In[•]:=
```
Clear["Global`*"];
```

1. Expressions for the V, Vz, Vzz, and Vzzz of a spherical zonal band in Deng (2022)

In[•]:=
```
OldZonalBandV = 2 G π ρ ( 1/2 Cos[θ1] (r1 - r Cos[θ1]) √(r^2 + r1^2 - 2 r r1 Cos[θ1]) +
    (r^2 + r1^2 - 2 r r1 Cos[θ1])^(3/2) / (3 r) +
    1/2 r^2 Cos[θ1] Log[r1 - r Cos[θ1] + √(r^2 + r1^2 - 2 r r1 Cos[θ1])] Sin[θ1]^2 ) -
  2 G π ρ ( 1/2 Cos[θ1] (r2 - r Cos[θ1]) √(r^2 + r2^2 - 2 r r2 Cos[θ1]) +
    (r^2 + r2^2 - 2 r r2 Cos[θ1])^(3/2) / (3 r) +
    1/2 r^2 Cos[θ1] Log[r2 - r Cos[θ1] + √(r^2 + r2^2 - 2 r r2 Cos[θ1])] Sin[θ1]^2 ) -
  2 G π ρ ( 1/2 Cos[θ2] (r1 - r Cos[θ2]) √(r^2 + r1^2 - 2 r r1 Cos[θ2]) +
    (r^2 + r1^2 - 2 r r1 Cos[θ2])^(3/2) / (3 r) +
    1/2 r^2 Cos[θ2] Log[r1 - r Cos[θ2] + √(r^2 + r1^2 - 2 r r1 Cos[θ2])] Sin[θ2]^2 ) +
  2 G π ρ ( 1/2 Cos[θ2] (r2 - r Cos[θ2]) √(r^2 + r2^2 - 2 r r2 Cos[θ2]) +
    (r^2 + r2^2 - 2 r r2 Cos[θ2])^(3/2) / (3 r) +
    1/2 r^2 Cos[θ2] Log[r2 - r Cos[θ2] + √(r^2 + r2^2 - 2 r r2 Cos[θ2])] Sin[θ2]^2 );
```

In[ ]:=
$$\begin{aligned}
\text{OldZonalBandVz} = {} & \frac{1}{3\,r^2}\,G\,\pi\,\rho\left(\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta 1]}\;\left(r^2-2\,r1^2-3\,r^2\,\text{Cos}[2\,\theta 1]\right)+\right.\\
& r\,\text{Cos}[\theta 1]\left(-2\,r1\,\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta 1]}\;+\right.\\
& \left.\left.3\,r^2\left(1+2\,\text{Log}\left[r1-r\,\text{Cos}[\theta 1]+\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta 1]}\,\right]\right)\text{Sin}[\theta 1]^2\right)\right)-\\
& \frac{1}{3\,r^2}\,G\,\pi\,\rho\left(\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta 1]}\;\left(r^2-2\,r2^2-3\,r^2\,\text{Cos}[2\,\theta 1]\right)+\right.\\
& r\,\text{Cos}[\theta 1]\left(-2\,r2\,\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta 1]}\;+\right.\\
& \left.\left.3\,r^2\left(1+2\,\text{Log}\left[r2-r\,\text{Cos}[\theta 1]+\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta 1]}\,\right]\right)\text{Sin}[\theta 1]^2\right)\right)-\\
& \frac{1}{3\,r^2}\,G\,\pi\,\rho\left(\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta 2]}\;\left(r^2-2\,r1^2-3\,r^2\,\text{Cos}[2\,\theta 2]\right)+\right.\\
& r\,\text{Cos}[\theta 2]\left(-2\,r1\,\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta 2]}\;+\right.\\
& \left.\left.3\,r^2\left(1+2\,\text{Log}\left[r1-r\,\text{Cos}[\theta 2]+\sqrt{r^2+r1^2-2\,r\,r1\,\text{Cos}[\theta 2]}\,\right]\right)\text{Sin}[\theta 2]^2\right)\right)+\\
& \frac{1}{3\,r^2}\,G\,\pi\,\rho\left(\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta 2]}\;\left(r^2-2\,r2^2-3\,r^2\,\text{Cos}[2\,\theta 2]\right)+\right.\\
& r\,\text{Cos}[\theta 2]\left(-2\,r2\,\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta 2]}\;+\right.\\
& \left.\left.3\,r^2\left(1+2\,\text{Log}\left[r2-r\,\text{Cos}[\theta 2]+\sqrt{r^2+r2^2-2\,r\,r2\,\text{Cos}[\theta 2]}\,\right]\right)\text{Sin}[\theta 2]^2\right)\right);
\end{aligned}$$

In[•]:=
```
OldZonalBandVzz =
  1/(3 r^3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) G π ρ (r^4 + 2 r^2 r1^2 + 4 r1^4 - 2 r^2 r1^2 Cos[θ1]^2 -
      3 r^4 Cos[2 θ1] - r Cos[θ1] (r^2 r1 + 4 r1^3 - 3 r^2 r1 Cos[2 θ1] -
        3 r^2 (-2 r1 + 3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]] + 2 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]
            Log[r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]]) Sin[θ1]^2)) -
  1/(3 r^3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]) G π ρ (r^4 + 2 r^2 r2^2 + 4 r2^4 - 2 r^2 r2^2 Cos[θ1]^2 -
      3 r^4 Cos[2 θ1] - r Cos[θ1] (r^2 r2 + 4 r2^3 - 3 r^2 r2 Cos[2 θ1] -
        3 r^2 (-2 r2 + 3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]] + 2 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]
            Log[r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]]) Sin[θ1]^2)) -
  1/(3 r^3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]) G π ρ (r^4 + 2 r^2 r1^2 + 4 r1^4 - 2 r^2 r1^2 Cos[θ2]^2 -
      3 r^4 Cos[2 θ2] - r Cos[θ2] (r^2 r1 + 4 r1^3 - 3 r^2 r1 Cos[2 θ2] -
        3 r^2 (-2 r1 + 3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]] + 2 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]
            Log[r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]]) Sin[θ2]^2)) +
  1/(3 r^3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]) G π ρ (r^4 + 2 r^2 r2^2 + 4 r2^4 - 2 r^2 r2^2 Cos[θ2]^2 -
      3 r^4 Cos[2 θ2] - r Cos[θ2] (r^2 r2 + 4 r2^3 - 3 r^2 r2 Cos[2 θ2] -
        3 r^2 (-2 r2 + 3 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]] + 2 Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]
            Log[r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]]) Sin[θ2]^2));
```

In[•]:=
```
OldZonalBandVzzz =
 -((G π ρ (4 r1 (r^6 + 3 r^4 r1^2 + 15 r^2 r1^4 + 4 r1^6) + (r^6 + 3 r^4 r1^2 + 36 r^2 r1^4 + 16 r1^6)
          Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]] -
        2 r Cos[θ1] (r^6 + 2 r^4 r1 (2 r1 + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) +
          8 r1^5 (4 r1 + 3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) +
          3 r^2 r1^3 (10 r1 + 3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])) +
        r^2 (4 r1^3 (2 r^2 + 9 r1^2 + 3 r1 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) Cos[2 θ1] +
          r (2 r^4 + 6 r^2 r1^2 - 4 r1^4 + 3 r^2 r1 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]] +
            2 r1^3 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) Cos[3 θ1] - r^2 (4 r^2 r1 + 4 r1^3 +
            r^2 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]] + 3 r1^2 Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])
```

$$\text{Cos}[4\,\theta 1] + r^3\,r1\left(2\,r1 + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 1]}\right)\text{Cos}[5\,\theta 1]\Big)\Big)\Big/$$
$$\left(4\,r^4\left(r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 1]\right)^{3/2}\left(r1 - r\,\text{Cos}[\theta 1] + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 1]}\right)\right) +$$
$$\Big(G\,\pi\,\rho\left(4\,r2\left(r^6 + 3\,r^4\,r2^2 + 15\,r^2\,r2^4 + 4\,r2^6\right) +\right.$$
$$\left(r^6 + 3\,r^4\,r2^2 + 36\,r^2\,r2^4 + 16\,r2^6\right)\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]} -$$
$$2\,r\,\text{Cos}[\theta 1]\left(r^6 + 2\,r^4\,r2\left(2\,r2 + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right) +\right.$$
$$8\,r2^5\left(4\,r2 + 3\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right) +$$
$$3\,r^2\,r2^3\left(10\,r2 + 3\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right)\Big) +$$
$$r^2\left(4\,r2^3\left(2\,r^2 + 9\,r2^2 + 3\,r2\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right)\text{Cos}[2\,\theta 1] +\right.$$
$$r\left(2\,r^4 + 6\,r^2\,r2^2 - 4\,r2^4 + 3\,r^2\,r2\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]} +\right.$$
$$\left.2\,r2^3\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right)\text{Cos}[3\,\theta 1] - r^2\left(4\,r^2\,r2 + 4\,r2^3 +\right.$$
$$\left.r^2\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]} + 3\,r2^2\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right)$$
$$\left.\text{Cos}[4\,\theta 1] + r^3\,r2\left(2\,r2 + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right)\text{Cos}[5\,\theta 1]\right)\Big)\Big)\Big/$$
$$\left(4\,r^4\left(r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]\right)^{3/2}\left(r2 - r\,\text{Cos}[\theta 1] + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 1]}\right)\right) +$$
$$\Big(G\,\pi\,\rho\left(4\,r1\left(r^6 + 3\,r^4\,r1^2 + 15\,r^2\,r1^4 + 4\,r1^6\right) +\right.$$
$$\left(r^6 + 3\,r^4\,r1^2 + 36\,r^2\,r1^4 + 16\,r1^6\right)\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]} -$$
$$2\,r\,\text{Cos}[\theta 2]\left(r^6 + 2\,r^4\,r1\left(2\,r1 + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right) +\right.$$
$$8\,r1^5\left(4\,r1 + 3\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right) +$$
$$3\,r^2\,r1^3\left(10\,r1 + 3\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right)\Big) +$$
$$r^2\left(4\,r1^3\left(2\,r^2 + 9\,r1^2 + 3\,r1\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right)\text{Cos}[2\,\theta 2] +\right.$$
$$r\left(2\,r^4 + 6\,r^2\,r1^2 - 4\,r1^4 + 3\,r^2\,r1\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]} +\right.$$
$$\left.2\,r1^3\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right)\text{Cos}[3\,\theta 2] - r^2\left(4\,r^2\,r1 + 4\,r1^3 +\right.$$
$$\left.r^2\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]} + 3\,r1^2\sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right)$$
$$\left.\text{Cos}[4\,\theta 2] + r^3\,r1\left(2\,r1 + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right)\text{Cos}[5\,\theta 2]\right)\Big)\Big)\Big/$$
$$\left(4\,r^4\left(r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]\right)^{3/2}\left(r1 - r\,\text{Cos}[\theta 2] + \sqrt{r^2 + r1^2 - 2\,r\,r1\,\text{Cos}[\theta 2]}\right)\right) -$$
$$\Big(G\,\pi\,\rho\left(4\,r2\left(r^6 + 3\,r^4\,r2^2 + 15\,r^2\,r2^4 + 4\,r2^6\right) +\right.$$
$$\left(r^6 + 3\,r^4\,r2^2 + 36\,r^2\,r2^4 + 16\,r2^6\right)\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 2]} -$$
$$2\,r\,\text{Cos}[\theta 2]\left(r^6 + 2\,r^4\,r2\left(2\,r2 + \sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 2]}\right) +\right.$$
$$8\,r2^5\left(4\,r2 + 3\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 2]}\right) +$$
$$3\,r^2\,r2^3\left(10\,r2 + 3\sqrt{r^2 + r2^2 - 2\,r\,r2\,\text{Cos}[\theta 2]}\right)\Big) +$$

$$r^2 \left(4\, r2^3 \left(2\, r^2 + 9\, r2^2 + 3\, r2\, \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]}\right) \text{Cos}[2\,\theta 2] + r \left(2\, r^4 + 6\, r^2\, r2^2 - 4\, r2^4 + 3\, r^2\, r2\, \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]} + 2\, r2^3\, \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]}\right) \text{Cos}[3\,\theta 2] - r^2 \left(4\, r^2\, r2 + 4\, r2^3 + r^2\, \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]} + 3\, r2^2\, \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]}\right) \text{Cos}[4\,\theta 2] + r^3\, r2 \left(2\, r2 + \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]}\right) \text{Cos}[5\,\theta 2]\right)\Big) \Big/ \left(4\, r^4 \left(r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]\right)^{3/2} \left(r2 - r\, \text{Cos}[\theta 2] + \sqrt{r^2 + r2^2 - 2\, r\, r2\, \text{Cos}[\theta 2]}\right)\right);$$

2. Functions for the V, Vz, Vzz, and Vzzz of a spherical zonal band in this paper

In[ ]:=
```
NewZonalBandV = 1/(6 r) G π ρ
  (Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]] (4 * r1^2 + r^2 * (1 - 3 * Cos[2 * θ1]) - 2 * r * r1 * Cos[θ1]) -
   Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]] (4 * r2^2 + r^2 * (1 - 3 * Cos[2 * θ1]) - 2 * r * r2 * Cos[θ1]) -
   Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]] (4 * r1^2 + r^2 * (1 - 3 * Cos[2 * θ2]) - 2 * r * r1 * Cos[θ2]) +
   Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]] (4 * r2^2 + r^2 * (1 - 3 * Cos[2 * θ2]) - 2 * r * r2 * Cos[θ2]) +
   6 r^3 Cos[θ1] Log[(r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])/(r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]])] Sin[θ1]^2 +
   6 r^3 Cos[θ2] Log[(r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]])/(r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]])] Sin[θ2]^2);

NewZonalBandVz =
 -1/(3 r^2) 2 G π ρ (Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]] (r1^2 + r r1 Cos[θ1] + r^2 (-2 + 3 Cos[θ1]^2)) -
   Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]] (r2^2 + r r2 Cos[θ1] + r^2 (-2 + 3 Cos[θ1]^2)) -
   Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]] (r1^2 + r r1 Cos[θ2] + r^2 (-2 + 3 Cos[θ2]^2)) +
   Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]] (r2^2 + r r2 Cos[θ2] + r^2 (-2 + 3 Cos[θ2]^2)) +
   3 r^3 Cos[θ1] Log[(r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]])/(r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])] Sin[θ1]^2 +
   3 r^3 Cos[θ2] Log[(r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]])/(r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]])] Sin[θ2]^2);
```

In[•]:=
```
NewZonalBandVzz =
 1/(3 r^3) G π ρ ((r^4 + r^2 r1^2 + 4 r1^4 - r r1 (r^2 + 4 r1^2) Cos[θ1] - r^2 (3 r^2 + r1^2) Cos[2 θ1] +
      3 r^3 r1 Cos[3 θ1]) / (Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) -
  (r^4 + r^2 r2^2 + 4 r2^4 - r r2 (r^2 + 4 r2^2) Cos[θ1] - r^2 (3 r^2 + r2^2) Cos[2 θ1] +
      3 r^3 r2 Cos[3 θ1]) / (Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]) -
  (r^4 + r^2 r1^2 + 4 r1^4 - r r1 (r^2 + 4 r1^2) Cos[θ2] - r^2 (3 r^2 + r1^2) Cos[2 θ2] +
      3 r^3 r1 Cos[3 θ2]) / (Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]) +
  (r^4 + r^2 r2^2 + 4 r2^4 - r r2 (r^2 + 4 r2^2) Cos[θ2] - r^2 (3 r^2 + r2^2) Cos[2 θ2] +
      3 r^3 r2 Cos[3 θ2]) / (Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]) +
  6 r^3 Cos[θ1] Log[(r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]) /
      (r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]])] Sin[θ1]^2 +
  6 r^3 Cos[θ2] Log[(r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]) /
      (r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]])] Sin[θ2]^2);
```

In[•]:=
```
NewZonalBandVzzz =
 -1/r^4 G π ρ ((r1^3 (9 r^2 r1 + 4 r1^3 - 4 r (r^2 + 3 r1^2) Cos[θ1] + 3 r^2 r1 Cos[2 θ1]) /
      (r^2 + r1^2 - 2 r r1 Cos[θ1])^(3/2) -
  r2^3 (9 r^2 r2 + 4 r2^3 - 4 r (r^2 + 3 r2^2) Cos[θ1] + 3 r^2 r2 Cos[2 θ1]) /
      (r^2 + r2^2 - 2 r r2 Cos[θ1])^(3/2) -
  r1^3 (9 r^2 r1 + 4 r1^3 - 4 r (r^2 + 3 r1^2) Cos[θ2] + 3 r^2 r1 Cos[2 θ2]) /
      (r^2 + r1^2 - 2 r r1 Cos[θ2])^(3/2) +
  r2^3 (9 r^2 r2 + 4 r2^3 - 4 r (r^2 + 3 r2^2) Cos[θ2] + 3 r^2 r2 Cos[2 θ2]) /
      (r^2 + r2^2 - 2 r r2 Cos[θ2])^(3/2));
```

3. Test the consistency of analytical expressions between the V, Vz, Vzz, and Vzzz of a spherical zonal band in Deng (2022) and this paper

In[•]:=
```
FullSimplify[
  (OldZonalBandV - NewZonalBandV) /. {Log[(r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])/(r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]])] →
     (Log[r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]] -
       Log[r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]]),
    Log[(r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]])/(r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]])] →
     (Log[r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]] -
       Log[r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]])}]
```

Out[•]=
```
0
```

In[•]:=
```
FullSimplify[(OldZonalBandVz - NewZonalBandVz) /.
  {Log[(r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]])/(r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])] →
     (Log[r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]] -
       Log[r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]]),
    Log[(r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]])/(r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]])] →
     (Log[r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]] -
       Log[r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]])}]
```

Out[•]=
```
0
```

In[ ]:=
```
FullSimplify[(OldZonalBandVzz - NewZonalBandVzz) /.
  {Log[(r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]])/
      (r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]])] →
    (Log[r1 - r Cos[θ1] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ1]]] -
      Log[r2 - r Cos[θ1] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ1]]]),
   Log[(r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]])/
      (r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]])] →
    (Log[r2 - r Cos[θ2] + Sqrt[r^2 + r2^2 - 2 r r2 Cos[θ2]]] -
      Log[r1 - r Cos[θ2] + Sqrt[r^2 + r1^2 - 2 r r1 Cos[θ2]]])}]
```

Out[ ]=
```
0
```

In[ ]:=
```
(*It will take ~18.5h to obtain the final result - zero*)
FullSimplify[(OldZonalBandVzzz - NewZonalBandVzzz)]
```

Out[ ]=
```
0
```

In[ ]:=
```
NotebookSave[EvaluationNotebook[]];
```