Lecture 6: Graphs

CS 5006/7: Algorithms, C and Systems

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February 12, 2019

Topics

- Intro/definitions
- Paths and Djikstra (REVIEW)
- Acyclic graphs and topological ordering
- Connectivity in Directed Graphs

What is a Graph?

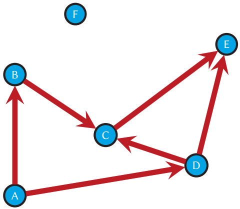
A graph is simply a collection of nodes plus edges

- Linked lists, trees, and heaps are all special cases of graphs
- The nodes are known as vertices (node = vertex)
- Formal Definition:
  - A graph G is a pair (V, E) where
  - V is a set of vertices or nodes
  - E is a set of edges that connect vertices

An Example

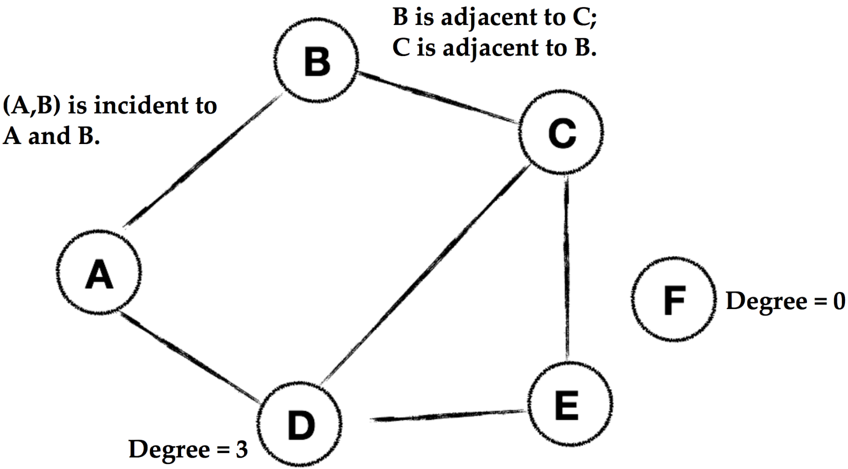
Here is a directed graph $G = (V, E)$

- Each edge is a pair (v_1, v_2), where v_1, v_2 are vertices in V
  - $V = \{A, B, C, D, E, F\}$
  - $E = \{(A, B), (A, D), (B, C), (C, D), (C, E), (D, E)\}$



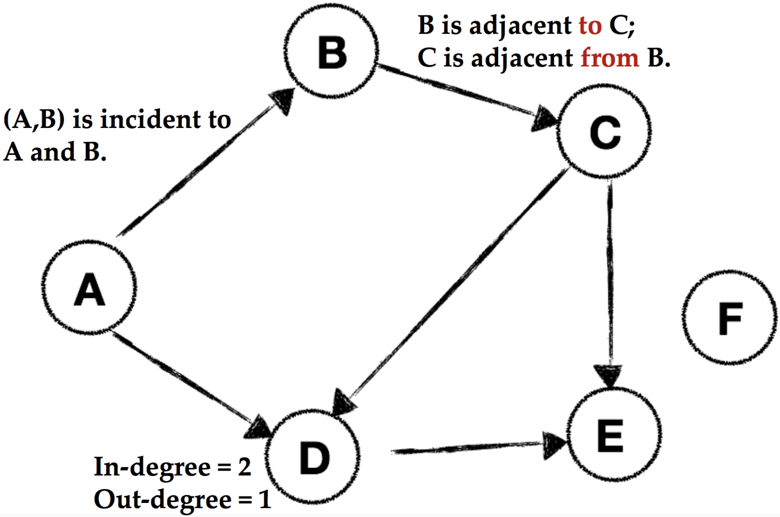
Terminology: Undirected Graph

- Two vertices u and v are ***adjacent*** in an undirected graph G if $\{u, v\}$ is an edge in G
  - edge $e = \{u, v\}$ is ***incident*** with vertex u and vertex v
- The ***degree*** of a vertex in an undirected graph is the number of edges incident with it
  - a self-loop counts twice (both ends count)
  - denoted with $deg(v)$


B is adjacent to C;
C is adjacent to B.
(A,B) is incident to
A and B.
B
C
A
F
Degree = 0
E
D
Degree = 3

Terminology: Directed Graph

- Vertex u is ***adjacent to*** vertex v in a directed graph G if (u, v) is an edge in G
  - vertex u is the initial vertex of (u, v)
- Vertex v is ***adjacent from*** vertex u
  - vertex v is the ***terminal*** (or end) vertex of (u, v)
- Degree
  - ***in-degree*** is the number of edges with the vertex as the terminal vertex
  - ***out-degree*** is the number of edges with the vertex as the initial vertex

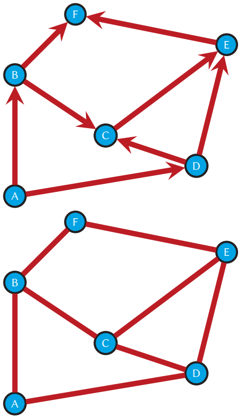

B is adjacent to C;
C is adjacent from B.
(A,B) is incident to
A and B.
In-degree = 2
Out-degree = 1
A
B
C
D
E
F

Kinds of Graphs

- directed vs undirected
- weighted vs unweighted
- simple vs non-simple
- sparse vs dense
- cyclic vs acyclic
- labeled vs unlabeled

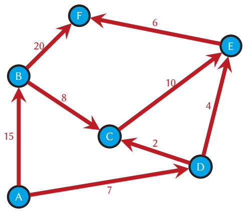
Directed vs Undirected

- Undirected if edge (x, y) implies edge (y, x).
  - otherwise directed
- Roads between cities are usually undirected (go both ways)
- Streets in cities tend to be directed (one-way)



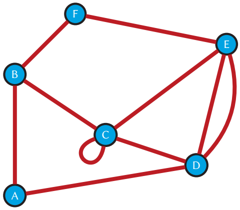
Weighted vs Unweighted

- Each edge or vertex is assigned a numerical value (weight).
- A road network might be labeled with:
    - length
    - drive-time
    - speed-limit
- In an unweighted graph, there is no distinction between edges.



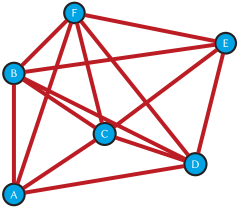
Simple vs Not simple

- Some kinds of edges make working with graphs complicated
- A ***self-loop*** is an edge (x, x) (one vertex).
- An edge (x, y) is a ***multiedge*** if it occurs more than once in a graph.



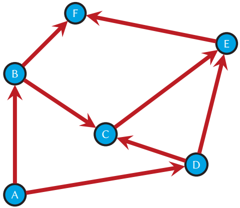
Sparse vs Dense

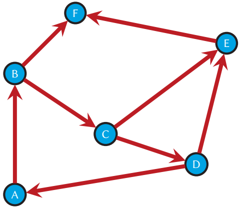
- Graphs are sparse when a small fraction of vertex pairs have edges between them
- Graphs are dense when a large fraction of vertex pairs have edges
- There's no formal distinction between sparse and dense



Cyclic vs Acyclic

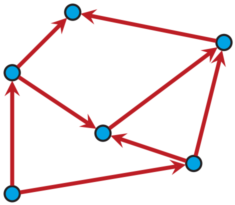
- An ***acyclic*** graph contains no cycles
- A ***cyclic*** graph contains a cycle
- Trees are *connected, acyclic, undirected* graphs
- Directed acyclic graphs are called ***DAGs***





Labeled vs Unlabeled

- Each vertex is assigned a unique name or identifier in a ***labeled*** graph
  - In an unlabeled graph, there are no named nodes
- Graphs usually have names—e.g., city names in a transportation network
- We might ignore names in graphs to determine if they are isomorphic (similar in structure)

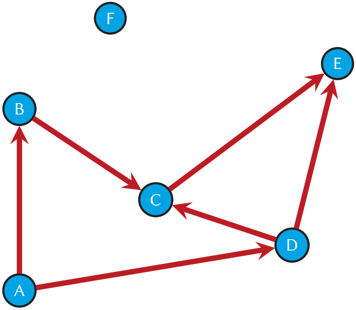


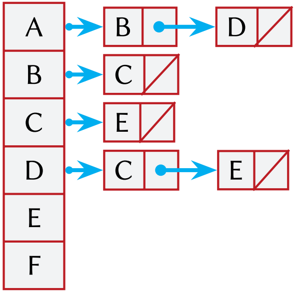
Graph Representation

Two ways to represent a graph in code:

- Adjacency List
  - A list of nodes
  - Every node has a list of adjacent nodes
- Adjacency Matrix
  - A matrix has a column and a row to represent every node
  - All entries are 0 by default
  - An entry $G[u, v]$ is 1 if there is an edge from node u to v

For each v in V, $L(v)$ = list of w such that (v, w) is in E:





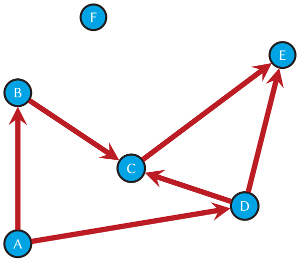
Storage space:

$$a|V| + b|E|$$

$$a = sizeof(node)$$

$$b = sizeof(\text{ linked list element})$$

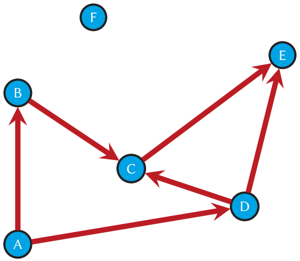
Adjacency Matrix



$$
\begin{array}{c|cccccc}
 & A & B & C & D & E & F \\
\hline
A & 0 & 1 & 0 & 1 & 0 & 0 \\
B & 1 & 0 & 1 & 0 & 0 & 0 \\
C & 0 & 1 & 0 & 1 & 1 & 0 \\
D & 1 & 0 & 1 & 0 & 1 & 0 \\
E & 0 & 0 & 1 & 1 & 0 & 0 \\
F & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Storage space: $|V|^2$

Adjacency Matrix



$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Storage space: $|V|^2$

Does this matrix represent a directed or undirected graph?

Comparing Matrix vs List

1. Faster to test if (x, y) is in a graph?

Comparing Matrix vs List

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1. adjacency matrix

Comparing Matrix vs List

1. Faster to test if (x, y) is in a graph?
2. Faster to find the degree of a vertex?

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3. Less memory on small graphs?

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7. Better for most problems?

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Analyzing Graph Algorithms

- Space and time are analyzed in terms of:
  - Number of vertices $m = |V|$
  - Number of edges $n = |E|$
- Aim for polynomial running times.
- But: is $O(m^2)$ or $O(n^3)$ a better running time?
  - depends on what the relation is between n and m
  - the number of edges m can be at most $n^2 \leq n^2$.
  - connected graphs have at least $m \geq n - 1$ edges
- Stil do not know which of two running times (such as m^2 and n^3) are better,
- Goal: implement the basic graph search algorithms in time $O(m + n)$.
  - This is linear time, since it takes $O(m + n)$ time simply to read the input.
- Note that when we work with connected graphs, a running time of $O(m + n)$ is the same as $O(m)$, since $m \geq n - 1$.

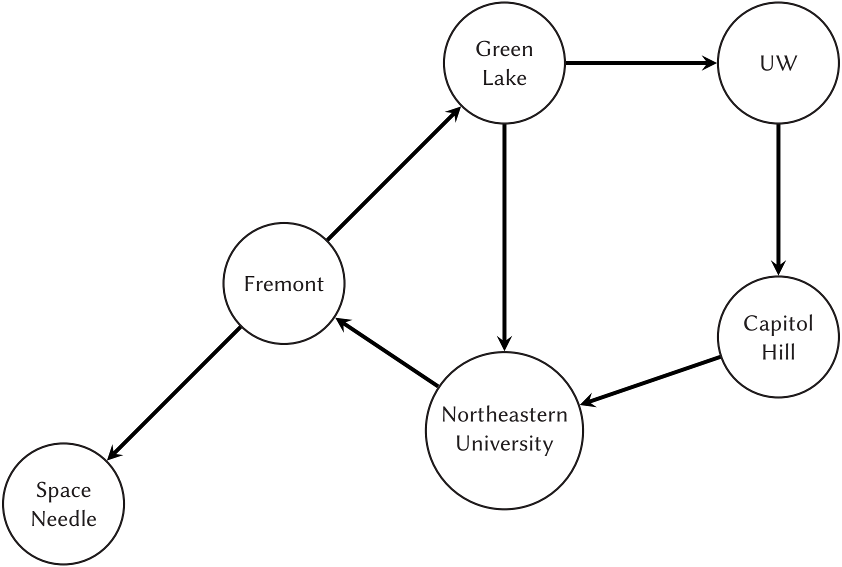
Graph Traversals

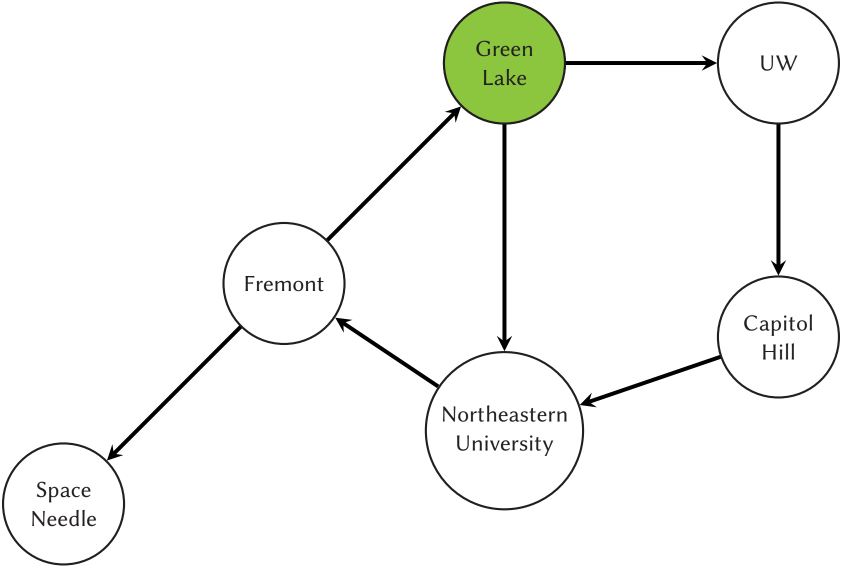
Two basic traversals:

- Breadth First Search (BFS)
- Depth First Search (DFS)

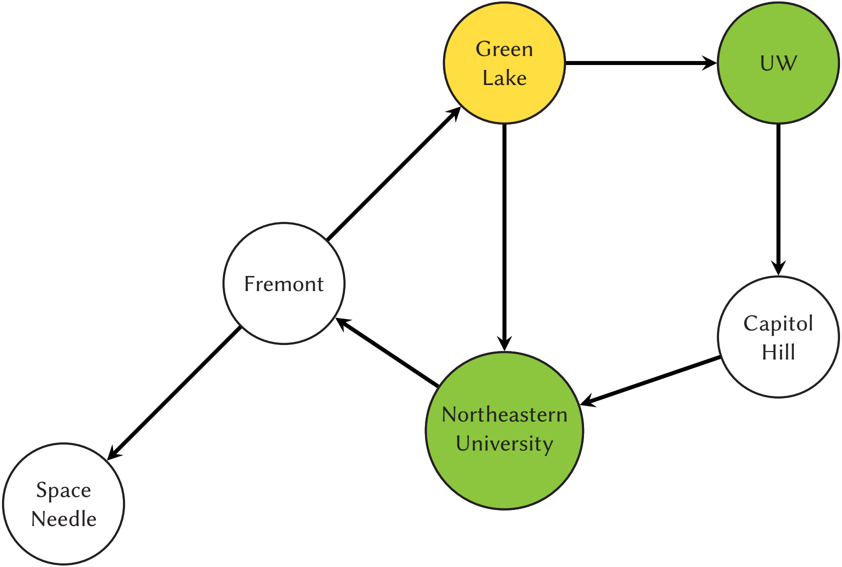
BFS

Example...


Green Lake
UW
Fremont
Capitol Hill
Northeastern University
Space Needle



What's the best way for me to get from Green Lake to Space Needle?



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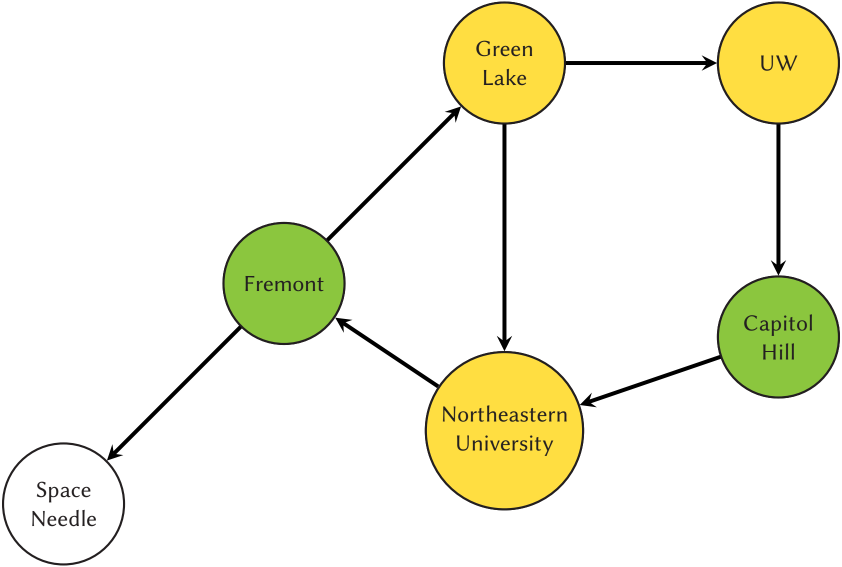

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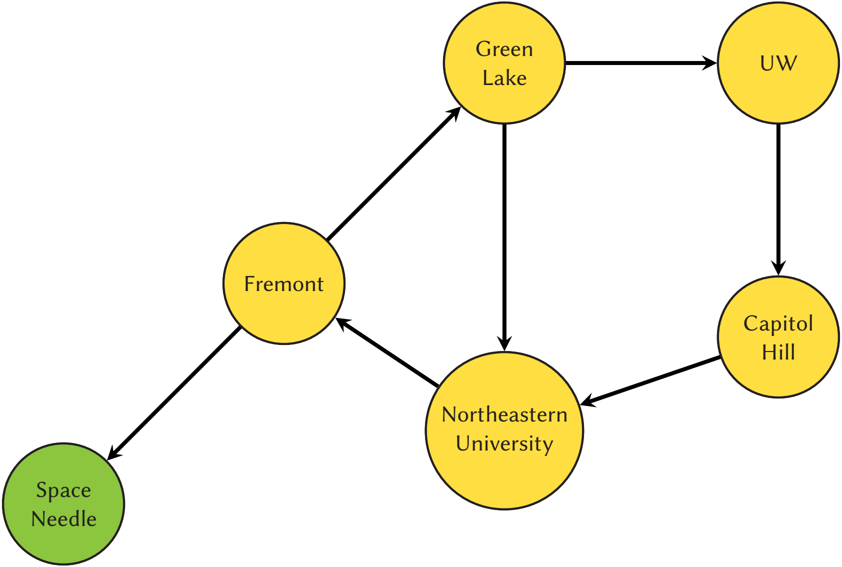

What's the best way for me to get from Green Lake to Space Needle?

BFS: The Algorithm

- Start at the start.
- Look at all the neighbors. Are any of them the destination?
- If no:
  - Look at all the neighbors of the neighbors. Are any of them the destination?
  - Look at all the neighbors of the neighbors of the neighbors. Are any of them the destination?

BFS: Runtime

- If you search the entire network, you traverse each edge at least once: $O(|E|)$
  - That is, O(number of edges)
- Keeping a queue of who to visit in order.
  - Add single node to queue: $O(1)$
  - For all nodes: O(number of nodes)
  - $O(|V|)$
- Together, it's $O(V + E)$

- Depth first search needs to check which nodes have been output or else it can get stuck in loops.
- In a connected graph, a BFS will print all nodes, but it will repeat if there are cycles and may not terminate
- As an aside, in-order, pre-order and postorder traversals only make sense in binary trees, so they aren't important for graphs. However, we do need some way to order our out-vertices (left and right in BST).

Traverse, psuedocode

```
void traverseGraph(Node start) {
  Set pending = emptySet()
  pending.add(start)
  mark start as visited
  while(pending is not empty) {
    next = pending.remove()
    for each node u adjacent to next
      if (u is not marked visited) {
        mark u
        pending.add(u)
      }
    }
  }
```

Using

- Assuming we can add and remove from our "pending" DS in $O(1)$ time, the entire traversal is $O(|E|)$
- Traversal order depends on what we use for our pending DS.
  - Stack : DFS
  - Queue: BFS
- These are the main traversal techniques in CS, but there are others!

- Breadth-first always finds shortest length paths, i.e., "optimal solutions"
- Better for "what is the shortest path from x to y"
  - But depth-first can use less space in finding a path
- If longest path in the graph is p and highest out- degree is d then DFS stack never has more than $d * p$ elements
- But a queue for BFS may hold $O(|V|)$ nodes

DFS

BFS vs DFS: Problems

BFS Applications

- Connected components
- Two-coloring graphs

DFS Applications

- Finding cycles
- Topological Sorting
- Strongly Connected Components

Single-Source Shortest Path

Input Directed graph with non-negative weighted edges, a starting node s and a destination node d

Problem Starting at the given node s, find the path with the lowest total edge weight to node d

Example A map with cities as nodes and the edges are distances between the cities. Find the shortest distance between city 1 and city 2.

Djikstra's Algorithm: Overview

- Find the "cheapest" node— the node you can get to in the shortest amount of time.
- Update the costs of the neighbors of this node.
- Repeat until you've done this for each node.
- Calculate the final path.

Djikstra's Algorithm: Formally

DJIKSTRA(G, w, s)

INITIALIZE-SINGLE-SOURCE(G, s)
$S = \emptyset$
$Q = G.V$
while $Q \neq \emptyset$
    $u =$ EXTRACT-MIN(Q)
    $S = S \cup \{u\}$
    for each vertex $v \in G.Adj[u]$
        RELAX (u, v, w)

DJIKSTRA(G, w, s)

$\triangleright$ G is a graph
$\triangleright$ w is the weighting function such that $w(u, v)$ returns the weight of the
$\triangleright$ s is the starting node
for each vertex $u \in G$
    $u.d = w(s, u)$ $\triangleright$ where $w(s, u) = \infty$ if there is no edge (s, u).
$S = \emptyset$ $\triangleright$ Nodes we know the distance to
$Q = G.V$ $\triangleright$ min-PriorityQueue starting with all our nodes, ordered by dis
while $Q \neq \emptyset$
    $u =$ EXTRACT-MIN(Q) $\triangleright$ Greedy step: get the closest node
    $S = S \cup \{u\}$ $\triangleright$ Set of nodes that have shortest-path-distance found
    for each vertex $v \in G.Adj[u]$
        RELAX (u, v, w)

RELAX(u, v, w)

$\triangleright$ u is the start node
$\triangleright$ v is the destination node
$\triangleright$ w is the weight function

Djikstra's: A walkthrough

- Find the "cheapest" node— the node you can get to in the shortest amount of time.
- Update the costs of the neighbors of this node.
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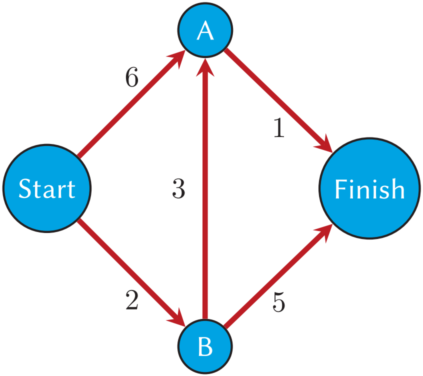
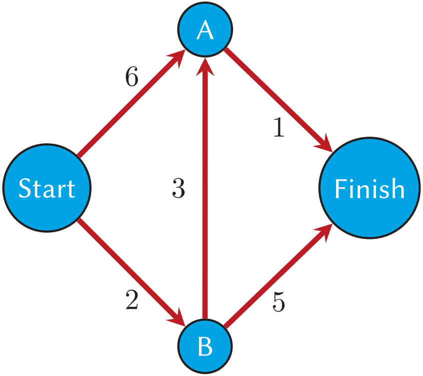


Breadth First Search: distance = 7

Step 1: Find the cheapest node

1. Should we go to A or B?
   - Make a table of how long it takes to get to each node from this node.
   - We don't know how long it takes to get to Finish, so we just say infinity for now.

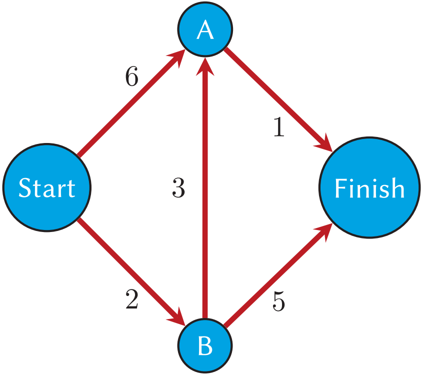
Node	Time to Node
A	6
B	2
Finish	∞



Step 2: Take the next step

1. Calculate how long it takes to get (from Start) to B's neighbors by following an edge from B
   - We chose B because it's the fastest to get to.
   - Assume we started at Start, went to B, and then now we're updating Time to Nodes.

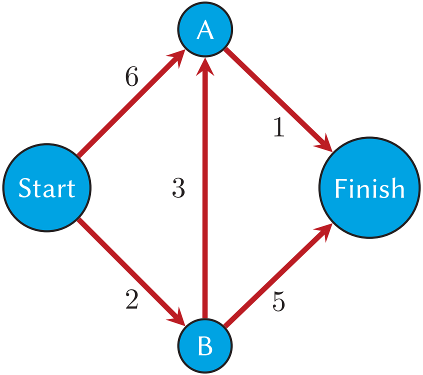
Node	Time to Node
A	~~6~~5
B	2
Finish	~~∞~~ 7



Step 3: Repeat!

1. Find the node that takes the least amount of time to get to.
   - We already did B, so let's do A.
   - Update the costs of A's neighbors
     - Takes 5 to get to A; 1 more to get to Finish

Node	Time to Node
A	~~6~~5
B	2
Finish	~~7~~6



Section 3

Topological Ordering

Topological Ordering

Input: Directed acyclic graph $G = (V, E)$

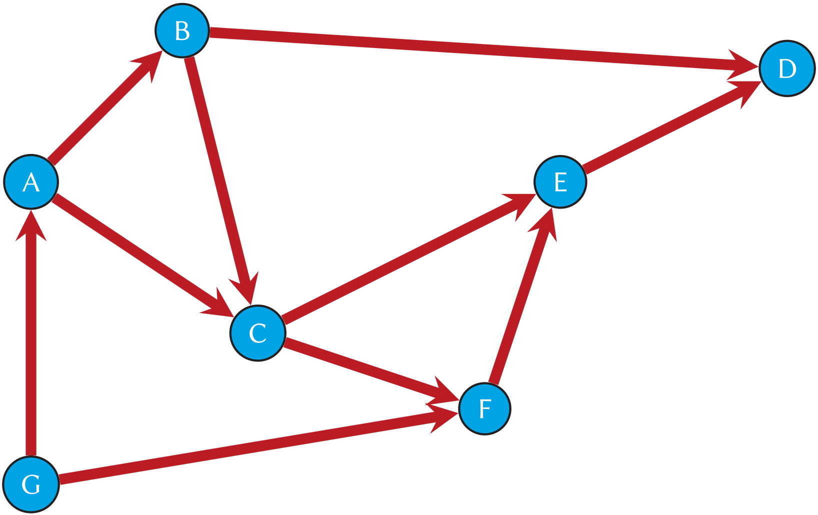
Problem: Find a linear ordering of the vertices V such that for each edge (i, j) in E, vertex i is to the left of j.

Example: Scheduling of tasks that have precedence constraints

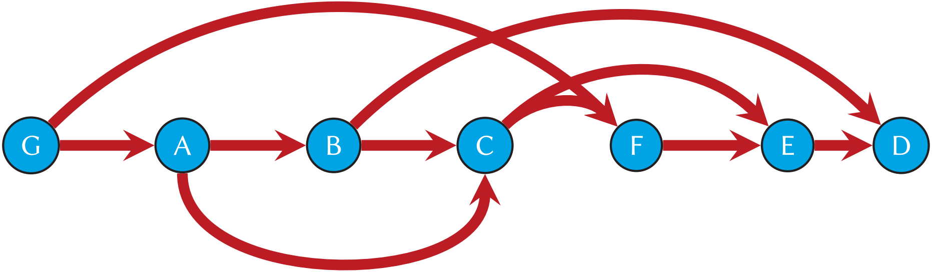
Example: Class Ordering

- Vertices are classes, and edges represent pre-reqs
- A ***topological ordering*** is any ordering that is a valid sequence of courses

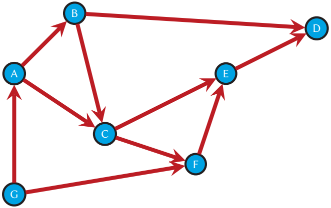
Toposort: Example Input

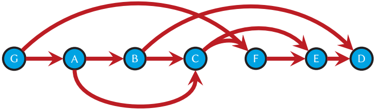


Toposort: Example Output



Toposort: Convince yourself





- Nodes are ordered in a linear fashion
- All the (directed) edges point to "future" nodes
- No edges are "pointing back"

Things to Know

- Why only DAGs?

Terminology: A DAG represents a ***partial order*** and a topological sort produces a ***total order*** that is consistent with it

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  - A cycle means there's no correct answer

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Things to Know

- Why only DAGs?
  - A cycle means there's no correct answer
- Can every DAG be topo sorted?

Terminology: A DAG represents a ***partial order*** and a topological sort produces a ***total order*** that is consistent with it

Things to Know

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Terminology: A DAG represents a ***partial order*** and a topological sort produces a ***total order*** that is consistent with it

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- Can every DAG be topo sorted?
  - Yes.
- Is there always a unique answer?
  - No, DAGs can be sorted in different ways
  - Especially when there are fewer constraints

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Using topological sort

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Using topological sort

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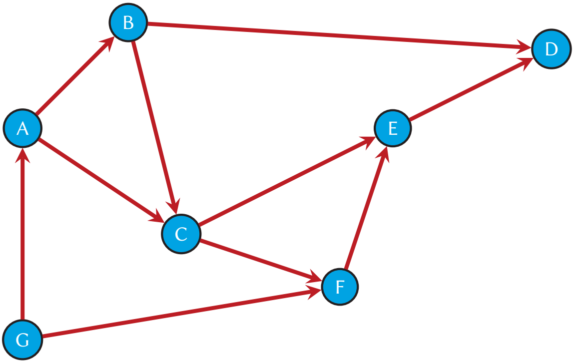
- Computing an order in which to recompute cells in a spreadsheet
- Determining an order to compile files using a Makefile
- In general, taking a dependency graph and finding an order of execution

Toposort: The Algorithm

- Mark each vertex with its in-degree
- While there are vertices not yet in the final output:
  - Choose a vertex v with labeled in-degree of 0
  - Output v and conceptually remove it from the graph
  - For each vertex u adjacent to v:
    - Decrement the in-degree of u

An Example...



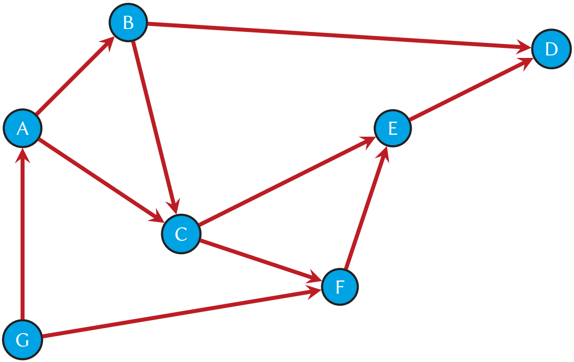
Node	In-Degree
A	1
B	1
C	2
D	2
E	2
F	2
G	0

Output order:

< >

First, calculate the in-degree of each node.

An Example...



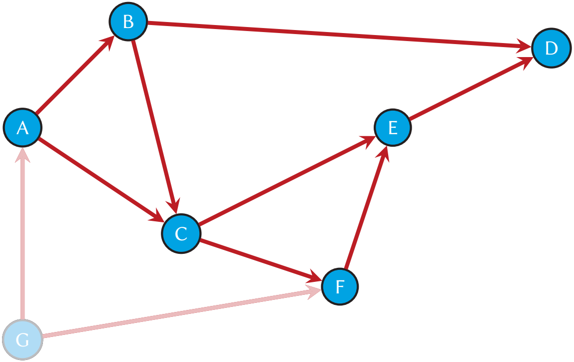
Node	In-Degree
A	1
B	1
C	2
D	2
E	2
F	2
G	0

Output order:

< >

Start with the node with in-degree of 0

An Example...



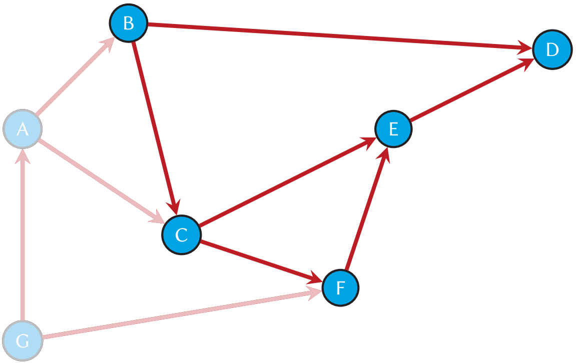
Node	In-Degree
A	~~1~~ 0
B	1
C	2
D	2
E	2
F	~~2~~ 1
G	0

Output order:

<G >

Output that node, “remove it from the graph”, decrementing the in-degree for each node it points to.

An Example...



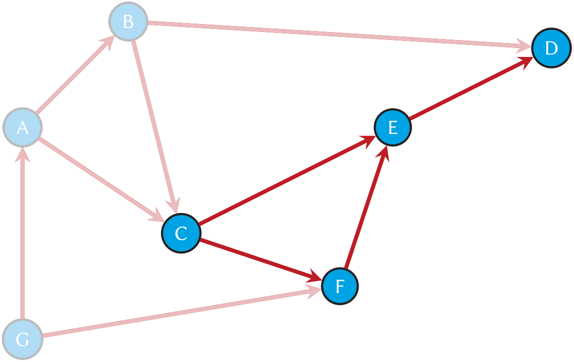
Node	In-Degree
A	~~1~~ 0
B	~~1~~ 0
C	~~2~~ 1
D	2
E	2
F	~~2~~ 1
G	0

Output order:

<G , A >

Repeat.

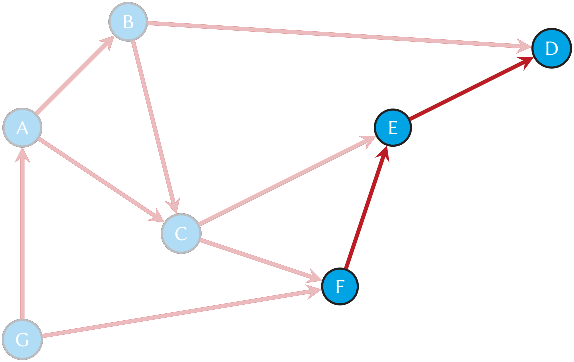
An Example...



Node	In-Degree
A	$\cancel{1}$ 0
B	$\cancel{1}$ 0
C	$\cancel{2}$ $\cancel{1}$ 0
D	$\cancel{2}$ 1
E	2
F	$\cancel{2}$ 1
G	0

Output order:

<G , A , B >

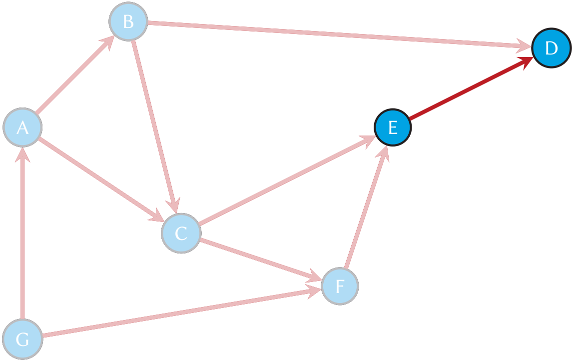


Node	In-Degree
A	$\cancel{1}$ 0
B	$\cancel{1}$ 0
C	$\cancel{2}$ $\cancel{1}$ 0
D	$\cancel{2}$ 1
E	$\cancel{2}$ 1
F	$\cancel{2}$ $\cancel{1}$ 0
G	0

Output order:

<G , A , B , C >

An Example...

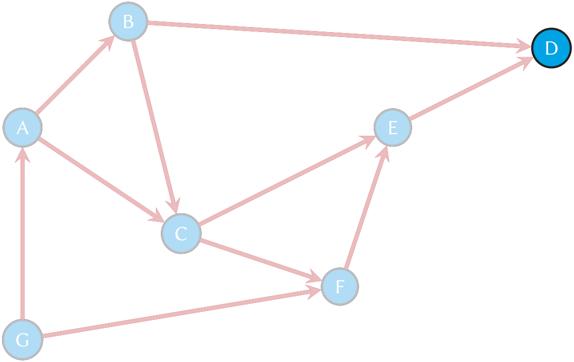


Node	In-Degree
A	$\cancel{1}$ 0
B	$\cancel{1}$ 0
C	$\cancel{2}$ $\cancel{1}$ 0
D	$\cancel{2}$ 1
E	$\cancel{2}$ 1 $\cancel{2}$ $\cancel{1}$ 0
F	$\cancel{2}$ $\cancel{1}$ 0
G	0

Output order:

<G , A , B , C , F >

An Example...

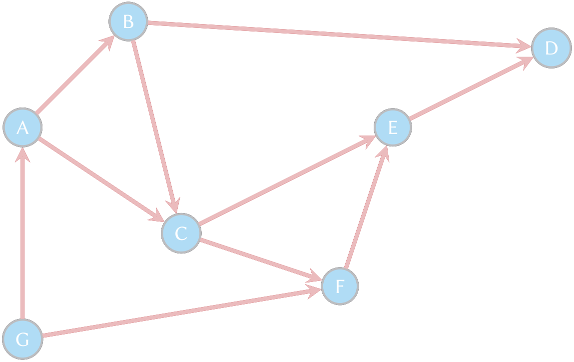


Node	In-Degree
A	$\cancel{1}$ 0
B	$\cancel{1}$ 0
C	$\cancel{2}$ $\cancel{1}$ 0
D	$\cancel{2}$ 1 $\cancel{2}$ $\cancel{1}$ 0
E	$\cancel{2}$ 1 $\cancel{2}$ $\cancel{1}$ 0
F	$\cancel{2}$ $\cancel{1}$ 0
G	0

Output order:

<G , A , B , C , F , E >

An Example...



Node	In-Degree
A	$\cancel{1}$ 0
B	$\cancel{1}$ 0
C	$\cancel{2}$ $\cancel{1}$ 0
D	$\cancel{2}$ 1 $\cancel{2}$ $\cancel{1}$0
E	$\cancel{2}$ 1 $\cancel{2}$ $\cancel{1}$ 0
F	$\cancel{2}$ $\cancel{1}$ 0
G	0

Output order:

<G , A , B , C , F , E , D >

Toposort Notes

- Always need a vertex with in-deg 0 to start
- And we will always have one because there are no cycles!
    - When we have more than one vertex with in-deg 0, it doesn?t matter which we choose.
- This is how we get more than one correct answer

Toposort: Implementation Details

- Don't want to have to search for a zero-degree node every time
- Keep the "pending" 0-deg nodes in a list/stack/queue/etc
- Note, your choice of data structure impacts order or output, but not correctness or efficiency
  - ...as long as push/pop $=O(1)$

Section 4

Strongly Connected Components

A directed graph is ***strongly connected*** if there is a directed path between any two vertices.

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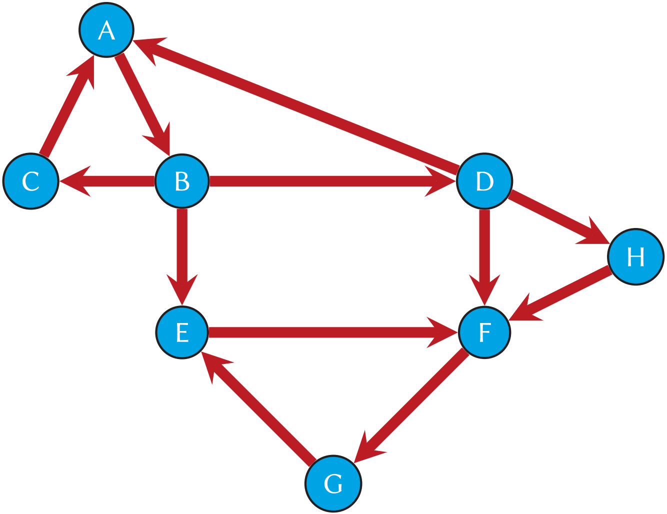
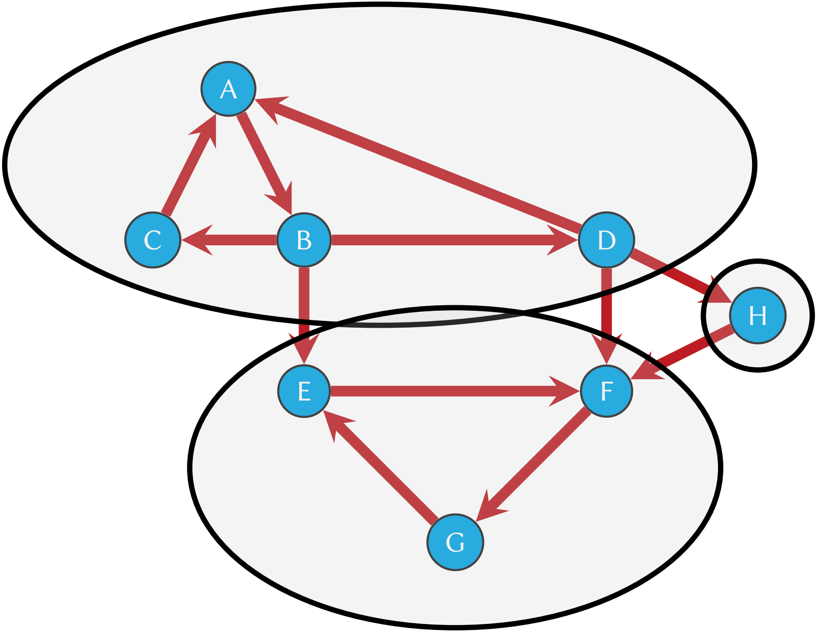
The ***strongly connected components*** of a graph is a partition of the vertices into subsets (maximal) such that each subset is strongly connected.


A
B
C
D
E
F
G
H


A
B
C
D
E
F
G
H

Strongly Connected Components

The Input: Directed or Undirected graph G

The Problem: Identify the components of G where vertices x and y are members of different components if no path exists from x to y in G.

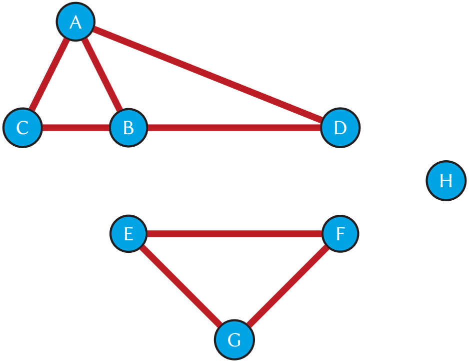
In an undirected graph G, components are ***connected*** or not.

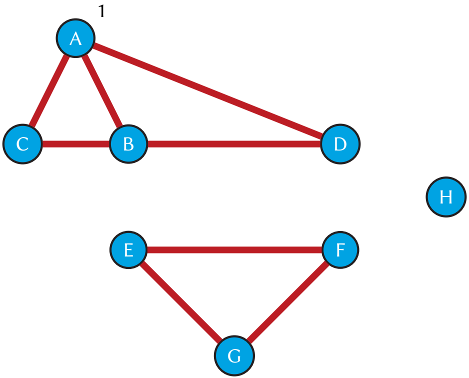
In a directed graph G, components can be ***strongly connected*** or not.

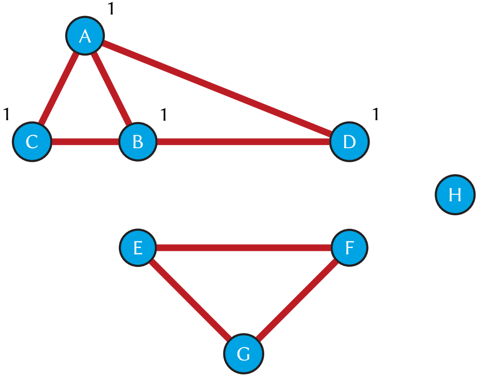
Connected Components: Algorithm Overview

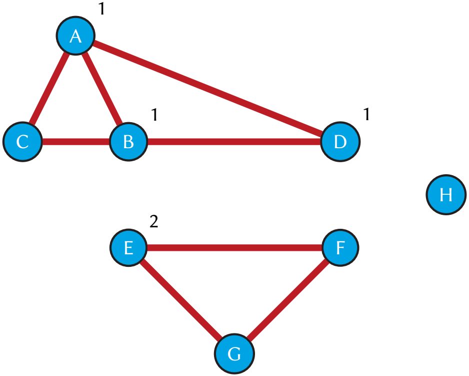
Connected Components in an undirected graph

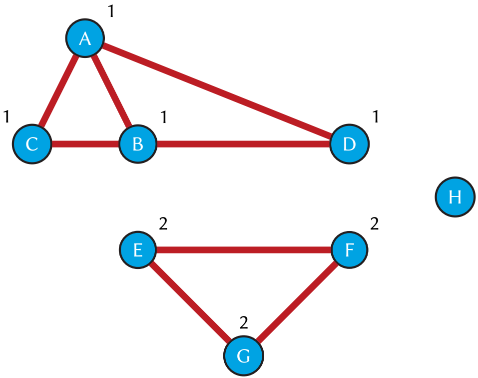
- Use either DFS or BFS
- Set the component_id for each node to 0.
- Set cur_component = 1.
- Start traversing:
  - For every node, set the component_id = cur_component.
  - When there are no more nodes to traverse, increment cur_component.

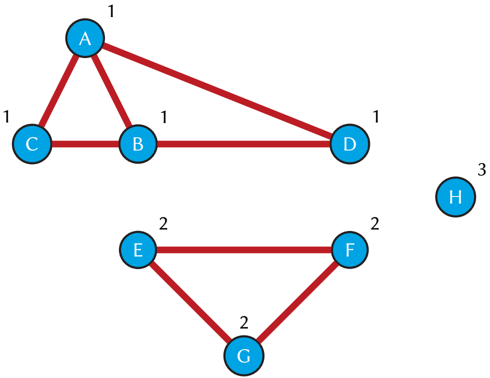

A
C
B
D
H
E
F
G


1
A
C
B
D
H
E
F
G


1
A
1
C
1
B
1
D
H
E
F
G


1
A
1
C
1
B
1
D
H
2
E
F
G


1
A
1
C
B
1
D
1
H
2
E
F
2
2
G

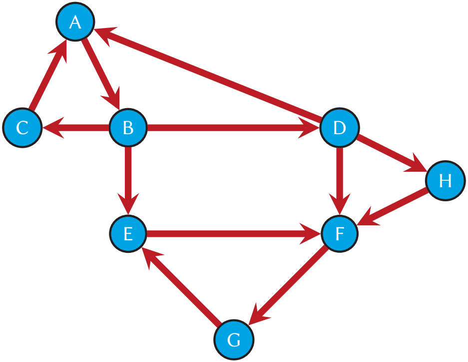

1
A
1
C
1
B
1
D
3
H
2
E
2
F
2
G

Simple for undirected... but what about directed??

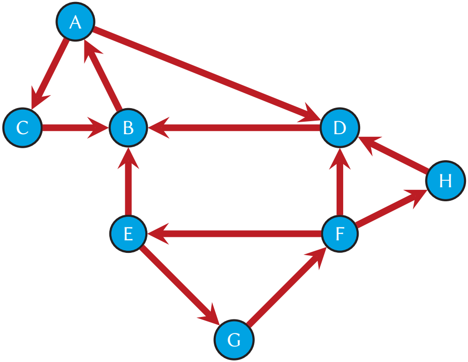
Transpose Graph

- $G^T = (V, E^T)$:
  - $E^T = \{(u, v) : (v, u) \text{ in } E\}$
  - E^T consists of the edges in G with their directions reversed
- G and G^T have the same strongly connected components

G and G^T



G



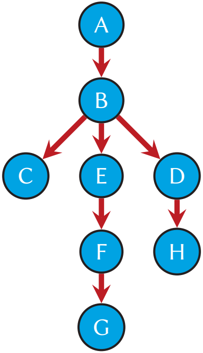
G^T

Strongly Connected Components: The Algorithm

1. Call $DFS(G)$ to compute finishing times $f[u]$ for each vertex u
2. Compute G^T
3. Call $DFS(G^T)$, but follow the vertices in order of decreasing $f[u]$
4. Output the vertices of each tree in the depth- first forest of step 3 as a separate component

Calculating Finish time

- When doing a graph traversal, there's a step that you start evaluating a node v and it's adjacency list.
- The finish time marks when the search finish's exploring v's adjacency list.



A DFS tree representing a traversal of the graph starting at Node A.

Node ID	$f(u)$
A	
B	
C	
D	
E	
F	
G	
H	

Calculating Finish time

- When doing a graph traversal, there's a step that you start evaluating a node v and it's adjacency list.
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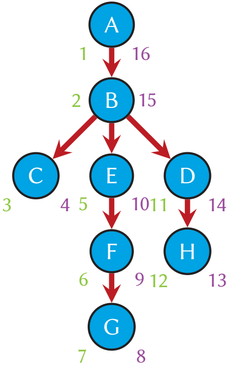
- In the following graph, the start times are marked in green; the finish times are marked in purple.



A DFS tree representing a traversal of the graph starting at Node A.

Node ID	$f(u)$
A	
B	
C	
D	
E	
F	
G	
H	

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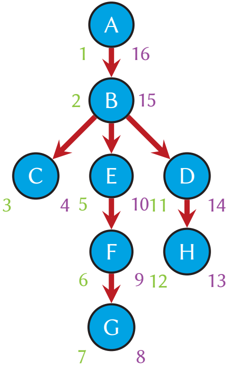


A DFS tree representing a traversal of the graph starting at Node A.

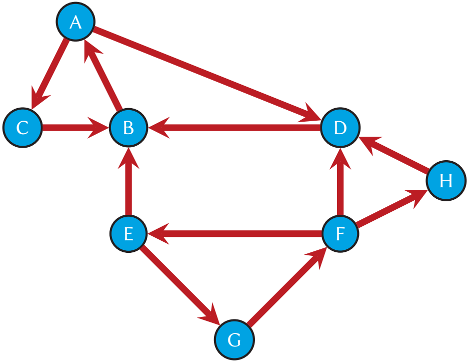
Node ID	$f(u)$
A	16
B	15
C	4
D	14
E	10
F	9
G	8
H	13

Step 2: Computer G^T

- A number of ways to do this, but whatever way, make sure the edges are reversed.

Step 3: Do DFS on G^T

- Find the node with the LAST finish time
- DFS to all the other nodes you can, without visiting a node twice
- Those are all Component 1.
- While there are more nodes that haven't been visited, repeat.



G^T

Node ID	$f(u)$
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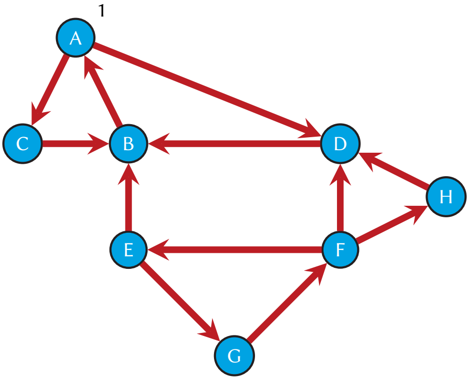
B	15
C	4
D	14
E	10
F	9
G	8
H	13



G^T

Node ID	$f(u)$
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C	4
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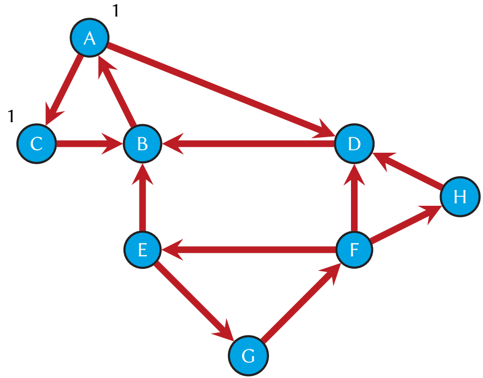
G	8
H	13



G^T

Node ID	$f(u)$
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B	15
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D	14
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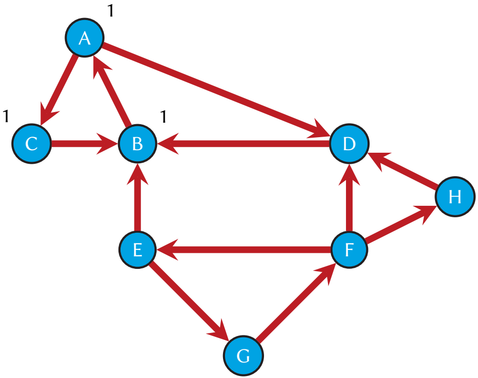

G^T

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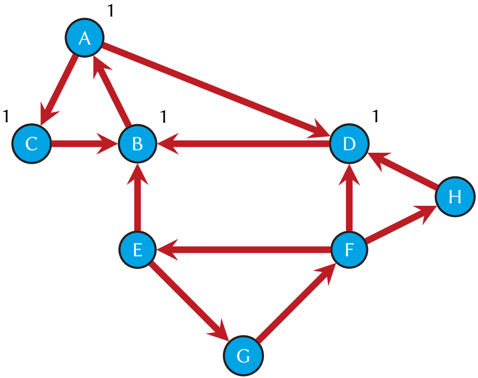

G^T

Node ID	$f(u)$
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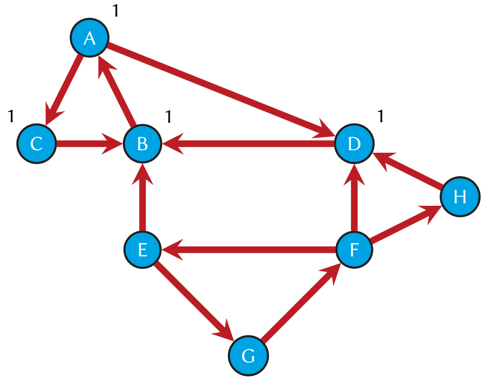
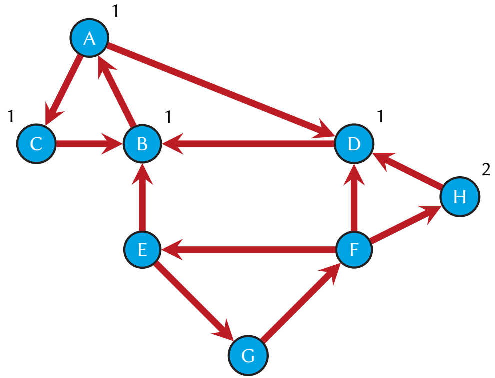
C	4
D	14
E	10
F	9
G	8
H	13



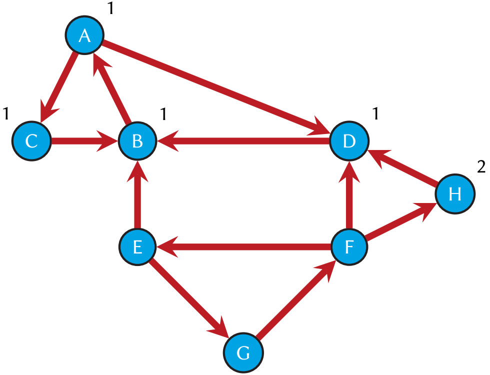
G^T

Node ID	$f(u)$
A	~~16~~-1
B	~~15~~-1
C	~~4~~-1
D	~~14~~-1
E	10
F	9
G	8
H	13



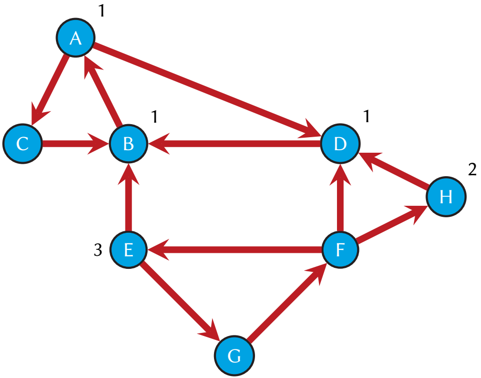
G^T

Node ID	$f(u)$
A	~~16~~-1
B	~~15~~-1
C	~~4~~-1
D	~~14~~-1
E	10
F	9
G	8
H	13



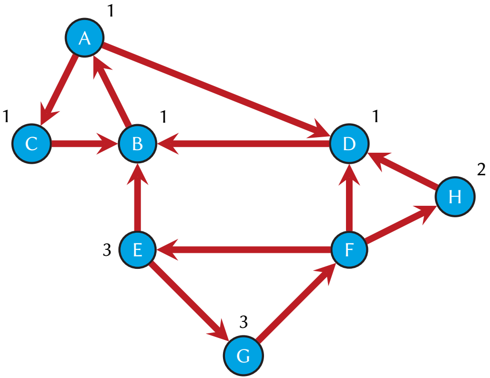
G^T

Node ID	$f(u)$
A	~~16~~-1
B	~~15~~-1
C	~~4~~-1
D	~~14~~-1
E	10
F	9
G	8
H	~~13~~-1



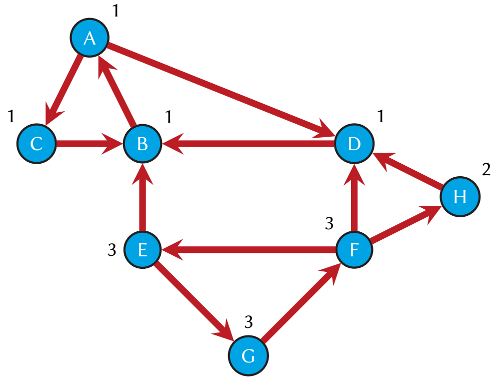
G^T

Node ID	$f(u)$
A	~~16~~-1
B	~~15~~-1
C	~~4~~-1
D	~~14~~-1
E	10
F	9
G	8
H	~~13~~-1



G^T

Node ID	$f(u)$
A	~~16~~-1
B	~~15~~-1
C	~~4~~-1
D	~~14~~-1
E	10
F	9
G	8
H	~~13~~-1



G^T

Node ID	$f(u)$
A	$\cancel{16}$-1
B	$\cancel{15}$-1
C	$\cancel{4}$-1
D	$\cancel{14}$-1
E	10
F	9
G	8
H	$\cancel{13}$-1

Section 5

Summary

Summary

What problems did we work on today?

- Sorting
- Strongly Connected Components