Contents

Abstract Algebra Toolbox Solution File

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1 Groups

Tool 1: Group Operation Checking

Solution 1.1: Exercise 1.1

1. Since $[(a_1, b_1) \star (a_2, b_2)] \star (a_3, b_3) = (a_1b_2 + b_1a_2, b_1b_2) \star (a_3, b_3) = (a_1b_2b_3 + b_1a_2b_3 + a_3b_1b_2, b_1b_2b_3)$, and $(a_1, b_1) \star [(a_2, b_2) \star (a_3, b_3)] = (a_1, b_1) \star (a_2b_3 + a_3b_2, b_2b_3) = (a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2, b_1b_2b_3)$, we know that $\star$ is associative.
2. Since $(a \star b) \star c = \frac{a}{b} \star c = \frac{a}{bc}$, and $a \star (b \star c) = a \star \frac{b}{c} = \frac{ac}{b} \neq \frac{a}{bc}$, we know that $\star$ is not associative.
3. From 1., we know that $(a_1, b_1) \star (a_2, b_2) = (a_1b_2 + b_1a_2, b_1b_2)$, and $(a_2, b_2) \star (a_1, b_1) = (a_1b_2 + b_1a_2, b_1b_2)$, hence $\star$ is commutative.
4. It is commutative, since $a \star b = \frac{a+b}{5} = \frac{b+a}{5} = b \star a$ for all $a, b \in \mathbb{Q}$.
5. Consider three residue classes $[a], [b], [c] \in \mathbb{Z}/n\mathbb{Z}$, then we know that $([a] + [b]) + [c] = [a + b + c] = [a] + ([b] + [c])$, hence addition is associative. Also, since $[a] + [b] = [a + b] = [b + a] = [b] + [a]$, we know that addition is also commutative.
6. Since $(ab)c = ac = a$ and $a(bc) = ab = a$, we know that this operation is associative. Since $ab = a \neq ba = b$, we know that it is not commutative.

Tool 2: Group or Not a Group?

Solution 1.2: Exercise 1.2

1. Since $e \cdot e = e$, we know that $e^{-1} = e$, e is invertible. Thus, $e \in S$. Also, for any invertible elements $a, b \in G$, $(ab) \star (b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = e$, we know that ab is invertible. Therefore, S is closed under $\star$. Since $\star$ is associative on G, it is also associative for S. Finally, for any invertible $a \in G$, we know that $a^{-1} \star a = e$, hence a^{-1} is also invertible.
2. For all $n > 1$, there is no multiplicative inverse for $0 \in \mathbb{Z}/n\mathbb{Z}$, hence $\mathbb{Z}/n\mathbb{Z}$ is not a group.
3.
(a) Not a group, since $\frac{1}{2} + \frac{1}{2} = 1$ which has absolute value equal to 1, hence this set is not closed under addition.
(b) Is a group, since addition in rational numbers is associative. The additive inverses of any rational number with denominator either 1 or 2 is another rational number with denominator either 1 or 2. Adding two rational numbers with denominator 1 or 2 results in another one with denominator 1 or 2. 0 can be think of $0 = \frac{0}{1}$, which is the identity element in this group.
(c) This is a group. Consider any two rational numbers $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ in lowest terms such that q_1, q_2 are odd. Then, $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$, which is a rational number with odd denominator, hence after reducing to lowest term, it still has odd denominator. Therefore, the set is closed under addition. Since the additive inverse of a rational with odd denominator is still a rational with odd denominator, and $0 = \frac{0}{1}$ has odd demonimator as above, also since addition is associative for $\mathbb{Q}$, we know that this set is a group.
4. Since given $0 \leq x, y < 1$, we know that $0 \leq x \star y < 1$, hence G is closed under $\star$. Next, since $x \star (1 - x) = 1 - \lfloor 1 \rfloor = 0$, we know that $x^{-1} = 1 - x$ for all $x \in G$, thus inverse exists for all $x \in G$ under $\star$. Then, for $x = 0$, $0 \star y = y - \lfloor y \rfloor = y - 0 = y$ for all $y \in G$, thus 0 is the identity element for operation $\star$. Finally, since $(x \star y) \star z = (x + y - \lfloor x + y \rfloor) \star z = (x + y - \lfloor x + y \rfloor) + z - \lfloor (x + y - \lfloor x + y \rfloor) + z \rfloor = x + y + z - \lfloor x + y + z \rfloor$ and $x \star (y \star z) = x \star (y + z - \lfloor y + z \rfloor) = x + (y + z - \lfloor y + z \rfloor) - \lfloor x + (y + z - \lfloor y + z \rfloor) \rfloor = x + y + z - \lfloor x + y + z \rfloor$, we know that $\star$ is associative. Therefore G is a group. Finally, since $x \star y = x + y - \lfloor x + y \rfloor = y + x - \lfloor y + x \rfloor = y \star x$, G is an abelian group.
5. Suppose $A \times B$ is an abelian group. Then, we know that $(a_1, b_1) \star (a_2, b_2) = (a_1 \star a_2, b_1 \star' b_2) = (a_2, b_2) \star (a_1, b_1) = (a_2 \star a_1, b_2 \star' b_1)$, hence $a_1 \star a_2 = a_2 \star a_1$ for all $a_1, a_2 \in A$, and $b_1 \star' b_2 = b_2 \star' b_1$ for all $b_1, b_2 \in B$, thus A, B are both abelian.
If A, B are both abelian, consider any $(a_1, b_1), (a_2, b_2) \in A \times B$, we know that $(a_1, b_1) \star (a_2, b_2) = (a_1 \star a_2, b_1 \star' b_2) = (a_2, b_2) \star (a_1, b_1) = (a_2 \star a_1, b_2 \star' b_1)$, hence $A \times B$ is abelian.
Note: in the above, we use $\star$ to symbolize operation on A and $A \times B$, and $\star'$ to denote operation on B.

Tool 3: Order of Group Elements

Solution 1.3: Exercise 1.3

1. Since $1^1 = 1$, $|1| = 1$. Since $2^n = 0 \in \mathbb{Z}/6\mathbb{Z}$, $|2| = \infty$. Since $3^n = 3 \in \mathbb{Z}/6\mathbb{Z}$, $|3| = \infty$. Since $4^n = 4 \in \mathbb{Z}/6\mathbb{Z}$, $|4| = \infty$. Since $5^2 = 1 \in \mathbb{Z}/6\mathbb{Z}$, $|5| = 2$.
2. Since $x^2 = e \in G$, we know that $|x| \leq 2$. Since if $x = e$, $e^2 = e$, and in this case x has order 1, $|x|$ is either 1 or 2.
3. Suppose $|x| = n$, i.e. $x^n = e$, and we know that $x^n \cdot (x^{-1})^n = e$, hence $(x^{-1})^n = e$, $|x^{-1}| \leq n$. If $|x^{-1}| = m < n$, we know that $(x^{-1})^m = e$, and since $(x^{-1})^m \cdot x^m = e$, $x^m = e$, which is impossible. Therefore, $|x^{-1}| = |x| = n$.
4. Firstly, since $x^n = x^{st} = (x^s)^t = e$, we know that $|x^s| \leq t$. Suppose $|x^s| = m < t$, then we know that $(x^s)^m = x^{sm} = e$, which is impossible since $sm < n = st$. Thus, $|x^s| = t$.
5. We simply need to show that $e, x, \ldots, x^{n-1}$ are all different to prove the claim. Suppose $x^i = x^j$ for some $0 \leq i, j \leq n-1$, suppose $i < j$. Then, we know that $x^{-i}x^i = x^{-i}x^j = x^{j-i} = e$, and we have $0 \leq j - i \leq n - 1 < n$, which is impossible.
6. Given any $a, b \in G$ where G is a group, suppose $|ab| = n < \infty$. Then, we know that $(ab)^n = a(ba)^{n-1}b = e$, thus $(ba)^{n-1} = a^{-1}b^{-1} = (ba)^{-1}$. Therefore, $(ba)^n = e$, $|ba| \leq n$. Suppose $|ba| = m < n$, then we know that $(ba)^m = b(ab)^{m-1}a = e$, similarly we know that $(ab)^m = e$, which is impossible since $m < n$. Hence, $|ba| = n = |ab|$.
Suppose $|ab| = \infty$ and $|ba| = n < \infty$, then from above, we know that $(ab)^n = e$, which is impossible. Hence, in this case, $|ab| = |ba| = \infty$.
7. Firstly we know that $(a, 1) \star (1, b) = (a \star 1, 1 \star b) = (1 \star a, b \star 1) = (1, b) \star (a, 1)$, hence they commutes in $A \times B$. Then, suppose $|(a, b)| = n$, thus $(a, b)^n = [(a, 1) \star (1, b)]^n = (a, 1)^n \star (1, b)^n = (a^n, b^n) = (1, 1)$. Therefore, we know that n is the smallest n satisfying $a^n = 1 = b^n$. Suppose $|a| = x$ and $|b| = y$. We know that for $m = lcm(x, y)$, $a^m = b^m = 1$, hence $n \leq m = lcm(x, y)$. Suppose $n < m$, then either $x \nmid n$ or $y \nmid n$. If $x \nmid n$, we can find $n = px + r$, where $r < x$, which means $a^n = a^{px+r} = a^{xp} \cdot a^r = a^r = 1$, which is impossible since $r < x$. Similarly for the case $y \nmid n$. Thus, we have shown that $n = lcm(x, y)$.
8. Suppose $x^i = x^j$ for some $i, j \in \mathbb{N}$, and suppose $i < j$. Then, we know that $x^{j-i} = e$, which means $|x| \leq j - i < \infty$, which is impossible.
9. Suppose the operation defined for G is $\star$. Then, G must be closed under $\star$. Consider $a \star b$, since $a \star b = a$ or $a \star b = b$ implies $b = 1$ or $a = 1$ respectively, which both are impossible, we know that $a \star b = c$. Similarly, $b \star a = c$, $a \star c = c \star a = b$, $b \star c = c \star b = a$. Consider a^2 under $\star$. $a^2 \neq a$ since $a \neq 1$. If $a^2 = b$ or $a^2 = c$, we know that $a^3 \neq 1$, which is impossible since $|a| \leq 3$. Therefore, $a^2 = 1 = b^2 = c^2$. Thus, the operation $\star$ is uniquely determined and from above we know that G is abelian under $\star$.

Tool 4: Arithmetic of Group Elements

Solution 1.4: Proof of Theorem 1.1

1. Suppose we have e, e' being identities of G. Then, we know that $ee' = e' = e$.
2. Suppose $g \in G$ has inverses $g^{-1}, g^{-1\prime}$. Then, we know that $g^{-1}g = e$, multiply both sides by $g^{-1\prime}$, we get $g^{-1}(gg^{-1\prime}) = g^{-1} = g^{-1\prime}$.
3. Since $g \cdot g^{-1} = e$, and by 2., we know that ${g^{-1}}^{-1} = g$.

Solution 1.5: Proof of Theorem 1.2

Suppose $g, x, y \in G$ for some group G and $gx = gy$, then multiply by g^{-1} on the left on both sides we get $x = y$. Similarly if $xg = yg$, multiply by g^{-1} on the right on both sides we get $x = y$.

Solution 1.6: Exercise 1.4

1. Since $(a_1 \star \cdots \star a_n) \star (a_n^{-1} \star \cdots a_1^{-1}) = e$ by associativity of $\star$, by Theorem 1.1, we know that $(a_1 \star \cdots \star a_n)^{-1} = (a_n^{-1} \star \cdots a_1^{-1})$.
2. Given group G, $x, y \in G$. Suppose $xy = yx$, then multiply both sides on the left by y^{-1}, we get $y^{-1}xy = x$. Multiply both sides on the left by $x^{-1}y^{-1}$, we get $x^{-1}y^{-1}xy = e$.
Suppose $y^{-1}xy = x$, multiply both sides on the left by y, we get $xy = yx$.
Suppose $x^{-1}y^{-1}xy = e$, multiply both sides on the left by yx, we get $xy = yx$.
3. Suppose $x^2 = e$ for all $x \in G$ for some group G. Hence, we know by Theorem 1.1 that $x = x^{-1}$ for all $x \in G$. Suppose $x, y \in G$. Then, $(xy)^2 = xyxy = e$, thus $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$. Therefore, G is abelian.
4. If $xyz = e$, then we know that $yz = x^{-1}$ by Theorem 1.1. Hence, $yzx = x^{-1}x = e$. However yxz is not always true. Pick any non-abelian group where $xy \neq yx$, then we know that $xyz \neq yxz$ by Theorem 1.2.

Tool 5: Dihedral Group Computation

Solution 1.7: Proof of Theorem 1.3

We are trying to show that the composition of two reflections is either identity or a rotation.

We have several cases. Given reflections s and t, suppose $s = t$, then $st = s^2 = e$.

Suppose n is odd. Label clockwise each vertex from $1, 2, \ldots n$. Then, we know that s fixes some point x and t fixes some point y. Suppose $x < y$ (case $x > y$ is exactly the same). Then, we know that $t \circ s : x \mapsto x \mapsto y+y-x = 2y-x$, $t \circ s : y \mapsto x-(y-x) = 2x-y \mapsto y-(2x-y)+y = 3y-2x$. Hence, both x and y rotates clockwise by $2y-2x$. We can repeat above and verify other vertex as well, in this case $t \circ s$ is a rotation.

Suppose n is even. Then, we have several cases. The case when both s and t are Type II reflections is the same as if n is odd, since there are fixed points, we can repeat arguments above. Suppose s is a Type I reflection and t is a Type II reflection. Pick one of the side cut through by the reflection axis of s, namely $(x, x+1)$, and pick a fixed point of t, namely y. Suppose $y \geq x+1$. ($y \leq x$ case is exactly the same) In this case, we know that $t \circ s : x \mapsto x+1 \mapsto y+y-(x+1) = 2y-x-1$, and $t \circ s : y \mapsto x-(y-(x+1)) = 2x-y+1 \mapsto y+y-(2x-y+1) = 3y-2x-1$. Here both x and y rotates clockwise by $2y-2x-1$ sides. Finally, if both s, t are Type I reflections, we can pick one of the sides cut through by the reflection axis of s, namely $(x, x+1)$, and one of the sides cut through by the reflection axis of t, namely $(y, y+1)$, and suppose $x < y$. In this case, $t \circ s : x \mapsto x+1 \mapsto y+1+y-(x+1) = 2y-x$, and $t \circ s : y \mapsto 2x-y+1 \mapsto y+1+2y-2x-1 = 3y-2x$, hence both x and y rotates clockwise by $2y-2x$.

With all cases being above, we have verified the claim.

Solution 1.8: Exercise 1.5

1. Since r is a rotation, we can write r as ρ^k for some $k \in \mathbb{N}$. Label the n-gon clockwise by $1, 2, \ldots, n$. If n is odd, suppose the reflection fixes a point a. Then, we know that for any point x, under rs, $x \mapsto a-(x-a) = 2a - x \mapsto 2a - x - k$, and under sr^{-1}, $x \mapsto x + k \mapsto a - (x + k - a) = 2a - x - k$, thus we know that in this case $rs = sr^{-1}$.
If n is even, suppose the reflection s is a Type II reflection, then this is the same case as if n is odd. If the reflection s is a Type I reflection, pick one side $(a, a + 1)$ where the axis of reflection goes through. Then, we know that for any point x, under rs, $x \mapsto a - (x - a - 1) = 2a - x + 1 \mapsto 2a-x+1-k$, and under sr^{-1}, $x \mapsto x+k \mapsto a-(x+k-a-1) = 2a-x+1-k$, thus in this case we also have $rs = sr^{-1}$.
2. Let s be any element in D_{2n} that is not a power of r, then we know that s is not a rotation, hence a reflection. Then, this follows from 1.
3. If n is odd, consider an element $a \neq e$ in D_{2n}. If a is a rotation, let us write $a = \rho^k$. Consider any reflection s, we then know that $as = sa^{-1}$, however since $a = \rho^k$, $a^{-1} = \rho^{n-k} \neq \rho^k$ since n is odd, $a \neq a^{-1}$, hence $as = sa^{-1} \neq sa$, thus a does not commute with any reflection.
If a is a reflection, consider any rotation $r = \rho^k$, then we know that $ar = a(r^{-1})^{-1} = r^{-1}a$, however since $r \neq r^{-1}$, we know that $ar \neq ra$, hence a does not commute with any rotation using similar reason as above. Therefore, e is the only element that commutes with all other elements in D_{2n}.
4. Since x, y are elements of order 2, we know that $x^2 = y^2 = e$, thus $x = x^{-1}$, $y = y^{-1}$. Thus, $tx = xyx$ and $xt^{-1} = x(xy)^{-1} = xy^{-1}x^{-1} = xyx$. Similar for yt.

Tool 6: Rigid Motion Group Order for 3D Spaces

Solution 1.9: Exercise 1.6

1. For any rigid motion on the cube, a face of the cube is shifted to some face on the cube (6 possibilities). Fixing where this face is shifted, we also have 4 choices of sending a particular edge on this face to an edge on the target face, but then due to the constraint of rigid motions, we have only 1 choice of sending other edges, since we can not ”twist” the cube. Therefore, the group of rigid motions of a cube has order $6 \cdot 4 = 24$.
2. For any rigid motion on a tetrahedron, a face of the tetrahedron is sent to some face of this tetrahedron (4 choices). Fixing where this face is sent, we also have 3 choices of sending a particular edge on this face to an edge on the target face. Due to constraint of rigid motions, we have only 1 choice of sending other edges, therefore, this group has order $4 \cdot 3 = 12$.
3. Repeat argument in 1., but replace with 8 faces and 3 edges, we know that the group order is $8 \cdot 3 = 24$. 4. Repeat above arguments with x faces and y edges, we get the answer.

Tool 7: Computations on Cycles and Permutations

Solution 1.10: Proof of Theorem 1.5

We know that S_n is the set of bijective permutations on the set $X = \{1, 2, \ldots, n\}$. There are n possibilities that 1 can be mapped to, but after 1 is mapped, 2 can’t map to the same place since it is a bijection, so we are left with $n-1$ choices for 2. In the end, we have only 1 choice of mapping n, thus the order of S_n is $n!$.

Solution 1.11: Exercise 1.7

1. We have $S_3 = \{id, (12), (23), (13), (123), (132)\}$. The orders are $1, 2, 2, 2, 3, 3$ respectively.
2. We do the computation from right to left, we have $1 \mapsto 2 \mapsto 2$, $2 \mapsto 3 \mapsto 5$, $3 \mapsto 1 \mapsto 3$, $5 \mapsto 5 \mapsto 1$, hence the result is (125).
3. We first observe that the order of any n-cycle is n. The order should be a multiple of 2 since we contain a 2-cycle, should be a multiple of 3, since we also contain a 3-cycle, and should be a multiple of 5 as well, since we contain a 5 cycle. Therefore, the order should be the least common multiple of $2, 3, 5$, which is 30. The proof of second claim follows exact same logic as above.
4. Firstly, there are $n \cdot (n-1) \cdots (n-m+1)$ ways to choose m elements in X into a m-cycle. However, for any m-cycle, there are m ways to write it in the cycle notation. So the total number of different m-cycles should be $\frac{n \cdot (n-1) \cdots (n-m+1)}{m}$.
5. Firstly, there are $\frac{n \cdot (n-1)}{2}$ ways to choose the first cycle, and after the first 2-cycle is chosen, there are only $n-2$ elements left in X, hence only $\frac{(n-2) \cdot (n-3)}{2}$ choices to choose the second cycle. Further more, we know that $(ab)(cd)$ and $(cd)(ab)$ is the same cycle, we need to further divide by 2. Therefore, the number of permutation who can be written as the multiplication of two disjoint 2-cycles is $\frac{n \cdot (n-1)}{2} \cdot \frac{(n-2) \cdot (n-3)}{2} \cdot \frac{1}{2} = \frac{n(n-1)(n-2)(n-3)}{8}$.

Tool 8: Order of Cycles

Solution 1.12: Exercise 1.8

1. Firstly, since the element has order 4 in S_4, its disjoint cycle decomposition has cycles with length's least common multiple 4, hence are either 4-cycles or product of 2 disjoint 2-cycles. Thus, we have $(1234), (1243), (1324), (1342), (1423), (1432)$ being the 4-cycles, and $(12)(34)$, $(13)(24)$, $(14)(23)$ being the products of 2 disjoint 2-cycles.
2. Firstly, $\sigma : a_x \mapsto a_{(x+1) \mod m}$. Suppose $\sigma^i : a_x \mapsto a_{(x+i) \mod m}$, then we know that $\sigma^{i+1} = \sigma \circ \sigma^i : a_x \mapsto a_{(x+i) \mod m} \mapsto a_{(x+i+1) \mod m}$. Thus, by induction the claim is proved.
3. Consider the disjoint cycle decomposition of this element. We know that the lengths of the disjont cycle decomposition of this element must have least common multiple 2, hence it must has the cycle decomposition of commuting(disjoint) 2-cycles, since we do not write 1-cycles in cycle decomposition.
4. Firstly we observe that this element is not identity, since identity has order 1. Thus, it must has a cycle decomposition with some n-cycles, $n \geq 2$. Consider the disjoint cycle decomposition of this element. We know that the lengths of the disjont cycle decomposition of this element must have least common multiple p, hence it must has the cycle decomposition of commuting(disjoint) p-cycles, since no other number $1 < x \leq n, x \neq p$ divide p and we do not write 1 cycles. Therefore, it must have a cycle decomposition of commuting p-cycles. This is wrong when p is not a prime. Consider $p = 6 = n$ and $x = (12)(345)$, x is still order 6 but does not have 6-cycles.
5. Suppose $n > 8$, then it is not possible that one of the element of the n-cycle "vanishes" as we compute σ^k to get τ while the other $1, 2, \ldots 8$ stays in the cycle decomposition. Thus, $n = 8$. Since a 8-cycle has order 8, but τ has order 2, we must have k being a multiple of 4, but not a multiple of 8, since otherwise this will result identity. Consider $k = 4$, then we can find $\sigma = (13572468)$.
6. Similar as in 5, $n = 5$. However, τ has order 6, but a 5-cycle has order 5, hence it is not possible that $\tau = \sigma^k$, since $|\sigma^k| \leq |\sigma| = 5$.

Tool 9: Usage of Various Examples of Groups

Solution 1.13: Exercise 1.9

1. We need the determinant of the matrix to be not equal 0, hence we have: $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$. Since $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$, and $\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$, we know that $GL_2(\mathbb{F}_2)$ is non-abelian. 2. Consider any matrix A such that $A_{11} = x \in F$, and $A_{jj} = 1$ for all $1 < j \leq n$, and all other entries not on the diagonal is 0. We know that $\det(A) \neq 0$, hence $A \in GL(n, F)$, but we have a infinite amount of choices for x if F is infinite. If F is finite, the total number of $n \times n$ matrices is finite, hence $GL(n, F)$ must be finite.
3. Firstly, the identity matrix is an upper triangular matrix. Secondly, we know that:

$$\begin{pmatrix}a & b\\0 & c\end{pmatrix} \cdot \begin{pmatrix}d & e\\0 & f\end{pmatrix} = \begin{pmatrix}ad & ae + bf\\0 & cf,\end{pmatrix}$$

which is also upper triangular. Given $\begin{pmatrix}a & b\\0 & c\end{pmatrix}$, we know that the matrix $\begin{pmatrix}\frac{1}{a} & -\frac{b}{ac}\\0 & \frac{1}{c}\end{pmatrix}$ is the inverse by above formula, which is also upper-triangular.
3. The order of 1 is 1, order of -1 is 2, order of i, j, k is 4.