An Introduction to Model Checking

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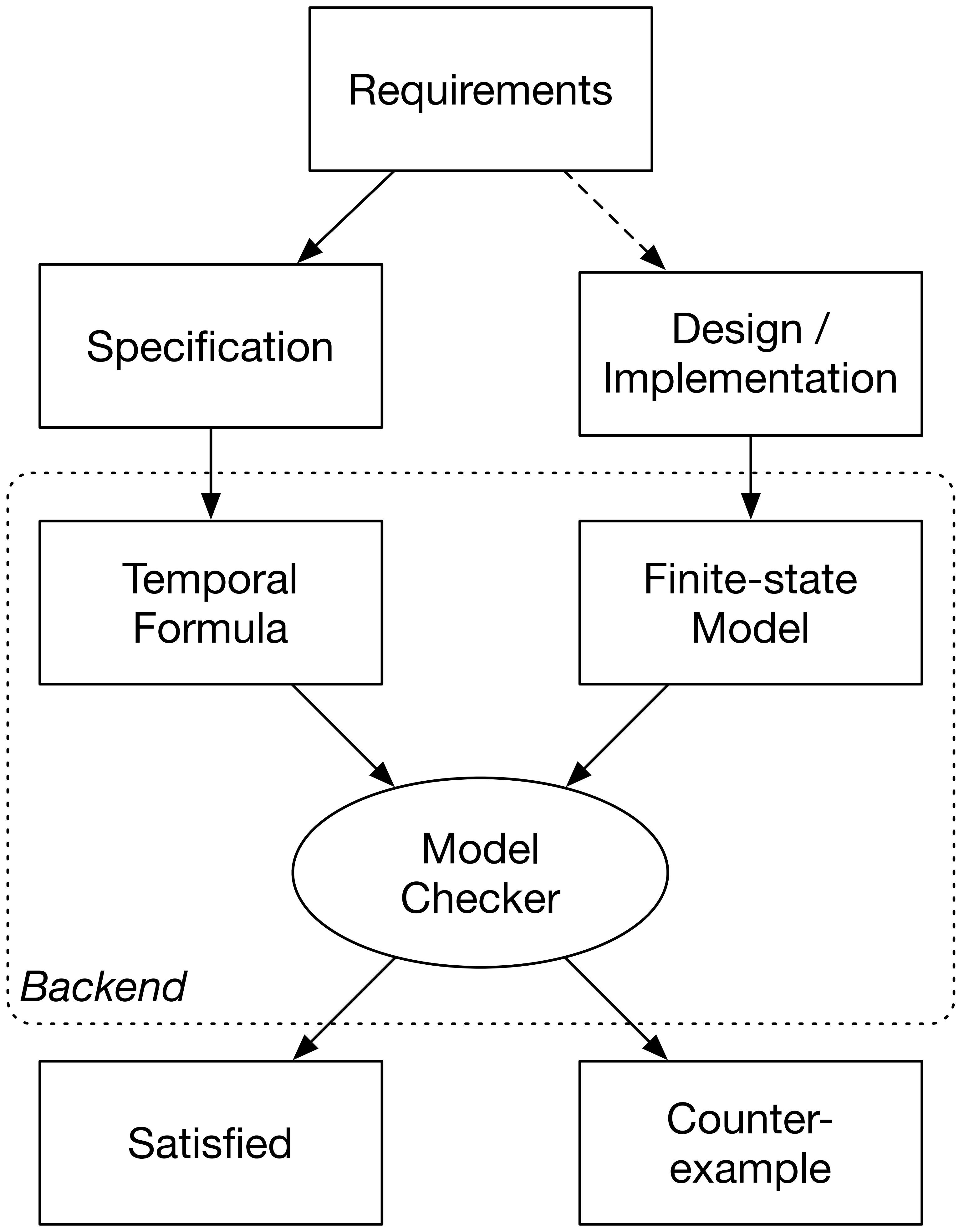
Model Checking

- A classic method to check whether a property holds in a reactive system
- System as a **finite-state machine**: all states exhaustively explored
- Provided in a higher-level description (design model, implementation, …)
- Requires a **bound** on the universe of discourse
- Properties in some kind of **propositional temporal logic**
- More expressive logics unfolded to propositional version within universe
- **Counter-examples** for unsatisfiable properties



System Model

- A model M is a transition system formalised as a Kripke structure (S,I,R,L), over a set of atomic propositions P:
  - S is a finite set of states
  - $I \subseteq S$ is the set of initial states
  - $R \subseteq S \times S$ is a left-total transition relation
  - $L : S \rightarrow 2^P$ is a function assigning atomic propositions to each state

System Paths

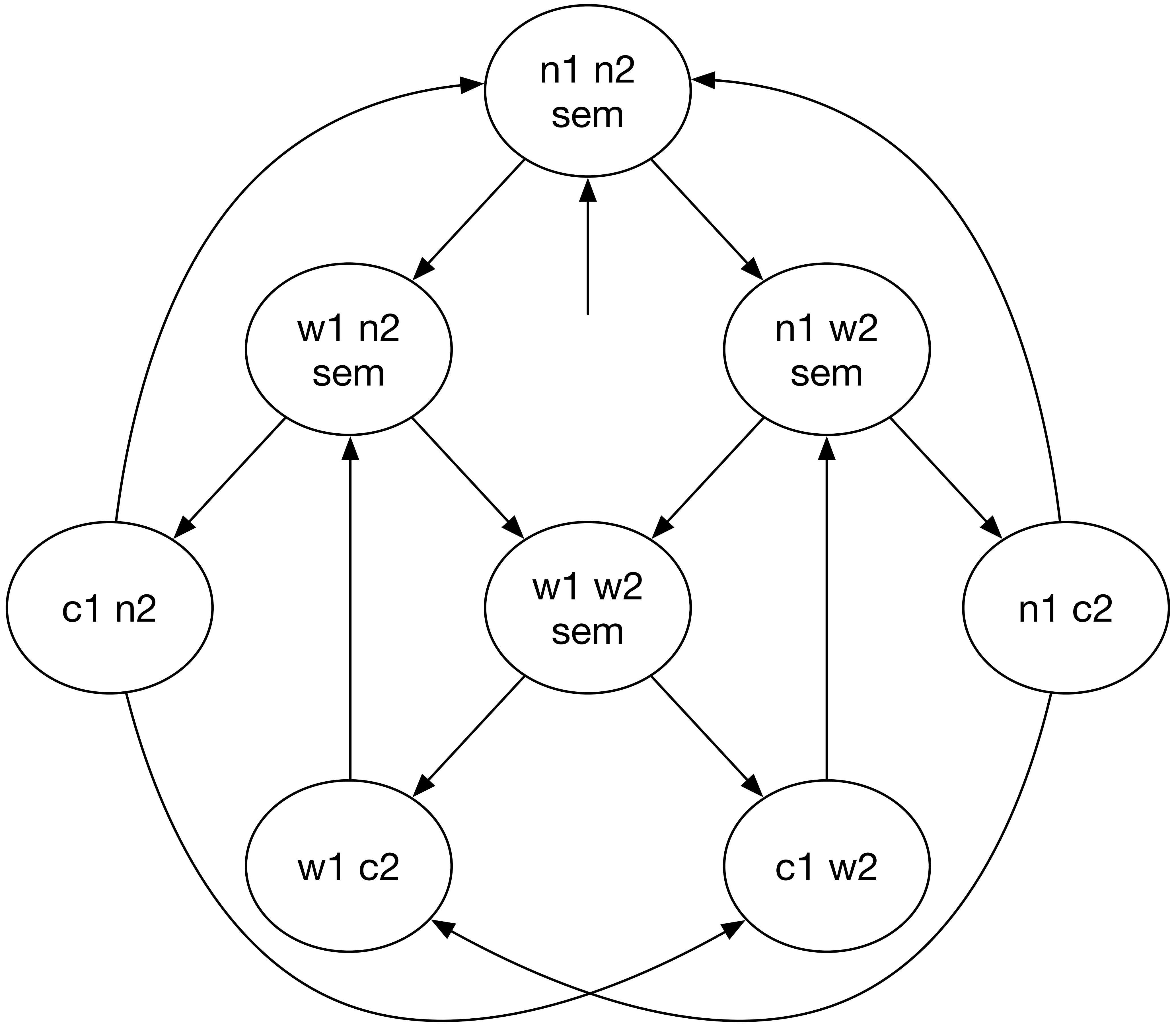
- A path π is an infinite sequence of states $s_0 s_1 s_2 \ldots$ where $\forall i \geq 0 : (s_i, s_{i+1}) \in R$
- For $i \geq 0$, π_i is the i-th state of the path, π^i is the suffix starting in i-th state
- We abuse notation and say $\pi \in M$ for any path in model M

Model example: Mutex

```
while true:
  n1: // noncritical actions
  w1: request semaphore
  c1: // critical section
      release semaphore
```

‖

```
while true:
  n2: // noncritical actions
  w2: request semaphore
  c2: // critical section
      release semaphore
```

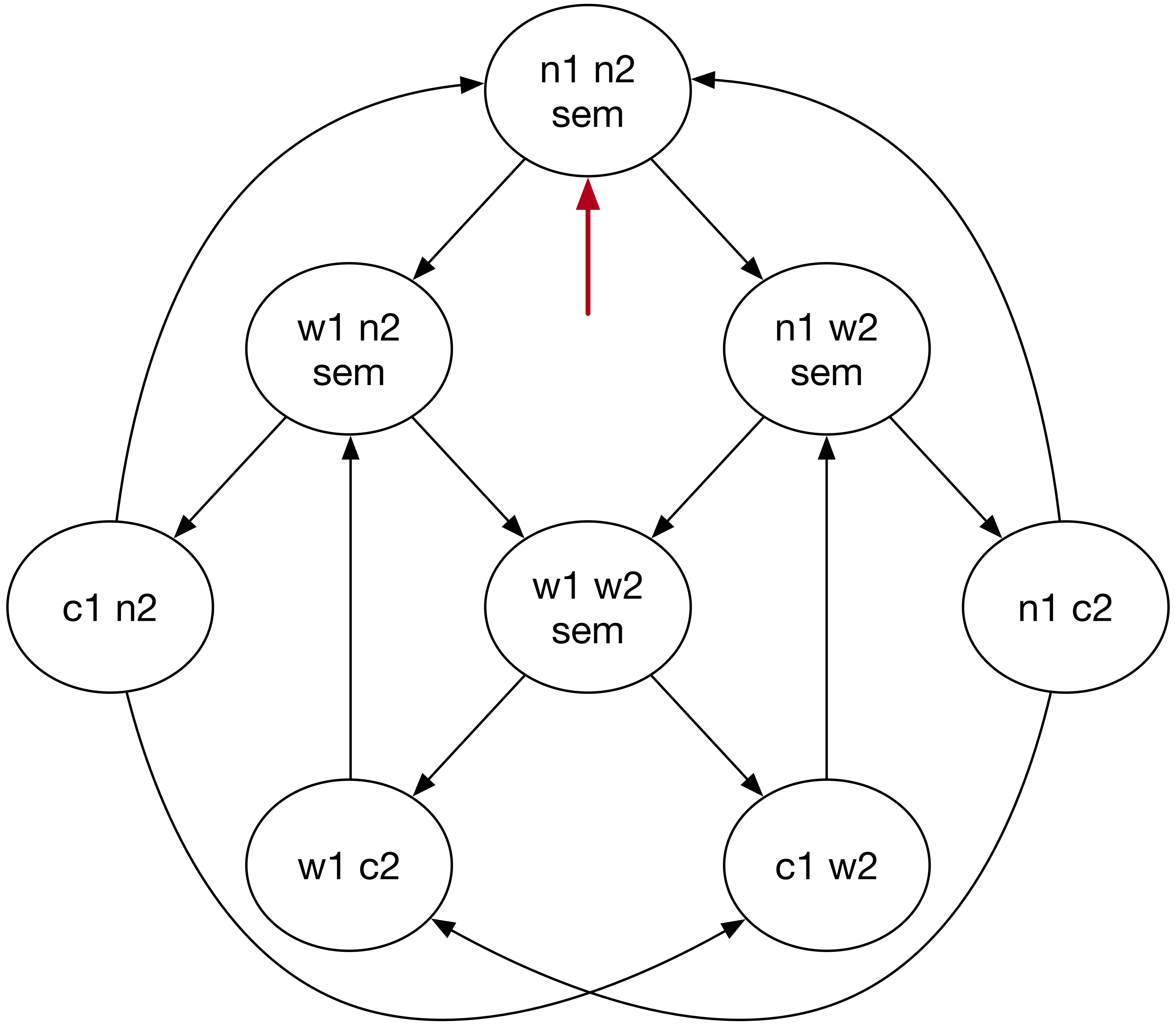


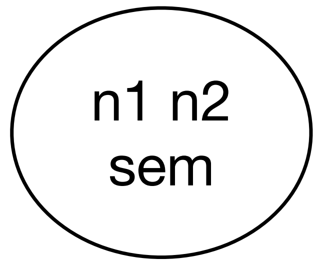
P = {n1,n2,w1,w2,c1,c2,sem}

Temporal Logic

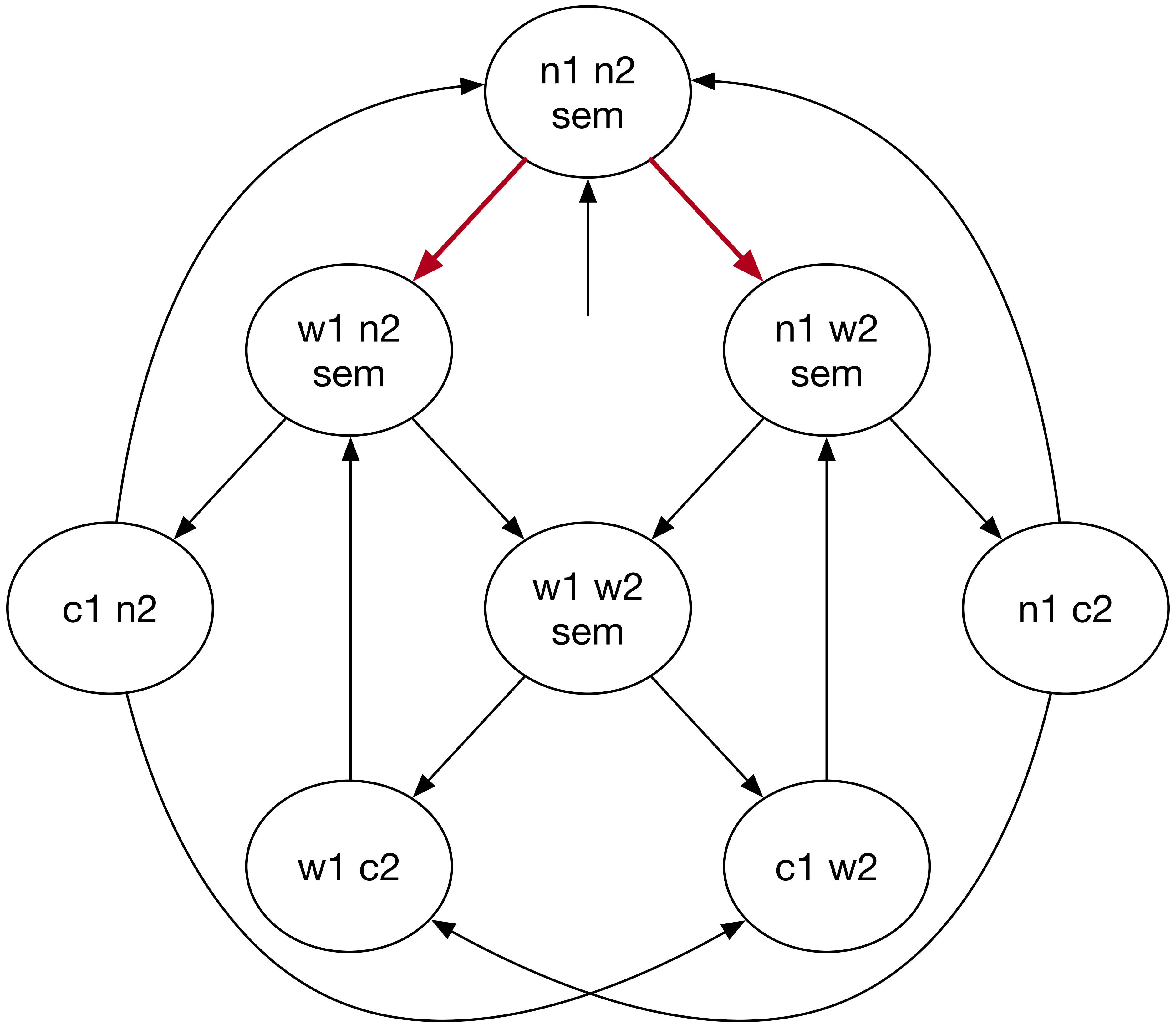
- Some properties can be checked by directly inspecting the Kripke structure (e.g., absence of deadlocks, or simple safety properties)
- However, to check more complex behaviours, relating multiple states, we need general-purpose **temporal logics**
- Standard temporal logics are **state-oriented** and **propositional**: they only consider the atomic propositions that are true in each state
- Two alternative models of time, supported by different logics:
    - **Linear Time**, the behaviour of a system is the set of all paths
    - **Branching Time**, the behaviour of the system is a set of computation trees

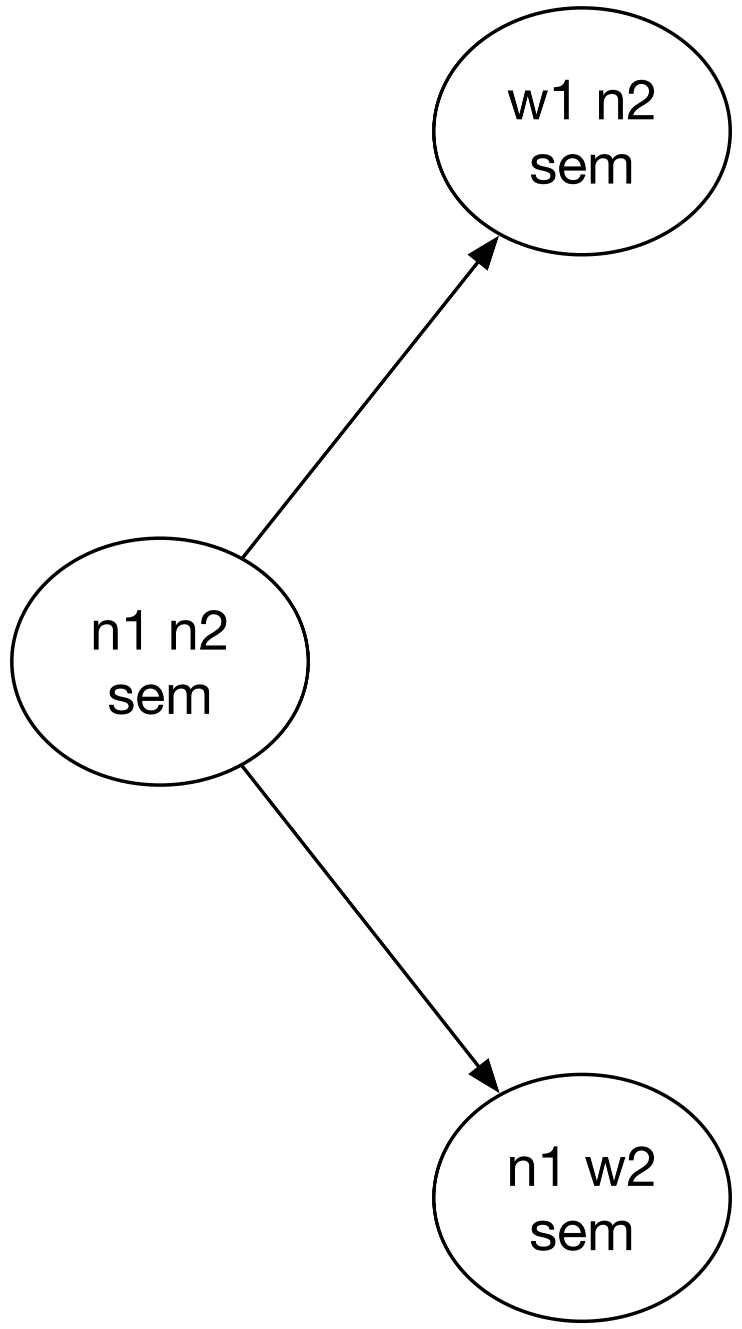
Branching Time



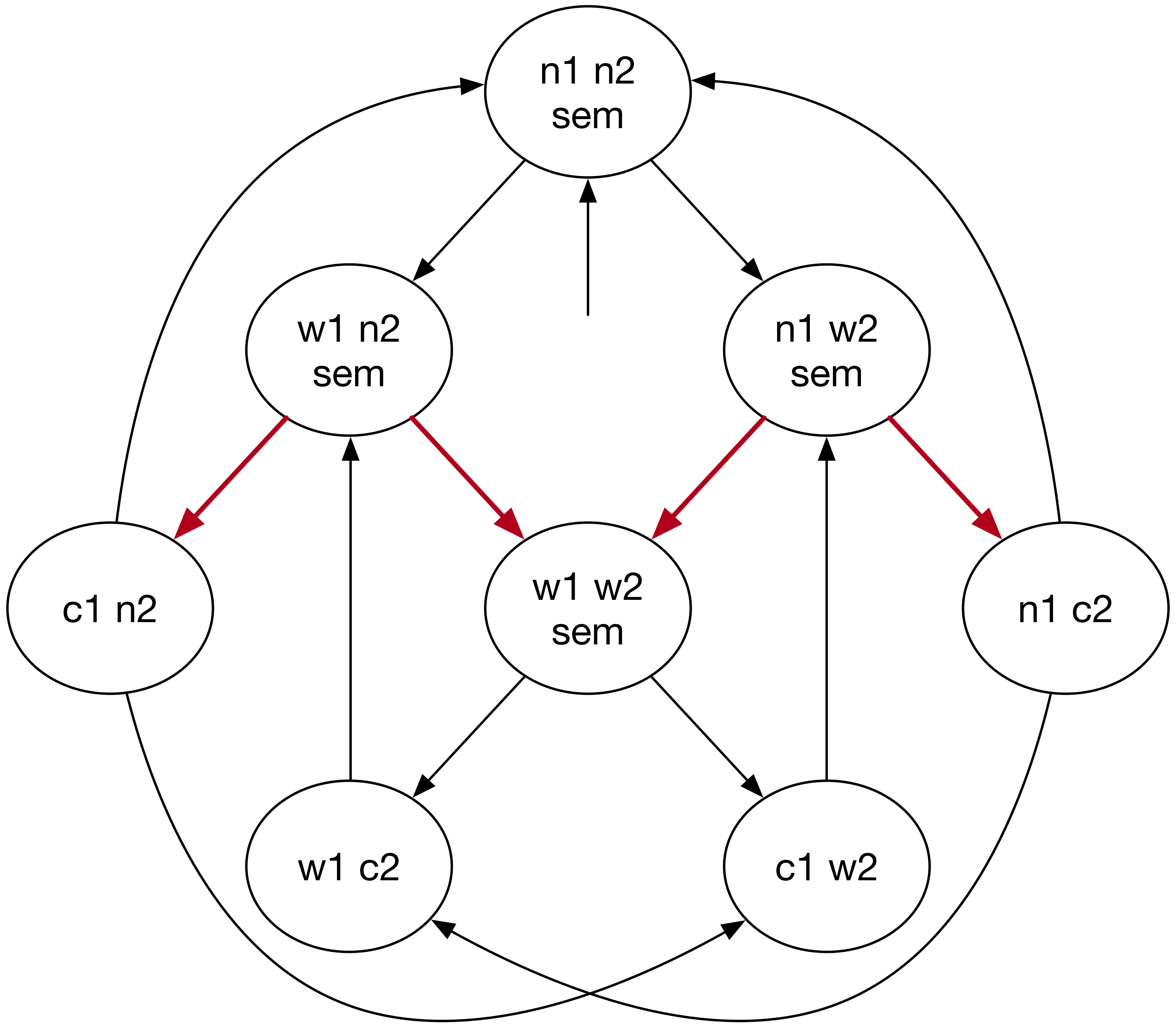


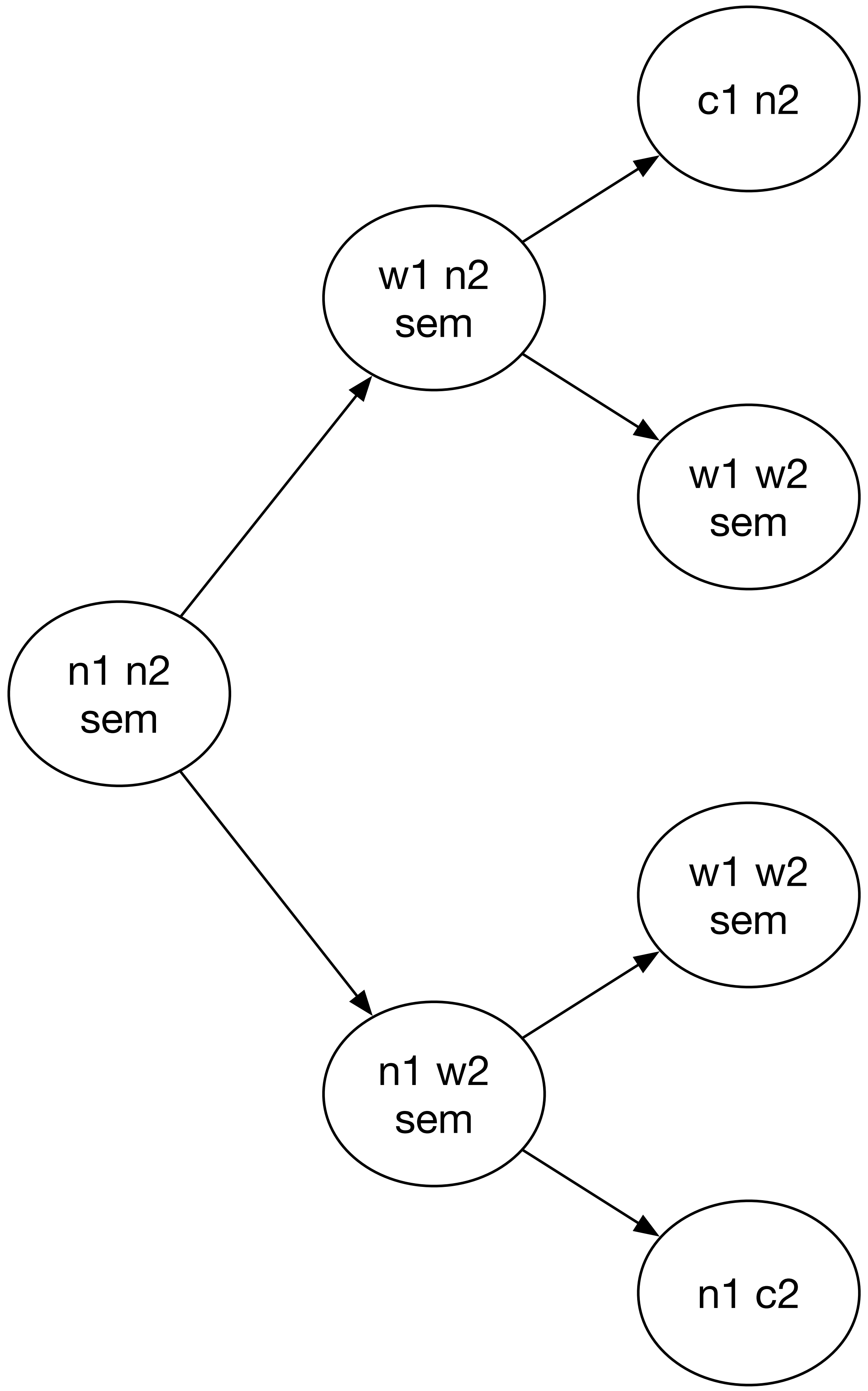
Branching Time



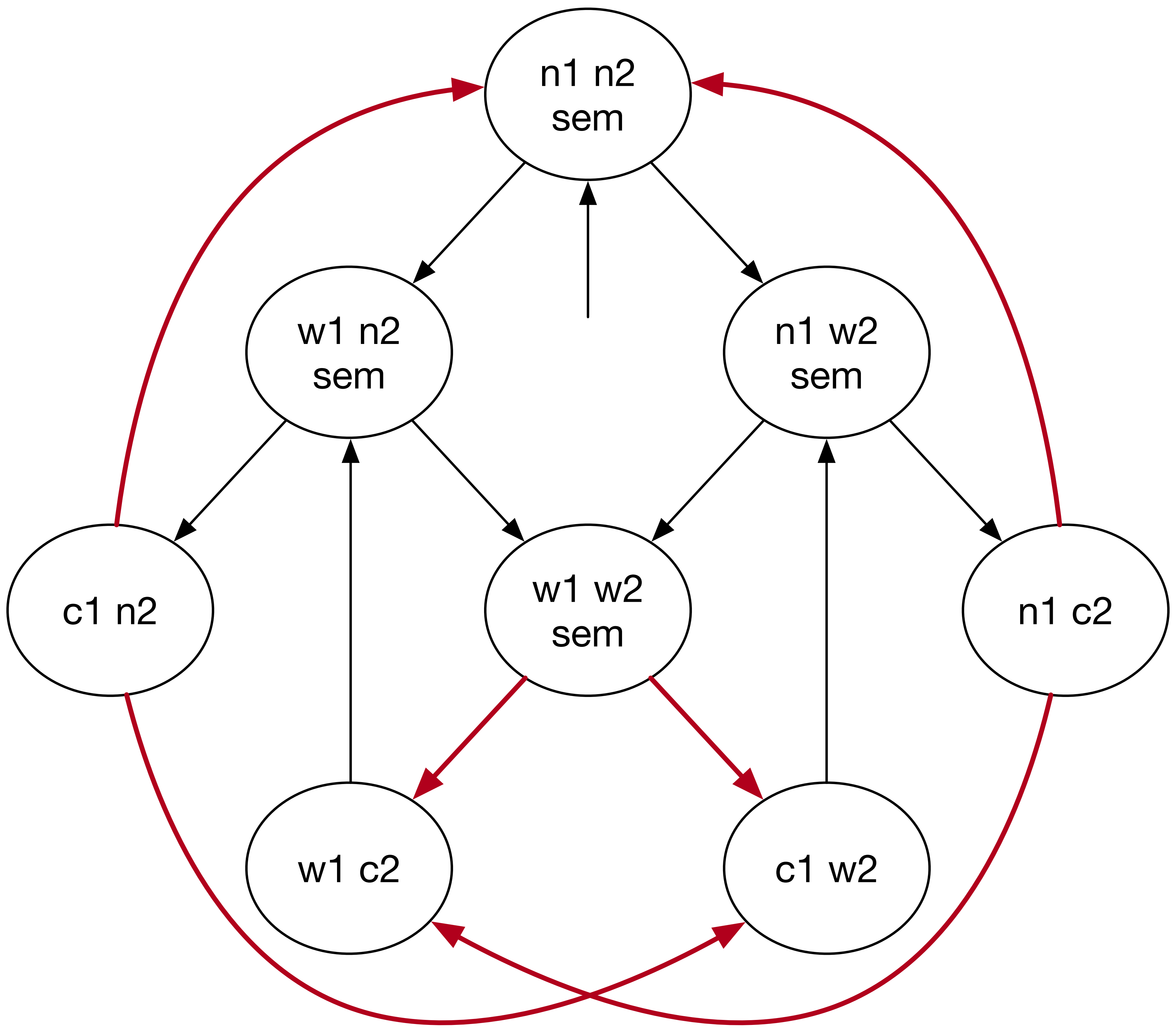


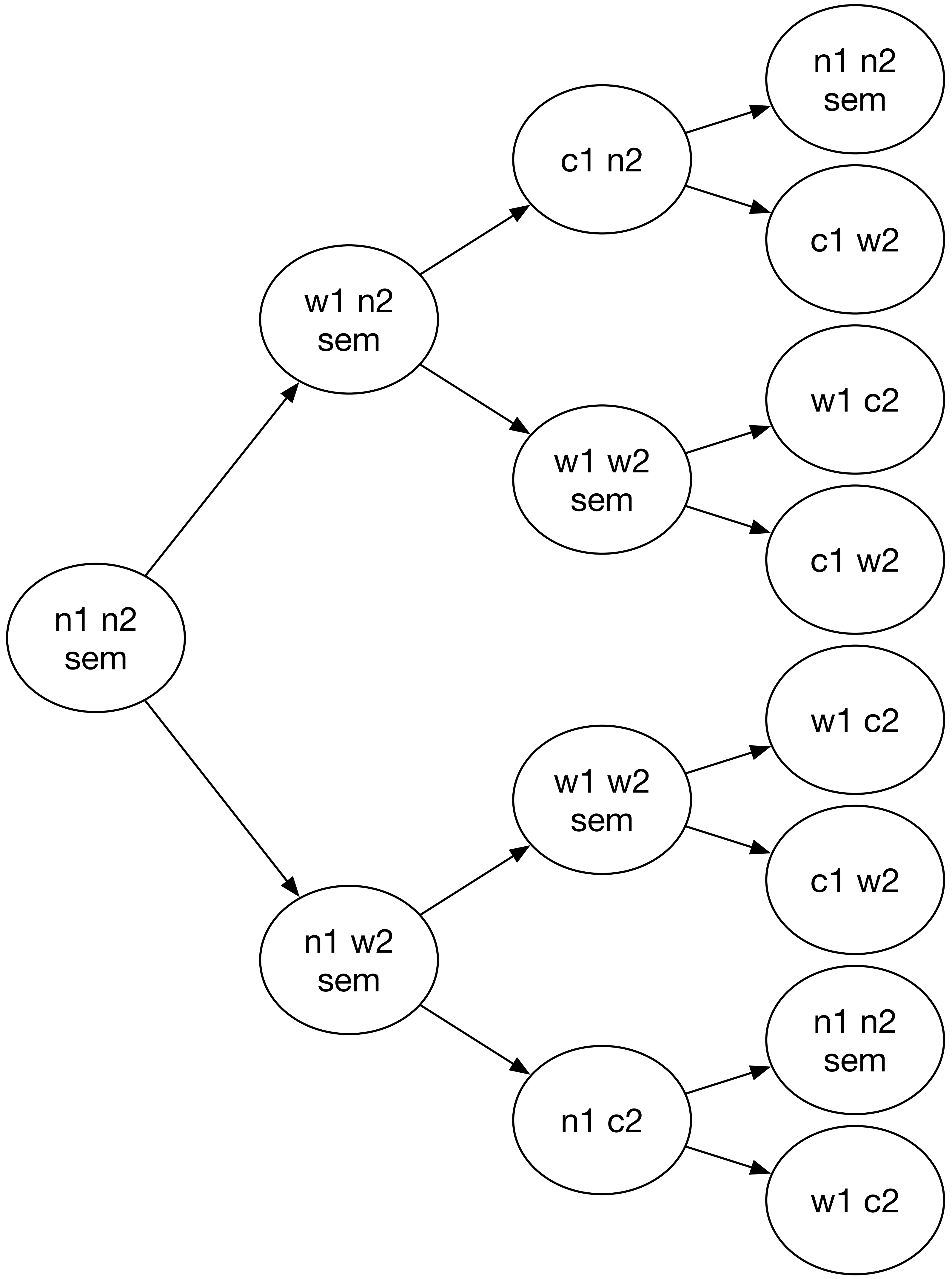
Branching Time



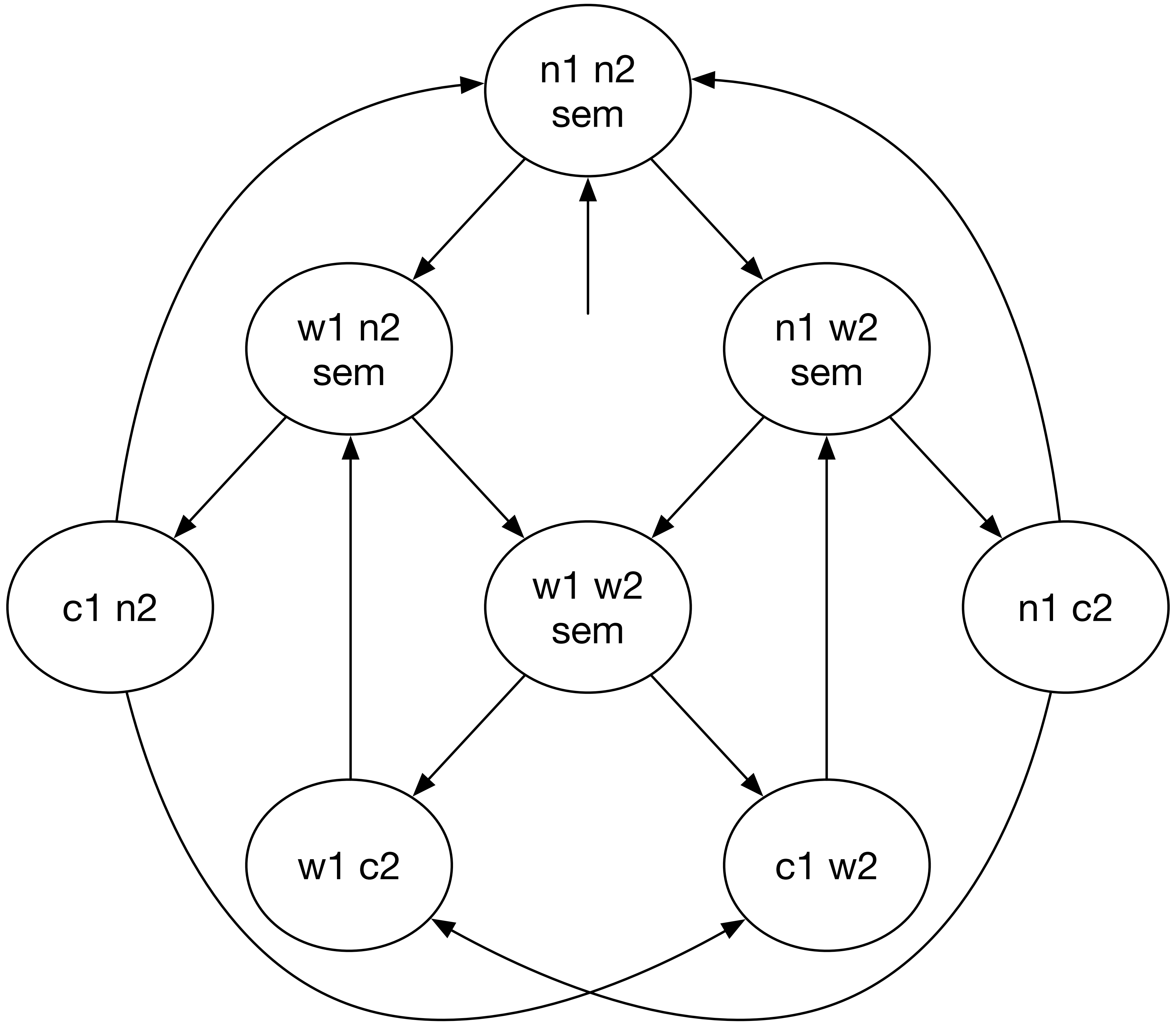


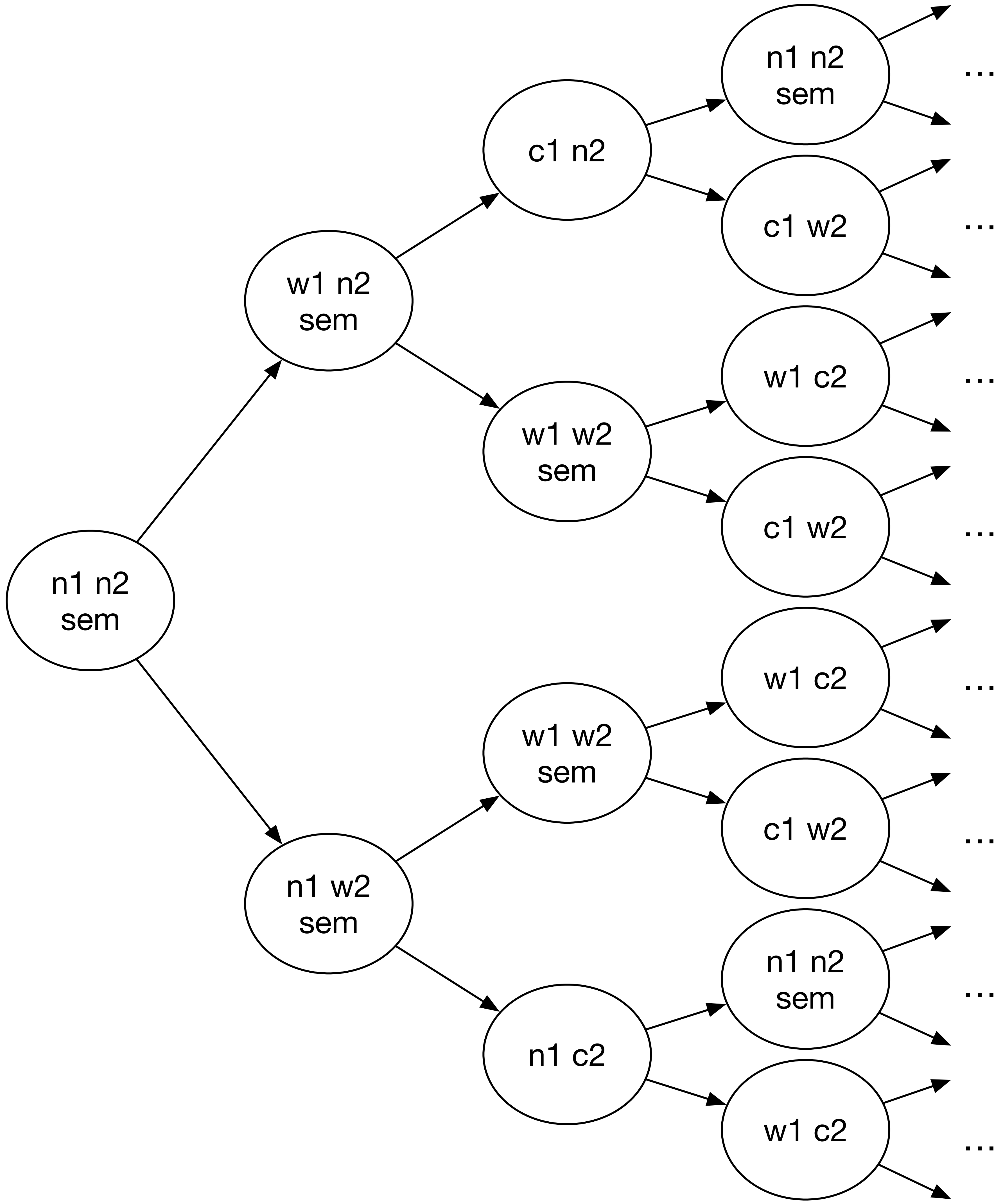
Branching Time





Branching Time





CTL

- **Computational Tree Logic** (CTL) is a branching time logic
- Besides the logical operators, CTL also has:
  - Path quantifiers
  - Temporal operators

CTL Syntax

- **X** (or $\circ$), **G** (or $\square$), **F** (or $\diamond$) and **U** are temporal operators:
  - **X**ϕ — ϕ holds in the ne**X**t state
  - **G**ϕ — ϕ always (or **G**lobally) holds
  - **F**ϕ — ϕ eventually (or in the **F**uture) holds
  - ϕ**U**ψ — ψ eventually holds and ϕ holds **U**ntil then
- **E** (or $\exists$) **A** (or $\forall$) are path quantifiers:
  - **A**ϕ — ϕ holds in **all** computation paths starting in the current state
  - **E**ϕ — ϕ holds in **some** computation path starting in the current state
- Path quantifiers must always be followed by a temporal operator (path formula)

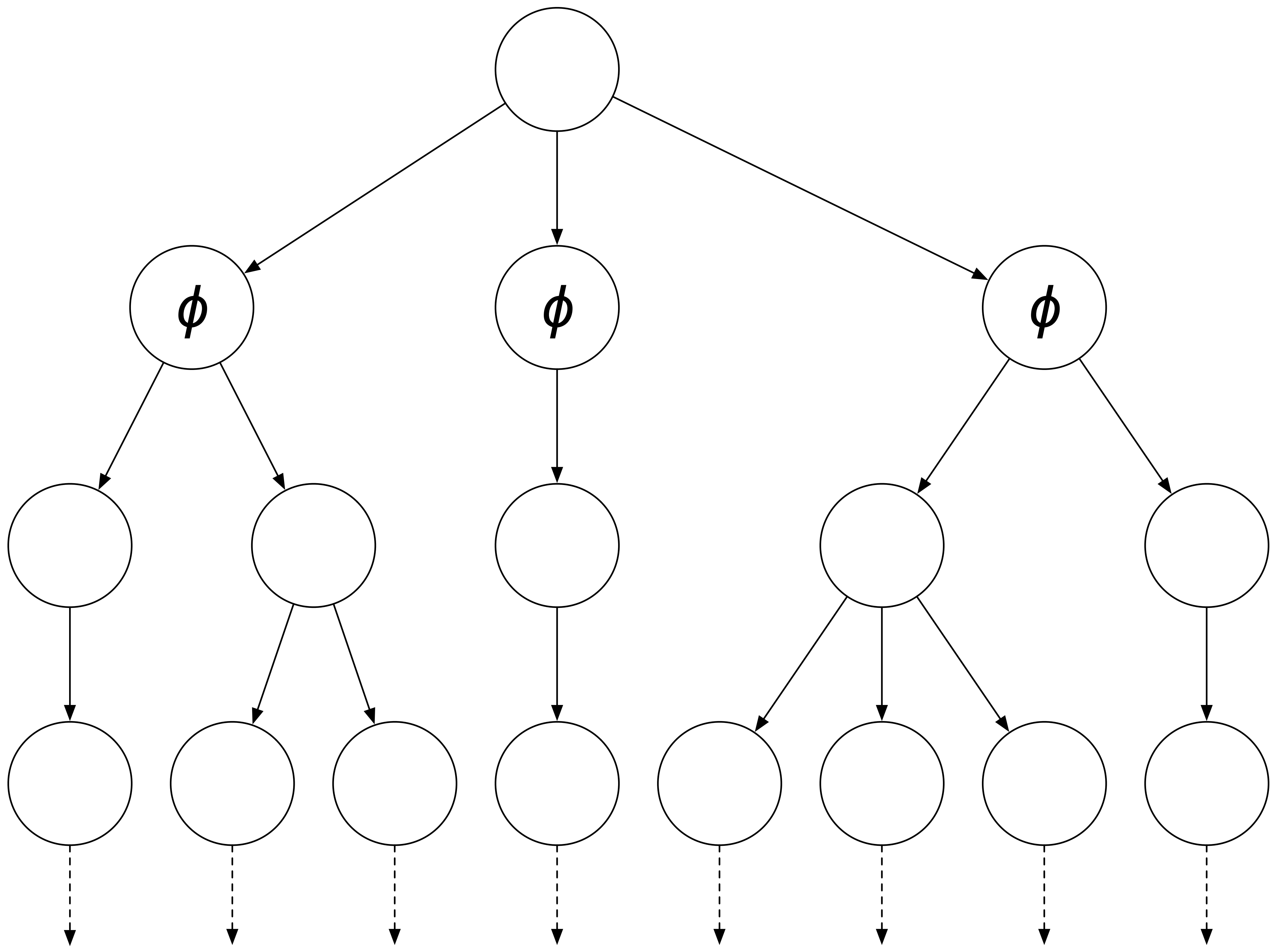
CTL Syntax

$$
\begin{aligned}
\phi, \psi ::= \;& \mathbf{AX}\,\phi \mid \mathbf{EX}\,\phi \\
\mid\;& \mathbf{AG}\,\phi \mid \mathbf{EG}\,\phi \\
\mid\;& \mathbf{AF}\,\phi \mid \mathbf{EF}\,\phi \\
\mid\;& \mathbf{A}[\phi\;\mathbf{U}\;\psi] \mid \mathbf{E}[\phi\;\mathbf{U}\;\psi] \\
\mid\;& p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \neg\phi \mid \top \mid \bot
\end{aligned}
$$

with $p \in P$ atomic propositions

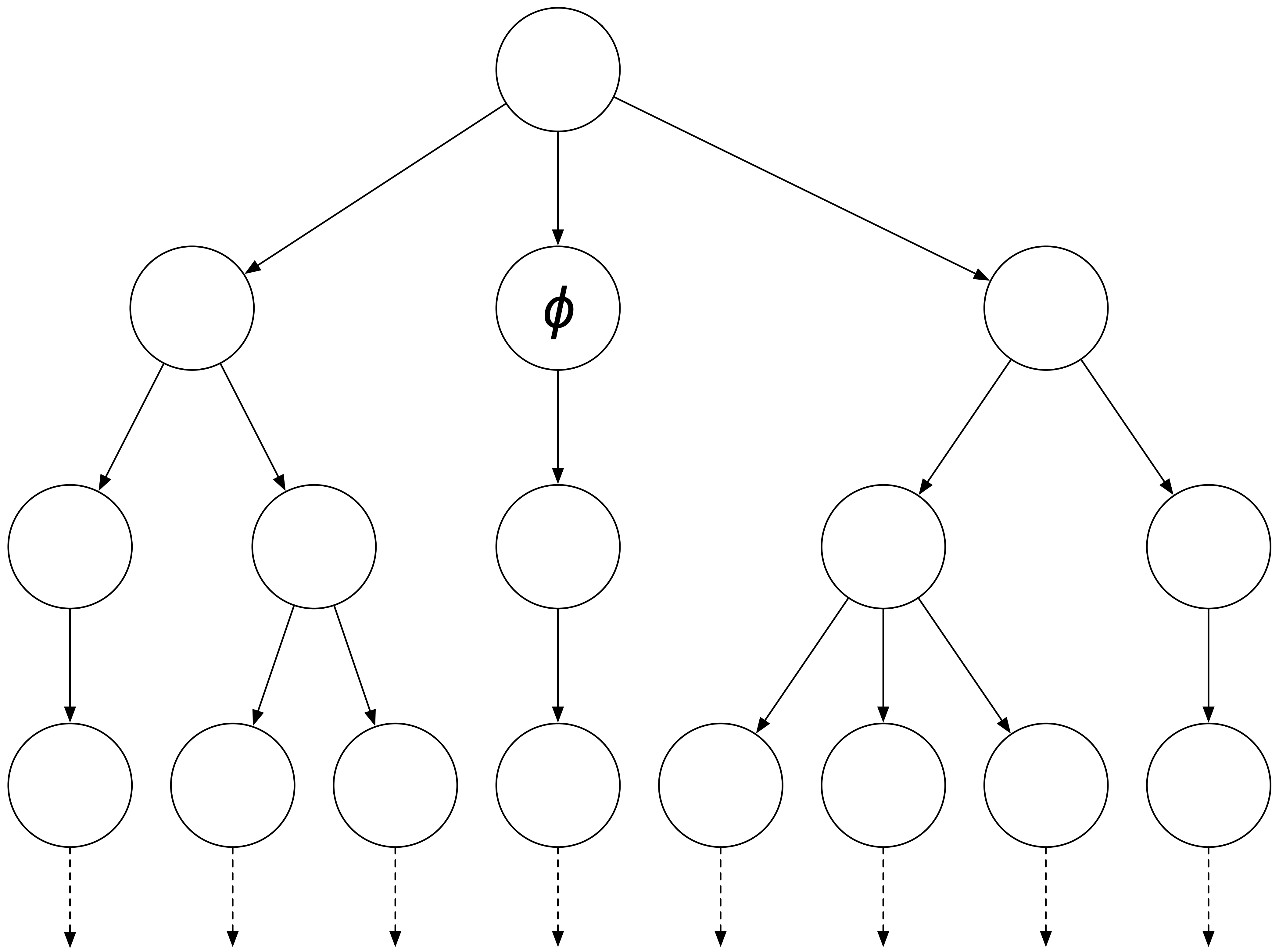
CTL Semantics

AX ϕ



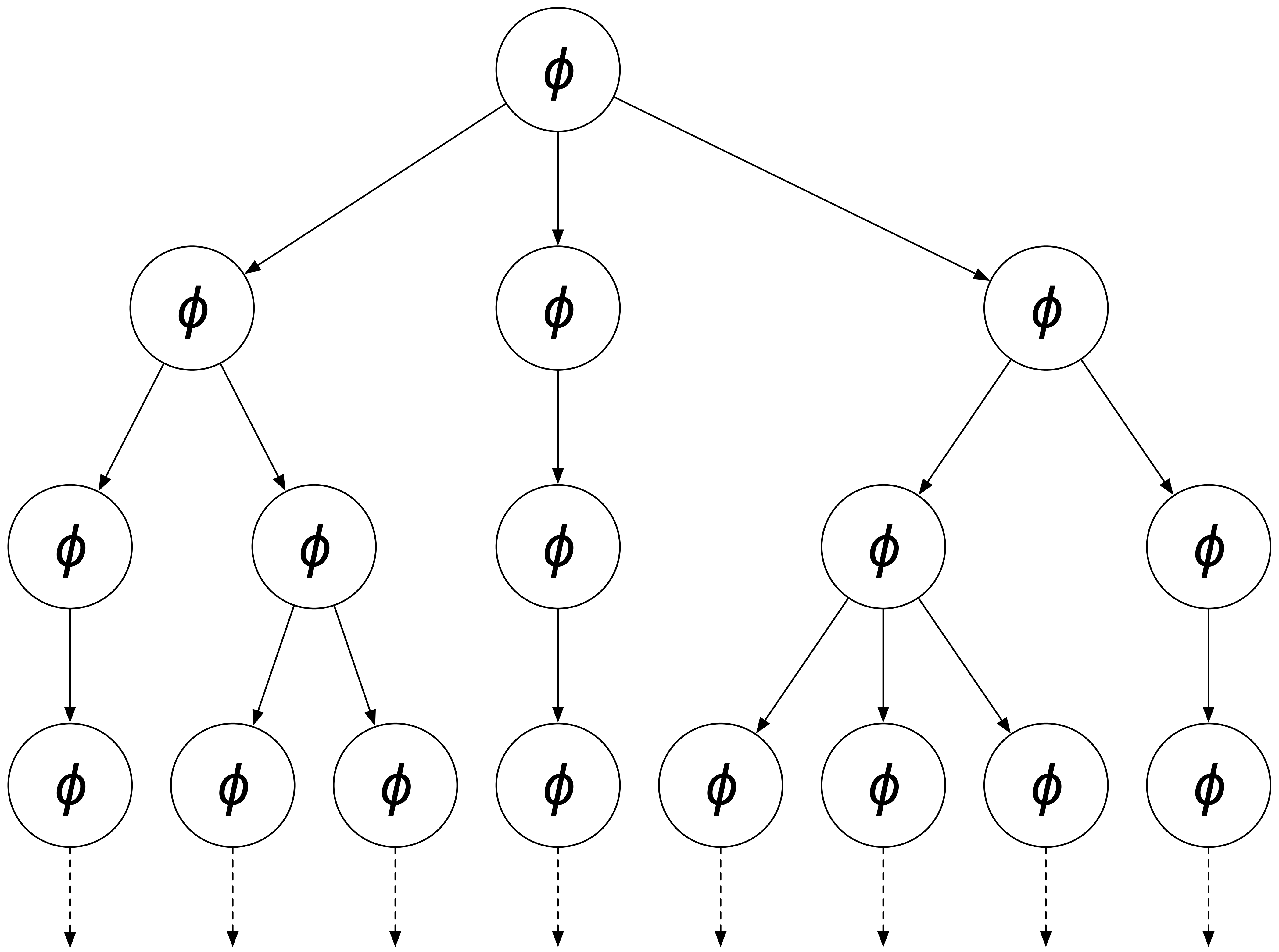
CTL Semantics

AF ϕ



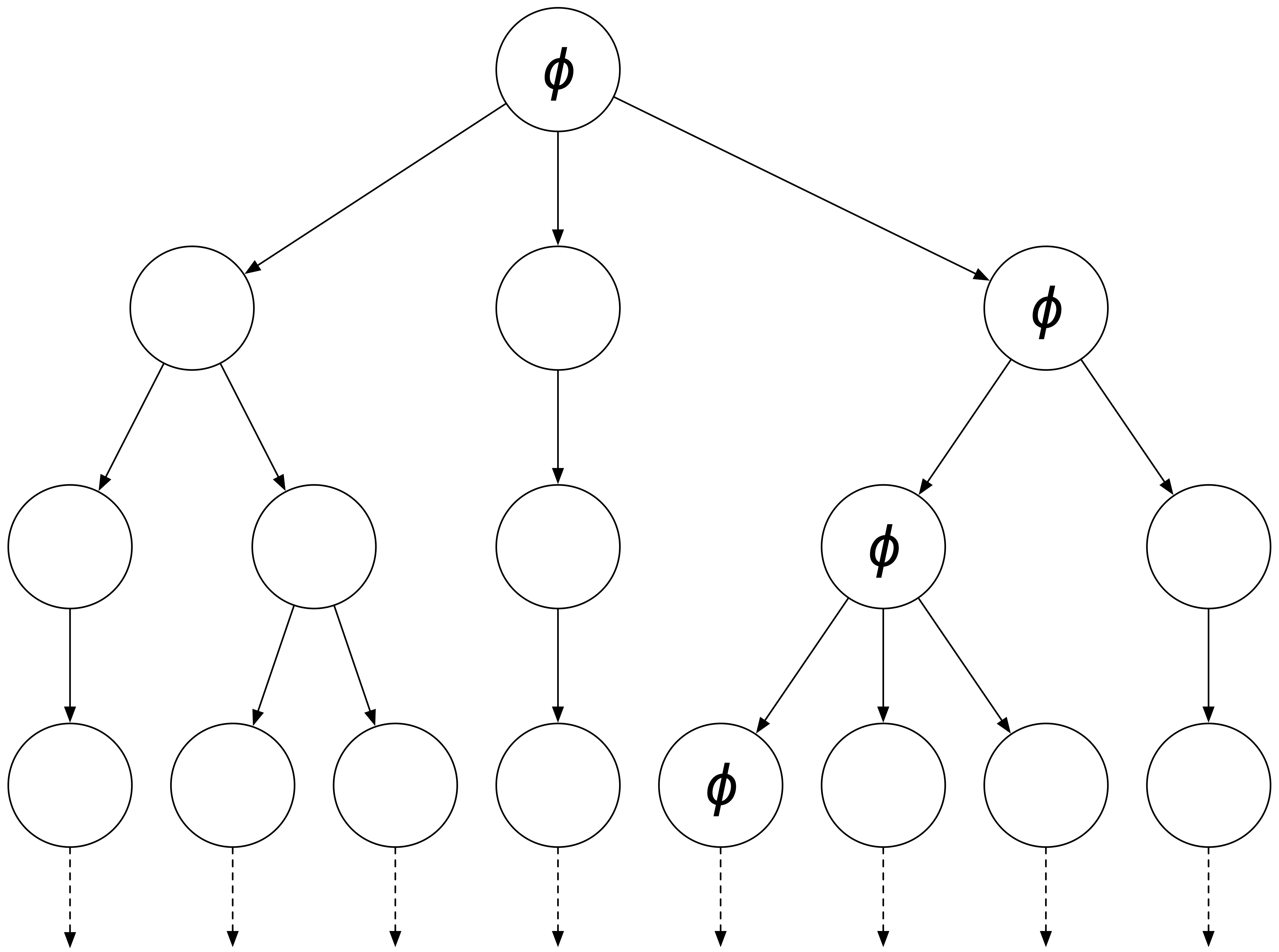
CTL Semantics

AG ϕ



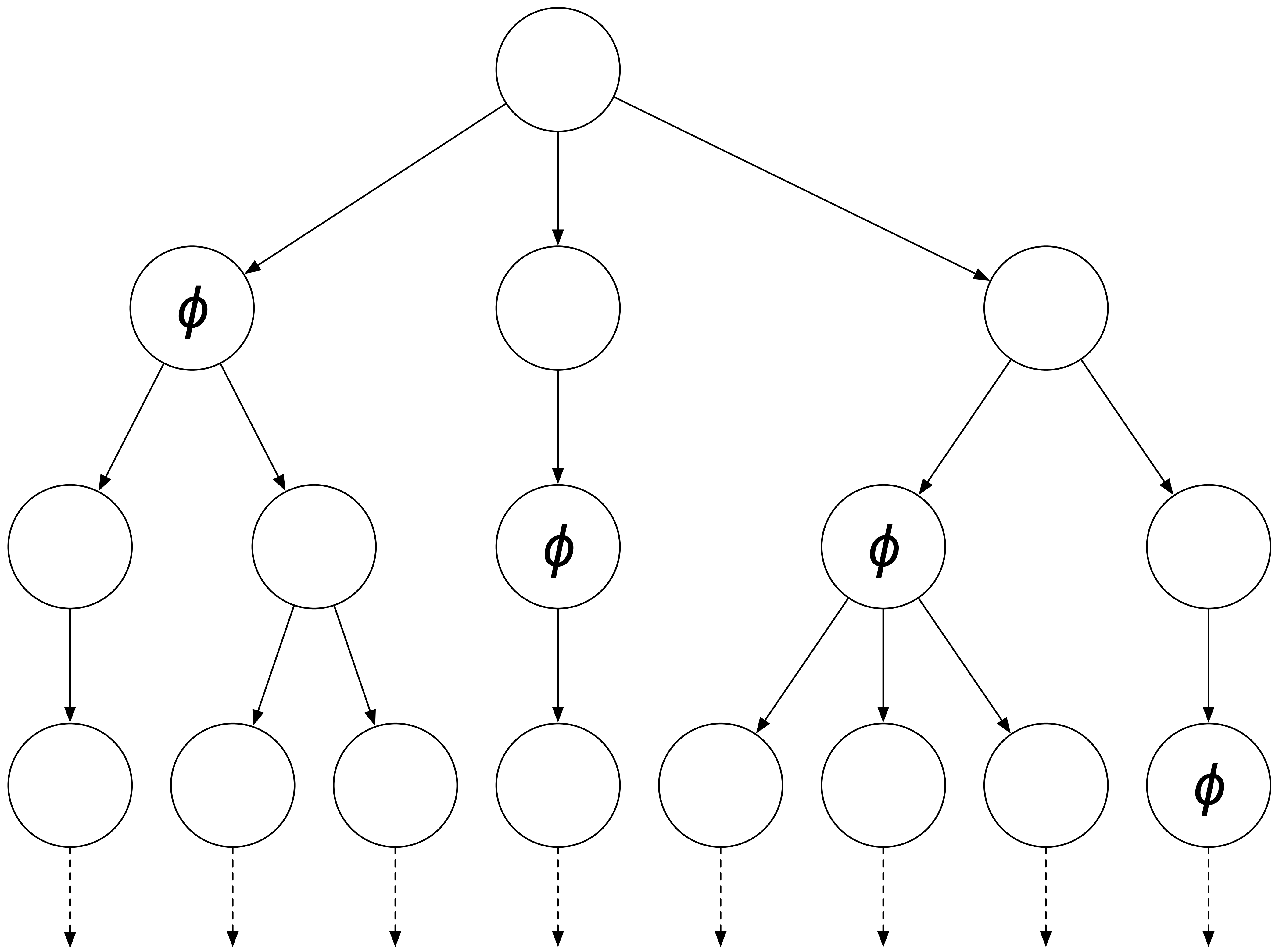
CTL Semantics

EG ϕ



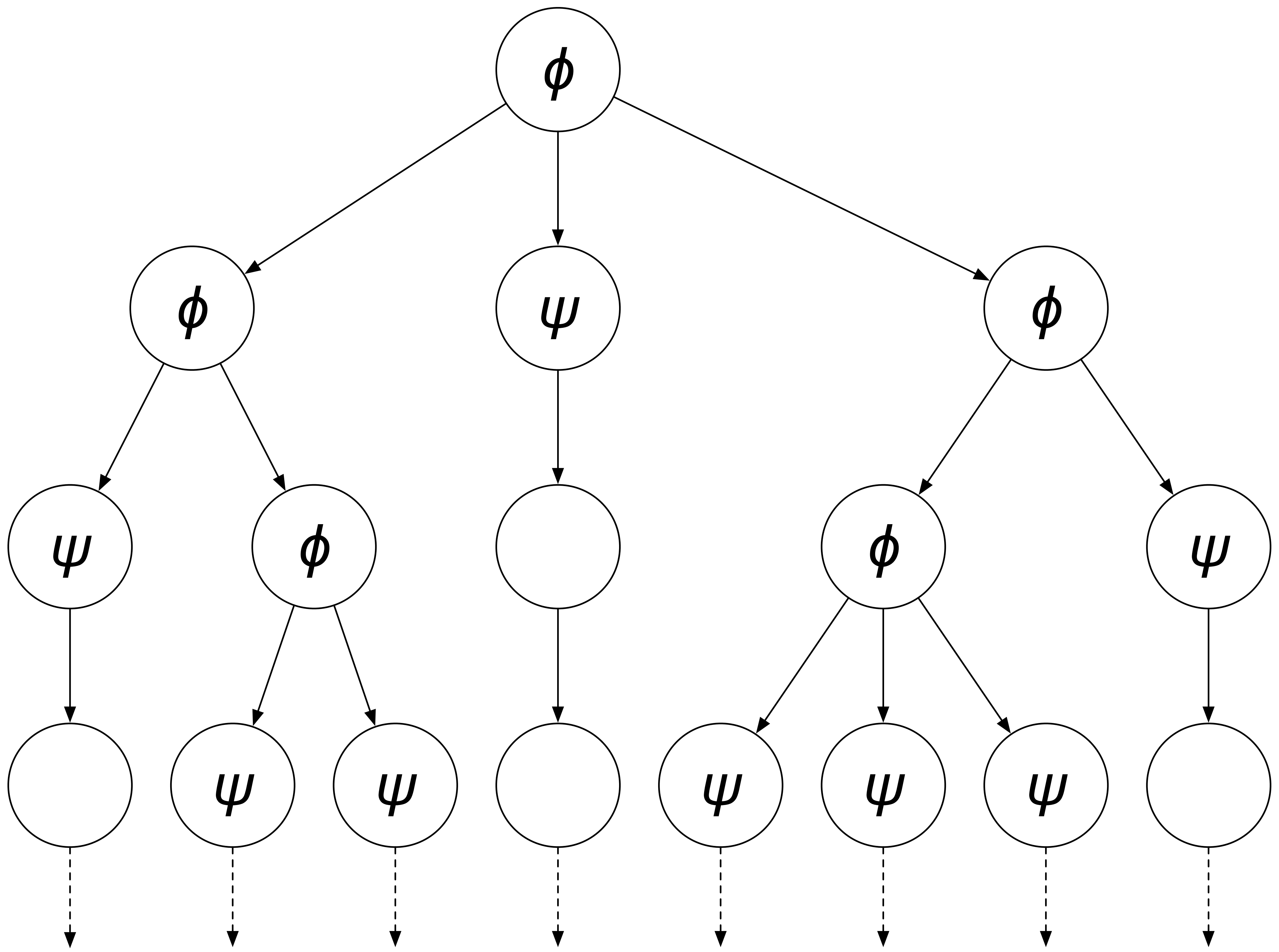
CTL Semantics

AF ϕ



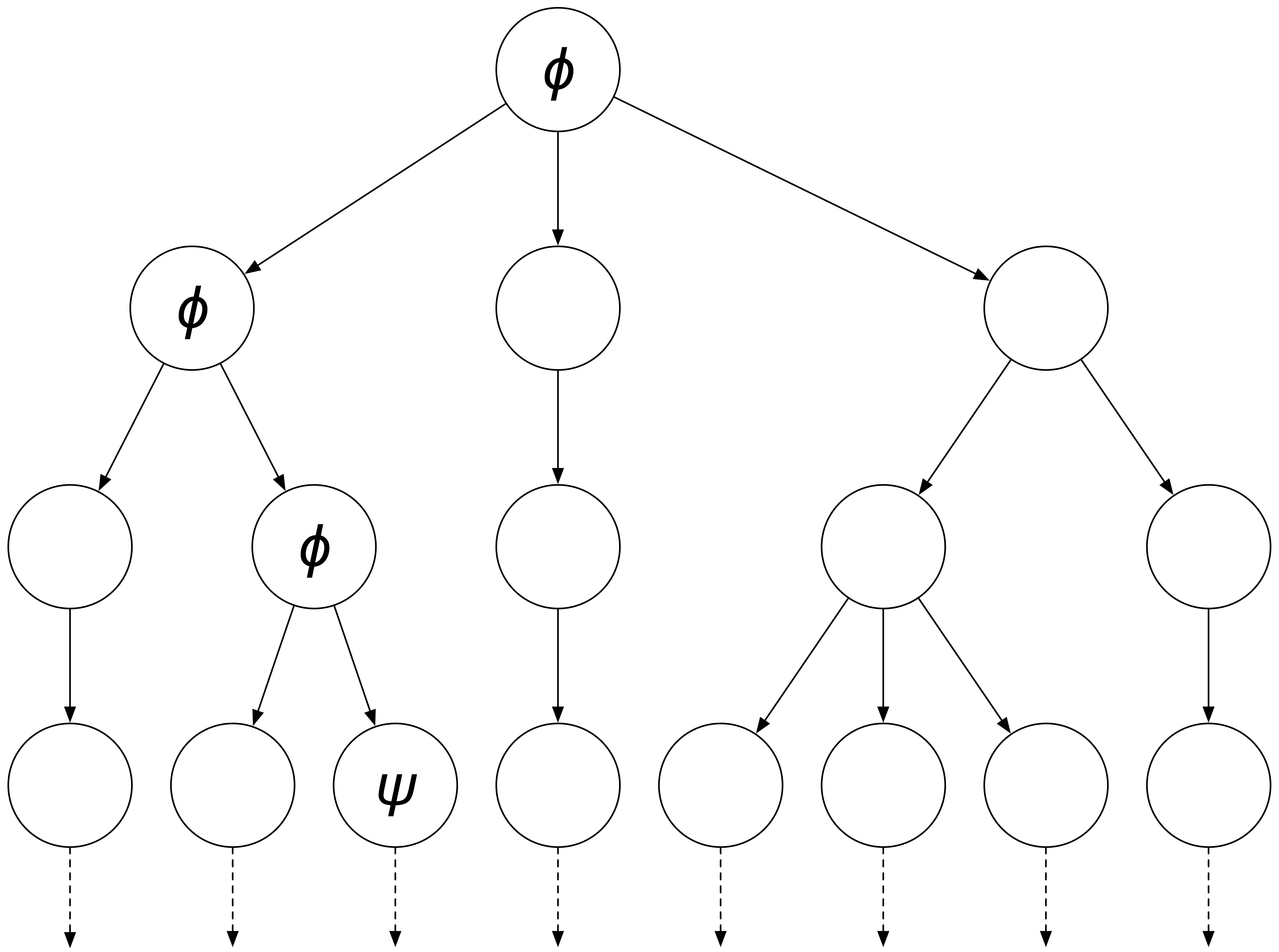
CTL Semantics

A[ϕ**U**ψ]



CTL Semantics

E[ϕ**U**ψ]



CTL Semantics

- If a CTL formula ϕ holds for a Kripke structure $M = (S,I,R,L)$ we say

$$M \vDash \phi$$

- $M \vDash \phi$ iff for all initial states $s \in I$ we have $M,s \vDash \phi$
- Minimal CTL subset: $\top$, $\vee$, $\neg$, **EG**, **EU**, **EX**
    - $\mathbf{EF}\phi \equiv \mathbf{E}[\top\mathbf{U}\phi]$
    - $\mathbf{AX}\phi \equiv \neg\mathbf{EX}(\neg\phi)$
    - $\mathbf{AG}\phi \equiv \neg\mathbf{EF}(\neg\phi)$
    - $\mathbf{AF}\phi \equiv \neg\mathbf{EG}(\neg\phi)$
    - $\mathbf{A}[\phi\mathbf{U}\psi] \equiv \neg(\mathbf{E}[(\neg\psi)\mathbf{U}\neg(\phi\vee\psi)] \vee \mathbf{EG}(\neg\psi))$

CTL Semantics

$M,s \models p \quad \equiv p \in L(s)$

$M,s \models \top \quad \equiv \top$

$M,s \models \neg\phi \quad \equiv M,s \not\models \phi$

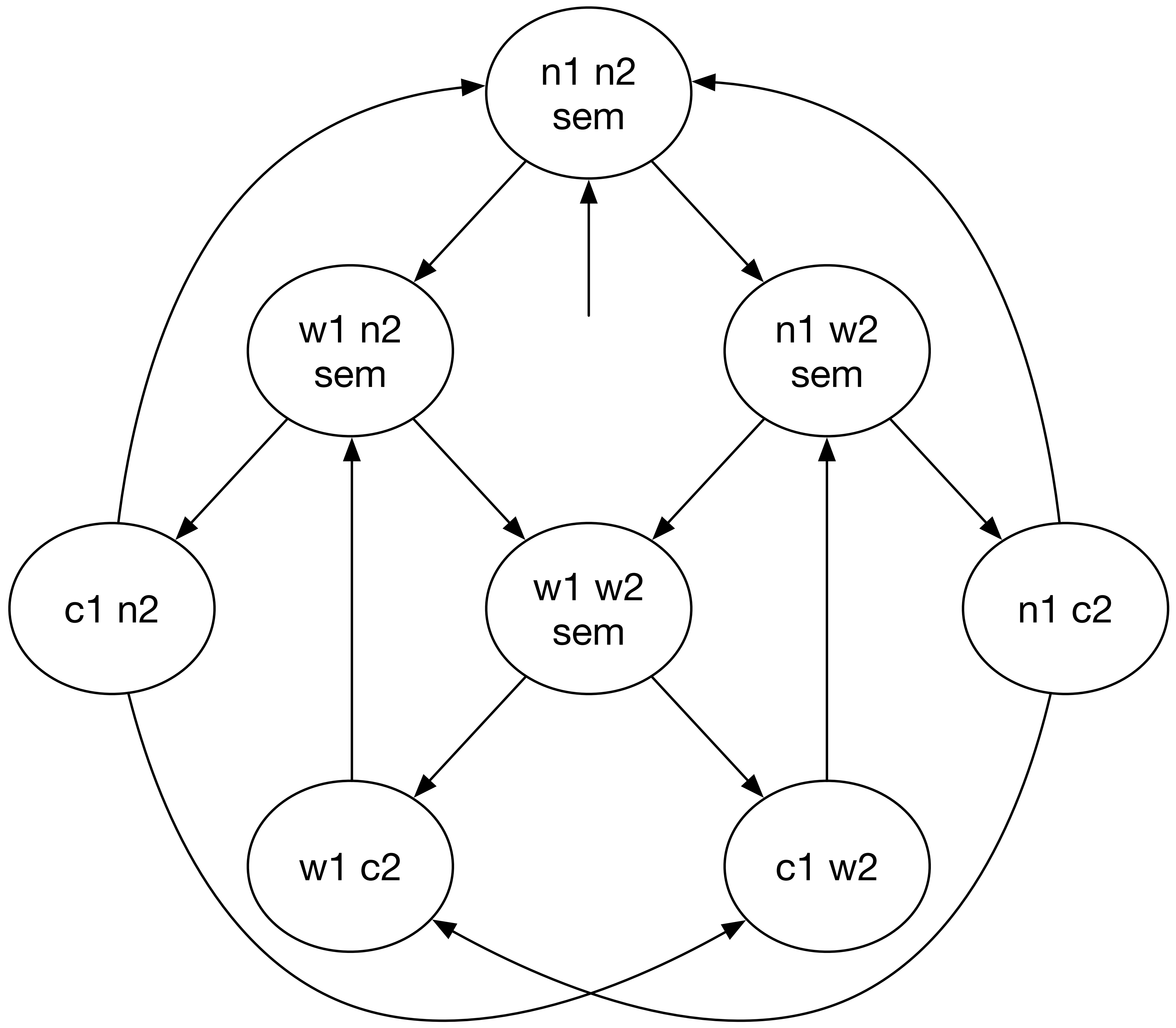
$M,s \models \phi \vee \psi \quad \equiv M,s \models \phi$ or $M,s \models \psi$

$M,s \models \mathbf{EX}\phi \quad \equiv \exists\pi \in M \,.\, \pi_0 = s$ and $M,\pi_1 \models \phi$

$M,s \models \mathbf{EG}\phi \quad \equiv \exists\pi \in M \,.\, \pi_0 = s$ and $\forall i{\geq}0 \,.\, M,\pi_i \models \phi$

$M,s \models \mathbf{E}[\phi\mathbf{U}\psi] \quad \equiv \exists\pi \in M \,.\, \pi_0 = s$ and $\exists i{\geq}0 \,.\, (M,\pi_i \models \psi$ and $\forall 0{\leq}j{<}i\ M \,.\, \pi_j \models \phi)$

CTL Examples

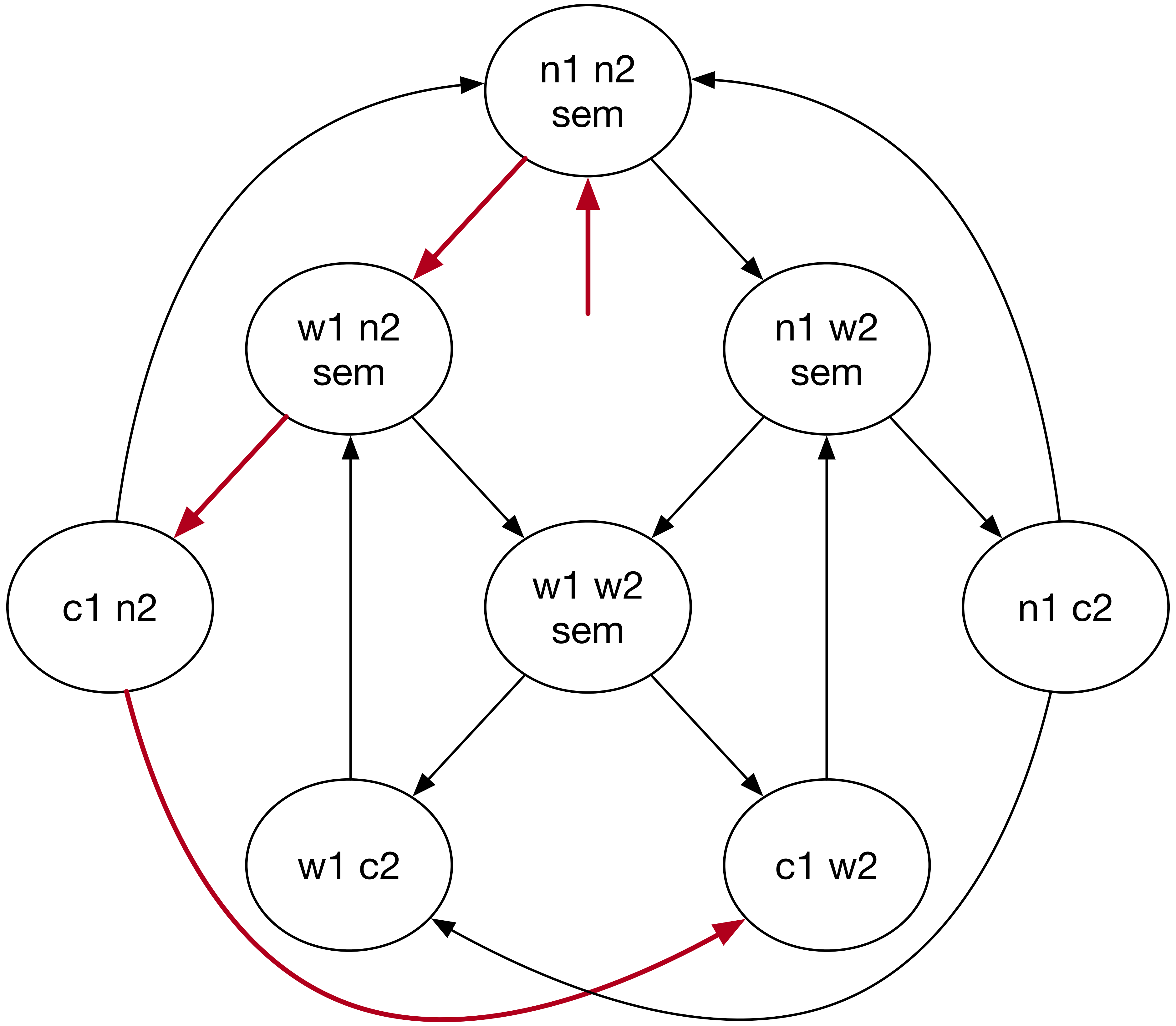


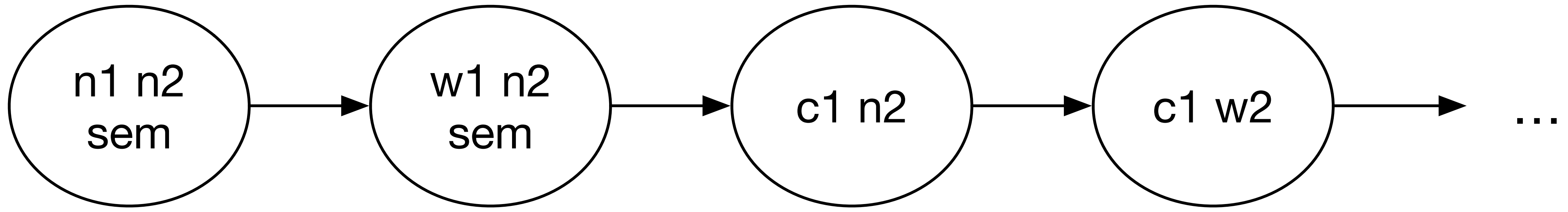
Mutual exclusion: AG ¬(c1 ∧ c2)

No starvation: AG (w1 → **AF** c1)

Reversibility: AG EF (n1 ∧ n2 ∧ sem)

Linear Time


n1 n2
sem
w1 n2
sem
n1 w2
sem
c1 n2
w1 w2
sem
n1 c2
w1 c2
c1 w2


n1 n2
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w1 n2
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c1 n2
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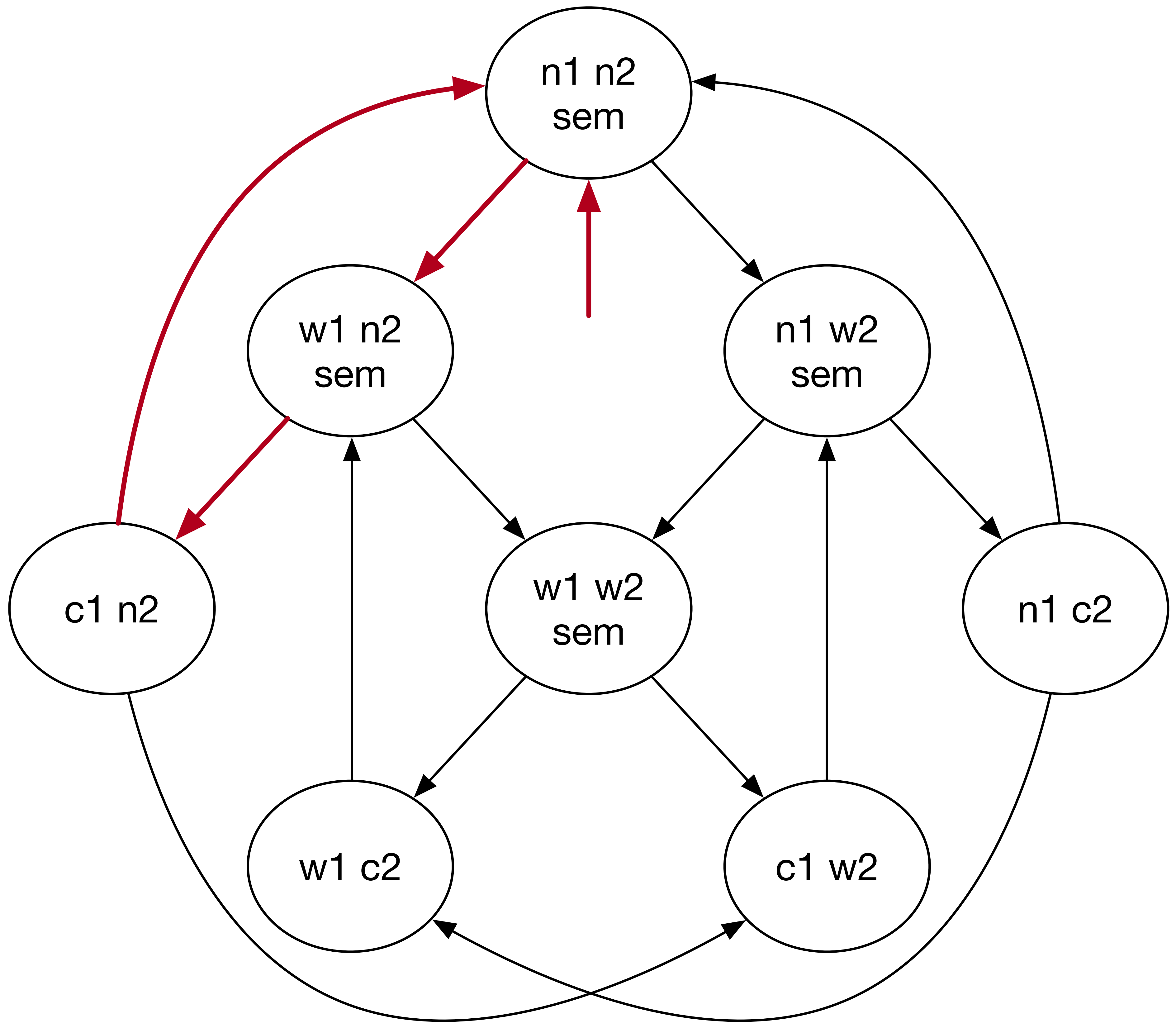
...

Linear Time



n1 n2
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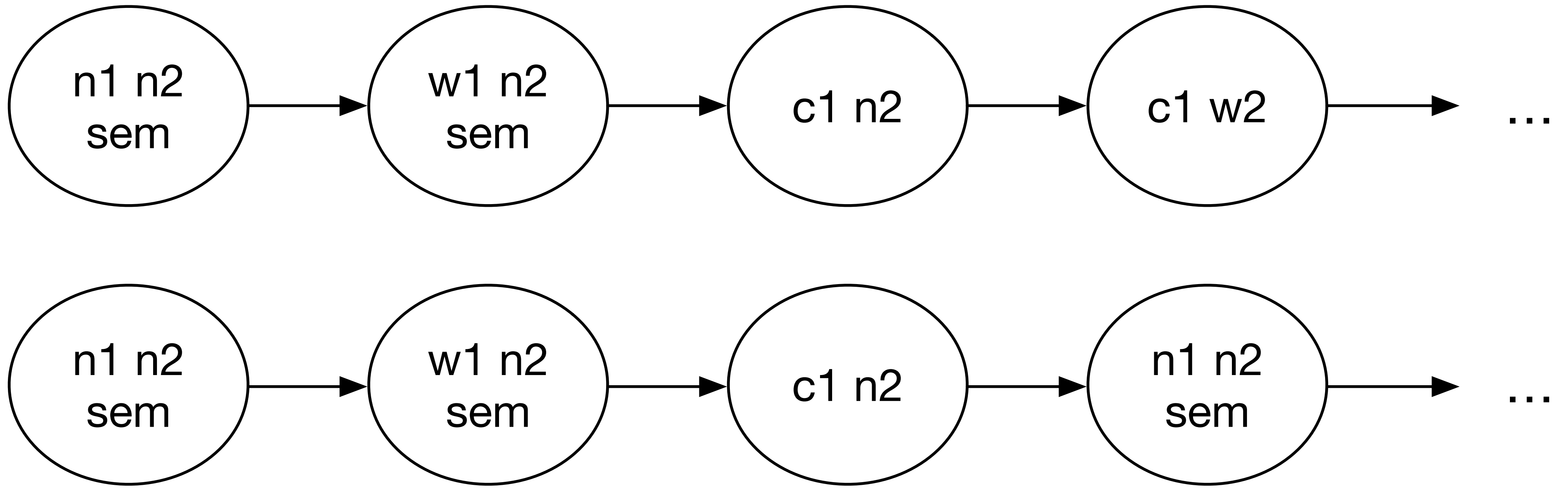
c1 n2
w1 w2
sem
n1 c2
w1 c2
c1 w2



n1 n2
sem
w1 n2
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c1 w2
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n1 n2
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w1 n2
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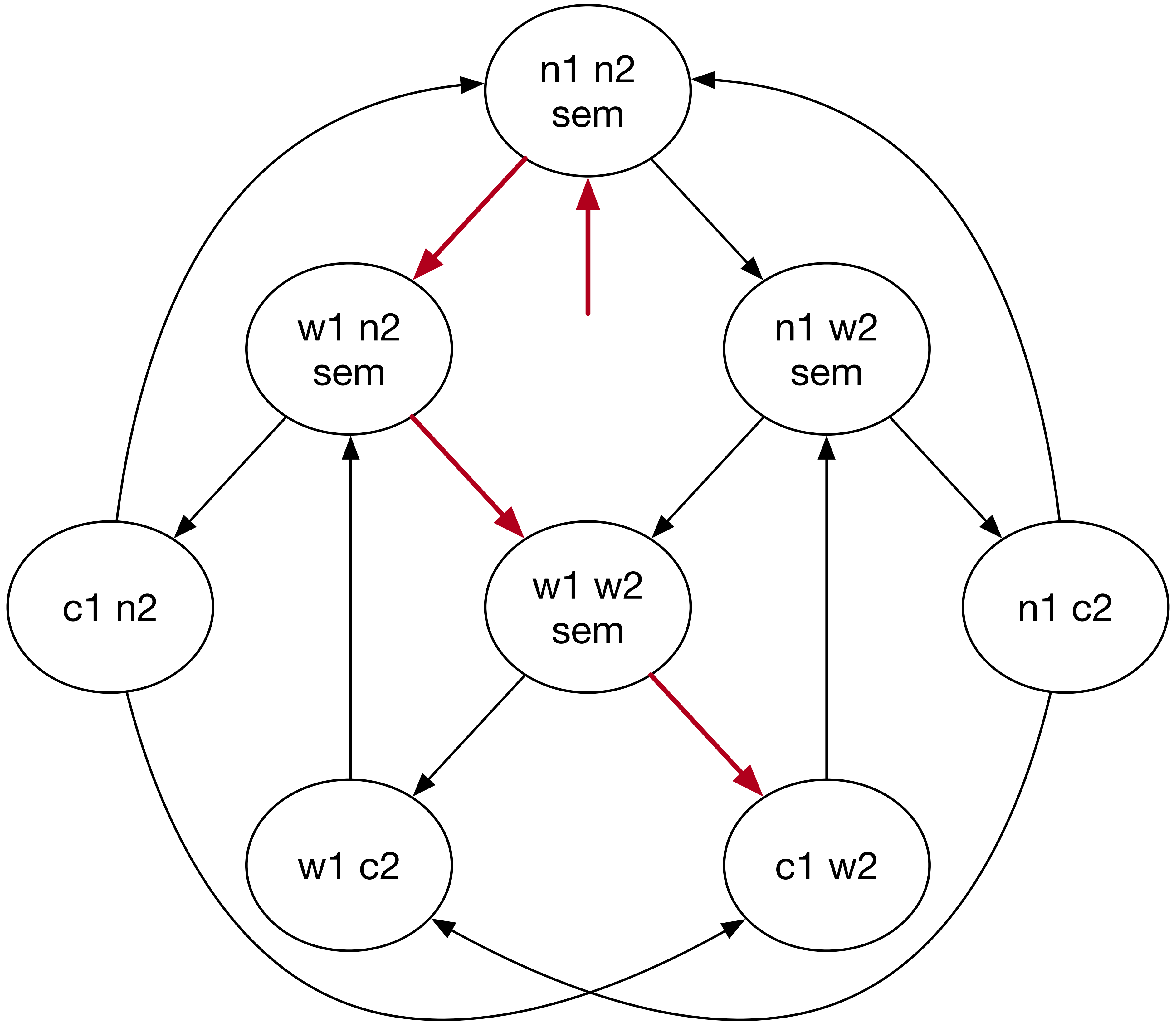
c1 n2
n1 n2
sem
...

Linear Time



n1 n2
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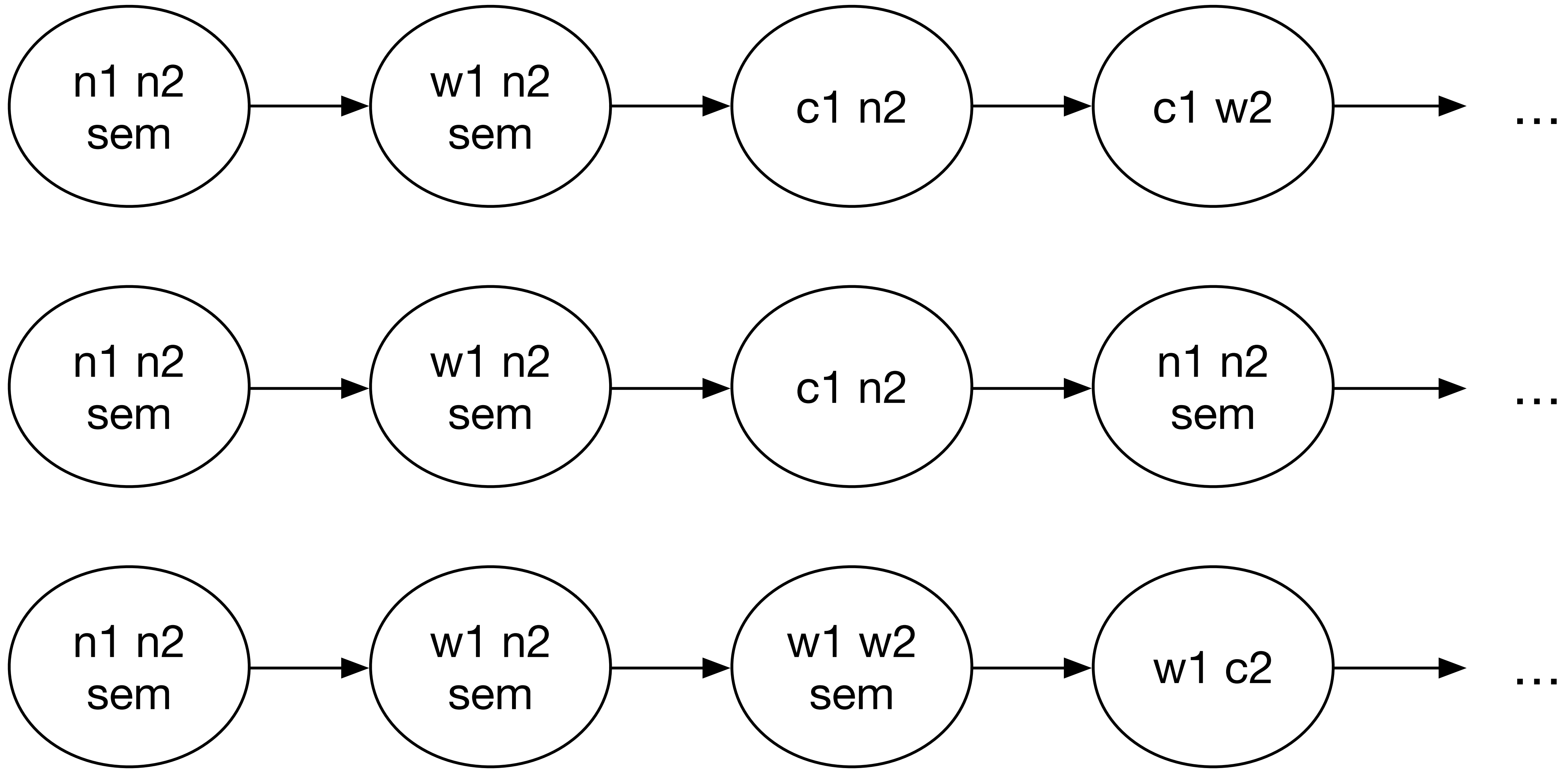
n1 c2
w1 c2
c1 w2



n1 n2
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n1 n2
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n1 n2
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w1 w2
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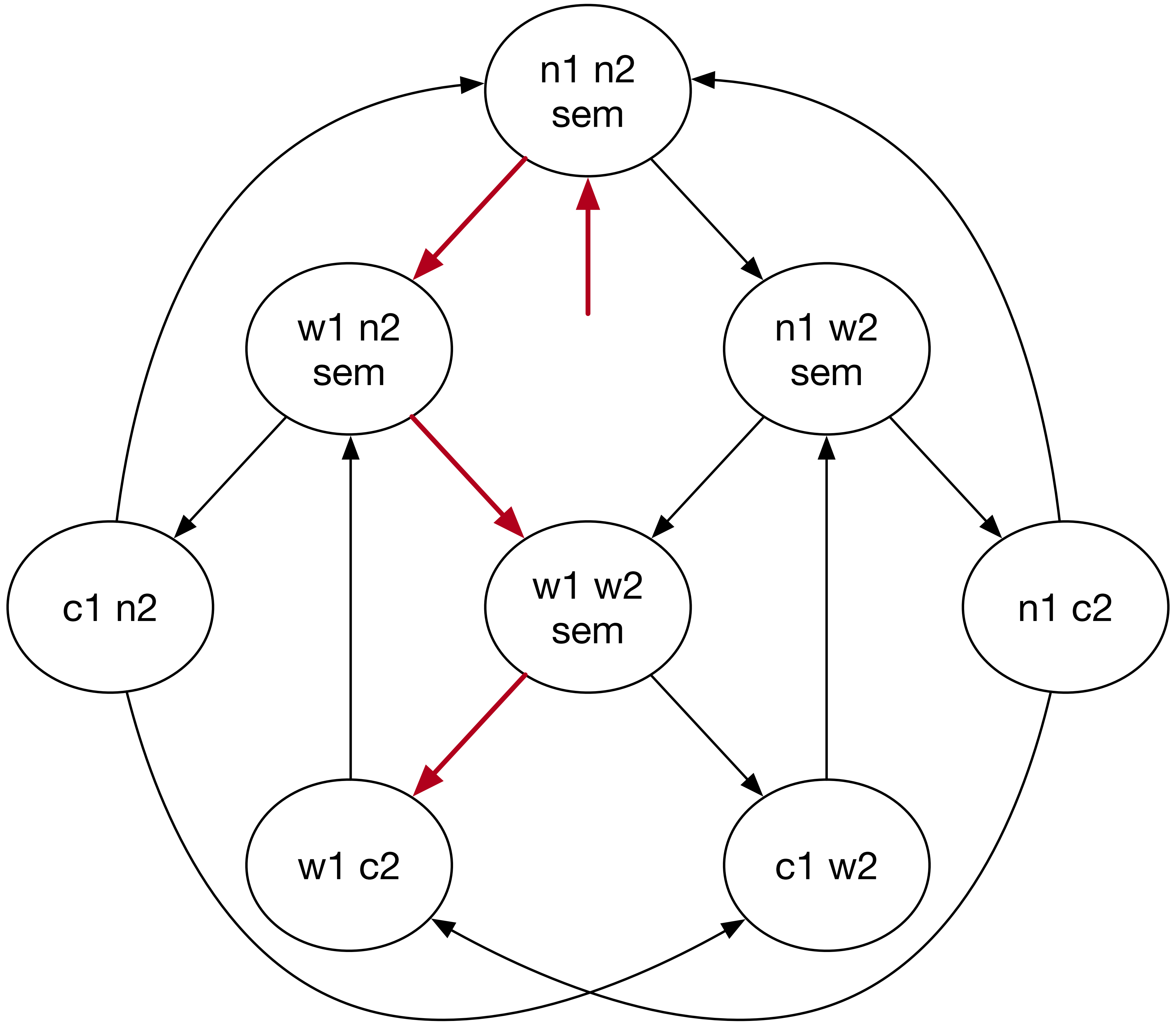
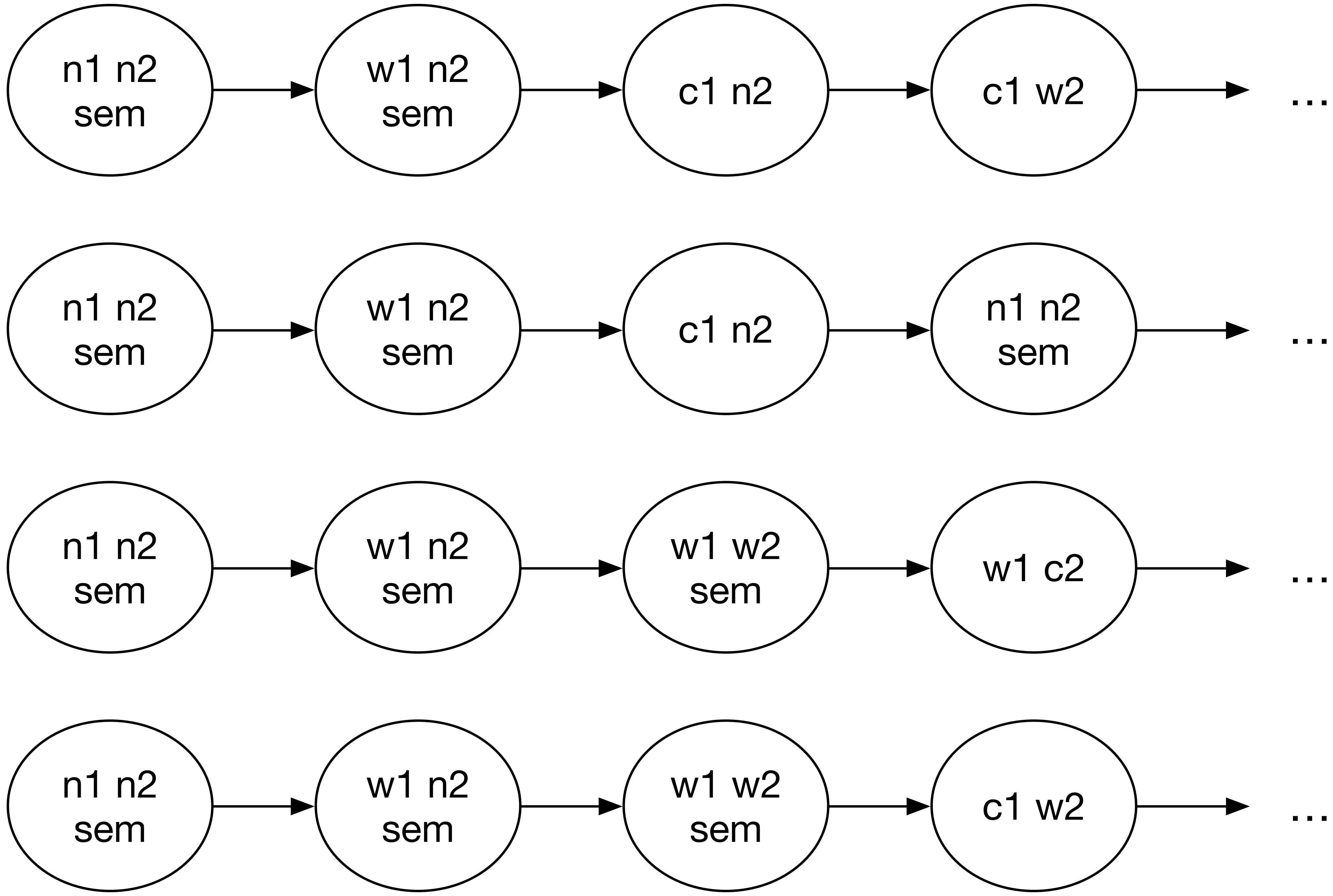
w1 c2
...

Linear Time


n1 n2 sem
w1 n2 sem
n1 w2 sem
c1 n2
w1 w2 sem
n1 c2
w1 c2
c1 w2


n1 n2 sem
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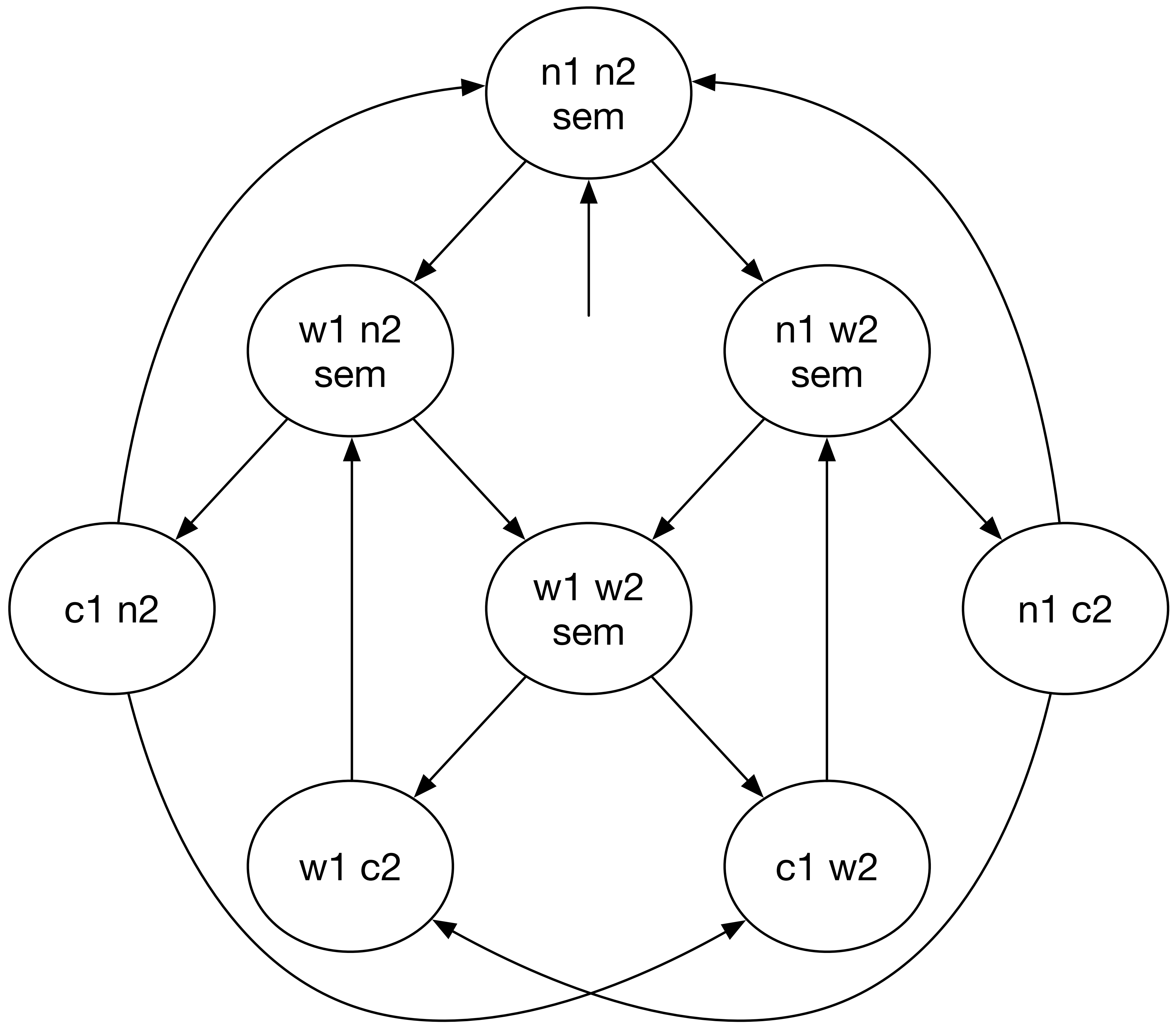
n1 n2 sem
w1 n2 sem
w1 w2 sem
w1 c2
…
n1 n2 sem
w1 n2 sem
w1 w2 sem
c1 w2
…

Linear Time


n1 n2
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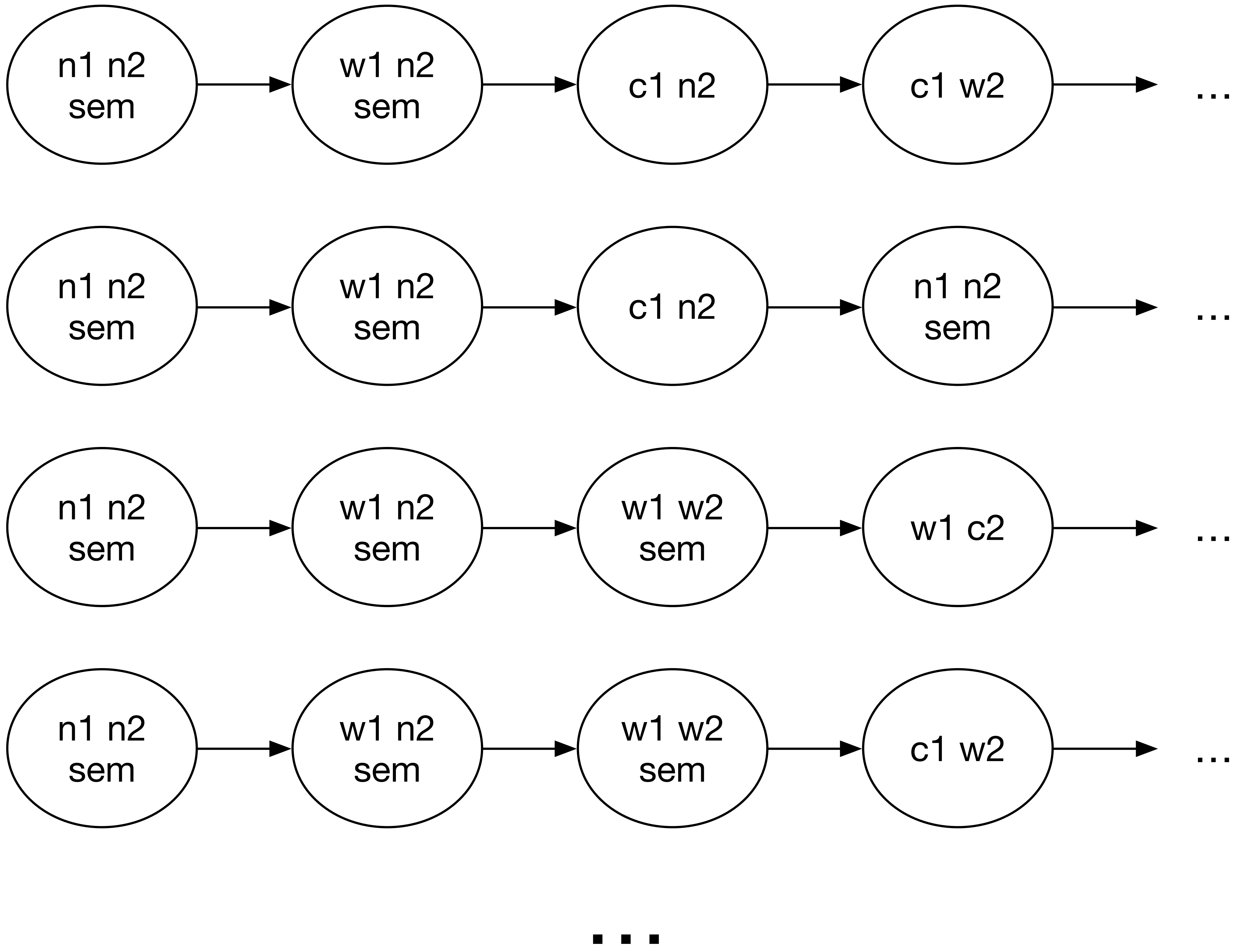
c1 w2


n1 n2
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c1 w2
…
n1 n2
sem
w1 n2
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c1 n2
n1 n2
sem
…
n1 n2
sem
w1 n2
sem
w1 w2
sem
w1 c2
…
n1 n2
sem
w1 n2
sem
w1 w2
sem
c1 w2
…
…

LTL

- **Linear Temporal Logic** (CTL) is a linear time logic
- LTL only has temporal operators

LTL Syntax

- **X** (or ○), **G** (or □), **F** (or ◇), **U** and **R** are temporal operators:
  - **X**ϕ — ϕ holds in the ne**X**t state
  - **G**ϕ — ϕ always (or **G**lobally) holds
  - **F**ϕ — ϕ eventually (or in the **F**uture) holds
  - ϕ**U**ψ — ψ eventually holds and ϕ holds **U**ntil then
  - ϕ**R**ψ — ψ always holds or until and ϕ holds (or **R**eleases)

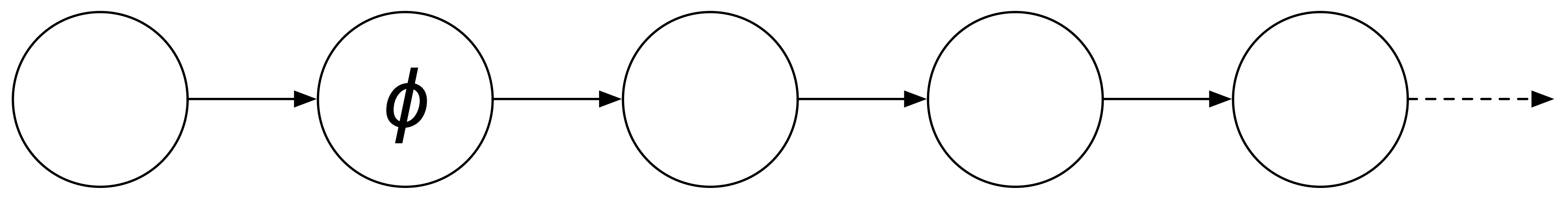
LTL Syntax

$$
\begin{aligned}
\phi, \psi ::= \; & \mathbf{X}\,\phi \mid \mathbf{G}\,\phi \mid \mathbf{F}\,\phi \\
\mid \; & \phi\,\mathbf{U}\,\psi \mid \phi\,\mathbf{R}\,\psi \\
\mid \; & p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \neg\phi \mid \top \mid \bot
\end{aligned}
$$

with $p \in P$ atomic propositions

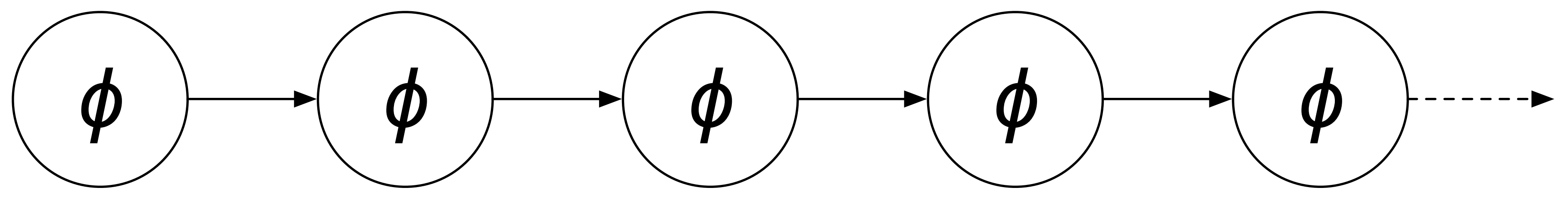
LTL Semantics

X ϕ



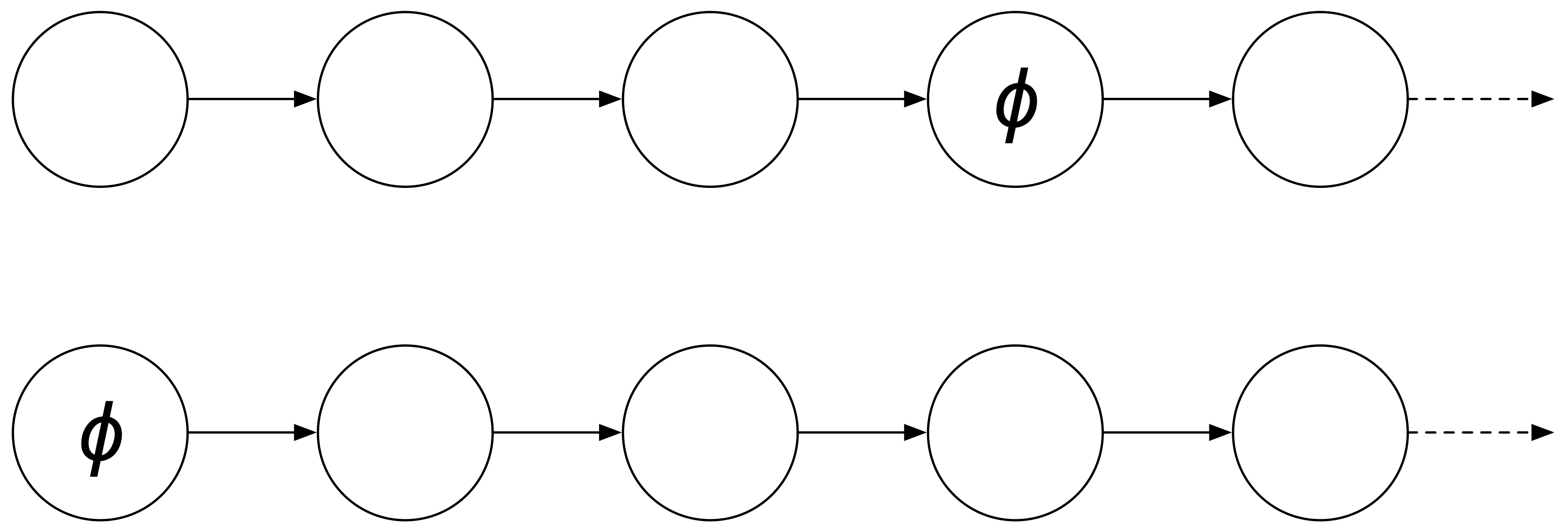
LTL Semantics

G ϕ



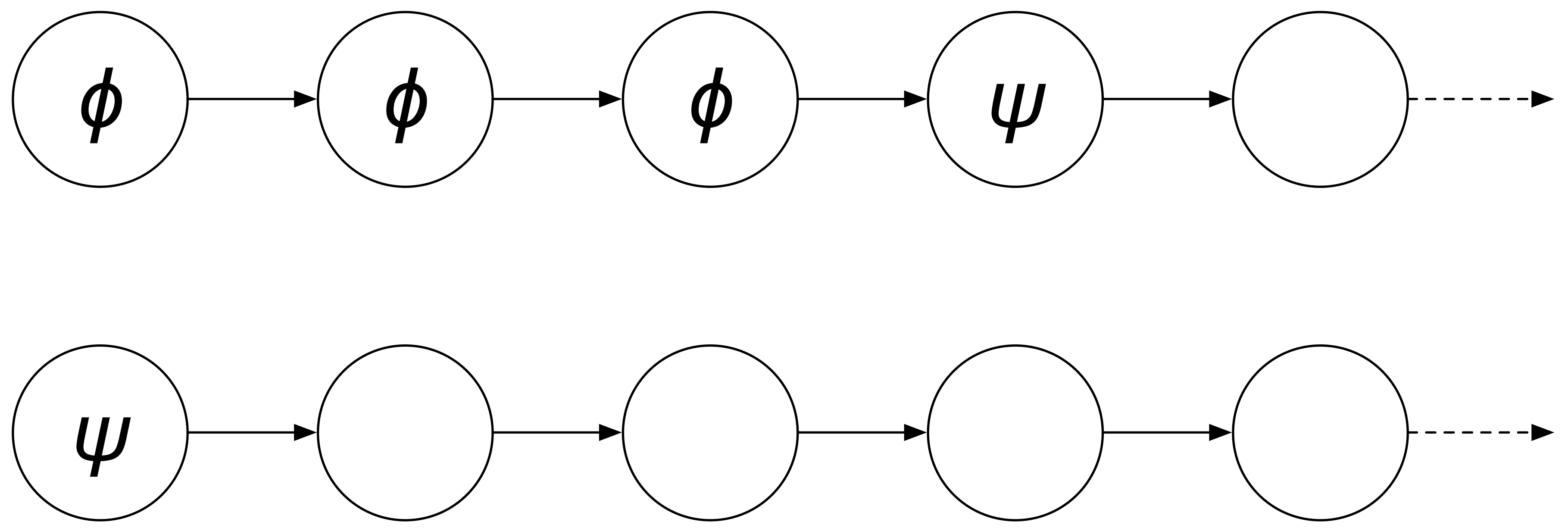
LTL Semantics

F ϕ



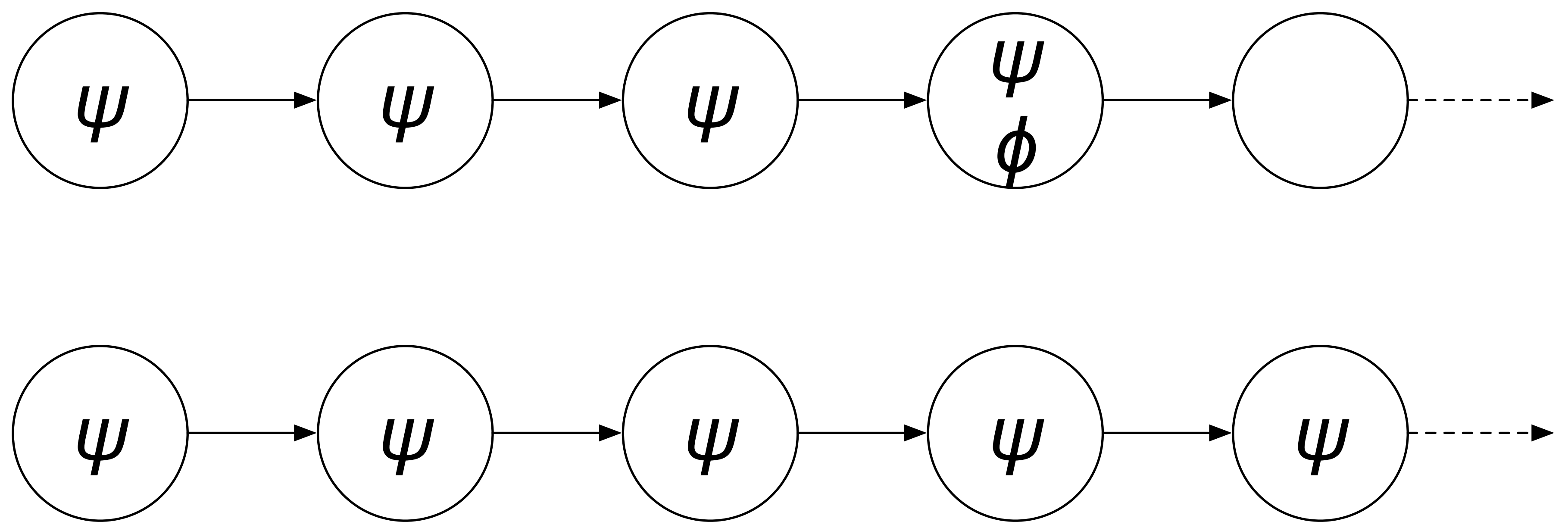
LTL Semantics

$\phi \mathbf{U} \psi$



LTL Semantics

$\phi \mathbf{R} \psi$



LTL Semantics

- If an LTL formula ϕ holds for a Kripke structure $M = (S,I,R,L)$ we say

$$M \vDash \phi$$

- $M \vDash \phi$ iff for all paths $\pi \in M$ such that $\pi_0 \in I$ we have $M,\pi \vDash \phi$
- Minimal CTL subset: $\top$, $\vee$, $\neg$, **U**, **X**
  - $\mathbf{F}\phi \quad \equiv \quad \top\mathbf{U}\phi$
  - $\mathbf{G}\phi \quad \equiv \quad \neg\mathbf{F}(\neg\phi)$
  - $\phi\mathbf{R}\psi \quad \equiv \quad \neg((\neg\phi)\mathbf{U}(\neg\psi))$

LTL Semantics

$M,\pi \models p \equiv p \in L(\pi_0)$

$M,\pi \models \top \equiv \top$

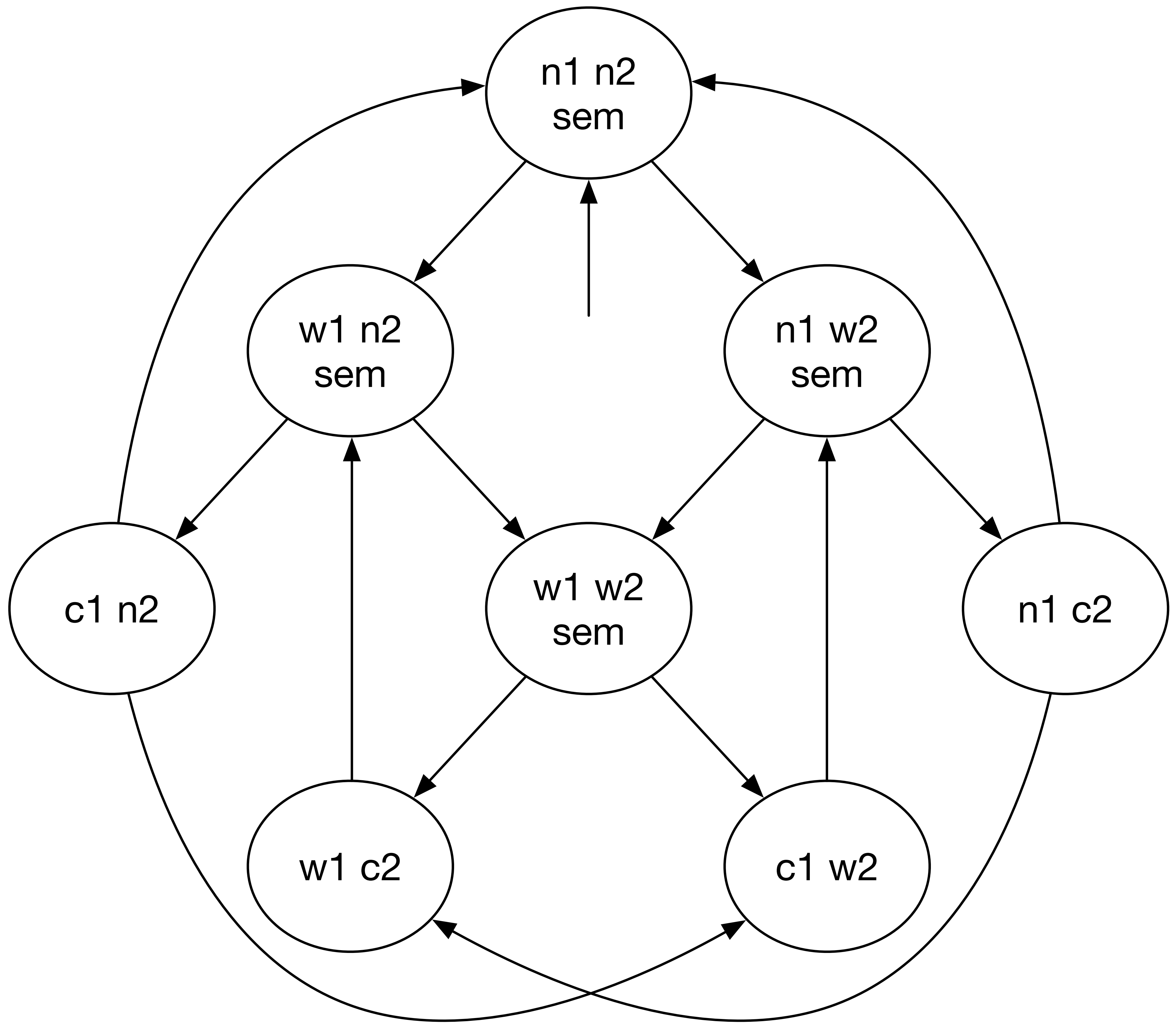
$M,\pi \models \neg\phi \equiv M,\pi \not\models \phi$

$M,\pi \models \phi \vee \psi \equiv M,\pi \models \phi$ or $M,\pi \models \psi$

$M,\pi \models \mathbf{X}\phi \equiv M,\pi^1 \models \phi$

$M,\pi \models \phi\mathbf{U}\psi \equiv \exists i \geq 0 \,.\, (M,\pi^i \models \psi$ and $\forall 0 \leq j < i \; M,\pi^j \models \phi)$

LTL Examples



Mutual exclusion: $\mathbf{G}\ \neg(c1 \wedge c2)$

No starvation: $\mathbf{G}\ (w1 \rightarrow \mathbf{F}\ c1)$

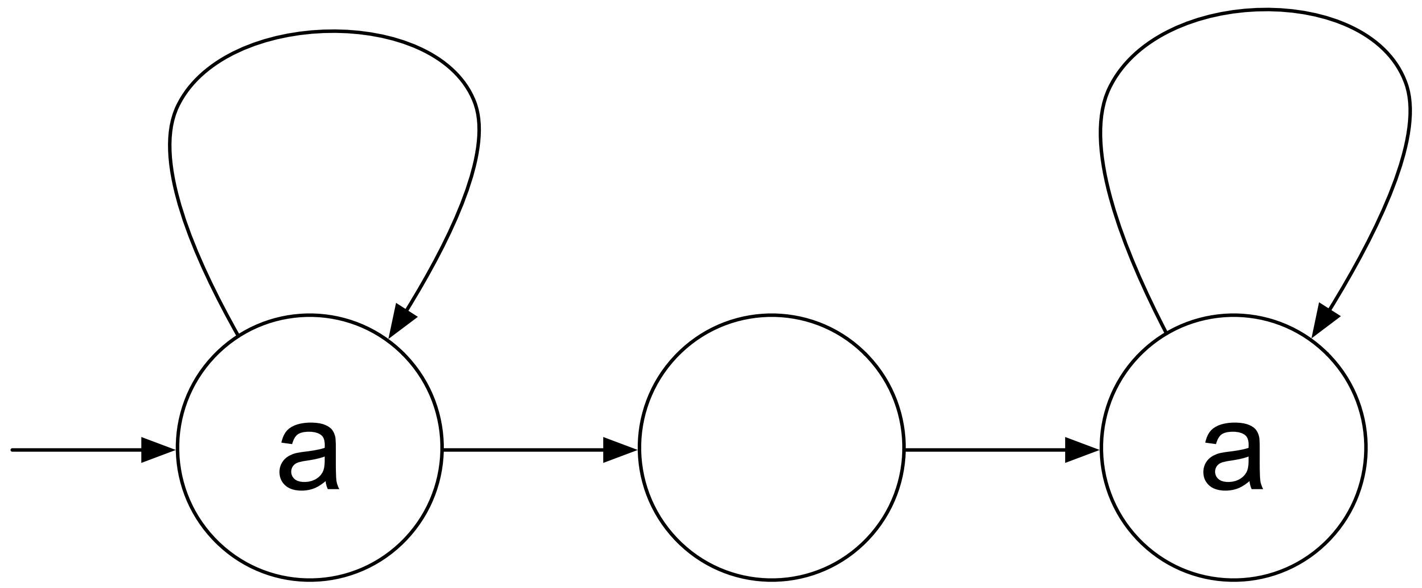
Reversibility: NA

LTL vs. CTL

- Many properties are expressible both in LTL and CTL
- LTL tends to be more intuitive, but CTL model checking is more efficient
- However, the two logics are **incomparable**: there are LTL properties not expressible in CTL, and vice-versa

LTL vs. CTL

- The CTL property **AG EF** ϕ is not expressible in LTL
- The LTL property **FG** ϕ (needed for *fairness*) is not expressible in CTL
- In general it is not sufficient to just add **A** quantifiers, e.g., **AF AG** a:



Model Checking Algorithms

- Two approaches to model checking:
  - **Complete**: considers paths of arbitrary length
  - **Bounded**: considers paths up to a certain length
- The latter may miss certain counter-examples but is more efficient
- We will focus on algorithms for model checking LTL

LTL Complete Model Checking

- The common approach is **automata-based**
- Create an automaton A_M whose language are the paths acceptable by M

$$L(A_M) = \{ w \mid w \in M \}$$

- Create an automaton A_ϕ whose language are the paths valid under ϕ

$$L(A_\phi) = \{ \pi \mid \pi \vDash \phi \}$$

- $M \vDash \phi$ iff $L(A_M) \subseteq L(A_\phi)$, or, more easily checked, iff

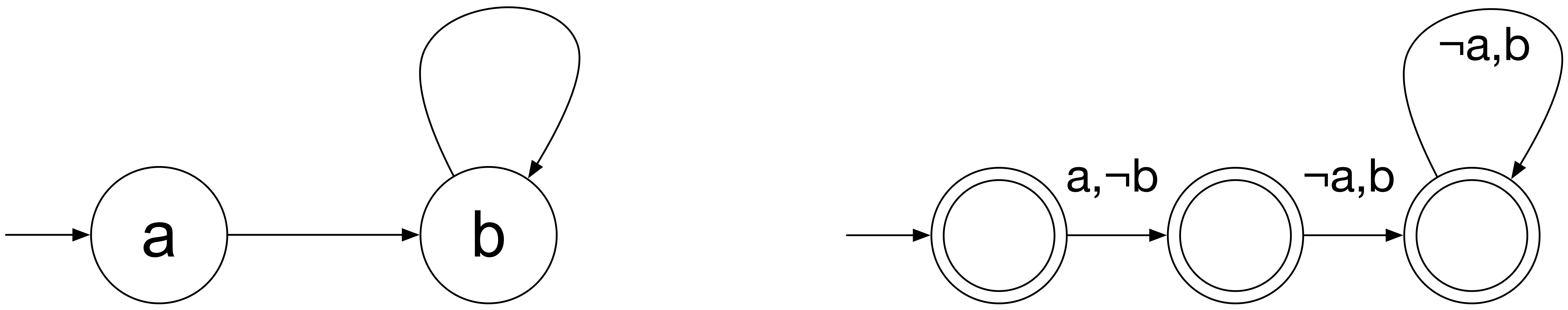
$$L(A_M) \cap L(A_{\neg\phi}) = \varnothing$$

Büchi Automata

- Finite-state automata cannot process infinite computations
- We need the notion of non-deterministic **Büchi automata** (NBA)
- An NBA $A = (Q,\Sigma,\delta,q_0,F)$ is:
  - Q is a set of states
  - Σ is the alphabet
  - $\delta : Q \rightarrow 2^Q$ is the transition function
  - $q_0 \in Q$ is the initial state
  - $F \subseteq Q$ is the set of accepting states
- A run is accepted by A if an accepting state is visited **infinitely often**

From *M* to Büchi Automata

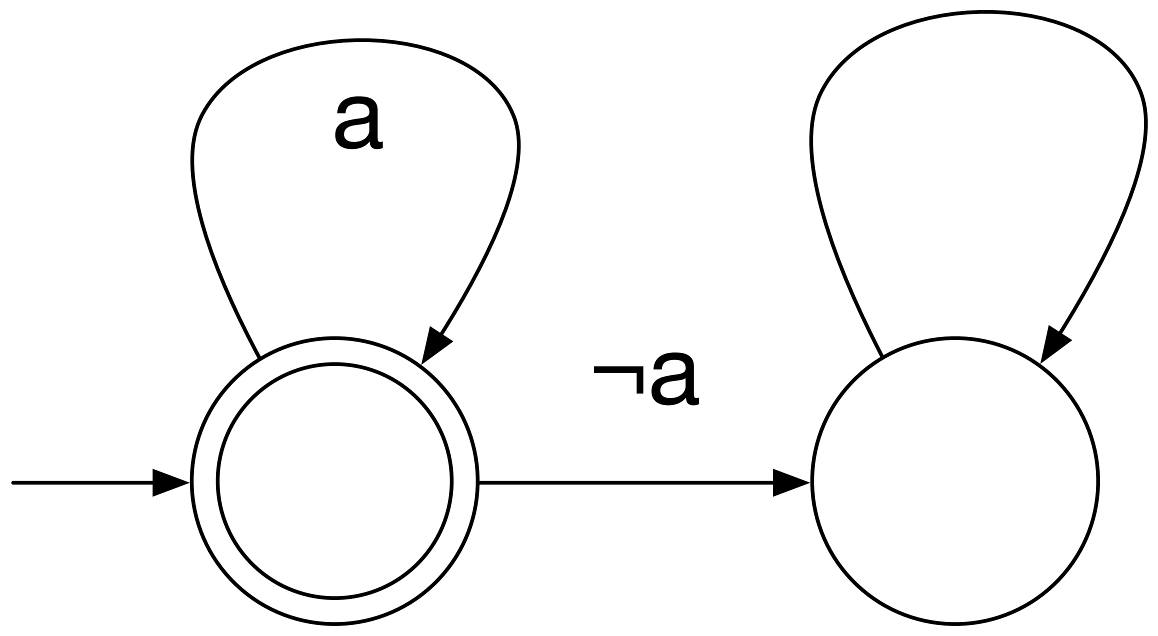
- Construct A_M from M:
  - $\Sigma = 2^P$, conjunctions of atomic propositions
  - Add an initial state to the existing Kripke structure M
  - $s_1 \in \delta(s_2, w)$ iff in M $(s_1, s_2) \in R$ and $L(s_2) = w$
  - $F = Q$, all states are accepting



M A_M

From ϕ to Büchi Automata

- Construct A_ϕ from ϕ:
  - $\Sigma = 2^P$, conjunctions of atomic propositions
  - Inductively create the states and transitions from ϕ (process omitted)



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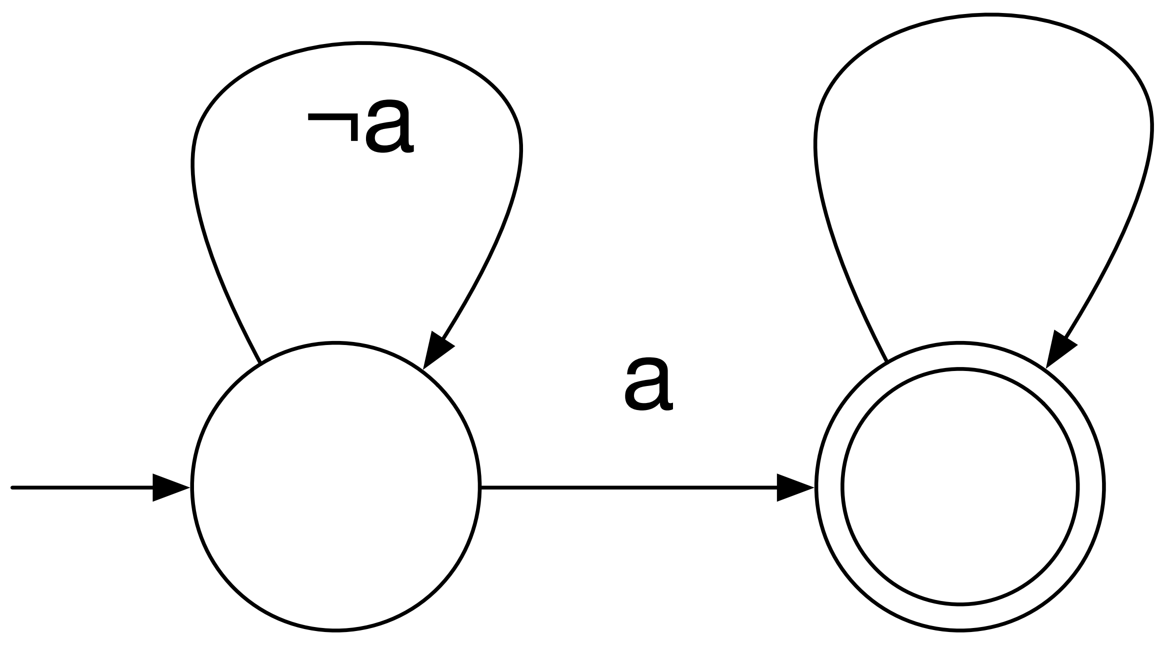

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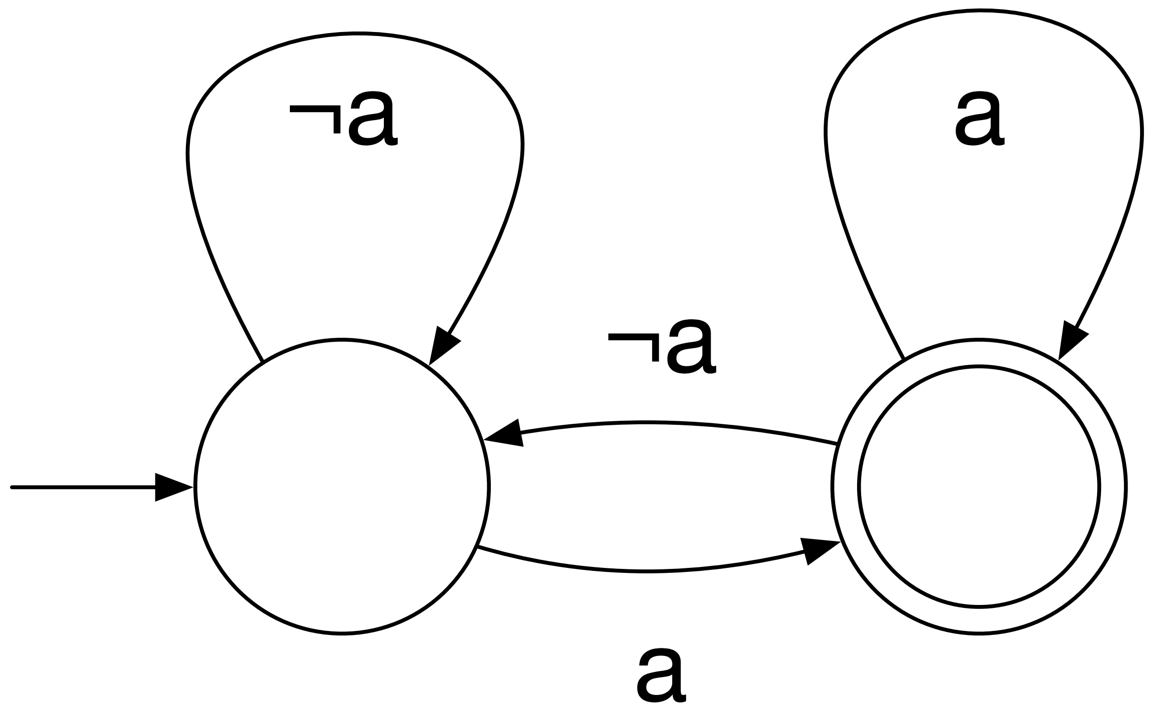


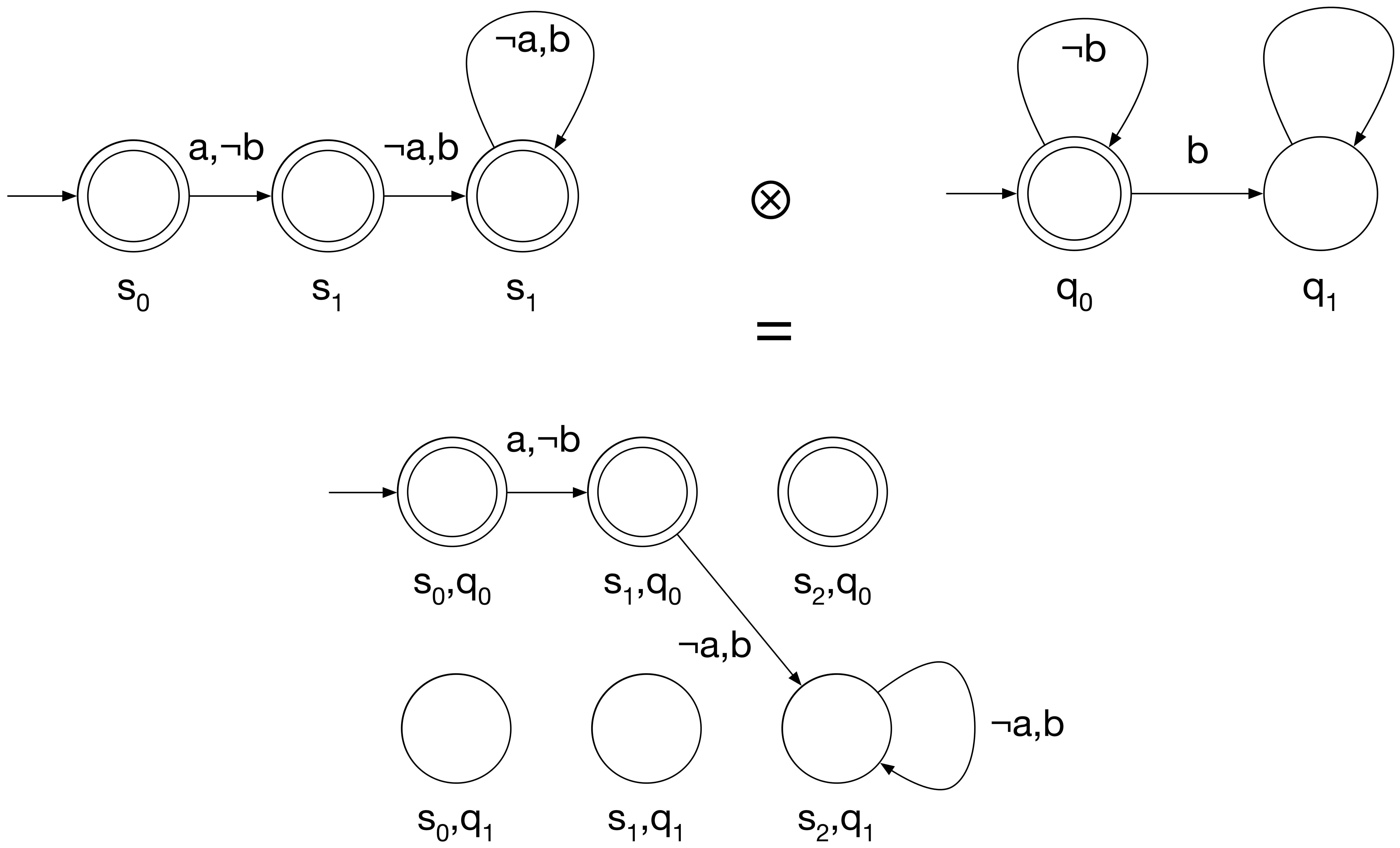
GFa

Checking for emptiness

- To test whether languages are disjoint, check whether the product automaton is empty
  - $M \vDash \phi$ iff $L(A_M) \cap L(A_\phi) = \varnothing$ iff $L(A_M \otimes A_\phi) = \varnothing$
- For $A_M = (Q_M,\Sigma,\delta_M,q_{0M},Q_M)$ and $A_{\neg\phi} = (Q_{\neg\phi},\Sigma,\delta_{\neg\phi},q_{0\neg\phi},F_{\neg\phi})$ we have
  - $A_M \otimes A_\phi = (Q_M \times Q_{\neg\phi},\Sigma,\delta,q_{0M} \times q_{0\neg\phi},Q_M \times F_{\neg\phi})$
  - $(s_2,q_2) \in \delta((s_1,q_1),w)$ iff $s_2 \in \delta(s_1,w)$ and $q_2 \in \delta(q_1,w)$

Checking for emptiness

$$M \models \mathbf{F}b \quad \text{iff} \quad L(A_M \otimes A_{\mathbf{G}\neg b}) = \emptyset$$



Checking for emptiness

- Compute Strongly Connected Components (SCC) and check if there is one with accepting state (reachable from initial state)
  - Requires creating the complete automaton
- Determine reachable states with Depth-First Search (DFS), and then do a nested DFS to check whether there is a cycle
  - Can be performed on-the-fly
- Use a **fair** CTL model checker and test for $\mathbf{EG}\top$
  - Benefit from advances in CTL model checkers (e.g., symbolic)

LTL Bounded Model Checking

- Checking a property for paths with a fixed length k simplifies the problem
  - We know exactly the size of the counter-examples ($k \times$ size of each state), so **SAT solvers** can be used
- To be sound, process should consider prefixes with **lassos**
- Some optimisations may ignore lassos, but adapted semantics:
  - $\mathbf{X}\phi$ false in last state
  - $\mathbf{G}\phi$ always false
  - for $\mathbf{F}\phi$, ϕ must occur in prefix

SAT Solving

- Procedures to solve the **Boolean satisfiability** problem (SAT)
- Is there a valuation for a set of Boolean variables that make a proposition true?
- Very low-level, problems in **conjunctive normal form**.
- Constant advances, techniques often use such solvers in the backend

From *M* to SAT

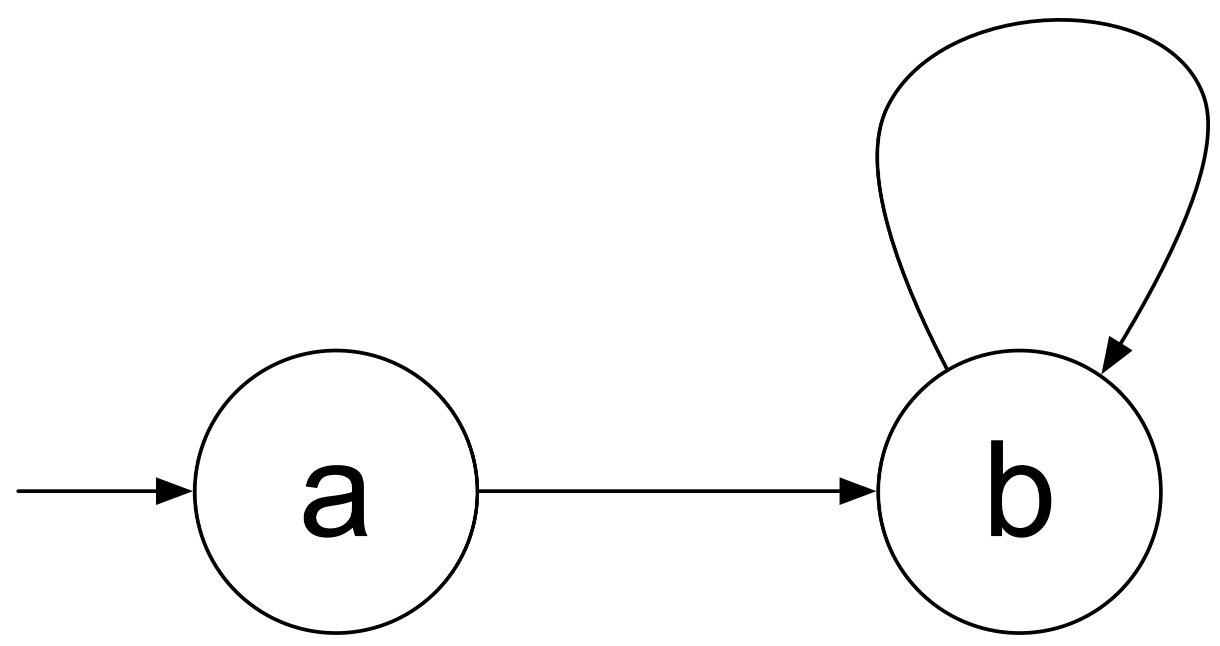
- For a length k
  - For each atomic proposition in $p \in P$ and each $0 \leq i < k$, create a Boolean variable p_i
  - Create a variable l_i for $0 \leq i < k$, to represent possible lassos
  - Let $I(s)$ be a predicate that holds when s is initial and $T(s_1, s_2)$ when there is a transition between s_1 and s_2 in R.
  - *M can* be encoded for a lasso l as:

$$[M]_l \equiv I(s_0) \wedge \bigwedge_{i=0}^{k-2} T(s_i, s_{i+1}) \wedge T(s_{k-1}, s_l)$$

From *M* to SAT

- For k = 3, variables are a_0, a_1, a_2, b_0, b_1, b_2 (since P = {a,b})
- $I(s_0) = a_0 \wedge \neg b_0$
- $T(s_i,s_{i+1}) = (a_i \wedge \neg b_i \wedge \neg a_{i+1} \wedge b_{i+1}) \vee (\neg a_i \wedge b_i \wedge \neg a_{i+1} \wedge b_{i+1})$
- For a particular lasso l, $[M]_l$ will be

$$
\begin{gathered}
(a_0 \wedge \neg b_0) \\
\wedge \\
((a_0 \wedge \neg b_0 \wedge \neg a1 \wedge b1) \vee (\neg a_0 \wedge b_0 \wedge \neg a_1 \wedge b_1)) \\
\wedge \\
((a_1 \wedge \neg b_1 \wedge \neg a_2 \wedge b_2) \vee (\neg a_1 \wedge b_1 \wedge \neg a_2 \wedge b_2)) \\
\wedge \\
((a_2 \wedge \neg b_2 \wedge \neg a_l \wedge b_l) \vee (\neg a_2 \wedge b_2 \wedge \neg a_l \wedge b_l))
\end{gathered}
$$



M

From ϕ to SAT

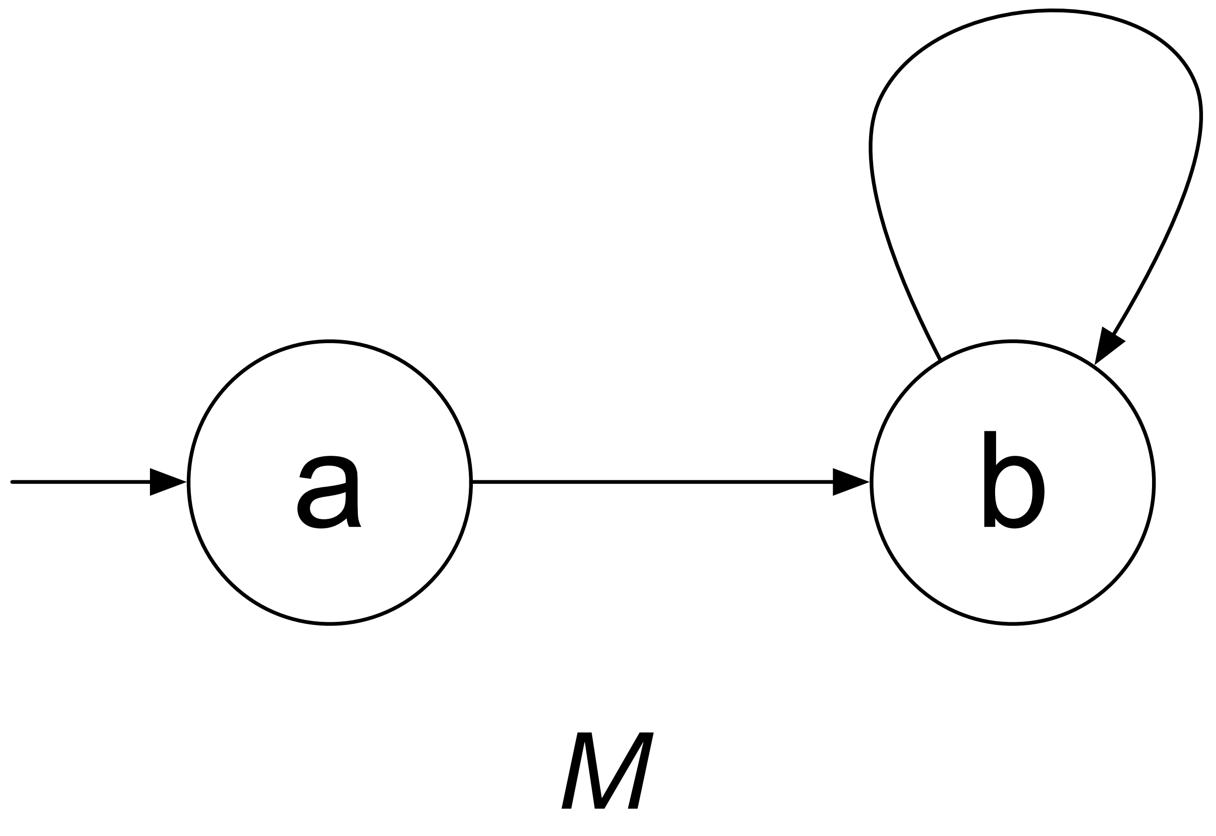
- For a length *k,* particular lasso *l*, translate ϕ for a state *i* as:
  - ${}_i[p]_l = p_i$
  - ${}_i[\neg p]_l = \neg p_i$
  - ${}_i[\phi \vee \psi]_l = {}_i[\phi]_l \vee {}_i[\psi]_l$
  - ${}_i[\mathbf{G}\phi]_l = {}_i[\phi]_l \wedge {}_{succ(i)}[\mathbf{G}\phi]_l$
  - ${}_i[\mathbf{F}\phi]_l = {}_i[\phi]_l \vee {}_{succ(i)}[\mathbf{F}\phi]_l$
  - ${}_i[\phi\mathbf{U}\psi]_l = {}_i[\psi]_l \vee ({}_i[\phi]_l \wedge {}_{succ(i)}[\phi\mathbf{U}\psi]_l)$
- where succ(*i*) = *i*+1 if *l* < *k,* and *l* otherwise

From ϕ to SAT

- For a particular lasso l, ${}_0[\mathbf{G}(b \rightarrow Xa)]_l$ will be

$$(\neg b_0 \vee a_1) \wedge (\neg b_1 \vee a_2) \wedge (\neg b_2 \vee a_l)$$

- Putting it all together, $M \vDash \phi$ would be translated as:



$$\bigvee_{l=0}^{l<k} ([M]_l \wedge {}_0[\neg \phi]_l)$$