JOURNAL OF REGIONAL SCIENCE, VOL. 22, NO. 2, 1982

A METHOD OF FITTING THE GRAVITY MODEL BASED ON THE POISSON DISTRIBUTION*

Robin Flowerdew and Murray Aitkin†

1. INTRODUCTION

It is frequently important to find a concise way of summarizing and describing large sets of data on interaction. Many such sets that are treated in geography demonstrate a clear relationship between the extent of interaction and the size of the objects, or places, which are interacting. It is common also to find an inverse relationship between interaction and the distance between objects or places. Models of interaction incorporating both relationships are referred to as gravity models, and such models, in many diverse forms, have been applied to many different kinds of data.

In this paper, we suggest an alternative method for fitting the gravity model. In this method, the interaction variable is treated as the outcome of a discrete probability process, whose mean is a function of the size and distance variables. This treatment seems appropriate when the dependent variable represents a count of the number of items (people, vehicles, shipments) moving from one place to another. It would seem to have special advantages where there are some pairs of places between which few items move. The argument will be illustrated with reference to data on the numbers of migrants moving in 1970–1971 between pairs of the 126 labor market areas defined for Great Britain [Flowerdew and Salt (1979) present some results from the analysis of this data set]. This data set includes a large number of zero and very small flows. The discussion is restricted to Newtonian gravity models rather than the origin- and destination-constrained forms developed by Wilson (1970).

2. THE DATA

The data set used here consists of observations on one-year migration flows between the 126 SMLA's (Standard Metropolitan Labor Areas) defined by Drewett et al. (1974) for Great Britain. The data were made available to us by Nigel Spence and Stephen Kennett of the Urban Change Study at the Department of Geography, London School of Economics. The SMLA's are intended to

*The authors would like to thank John Salt, John Nelder, and Peter Vincent for their comments and help, and to clear them of responsibility for any remaining errors. The first author would also like to thank the faculty and staff of the Department of Geography, Rutgers University, where the final version of the paper was produced.

†Lecturer in Geography, University of Lancaster and Professor of Statistics at the Centre for Applied Statistics, University of Lancaster, England, respectively.

Date received: June, 1980; revised, May, 1981 and September, 1981.

represent functional labor markets, and consist of core cities of a certain size together with local authority areas linked to them by commuting flows in excess of a critical percentage. SMLA's contain 79.3 percent of the population of Great Britain, and form a continuous cover over most of southeast England, but are patchier in their distribution elsewhere (see Figure 1). Migration between SMLA's constitutes a large proportion of internal migration within Great Britain (73.6 percent of migrants from SMLA's went to other SMLA's). The units were chosen in order that moves between them should reflect as far as possible migration related to job change rather than residential mobility as a result of housing factors.

The data were assembled from the 1971 Census and are based on responses to questions given to a 10 percent sample of households in Great Britain. The analysis was conducted using sample data, and therefore estimates, observations, and residuals must be interpreted accordingly. Migrant flows for the whole population should be approximately 10 times larger than the figures quoted. As movements within SMLA's are excluded from the data set, there are 125 flows from each of the 126 SMLA's, making 15,750 flows in all, with a total of 89,101 inter-SMLA movers. The mean size of flow is thus only 5.7. The largest, from London to Brighton, involved 681 migrants, 249 others involved more than 60 migrants, and 8,150 had no migrants at all. In addition to these zero flows, there is a very large number of very small flows. Although this data set may have an abnormally high proportion of zero and small flows, many such flows may often be found in interaction matrices encountered in geography and regional science.

3. GRAVITY MODELS

The history of the gravity model has been reviewed by Carrothers (1956) and Olsson (1965). Although forerunners of the gravity model can be picked out in the mid-nineteenth century [Carey (1858)], its modern use was popularized by Stewart (1948), who suggested that a force of interaction, F_{ij}, was exerted between two places, i and j, proportional to their masses and inversely proportional to the square of the distance between them. The populations of i and j, P_i and P_j, were regarded as the demographic equivalents of the mass terms. Thus

$$F_{ij} = k\frac{P_iP_j}{d_{ij}^2} \tag{1}$$

where d_{ij} represents the distance from i to j.

Several authors [see, e.g., Isard (1960)] suggested that the distance exponent in Stewart's equation need not necessarily be 2, mainly on the pragmatic grounds that other values appeared to fit observed data better. It was also argued that the relationship of interaction to population might not be linear—accordingly a more general model was introduced:

$$F_{ij} = \frac{kP_i^{\beta_1}P_j^{\beta_2}}{d_{ij}^{\beta_3}} \tag{2}$$

Further generalizations, incorporating different functional forms for the equation, have been made, but the subsequent discussion is based on Equation (2), which is still frequently used.

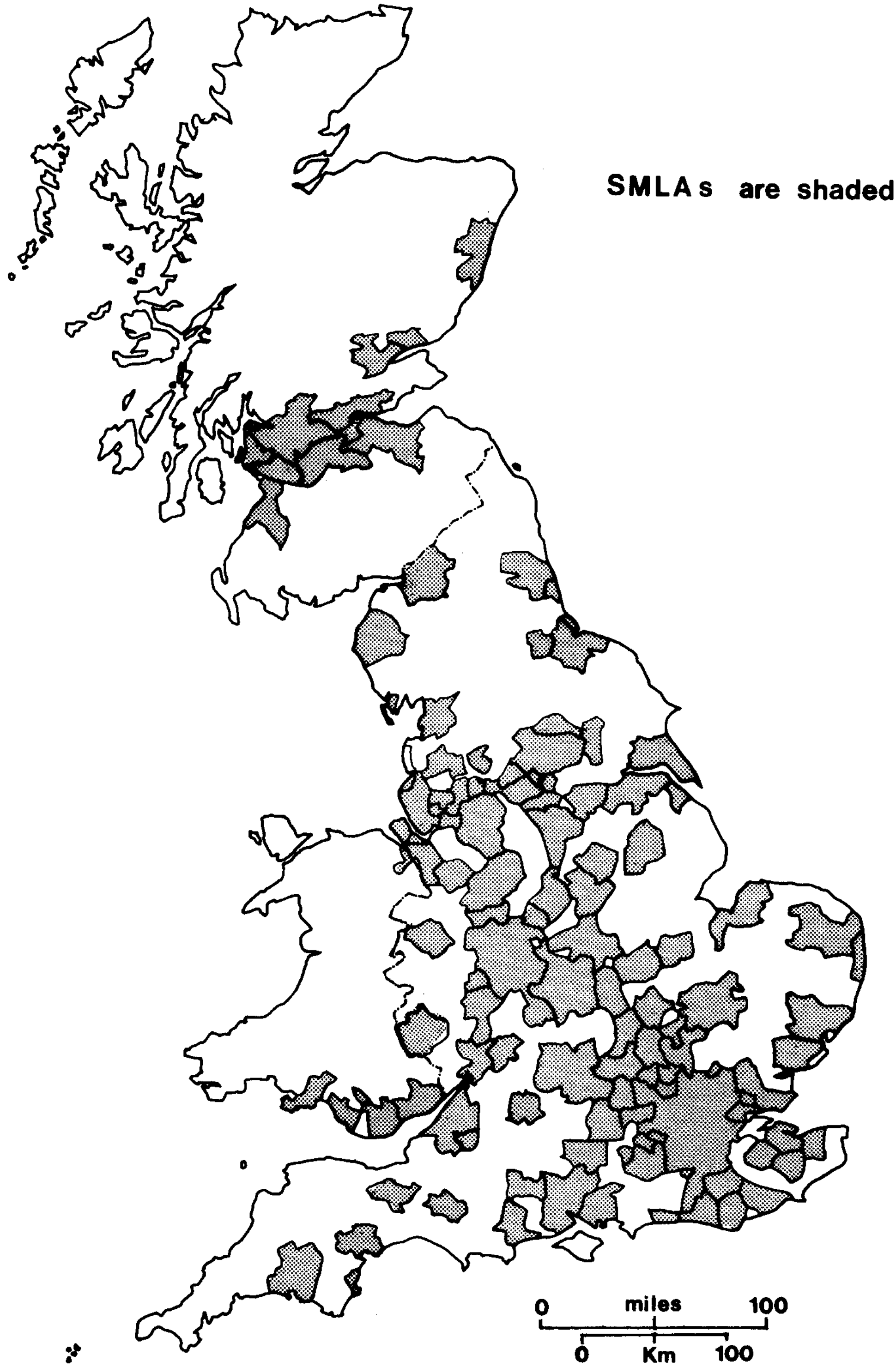

FIGURE 1: Distribution of SMLA's in Great Britain, 1970–1971.

Rewriting the left-hand side of Equation (2) as μ_{ij}, the mean of the random variable F_{ij}, and taking logarithms gives the equation

$$\ln \mu_{ij} = \ln k + \beta_1 \ln P_i + \beta_2 \ln P_j - \beta_3 \ln d_{ij} \tag{3}$$

It is then usual to use $\ln n_{ij}$, the logarithm of the number of migrants recorded as moving from i to j, to estimate the values of $\ln k$, β_{ar}, β_2, and β_3 in an ordinary least-squares multiple regression analysis. This assumes that

$$\ln n_{ij} = \ln k + \beta_1 \ln P_i + \beta_2 \ln P_j - \beta_3 \ln d_{ij} + u_{ij} \tag{4}$$

where the u_{ij}'s are assumed to be independent random variables which are normally distributed with zero mean and identical variance σ^2. This form of the gravity model will be referred to subsequently as the log-normal model.

4. PROBLEMS WITH THE LOG-NORMAL MODEL

The use of the log-normal model in the context of this analysis raises several problems. First, the use of the logarithmic transformation affects the nature of the estimates produced. The regression produces estimates of the logarithms of μ_{ij}, not of the μ_{ij}'s themselves. The antilogarithms of these estimates are biased estimates of μ_{ij} [see Haworth and Vincent (1979)]. One of the effects of this is to underpredict large flows, and to underpredict the total flow. The sum of the estimated flows is substantially less than the sum of the observed flows (45,467 against 89,101 for this data set). As Senior (1979) and others have remarked, this is a severe weakness in some applications of the gravity model.

A second problem concerns the relationship between the estimated mean of the random variable and the values that the variable takes on. The assumption that the u_{ij}'s are normally distributed implies that the possible values that can be taken on by the F_{ij}'s are log-normally distributed around the estimate. In fact, there is little reason to suppose that these values are log-normal; indeed, they must be nonnegative integers because only a whole number of people can be counted as movers.

Thirdly, the log-normal model assumes that the variances of the u_{ij}'s are identical, and hence that the expected difference between the estimate of $\ln \mu_{ij}$ and $\ln n_{ij}$ is the same for all pairs of origins and destinations. This condition applies to the logarithms of the observations. It means that an observed flow of two in relation to an estimate of one has exactly the same probability as an observed flow of 200 in relation to an estimate of 100. As there are a large number of cases where estimated and observed flows are very low, small absolute differences may result in a large difference between the two when compared in logarithmic form.

A fourth difficulty arises from the use of the logarithmic transformation when some of the flows are zero. The logarithm of zero cannot be computed and in fitting the model, therefore, a small positive number (often 0.5) is usually added to all observations. When there are many zero flows, the choice of this number can have a considerable impact on the coefficients of the model and on its explanatory power.

5. THE LOG-NORMAL MODEL

A regression model was fitted according to Equation (4) to the data described above. All values were increased by 0.5. The following equation was obtained, where $\widehat{\ln \mu_{ij}}$ is an estimate of the mean of the distribution of the logged random variable, F_{ij}.

$$\widehat{\ln \mu_{ij}} = -12.66 + 0.723 \ln P_i + 0.666 \ln P_j - 0.721 \ln d_{ij} \tag{5}$$

or

$$\hat{\mu}_{ij} = e^{-12.66} \frac{P_i^{0.723} P_j^{0.666}}{d_{ij}^{0.721}} \tag{6}$$

where $\hat{\mu}_{ij}$ is the antilogarithm of $\widehat{\ln \mu_{ij}}$. The coefficient of determination, R^2, was 0.521. As another measure of goodness of fit, a chi-squared statistic was calculated as a comparison between observed and estimated flows

$$\chi^2 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(n_{ij} - \hat{\mu}_{ij})^2}{\hat{\mu}_{ij}} \tag{7}$$

where n is the number of SMLA's. This statistic has the value 284,663.

Examination of the residuals $(\ln n_{ij} - \widehat{\ln \mu_{ij}})/(\widehat{\ln \mu_{ij}})$ showed that a high proportion of the largest residual values occurred in cases where the estimated flow was very low, and the observed flow was also too low to justify putting much confidence in its empirical significance. The tenth largest positive residual, for example, was movement from Perth to Basildon, where the estimated flow was 0.23 and the observed flow 5. Of the one hundred largest positive residuals, the observed flow in 37 cases was less than 10. It is difficult to justify the importance that the model ascribes to these cases in evaluating the fit when such small discrepancies could easily be a product of sampling variability.

The arbitrary nature of the decision to choose 0.5 as the constant to be added to all values is illustrated by the widely varying nature of the results when other values are used. From Table 1, it is clear that there are major differences in the

TABLE 1: Response of Gravity Model Coefficients to Treatment of Zero Values in the Normal Model

Value Added to Observations	Coefficients				R^2
	Constant	$\ln P_i$	$\ln P_j$	$\ln d_{ij}$	
0.01	−28.64	1.472	1.386	−1.471	.420
0.1	−19.21	1.028	0.959	−1.027	.474
0.3	−14.73	0.818	0.758	−0.817	.507
0.5	−12.66	0.723	0.666	−0.721	.521
0.6	−11.93	0.689	0.634	−0.687	.526
0.7	−11.31	0.661	0.607	−0.659	.529
0.9	−10.31	0.615	0.564	−0.613	.534
1.0	−9.90	0.596	0.546	−0.594	.536

values of the regression coefficients and the coefficient of determination, with the latter generally increasing with the constant selected, and the absolute size of the coefficients declining. It can be seen that the interpretations and conclusions drawn from the analysis could differ considerably. For example, migration between two cities of 450,000 and 1,200,000, twenty miles apart, would be estimated at 90 if the model were based on addition of a constant of 0.1; if the constant were 0.5, the prediction would be 43, and if the constant were 0.9, it would be only 33. The total number of migrants predicted by the model was 45,467, compared to 89,101 observed. Although 67 of the observed flows were over 200, the largest flow predicted was only 168.

6. THE POISSON MODEL

Several of the problems discussed in connection with the use of the log-normal model stem from the assumption that the random variables F_{ij} are log-normally distributed. These problems can be overcome by recognizing that the number of people recorded as moving between i and j must be a nonnegative integer, and hence that each F_{ij} variable should be regarded as having a discrete probability distribution. If there is a small constant probability P_{ij} that any individual in place i moves to place j, and if movements of individuals are independent, then if the population of i is large, the number of individuals recorded as moving from i to j will have a Poisson distribution with mean λ_{ij}. This implies that the probability that k people will have been recorded as moving is

$$Pr(n_{ij} = k) = \frac{e^{-\lambda_{ij}} \lambda_{ij}^k}{k!} \tag{8}$$

The nature of the dependence of migration on population size and distance can then be investigated by deriving a relationship between λ_{ij} and P_i, P_j, and d_{ij}. It is assumed that the parameter λ_{ij} is logarithmically linked to a linear combination of the logged independent variables

$$\lambda_{ij} = \exp(\beta_0 + \beta_1 \ln P_i + \beta_2 \ln P_j + \beta_3 \ln d_{ij}) \tag{9}$$

As λ_{ij} is the parameter of a Poisson distribution, differences between λ_{ij} and the observed flow are explained as results of the particular realization of the Poisson process [the particular value taken by n_{ij} in (8)]. It should be noted that this difference is measured on the scale on which n_{ij} is recorded, whereas the disturbance term in the log-normal distribution (4) is measured on a logarithmic scale.

In contrast to the log-normal case, the observations will not have constant variance (their variance is equal to the mean, as they have Poisson distributions). It is therefore appropriate to use weighted least squares with a weight function equal to λ_{ij} [Nelder and Wedderburn (1972)]. However, λ_{ij} is unknown, but it can be estimated from the fitted values using regression. An iteratively reweighted least squares procedure may then be used, in which initial variance estimates are made from the observed flows (adding 0.5 to all observations) and used to estimate the parameters by weighted least squares. The estimated regression function is then used to produce fitted values $\hat{\lambda}_{ij}$ from which weights are obtained for the second iteration; at each stage, the independent variables are multiplied by the square root of $\hat{\lambda}_{ij}$, and the dependent variable is taken to be $(n_{ij} - \hat{\lambda}_{ij} + \hat{\lambda}_{ij} \ln \hat{\lambda}_{ij})/\sqrt{\hat{\lambda}_{ij}}$. This process is repeated until the parameter estimates converge. This procedure is described by Nelder and Wedderburn (1972), who show that it is equivalent to maximum likelihood estimation for Poisson-distributed random variables.

Unlike the log-normal model, the Poisson model is based on the premise that the dependent variable is a nonnegative integer; where the mean is small, a zero value is not unlikely to occur. Although an adjustment to avoid zeros must be made in using observed flows as initial variance estimates, the fitted values obtained in the first and subsequent iterations are always positive, and no further adjustments are necessary. If 1.0, 0.1, or any other small positive constant is added to observed flows at the start of the iterative procedure, the solution on which the procedure converges is unchanged. The method thus circumvents the problem of how to handle zero flows.

Whereas the log-normal model produces estimates which consistently under-predict large flows, estimates derived from the Poisson model are usually of the same order of magnitude as observed flows. This is because the disturbance term is related directly to the estimate, rather than to the logarithm of the estimate. In addition, the sums of observed and estimated flows are approximately equal, and the size distributions of observed and estimated flows are similar.

The Poisson model discussed above can be fitted in the GLIM (General Linear Interactive Modeling) statistical package [see Baker and Nelder (1978)]. It is one of a class of generalized linear models described by Nelder and Wedderburn (1972), who derive iterative procedures which give maximum likelihood estimates for the coefficients. The models in this class are distinguished according to the sugggested distribution of values around the parameter which is being estimated by regression. In addition to the usual model based on the Normal distribution, Poisson, Binomial, and Gamma distributions can be fitted. In the Normal case, the procedure reduces to ordinary least squares. In the Poisson case, the appropriate procedure is the iteratively reweighted method described above.

Nelder and Wedderburn suggest a statistic, the deviance, for the measurement of goodness-of-fit for generalized linear models. They argue that models for any set of data can range between two extremes, a null model in which the overall mean is used as the estimate for all cases, and a complete model, in which sufficient parameters are fitted to give a perfect fit to the data. As more parameters are fitted, the model becomes more accurate at the expense of increased complexity. The lack of fit of a particular model to data is measured as minus twice the maximum log-likelihood, and the deviance of a particular model is the difference in this quantity between the model concerned and the complete model.

The likelihood function, and hence the deviance, differ according to the distribution being fitted. For the Normal distribution, the deviance is linearly related to the residual sum of squares. For the Poisson, its formula is

$$(10) \qquad 2\left\{ \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} n_{ij} \ln\left(\frac{n_{ij}}{\hat{\lambda}_{ij}}\right) - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} (n_{ij} - \hat{\lambda}_{ij}) \right\}$$

This quantity measures the variation in the data that the model fails to explain. Under a null hypothesis that the model is correct, the deviance is distributed as χ^2, with $n - p$ degrees of freedom, where p is the number of parameters fitted. The proportion of deviance unexplained by the model may be measured by comparing this quantity with the deviance of a null model, in which the mean observed flow $\bar{n}$ is used as the estimate of λ_{ij}

$$(11) \qquad 2\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} n_{ij}\ln\left(\frac{n_{ij}}{\bar{n}}\right) - \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(n_{ij}-\bar{n})\right\}$$

7. RESULTS FOR POISSON MODEL

The Poisson model was estimated for the inter-SMLA migration data, giving the following estimating equation:

$$(12) \qquad \lambda_{ij} = e^{-14.94+0.954\ln P_i+0.804\ln P_j-1.134\ln d_{ij}}$$

The results of the analysis of deviance were as follows:

Deviance of null model	368,787
Deviance of gravity model	77,188
Degrees of freedom	15,746
Proportion of deviance explained	0.791

The coefficients are somewhat different from those estimated for the log-normal model, being a little closer to those predicted by the original gravity analogy. The proportion of deviance accounted for by the model is just under 0.8; however, this figure is not directly comparable with R^2 for the log-normal model. The log-normal model for each observation has two parameters, the mean and the variance (constant for all observations); a low value for R^2 may have either of two causes—the model may not be correctly specified, or the variance may be large. In the Poisson case, however, mean and variance are equal, and there is only one parameter. In this case, the deviance unaccounted for by the model, 77,188, is nearly five times greater than the number of degrees of freedom. The high deviance cannot be explained by the size of the variance, and so, despite the improved fit of the Poisson model, it cannot be a completely satisfactory representation of the data. A more appropriate model, based on the compound Poisson distribution, is discussed below.

A statistic which can be used as a comparison between the log-normal and the Poisson models is the chi-squared statistic of Equation (7). A low value of this statistic indicates a relatively good fit for the model. The respective values, 284,663 and 117,950, are strikingly different, clearly showing the superiority of the Poisson. Examination of the residuals $[(\ln n_{ij} - \ln \hat{\mu}_{ij})/\sqrt{\ln \hat{\mu}_{ij}}]$ from the Poisson model shows, as is to be expected, that the same cases have high residuals in both the log-normal and Poisson regressions (see Table 2). One important difference, however, is that far fewer very small flows are prominent as residuals in the Poisson model. Only six of the 100 largest positive residuals occur when the observed flow is under 10; the highest-ranked again being the Perth-Basildon flow,

TABLE 2: Gravity Model Residuals

Normal					Poisson				
	Origin	Destination	Flow[a]	Estimate		Origin	Destination	Flow[a]	Estimate
1	Plymouth	Portsmouth	211	3.8	1	Plymouth	Portsmouth	211	4.8
2	Plymouth	Dunfermline	35	0.7	2	Portsmouth	Plymouth	145	5.2
3	Portsmouth	Dunfermline	41	1.1	3	Plymouth	Dunfermline	35	0.5
4	Dunfermline	Portsmouth	37	1.0	4	Dunfermline	Portsmouth	37	0.7
5	Portsmouth	Plymouth	145	3.9	5	Portsmouth	Dunfermline	41	0.8
6	Dundee	Burnley	21	0.8	6	Liverpool	Wigan	514	114.9
7	Sheffield	Bolton	0	13.0	7	Glasgow	Corby	69	3.9
8	Aberdeen	Corby	15	0.7	8	Glasgow	London	376	83.8
9	Perth	Basildon	5	0.2	9	Chatham	Portsmouth	81	7.0
10	Glasgow	Corby	69	3.3	10	Edinburgh	London	195	35.3
11	Motherwell	Corby	19	0.9	11	Slough	Reading	170	29.7
12	Perth	Ellesmere Port	4	0.2	12	Norwich	Gt Yarmouth	108	13.6
13	Yeovil	Taunton	22	1.2	13	Aberdeen	Edinburgh	72	6.8
14	Corby	Aberdeen	12	0.7	14	Dundee	Burnley	21	0.7
15	Dunfermline	Plymouth	12	0.7	15	London	Basildon	651	262.9
16	Norwich	Gt Yarmouth	108	5.9	16	London	Basingstoke	364	114.2
17	Ellesmere Port	Manchester	0	9.2	17	Luton	Bedford	113	17.8
18	Rochdale	Kilmarnock	10	0.6	18	Gt Yarmouth	Norwich	90	12.1
19	Barnsley	Liverpool	0	8.9	19	Manchester	Blackpool	241	63.7
20	Letchworth	Bedford	58	3.4	20	London	Brighton	681	298.9

[a]This is the number recorded in the 10 percent sample. The residuals are ranked according to the size of the difference between the logarithm of the flow and the logarithm of the estimate (produced from the gravity model).

now only 66th. Considerably more prominence is given to the larger flows by this analysis: the 20 largest residuals from the log-normal model contain only three flows over 100, whereas 12 of the 20 largest Poisson residuals are in this category. As these flows represent a far greater number of individual migrants, it can be argued that a model which highlights these flows is more useful in policy terms than one where small flows are picked out. It may also be noted that the Poisson model gives considerably better estimates for large flows than does the log-normal model (see Figure 2). The flow from Glasgow to London, for example, with an observed value of 376, was estimated at 84 by the Poisson model (the eighth largest residual in this model), but at only 45 by the normal model.

8. DEFICIENCIES OF THE POISSON MODEL

The Poisson model is more appropriate than the log-normal model which is usually employed in fitting a gravity model to data of this type. Its performance, as measured by the deviance statistic, represents a dramatic improvement over that of the log-normal. Nonetheless, the deviance is still too great for the Poisson model to be accepted as an adequate description of the data. There are two reasons why this conclusion might be expected, related to the limited set of independent variables used and to the form of the model.

The only independent variables used in the analysis were the origin and destination populations and distance. A vast body of work exists on the importance of other variables, such as wage rates, employment rates, housing availabili-

The twenty five largest flows as estimated by the POISSON MODEL are shown, arranged according to the size of estimate.

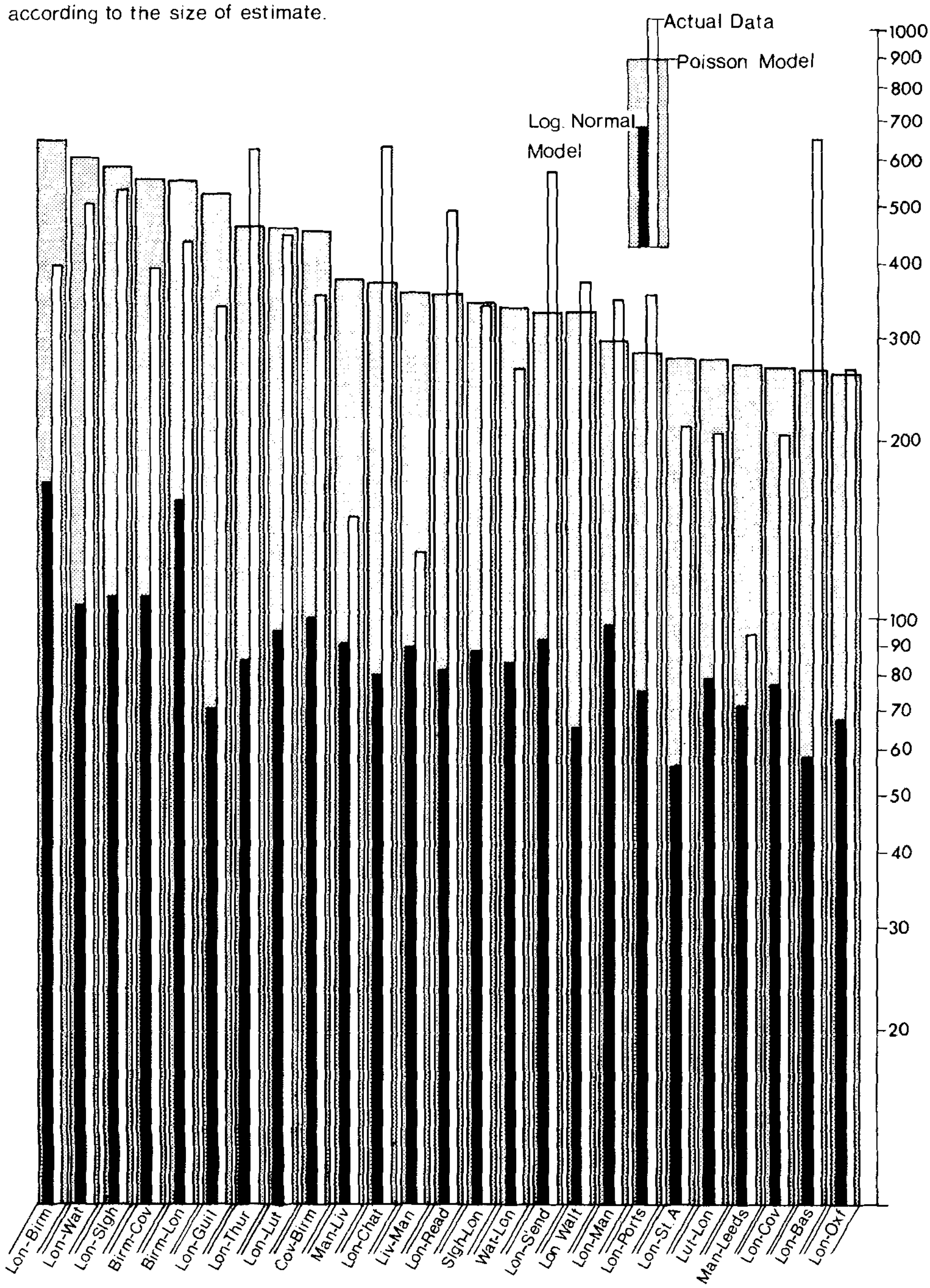

FIGURE 2: Comparison of Residuals from the Two Models for the Largest Flows.

ty, and population characteristics at both origin and destination. A look at the residuals makes it obvious that these factors and others do affect migration. The five largest residuals in both models are for flows between naval bases, and their size must clearly be connected with troop movements; other large residuals can be linked to planned overspill from some of the major cities (Liverpool-Wigan, London-Basingstoke), to the recruitment policies of particular employers (Glasgow-Corby, Dundee-Burnley), or to the well-known drift to the South, from the relatively depressed North of England and Scotland to the relatively prosperous South (Glasgow-London). Other large residuals relate to contiguous SMLA's (Slough-Reading, Luton-Bedford), and may be a product of difficulties in specifying the distance term in the model: some short-distance moves across a boundary may have been treated as if they were between the main population centers of the SMLA's.

The second reason why one might expect the Poisson model to be inadequate lies in the nature of the migration process. Most households involved in interurban migration have more than one member, but the whole household is likely to move in response to a change of job or of other circumstances of one member only. If the number of household heads migrating has a Poisson distribution as described above, then the number of individual migrants should have a compound Poisson distribution [Feller (1957, pp. 268–271)], related to the probability of moving and to the distribution of household size. For any distribution of household size, the variance of the number of migrants will exceed the mean. Fitting the compound Poisson distribution does not affect the parameters of the model or the estimates of the individual flows, but it does change the form of the likelihood function and, hence, the value of the deviance statistic. Preliminary results indicate that this model provides a far more satisfactory fit to the data.

9. CONCLUSION

The traditional approach to estimating a gravity model has been to take logarithms of all the variables, and to assume that the logarithm of the interaction variable is equal to a linear combination of the logarithms of the size and distance variables plus a normally distributed error term. The assumption that this term is normally distributed is not warranted when the interaction variable takes integer values; when many of these values are low, the log-normal model does not give an acceptable approximation. Four specific problems were identified, concerning bias in estimated flows introduced by fitting the model in logarithmic form, the failure of the assumption that the error terms are normally distributed, unequal variance in the error terms, and the sensitivity of model results to the treatment of zero flows. The availability of programs for generalized linear models through the GLIM and GENSTAT packages allows a more appropriate regression model to be fitted. It was assumed that the number of migrants between i and j has a Poisson distribution, and that the logarithm of the Poisson parameter can be estimated as a linear combination of the logarithms of the size and distance variables. A Poisson model was then fitted, a model which appears far superior to the log-normal model in terms of goodness of fit and in the interpretability of the largest residuals, as well as in its applicability to the situation under study. The Poisson model deals directly with the second and third problems mentioned above. It also avoids the bias of log-normal flow estimates, and can successfully handle zero flows.

Although it appears that the Poisson model is not the correct one for the data, it is clearly better than the log-normal model. The facilities in GLIM for users to define their own models allow further experimentation in which appropriate interpretations of the dependent variable can be tried. The Poisson method is applicable not just to gravity models but to the analysis of interaction matrices in terms of any attributes of the interacting units, provided only that the dependent variable is a nonnegative integer.

REFERENCES

Baker, Robert J. and John A. Nelder. *The GLIM System Release 3: Generalised Linear Interactive Modelling.* Oxford: Numerical Algorithms Group, 1978.

Carey, Henry C. *Principles of Social Science.* Philadelphia: Lippincott, 1858.

Carrothers, Gerald A. P. "An Historical Review of the Gravity and Potential Concepts of Human Interaction," *Journal of the American Institute of Planners,* 22 (1956), 94–102.

Drewett, Roy et al. "Urban Change in Britain: 1961–71," Working Reports 1 and 2, Department of Geography, London School of Economics, London, 1974.

Feller, William. *An Introduction to Probability Theory and Its Applications,* Volume 1. New York: Wiley, 1957.

Flowerdew, Robin and John Salt. "Migration Betwen Labour Market Areas in Great Britain, 1970–71," *Regional Studies,* 13 (1979), 211–231.

Haworth, Jean M. and Peter J. Vincent. "The Stochastic Disturbance Specification and its Implications for Log-Linear Regression," *Environment and Planning A,* 11 (1979), 781–790.

Isard, Walter. *Methods of Regional Analysis.* Cambridge: M.I.T. Press, 1960.

Nelder, John A. and Robert W. M. Wedderburn. "Generalised Linear Models," *Journal of the Royal Statistical Society A,* 135 (1972), 370–384.

Olsson, Gunnar. *Distance and Human Interaction: A Review and Bibliography.* Philadelphia: Regional Science Research Institute, 1965.

Senior, Martyn L. "From Gravity Modelling to Entropy Maximizing: A Pedagogic Guide," *Progress in Human Geography,* 3 (1979), 175–210.

Stewart, John Q. "Demographic Gravitation: Evidence and Applications," *Sociometry,* 11 (1948), 31–58.

Wilson, Alan G. *Entropy in Urban and Regional Modelling.* London: Pion, 1970.