Q-Learning

CMPT 729 G100

Jason Peng

Overview

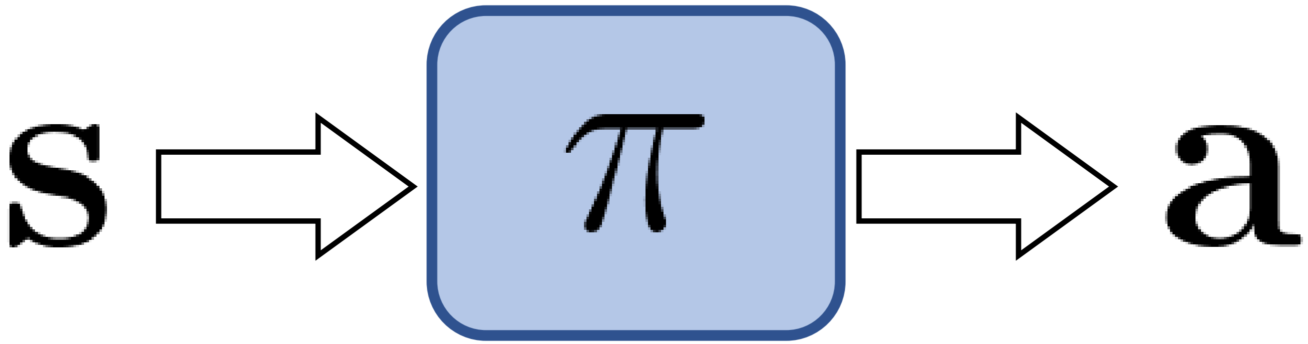
- Q-Function
- Q-Learning
- Exploration

Taxonomy of RL Algorithms

- Policy-Based Methods
- Value-Based Methods
- Actor-Critic Methods
- Model-Based Methods

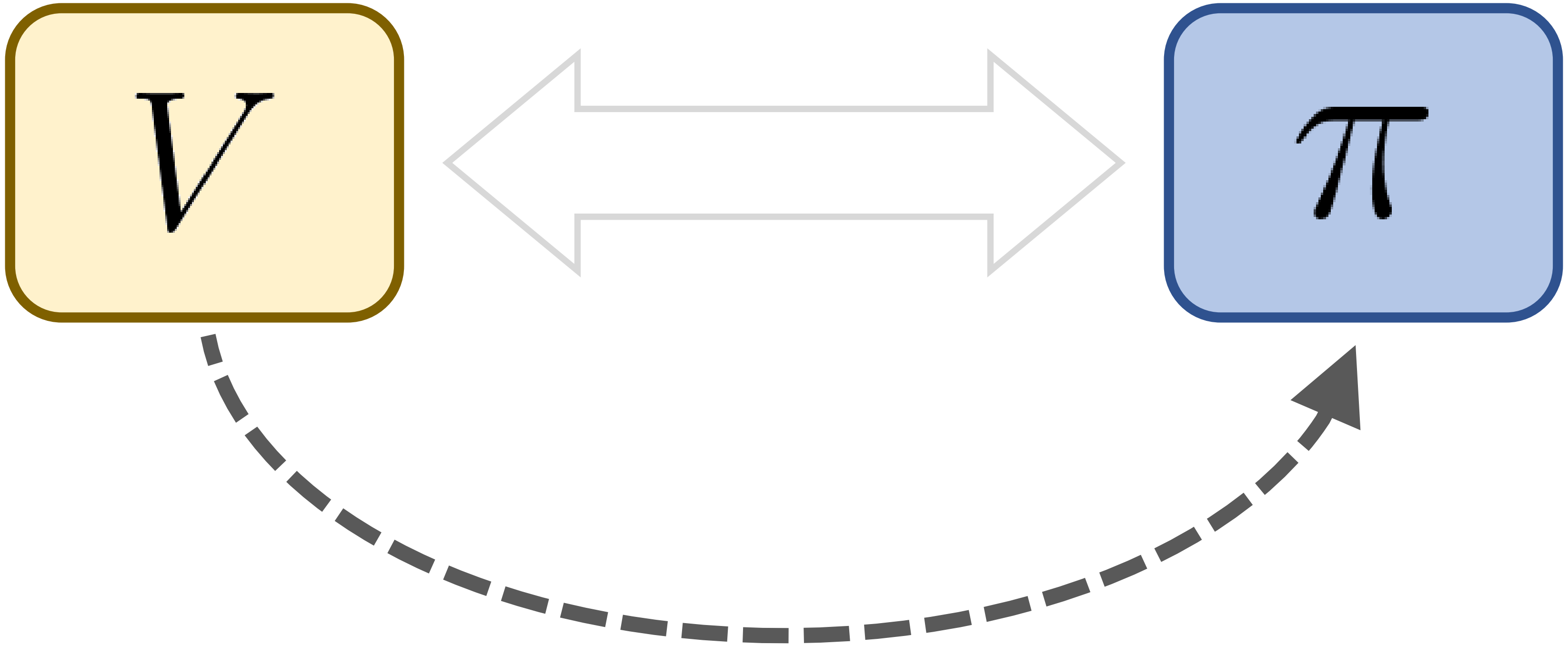
Policy-Based Methods

$$\pi(\mathbf{a}|\mathbf{s})$$



Value-Based Methods

Value-Based Methods



Policy Gradient

$$\nabla_{\pi} J(\pi) = \mathbb{E}_{\mathbf{s}\sim d_{\pi}(\mathbf{s})} \mathbb{E}_{\mathbf{a}\sim \pi(\mathbf{a}|\mathbf{s})} \mathbb{E}_{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s},\mathbf{a}_0=\mathbf{a})} \left[\nabla_{\pi} \log \pi(\mathbf{a}|\mathbf{s}) \left(\sum_{t=0} \gamma^{t} r_{t} - \cancel{V^{\pi}}(\mathbf{s}) \right) \right]$$

Policy Gradient

$$\nabla_\pi J(\pi) = \mathbb{E}_{\mathbf{s}\sim d_\pi(\mathbf{s})}\mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\underline{\mathbb{E}_{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s},\mathbf{a}_0=\mathbf{a})}}\left[\nabla_\pi \log \pi(\mathbf{a}|\mathbf{s})\left(\sum_{t=0}\gamma^t r_t\right)\right]$$

Policy Gradient

$$\nabla_\pi J(\pi) = \mathbb{E}_{\mathbf{s}\sim d_\pi(\mathbf{s})}\mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[\nabla_\pi \log\pi(\mathbf{a}|\mathbf{s}) \underbrace{\mathbb{E}_{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s},\mathbf{a}_0=\mathbf{a})}\left(\sum_{t=0}\gamma^t r_t\right)}_{\text{“reward-to-go”}}\right]$$

$$= Q^\pi(\mathbf{s},\mathbf{a})$$

reward-to-go: expected return of taking an action $\mathbf{a}$ in state $\mathbf{s}$

Policy Gradient

$$\nabla_\pi J(\pi) = \mathbb{E}_{\mathbf{s}\sim d_\pi(\mathbf{s})}\mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[\nabla_\pi \mathrm{log}\pi(\mathbf{a}|\mathbf{s})\underline{Q^\pi(\mathbf{s},\mathbf{a})}\right]$$

reward-to-go: expected return of taking an action $\mathbf{a}$ in state $\mathbf{s}$

Policy Gradient

$$\nabla_\pi J(\pi) = \mathbb{E}_{\underline{\mathbf{s}\sim d_\pi(\mathbf{s})}}\mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[\nabla_\pi \log \pi(\mathbf{a}|\mathbf{s}) Q^\pi(\mathbf{s},\mathbf{a})\right]$$

Policy Gradient

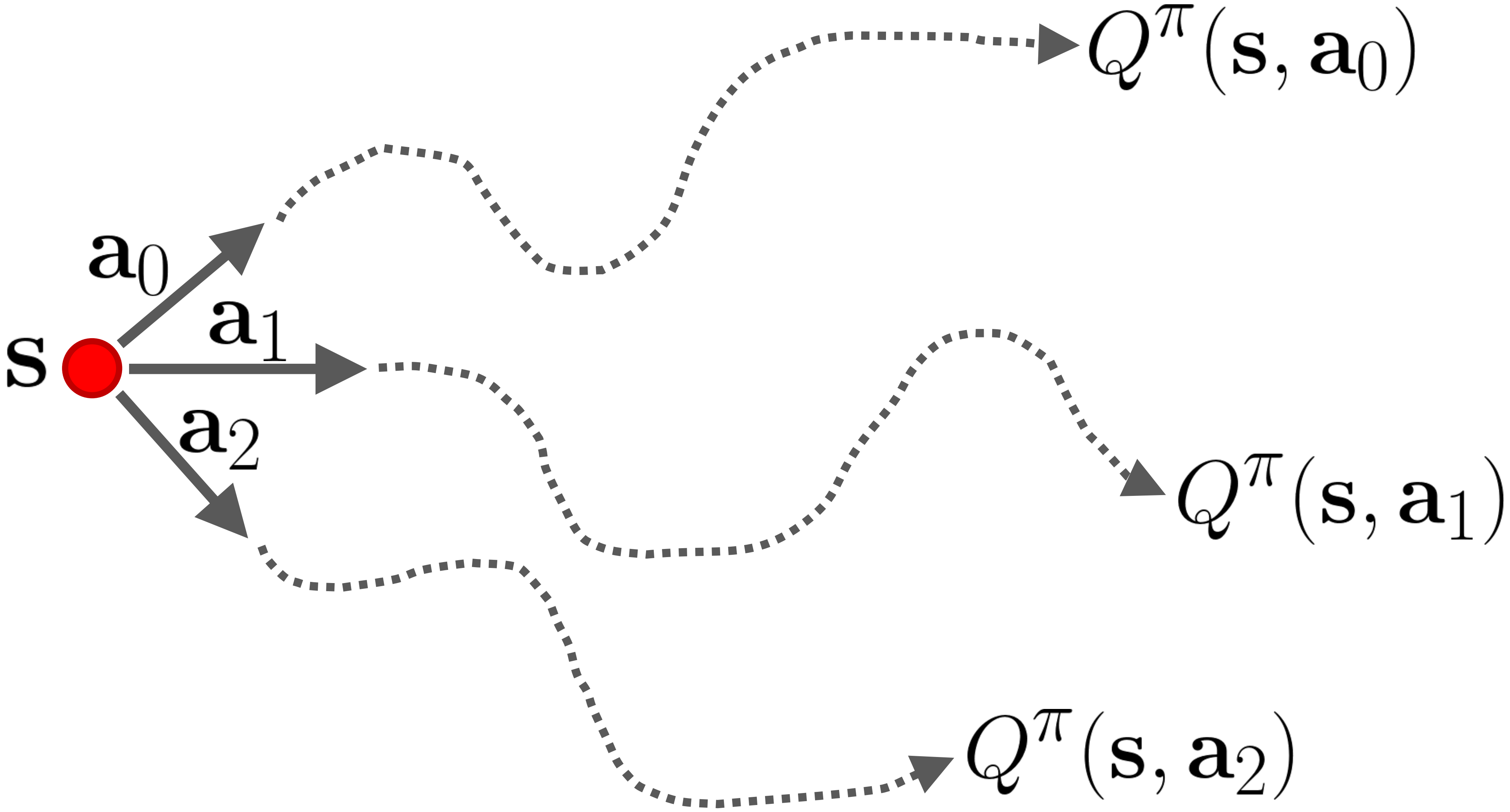
$$\nabla_\pi J(\pi) = \mathbb{E}_{\mathbf{s}\sim d_\pi(\mathbf{s})} \boxed{\mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[\nabla_\pi \log \pi(\mathbf{a}|\mathbf{s}) Q^\pi(\mathbf{s},\mathbf{a})\right]}$$

$$\Downarrow$$

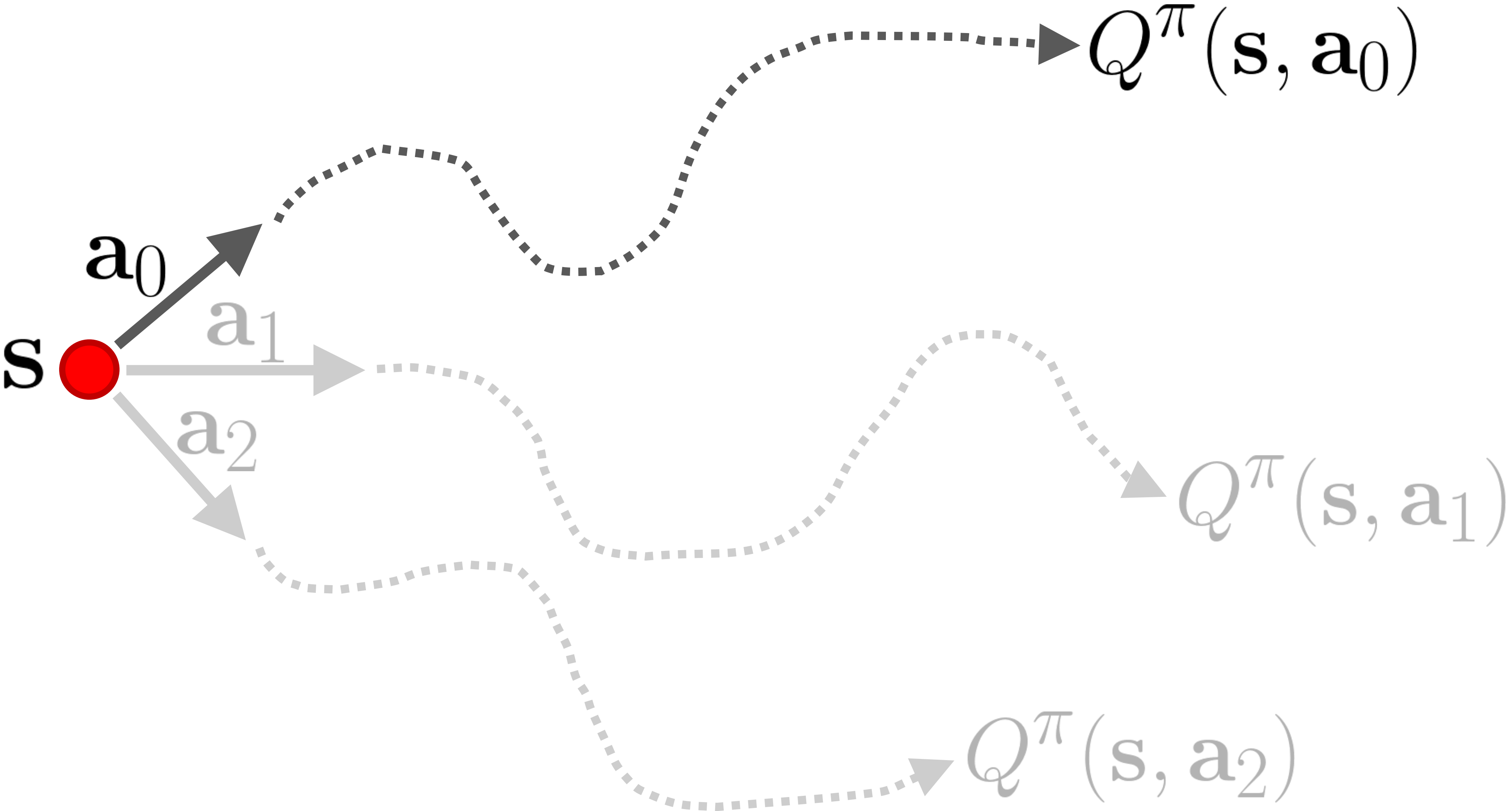
$$\max_\pi \; \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[Q^\pi(\mathbf{s},\mathbf{a})\right]$$

Per-state objective: pick actions that maximize the expected return at each state (i.e. Q-function)

Q-Function


$Q^{\pi}(\mathbf{s}, \mathbf{a}_0)$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{s}$
$\mathbf{a}_2$
$Q^{\pi}(\mathbf{s}, \mathbf{a}_1)$
$Q^{\pi}(\mathbf{s}, \mathbf{a}_2)$

Q-Function


$Q^\pi(\mathbf{s}, \mathbf{a}_0)$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{s}$
$\mathbf{a}_2$
$Q^\pi(\mathbf{s}, \mathbf{a}_1)$
$Q^\pi(\mathbf{s}, \mathbf{a}_2)$

Value Functions

Value Function

“State Value Function”

$$V^{\pi}(\mathbf{s}) = \mathbb{E}_{\underline{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s})}}\left[\sum_{t=0}\gamma^t r_t\right]$$

Likelihood of a trajectory starting at state $\mathbf{s}$ and then following π for all future timesteps

Q-Function

“State-Action Value Function”

$$Q^{\pi}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\underline{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s},\mathbf{a}_0=\mathbf{a})}}\left[\sum_{t=0}\gamma^t r_t\right]$$

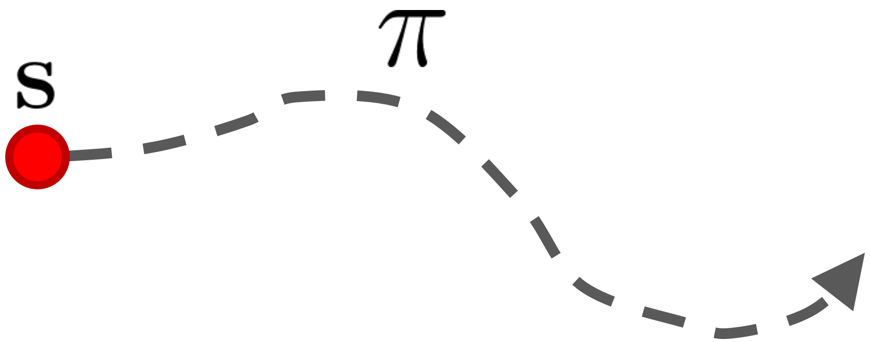
Likelihood of a trajectory after taking action $\mathbf{a}$ in state $\mathbf{s}$ and then following π for all future timesteps

Value Functions

Value Function

“State Value Function”

$$V^{\pi}(\mathbf{s}) = \mathbb{E}_{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s})}\left[\sum_{t=0}\gamma^t r_t\right]$$

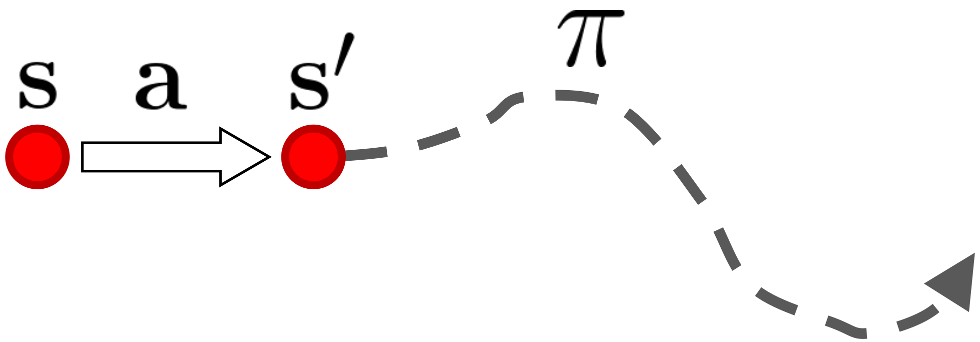


Q-Function

“State-Action Value Function”

$$Q^{\pi}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\tau\sim p(\tau|\pi,\mathbf{s}_0=\mathbf{s},\mathbf{a}_0=\mathbf{a})}\left[\sum_{t=0}\gamma^t r_t\right]$$

“Quality”



Recursive Definition

Value Function

$$V^{\pi}(\mathbf{s}) = \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma V^{\pi}(\mathbf{s}')\right]$$

Recursive Definition

Value Function

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Recursive Definition

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Q-Function

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Recursive Definition

Value Function

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Q-Function

$$Q^\pi(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma\mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[\underline{Q^\pi(\mathbf{s}',\mathbf{a}')}\right]\right]$$

Recursive Definition

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Recursive Definition

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Q-Function

$$Q^\pi(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma\mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[Q^\pi(\mathbf{s}',\mathbf{a}')\right]\right]$$

Recursive Definition

Value Function

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$$= \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[Q^{\pi}(\mathbf{s},\mathbf{a})\right]$$

Q-Function

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Value Function

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$$= \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[Q^{\pi}(\mathbf{s},\mathbf{a})\right]$$

Q-Function

$$Q^{\pi}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[Q^{\pi}(\mathbf{s}',\mathbf{a}')\right]\right]$$

Recursive Definition

Value Function

$$\begin{aligned} V^{\pi}(\mathbf{s}) &= \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma V^{\pi}(\mathbf{s}')\right] \\ &= \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[Q^{\pi}(\mathbf{s},\mathbf{a})\right] \end{aligned}$$

Q-Function

$$Q^{\pi}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \underbrace{\mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[Q^{\pi}(\mathbf{s}',\mathbf{a}')\right]}_{= V^{\pi}(\mathbf{s}')}\right]$$

Recursive Definition

Value Function

$$\begin{aligned} V^\pi(\mathbf{s}) &= \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma V^\pi(\mathbf{s}')\right] \\ &= \mathbb{E}_{\mathbf{a}\sim\pi(\mathbf{a}|\mathbf{s})}\left[Q^\pi(\mathbf{s},\mathbf{a})\right] \end{aligned}$$

Q-Function

$$\begin{aligned} Q^\pi(\mathbf{s},\mathbf{a}) &= \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[Q^\pi(\mathbf{s}',\mathbf{a}')\right]\right] \\ &= \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma V^\pi(\mathbf{s}')\right] \end{aligned}$$

Q-Function

$$\xcancel{\pi^*(\mathbf{a}|\mathbf{s})} \Longrightarrow Q^*(\mathbf{s}, \mathbf{a})$$

Recover optimal policy:

$$\pi^*(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} \; Q^*(\mathbf{s}, \mathbf{a}') \\ 0 & \text{otherwise} \end{cases}$$

Instead of learning policy, just learn Q-function.

Q-Function

$$\pi(\mathbf{a}|\mathbf{s}) \Longrightarrow Q^{\pi}(\mathbf{s},\mathbf{a})$$

Recover a policy:

“arg max policy”

$$\pi'(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} \; Q^{\pi}(\mathbf{s},\mathbf{a}') \\ 0 & \text{otherwise} \end{cases}$$

New policy is at least as good as the old policy.

$$J(\pi') \geq J(\pi) \qquad Q^{\pi'}(\mathbf{s},\mathbf{a}) \geq Q^{\pi}(\mathbf{s},\mathbf{a})$$

Q-Learning

Key idea:

- Instead of trying to learn the optimal policy, just learn optimal Q-function
- Then recover policy from Q-function

Q-Learning

Recursive definition

$$Q^{\pi}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma\mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[Q^{\pi}(\mathbf{s}',\mathbf{a}')\right]\right]$$

Optimal policy

$$Q^{*}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma\mathbb{E}_{\underline{\mathbf{a}'\sim\pi^{*}(\mathbf{a}'|\mathbf{s}')}}\left[Q^{*}(\mathbf{s}',\mathbf{a}')\right]\right]$$

$$\pi^{*}(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} Q^{*}(\mathbf{s},\mathbf{a}) \\ 0 & \text{otherwise} \end{cases}$$

Q-Learning

Recursive definition

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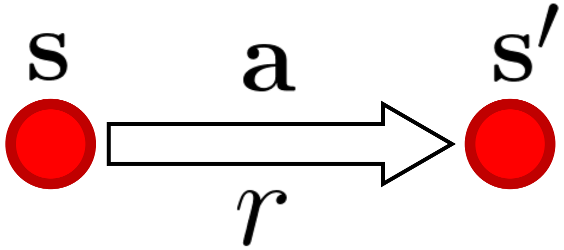
Optimal policy

$$Q^{*}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^{*}(\mathbf{s}',\mathbf{a}')\right)\right]$$

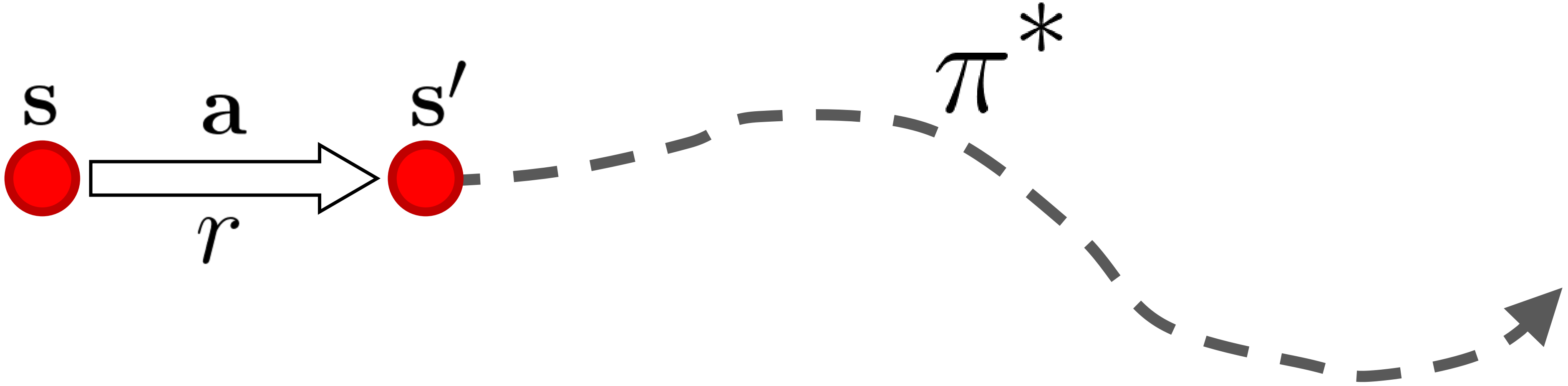
Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)\right]$$



Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \underline{\gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)}\right]$$



Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)\right]$$

Not true for non-optimal policies

$$Q^\pi(\mathbf{s},\mathbf{a}) \cancel{=} \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^\pi(\mathbf{s}',\mathbf{a}')\right)\right]$$

Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)\right]$$

Not true for non-optimal policies

$$Q^\pi(\mathbf{s},\mathbf{a}) \leq \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \underline{\max_{\mathbf{a}'}}\left(Q^\pi(\mathbf{s}',\mathbf{a}')\right)\right]$$

$$\geq$$

$$Q^\pi(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \underline{\mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}}\left[Q^\pi(\mathbf{s}',\mathbf{a}')\right]\right]$$

Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)\right]$$

Not true for non-optimal policies

$$Q^\pi(\mathbf{s},\mathbf{a}) \leq \underbrace{\mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^\pi(\mathbf{s}',\mathbf{a}')\right)\right]}_{}$$

Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)\right]$$

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Q-Learning

$$Q^*(\mathbf{s}, \mathbf{a}) = \mathbb{E}_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma \max_{\mathbf{a}'} \left(Q^*(\mathbf{s}', \mathbf{a}') \right) \right]$$

Not true for non-optimal policies

$$Q^\pi(\mathbf{s}, \mathbf{a}) \leq \mathbb{E}_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma \underbrace{\max_{\mathbf{a}'} \left(Q^\pi(\mathbf{s}', \mathbf{a}') \right)}_{\text{arg max policy}} \right]$$

Q-Learning

$$Q^*(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^*(\mathbf{s}',\mathbf{a}')\right)\right]$$

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$$\Downarrow$$

$$Q^\pi(\mathbf{s},\mathbf{a}) \leq Q^{\pi'}(\mathbf{s},\mathbf{a})$$

Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\underline{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \max_{\mathbf{a}'}\left(Q^{k}(\mathbf{s}',\mathbf{a}')\right)\right]$$

Q-Learning

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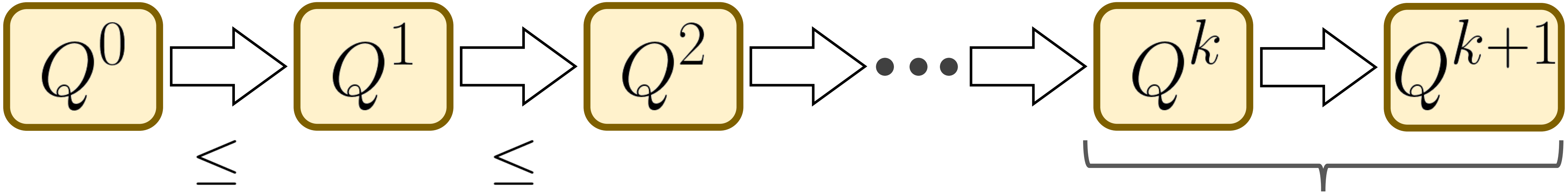
Q-Learning

$$Q^{k+1}(\mathbf{s}, \mathbf{a}) = \mathbb{E}_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma \max_{\mathbf{a}'} \left(Q^k(\mathbf{s}', \mathbf{a}') \right) \right]$$

$$Q^{k+1}(\mathbf{s}, \mathbf{a}) \geq Q^k(\mathbf{s}, \mathbf{a})$$



$$Q^0 \Rightarrow Q^1 \Rightarrow Q^2 \Rightarrow \cdots \Rightarrow \underbrace{Q^k \Rightarrow Q^{k+1}}$$

$$\leq \qquad \leq$$

$$Q^k(\mathbf{s}, \mathbf{a}) = Q^{k+1}(\mathbf{s}, \mathbf{a})$$

$$Q^k(\mathbf{s}, \mathbf{a}) = Q^*(\mathbf{s}, \mathbf{a})$$

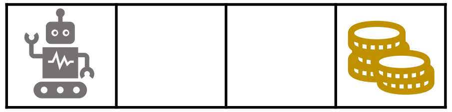
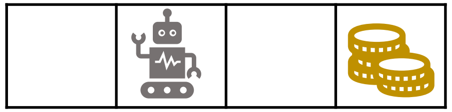
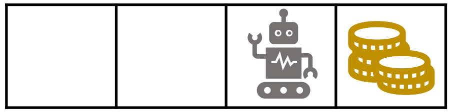
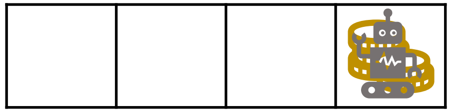
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

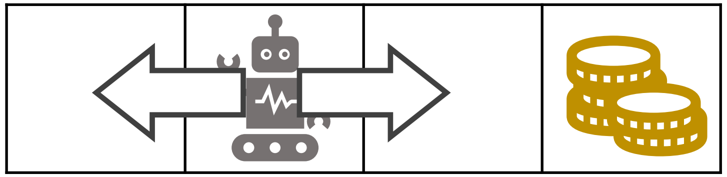
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

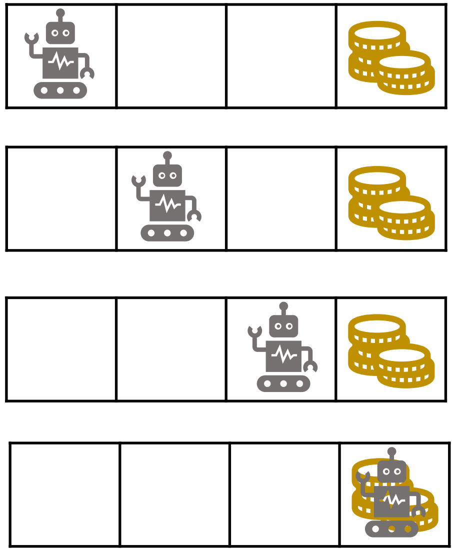
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \max_{\mathbf{a}'}\left(Q^{k}(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:

Action

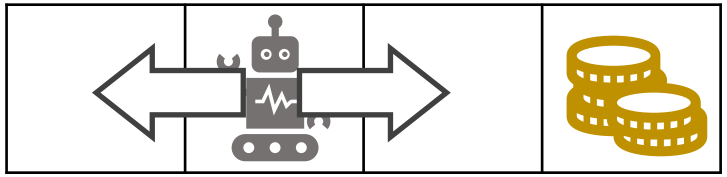
Q

State



←	□	→
0	0	0
0	0	0
0	0	0
0	0	0

Environment



$$\gamma = 1/2$$

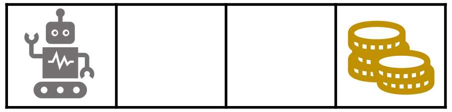
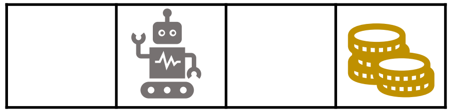
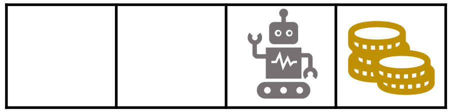
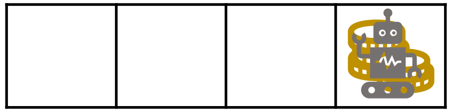
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

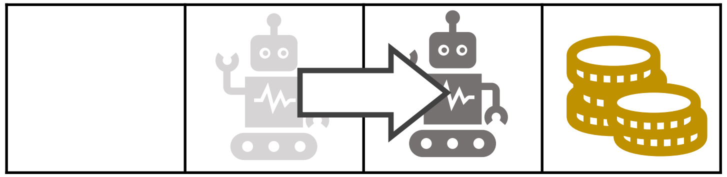
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

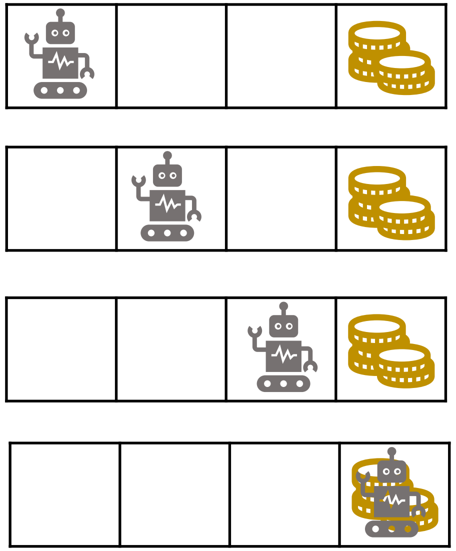
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \max_{\mathbf{a}'}\left(Q^{k}(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:

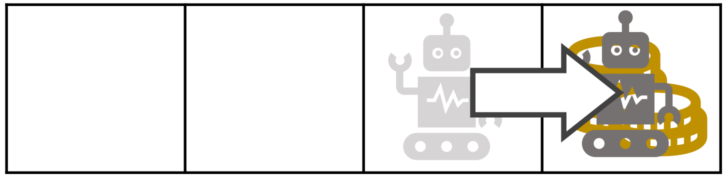
Action

State



Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

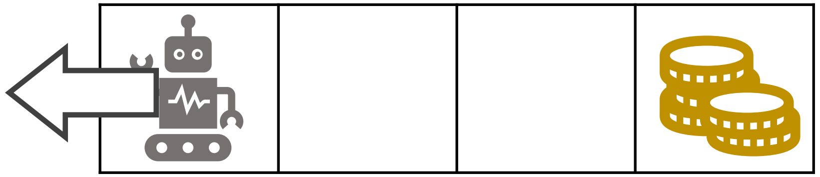
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

$$= 0$$

Iteration 0:

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

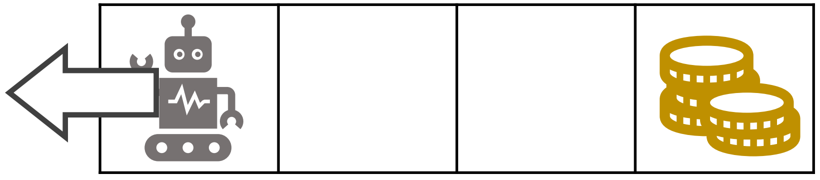
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

Iteration 0:

←	□	→
0	0	0
0	0	0
0	0	0
0	0	0

Environment

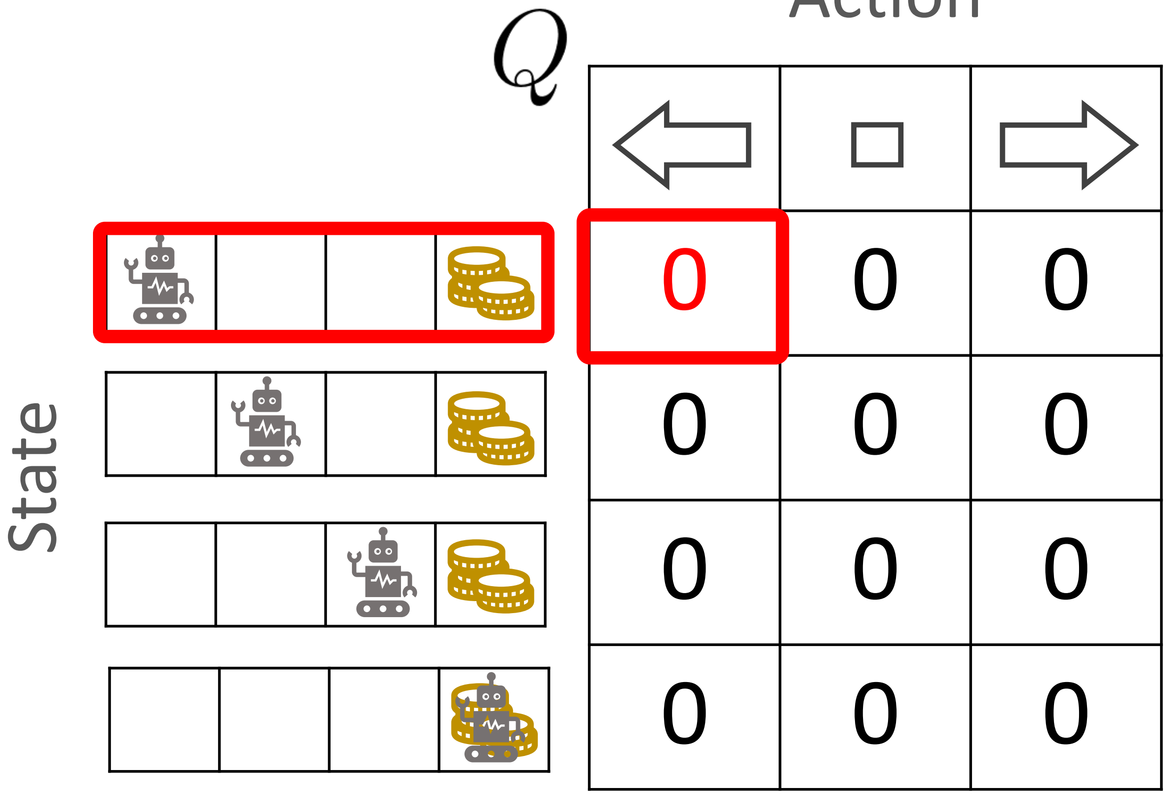


$$\gamma = 1/2$$

Tabular Q-Learning

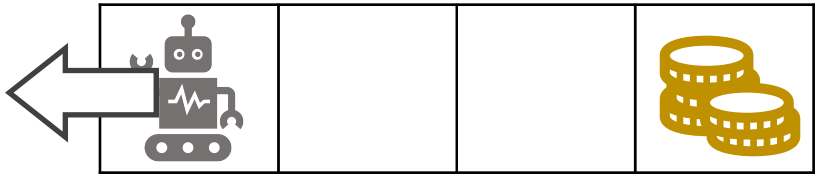
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

Iteration 0:



Q	←	□	→
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



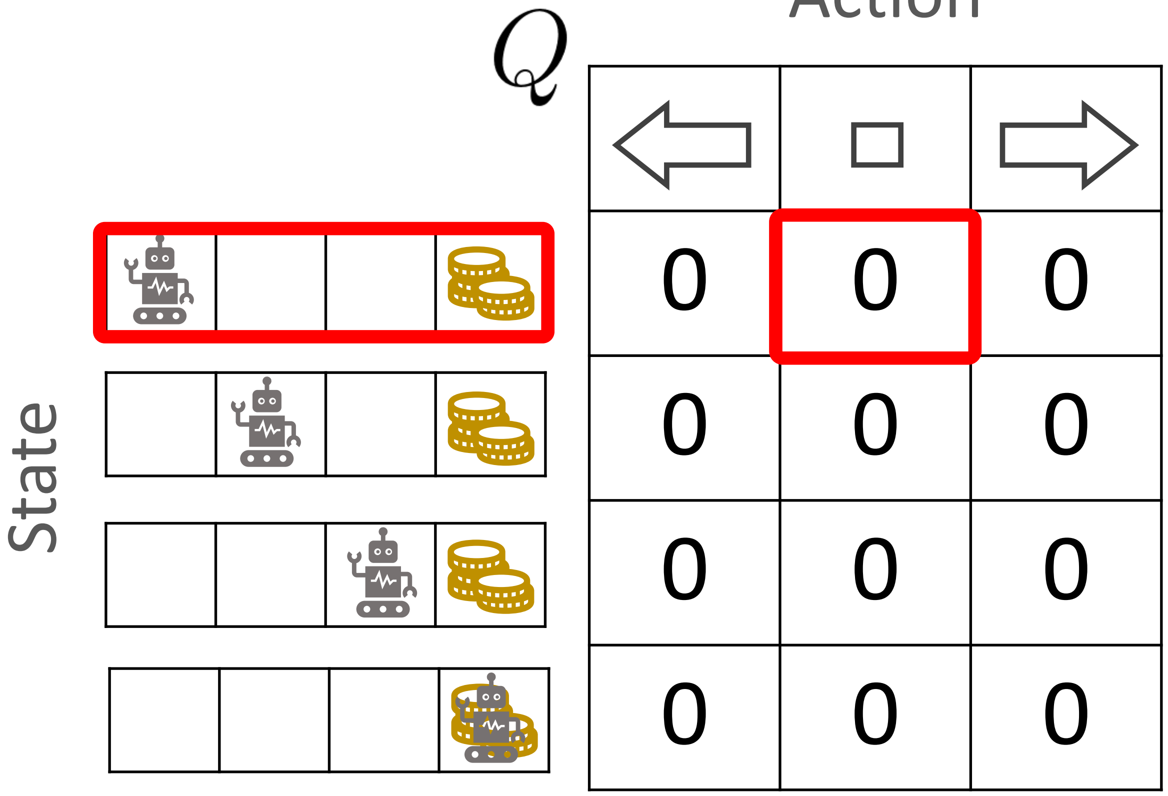
$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma\,\underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

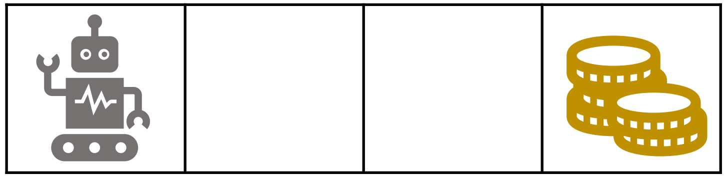
$$= 0$$

Iteration 0:



←	□	→
0	0	0
0	0	0
0	0	0
0	0	0

Environment



$$\gamma = 1/2$$

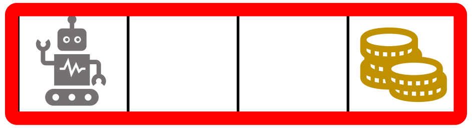
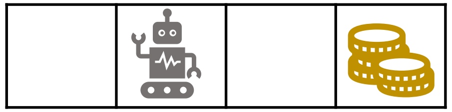
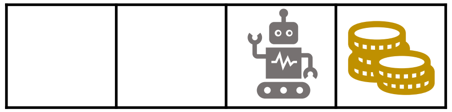
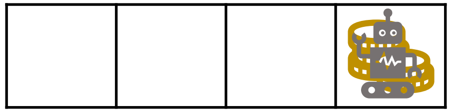
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

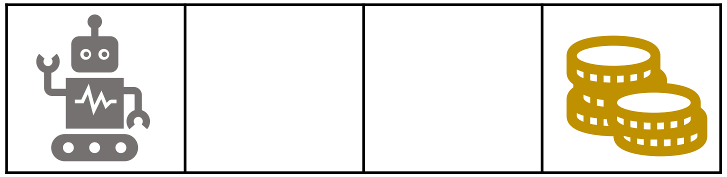
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

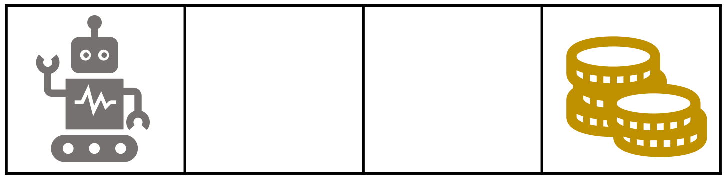
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

Iteration 0:

Q	←	□	→
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

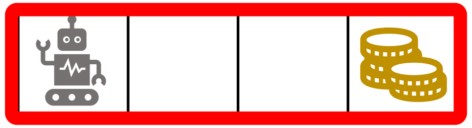
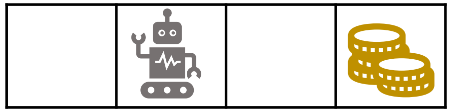
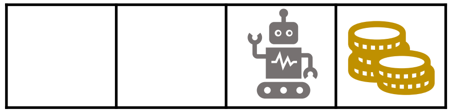
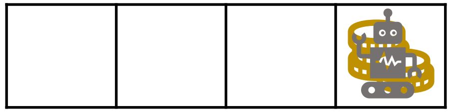
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

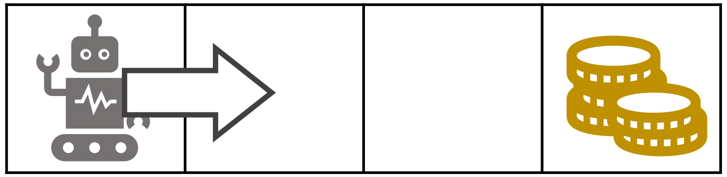
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

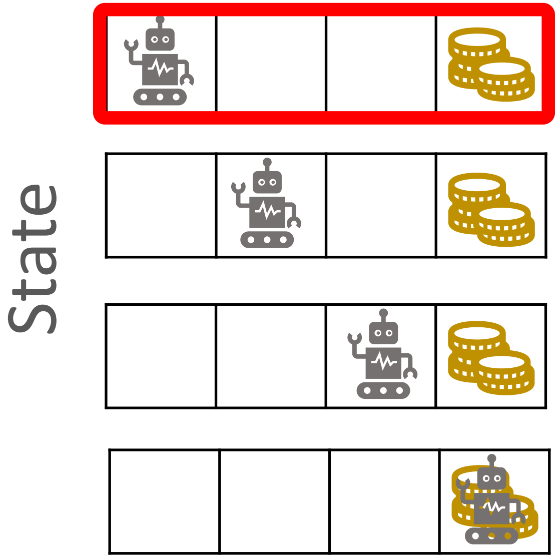
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

$$= 0$$

Iteration 0:

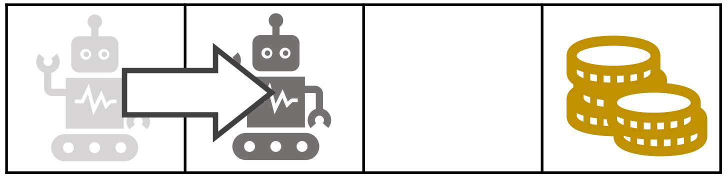
Action

Q



	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment

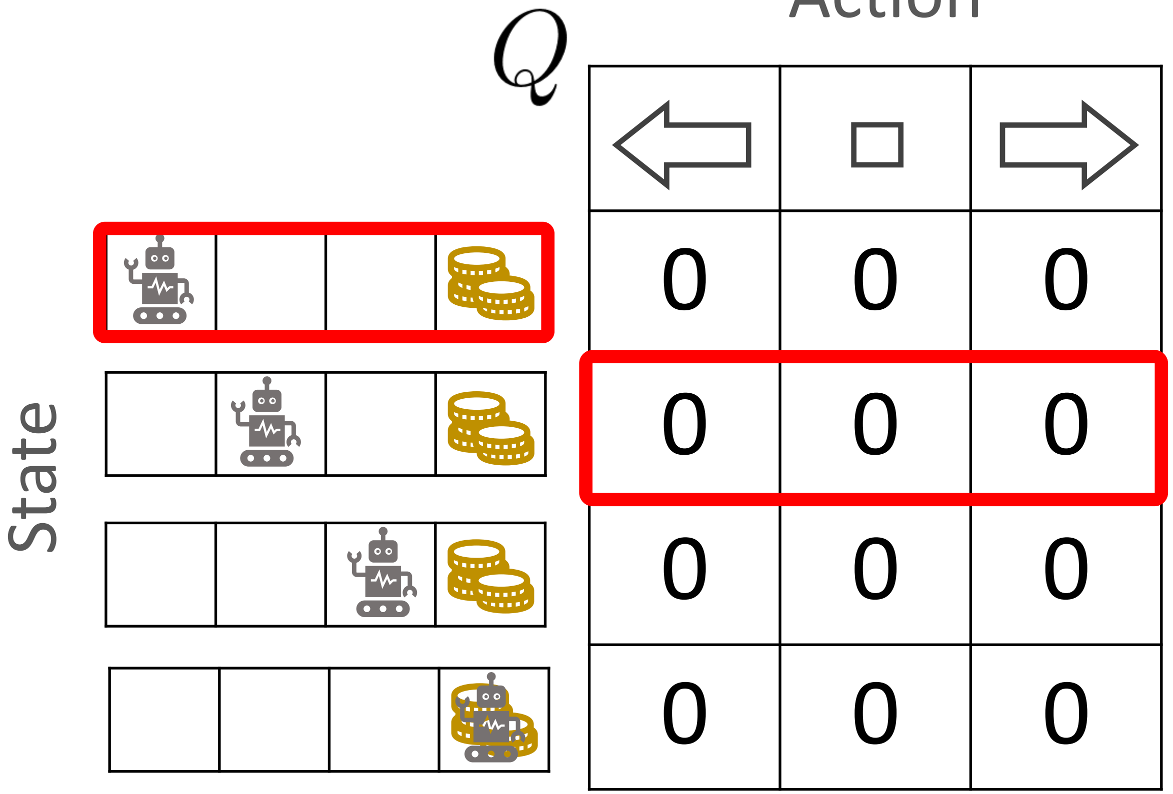


$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

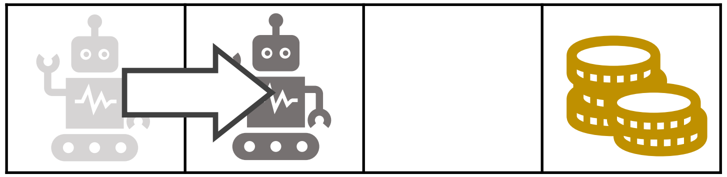
Iteration 0:



Action

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

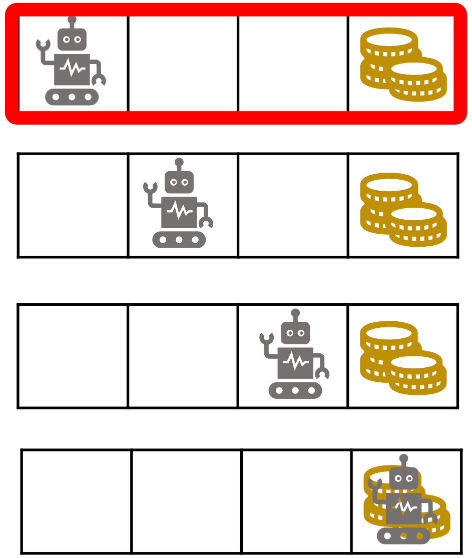
Iteration 0:

Action

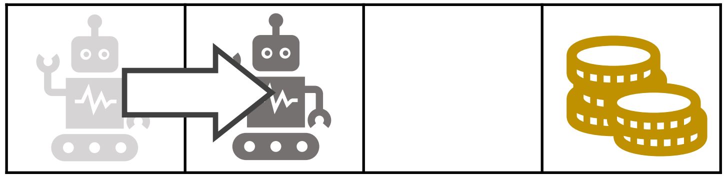
Q

	←	□	→
	0	0	0
	0	0	0
	0	0	0
	0	0	0

State



Environment

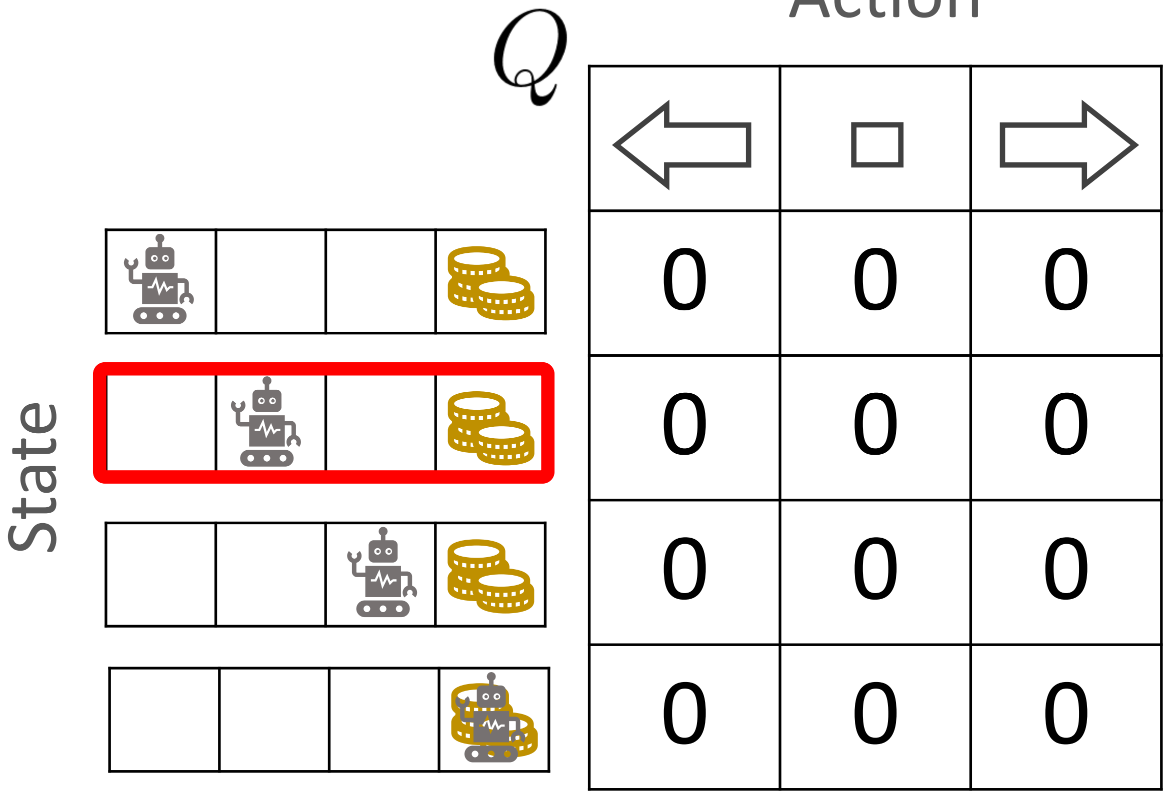


$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:

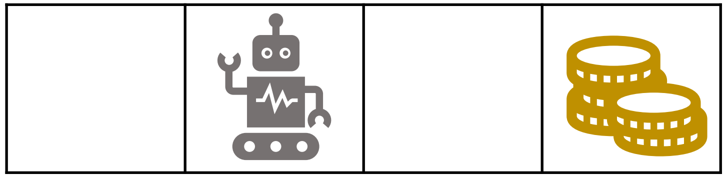


Action

Q

←	□	→
0	0	0
0	0	0
0	0	0
0	0	0

Environment

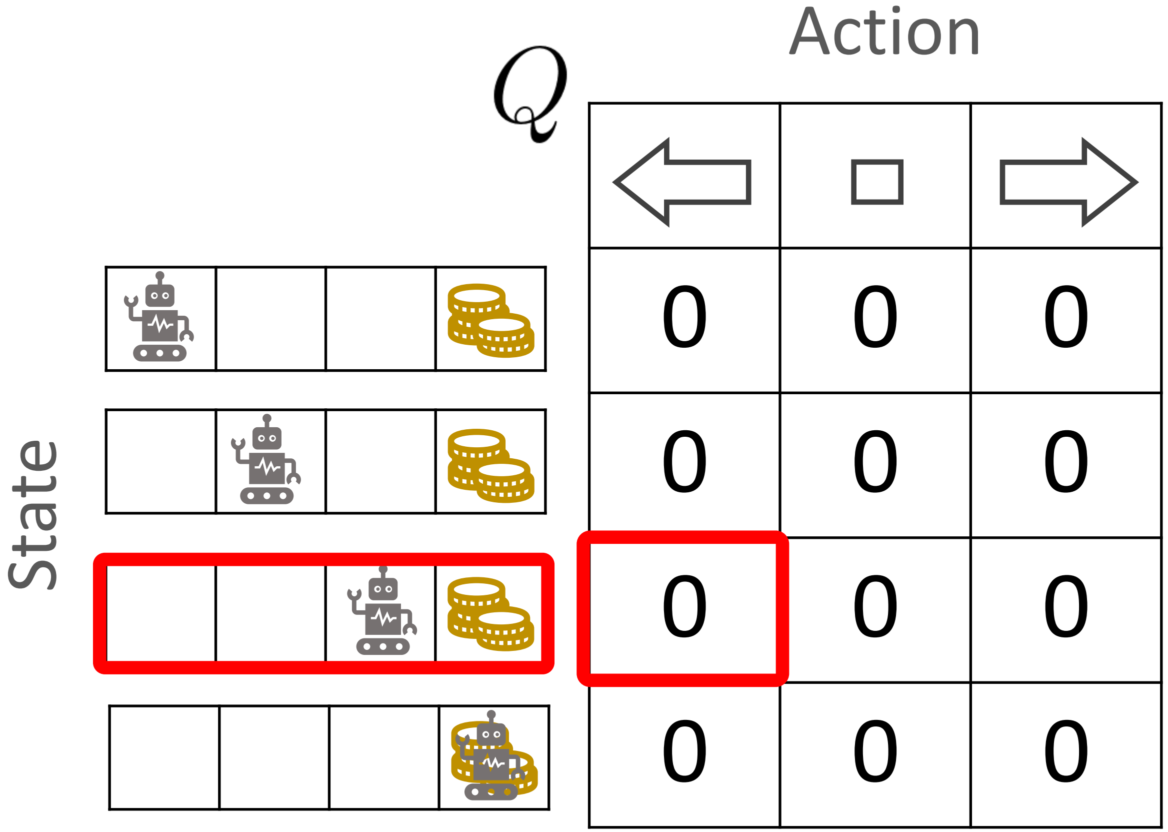


$$\gamma = 1/2$$

Tabular Q-Learning

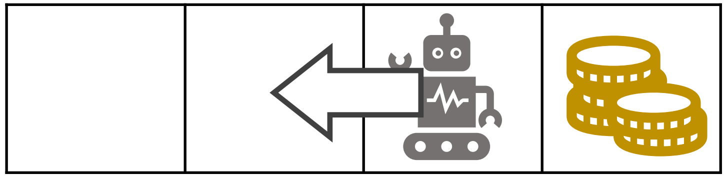
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:



←	□	→
0	0	0
0	0	0
0	0	0
0	0	0

Environment

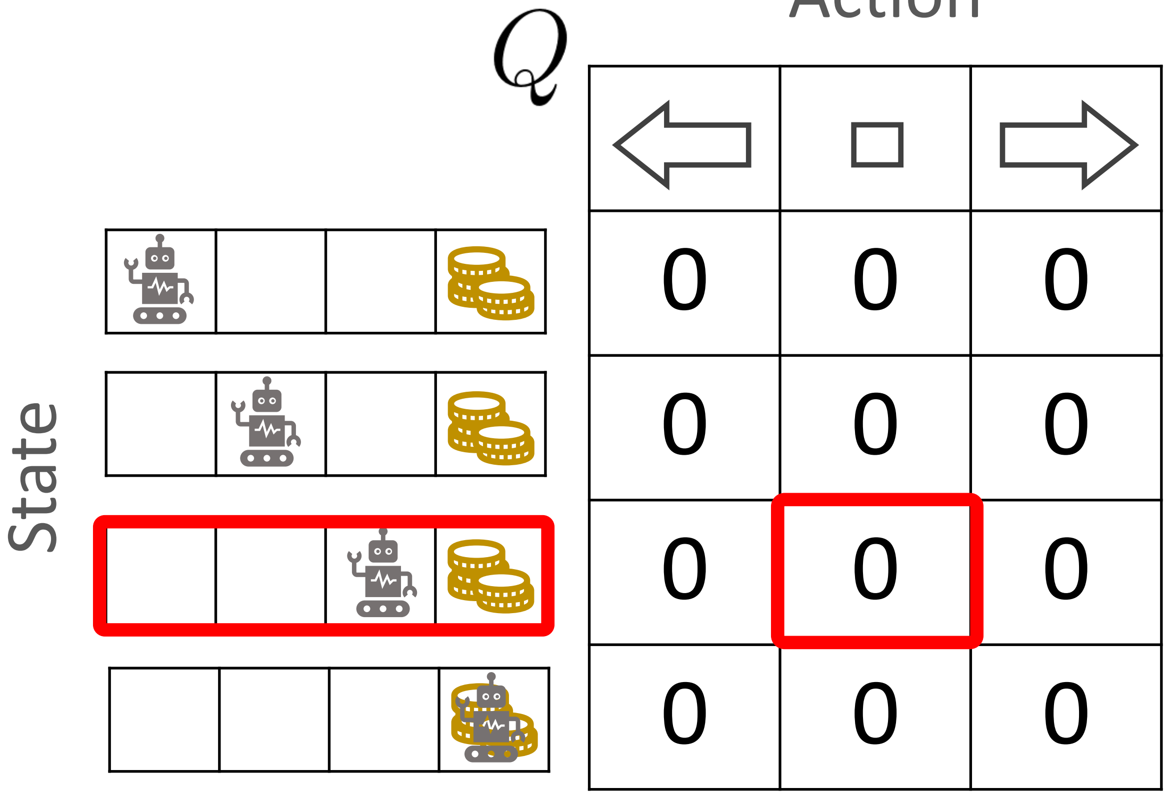


$$\gamma = 1/2$$

Tabular Q-Learning

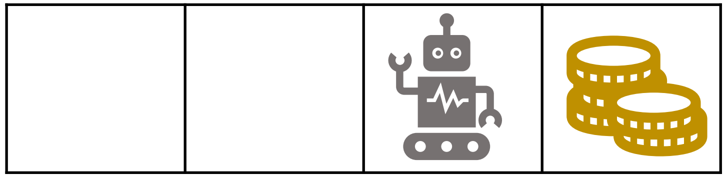
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:



←	□	→
0	0	0
0	0	0
0	0	0
0	0	0

Environment

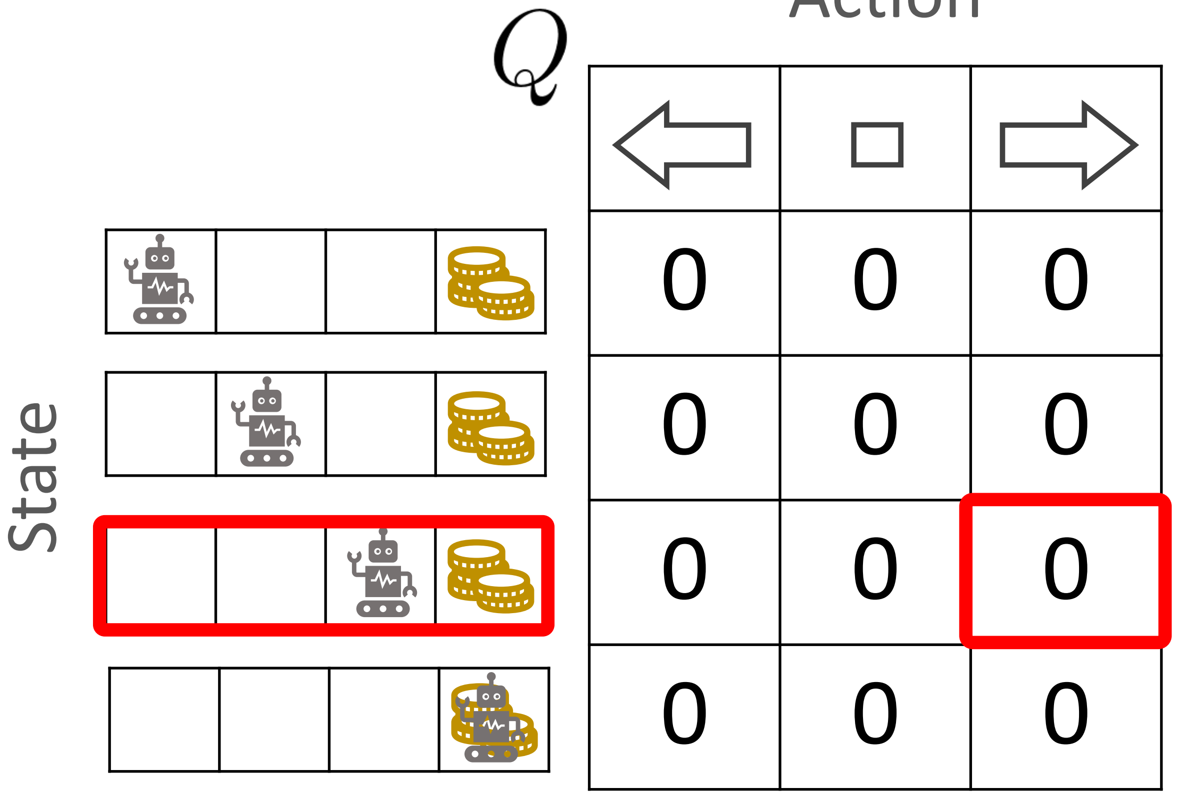


$$\gamma = 1/2$$

Tabular Q-Learning

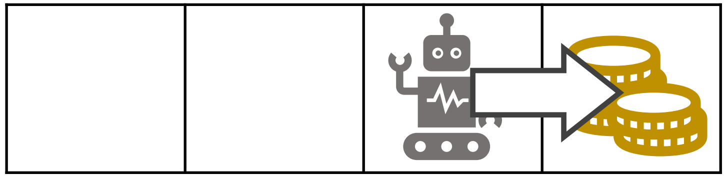
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:



⇦	□	⇨
0	0	0
0	0	0
0	0	0
0	0	0

Environment



$$\gamma = 1/2$$

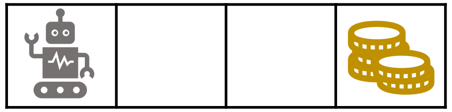
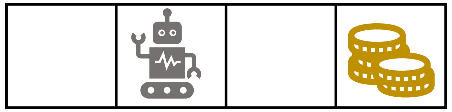
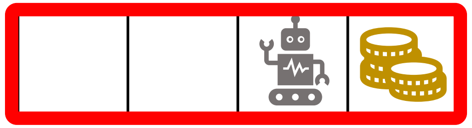
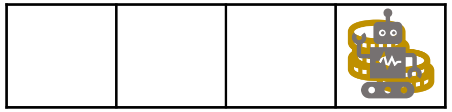
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

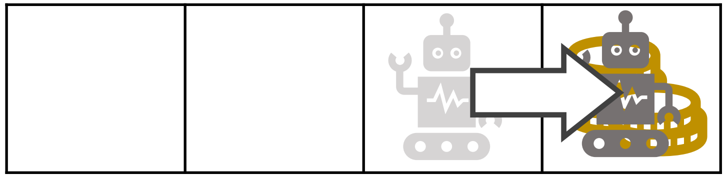
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	0
	0	0	0

Environment

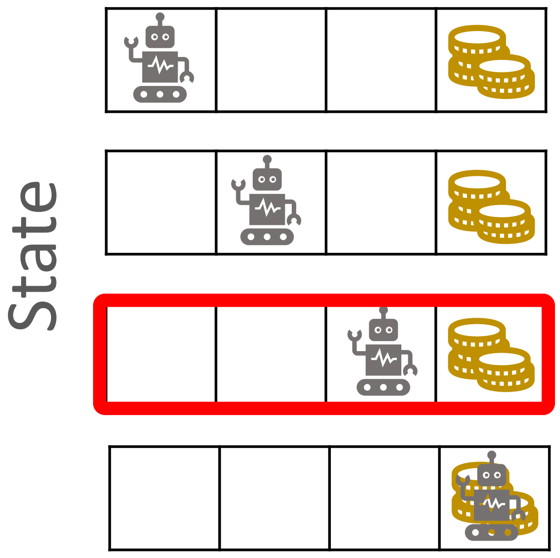


$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

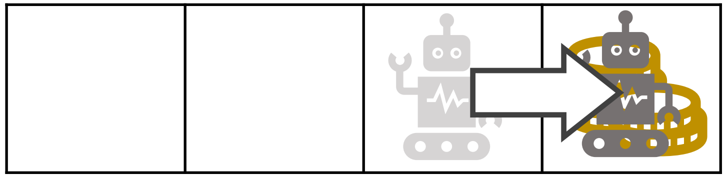
Iteration 0:



Q Action

⇦	□	⇨
0	0	0
0	0	0
0	0	$\boxed{0}$
0	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

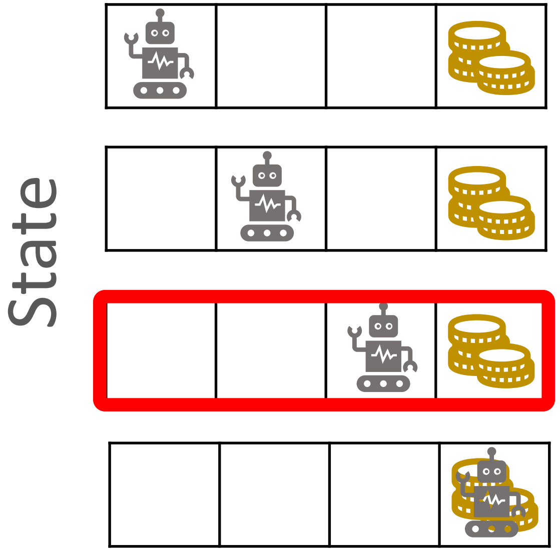
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

Iteration 0:

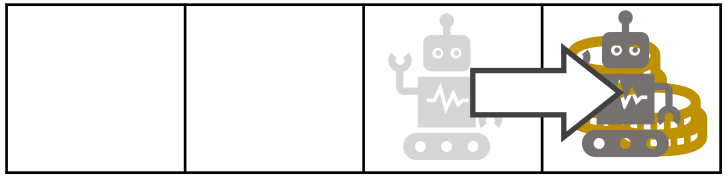
Action

Q

←	□	→
0	0	0
0	0	0
0	0	1
0	0	0



Environment



$$\gamma = 1/2$$

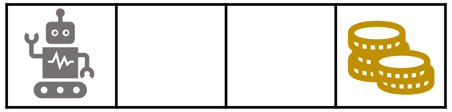
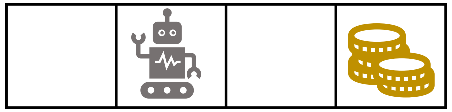
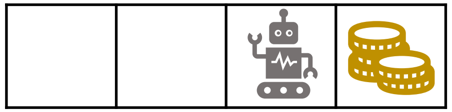
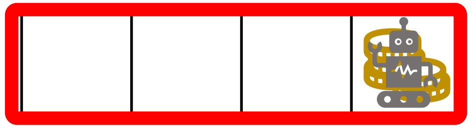
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma\, \underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

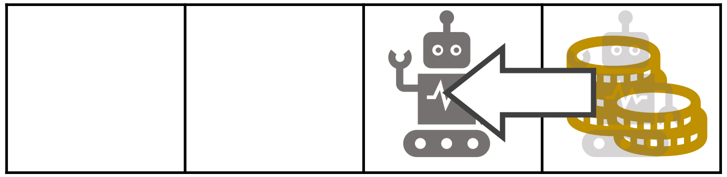
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	1
	0	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

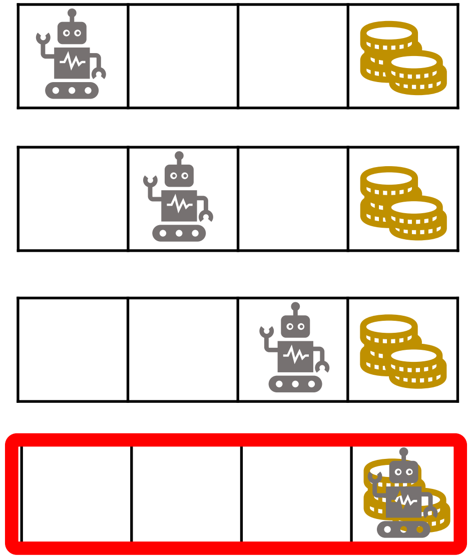
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

Iteration 0:

Action

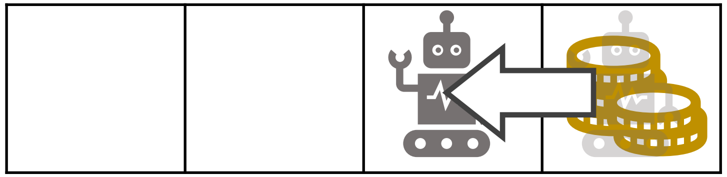
Q

State



←	□	→
0	0	0
0	0	0
0	0	1
0	0	0

Environment



$$\gamma = 1/2$$

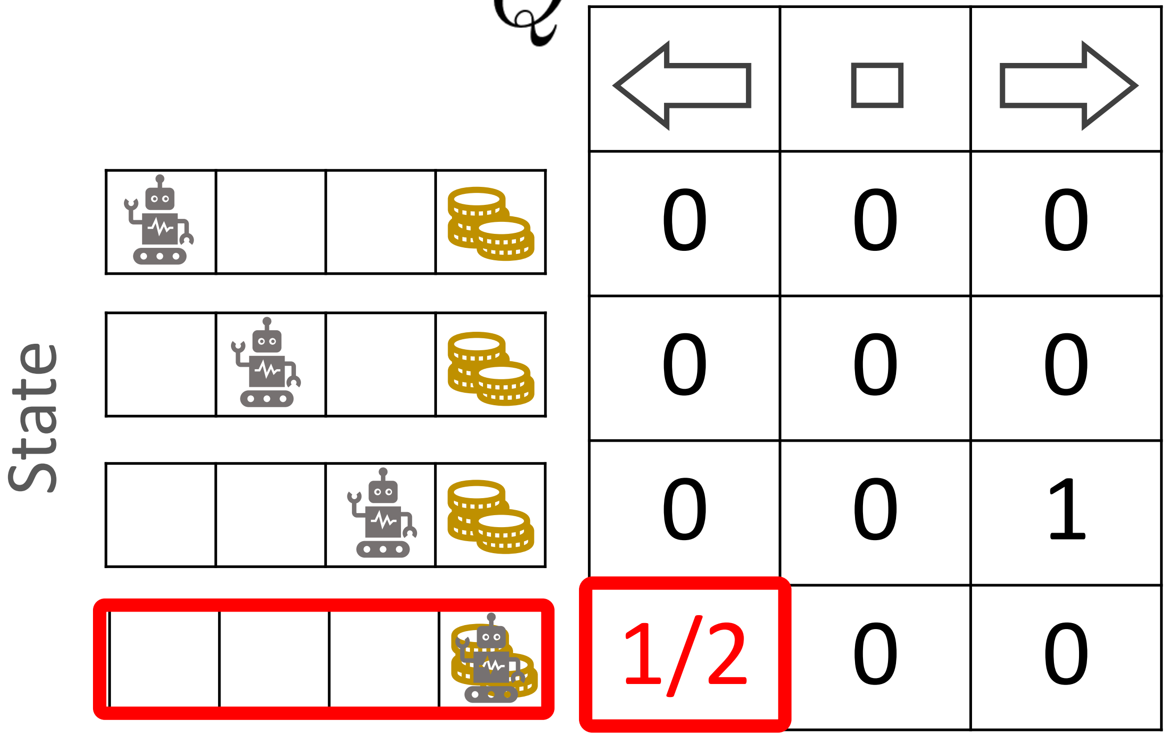
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

Iteration 0:

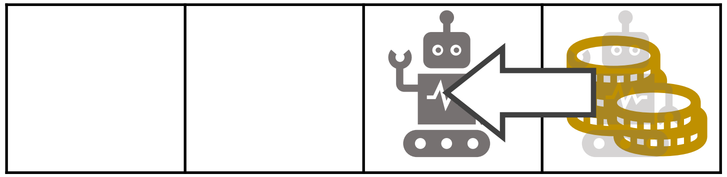
Action

Q



	←	□	→
	0	0	0
	0	0	0
	0	0	1
	1/2	0	0

Environment

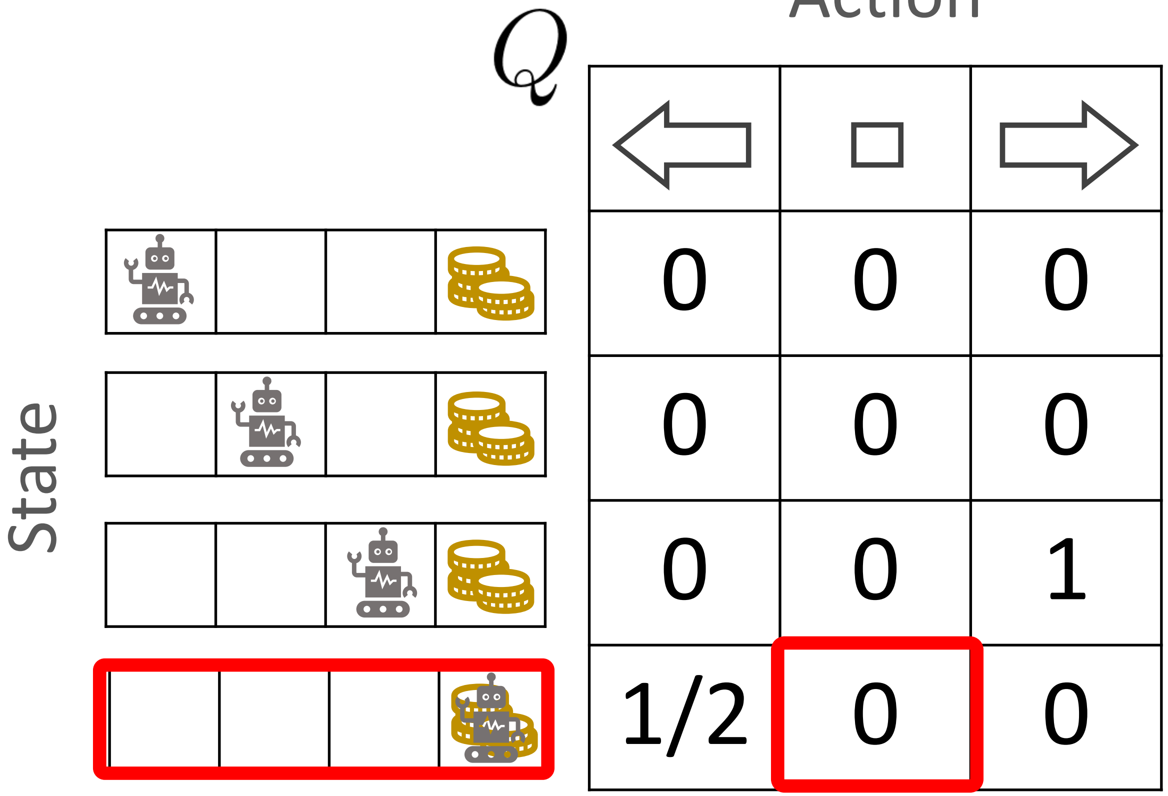


$$\gamma = 1/2$$

Tabular Q-Learning

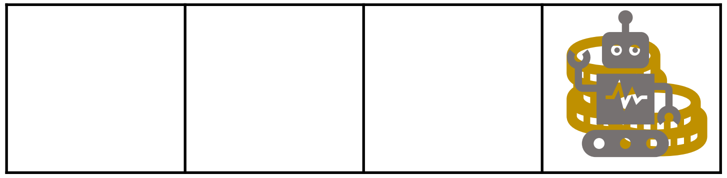
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:



←	□	→
0	0	0
0	0	0
0	0	1
1/2	0	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

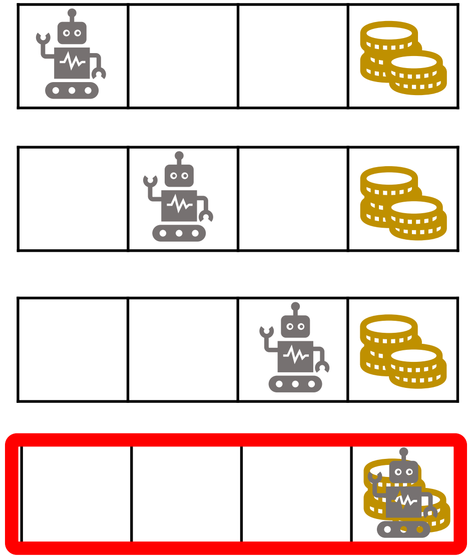
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underset{=1}{\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')}} + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:

Q

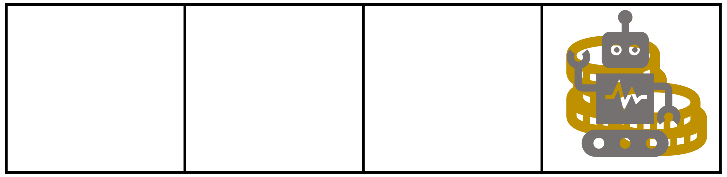
Action

State



⇦	□	⇨
0	0	0
0	0	0
0	0	1
1/2	$\boxed{0}$	0

Environment



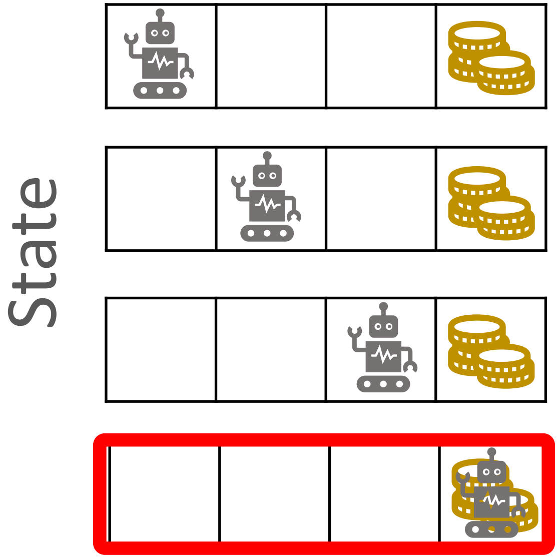
$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \underbrace{\gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

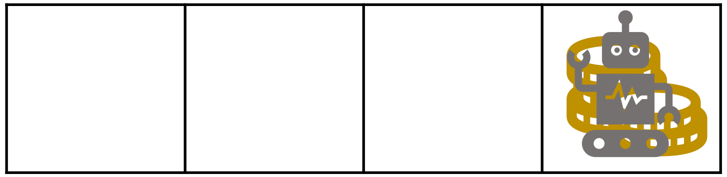
Iteration 0:

Q Action



←	□	→
0	0	0
0	0	0
0	0	1
1/2	0	0

Environment

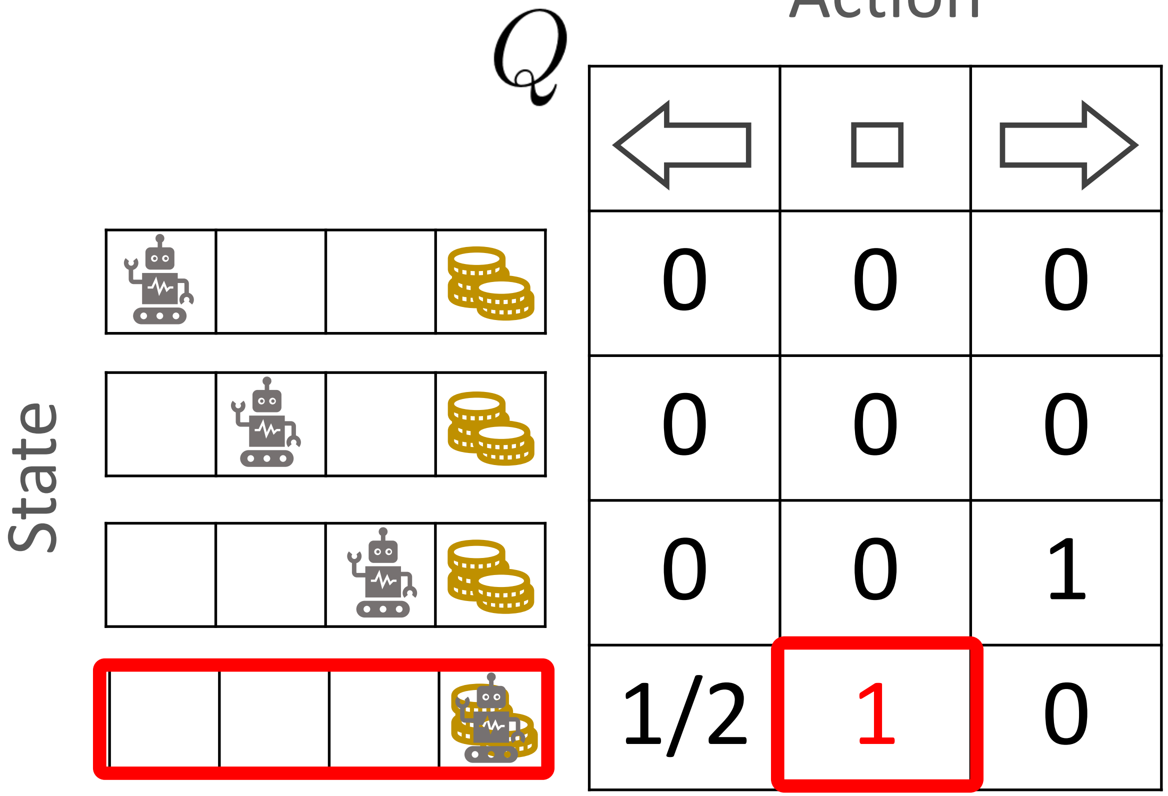


$$\gamma = 1/2$$

Tabular Q-Learning

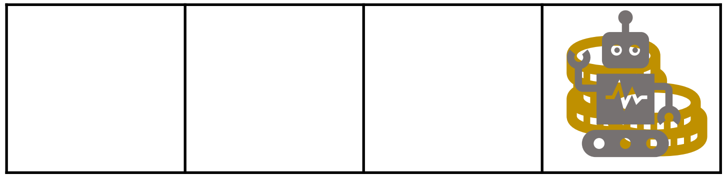
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

Iteration 0:



⇦	□	⇨
0	0	0
0	0	0
0	0	1
1/2	1	0

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

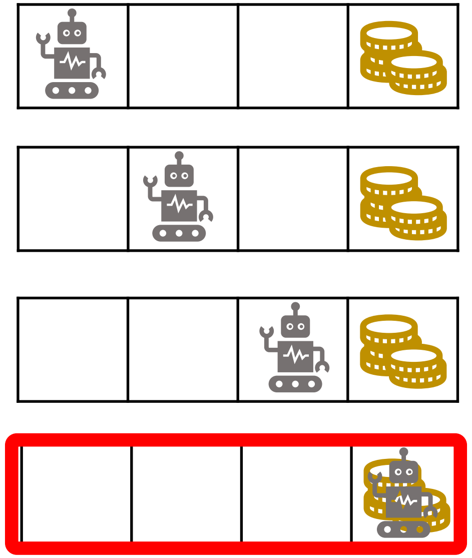
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

Iteration 0:

Action

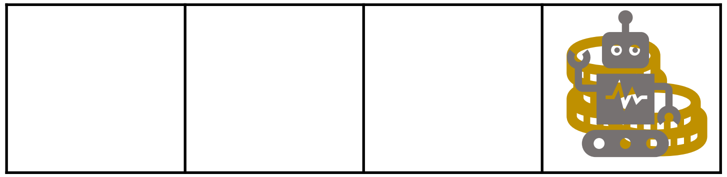
Q

State



⇦	□	⇨
0	0	0
0	0	0
0	0	1
1/2	1	0

Environment



$$\gamma = 1/2$$

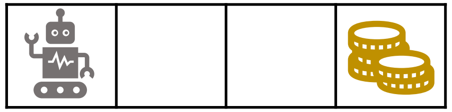
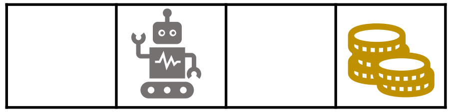
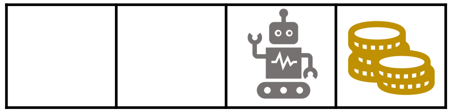
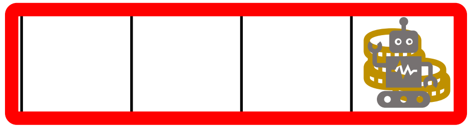
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=1} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=0}\right]$$

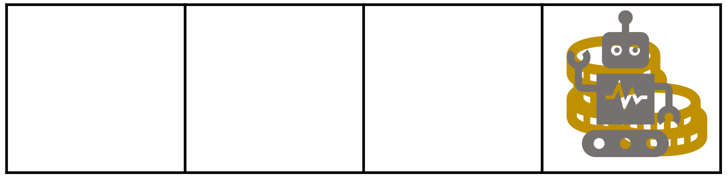
Iteration 0:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	1
	1/2	1	1

Environment

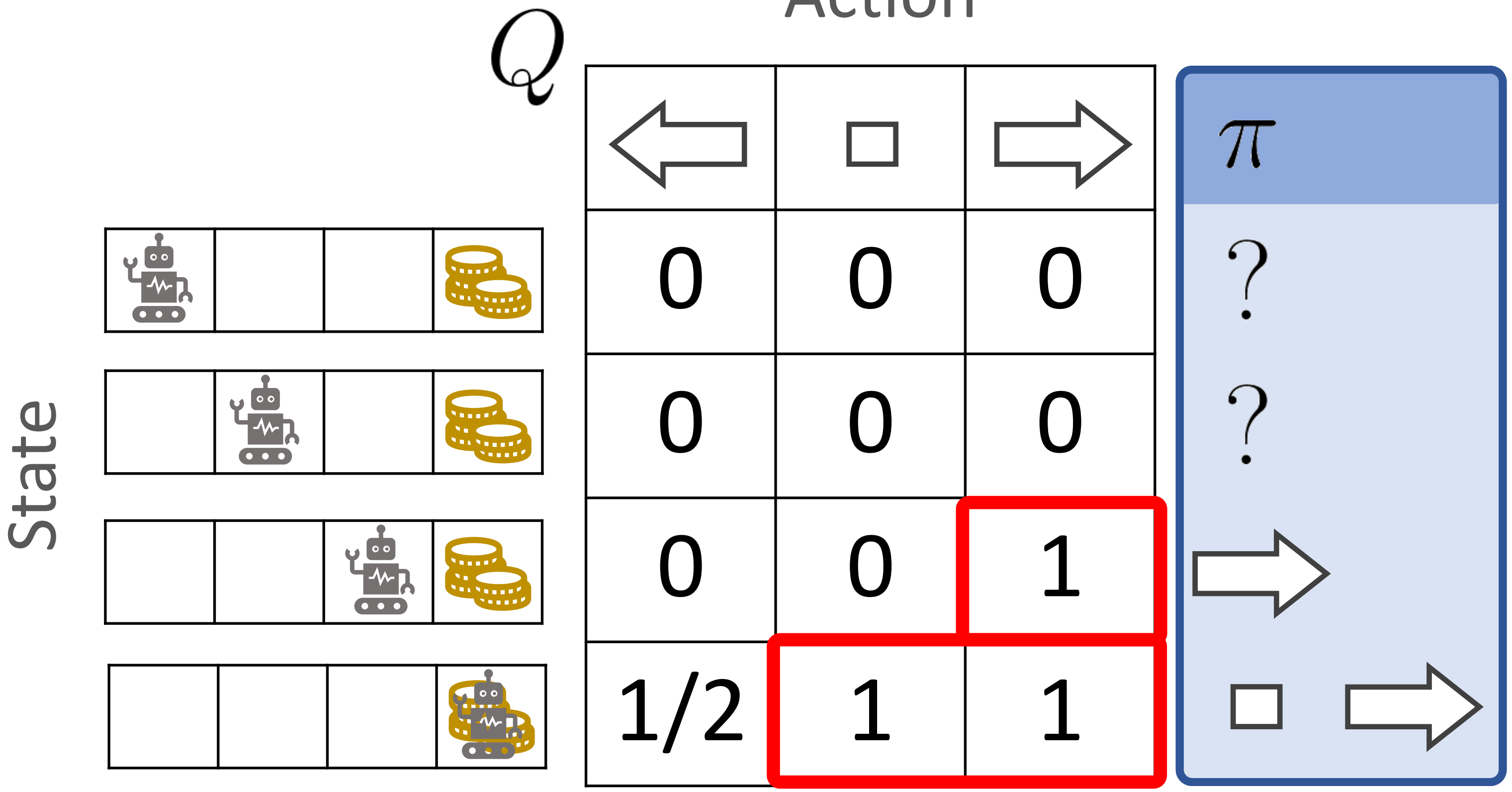


$$\gamma = 1/2$$

Tabular Q-Learning

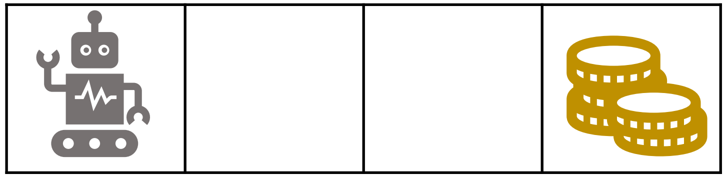
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 0:



⇦	□	⇨	π
0	0	0	?
0	0	0	?
0	0	1	⇨
1/2	1	1	□ ⇨

Environment

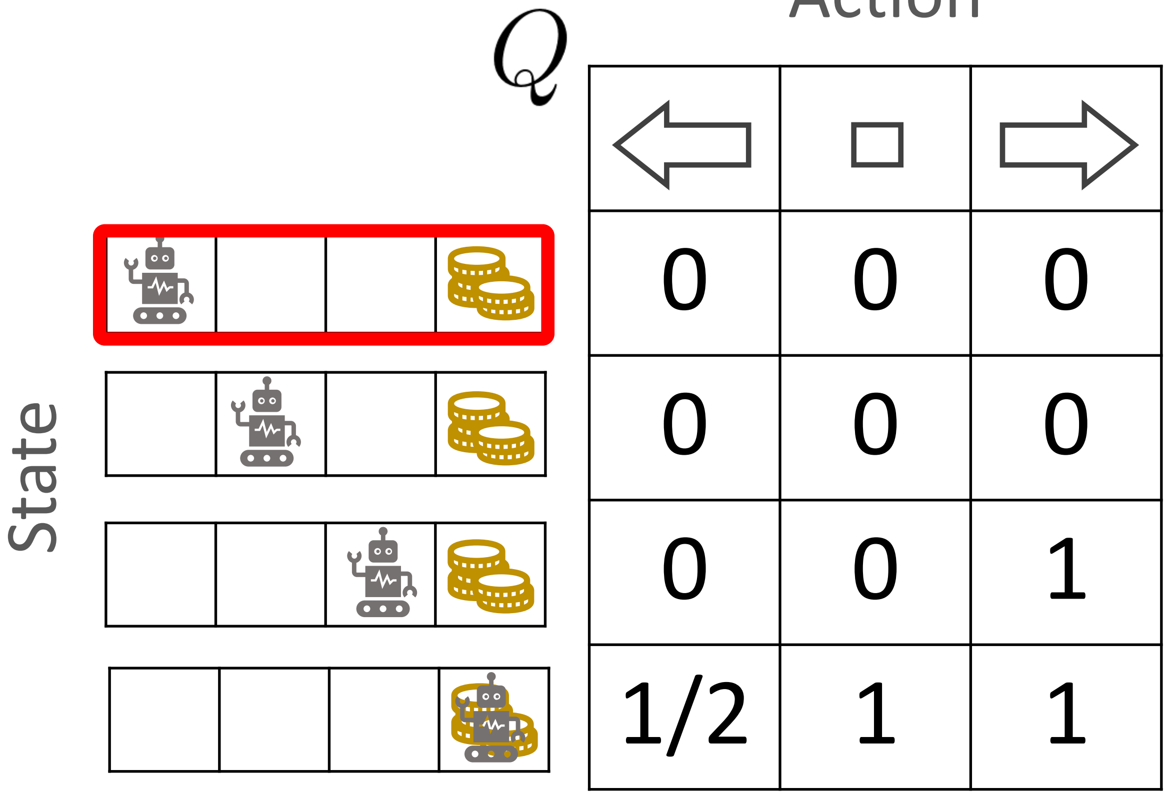


$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 1:



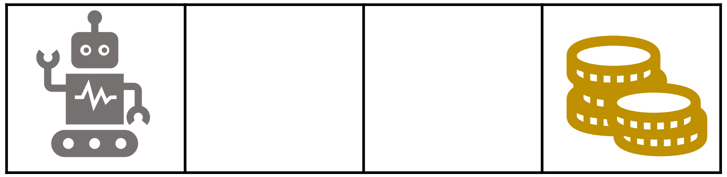
Action

Q

State

←	□	→
0	0	0
0	0	0
0	0	1
1/2	1	1

Environment

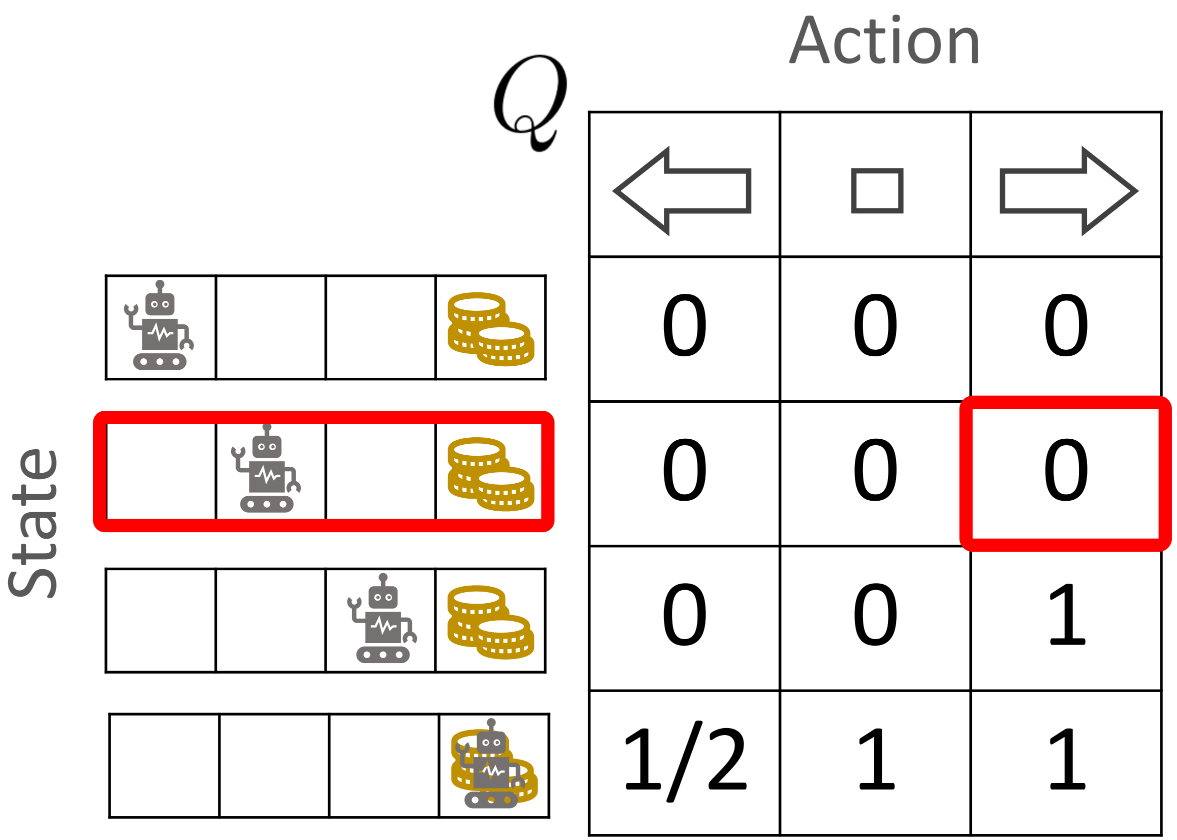


$$\gamma = 1/2$$

Tabular Q-Learning

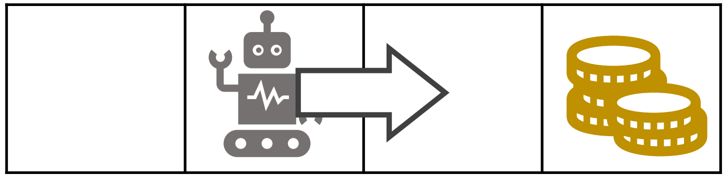
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration 1:



←	□	→
0	0	0
0	0	0
0	0	1
1/2	1	1

Environment



$$\gamma = 1/2$$

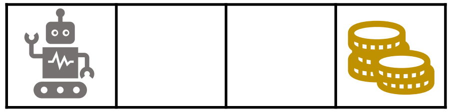
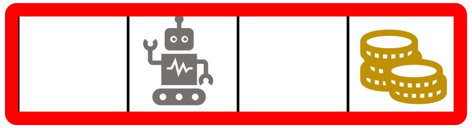
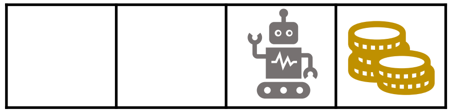
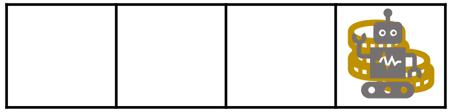
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

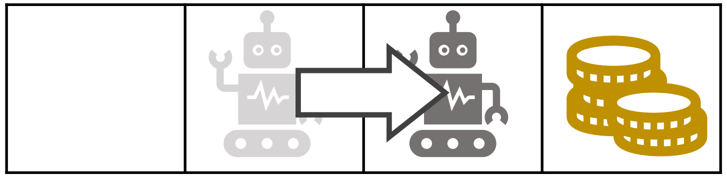
Iteration 1:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	$\boxed{0}$
	0	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

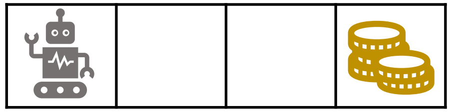
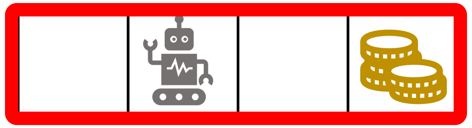
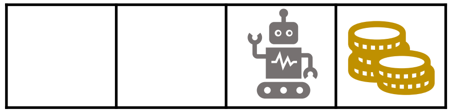
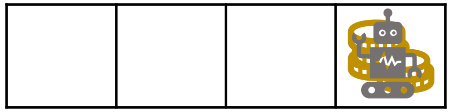
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

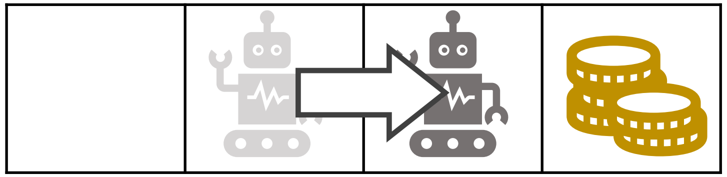
Iteration 1:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	0
	0	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

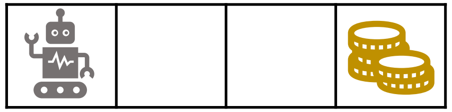
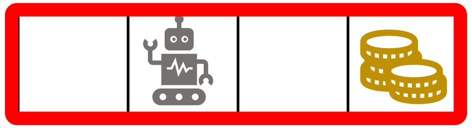
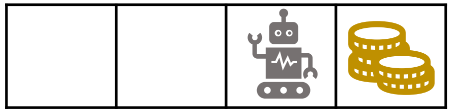
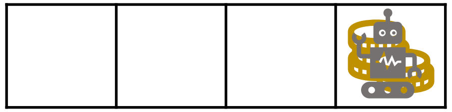
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

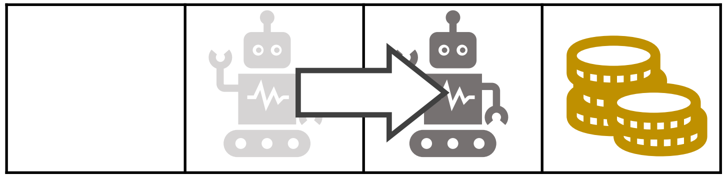
Iteration 1:

Action

State

Q	←	□	→
	0	0	0
	0	0	1/2
	0	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

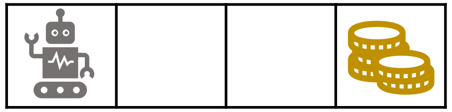
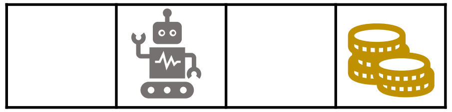
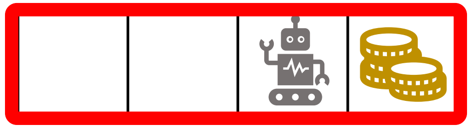
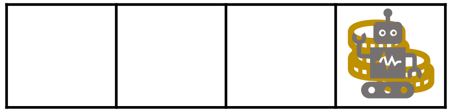
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

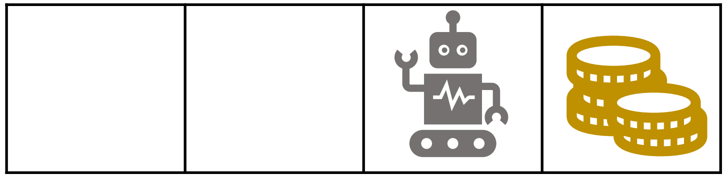
Iteration 1:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	1/2
	0	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

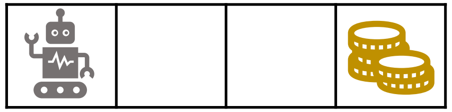
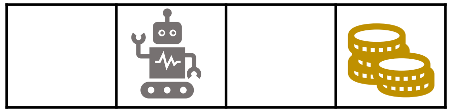
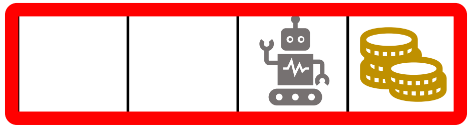
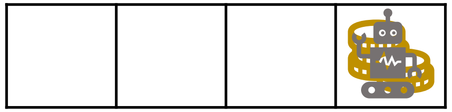
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underline{r(\mathbf{s},\mathbf{a},\mathbf{s}')} + \gamma \underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

$$= 0$$

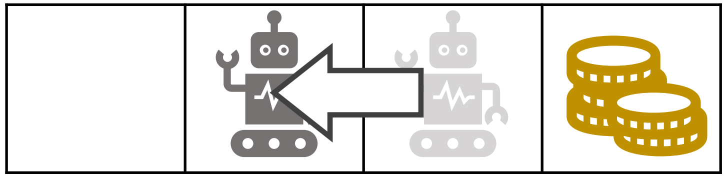
Iteration 1:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	1/2
	0	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

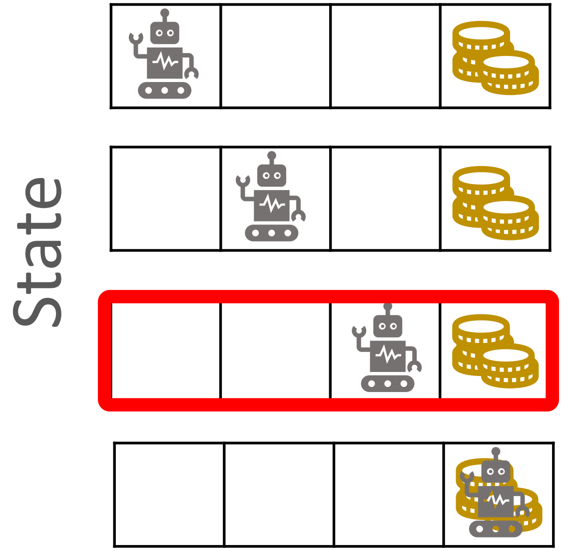
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1/2}\right]$$

Iteration 1:

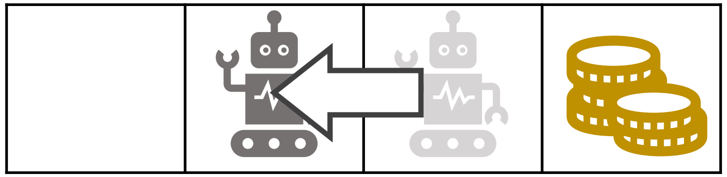
Action

Q



←	□	→
0	0	0
0	0	1/2
0	0	1
1/2	1	1

Environment

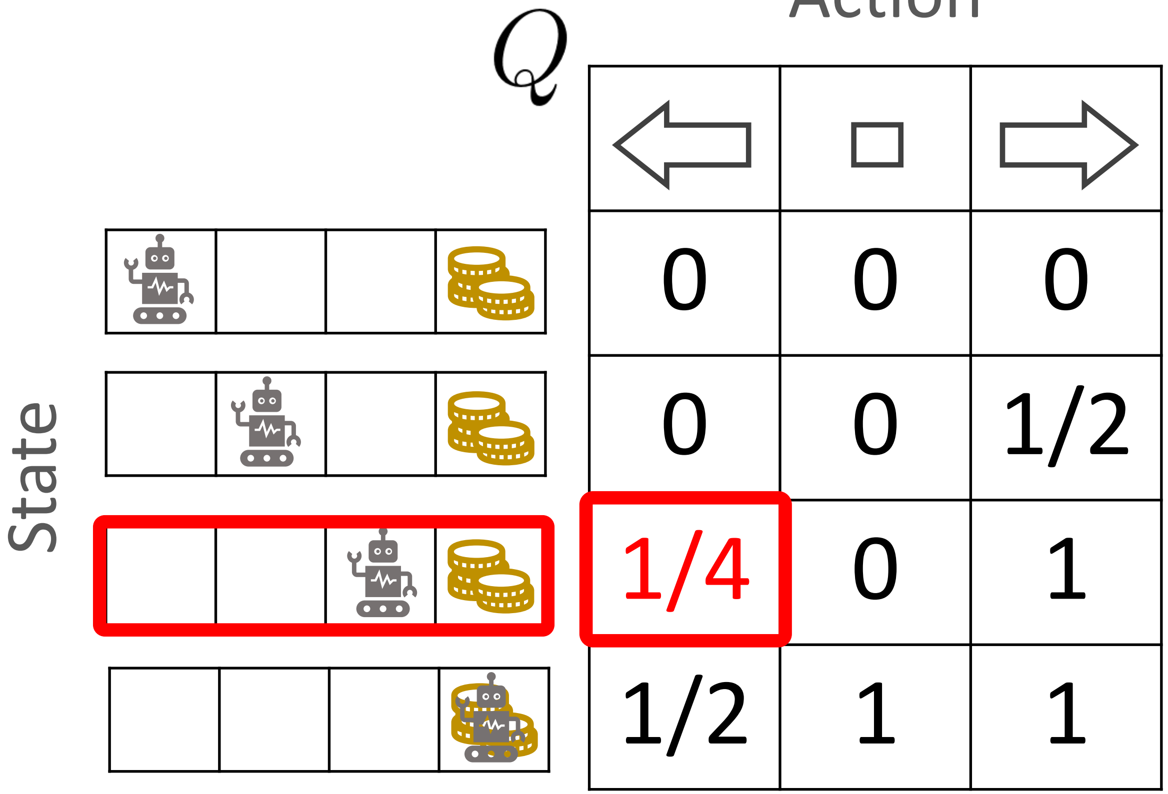


$$\gamma = 1/2$$

Tabular Q-Learning

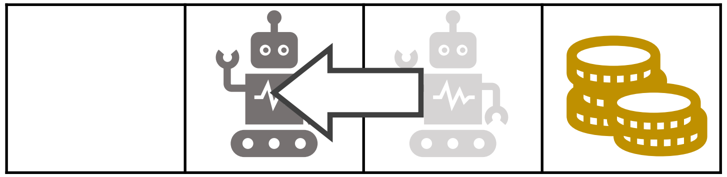
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \underbrace{\gamma \max_{\mathbf{a}'}\left(Q^{k}(\mathbf{s}',\mathbf{a}')\right)}_{=1/2}\right]$$

Iteration 1:



⇦	□	⇨
0	0	0
0	0	1/2
1/4	0	1
1/2	1	1

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

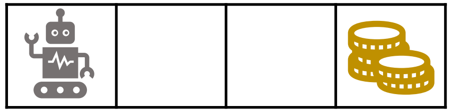
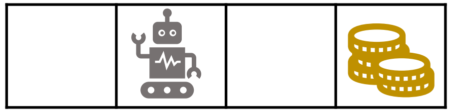
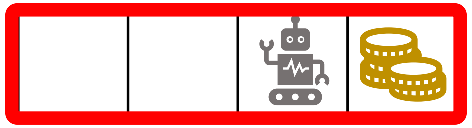
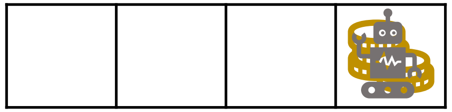
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underline{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}\right]$$

Iteration 1:

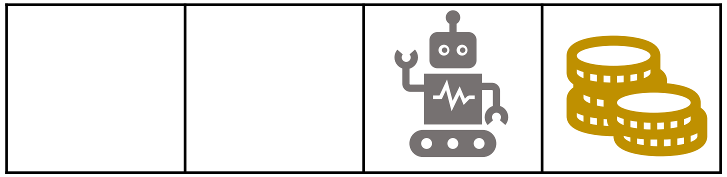
Action

Q

State

	←	□	→
	0	0	0
	0	0	1/2
	1/4	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

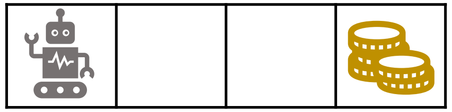
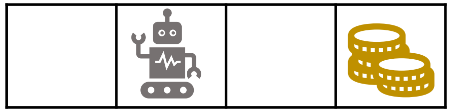
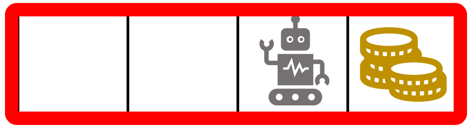
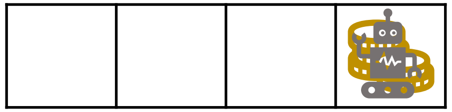
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

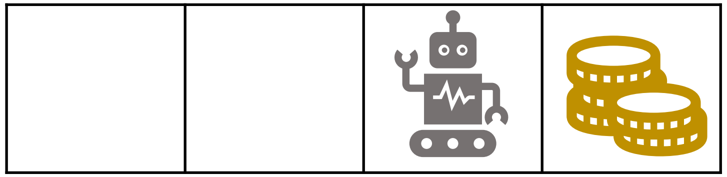
Iteration 1:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	1/2
	1/4	0	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

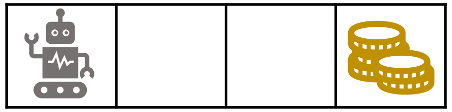
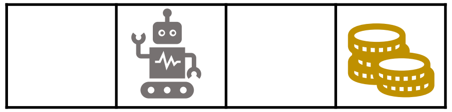
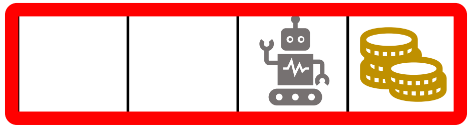
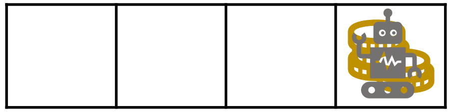
Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \underbrace{\gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

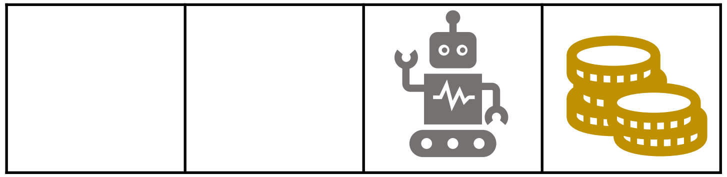
Iteration 1:

Action

State

Q	⇦	□	⇨
	0	0	0
	0	0	1/2
	1/4	1/2	1
	1/2	1	1

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

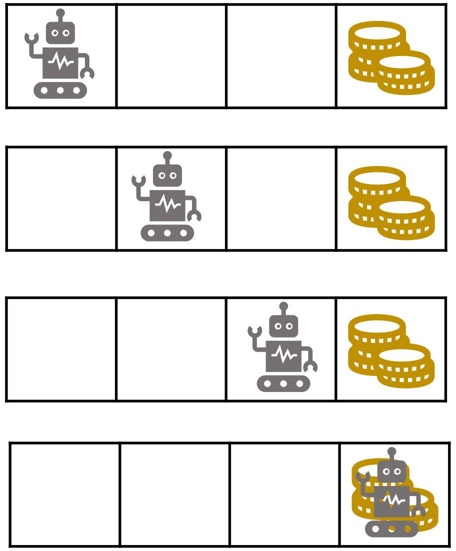
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[\underbrace{r(\mathbf{s},\mathbf{a},\mathbf{s}')}_{=0} + \gamma \underbrace{\max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)}_{=1}\right]$$

Iteration 1:

Action

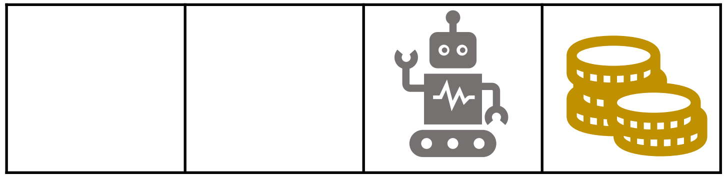
Q

State



	←	□	→	π
	0	0	0	?
	0	0	1/2	→
	1/4	1/2	1	→
	1/2	1	1	□ →

Environment

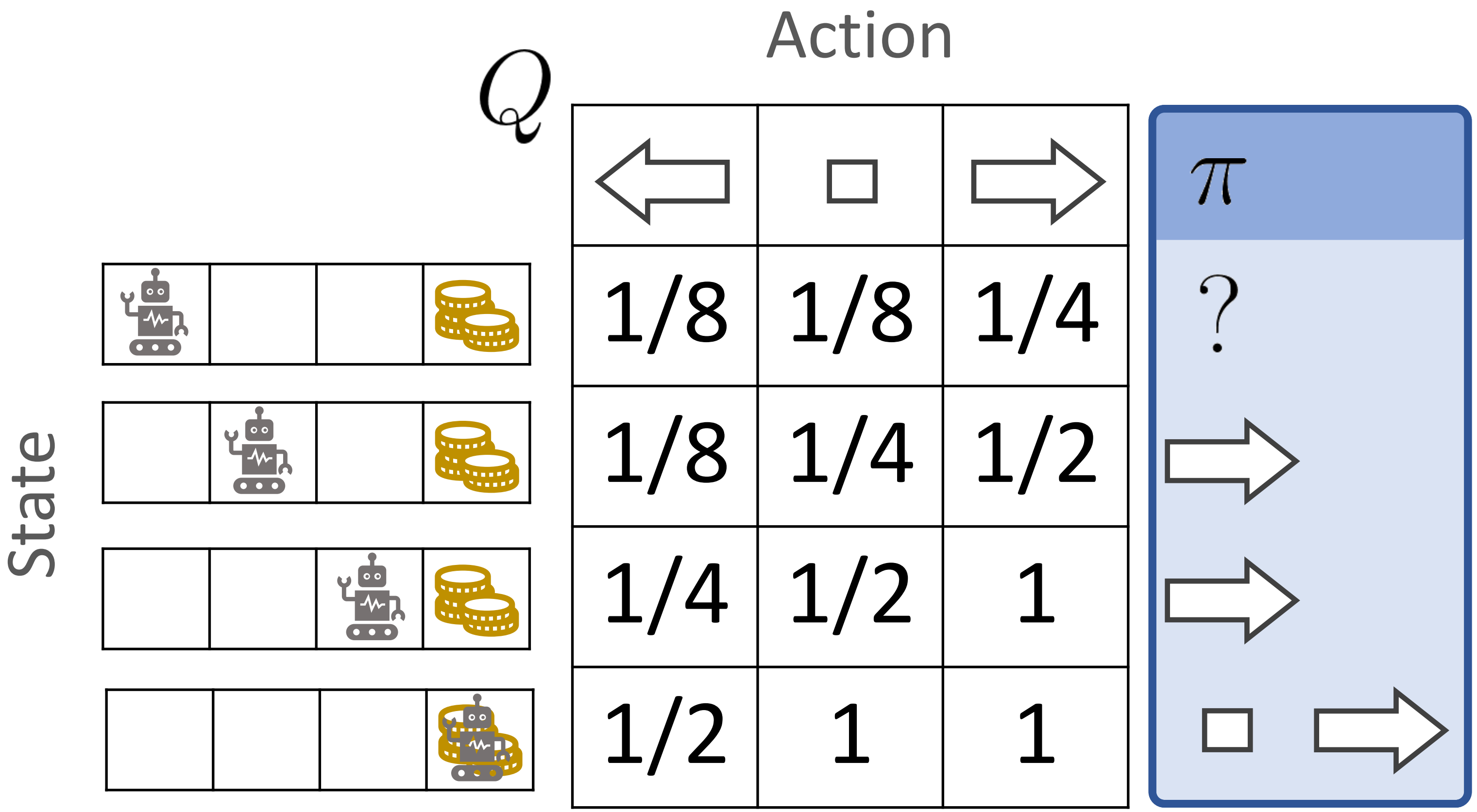


$$\gamma = 1/2$$

Tabular Q-Learning

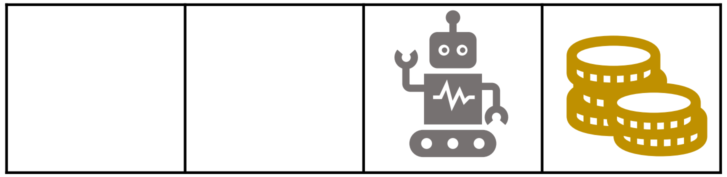
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration k:



Q	⇦	□	⇨	π
	1/8	1/8	1/4	?
	1/8	1/4	1/2	⇨
	1/4	1/2	1	⇨
	1/2	1	1	□ ⇨

Environment

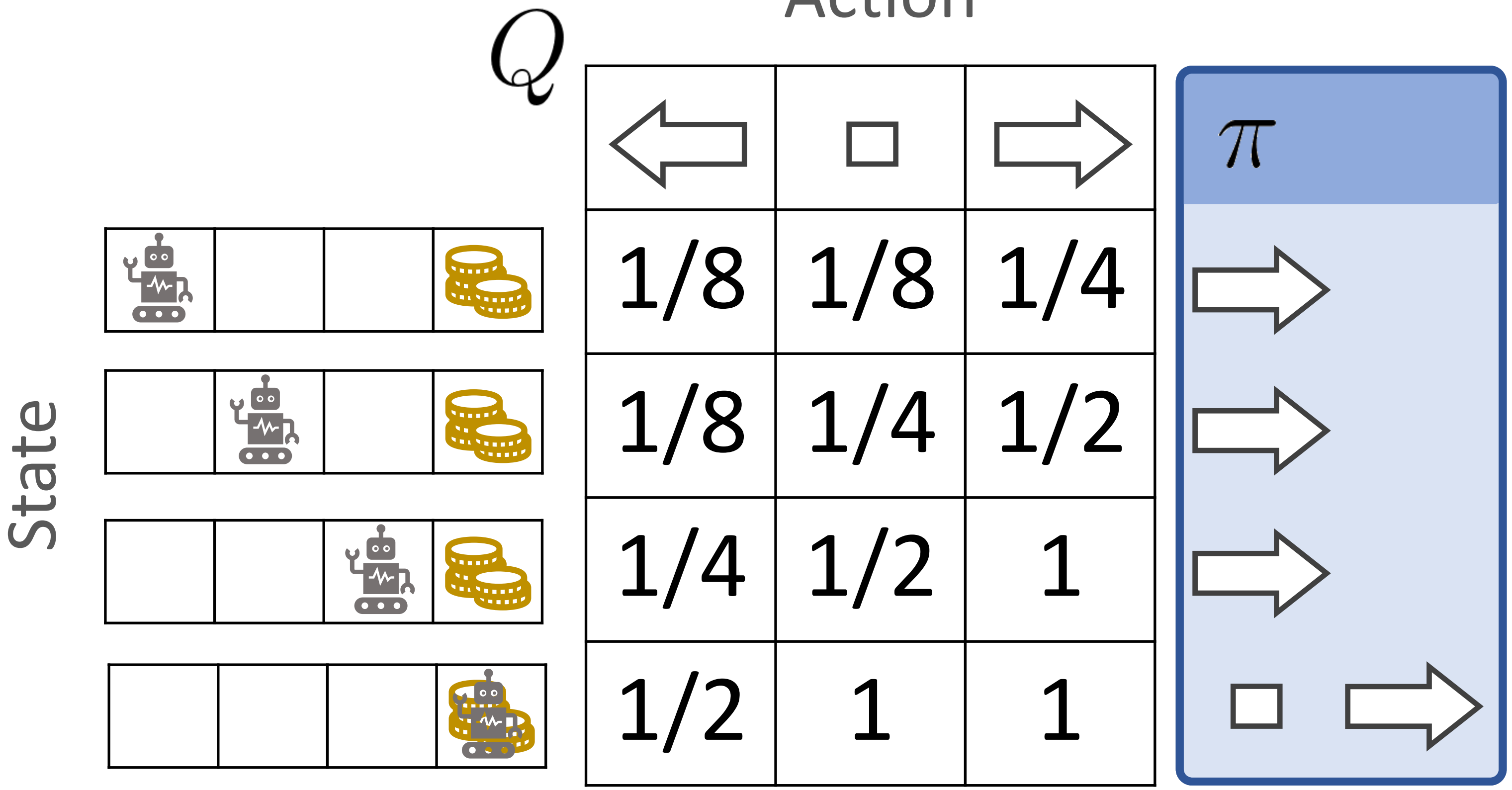


$$\gamma = 1/2$$

Tabular Q-Learning

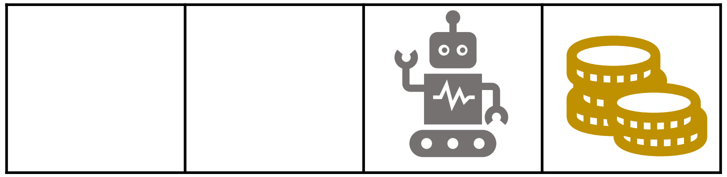
$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

Iteration k:



	⇦	□	⇨	π
	1/8	1/8	1/4	⇨
	1/8	1/4	1/2	⇨
	1/4	1/2	1	⇨
	1/2	1	1	□ ⇨

Environment

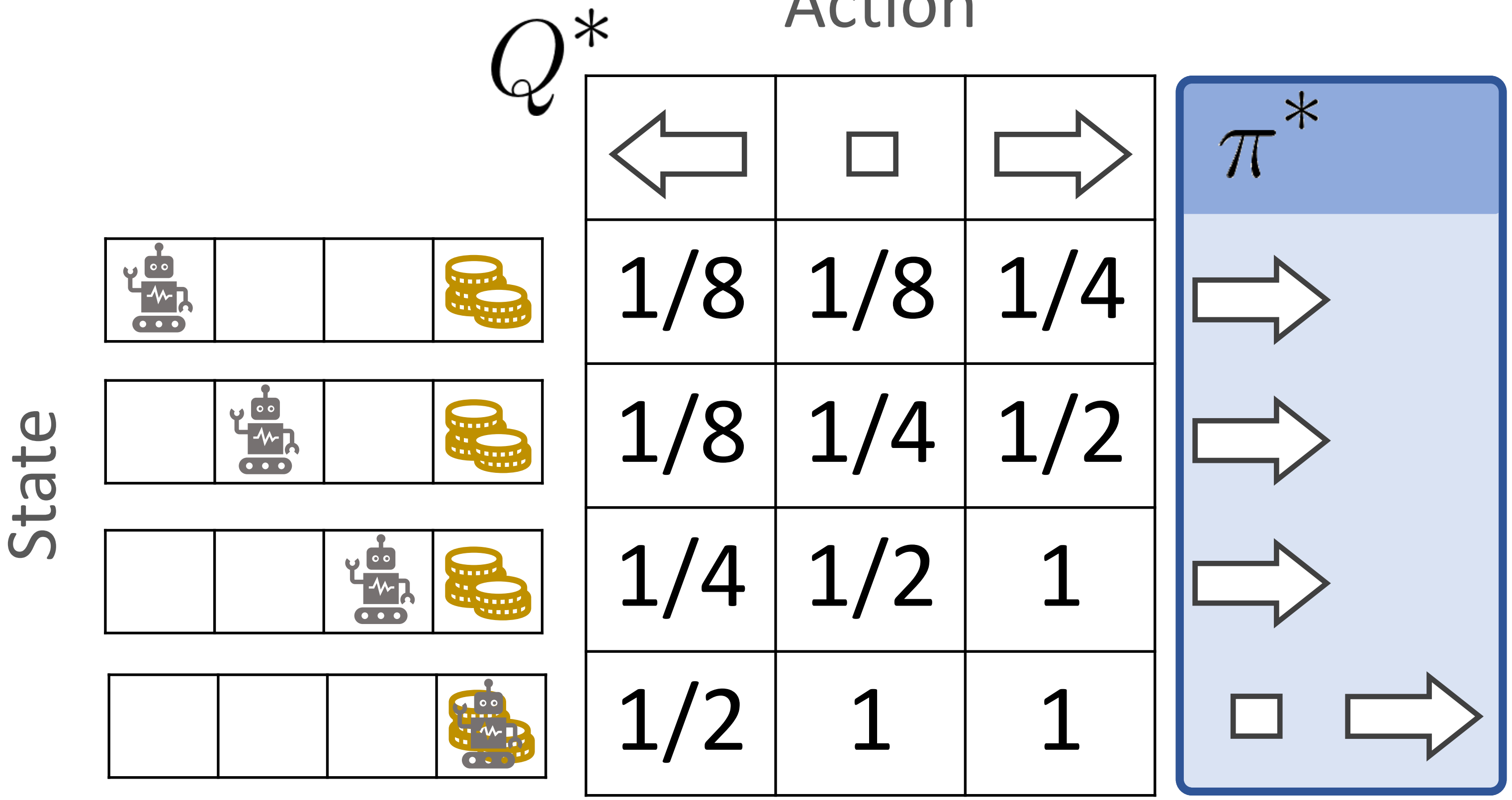


$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

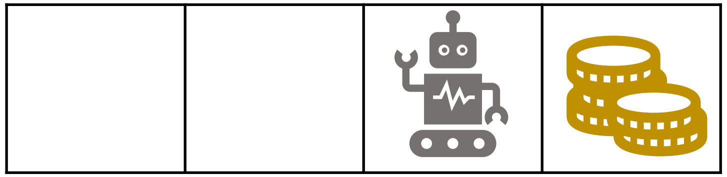
Iteration k:



Q^*

	⇦	□	⇨	π^*
	1/8	1/8	1/4	⇨
	1/8	1/4	1/2	⇨
	1/4	1/2	1	⇨
	1/2	1	1	□ ⇨

Environment



$$\gamma = 1/2$$

Tabular Q-Learning

$$Q^{k+1}(\mathbf{s}, \mathbf{a}) = \mathbb{E}_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma \max_{\mathbf{a}'} \left(Q^k(\mathbf{s}', \mathbf{a}') \right) \right]$$

In tabular setting:

- Every iteration leads to a better Q-function + policy
- Converges to optimal Q-function + policy

Limitations:

- Can only be applied to discrete states and actions
- Need to enumerate over all states and actions every iteration

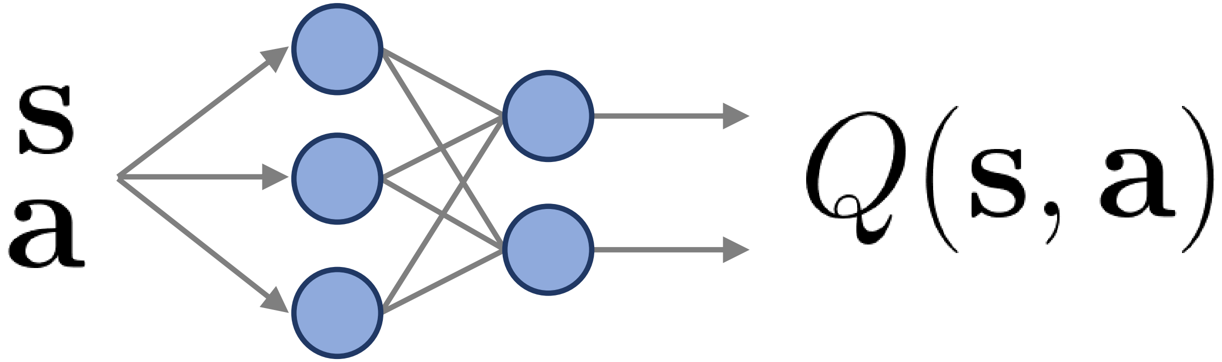
Large State Spaces

Observation:

- 64 x 64 image
- 8 bits per pixel
- $2^{8\times64\times64}$ different states!

Large/Continuous State Spaces

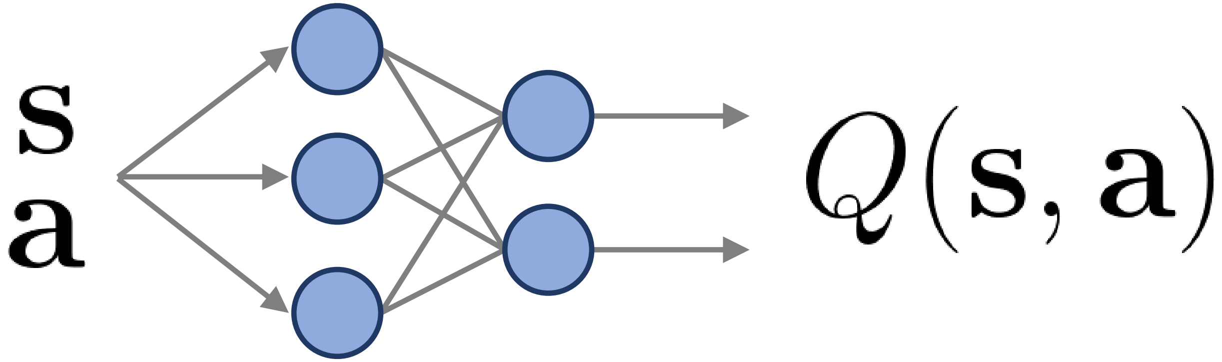
$$\underline{Q^{k+1}(\mathbf{s},\mathbf{a})} = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^{k}(\mathbf{s}',\mathbf{a}')\right)\right]$$



Large/Continuous State Spaces

$$\underline{Q^{k+1}(\mathbf{s},\mathbf{a})} = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\Big(Q^k(\mathbf{s}',\mathbf{a}')\Big)\right]$$

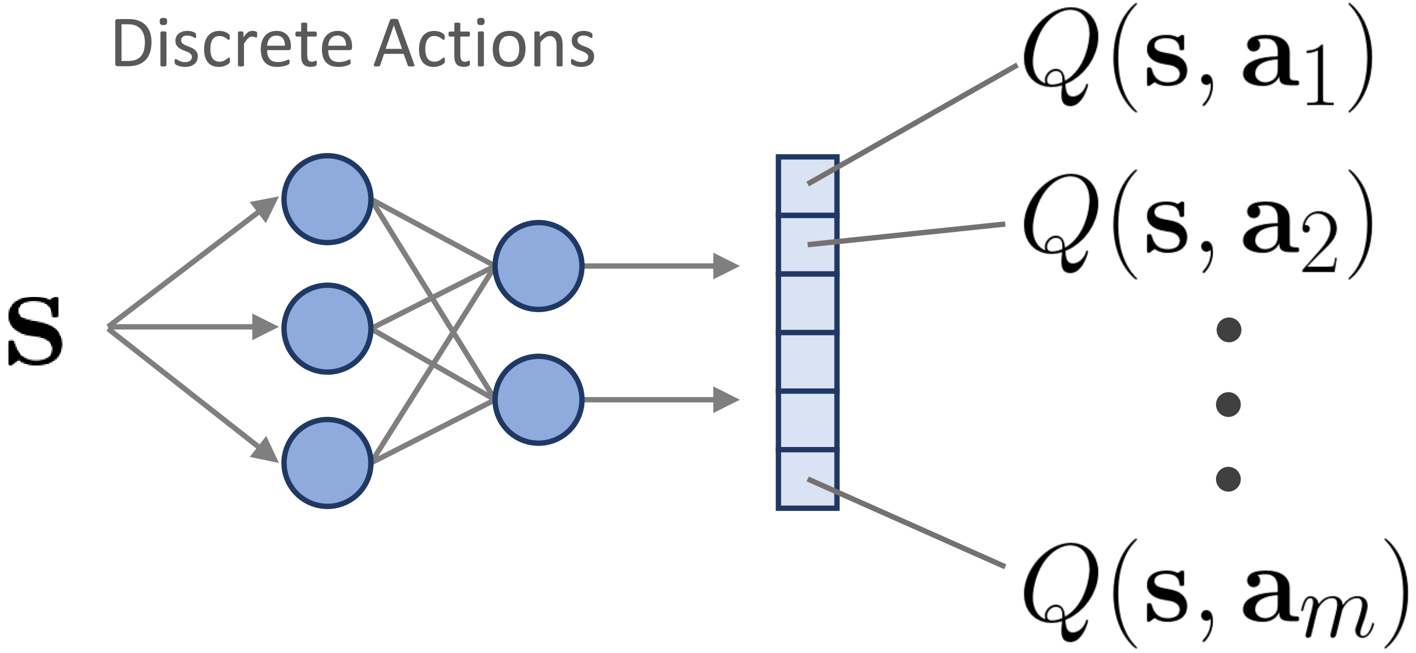
Discrete Actions



Large/Continuous State Spaces

$$\underline{Q^{k+1}(\mathbf{s},\mathbf{a})} = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^{k}(\mathbf{s}',\mathbf{a}')\right)\right]$$

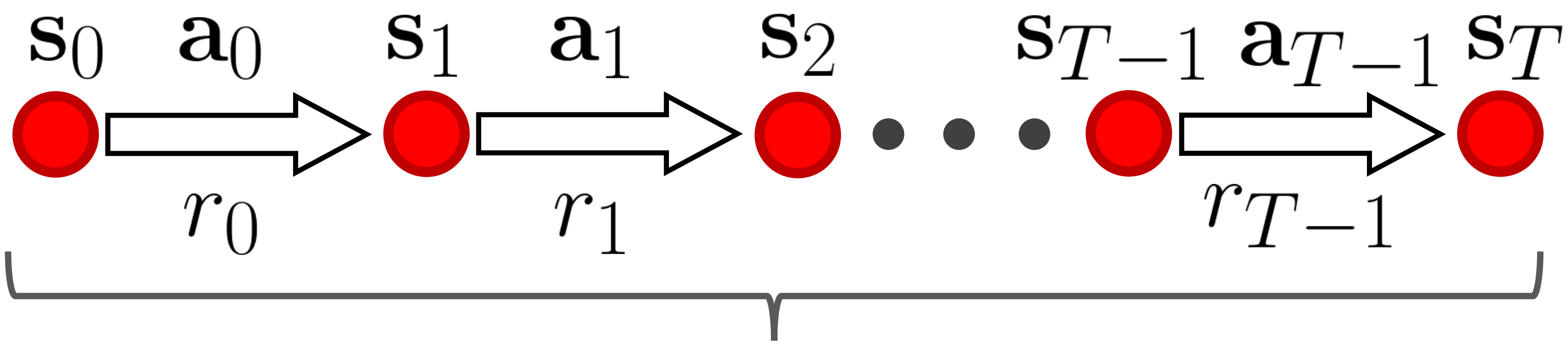
Discrete Actions



Large/Continuous State Spaces

$$Q^{k+1}\underline{(\mathbf{s},\mathbf{a})} = \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]$$

$$Q^k(\mathbf{s},\mathbf{a}) \Longrightarrow \pi^k(\mathbf{a}|\mathbf{s})$$



$$\mathcal{D} = \{(\mathbf{s}_i,\mathbf{a}_i,r_i,\mathbf{s}'_i)\}$$

Large/Continuous State Spaces

$$Q^{k+1}(\mathbf{s},\mathbf{a}) = \underline{\mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \max_{\mathbf{a}'}\left(Q^k(\mathbf{s}',\mathbf{a}')\right)\right]}$$

$$\mathcal{D} = \{(\mathbf{s}_i, \mathbf{a}_i, r_i, \mathbf{s}'_i)\}$$

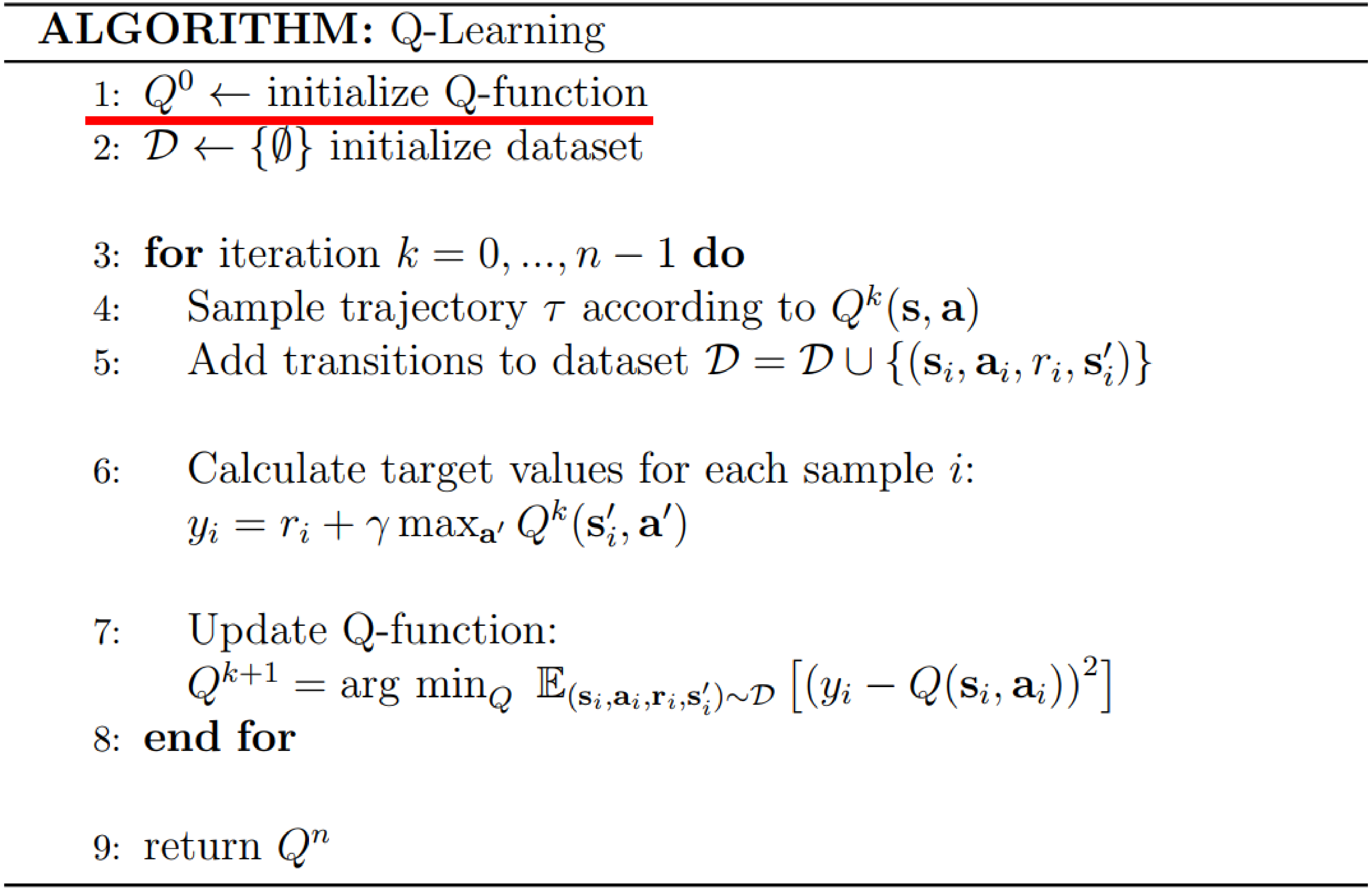
Compute target values for each sample i

$$y_i = r_i + \gamma \max_{\mathbf{a}'} \; Q^k(\mathbf{s}'_i, \mathbf{a}')$$

Fit new Q-function

$$Q^{k+1} = \arg\min_{Q} \; \underbrace{\mathbb{E}_{(\mathbf{s}_i,\mathbf{a}_i,r_i,\mathbf{s}'_i)\sim\mathcal{D}}\left[(y_i - Q(\mathbf{s}_i,\mathbf{a}_i))^2\right]}_{\text{“Bellman error”}}$$

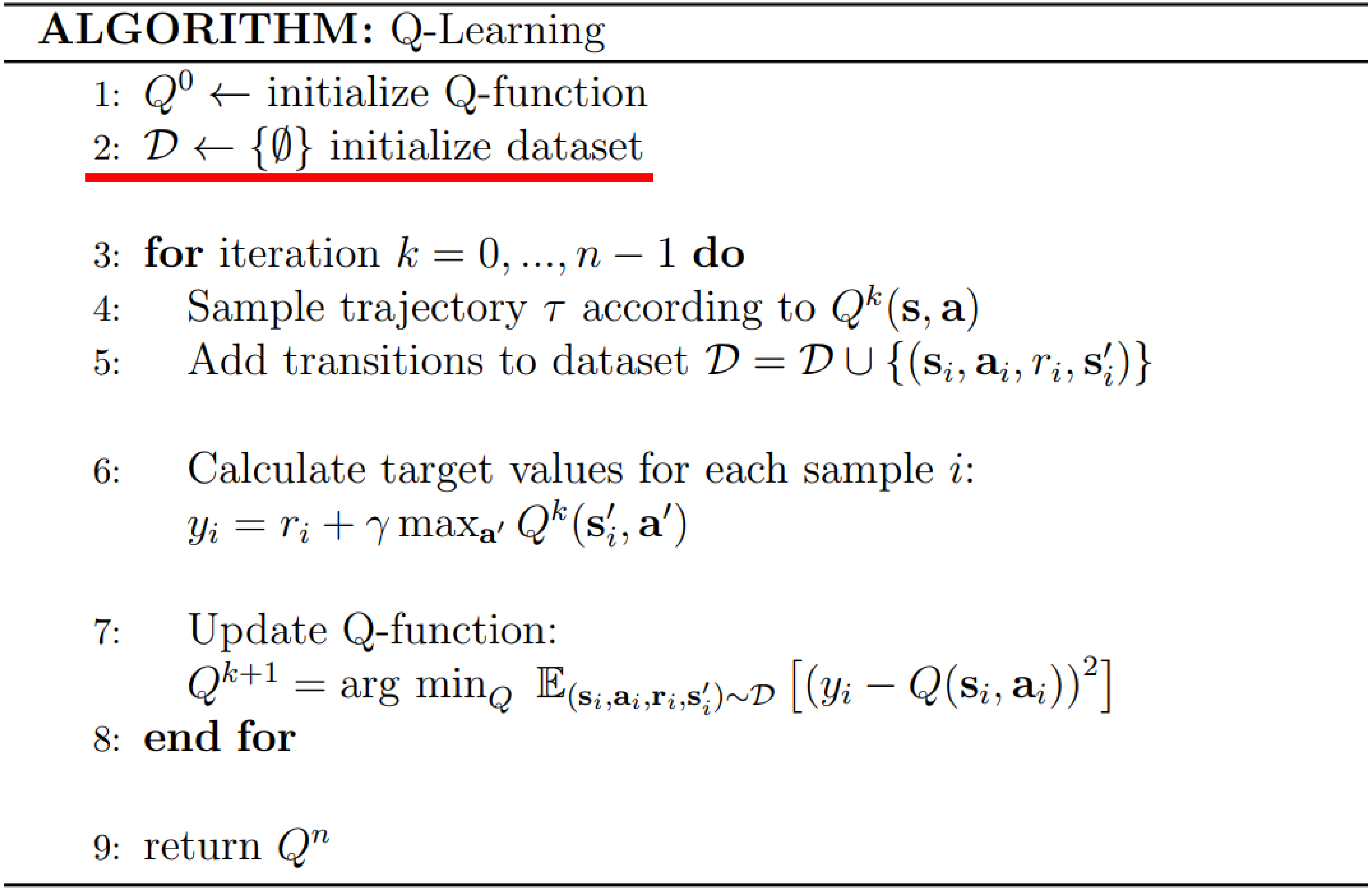
Q-Learning



ALGORITHM: Q-Learning

```
$Q^0 \leftarrow$ initialize Q-function
$\mathcal{D} \leftarrow \{\emptyset\}$ initialize dataset

for iteration $k = 0, ..., n-1$ do
    Sample trajectory $\tau$ according to $Q^k(\mathbf{s}, \mathbf{a})$
    Add transitions to dataset $\mathcal{D} = \mathcal{D} \cup \{(\mathbf{s}_i, \mathbf{a}_i, r_i, \mathbf{s}'_i)\}$

    Calculate target values for each sample $i$:
    $y_i = r_i + \gamma \max_{\mathbf{a}'} Q^k(\mathbf{s}'_i, \mathbf{a}')$

    Update Q-function:
    $Q^{k+1} = \arg\min_Q \mathbb{E}_{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{r}_i, \mathbf{s}'_i) \sim \mathcal{D}} \left[(y_i - Q(\mathbf{s}_i, \mathbf{a}_i))^2\right]$
end for

return $Q^n$
```

Q-Learning



ALGORITHM: Q-Learning

```
$Q^0 \leftarrow$ initialize Q-function
$\mathcal{D} \leftarrow \{\emptyset\}$ initialize dataset

for iteration $k = 0, ..., n-1$ do
    Sample trajectory $\tau$ according to $Q^k(\mathbf{s}, \mathbf{a})$
    Add transitions to dataset $\mathcal{D} = \mathcal{D} \cup \{(\mathbf{s}_i, \mathbf{a}_i, r_i, \mathbf{s}'_i)\}$

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```
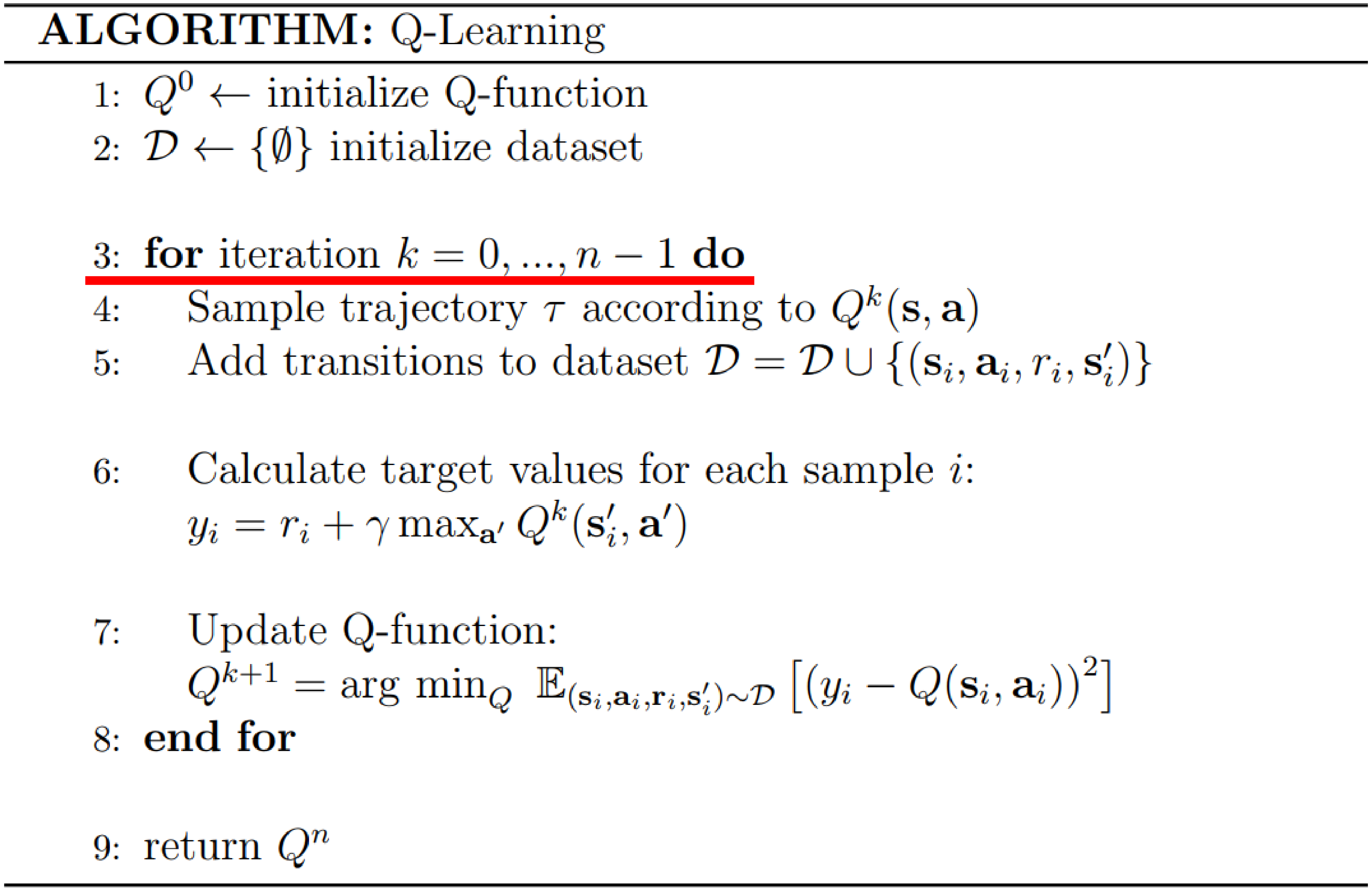

Q-Learning



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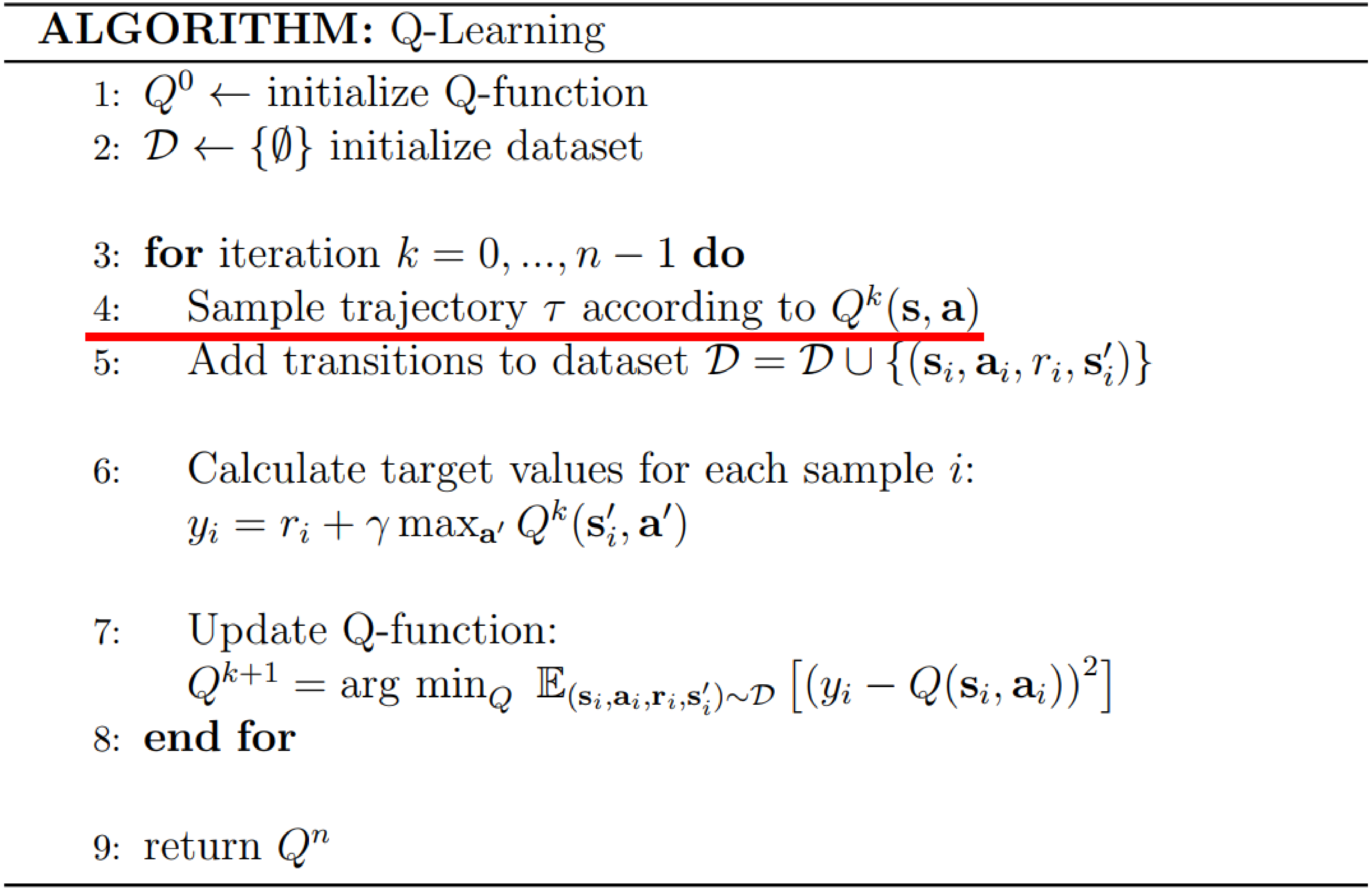
Q-Learning



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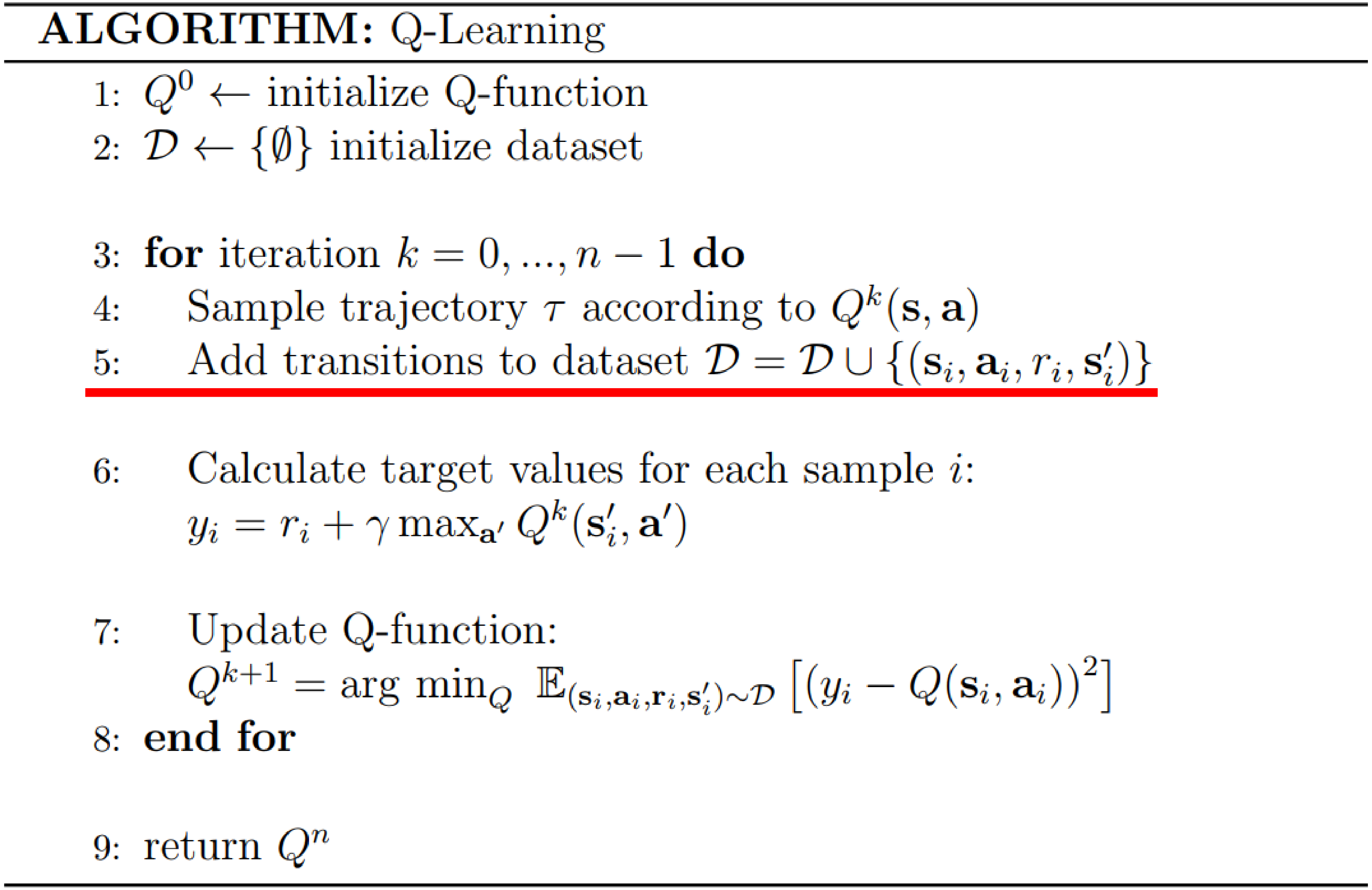
Q-Learning



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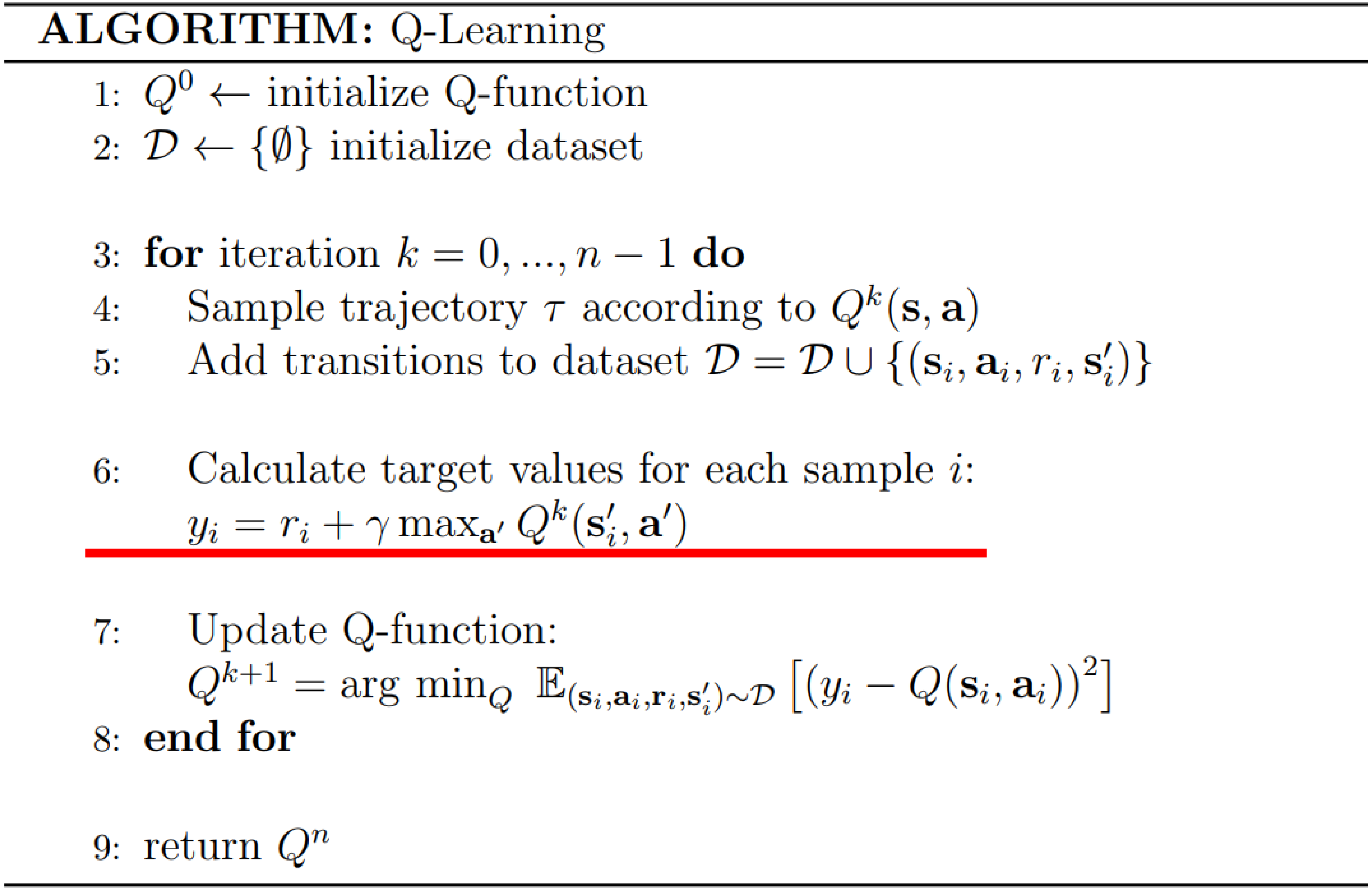
Q-Learning



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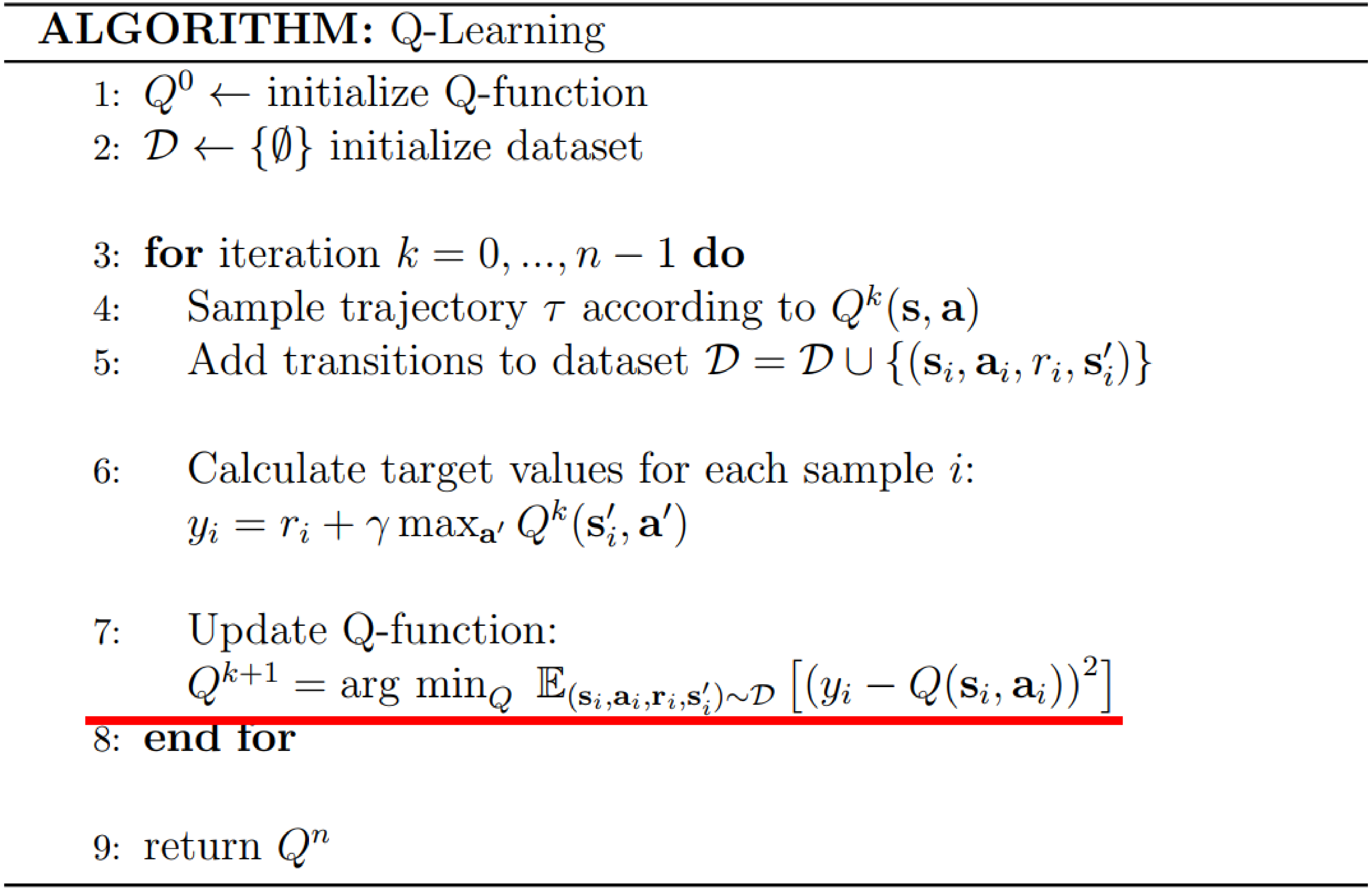
Q-Learning



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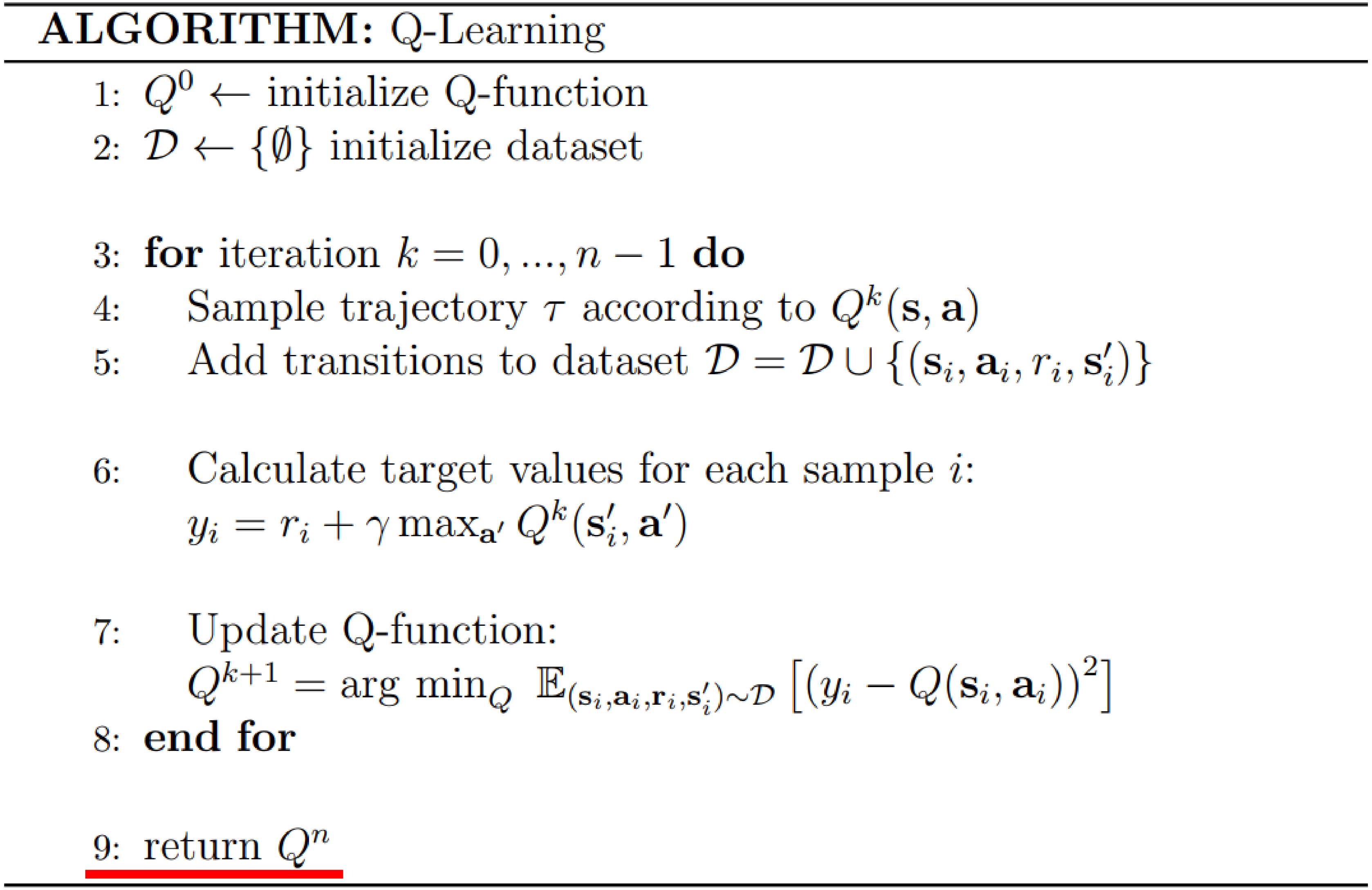

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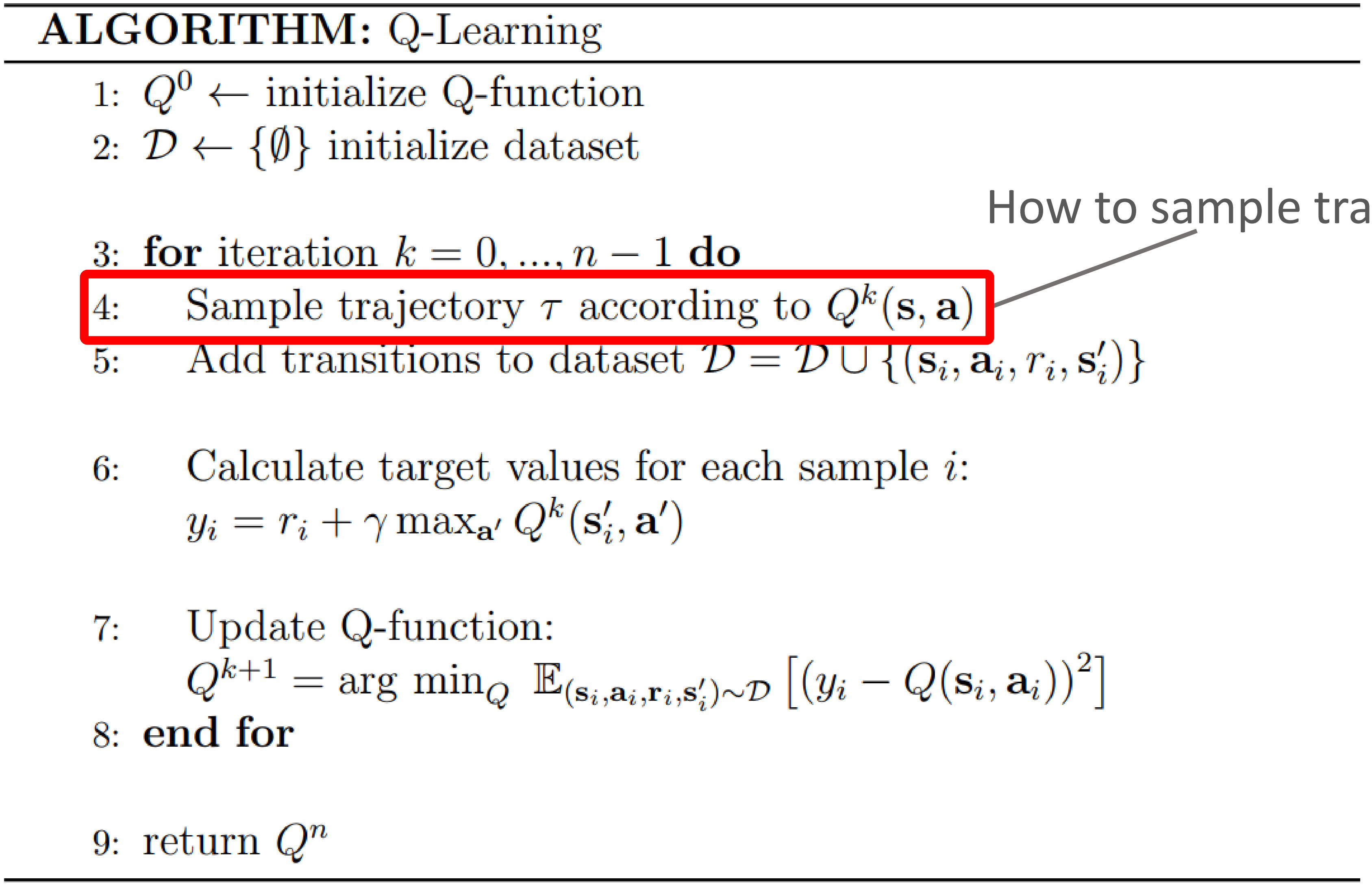
Q-Learning



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end for

return $Q^n$
```

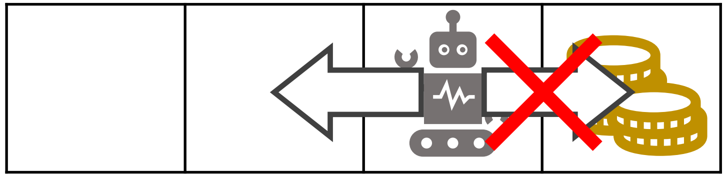
How to sample trajectories?

Sampling

$$Q^k(\mathbf{s}, \mathbf{a}) \Longrightarrow \pi^k(\mathbf{a}|\mathbf{s})$$

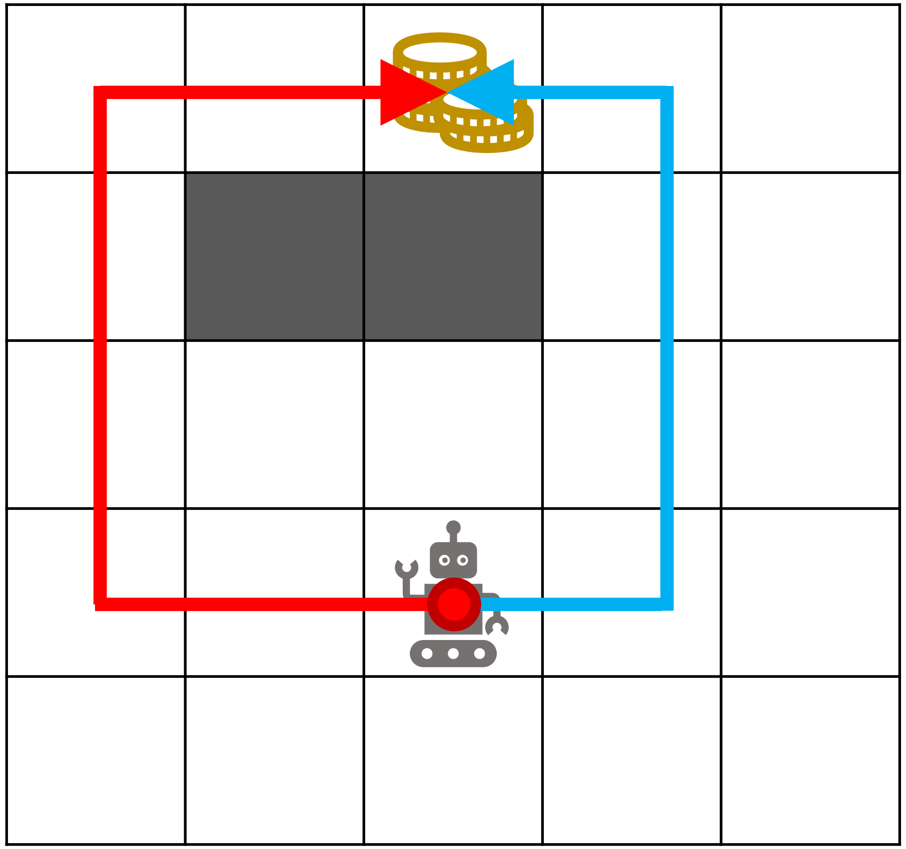
$$\pi^k(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} \ Q^k(\mathbf{s}, \mathbf{a}') \\ 0 & \text{otherwise} \end{cases}$$

✗ Insufficient exploration



Exploration-Exploitation

Need to try new actions in case they are better



Exploration-Exploitation

Need to try new actions in case they are better

Keep going to the same restaurant

Try new restaurant

Exploration-Exploitation

Need to try new actions in case they are better

$$\pi^k(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} \; Q^k(\mathbf{s}, \mathbf{a}') \\ 0 & \text{otherwise} \end{cases}$$

Exploration-Exploitation

Need to try new actions in case they are better

$$\pi^k(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 - \boxed{\epsilon} & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} Q^k(\mathbf{s}, \mathbf{a}') \\ \boxed{\epsilon} & \text{otherwise} \end{cases}$$

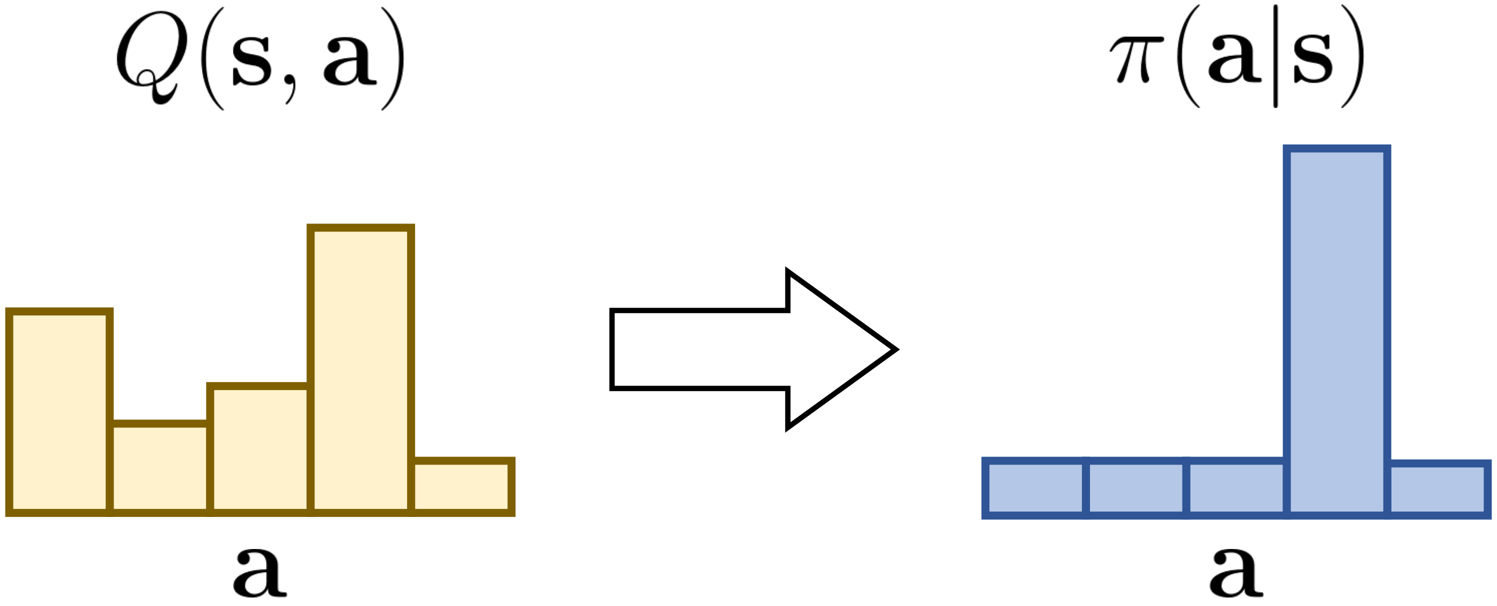
Epsilon-greedy exploration

- With probability $1 - \epsilon$ exploit current best action
- With probability ϵ explore new action by sampling a random action
- Start with $\epsilon = 1$ and then anneal to lower value (e.g. $\epsilon \rightarrow 0.1$)

Epsilon-Greedy Exploration

$$\pi^k(\mathbf{a}|\mathbf{s}) = \begin{cases} 1-\epsilon & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} \; Q^k(\mathbf{s},\mathbf{a}') \\ \epsilon & \text{otherwise} \end{cases}$$

$\epsilon \to 1$



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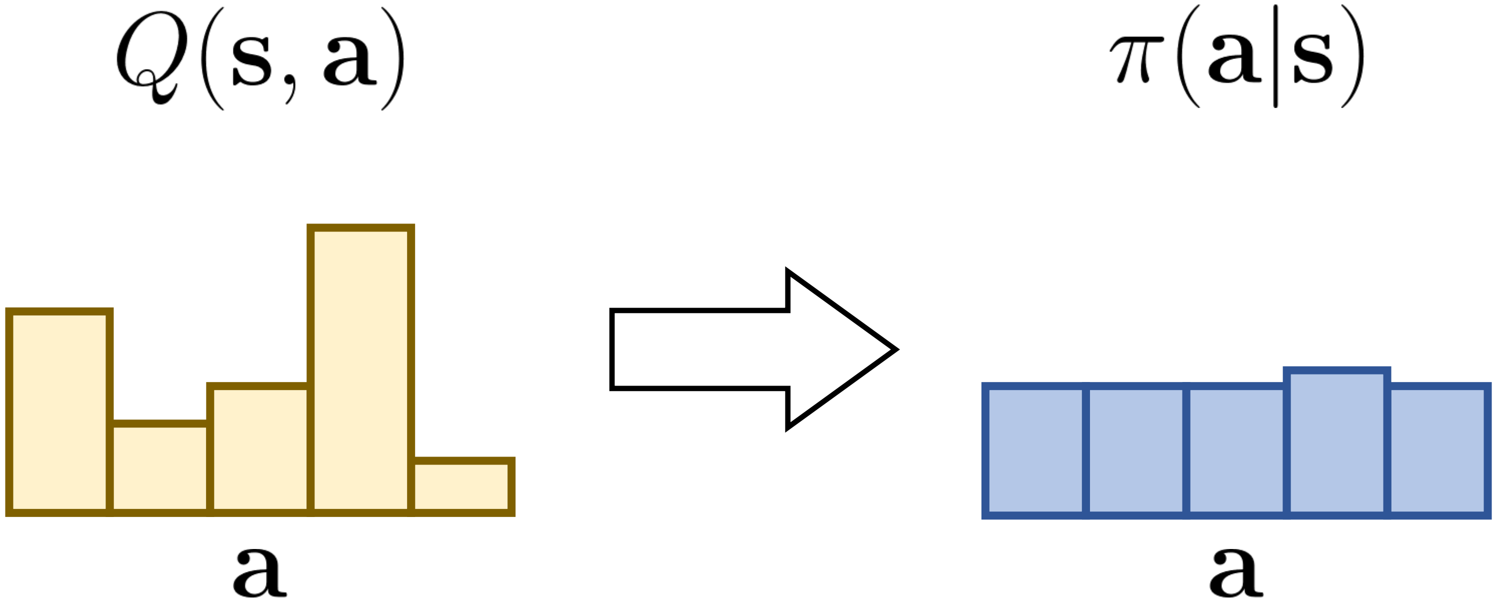

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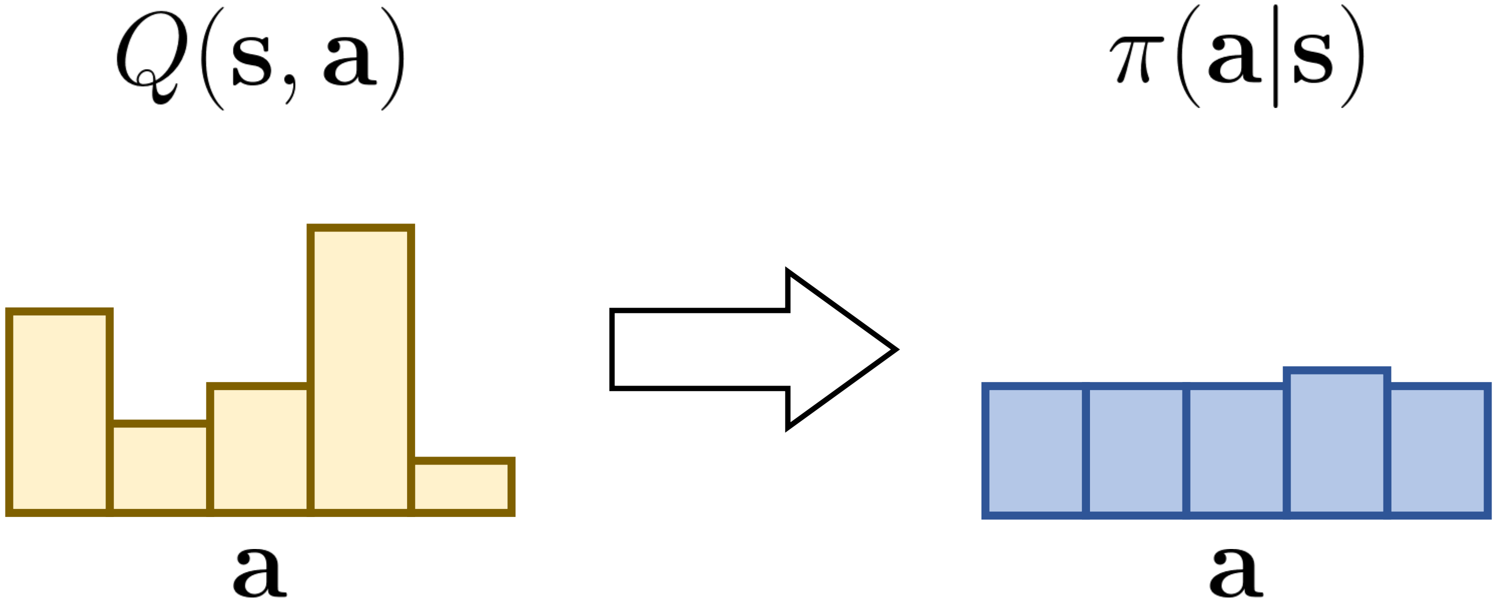
$\epsilon \rightarrow 0$



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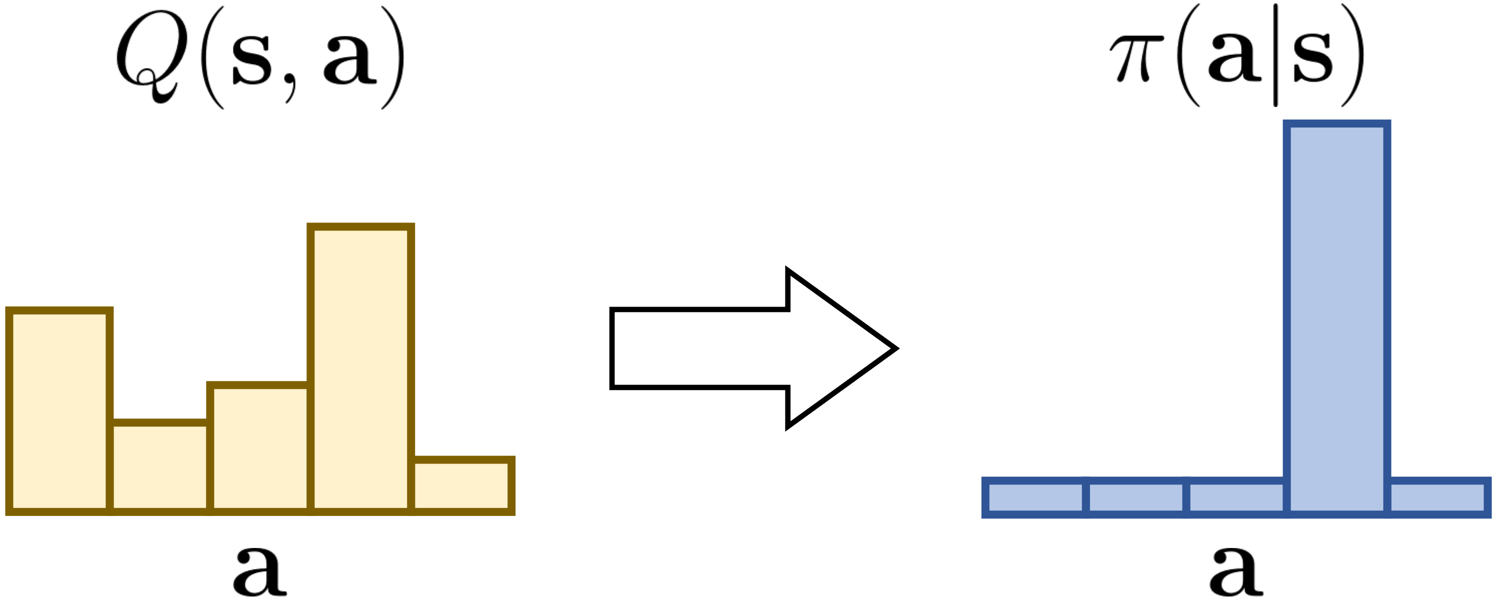


Boltzmann Exploration

Probability of an action is proportion to its “goodness”

$$\pi^k(\mathbf{a}|\mathbf{s}) = \frac{1}{Z}\exp\left(\frac{1}{\beta}Q^k(\mathbf{s},\mathbf{a})\right)$$

where,

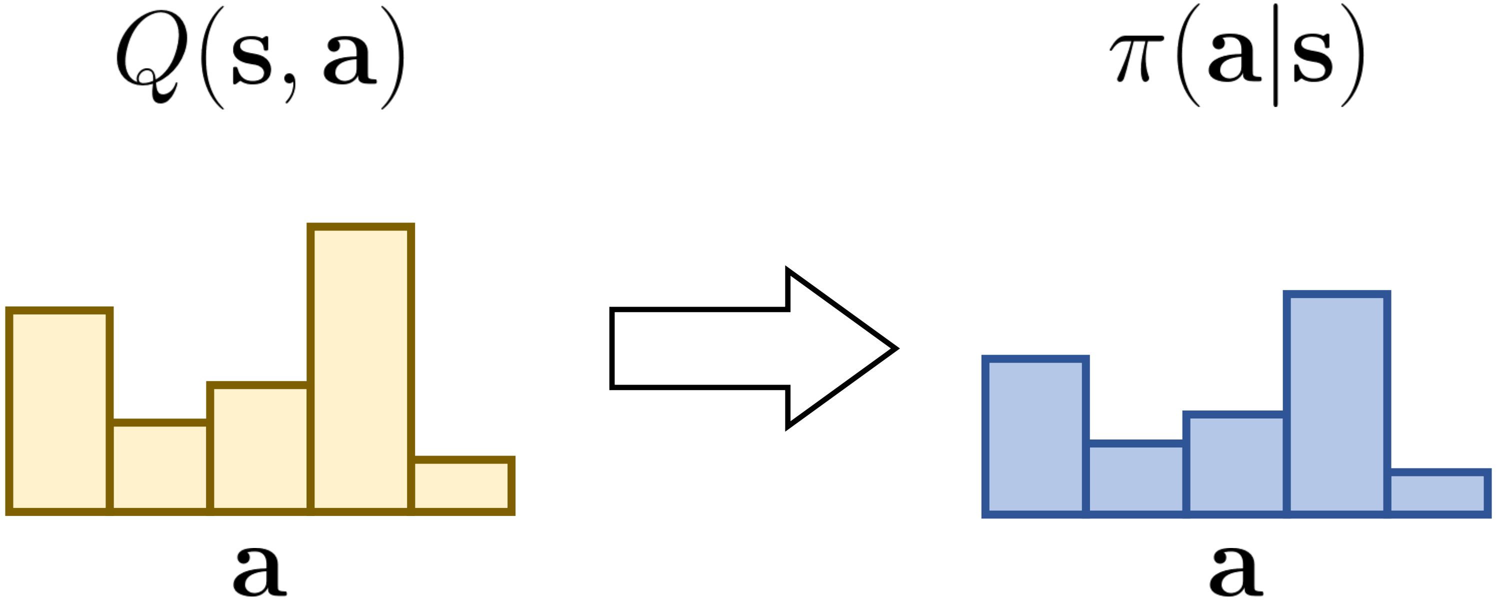
temperature parameters: $\beta \in \mathbb{R}$

normalization factor: $Z = \sum_{\mathbf{a}'} \exp\left(\frac{1}{\beta}Q^k(\mathbf{s},\mathbf{a}')\right)$

Boltzmann Exploration

$$\pi^k(\mathbf{a}|\mathbf{s}) = \frac{1}{Z}\text{exp}\left(\frac{1}{\beta}Q^k(\mathbf{s},\mathbf{a})\right)$$

$\beta \rightarrow \infty$



Boltzmann Exploration

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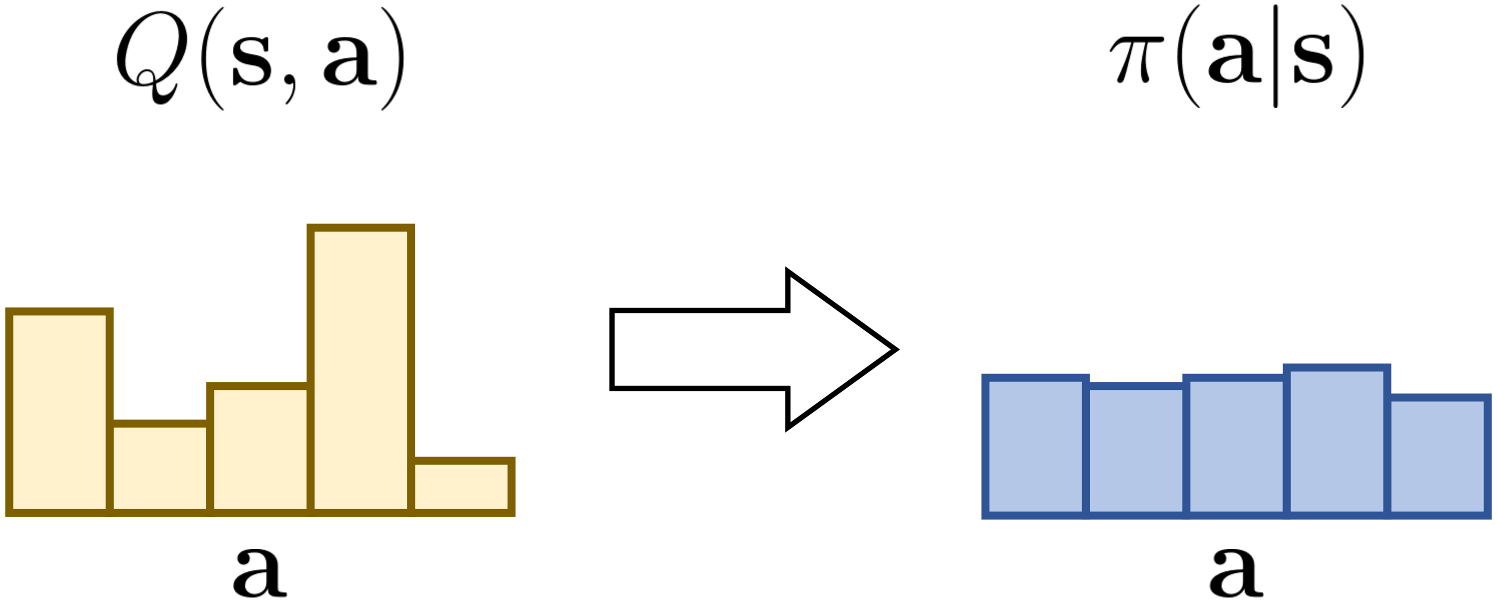

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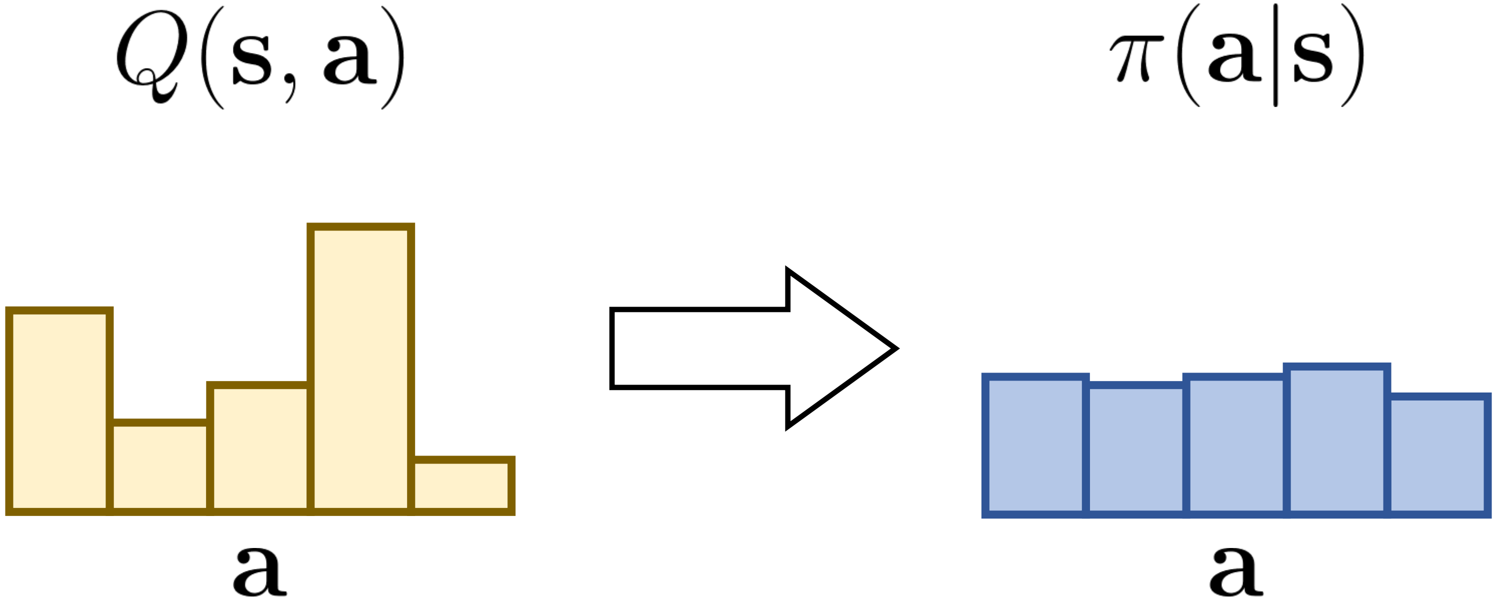

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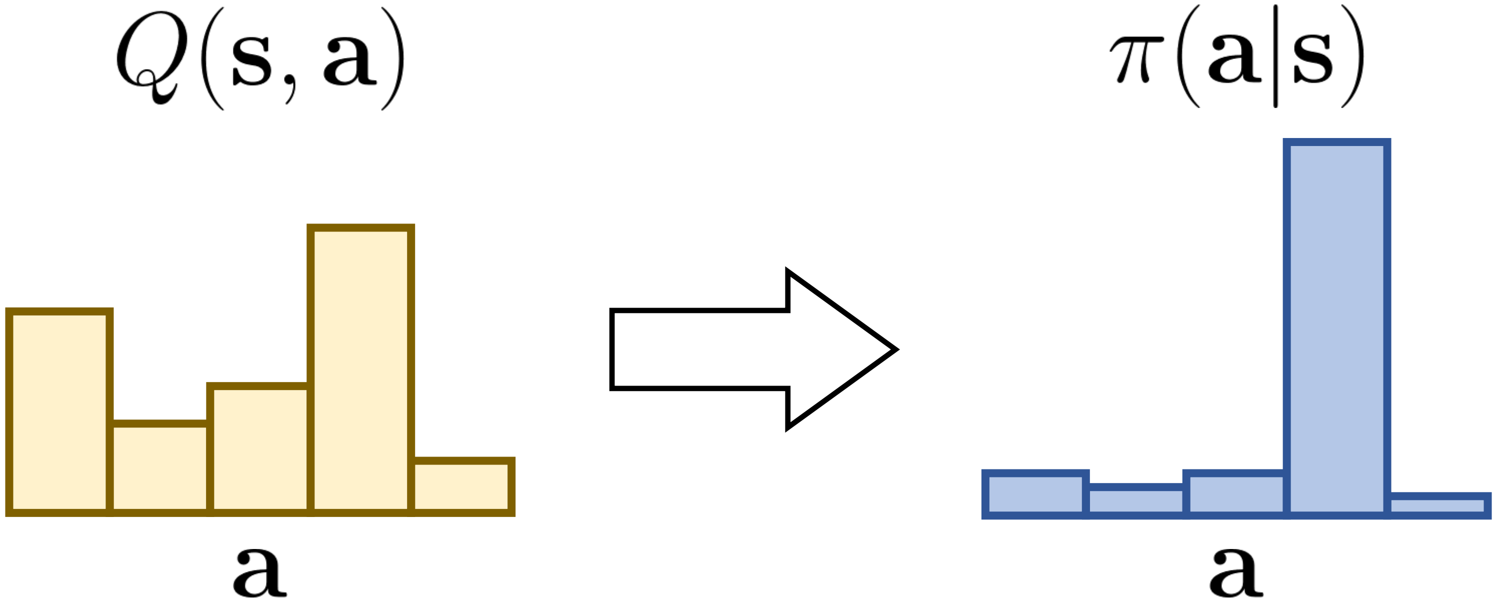
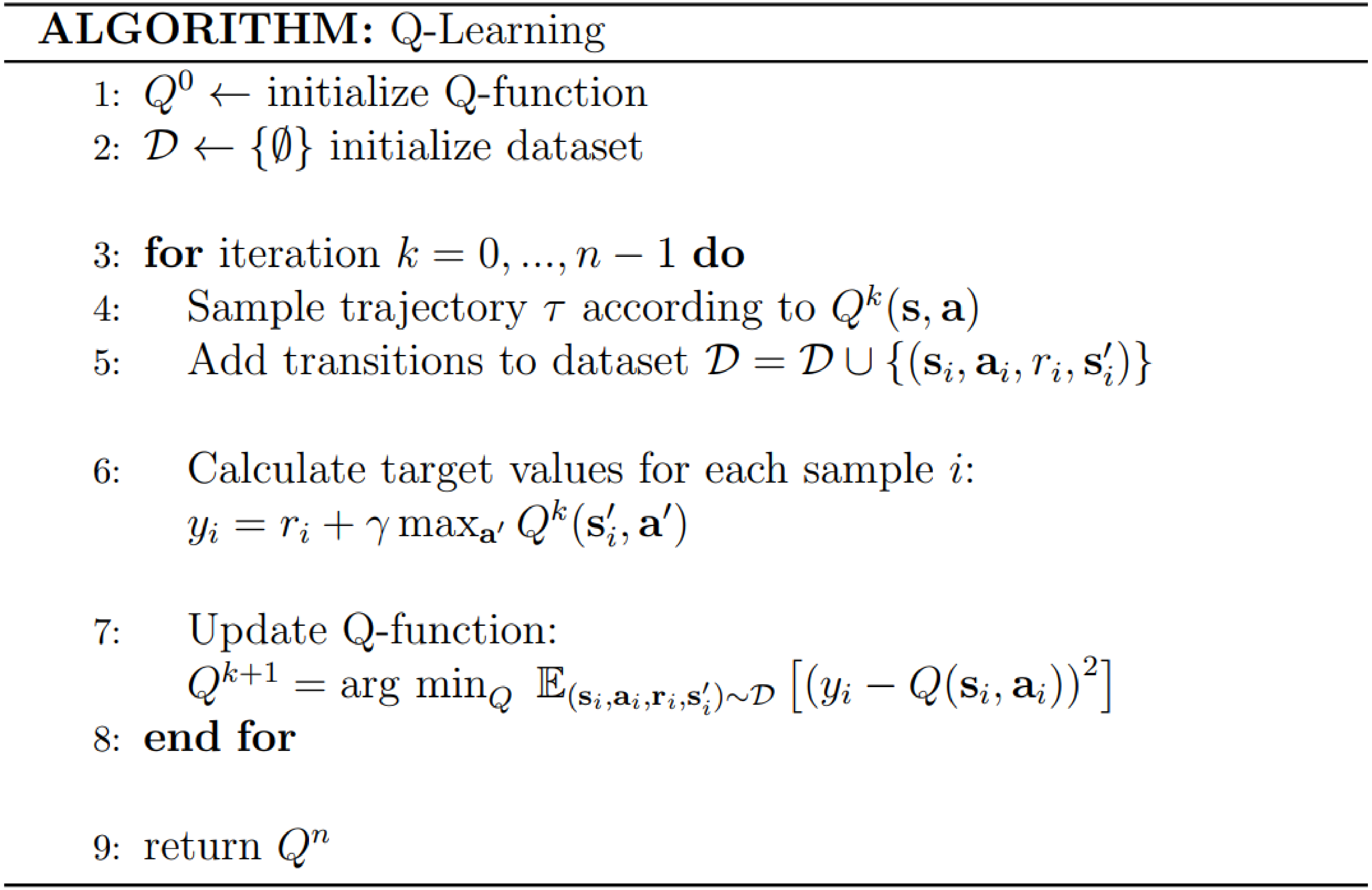


Testing

After training, test with greedy policy

$$\pi^k(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a'}} \; Q^k(\mathbf{s}, \mathbf{a'}) \\ 0 & \text{otherwise} \end{cases}$$

Q-Learning



ALGORITHM: Q-Learning

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    Calculate target values for each sample $i$:
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end for

return $Q^n$
```

Q-Learning with Function Approximators

- No improvement guarantees

$$Q^{k+1}(\mathbf{s}, \mathbf{a}) \xcancel{\geq} Q^{k}(\mathbf{s}, \mathbf{a}) \qquad J(\pi^{k+1}) \xcancel{\geq} J(\pi^{k})$$

- No convergence guarantees

$$Q^{k} \xcancel{\Longrightarrow} Q^{*}$$

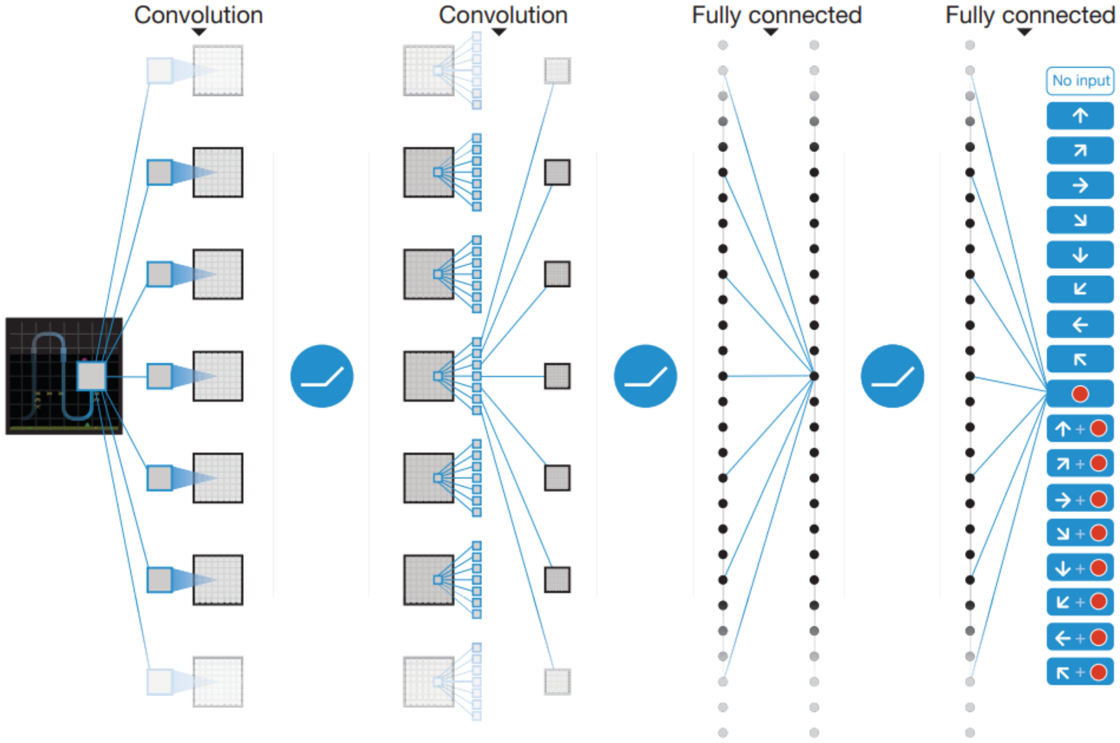
- But in practice, it works!

Deep Q-Networks (DQN)

Human-Level Control Through Deep Reinforcement Learning
[Mnih et al. 2015]

Deep Q-Networks (DQN)

Input: 84 x 84 images



Output: Q-values for joystick controls

Human-Level Control Through Deep Reinforcement Learning
[Mnih et al. 2015]

Deep Q-Networks (DQN)

Human-Level Control Through Deep Reinforcement Learning
[Mnih et al. 2015]

Q-Learning

✓ Often much more sample efficient than policy gradient

✓ Off-policy learning

✗ Limited to relatively small discrete action spaces

✗ Does not directly optimize performance

- Lower Bellman error ≠ better performance

✗ No convergence guarantees with function approximators

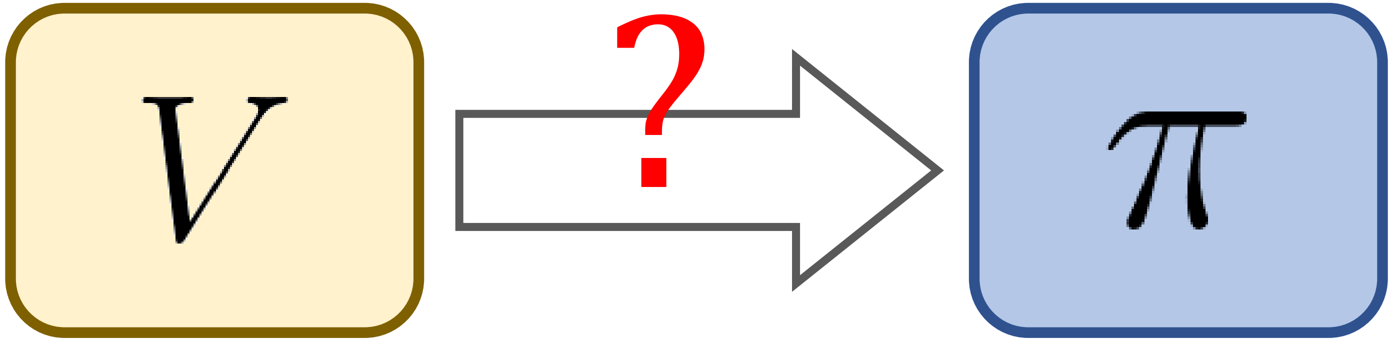
$$Q^{k+1} = \arg\min_{Q} \mathbb{E}_{(\mathbf{s},\mathbf{a},r,\mathbf{s}')\sim\mathcal{D}}\left[\left(\left(\underline{r + \gamma \max_{\mathbf{a}'} Q^k(\mathbf{s}', \mathbf{a}')}\right) - Q(\mathbf{s}, \mathbf{a})\right)^2\right]$$

Intractable in large/continuous action spaces

Value Functions

$$\pi(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \arg\max_{\mathbf{a}'} Q(\mathbf{s}, \mathbf{a}') \\ 0 & \text{otherwise} \end{cases}$$

What about $V(\mathbf{s})$?

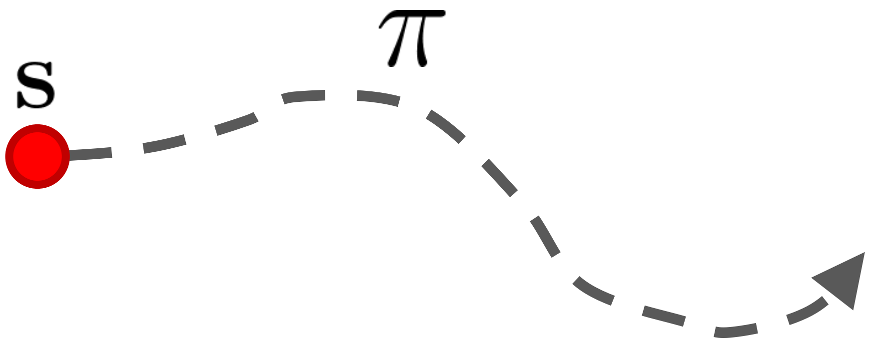


Value Functions

Value Function

"State Value Function"

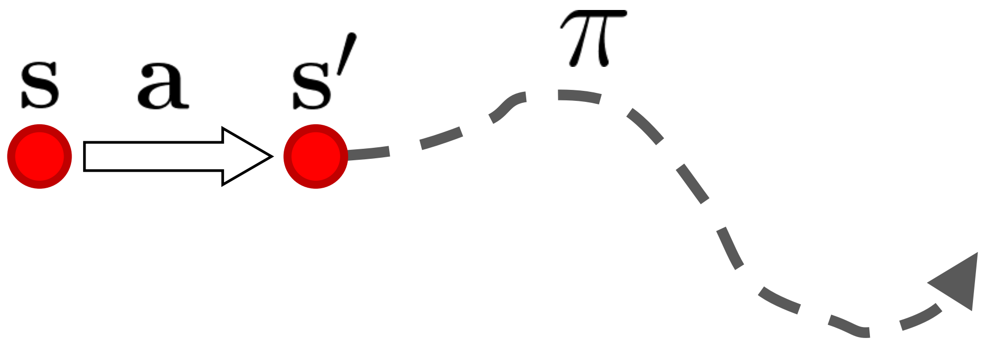
$$V^{\pi}(\mathbf{s}) = \mathbb{E}_{\tau \sim p(\tau | \pi, \mathbf{s}_0 = \mathbf{s})} \left[\sum_{t=0} \gamma^t r_t \right]$$



Q-Function

"State-Action Value Function"

$$Q^{\pi}(\mathbf{s}, \underline{\mathbf{a}}) = \mathbb{E}_{\tau \sim p(\tau | \pi, \mathbf{s}_0 = \mathbf{s}, \mathbf{a}_0 = \mathbf{a})} \left[\sum_{t=0} \gamma^t r_t \right]$$



Value Functions

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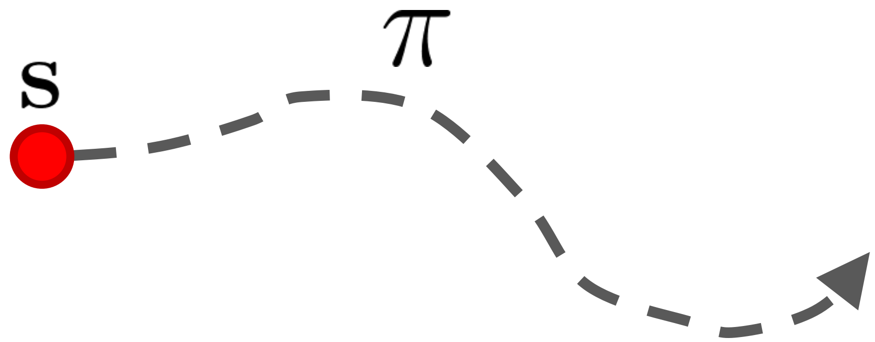

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"State-Action Value Function"

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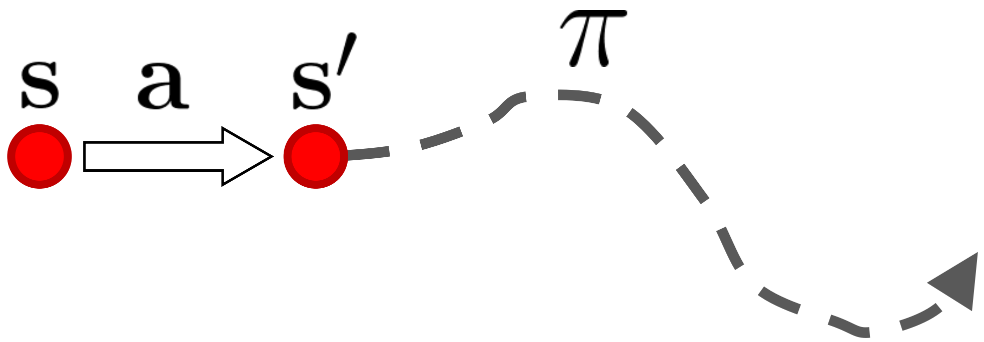
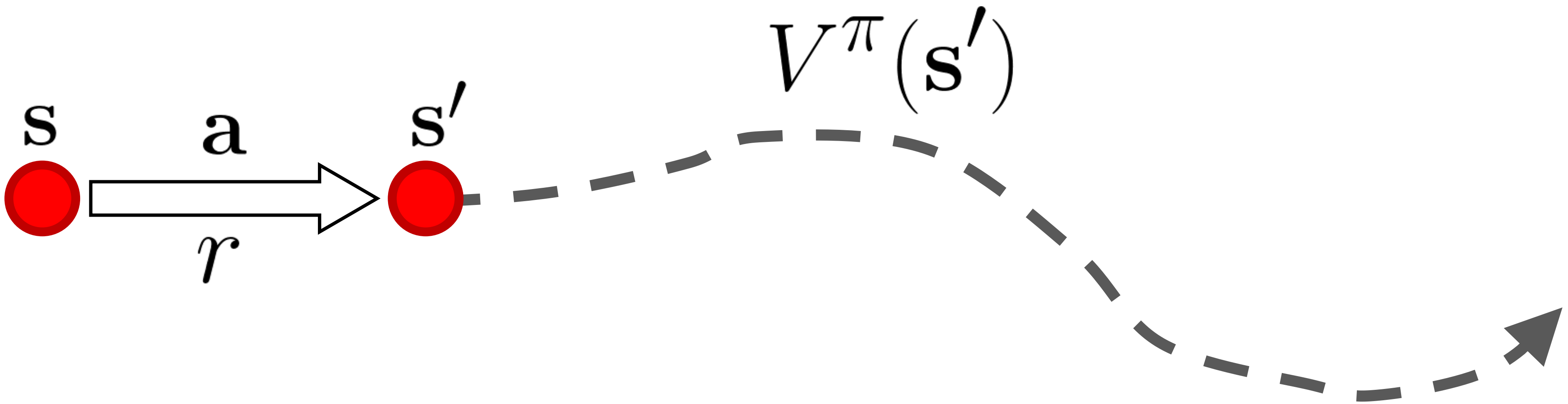


Recursive definition

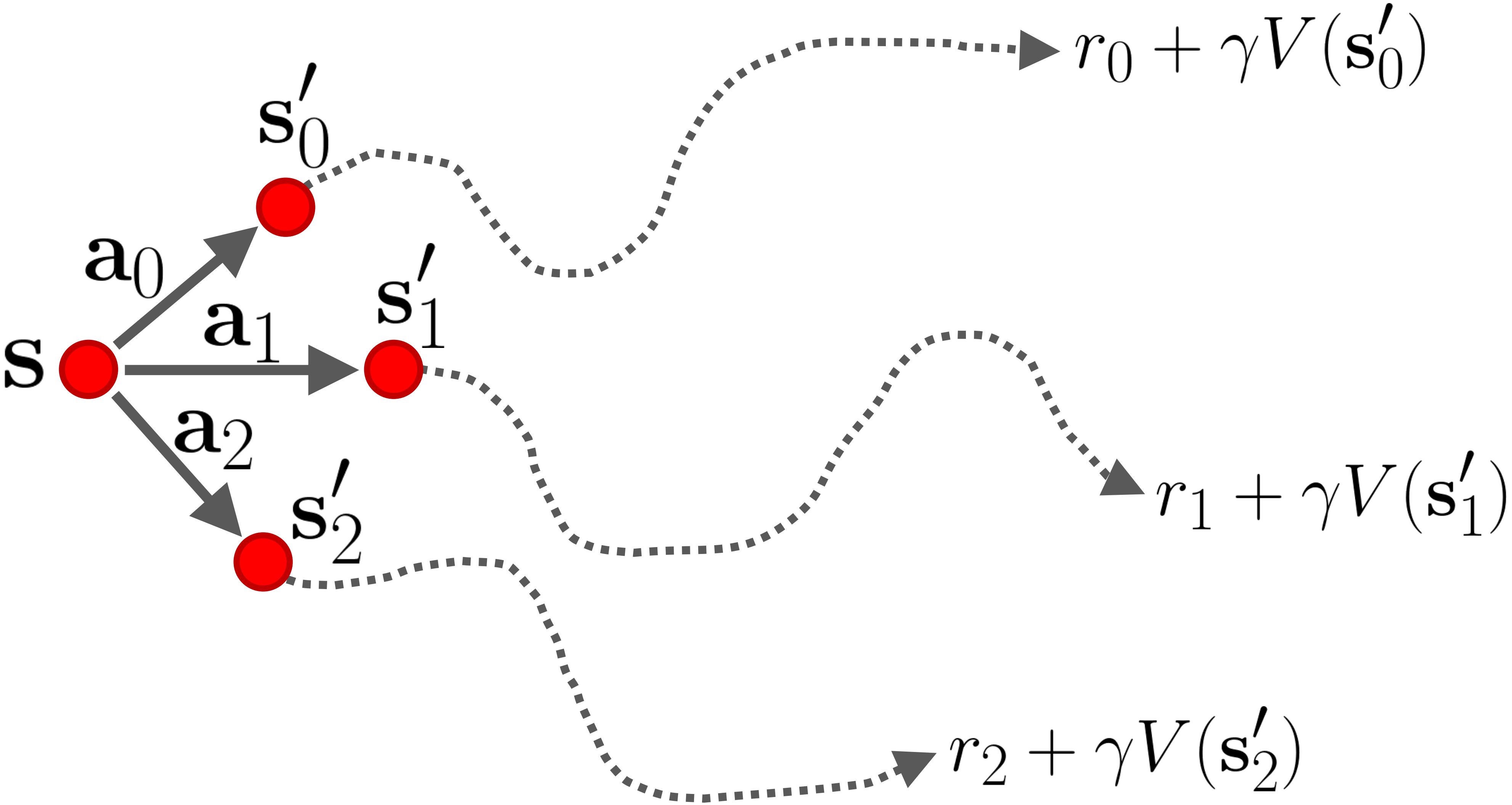
$$Q^{\pi}(\mathbf{s}, \mathbf{a}) = \mathbb{E}_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma \underline{\mathbb{E}_{\mathbf{a}' \sim \pi(\mathbf{a}'|\mathbf{s}')} \left[Q^{\pi}(\mathbf{s}', \mathbf{a}') \right]} \right]$$

Recursive definition

$$
\begin{aligned}
Q^\pi(\mathbf{s},\mathbf{a}) &= \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \mathbb{E}_{\mathbf{a}'\sim\pi(\mathbf{a}'|\mathbf{s}')}\left[Q^\pi(\mathbf{s}',\mathbf{a}')\right]\right] \\
&= \mathbb{E}_{\mathbf{s}'\sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}\left[r(\mathbf{s},\mathbf{a},\mathbf{s}') + \gamma \underline{V^\pi(\mathbf{s}')}\right]
\end{aligned}
$$



Value Function



Value Function

Value-function:

$$\arg\max_{\mathbf{a}} \mathbb{E}_{\underline{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})}} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma V(\mathbf{s}') \right]$$

Need access to dynamics

Value Function

Value-function:

$$\arg\max_{\mathbf{a}} \mathbb{E}_{\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s},\mathbf{a})} \left[r(\mathbf{s}, \mathbf{a}, \mathbf{s}') + \gamma V(\mathbf{s}') \right]$$

Q-function:

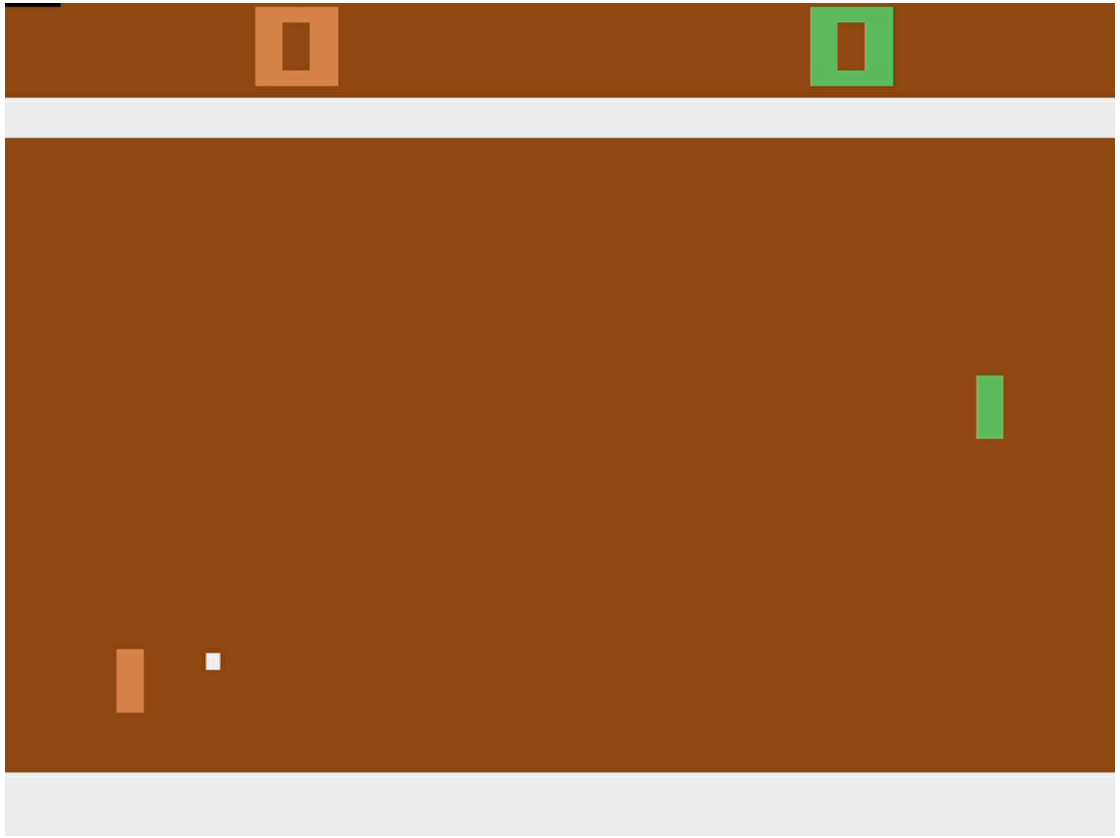
$$\arg\max_{\mathbf{a}} \underline{Q(\mathbf{s}, \mathbf{a})}$$

Does not need dynamics

Summary

- Q-Function
- Q-Learning
- Exploration

Assignment 3: Q-Learning



Pong

Breakout