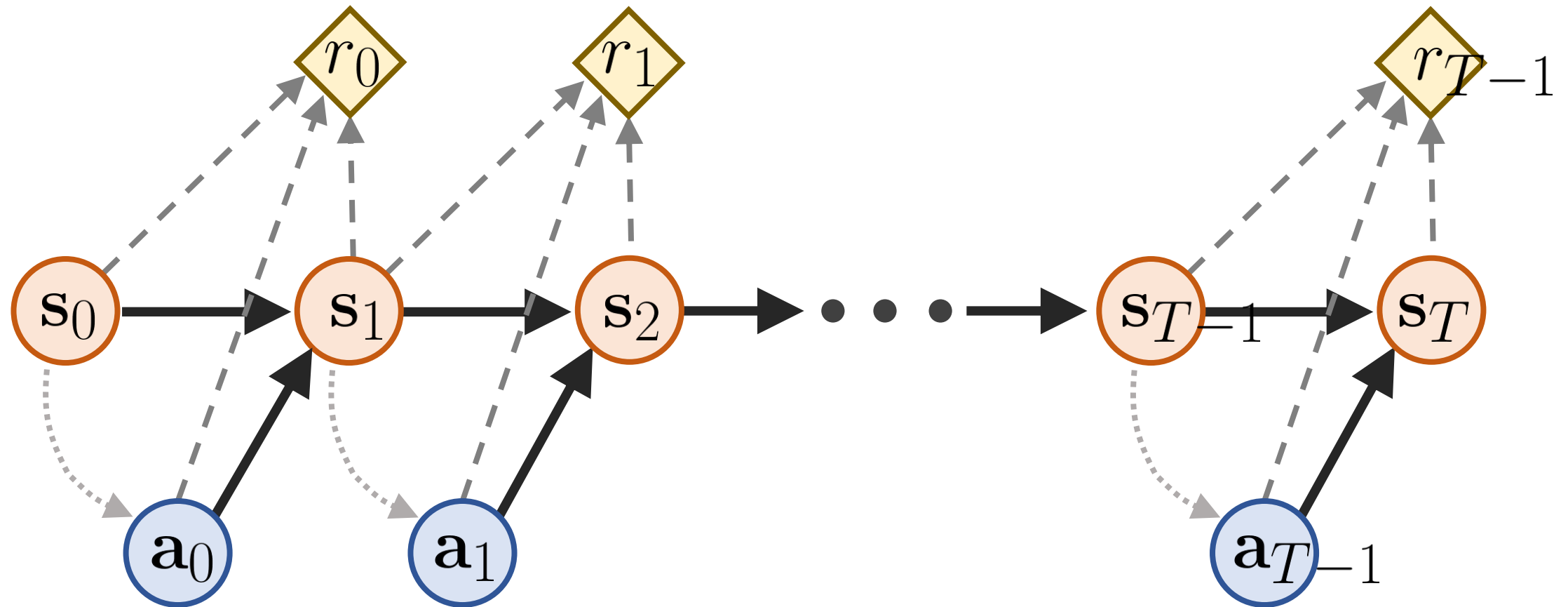
- image of the game screen



Pong [Atari]

# MDP

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# Summary

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- Agent-Environment Interface
- Markov Decision Processes
- Partially Observable Markov Decision Processes