CMSC 35401:The Interplay of Learning and Game Theory (Autumn 2022)

Pricing of Information

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Outline

Bayesian Persuasion and Information Selling

> Sell to a Single Decision Maker

> Sell to Multiple Decision Makers

Motivation: Selling Information

Car/house inspections



> Financial advices



Credit report



Consumer data



Motivation: Selling Information

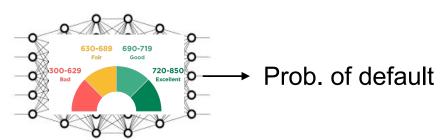
Car/house inspections



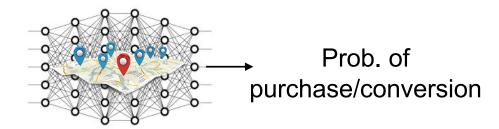
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Consumer data



Persuasion vs Information Selling

➤ In persuasion, we selectively reveal information to induce actions that we like

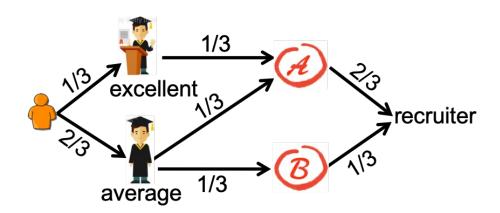




When selling information, we reveal information for a profit

Recap: Model of Bayesian Persuasion

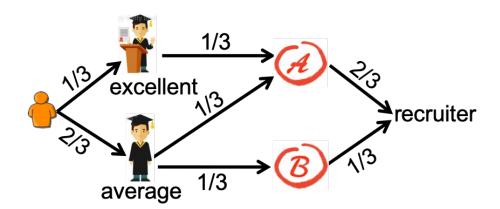
- > Two players: persuader (Sender, she), decision maker (Receiver he)
 - Example: advisor = sender, recruiter = receiver
- \triangleright Receiver looks to take an action $i \in [n] = \{1, 2, ..., n\}$
 - Receiver utility $r(i, \theta)$ $\theta \in \Theta$ is a random state of nature
 - Sender utility $s(i, \theta)$
- \blacktriangleright Both players know $\theta \sim prior\ dist.\ \mu$, but Sender has an informational advantage she can observe realization of θ
- Sender reveal partial information via a signaling scheme



(Simplified) Model of Selling Information

seller

- > Two players: persuader (Sender, she), decision maker (Receiver he)
 - Example: advisor = sender, recruiter = receiver
- \triangleright Receiver looks to take an action $i \in [n] = \{1, 2, ..., n\}$
 - Receiver utility $r(i, \theta)$ $\theta \in \Theta$ is a random state of nature
 - Sender utility $s(i,\theta)$ payment from the receiver
- \blacktriangleright Both players know $\theta \sim prior\ dist.\ \mu$, but Sender has an informational advantage she can observe realization of θ
- Sender reveal partial information via a signaling scheme



How to Sell Information Optimally?

- For any signaling scheme, seller knows how much it improves buyer's expected utility
 - The value of any signaling scheme is known
 - 1. Receiver utility under no information: $\max_{i} \sum_{\theta \in \Theta} r(i, \theta) \cdot \mu(\theta)$
 - 2. Receiver utility under any π : $\sum_{\sigma} \Pr(\sigma) \cdot R(\sigma)$

where
$$R(\sigma) = \max_{i \in [n]} \left[\sum_{\theta \in \Theta} r(i, \theta) \cdot \frac{\pi(\sigma, \theta) \cdot \mu(\theta)}{\Pr(\sigma)} \right]$$

- How to maximize revenue?
 - Reveal full information helps the buyer the most. Why?
 - So OPT is to charge him following amount and then reveal θ directly

Payment =
$$\sum_{\theta \in \Theta} p(\theta) \cdot \left[\max_{i} u(i, \theta; \omega) \right] - \max_{i} \sum_{\theta \in \Theta} p(\theta) \cdot u(i, \theta; \omega)$$

Buyer expected utility if learns θ precisely

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Q: Are we done?

No – in pricing problems, we typically do not know how much buyer values our "product"

Outline

> Bayesian Persuasion and Information Selling

> Sell to a Single Decision Maker

> Sell to Multiple Decision Makers

(True) Model of Selling Information

- ➤ Sender = seller, Receiver = buyer who is a decision maker
- ▶ Buyer takes an action $i \in [n] = \{1, \dots, n\}$
- > Buyer has a utility function $u(i, \theta; t)$ where
 - $\theta \sim dist.$ p is a random state of nature
 - t ~ dist. f captures buyer's (private) utility type

Remarks:

- $\triangleright u, p, f$ are public knowledge
- \triangleright Assume θ , t are independent
- \triangleright Seller observes θ but does not know buyer's type t
- \triangleright Buyer knows his own type t but does not know θ

Key Challenge

The class of mechanisms is too broad

- The mechanism will: (1) elicit private info from buyer; (2) reveal info based on realized θ ; (3) charge buyer
- ➤ May interact with buyer for many rounds
- ➤ Buyer may misreport his private type *t*

Key Challenge

The class of mechanisms is too broad

. . . but, at the end of the day, the buyer of type t is charged some amount x_t in expectation and learns a posterior belief about θ

Theorem (Revelation Principle). Any information selling mechanism is "equivalent" to a direct and truthful revelation mechanism:

- 1. Ask buyer to report type *t*
- 2. Charge buyer x_t and reveal info to buyer via signaling scheme π_t that use n signals (as action recommendations)

 Moreover, the mechanism is incentive compatible (IC) it is the buyer's best interest to truthfully report t
- \triangleright Optimal mechanism reduces to computing an IC menu $\{x_t, \pi_t\}_t$
- Proof omitted here

The Consulting Mechanism [CXZ, SODA'20]

- Elicit buyer type t
- 2. Charge buyer x_t
- 3. Observe realized state θ and recommend action i to the buyer with probability $\pi_t(\sigma_i, \theta)$
- \triangleright Will be incentive compatible reporting true t is optimal
- ➤ The recommended action is guaranteed to be the optimal action for buyer *t* given his information
- $> \{x_t, \pi_t\}_t$ is public knowledge, and computed by LP

Theorem. Consulting mechanism with $\{x_t, \pi_t\}_t$ computed by the following program is the optimal mechanism.

Optimal $\{x_t, \pi_t\}_t$ can be computed by a convex program

- Variables: $\pi_t(\sigma_i, \theta)$ = prob of sending σ_i conditioned on θ for each t
- Variable x_t is the payment from buyer type t

Expected revenue

 $\max \left[\begin{array}{c|c} \sum_{t} f(t) \cdot x_{t} \\ \text{s.t.} & \sum_{i} \left[\sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(i, \theta; t) \right] - x_{t} \\ & \geq \sum_{i} \max_{j} \left[\sum_{\theta} p(\theta) \pi_{t'}(\sigma_{i}, \theta) u(j, \theta; t) \right] - x_{t'}, & \text{for } t' \neq t \\ & \sum_{i} \left[\sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(i, \theta; t) \right] - x_{t} \geq \max_{i} \sum_{\theta} p(\theta) u(i, \theta; t), & \text{for } t \\ & \sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(i, \theta; t) \geq \sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(j, \theta; t), & \text{for } i \neq j, t \\ & \sum_{i} \pi_{t}(\sigma_{i}, \theta) = 1, & \text{for } \theta, t \\ & \pi_{t}(\sigma_{i}, \theta) \geq 0, & \text{for } t, \sigma_{i}, \theta \end{array} \right]$

Optimal $\{x_t, \pi_t\}_t$ can be computed by a convex program

- Variables: $\pi_t(\sigma_i, \theta)$ = prob of sending σ_i conditioned on θ for each t
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Truthfully reporting true *t* is optimal

```
\max \sum_{t} f(t) \cdot x_{t}
s.t. \sum_{i} \left[ \sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(i, \theta; t) \right] - x_{t}
\geq \sum_{i} \max_{j} \left[ \sum_{\theta} p(\theta) \pi_{t'}(\sigma_{i}, \theta) u(j, \theta; t) \right] - x_{t'}, \qquad \text{for } t' \neq t
\sum_{i} \left[ \sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(i, \theta; t) \right] - x_{t} \geq \max_{i} \sum_{\theta} p(\theta) u(i, \theta; t), \qquad \text{for } t
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Participation is no worse than not

```
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- Variables: $\pi_t(\sigma_i, \theta)$ = prob of sending σ_i conditioned on θ for each t
- Variable x_t is the payment from buyer type t
 - A convex function of variables
 - Can be converted to an LP

```
\max \quad \sum_{t} f(t) \cdot x_{t}
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\geq \left[ \sum_{i} \max_{j} \left[ \sum_{\theta} p(\theta) \pi_{t'}(\sigma_{i}, \theta) u(j, \theta; t) \right] - x_{t'}, \right] \quad \text{for } t' \neq t
\sum_{i} \left[ \sum_{\theta} p(\theta) \pi_{t}(\sigma_{i}, \theta) u(i, \theta; t) \right] - x_{t} \geq \max_{i} \sum_{\theta} p(\theta) u(i, \theta; t), \quad \text{for } t
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```

Practical Mechanisms?

What the mechanism is like?

- ➤ Generally, the optimal solution to the previous LP has no structure neither any interpretation
- Nevertheless, closed-form optimal solution is possible for more structured problems

Recall Model of Selling Information

- ➤ Sender = seller, Receiver = buyer who is a decision maker
- ▶ Buyer takes an action $i \in [n] = \{1, \dots, n\}$
- \triangleright Buyer has a utility function u(i,q;t) where
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Remarks:

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Selling Information to a Binary DM

- ➤ Sender = seller, Receiver = buyer who is a decision maker
- ▶ Buyer takes an action $i \in \{0,1\}$: an active action 1 and a passive action 0
 - Active action: approve loan, buyer a car, invest stock X, etc.
- > Buyer has a utility function u(i,q;t) where $\begin{cases} u(0,q;t) \equiv 0 \\ u(1,q;t) = v(q,t) \end{cases}$
 - t ~ dist. f captures buyer's (private) utility type
- Further assume v(q,t) is linear and non-decreasing in t

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 - $t \sim dist. f$ captures buyer's (private) utility type
- Further assume v(q,t) is linear and non-decreasing in t

That is:
$$v(q,t) = v_1(q)[t + \rho(q)]$$
 for some $v_1(q) \ge 0$

Can generalize to convex v(q, t)

Selling Information to a Binary DM

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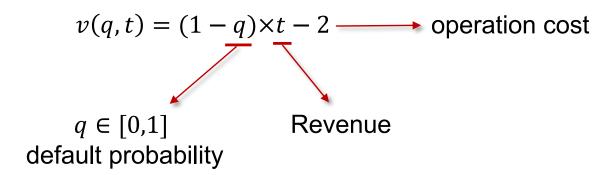
What is the optimal mechanism for this more structured problem?

An Example





- > Buyer is a loan company; action is to approve a loan or not
 - If not approving (action 0), payoff is 0
 - If approving (action 1), payoff is



Threshold experiments turn out to suffice

Recall
$$v(q,t) = v_1(q)[t + \rho(q)]$$

(q is the state unknown to buyer)

Def. π_t is a threshold experiment if π_t simply reveals $\rho(q) \ge \theta(t)$ or not for some buyer-type-dependent threshold $\theta(t)$

 \triangleright Threshold is on $\rho(q)$

Virtual Value Functions

> Recall virtual value function in [Myerson'81]: $\phi(t) = t - \frac{1 - F(t)}{f(t)}$

Def. Lower virtual value function: $\underline{\phi}(t) = t - \frac{1 - F(t)}{f(t)}$

Virtual Value Functions

> Recall virtual value function in [Myerson'81]: $\phi(t) = t - \frac{1 - F(t)}{f(t)}$

Def. Lower virtual value function: $\underline{\phi}(t) = t - \frac{1 - F(t)}{f(t)}$

Upper virtual value function: $\bar{\phi}(t) = t + \frac{F(t)}{f(t)}$

Mixed virtual value function: $\phi_c(t) = c\phi(t) + (1-c)\overline{\phi}(t)$

Note: "upper" or "lower" is due to

$$\underline{\phi}(t) \le t \le \bar{\phi}(t)$$

Virtual Value Functions

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Def. Lower virtual value function: $\underline{\phi}(t) = t - \frac{1 - F(t)}{f(t)}$

Upper virtual value function: $\bar{\phi}(t) = t + \frac{F(t)}{f(t)}$

Mixed virtual value function: $\phi_c(t) = c\phi(t) + (1-c)\overline{\phi}(t)$

- Will assume the virtual value function $\phi(t)$ is monotone (weakly) increasing in t (known as the regularity assumption)
 - Not crucial, since if not monotone, there is an "ironing" procedure to make it monotone

Depend on two problem-related constants:

$$V_{L} = \max\{v(t_{1}), 0\} + \int_{t_{1}}^{t_{2}} \int_{q:\rho(q) \geq -\underline{\phi}^{+}(x)} g(q)v_{1}(q) \, dq dx,$$

$$V_{H} = \max\{v(t_{1}), 0\} + \int_{t_{1}}^{t_{2}} \int_{q:\rho(q) \geq -\overline{\phi}^{+}(x)} g(q)v_{1}(q) \, dq dx,$$

Note: $V_L < V_H$

Theorem.

1. If $\bar{v}(t_2) \leq V_L$, the mechanism with threshold experiments $\theta^*(t) = -\phi^+(t)$ and following payment function represents an optimal mechanism:

$$p^*(t) = \int_{q \in Q} \pi^*(q, t) g(q) v(q, t) dq - \int_{t_1}^t \int_{q \in Q} \pi^*(q, x) g(q) v_1(q) dq dx$$

Theorem.

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2. If $\bar{v}(t_2) \ge V_H$, the mechanism with threshold experiments $\theta^*(t) = -\bar{\phi}^+(t)$ and following payment function represents an optimal mechanism:

$$p^*(t) = \int_{q \in Q} \pi^*(q, t) g(q) v(q, t) dq + \int_{t}^{t_2} \int_{q \in Q} \pi^*(q, x) g(q) v_1(q) dq \ dx - \bar{v}(t_2)$$

Theorem (cont'd).

3. If $V_L \le \bar{v}(t_2) \le V_H$, the mechanism with threshold experiments $\theta^*(t) = -\phi_c^+(t)$ and following payment function represents an optimal mechanism:

$$p^*(t) = \int_{q \in Q} \pi^*(q, t) g(q) v(q, t) dq - \int_{t_1}^t \int_{q \in Q} \pi^*(q, x) g(q) v_1(q) dq dx$$

where constant *c* is chosen such that

$$\int_{t_1}^{t_2} \int_{q:\rho(q) \ge \phi_c^+(x)} g(q) v_1(q) dq \ dx = \bar{v}(t_2)$$

This is the technically more involved part to prove (see [LSX, EC'21])

Remarks

- Threshold mechanisms are common in real life
 - House/car inspections, stock recommendations: information seller only need to reveal it "passed" or "deserves a buy" or not
- Proproximality of threshold mechanisms relies on monotonicity of v(q,t) in t, not true in general
- Optimal mechanism has personalized thresholds and payments, tailored to accommodate different level of risk each buyer type can take
 - Different from optimal pricing of physical goods







Remarks

What if seller is restricted to sell the same information to every buyer? How will revenue change?

- Revenue can be arbitrarily worse
- \blacktriangleright 1/e-approximation of optimal revenue if the *value of full information* as a function of t is "heavy tail"

Outline

Bayesian Persuasion and Information Selling

> Sell to a Single Decision Maker

➤ Sell to Multiple Decision Makers

Challenges

- > For single decision maker, more information always helps
 - Recall in persuasion, receiver always benefits from signaling scheme
- ➤ A fundamental challenge for selling to multiple buyers is that information does not necessarily help them

- ➤ Insurance industry: *insurance company* and *customer*
 - Both are potential information buyers
- > Two types of customers: Healthy and Unhealthy
 - Publicly know, Pr(Healthy) = 0.9
- ➤ Seller is an information holder, who knows whether any customer is healthy or not

Insurance company

	Sell	Not Sell
Buy	(-10, 10)	(-0, 0)
Not Buy	(0,0)	(0,0)

customer

Healthy customer

Insurance company

	Sell	Not Sell
Buy	(-10, -50)	(-110, 0)
Not Buy	(-111 , 0)	(-111, 0)

Unhealthy customer

Insurance company

customer

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Buy	(-10, 10)	(-0, 0)
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Healthy customer, prob = 0.9

Insurance company

	Sell	Not Sell
Buy	(-10, -50)	(-110, 0)
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Unhealthy customer

Q: What happens without seller's information?

- > Customer and insurance company will look at expectation
 - Dominant strategy equilibrium is (Buy, Sell)

	Sell	Not Sell
Buy	(-10, 4)	(-11, 0)
Not Buy	(-11.1, 0)	(-11.1, 0)

Insurance company

customer

	Sell	Not Sell
Buy	(-10, 10)	(-0, 0)
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Healthy customer, prob = 0.9

Insurance company

	Sell	Not Sell
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Unhealthy customer

Q: What if seller tells (even only) customer her health status?

- > If Healthy, customer will not buy → utility (0,0) for both
- ➤If Unhealthy, customer will buy → Will not sell, utility (-110,0)
- >Customer's reaction reveals his healthy status
- ➤ In expectation (-11, 0), and no insurance was sold ever

Recall previous utilities (-10,4)

Insurance company

customer

	Sell	Not Sell
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Healthy customer, prob = 0.9

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Unhealthy customer

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Lessons Learned

- Existence of insurance is due to ignorance to our health condition
- Such ignorance benefits both us and insurance companies

Thank You

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