In[ ]:=

```
Clear["Global`*"];
```

0. Readme

1. The below derivations of the Vxy, Vxz, and Vyz of a tesseroid are based on Eqs. (A.6), (A.7) and (A.9) of Deng and Shen (2019) SGG, where $\lambda 3$, $\theta 3$, and r3 represent λ', θ', and r' of the integration point.
2. VxySphericalZonalBand, VxzSphericalZonalBand, and VyzSphericalZonalBand are the Vxy, Vxz, and Vyz of a spherical zonal band, which are equal to zero.
3. VxySphericalShell, VxzSphericalShell and VyzSphericalShell are the Vxy, Vxz, and Vyz of a spherical shell, which are equal to zero.

In[ ]:=

```
φ = Pi / 2; (*φ=Pi/2-θ, θ=0*)
```

In[ ]:=

```
φ3 = Pi / 2 - θ3;
```

In[ ]:=

```
cosPhi = Sin[φ] * Sin[φ3] + Cos[φ] * Cos[φ3] * Cos[λ3 - λ]
```

Out[ ]=

```
Cos[θ3]
```

In[ ]:=

```
Δx = r3 * (Cos[φ] * Sin[φ3] - Sin[φ] * Cos[φ3] * Cos[λ3 - λ])
```

Out[ ]=

```
-r3 Cos[λ - λ3] Sin[θ3]
```

In[ ]:=

```
Δy = r3 * Cos[φ3] * Sin[λ3 - λ]
```

Out[ ]=

```
-r3 Sin[θ3] Sin[λ - λ3]
```

In[ ]:=

```
Δz = r3 * cosPhi - r
```

Out[ ]=

```
-r + r3 Cos[θ3]
```

1. Vxy

In[ ]:=

$$\text{FullSimplify}\left[\int r3^2\,\text{Sin}[\theta 3]\left(\frac{3*(-r3\,\text{Cos}[\lambda-\lambda 3]\,\text{Sin}[\theta 3])*(-r3\,\text{Sin}[\theta 3]\,\text{Sin}[\lambda-\lambda 3])}{\left(\sqrt{r^2+r3^2-2*r*r3*\text{Cos}[\theta 3]}\right)^5}\right)d\lambda 3\right]$$

Out[ ]=

$$\frac{3\,r3^4\,\text{Cos}[2\,(\lambda-\lambda 3)]\,\text{Sin}[\theta 3]^3}{4\left(r^2+r3^2-2\,r\,r3\,\text{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:=

$$\text{Vxy}\lambda 3[\lambda 3\_] := \frac{3\,r3^4\,\text{Cos}[2\,(\lambda-\lambda 3)]\,\text{Sin}[\theta 3]^3}{4\left(r^2+r3^2-2\,r\,r3\,\text{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:=

```
FullSimplify[Vxyλ3[λ2] - Vxyλ3[λ1]]
```

Out[ ]=

$$-\frac{3\,r3^4\,\text{Sin}[\theta 3]^3\,\text{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\text{Sin}[\lambda 1-\lambda 2]}{2\left(r^2+r3^2-2\,r\,r3\,\text{Cos}[\theta 3]\right)^{5/2}}$$

In[∘]:= FullSimplify$\left[\int\left(-\frac{3\,\text{r3}^4\,\text{Sin}[\theta 3]^3\,\text{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\text{Sin}[\lambda 1-\lambda 2]}{2\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{5/2}}\right)d\theta 3\right]$

Out[∘]=

$$\left(\text{r3}\left(8\,\text{r}^4+26\,\text{r}^2\,\text{r3}^2+8\,\text{r3}^4-3\left(\text{r}^2+\text{r3}^2\right)^2\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]}{\text{r}^2+\text{r3}^2}}+6\,\text{r}\,\text{r3}\left(\text{r}^2+\text{r3}^2\right)\text{Cos}[\theta 3]\left(-4+\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]}{\text{r}^2+\text{r3}^2}}\right)+6\,\text{r}^2\,\text{r3}^2\,\text{Cos}[2\,\theta 3]\right)\text{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\text{Sin}[\lambda 1-\lambda 2]\right)\Big/\left(8\,\text{r}^3\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{3/2}\right)$$

In[∘]:= Vxyθ3[θ3_] :=

$$\left(\text{r3}\left(8\,\text{r}^4+26\,\text{r}^2\,\text{r3}^2+8\,\text{r3}^4-3\left(\text{r}^2+\text{r3}^2\right)^2\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]}{\text{r}^2+\text{r3}^2}}+6\,\text{r}\,\text{r3}\left(\text{r}^2+\text{r3}^2\right)\text{Cos}[\theta 3]\left(-4+\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]}{\text{r}^2+\text{r3}^2}}\right)+6\,\text{r}^2\,\text{r3}^2\,\text{Cos}[2\,\theta 3]\right)\text{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\text{Sin}[\lambda 1-\lambda 2]\right)\Big/\left(8\,\text{r}^3\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{3/2}\right)$$

In[∘]:= Vxyθ3[θ2] - Vxyθ3[θ1]

Out[∘]=

$$-\left(\left(\text{r3}\left(8\,\text{r}^4+26\,\text{r}^2\,\text{r3}^2+8\,\text{r3}^4-3\left(\text{r}^2+\text{r3}^2\right)^2\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 1]}{\text{r}^2+\text{r3}^2}}+6\,\text{r}\,\text{r3}\left(\text{r}^2+\text{r3}^2\right)\text{Cos}[\theta 1]\left(-4+\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 1]}{\text{r}^2+\text{r3}^2}}\right)+6\,\text{r}^2\,\text{r3}^2\,\text{Cos}[2\,\theta 1]\right)\text{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\text{Sin}[\lambda 1-\lambda 2]\right)\Big/\left(8\,\text{r}^3\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 1]\right)^{3/2}\right)\right)+$$

$$\left(\text{r3}\left(8\,\text{r}^4+26\,\text{r}^2\,\text{r3}^2+8\,\text{r3}^4-3\left(\text{r}^2+\text{r3}^2\right)^2\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 2]}{\text{r}^2+\text{r3}^2}}+6\,\text{r}\,\text{r3}\left(\text{r}^2+\text{r3}^2\right)\text{Cos}[\theta 2]\left(-4+\sqrt{1-\frac{2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 2]}{\text{r}^2+\text{r3}^2}}\right)+6\,\text{r}^2\,\text{r3}^2\,\text{Cos}[2\,\theta 2]\right)\text{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\text{Sin}[\lambda 1-\lambda 2]\right)\Big/\left(8\,\text{r}^3\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 2]\right)^{3/2}\right)$$

In[∘]:= Simplify[

$$\int\left(-\left(\left(r3\left(8r^4+26r^2r3^2+8r3^4-3\left(r^2+r3^2\right)^2\sqrt{1-\frac{2\,r\,r3\,\mathrm{Cos}[\theta1]}{r^2+r3^2}}+6\,r\,r3\left(r^2+r3^2\right)\mathrm{Cos}[\theta1]\left(-4+\sqrt{1-\frac{2\,r\,r3\,\mathrm{Cos}[\theta1]}{r^2+r3^2}}\right)+6r^2r3^2\,\mathrm{Cos}[2\,\theta1]\right)\mathrm{Sin}[2\lambda-\lambda1-\lambda2]\,\mathrm{Sin}[\lambda1-\lambda2]\right)\Big/\left(8r^3\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta1]\right)^{3/2}\right)\right)+\left(r3\left(8r^4+26r^2r3^2+8r3^4-3\left(r^2+r3^2\right)^2\sqrt{1-\frac{2\,r\,r3\,\mathrm{Cos}[\theta2]}{r^2+r3^2}}+6\,r\,r3\left(r^2+r3^2\right)\mathrm{Cos}[\theta2]\left(-4+\sqrt{1-\frac{2\,r\,r3\,\mathrm{Cos}[\theta2]}{r^2+r3^2}}\right)+6r^2r3^2\,\mathrm{Cos}[2\,\theta2]\right)\mathrm{Sin}[2\lambda-\lambda1-\lambda2]\,\mathrm{Sin}[\lambda1-\lambda2]\right)\Big/\left(8r^3\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta2]\right)^{3/2}\right)\right)dr3\Big]$$

Out[∘]=

$$\frac{1}{8\,r^3}\left(\frac{\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta1]\right)^{3/2}}{\left(\frac{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta1]}{r^2+r3^2}\right)^{3/2}}-\left(2\left(13r^4+13r^2r3^2+4r3^4-r\,r3\left(25r^2+4r3^2\right)\mathrm{Cos}[\theta1]-r^2\left(3r^2+r3^2\right)\mathrm{Cos}[2\,\theta1]+3r^3\,r3\,\mathrm{Cos}[3\,\theta1]\right)\right)\Big/\left(3\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta1]}\right)-\frac{\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta2]\right)^{3/2}}{\left(\frac{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta2]}{r^2+r3^2}\right)^{3/2}}+\left(2\left(13r^4+13r^2r3^2+4r3^4-r\,r3\left(25r^2+4r3^2\right)\mathrm{Cos}[\theta2]-r^2\left(3r^2+r3^2\right)\mathrm{Cos}[2\,\theta2]+3r^3\,r3\,\mathrm{Cos}[3\,\theta2]\right)\right)\Big/\left(3\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta2]}\right)+2r^3\,\mathrm{Cos}[\theta1]\left(-5+\mathrm{Cos}[2\,\theta1]\right)\mathrm{Log}\left[r3-r\,\mathrm{Cos}[\theta1]+\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta1]}\right]-2r^3\,\mathrm{Cos}[\theta2]\left(-5+\mathrm{Cos}[2\,\theta2]\right)\mathrm{Log}\left[r3-r\,\mathrm{Cos}[\theta2]+\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta2]}\right]\right)\mathrm{Sin}[2\lambda-\lambda1-\lambda2]\,\mathrm{Sin}[\lambda1-\lambda2]$$

In[ ]:= Vxyr3[r3_] := $\frac{1}{8\, r^3}$

$$\left(\frac{\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 1]\right)^{3/2}}{\left(\frac{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 1]}{r^2+r3^2}\right)^{3/2}} - \left(2\left(13\,r^4+13\,r^2\,r3^2+4\,r3^4-r\,r3\left(25\,r^2+4\,r3^2\right)\mathrm{Cos}[\theta 1] - r^2\left(3\,r^2+r3^2\right)\mathrm{Cos}[2\,\theta 1]+3\,r^3\,r3\,\mathrm{Cos}[3\,\theta 1]\right)\right)\Big/\left(3\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 1]}\right) - \frac{\left(r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 2]\right)^{3/2}}{\left(\frac{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 2]}{r^2+r3^2}\right)^{3/2}} + \left(2\left(13\,r^4+13\,r^2\,r3^2+4\,r3^4-r\,r3\left(25\,r^2+4\,r3^2\right)\mathrm{Cos}[\theta 2]-r^2\left(3\,r^2+r3^2\right)\mathrm{Cos}[2\,\theta 2]+3\,r^3\,r3\,\mathrm{Cos}[3\,\theta 2]\right)\right)\Big/\left(3\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 2]}\right) + 2\,r^3\,\mathrm{Cos}[\theta 1]\,(-5+\mathrm{Cos}[2\,\theta 1])\,\mathrm{Log}\left[r3-r\,\mathrm{Cos}[\theta 1]+\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 1]}\right] - 2\,r^3\,\mathrm{Cos}[\theta 2]\,(-5+\mathrm{Cos}[2\,\theta 2])\,\mathrm{Log}\left[r3-r\,\mathrm{Cos}[\theta 2]+\sqrt{r^2+r3^2-2\,r\,r3\,\mathrm{Cos}[\theta 2]}\right]\right)$$

$$\mathrm{Sin}[2\,\lambda-\lambda 1-\lambda 2]\,\mathrm{Sin}[\lambda 1-\lambda 2]$$

In[ ]:= G * rho * (Vxyr3[r2] - Vxyr3[r1])

Out[ ]=

$$G\,rho\left(-\frac{1}{8\,r^3}\left(\frac{\left(r^2+r1^2-2\,r\,r1\,Cos[\theta 1]\right)^{3/2}}{\left(\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 1]}{r^2+r1^2}\right)^{3/2}}-\left(2\left(13\,r^4+13\,r^2\,r1^2+4\,r1^4-r\,r1\left(25\,r^2+4\,r1^2\right)Cos[\theta 1]-r^2\left(3\,r^2+r1^2\right)Cos[2\,\theta 1]+3\,r^3\,r1\,Cos[3\,\theta 1]\right)\right)\Big/\left(3\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 1]}\right)-\frac{\left(r^2+r1^2-2\,r\,r1\,Cos[\theta 2]\right)^{3/2}}{\left(\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}{r^2+r1^2}\right)^{3/2}}+\left(2\left(13\,r^4+13\,r^2\,r1^2+4\,r1^4-r\,r1\left(25\,r^2+4\,r1^2\right)Cos[\theta 2]-r^2\left(3\,r^2+r1^2\right)Cos[2\,\theta 2]+3\,r^3\,r1\,Cos[3\,\theta 2]\right)\right)\Big/\left(3\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}\right)+2\,r^3\,Cos[\theta 1]\,(-5+Cos[2\,\theta 1])\,Log\left[r1-r\,Cos[\theta 1]+\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 1]}\right]-2\,r^3\,Cos[\theta 2]\,(-5+Cos[2\,\theta 2])\,Log\left[r1-r\,Cos[\theta 2]+\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}\right]\right)Sin[2\,\lambda-\lambda 1-\lambda 2]\,Sin[\lambda 1-\lambda 2]+\frac{1}{8\,r^3}\left(\frac{\left(r^2+r2^2-2\,r\,r2\,Cos[\theta 1]\right)^{3/2}}{\left(\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}{r^2+r2^2}\right)^{3/2}}-\left(2\left(13\,r^4+13\,r^2\,r2^2+4\,r2^4-r\,r2\left(25\,r^2+4\,r2^2\right)Cos[\theta 1]-r^2\left(3\,r^2+r2^2\right)Cos[2\,\theta 1]+3\,r^3\,r2\,Cos[3\,\theta 1]\right)\right)\Big/\left(3\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}\right)-\frac{\left(r^2+r2^2-2\,r\,r2\,Cos[\theta 2]\right)^{3/2}}{\left(\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}{r^2+r2^2}\right)^{3/2}}+\left(2\left(13\,r^4+13\,r^2\,r2^2+4\,r2^4-r\,r2\left(25\,r^2+4\,r2^2\right)Cos[\theta 2]-r^2\left(3\,r^2+r2^2\right)Cos[2\,\theta 2]+3\,r^3\,r2\,Cos[3\,\theta 2]\right)\right)\Big/\left(3\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}\right)+2\,r^3\,Cos[\theta 1]\,(-5+Cos[2\,\theta 1])\,Log\left[r2-r\,Cos[\theta 1]+\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}\right]-2\,r^3\,Cos[\theta 2]\,(-5+Cos[2\,\theta 2])\,Log\left[r2-r\,Cos[\theta 2]+\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}\right]\right)Sin[2\,\lambda-\lambda 1-\lambda 2]\,Sin[\lambda 1-\lambda 2]\right)$$

In[ ]:=
```
Vxy[λ1_, λ2_, θ1_, θ2_] :=
 G rho (-1/(8 r^3) ((r^2 + r1^2 - 2 r r1 Cos[θ1])^(3/2)/((r^2+r1^2-2 r r1 Cos[θ1])/(r^2+r1^2))^(3/2) - (2 (13 r^4 + 13 r^2 r1^2 + 4 r1^4 -
        r r1 (25 r^2 + 4 r1^2) Cos[θ1] - r^2 (3 r^2 + r1^2) Cos[2 θ1] + 3 r^3 r1 Cos[3 θ1]))/
      (3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) - (r^2 + r1^2 - 2 r r1 Cos[θ2])^(3/2)/((r^2+r1^2-2 r r1 Cos[θ2])/(r^2+r1^2))^(3/2) +
     (2 (13 r^4 + 13 r^2 r1^2 + 4 r1^4 - r r1 (25 r^2 + 4 r1^2) Cos[θ2] - r^2 (3 r^2 + r1^2)
          Cos[2 θ2] + 3 r^3 r1 Cos[3 θ2]))/(3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]) +
     2 r^3 Cos[θ1] (-5 + Cos[2 θ1]) Log[r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]] -
     2 r^3 Cos[θ2] (-5 + Cos[2 θ2]) Log[r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]])
    Sin[2 λ - λ1 - λ2] Sin[λ1 - λ2] +
   1/(8 r^3) ((r^2 + r2^2 - 2 r r2 Cos[θ1])^(3/2)/((r^2+r2^2-2 r r2 Cos[θ1])/(r^2+r2^2))^(3/2) - (2 (13 r^4 + 13 r^2 r2^2 + 4 r2^4 -
        r r2 (25 r^2 + 4 r2^2) Cos[θ1] - r^2 (3 r^2 + r2^2) Cos[2 θ1] + 3 r^3 r2 Cos[3 θ1]))/
      (3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]) - (r^2 + r2^2 - 2 r r2 Cos[θ2])^(3/2)/((r^2+r2^2-2 r r2 Cos[θ2])/(r^2+r2^2))^(3/2) +
     (2 (13 r^4 + 13 r^2 r2^2 + 4 r2^4 - r r2 (25 r^2 + 4 r2^2) Cos[θ2] - r^2 (3 r^2 + r2^2)
          Cos[2 θ2] + 3 r^3 r2 Cos[3 θ2]))/(3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]) +
     2 r^3 Cos[θ1] (-5 + Cos[2 θ1]) Log[r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]] -
     2 r^3 Cos[θ2] (-5 + Cos[2 θ2]) Log[r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]])
    Sin[2 λ - λ1 - λ2] Sin[λ1 - λ2])
```

In[ ]:=
```
VxySphericalZonalBand = Vxy[0, 2 * Pi, θ1, θ2]
```
Out[ ]=
```
0
```

In[ ]:=
```
VxySphericalShell = Vxy[0, 2 * Pi, 0, Pi]
```
Out[ ]=
```
0
```

2. Vxz

In[ ]:=
$$\text{FullSimplify}\left[\int \text{r3}^2\,\text{Sin}[\theta 3]\left(\frac{3*(-\text{r3}\,\text{Cos}[\lambda-\lambda 3]\,\text{Sin}[\theta 3])*(-\text{r}+\text{r3}\,\text{Cos}[\theta 3])}{\left(\sqrt{\text{r}^2+\text{r3}^2-2*\text{r}*\text{r3}*\text{Cos}[\theta 3]}\right)^5}\right)d\lambda 3\right]$$

Out[ ]=
$$\frac{3\,\text{r3}^3\,(-\text{r}+\text{r3}\,\text{Cos}[\theta 3])\,\text{Sin}[\theta 3]^2\,\text{Sin}[\lambda-\lambda 3]}{\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:=
$$\text{Vxz}\lambda 3[\lambda 3\_] := \frac{3\,\text{r3}^3\,(-\text{r}+\text{r3}\,\text{Cos}[\theta 3])\,\text{Sin}[\theta 3]^2\,\text{Sin}[\lambda-\lambda 3]}{\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:= FullSimplify[Vxzλ3[λ2] - Vxzλ3[λ1]]

Out[ ]=
$$\frac{6\,\text{r3}^3\,(-\text{r}+\text{r3}\,\text{Cos}[\theta 3])\,\text{Cos}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\,\text{Sin}[\theta 3]^2\,\text{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]}{\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:=
$$\int\left(\frac{6\,\text{r3}^3\,(-\text{r}+\text{r3}\,\text{Cos}[\theta 3])\,\text{Cos}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\,\text{Sin}[\theta 3]^2\,\text{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]}{\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{5/2}}\right)d\text{r3}$$

Out[ ]=
$$6\,\text{Cos}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\left(\text{ArcTanh}\left[\frac{\text{r3}-\text{r}\,\text{Cos}[\theta 3]}{\sqrt{\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]}}\right]\text{Cos}[\theta 3]+\left(\left(\text{r}^3+3\,\text{r}\,\text{r3}^2-2\,\text{r3}^3\,\text{Cos}[\theta 3]-3\,\text{r}\left(\text{r}^2+2\,\text{r3}^2\right)\text{Cos}[2\,\theta 3]+6\,\text{r}^2\,\text{r3}\,\text{Cos}[3\,\theta 3]+4\,\text{r3}^3\,\text{Cos}[3\,\theta 3]-3\,\text{r}\,\text{r3}^2\,\text{Cos}[4\,\theta 3]\right)\text{Csc}[\theta 3]^2\right)\Big/\left(6\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{3/2}\right)\right)\text{Sin}[\theta 3]^2\,\text{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]$$

In[ ]:=
$$\text{Vxzr3}[\text{r3}\_] := 6\,\text{Cos}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\left(\text{ArcTanh}\left[\frac{\text{r3}-\text{r}\,\text{Cos}[\theta 3]}{\sqrt{\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]}}\right]\text{Cos}[\theta 3]+\left(\left(\text{r}^3+3\,\text{r}\,\text{r3}^2-2\,\text{r3}^3\,\text{Cos}[\theta 3]-3\,\text{r}\left(\text{r}^2+2\,\text{r3}^2\right)\text{Cos}[2\,\theta 3]+6\,\text{r}^2\,\text{r3}\,\text{Cos}[3\,\theta 3]+4\,\text{r3}^3\,\text{Cos}[3\,\theta 3]-3\,\text{r}\,\text{r3}^2\,\text{Cos}[4\,\theta 3]\right)\text{Csc}[\theta 3]^2\right)\Big/\left(6\left(\text{r}^2+\text{r3}^2-2\,\text{r}\,\text{r3}\,\text{Cos}[\theta 3]\right)^{3/2}\right)\right)\text{Sin}[\theta 3]^2\,\text{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]$$

In[ ]:= Vxzr3[r2] - Vxzr3[r1]

Out[ ]=

$$-6\,\mathrm{Cos}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]\left(\mathrm{ArcTanh}\left[\frac{r1-r\,\mathrm{Cos}[\theta3]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta3]}}\right]\mathrm{Cos}[\theta3]+\left(\left(r^3+3\,r\,r1^2-2\,r1^3\,\mathrm{Cos}[\theta3]-3\,r\left(r^2+2\,r1^2\right)\mathrm{Cos}[2\,\theta3]+6\,r^2\,r1\,\mathrm{Cos}[3\,\theta3]+4\,r1^3\,\mathrm{Cos}[3\,\theta3]-3\,r\,r1^2\,\mathrm{Cos}[4\,\theta3]\right)\mathrm{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta3]\right)^{3/2}\right)\right)\mathrm{Sin}[\theta3]^2\,\mathrm{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]+6\,\mathrm{Cos}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]\left(\mathrm{ArcTanh}\left[\frac{r2-r\,\mathrm{Cos}[\theta3]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta3]}}\right]\mathrm{Cos}[\theta3]+\left(\left(r^3+3\,r\,r2^2-2\,r2^3\,\mathrm{Cos}[\theta3]-3\,r\left(r^2+2\,r2^2\right)\mathrm{Cos}[2\,\theta3]+6\,r^2\,r2\,\mathrm{Cos}[3\,\theta3]+4\,r2^3\,\mathrm{Cos}[3\,\theta3]-3\,r\,r2^2\,\mathrm{Cos}[4\,\theta3]\right)\mathrm{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta3]\right)^{3/2}\right)\right)\mathrm{Sin}[\theta3]^2\,\mathrm{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]$$

In[ ]:=

$$\mathbf{Simplify}\left[\int\left(-6\,\mathrm{Cos}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]\left(\mathrm{ArcTanh}\left[\frac{r1-r\,\mathrm{Cos}[\theta3]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta3]}}\right]\mathrm{Cos}[\theta3]+\left(\left(r^3+3\,r\,r1^2-2\,r1^3\,\mathrm{Cos}[\theta3]-3\,r\left(r^2+2\,r1^2\right)\mathrm{Cos}[2\,\theta3]+6\,r^2\,r1\,\mathrm{Cos}[3\,\theta3]+4\,r1^3\,\mathrm{Cos}[3\,\theta3]-3\,r\,r1^2\,\mathrm{Cos}[4\,\theta3]\right)\mathrm{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta3]\right)^{3/2}\right)\right)\mathrm{Sin}[\theta3]^2\,\mathrm{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]+6\,\mathrm{Cos}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]\left(\mathrm{ArcTanh}\left[\frac{r2-r\,\mathrm{Cos}[\theta3]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta3]}}\right]\mathrm{Cos}[\theta3]+\left(\left(r^3+3\,r\,r2^2-2\,r2^3\,\mathrm{Cos}[\theta3]-3\,r\left(r^2+2\,r2^2\right)\mathrm{Cos}[2\,\theta3]+6\,r^2\,r2\,\mathrm{Cos}[3\,\theta3]+4\,r2^3\,\mathrm{Cos}[3\,\theta3]-3\,r\,r2^2\,\mathrm{Cos}[4\,\theta3]\right)\mathrm{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta3]\right)^{3/2}\right)\right)\mathrm{Sin}[\theta3]^2\,\mathrm{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]\right)d\theta3\right]$$

Out[ ]=

$$\frac{2}{3}\,\text{Cos}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\left(-\frac{2\,\text{r1}^2\,\sqrt{\frac{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta 3]}{(r-\text{r1})^2}}\,\text{EllipticF}\left[\frac{\theta 3}{2},-\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]}{r\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta 3]}}-\right.$$

$$\left(2\,(r^2+\text{r1}^2)\,\sqrt{\frac{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta 3]}{(r-\text{r1})^2}}\right.$$

$$\left.\left((r-\text{r1})^2\,\text{EllipticE}\left[\frac{\theta 3}{2},-\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]-(r^2+\text{r1}^2)\,\text{EllipticF}\left[\frac{\theta 3}{2},-\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]\right)\right)\Big/$$

$$\left(r^3\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta 3]}\right)+\frac{2\,\text{r2}^2\,\sqrt{\frac{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta 3]}{(r-\text{r2})^2}}\,\text{EllipticF}\left[\frac{\theta 3}{2},-\frac{4\,r\,\text{r2}}{(r-\text{r2})^2}\right]}{r\,\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta 3]}}+$$

$$\left(2\,(r^2+\text{r2}^2)\,\sqrt{\frac{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta 3]}{(r-\text{r2})^2}}\,\left((r-\text{r2})^2\,\text{EllipticE}\left[\frac{\theta 3}{2},-\frac{4\,r\,\text{r2}}{(r-\text{r2})^2}\right]-\right.\right.$$

$$\left.\left.(r^2+\text{r2}^2)\,\text{EllipticF}\left[\frac{\theta 3}{2},-\frac{4\,r\,\text{r2}}{(r-\text{r2})^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta 3]}\right)+$$

$$\frac{\left(-2\,\text{r1}\,(r^2+\text{r1}^2)+r\,(3\,r^2+\text{r1}^2)\,\text{Cos}[\theta 3]-3\,r^2\,\text{r1}\,\text{Cos}[2\,\theta 3]\right)\,\text{Sin}[\theta 3]}{r^2\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta 3]}}+$$

$$\frac{\left(2\,\text{r2}\,(r^2+\text{r2}^2)-r\,(3\,r^2+\text{r2}^2)\,\text{Cos}[\theta 3]+3\,r^2\,\text{r2}\,\text{Cos}[2\,\theta 3]\right)\,\text{Sin}[\theta 3]}{r^2\,\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta 3]}}-$$

$$3\,\text{ArcTanh}\left[\frac{\text{r1}-r\,\text{Cos}[\theta 3]}{\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta 3]}}\right]\,\text{Sin}[\theta 3]^3+$$

$$\left.3\,\text{ArcTanh}\left[\frac{\text{r2}-r\,\text{Cos}[\theta 3]}{\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta 3]}}\right]\,\text{Sin}[\theta 3]^3\right)\,\text{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]$$

In[∘]:=

$$\text{Vxz}\theta3[\theta3\_] := \frac{2}{3}\,\text{Cos}\left[\lambda - \frac{\lambda1}{2} - \frac{\lambda2}{2}\right]\left(-\frac{2\,\text{r1}^2\,\sqrt{\frac{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta3]}{(r-\text{r1})^2}}\;\text{EllipticF}\left[\frac{\theta3}{2}, -\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]}{r\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta3]}} - \left(2\,(r^2+\text{r1}^2)\,\sqrt{\frac{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta3]}{(r-\text{r1})^2}}\left((r-\text{r1})^2\,\text{EllipticE}\left[\frac{\theta3}{2}, -\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right] - (r^2+\text{r1}^2)\,\text{EllipticF}\left[\frac{\theta3}{2}, -\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta3]}\right) + \frac{2\,\text{r2}^2\,\sqrt{\frac{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta3]}{(r-\text{r2})^2}}\;\text{EllipticF}\left[\frac{\theta3}{2}, -\frac{4\,r\,\text{r2}}{(r-\text{r2})^2}\right]}{r\,\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta3]}} + \left(2\,(r^2+\text{r2}^2)\,\sqrt{\frac{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta3]}{(r-\text{r2})^2}}\left((r-\text{r2})^2\,\text{EllipticE}\left[\frac{\theta3}{2}, -\frac{4\,r\,\text{r2}}{(r-\text{r2})^2}\right] - (r^2+\text{r2}^2)\,\text{EllipticF}\left[\frac{\theta3}{2}, -\frac{4\,r\,\text{r2}}{(r-\text{r2})^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta3]}\right) + \frac{\left(-2\,\text{r1}\,(r^2+\text{r1}^2) + r\,(3\,r^2+\text{r1}^2)\,\text{Cos}[\theta3] - 3\,r^2\,\text{r1}\,\text{Cos}[2\,\theta3]\right)\text{Sin}[\theta3]}{r^2\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta3]}} + \frac{\left(2\,\text{r2}\,(r^2+\text{r2}^2) - r\,(3\,r^2+\text{r2}^2)\,\text{Cos}[\theta3] + 3\,r^2\,\text{r2}\,\text{Cos}[2\,\theta3]\right)\text{Sin}[\theta3]}{r^2\,\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta3]}} - 3\,\text{ArcTanh}\left[\frac{\text{r1}-r\,\text{Cos}[\theta3]}{\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta3]}}\right]\text{Sin}[\theta3]^3 + 3\,\text{ArcTanh}\left[\frac{\text{r2}-r\,\text{Cos}[\theta3]}{\sqrt{r^2+\text{r2}^2-2\,r\,\text{r2}\,\text{Cos}[\theta3]}}\right]\text{Sin}[\theta3]^3\right)\text{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]$$

In[∘]:= $G * \text{rho} * (\text{Vxz}\theta3[\theta2] - \text{Vxz}\theta3[\theta1])$

Out[∘]=

$$G\,\text{rho}\left(-\frac{2}{3}\,\text{Cos}\left[\lambda - \frac{\lambda1}{2} - \frac{\lambda2}{2}\right]\left(-\frac{2\,\text{r1}^2\,\sqrt{\frac{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta1]}{(r-\text{r1})^2}}\;\text{EllipticF}\left[\frac{\theta1}{2}, -\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]}{r\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta1]}} - \left(2\,(r^2+\text{r1}^2)\,\sqrt{\frac{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta1]}{(r-\text{r1})^2}}\left((r-\text{r1})^2\,\text{EllipticE}\left[\frac{\theta1}{2}, -\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right] - (r^2+\text{r1}^2)\,\text{EllipticF}\left[\frac{\theta1}{2}, -\frac{4\,r\,\text{r1}}{(r-\text{r1})^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+\text{r1}^2-2\,r\,\text{r1}\,\text{Cos}[\theta1]}\right) + \right.$$

$$\frac{2\,r2^2\,\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}{(r-r2)^2}}\;EllipticF\left[\frac{\theta 1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\,\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}}+$$

$$\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}{(r-r2)^2}}\left((r-r2)^2\,EllipticE\left[\frac{\theta 1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r2^2\right)EllipticF\left[\frac{\theta 1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}\right)+$$

$$\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)Cos[\theta 1]-3\,r^2\,r1\,Cos[2\,\theta 1]\right)Sin[\theta 1]}{r^2\,\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 1]}}+$$

$$\frac{\left(2\,r2\left(r^2+r2^2\right)-r\left(3\,r^2+r2^2\right)Cos[\theta 1]+3\,r^2\,r2\,Cos[2\,\theta 1]\right)Sin[\theta 1]}{r^2\,\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}}-$$

$$3\,ArcTanh\left[\frac{r1-r\,Cos[\theta 1]}{\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 1]}}\right]Sin[\theta 1]^3+$$

$$\left.3\,ArcTanh\left[\frac{r2-r\,Cos[\theta 1]}{\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 1]}}\right]Sin[\theta 1]^3\right)Sin\left[\frac{\lambda 1-\lambda 2}{2}\right]+$$

$$\frac{2}{3}\,Cos\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\left(-\frac{2\,r1^2\,\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}{(r-r1)^2}}\;EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\,\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}}-\right.$$

$$\left(2\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}{(r-r1)^2}}\left((r-r1)^2\,EllipticE\left[\frac{\theta 2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r1^2\right)EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}\right)+$$

$$\frac{2\,r2^2\,\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}{(r-r2)^2}}\;EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\,\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}}+$$

$$\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}{(r-r2)^2}}\left((r-r2)^2\,EllipticE\left[\frac{\theta 2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r2^2\right)EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}\right)+$$

$$\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)Cos[\theta 2]-3\,r^2\,r1\,Cos[2\,\theta 2]\right)Sin[\theta 2]}{r^2\,\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}}+$$

$$\frac{\left(2\ r2\ \left(r^2+r2^2\right)-r\ \left(3\ r^2+r2^2\right)\ Cos[\theta 2]+3\ r^2\ r2\ Cos[2\ \theta 2]\right)\ Sin[\theta 2]}{r^2\ \sqrt{r^2+r2^2-2\ r\ r2\ Cos[\theta 2]}}-$$

$$3\ ArcTanh\left[\frac{r1-r\ Cos[\theta 2]}{\sqrt{r^2+r1^2-2\ r\ r1\ Cos[\theta 2]}}\right]\ Sin[\theta 2]^3+$$

$$\left.\left.3\ ArcTanh\left[\frac{r2-r\ Cos[\theta 2]}{\sqrt{r^2+r2^2-2\ r\ r2\ Cos[\theta 2]}}\right]\ Sin[\theta 2]^3\right)\ Sin\left[\frac{\lambda 1-\lambda 2}{2}\right]\right)$$

In[]:= **Vxz[λ1_, λ2_, θ1_, θ2_] :=**

$$G\ rho\left(-\frac{2}{3}\ Cos\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\left(-\frac{2\ r1^2\ \sqrt{\frac{r^2+r1^2-2\ r\ r1\ Cos[\theta 1]}{(r-r1)^2}}\ EllipticF\left[\frac{\theta 1}{2},-\frac{4\ r\ r1}{(r-r1)^2}\right]}{r\ \sqrt{r^2+r1^2-2\ r\ r1\ Cos[\theta 1]}}-\right.\right.$$

$$\left(2\ \left(r^2+r1^2\right)\ \sqrt{\frac{r^2+r1^2-2\ r\ r1\ Cos[\theta 1]}{(r-r1)^2}}\ \left((r-r1)^2\ EllipticE\left[\frac{\theta 1}{2},-\frac{4\ r\ r1}{(r-r1)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r1^2\right)\ EllipticF\left[\frac{\theta 1}{2},-\frac{4\ r\ r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\ \sqrt{r^2+r1^2-2\ r\ r1\ Cos[\theta 1]}\right)+$$

$$\frac{2\ r2^2\ \sqrt{\frac{r^2+r2^2-2\ r\ r2\ Cos[\theta 1]}{(r-r2)^2}}\ EllipticF\left[\frac{\theta 1}{2},-\frac{4\ r\ r2}{(r-r2)^2}\right]}{r\ \sqrt{r^2+r2^2-2\ r\ r2\ Cos[\theta 1]}}+$$

$$\left(2\ \left(r^2+r2^2\right)\ \sqrt{\frac{r^2+r2^2-2\ r\ r2\ Cos[\theta 1]}{(r-r2)^2}}\ \left((r-r2)^2\ EllipticE\left[\frac{\theta 1}{2},-\frac{4\ r\ r2}{(r-r2)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r2^2\right)\ EllipticF\left[\frac{\theta 1}{2},-\frac{4\ r\ r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\ \sqrt{r^2+r2^2-2\ r\ r2\ Cos[\theta 1]}\right)+$$

$$\frac{\left(-2\ r1\ \left(r^2+r1^2\right)+r\ \left(3\ r^2+r1^2\right)\ Cos[\theta 1]-3\ r^2\ r1\ Cos[2\ \theta 1]\right)\ Sin[\theta 1]}{r^2\ \sqrt{r^2+r1^2-2\ r\ r1\ Cos[\theta 1]}}+$$

$$\frac{\left(2\ r2\ \left(r^2+r2^2\right)-r\ \left(3\ r^2+r2^2\right)\ Cos[\theta 1]+3\ r^2\ r2\ Cos[2\ \theta 1]\right)\ Sin[\theta 1]}{r^2\ \sqrt{r^2+r2^2-2\ r\ r2\ Cos[\theta 1]}}-$$

$$3\ ArcTanh\left[\frac{r1-r\ Cos[\theta 1]}{\sqrt{r^2+r1^2-2\ r\ r1\ Cos[\theta 1]}}\right]\ Sin[\theta 1]^3+$$

$$\left.3\ ArcTanh\left[\frac{r2-r\ Cos[\theta 1]}{\sqrt{r^2+r2^2-2\ r\ r2\ Cos[\theta 1]}}\right]\ Sin[\theta 1]^3\right)\ Sin\left[\frac{\lambda 1-\lambda 2}{2}\right]+$$

$$\frac{2}{3}\,\text{Cos}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\left(-\frac{2\,\text{r1}^2\,\sqrt{\frac{\text{r}^2+\text{r1}^2-2\,\text{r}\,\text{r1}\,\text{Cos}[\theta 2]}{(\text{r}-\text{r1})^2}}\;\text{EllipticF}\left[\frac{\theta 2}{2},-\frac{4\,\text{r}\,\text{r1}}{(\text{r}-\text{r1})^2}\right]}{\text{r}\,\sqrt{\text{r}^2+\text{r1}^2-2\,\text{r}\,\text{r1}\,\text{Cos}[\theta 2]}}-\right.$$

$$\left(2\left(\text{r}^2+\text{r1}^2\right)\sqrt{\frac{\text{r}^2+\text{r1}^2-2\,\text{r}\,\text{r1}\,\text{Cos}[\theta 2]}{(\text{r}-\text{r1})^2}}\left((\text{r}-\text{r1})^2\,\text{EllipticE}\left[\frac{\theta 2}{2},-\frac{4\,\text{r}\,\text{r1}}{(\text{r}-\text{r1})^2}\right]-\right.\right.$$

$$\left.\left.\left(\text{r}^2+\text{r1}^2\right)\text{EllipticF}\left[\frac{\theta 2}{2},-\frac{4\,\text{r}\,\text{r1}}{(\text{r}-\text{r1})^2}\right]\right)\right)\Big/\left(\text{r}^3\,\sqrt{\text{r}^2+\text{r1}^2-2\,\text{r}\,\text{r1}\,\text{Cos}[\theta 2]}\right)+$$

$$\frac{2\,\text{r2}^2\,\sqrt{\frac{\text{r}^2+\text{r2}^2-2\,\text{r}\,\text{r2}\,\text{Cos}[\theta 2]}{(\text{r}-\text{r2})^2}}\;\text{EllipticF}\left[\frac{\theta 2}{2},-\frac{4\,\text{r}\,\text{r2}}{(\text{r}-\text{r2})^2}\right]}{\text{r}\,\sqrt{\text{r}^2+\text{r2}^2-2\,\text{r}\,\text{r2}\,\text{Cos}[\theta 2]}}+$$

$$\left(2\left(\text{r}^2+\text{r2}^2\right)\sqrt{\frac{\text{r}^2+\text{r2}^2-2\,\text{r}\,\text{r2}\,\text{Cos}[\theta 2]}{(\text{r}-\text{r2})^2}}\left((\text{r}-\text{r2})^2\,\text{EllipticE}\left[\frac{\theta 2}{2},-\frac{4\,\text{r}\,\text{r2}}{(\text{r}-\text{r2})^2}\right]-\right.\right.$$

$$\left.\left.\left(\text{r}^2+\text{r2}^2\right)\text{EllipticF}\left[\frac{\theta 2}{2},-\frac{4\,\text{r}\,\text{r2}}{(\text{r}-\text{r2})^2}\right]\right)\right)\Big/\left(\text{r}^3\,\sqrt{\text{r}^2+\text{r2}^2-2\,\text{r}\,\text{r2}\,\text{Cos}[\theta 2]}\right)+$$

$$\frac{\left(-2\,\text{r1}\left(\text{r}^2+\text{r1}^2\right)+\text{r}\left(3\,\text{r}^2+\text{r1}^2\right)\text{Cos}[\theta 2]-3\,\text{r}^2\,\text{r1}\,\text{Cos}[2\,\theta 2]\right)\text{Sin}[\theta 2]}{\text{r}^2\,\sqrt{\text{r}^2+\text{r1}^2-2\,\text{r}\,\text{r1}\,\text{Cos}[\theta 2]}}+$$

$$\frac{\left(2\,\text{r2}\left(\text{r}^2+\text{r2}^2\right)-\text{r}\left(3\,\text{r}^2+\text{r2}^2\right)\text{Cos}[\theta 2]+3\,\text{r}^2\,\text{r2}\,\text{Cos}[2\,\theta 2]\right)\text{Sin}[\theta 2]}{\text{r}^2\,\sqrt{\text{r}^2+\text{r2}^2-2\,\text{r}\,\text{r2}\,\text{Cos}[\theta 2]}}-$$

$$3\,\text{ArcTanh}\left[\frac{\text{r1}-\text{r}\,\text{Cos}[\theta 2]}{\sqrt{\text{r}^2+\text{r1}^2-2\,\text{r}\,\text{r1}\,\text{Cos}[\theta 2]}}\right]\text{Sin}[\theta 2]^3+$$

$$\left.\left.3\,\text{ArcTanh}\left[\frac{\text{r2}-\text{r}\,\text{Cos}[\theta 2]}{\sqrt{\text{r}^2+\text{r2}^2-2\,\text{r}\,\text{r2}\,\text{Cos}[\theta 2]}}\right]\text{Sin}[\theta 2]^3\right)\text{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]\right)$$

In[ ]:=
```
VxzSphericalZonalBand = Vxz[0, 2 * Pi, θ1, θ2]
```
Out[ ]=
```
0
```

In[ ]:=
```
VxzSphericalSphericalShell = Vxz[0, 2 * Pi, 0, Pi]
```
Out[ ]=
```
0
```

3. Vyz

In[ ]:= FullSimplify[∫ r3^2 Sin[θ3] ((3 * (-r3 Sin[θ3] Sin[λ - λ3]) * (-r + r3 Cos[θ3]))/((Sqrt[r^2 + r3^2 - 2 * r * r3 * Cos[θ3]])^5)) dλ3]

Out[ ]=

$$\frac{3\, r3^3\, (r - r3\, \mathrm{Cos}[\theta 3])\, \mathrm{Cos}[\lambda - \lambda 3]\, \mathrm{Sin}[\theta 3]^2}{\left(r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:= Vyzλ3[λ3_] :=

$$\frac{3\, r3^3\, (r - r3\, \mathrm{Cos}[\theta 3])\, \mathrm{Cos}[\lambda - \lambda 3]\, \mathrm{Sin}[\theta 3]^2}{\left(r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:= FullSimplify[Vyzλ3[λ2] - Vyzλ3[λ1]]

Out[ ]=

$$\frac{6\, r3^3\, (-r + r3\, \mathrm{Cos}[\theta 3])\, \mathrm{Sin}[\theta 3]^2\, \mathrm{Sin}\left[\frac{\lambda 1 - \lambda 2}{2}\right]\, \mathrm{Sin}\left[\lambda - \frac{\lambda 1}{2} - \frac{\lambda 2}{2}\right]}{\left(r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]\right)^{5/2}}$$

In[ ]:=

$$\int \left(\frac{6\, r3^3\, (-r + r3\, \mathrm{Cos}[\theta 3])\, \mathrm{Sin}[\theta 3]^2\, \mathrm{Sin}\left[\frac{\lambda 1 - \lambda 2}{2}\right]\, \mathrm{Sin}\left[\lambda - \frac{\lambda 1}{2} - \frac{\lambda 2}{2}\right]}{\left(r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]\right)^{5/2}}\right) dr3$$

Out[ ]=

$$6 \left(\mathrm{ArcTanh}\left[\frac{r3 - r\, \mathrm{Cos}[\theta 3]}{\sqrt{r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]}}\right] \mathrm{Cos}[\theta 3] + \left(\left(r^3 + 3\, r\, r3^2 - 2\, r3^3\, \mathrm{Cos}[\theta 3] - 3\, r\, \left(r^2 + 2\, r3^2\right) \mathrm{Cos}[2\, \theta 3] + 6\, r^2\, r3\, \mathrm{Cos}[3\, \theta 3] + 4\, r3^3\, \mathrm{Cos}[3\, \theta 3] - 3\, r\, r3^2\, \mathrm{Cos}[4\, \theta 3]\right) \mathrm{Csc}[\theta 3]^2\right) \Big/ \left(6\, \left(r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]\right)^{3/2}\right)\right) \mathrm{Sin}[\theta 3]^2\, \mathrm{Sin}\left[\frac{\lambda 1 - \lambda 2}{2}\right] \mathrm{Sin}\left[\lambda - \frac{\lambda 1}{2} - \frac{\lambda 2}{2}\right]$$

In[ ]:= Vyzr3[r3_] :=

$$6 \left(\mathrm{ArcTanh}\left[\frac{r3 - r\, \mathrm{Cos}[\theta 3]}{\sqrt{r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]}}\right] \mathrm{Cos}[\theta 3] + \left(\left(r^3 + 3\, r\, r3^2 - 2\, r3^3\, \mathrm{Cos}[\theta 3] - 3\, r\, \left(r^2 + 2\, r3^2\right) \mathrm{Cos}[2\, \theta 3] + 6\, r^2\, r3\, \mathrm{Cos}[3\, \theta 3] + 4\, r3^3\, \mathrm{Cos}[3\, \theta 3] - 3\, r\, r3^2\, \mathrm{Cos}[4\, \theta 3]\right) \mathrm{Csc}[\theta 3]^2\right) \Big/ \left(6\, \left(r^2 + r3^2 - 2\, r\, r3\, \mathrm{Cos}[\theta 3]\right)^{3/2}\right)\right) \mathrm{Sin}[\theta 3]^2\, \mathrm{Sin}\left[\frac{\lambda 1 - \lambda 2}{2}\right] \mathrm{Sin}\left[\lambda - \frac{\lambda 1}{2} - \frac{\lambda 2}{2}\right]$$

In[∘]:= **Vyzr3[r2] - Vyzr3[r1]**

Out[∘]=

$$-6\left(\text{ArcTanh}\left[\frac{r1-r\,\text{Cos}[\theta3]}{\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}}\right]\text{Cos}[\theta3]+\left(\left(r^3+3\,r\,r1^2-2\,r1^3\,\text{Cos}[\theta3]-3\,r\left(r^2+2\,r1^2\right)\text{Cos}[2\,\theta3]+6\,r^2\,r1\,\text{Cos}[3\,\theta3]+4\,r1^3\,\text{Cos}[3\,\theta3]-3\,r\,r1^2\,\text{Cos}[4\,\theta3]\right)\text{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]\right)^{3/2}\right)\right)\text{Sin}[\theta3]^2\,\text{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]\text{Sin}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]+6\left(\text{ArcTanh}\left[\frac{r2-r\,\text{Cos}[\theta3]}{\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}}\right]\text{Cos}[\theta3]+\left(\left(r^3+3\,r\,r2^2-2\,r2^3\,\text{Cos}[\theta3]-3\,r\left(r^2+2\,r2^2\right)\text{Cos}[2\,\theta3]+6\,r^2\,r2\,\text{Cos}[3\,\theta3]+4\,r2^3\,\text{Cos}[3\,\theta3]-3\,r\,r2^2\,\text{Cos}[4\,\theta3]\right)\text{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]\right)^{3/2}\right)\right)\text{Sin}[\theta3]^2\,\text{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]\text{Sin}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]$$

In[∘]:= **Simplify[**

$$\int\left(-6\left(\text{ArcTanh}\left[\frac{r1-r\,\text{Cos}[\theta3]}{\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}}\right]\text{Cos}[\theta3]+\left(\left(r^3+3\,r\,r1^2-2\,r1^3\,\text{Cos}[\theta3]-3\,r\left(r^2+2\,r1^2\right)\text{Cos}[2\,\theta3]+6\,r^2\,r1\,\text{Cos}[3\,\theta3]+4\,r1^3\,\text{Cos}[3\,\theta3]-3\,r\,r1^2\,\text{Cos}[4\,\theta3]\right)\text{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]\right)^{3/2}\right)\right)\text{Sin}[\theta3]^2\,\text{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]\text{Sin}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]+6\left(\text{ArcTanh}\left[\frac{r2-r\,\text{Cos}[\theta3]}{\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}}\right]\text{Cos}[\theta3]+\left(\left(r^3+3\,r\,r2^2-2\,r2^3\,\text{Cos}[\theta3]-3\,r\left(r^2+2\,r2^2\right)\text{Cos}[2\,\theta3]+6\,r^2\,r2\,\text{Cos}[3\,\theta3]+4\,r2^3\,\text{Cos}[3\,\theta3]-3\,r\,r2^2\,\text{Cos}[4\,\theta3]\right)\text{Csc}[\theta3]^2\right)\Big/\left(6\left(r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]\right)^{3/2}\right)\right)\text{Sin}[\theta3]^2\,\text{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]\text{Sin}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]\right)d\theta3\Big]$$

Out[ ]=

$$\frac{2}{3}\left(-\frac{2\,r1^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 3]}{(r-r1)^2}}\;EllipticF\left[\frac{\theta 3}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 3]}}-\left(2\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 3]}{(r-r1)^2}}\left((r-r1)^2\,EllipticE\left[\frac{\theta 3}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\left(r^2+r1^2\right)EllipticF\left[\frac{\theta 3}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 3]}\right)+\frac{2\,r2^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 3]}{(r-r2)^2}}\;EllipticF\left[\frac{\theta 3}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 3]}}+\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 3]}{(r-r2)^2}}\left((r-r2)^2\,EllipticE\left[\frac{\theta 3}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\left(r^2+r2^2\right)EllipticF\left[\frac{\theta 3}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 3]}\right)+\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)Cos[\theta 3]-3\,r^2\,r1\,Cos[2\,\theta 3]\right)Sin[\theta 3]}{r^2\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 3]}}+\frac{\left(2\,r2\left(r^2+r2^2\right)-r\left(3\,r^2+r2^2\right)Cos[\theta 3]+3\,r^2\,r2\,Cos[2\,\theta 3]\right)Sin[\theta 3]}{r^2\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 3]}}-3\,ArcTanh\left[\frac{r1-r\,Cos[\theta 3]}{\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 3]}}\right]Sin[\theta 3]^3+3\,ArcTanh\left[\frac{r2-r\,Cos[\theta 3]}{\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 3]}}\right]Sin[\theta 3]^3\right)$$

$$Sin\left[\frac{\lambda 1-\lambda 2}{2}\right]\,Sin\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]$$

In[∘]:=

$$\text{Vyz}\theta3[\theta3\_] := \frac{2}{3}\left(-\frac{2\,r1^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}{(r-r1)^2}}\ \text{EllipticF}\left[\frac{\theta3}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}}-\left(2\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}{(r-r1)^2}}\left((r-r1)^2\,\text{EllipticE}\left[\frac{\theta3}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\left(r^2+r1^2\right)\text{EllipticF}\left[\frac{\theta3}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}\right)+\frac{2\,r2^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}{(r-r2)^2}}\ \text{EllipticF}\left[\frac{\theta3}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}}+\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}{(r-r2)^2}}\left((r-r2)^2\,\text{EllipticE}\left[\frac{\theta3}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\left(r^2+r2^2\right)\text{EllipticF}\left[\frac{\theta3}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}\right)+\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)\text{Cos}[\theta3]-3\,r^2\,r1\,\text{Cos}[2\,\theta3]\right)\text{Sin}[\theta3]}{r^2\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}}+\frac{\left(2\,r2\left(r^2+r2^2\right)-r\left(3\,r^2+r2^2\right)\text{Cos}[\theta3]+3\,r^2\,r2\,\text{Cos}[2\,\theta3]\right)\text{Sin}[\theta3]}{r^2\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}}-3\,\text{ArcTanh}\left[\frac{r1-r\,\text{Cos}[\theta3]}{\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta3]}}\right]\text{Sin}[\theta3]^3+3\,\text{ArcTanh}\left[\frac{r2-r\,\text{Cos}[\theta3]}{\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta3]}}\right]\text{Sin}[\theta3]^3\right)\text{Sin}\left[\frac{\lambda1-\lambda2}{2}\right]\text{Sin}\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]$$

In[∘]:= G * rho * (Vyzθ3[θ2] - Vyzθ3[θ1])

Out[∘]=

$$\text{G rho}\left(-\frac{2}{3}\left(-\frac{2\,r1^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta1]}{(r-r1)^2}}\ \text{EllipticF}\left[\frac{\theta1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta1]}}-\left(2\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta1]}{(r-r1)^2}}\left((r-r1)^2\,\text{EllipticE}\left[\frac{\theta1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\left(r^2+r1^2\right)\text{EllipticF}\left[\frac{\theta1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta1]}\right)+\right.\right.$$

$$\frac{2\,r2^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta1]}{(r-r2)^2}}\;EllipticF\left[\frac{\theta1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta1]}}+$$

$$\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta1]}{(r-r2)^2}}\left((r-r2)^2\,EllipticE\left[\frac{\theta1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r2^2\right)EllipticF\left[\frac{\theta1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta1]}\right)+$$

$$\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)Cos[\theta1]-3\,r^2\,r1\,Cos[2\,\theta1]\right)Sin[\theta1]}{r^2\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta1]}}+$$

$$\frac{\left(2\,r2\left(r^2+r2^2\right)-r\left(3\,r^2+r2^2\right)Cos[\theta1]+3\,r^2\,r2\,Cos[2\,\theta1]\right)Sin[\theta1]}{r^2\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta1]}}-$$

$$3\,ArcTanh\left[\frac{r1-r\,Cos[\theta1]}{\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta1]}}\right]Sin[\theta1]^3+$$

$$\left.3\,ArcTanh\left[\frac{r2-r\,Cos[\theta1]}{\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta1]}}\right]Sin[\theta1]^3\right)Sin\left[\frac{\lambda1-\lambda2}{2}\right]Sin\left[\lambda-\frac{\lambda1}{2}-\frac{\lambda2}{2}\right]+$$

$$\frac{2}{3}\left(-\frac{2\,r1^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta2]}{(r-r1)^2}}\;EllipticF\left[\frac{\theta2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta2]}}-\right.$$

$$\left(2\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta2]}{(r-r1)^2}}\left((r-r1)^2\,EllipticE\left[\frac{\theta2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r1^2\right)EllipticF\left[\frac{\theta2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta2]}\right)+$$

$$\frac{2\,r2^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta2]}{(r-r2)^2}}\;EllipticF\left[\frac{\theta2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta2]}}+$$

$$\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta2]}{(r-r2)^2}}\left((r-r2)^2\,EllipticE\left[\frac{\theta2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r2^2\right)EllipticF\left[\frac{\theta2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta2]}\right)+$$

$$\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)Cos[\theta2]-3\,r^2\,r1\,Cos[2\,\theta2]\right)Sin[\theta2]}{r^2\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta2]}}+$$

$$\frac{\left(2\,r2\,\left(r^2+r2^2\right)-r\,\left(3\,r^2+r2^2\right)\,\mathrm{Cos}[\theta 2]+3\,r^2\,r2\,\mathrm{Cos}[2\,\theta 2]\right)\,\mathrm{Sin}[\theta 2]}{r^2\,\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 2]}}-$$

$$3\,\mathrm{ArcTanh}\left[\frac{r1-r\,\mathrm{Cos}[\theta 2]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 2]}}\right]\,\mathrm{Sin}[\theta 2]^3+$$

$$\left.3\,\mathrm{ArcTanh}\left[\frac{r2-r\,\mathrm{Cos}[\theta 2]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 2]}}\right]\,\mathrm{Sin}[\theta 2]^3\right)\,\mathrm{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]\,\mathrm{Sin}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\Bigg)$$

In[ ]:= **Vyz[λ1_, λ2_, θ1_, θ2_] :=**

$$\mathrm{G\,rho}\left(-\frac{2}{3}\left(-\frac{2\,r1^2\,\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 1]}{(r-r1)^2}}\,\mathrm{EllipticF}\left[\frac{\theta 1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\,\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 1]}}-\right.\right.$$

$$\left(2\,\left(r^2+r1^2\right)\,\sqrt{\frac{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 1]}{(r-r1)^2}}\,\left((r-r1)^2\,\mathrm{EllipticE}\left[\frac{\theta 1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r1^2\right)\,\mathrm{EllipticF}\left[\frac{\theta 1}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 1]}\right)+$$

$$\frac{2\,r2^2\,\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 1]}{(r-r2)^2}}\,\mathrm{EllipticF}\left[\frac{\theta 1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\,\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 1]}}+$$

$$\left(2\,\left(r^2+r2^2\right)\,\sqrt{\frac{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 1]}{(r-r2)^2}}\,\left((r-r2)^2\,\mathrm{EllipticE}\left[\frac{\theta 1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\right.\right.$$

$$\left.\left.\left(r^2+r2^2\right)\,\mathrm{EllipticF}\left[\frac{\theta 1}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\,\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 1]}\right)+$$

$$\frac{\left(-2\,r1\,\left(r^2+r1^2\right)+r\,\left(3\,r^2+r1^2\right)\,\mathrm{Cos}[\theta 1]-3\,r^2\,r1\,\mathrm{Cos}[2\,\theta 1]\right)\,\mathrm{Sin}[\theta 1]}{r^2\,\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 1]}}+$$

$$\frac{\left(2\,r2\,\left(r^2+r2^2\right)-r\,\left(3\,r^2+r2^2\right)\,\mathrm{Cos}[\theta 1]+3\,r^2\,r2\,\mathrm{Cos}[2\,\theta 1]\right)\,\mathrm{Sin}[\theta 1]}{r^2\,\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 1]}}-$$

$$3\,\mathrm{ArcTanh}\left[\frac{r1-r\,\mathrm{Cos}[\theta 1]}{\sqrt{r^2+r1^2-2\,r\,r1\,\mathrm{Cos}[\theta 1]}}\right]\,\mathrm{Sin}[\theta 1]^3+$$

$$\left.3\,\mathrm{ArcTanh}\left[\frac{r2-r\,\mathrm{Cos}[\theta 1]}{\sqrt{r^2+r2^2-2\,r\,r2\,\mathrm{Cos}[\theta 1]}}\right]\,\mathrm{Sin}[\theta 1]^3\right)\,\mathrm{Sin}\left[\frac{\lambda 1-\lambda 2}{2}\right]\,\mathrm{Sin}\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]+$$

$$\frac{2}{3}\left(-\frac{2\,r1^2\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}{(r-r1)^2}}\;EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]}{r\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}}-\left(2\left(r^2+r1^2\right)\sqrt{\frac{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}{(r-r1)^2}}\left((r-r1)^2\,EllipticE\left[\frac{\theta 2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]-\left(r^2+r1^2\right)EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r1}{(r-r1)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}\right)+\frac{2\,r2^2\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}{(r-r2)^2}}\;EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]}{r\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}}+\left(2\left(r^2+r2^2\right)\sqrt{\frac{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}{(r-r2)^2}}\left((r-r2)^2\,EllipticE\left[\frac{\theta 2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]-\left(r^2+r2^2\right)EllipticF\left[\frac{\theta 2}{2},-\frac{4\,r\,r2}{(r-r2)^2}\right]\right)\right)\Big/\left(r^3\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}\right)+\frac{\left(-2\,r1\left(r^2+r1^2\right)+r\left(3\,r^2+r1^2\right)Cos[\theta 2]-3\,r^2\,r1\,Cos[2\,\theta 2]\right)Sin[\theta 2]}{r^2\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}}+\frac{\left(2\,r2\left(r^2+r2^2\right)-r\left(3\,r^2+r2^2\right)Cos[\theta 2]+3\,r^2\,r2\,Cos[2\,\theta 2]\right)Sin[\theta 2]}{r^2\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}}-3\,ArcTanh\left[\frac{r1-r\,Cos[\theta 2]}{\sqrt{r^2+r1^2-2\,r\,r1\,Cos[\theta 2]}}\right]Sin[\theta 2]^3+3\,ArcTanh\left[\frac{r2-r\,Cos[\theta 2]}{\sqrt{r^2+r2^2-2\,r\,r2\,Cos[\theta 2]}}\right]Sin[\theta 2]^3\right)Sin\left[\frac{\lambda 1-\lambda 2}{2}\right]Sin\left[\lambda-\frac{\lambda 1}{2}-\frac{\lambda 2}{2}\right]\Bigg)$$

In[ ]:=
```
VyzSphericalZonalBand = Vyz[0, 2 * Pi, θ1, θ2]
```
Out[ ]=
```
0
```

In[ ]:=
```
VyzSphericalShell = Vyz[0, 2 * Pi, 0, Pi]
```
Out[ ]=
```
0
```