定义1. $\underline{A}: V \to V'$, $\operatorname{Ker} A := \{\alpha \in V \mid \underline{A}\alpha = 0'\}$ 称为 $\underline{A}$ 的核

性质1. $\operatorname{Ker} A$ 是 V 的一个子空间

证：由于 $A(0) = 0'$，$\therefore$ $\operatorname{Ker} A$ 非空

任取 $\alpha, \beta \in \operatorname{Ker} A$，则 $\underline{A}\alpha = 0'$, $\underline{A}\beta = 0'$

$\therefore \underline{A}(\alpha+\beta) = \underline{A}\alpha + \underline{A}\beta = 0' + 0' = 0'$ $\therefore \alpha+\beta \in \ker$

同理 $\underline{A}(k\alpha) = 0'$ $\therefore k\alpha \in \operatorname{Ker} A$

性质2: $\underline{A}$ 是单射 $\Longleftrightarrow \operatorname{Ker} \underline{A} = 0$

证："$\Rightarrow$" $\forall \alpha \in \ker A$，有 $\underline{A}\alpha = 0' = \underline{A}(0)$ $\therefore \alpha = 0$

"$\Leftarrow$" 设 $\alpha_1, \alpha_2 \in V$, s.t. $\underline{A}\alpha_1 = \underline{A}\alpha_2$

则 $\underline{A}(\alpha_1 - \alpha_2) = \underline{A}\alpha_1 - \underline{A}\alpha_2 = 0'$

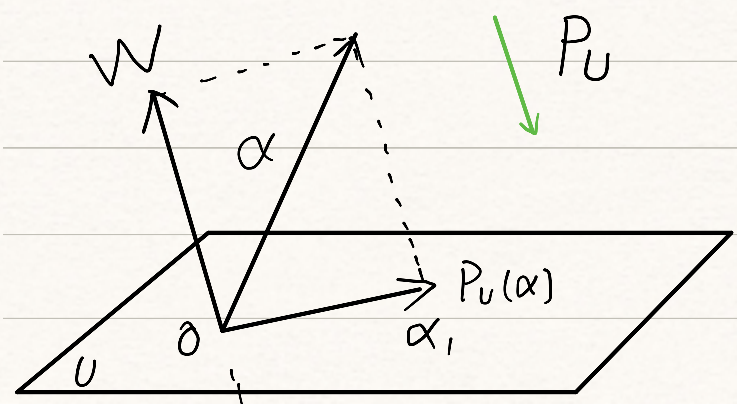
$\therefore \alpha_1 - \alpha_2 = 0$，即 $\alpha_1 = \alpha_2$

即 $\underline{A}$ 为单射

易知，$\operatorname{Im} A$ ($\underline{A}V$) 是 V' 的子空间，$\underline{A}$ 是满射 $\Longleftrightarrow \operatorname{Im} A = V'$

象

几何空间 $V = U \oplus W$



显然 $W\subseteq \mathrm{Ker}\,P_U$

任取 $\alpha\in \mathrm{Ker}P_U$

$\because \alpha\in V$, $\therefore \alpha=\alpha_1+\alpha_2$, $\alpha_1\in U,\alpha_2\in W$

则 $P_U(\alpha)=P_U(\alpha_1)+P_U(\alpha_2)=\alpha_1=0$

$\therefore \alpha=\alpha_2$ $\therefore \alpha\in W$

$\therefore \mathrm{ker}P_U\subseteq W$

$\therefore \mathrm{Ker}P_U=W$

显然 $\mathrm{Im}P_U\subseteq U$

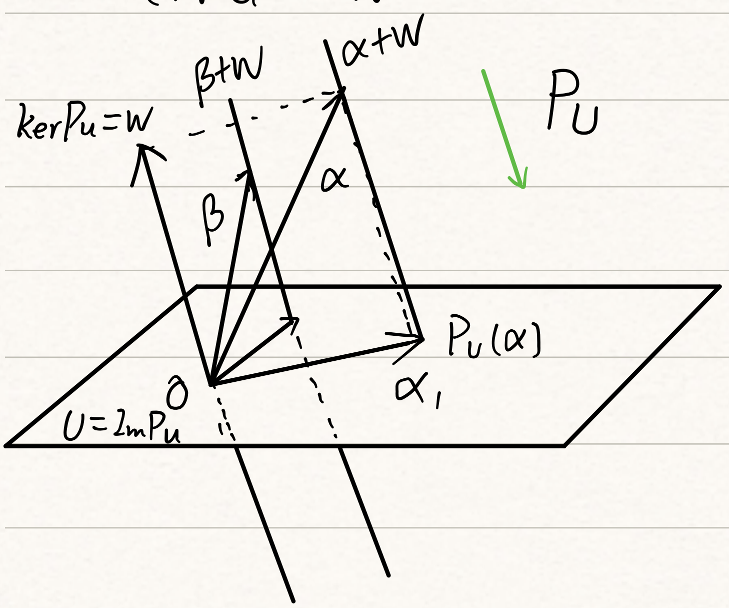
任取 $\alpha\in U$，设 $\beta\in W$

则 $\alpha=P_U(\alpha+\beta)$

$\therefore \alpha\in \mathrm{Im}P_U$

$\therefore U\subseteq \mathrm{Im}P_U$

$\therefore U=\mathrm{Im}P_U$



$$\sigma \quad V/W \longrightarrow U$$
$$\alpha+W \longmapsto P_U(\alpha)$$

易证：σ 即为单射也为满射

σ 保持加法与数乘

$\therefore$ σ 为同构映射

从而 $V/W\cong \mathrm{Im}P_U$

即 $V/\mathrm{ker}P_U\cong \mathrm{Im}P_U$

定理1. 设 $\underline{A}\in \mathrm{Hom}(V,V')$，则 $V/\mathrm{ker}\underline{A}\cong \mathrm{Im}\underline{A}$

证：记 $W=\mathrm{Ker}\,\underline{A}$

$$\sigma: \quad V/W \longrightarrow \mathrm{Im}\underline{A}$$
$$\alpha+W \longmapsto \underline{A}(\alpha)$$

$$\alpha+W=\beta+W \Longleftrightarrow \alpha-\beta\in W=\mathrm{Ker}\underline{A}$$
$$\Longleftrightarrow \underline{A}(\alpha-\beta)=0$$
$$\Longleftrightarrow \underline{A}(\alpha)=\underline{A}(\beta)$$

$\therefore$ σ 为一映射且为单射

又 显然 σ 为满射

$\therefore$ σ 为双射

$\because$ $\sigma((\alpha+W)+(\beta+W)) = \sigma(\alpha+\beta+W) = \underline{A}(\alpha+\beta) = \underline{A}\alpha+\underline{A}\beta$

$\sigma(k(\alpha+W)) = \sigma(k\alpha+W) = \underline{A}(k\alpha) = k\underline{A}(\alpha)$

$\therefore$ σ 保加法与数乘

$\therefore$ σ 为同构映射

$\therefore$ $V/\ker A \cong \operatorname{Im}\underline{A}$

定义，称 $\dim(\operatorname{Ker}\underline{A})$ 为 $\underline{A}$ 的零度，$\dim(\operatorname{Im}A)$ 为 $\underline{A}$ 的秩，记作 $\operatorname{rank}(\underline{A})$

定理2：设 $\underline{A} \in \operatorname{Hom}(V, V')$，$\dim V = n$，则

$$\dim V = \dim \operatorname{Ker} A + \dim \operatorname{Im} \underline{A}$$

证 $\because \dim(V/\ker A) = \dim(\operatorname{Im}\underline{A})$，

$\dim(V/\ker A) = \dim V - \dim \operatorname{Ker} A$

$\therefore \dim V = \dim \operatorname{Ker}\underline{A} + \dim \operatorname{Im}\underline{A}$

推论1. $\underline{A}$ 是单射 $\Longleftrightarrow$ $\underline{A}$ 是满射

$\Updownarrow$ $\Updownarrow$

$\ker\underline{A} = 0$，$\dim(\operatorname{Ker}A) = 0 \Longleftrightarrow \dim(\operatorname{Im}\underline{A}) = \dim V = n$，$\operatorname{Im}\underline{A} = V$