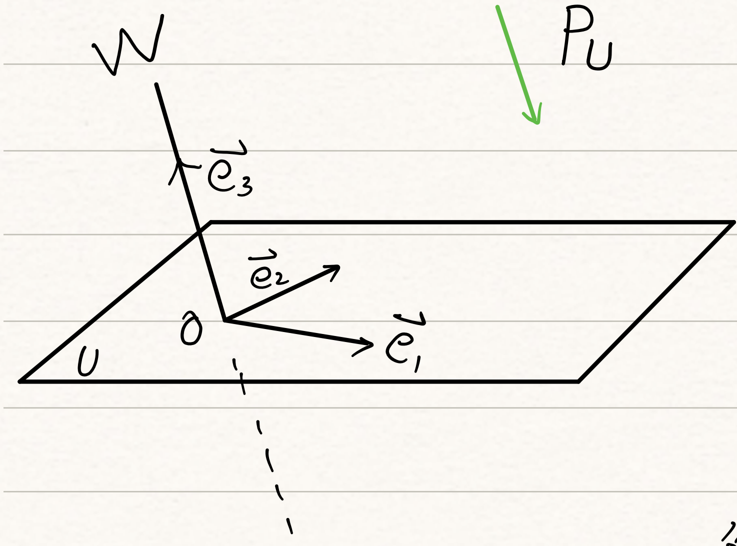

$$P_U(\vec{e}_1)=\vec{e}_1=1\vec{e}_1+0\vec{e}_2+0\vec{e}_3$$
$$P_U(\vec{e}_2)=\vec{e}_2=0\vec{e}_1+1\vec{e}_2+0\vec{e}_3$$
$$P_U(\vec{e}_3)=0=0\vec{e}_1+0\vec{e}_2+0\vec{e}_3$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

称为 P_U 在基 $\vec{e}_1,\vec{e}_2,\vec{e}_3$ 下的矩阵

设 V 为域 F 上线性空间

$$V^n:=V\times V\times\cdots\times V:=\{(\alpha_1,\cdots,\alpha_n)\mid \alpha_i\in V,i=1,\cdots,n\}$$

$$(\alpha_1,\cdots,\alpha_n)=(\beta_1,\cdots,\beta_n):\Longleftrightarrow \alpha_i=\beta_i\quad i=1,\cdots,n$$

$$(\alpha_1,\cdots,\alpha_n)+(\beta_1,\cdots,\beta_n):=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n),$$

$$k(\alpha_1,\cdots,\alpha_n):=(k\alpha_1,\cdots,k\alpha_n)$$

(1) 易验证 V^n 为域 F 上的一个线性空间

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}a_{11}&\cdots&a_{1m}\\a_{21}&\cdots&a_{2m}\\\vdots&&\vdots\\a_{n1}&\cdots&a_{nm}\end{pmatrix}$$

$$:=(a_{11}\alpha_1+a_{21}\alpha_2+\cdots+a_{n1}\alpha_n,\cdots,a_{1m}\alpha_1+a_{2m}\alpha_2+\cdots+a_{nm}\alpha_n)\in V^n$$

易知，满足：

$$[(\alpha_1,\cdots,\alpha_m)A]B=(\alpha_1,\cdots,\alpha_n)(AB)$$

$$(\alpha_1,\cdots,\alpha_n)A+(\alpha_1,\cdots,\alpha_n)B=(\alpha_1,\cdots,\alpha_n)(A+B)$$

$$(\alpha_1,\cdots,\alpha_n)A+(\beta_1,\cdots,\beta_n)A=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n)A$$

$$k[(\alpha_1,\cdots,\alpha_n)A]=[k(\alpha_1,\cdots,\alpha_n)]A=(\alpha_1,\cdots,\alpha_n)(kA)$$

设 V 是域 F 上的 n 维线性空间，$\underline{A}$ 是 V 上一个线性变换.

V 中一基为 $\alpha_1, \cdots, \alpha_n$

$$\begin{cases} \underline{A}\alpha_1 = a_{11}\alpha_1 + \cdots + a_{n1}\alpha_n \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \underline{A}\alpha_n = a_{1n}\alpha_1 + \cdots + a_{nn}\alpha_n \end{cases}$$

$$\Rightarrow (\underline{A}\alpha_1, \underline{A}\alpha_2, \cdots, \underline{A}\alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

记为 A

即 $(\underline{A}\alpha_1, \underline{A}\alpha_2, \cdots, \underline{A}\alpha_n) = (\alpha_1, \cdots, \alpha_n)A$

$\underline{A}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n)A$

把 A 称为 $\underline{A}$ 在基下线性变换的矩阵

A 的第 j 列为 $\underline{A}\alpha_j$ 在基下的坐标 $(j=1, \cdots, n)$

设 $\underline{A} \in Hom(V, V')$，$\dim V = n$，$\dim V' = s$

取 V 中一基 $\alpha_1, \cdots, \alpha_n$；$V'$ 中一基 $\eta_1, \cdots, \eta_s$

$$(\underline{A}\alpha_1, \cdots, \underline{A}\alpha_n) = (\eta_1, \cdots, \eta_s)\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{sn} \end{bmatrix}$$

即 $\underline{A}(\alpha_1, \cdots, \alpha_n) = (\eta_1, \cdots, \eta_s)A$

把 A 称为线性映射 $\underline{A}$ 在 V 的基 $\alpha_1, \cdots, \alpha_n$ 与 V' 的基 $\eta_1, \cdots, \eta_s$ 下矩阵

设 $\underline{B} \in Hom(V, V')$，$\underline{B}(\alpha_1, \cdots, \alpha_n) = (\eta_1, \cdots, \eta_s)B$

$(\underline{A}+\underline{B})(\alpha_1, \cdots, \alpha_n) = ((\underline{A}+\underline{B})\alpha_1, \cdots, (\underline{A}+\underline{B})\alpha_n)$

$= (\underline{A}\alpha_1+\underline{B}\alpha_1,\cdots,\underline{A}\alpha_n+\underline{B}\alpha_n)$

$= (\underline{A}\alpha_1,\cdots,\underline{A}\alpha_n)+(\underline{B}\alpha_1,\cdots,\underline{B}\alpha_n)$

$=(\eta_1,\cdots,\eta_s)A+(\eta_1,\cdots,\eta_s)B$

$=(\eta_1,\cdots,\eta_s)(A+B)$

$\therefore$ $\underline{A}+\underline{B}$ 在 V 的基 $\alpha_1,\cdots,\alpha_n$ 和 V' 的基 $\eta_1,\cdots,\eta_s$ 下矩阵为 $A+B$

同理 $k\underline{A}$ 在 V 的基 $\alpha_1,\cdots,\alpha_n$ 和 V' 的基 $\eta_1,\cdots,\eta_s$ 下矩阵为 kA

令 $\sigma: Hom(V,V') \longrightarrow M(F)_{s\times n}$

$\underline{A} \longmapsto A$，其中 $\underline{A}(\alpha_1,\cdots,\alpha_n)=(\eta_1,\cdots,\eta_s)A$

设 $\underline{B}\longmapsto B$ 满足 $\underline{B}(\alpha_1,\cdots,\alpha_n)=(\eta_1,\cdots,\eta_s)B$

则易知 $\sigma(\underline{A}+\underline{B})=A+B=\sigma(\underline{A})+\sigma(\underline{B})$

$\sigma(k\underline{A})=kA=k\sigma(\underline{A})$

$\therefore$ σ 为线性映射

任给 $C=(C_1,\cdots,C_n)\in M(F)_{s\times n}$

则存在唯一的线性映射 $\underline{C}$，s.t.

$\underline{C}(\alpha_1,\cdots,\alpha_n)=(\eta_1,\cdots,\eta_s)(C_1,\cdots,C_n)$

(注：$\underline{C}: \alpha_i \longmapsto (\eta_1,\cdots,\eta_s)C_i$)

$\therefore$ σ 为双射

$\therefore$ σ 为 $Hom(V,V')$ 到 $M_{s\times n}(F)$ 的一个同构映射

$Hom(V,V')\cong M_{s\times n}(F)$

$\therefore$ $\dim(Hom(V,V'))=\dim(M_{s\times n})=sn=\dim V\cdot\dim V'$

特别地 $Hom(V,V)\cong M_{n\times n}(F)$

$\dim(Hom(V,V))=n^2$

设 $\underline{A},\underline{B}\in Hom(V,V)$，$\alpha_1,\cdots,\alpha_n$ 为 V 一基

$\underline{A}(\alpha_1,\cdots,\alpha_n)=(\alpha_1,\cdots,\alpha_n)A$; $\underline{B}(\alpha_1,\cdots,\alpha_n)=(\alpha_1,\cdots,\alpha_n)B$

$$
\begin{aligned}
(\underline{A}\,\underline{B})(\alpha_1,\cdots,\alpha_n) &= (\underline{A}\,\underline{B}\alpha_1,\ \underline{A}\,\underline{B}\alpha_2,\cdots,\ \underline{A}\,\underline{B}\alpha_n)\\
&= \underline{A}[(\alpha_1,\cdots,\alpha_n)B]\\
&= \underline{A}(\alpha\cdot b_1,\ \alpha\cdot b_2,\cdots,\alpha b_n) \quad (b_i\text{为}B\ i\text{列向量})\\
&= (\underline{A}\alpha b_1,\ \underline{A}\alpha b_2,\cdots,\ \underline{A}\alpha b_n)\\
&= ((\underline{A}\alpha)b_1,\ (\underline{A}\alpha)b_2,\cdots,(\underline{A}\alpha)b_n)\\
&= (\underline{A}\alpha_1,\ \underline{A}\alpha_2,\cdots\ \underline{A}\alpha_n)B\\
&= [\underline{A}(\alpha_1,\cdots,\alpha_n)]B\\
&= [(\alpha_1,\cdots,\alpha_n)A]B = (\alpha_1,\cdots,\alpha_n)(AB)
\end{aligned}
$$

设 $\sigma:\ Hom(V,V') \longrightarrow M_{n\times n}(F)$

$\underline{A} \longmapsto A \quad (\underline{A}(\alpha_1,\cdots,\alpha_n)=(\alpha_1,\cdots,\alpha_n)A)$

$\therefore\ \sigma$ 保持乘法

且 (1) $\underline{A}$ 可逆 $\Longleftrightarrow$ A 可逆

$\underline{A}^{-1}$在 $(\alpha_1,\cdots,\alpha_n)$下矩阵为 A^{-1}

(2) $\underline{A}$ 为幂等变换 $\Longleftrightarrow$ A为幂等矩阵

设 $\alpha=(\alpha_1,\cdots,\alpha_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=(\alpha_1,\cdots,\alpha_n)X$

则 $\underline{A}\alpha=\underline{A}[(\alpha_1,\cdots,\alpha_n)X]=[\underline{A}(\alpha_1,\cdots,\alpha_n)]X=(\alpha_1,\cdots,\alpha_n)(AX)$

即 $\underline{A}\alpha$ 在基 $\alpha_1,\cdots,\alpha_n$ 下坐标为 (AX)