定义1. 设$\underline{A}$为实内积空间V上一个变换. 若$\underline{A}$是满射,且满足

$$(\underline{A}\alpha, \underline{A}\beta)=(\alpha,\beta), \quad \forall \alpha,\beta\in V$$

则称$\underline{A}$是V上一正交变换 (类似于几何空间中"旋转")

性质1. 设$\underline{A}$为为实内积空间V上的正交变换,则

(1) $|\underline{A}\alpha|=|\alpha|, \quad \forall \alpha\in V$;

$$|\underline{A}\alpha|^2=(\underline{A}\alpha,\underline{A}\alpha)=(\alpha,\alpha)=|\alpha|^2$$

(2) $\langle \underline{A}\alpha, \underline{A}\beta\rangle=\langle\alpha,\beta\rangle \quad \forall \alpha,\beta\in V, \alpha\neq 0, \beta\neq 0$

$$\cos\langle \underline{A}\alpha, \underline{A}\beta\rangle=\frac{(\underline{A}\alpha,\underline{A}\beta)}{|\underline{A}\alpha|\cdot|\underline{A}\beta|}=\frac{(\alpha,\beta)}{|\alpha|\cdot|\beta|}=\cos\langle\alpha,\beta\rangle$$

(3) $\underline{A}\alpha\perp\underline{A}\beta \Leftrightarrow \alpha\perp\beta, \quad \forall\alpha,\beta\in V$

$$(\underline{A}\alpha,\underline{A}\beta)=(\alpha,\beta)=0$$

(4) $\underline{A}$为线性变换.

利用 $|\alpha|^2=(\alpha,\alpha)=0 \Leftrightarrow \alpha=0$

$$\begin{aligned}
&|\underline{A}(\alpha+\beta)-(\underline{A}\alpha+\underline{A}\beta)|^2\\
=&(\underline{A}(\alpha+\beta)-(\underline{A}\alpha+\underline{A}\beta), \underline{A}(\alpha+\beta)-(\underline{A}\alpha+\underline{A}\beta))\\
=&|\underline{A}(\alpha+\beta)|^2-2(\underline{A}(\alpha+\beta),\underline{A}\alpha+\underline{A}\beta)+|\underline{A}\alpha+\underline{A}\beta|^2\\
=&|\alpha+\beta|^2-2[(\underline{A}(\alpha+\beta),\underline{A}\alpha)+(\underline{A}(\alpha+\beta),\underline{A}\beta)]+|\alpha|^2+|\beta|^2+2(\alpha,\beta)\\
=&|\alpha+\beta|^2-2(\alpha+\beta,\alpha+\beta)+(\alpha+\beta,\alpha+\beta)\\
=&0
\end{aligned}$$

$$\therefore \underline{A}(\alpha+\beta)=\underline{A}\alpha+\underline{A}\beta$$

$$\begin{aligned}
&|\underline{A}(k\alpha)-k\underline{A}\alpha|^2\\
=&(\underline{A}(k\alpha),\underline{A}(k\alpha))-2(\underline{A}(k\alpha),k\underline{A}\alpha)+(k\underline{A}\alpha,k\underline{A}\alpha)\\
=&(k\alpha,k\alpha)-2k(\underline{A}(k\alpha),\underline{A}\alpha)+k^2(\underline{A}\alpha,\underline{A}\alpha)\\
=&k^2|\alpha|^2-2k^2|\alpha|^2+k^2|\alpha|^2\\
=&0
\end{aligned}$$

$\therefore \underline{A}(k\alpha) = k\underline{A}\alpha$

$\therefore \underline{A}$为线性变换

(5) $\underline{A}$为单射，从而$\underline{A}$为双射，即$\underline{A}$可逆

设$\alpha \in Ker\,\underline{A} \Leftrightarrow \underline{A}\alpha = 0 \Rightarrow |\underline{A}\alpha| = 0 \Rightarrow |\alpha| = 0 \Rightarrow \alpha = 0$

$\therefore Ker\,\underline{A} = \{0\}$ $\quad\therefore \underline{A}$为单射

(6) $d(\underline{A}\alpha, \underline{A}\beta) = d(\alpha, \beta)$, $\forall \alpha, \beta \in V$

$d(\underline{A}\alpha, \underline{A}\beta) = |\underline{A}\alpha - \underline{A}\beta| = |\underline{A}(\alpha - \beta)| = |\alpha - \beta| = d(\alpha, \beta)$

命题 1. 实内积空间V上的变换$\underline{A}$是正交变换

$\Leftrightarrow$ $\underline{A}$是V到自身的一个保距同构

2. 正交变换的逆变换、乘积、也是正交变换

3. 欧几里得空间V上变换$\underline{A}$若满足

$(\underline{A}\alpha, \underline{A}\beta) = (\alpha, \beta)$, $\forall \alpha, \beta \in V$

则$\underline{A}$为上一正交变换

由(4)可知$\underline{A}$为线性变换 (5)可知$\underline{A}$为单射

故$\underline{A}$为满射，从而$\underline{A}$为正交变换.

设$\underline{A}$是n维欧几里得空间V上线性变换，则下列命题等价

(1) $\underline{A}$为正交变换

(2) $\underline{A}$把V的标准正交基映成标准正交基

(3) $\underline{A}$在V任一标准正交基$\eta_1, \cdots, \eta_n$下矩阵A为正交矩阵

$(\underline{A}\eta_1, \cdots, \underline{A}\eta_n) = (\eta_1, \cdots, \eta_n)A$, $\underline{A}\eta_1, \cdots, \underline{A}\eta_n$为标准正交基，

A为两标准正交基过度矩阵，故A为正交矩阵

(3)→(1) 设$\alpha = (\eta_1, \cdots, \eta_n)X$, $\beta = (\eta_1, \cdots, \eta_n)Y$

$\underline{A}\alpha=(\eta_1,\cdots,\eta_n)AX$, $\underline{A}\beta=(\eta_1,\cdots,\eta_n)AY$

$(\underline{A}\alpha,\underline{A}\beta)=(AX)'AY=X'Y=(\alpha,\beta)$

推论1、n维欧几里得空间V上的正交变换 $\underline{A}$ 的行列式为1或-1

若 $|\underline{A}|=1$, 则称为第一类正交变换，否则为第二类正交变换

定理2. 设 $\underline{A}$ 是n维欧氏空间V上的正交变换. 则V中存在一标准正交基，使得 $\underline{A}$ 在此基下矩阵为形如下式的分块对角矩阵

$$\begin{bmatrix} A_1 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & & \\ 0 & \cdots & A_n & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_s \end{bmatrix} \qquad A_i=\begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$$

$0<\theta_i<\pi$, $i=1,\cdots,n$

$\lambda_j=1$ 或 -1 $j=1,\cdots,S$