

CS6501:Topics in Learning and Game Theory (Fall 2019)

Prediction Markets and Scoring Rules

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Outline

- Recap: Scoring Rule and Information Elicitation
- Connection to Prediction Markets
- Manipulations in Prediction Markets

Information Elicitation from A Single Expert

- We (designer) want to learn the distribution of random var $E \in [n]$
 - E will be sampled in the future
- An expert/predictor has a predicted distribution $\lambda \in \Delta_n$
- Want to incentivize the expert to truthfully report λ

Idea: reward expert by designing a **scoring rule** $S(i; p)$ where:

- (1) p is the expert's report (may not equal λ);
- (2) $i \in [n]$ is the event realization

Definition. The “**scoring rule**” $S(i; p)$ is **[strictly]** proper if truthful report $p = \lambda$ **[uniquely]** maximizes expected utility $S(\lambda; p)$.

Proper Scoring Rules

Example 1 [Log Scoring Rule]

$$\triangleright S(i; p) = \log p_i$$

Example 2 [Quadratic Scoring Rule]

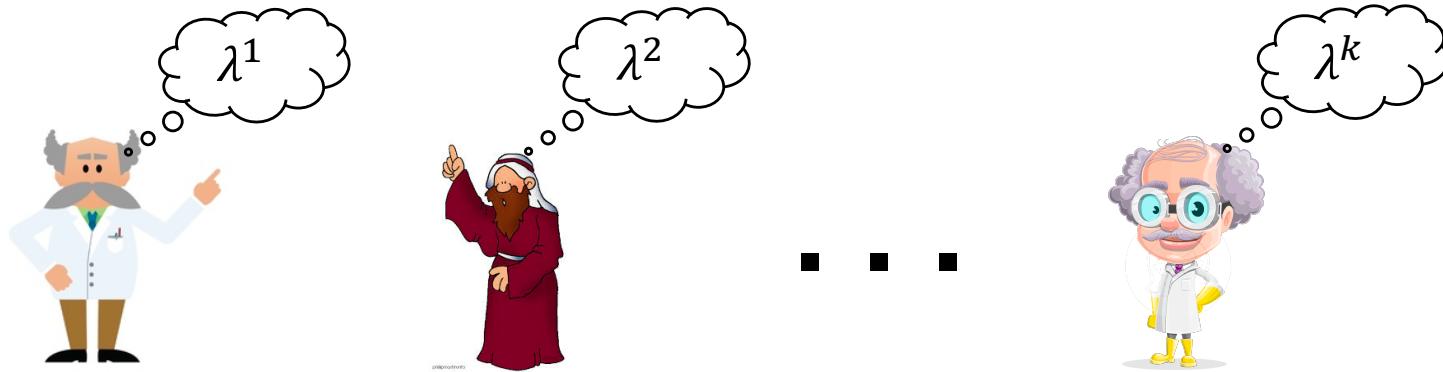
$$\triangleright S(i; p) = 2p_i - \sum_{j \in [n]} p_j^2$$

Theorem. The scoring rule $S(i; p)$ is (strictly) proper if and only if there exists a (strictly) convex function $G: \Delta_n \rightarrow \mathbb{R}$ such that

$$S(i; p) = G(p) + \nabla G(p)(e_i - p)$$

basis vector

Information Elicitation from Many Experts



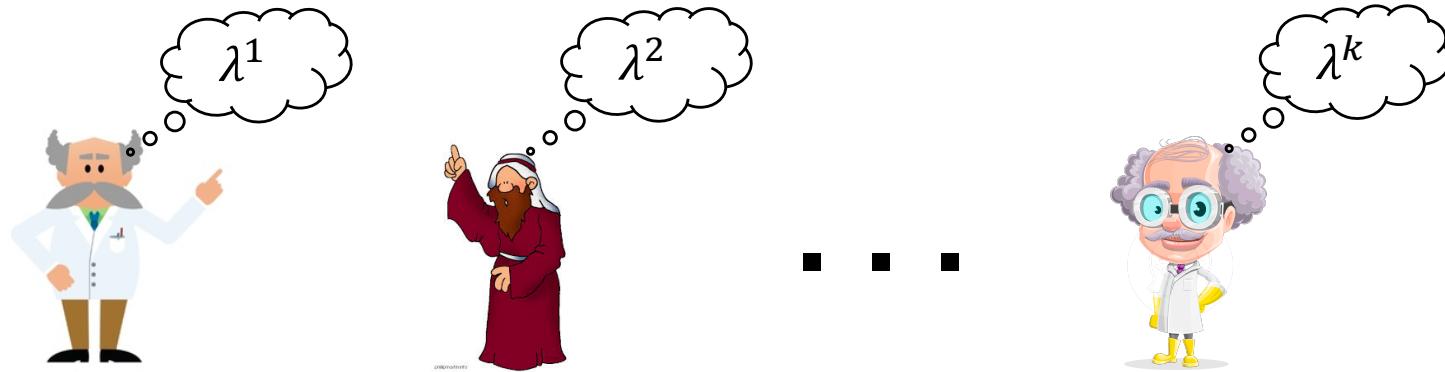
Idea: sequential elicitation – experts make predictions in sequence

- Reward for expert k 's prediction p^k is

$$S(i; p^k) - S(i; p^{k-1})$$

- I.e., experts are paid based on how much they improved the prediction

Information Elicitation from Many Experts

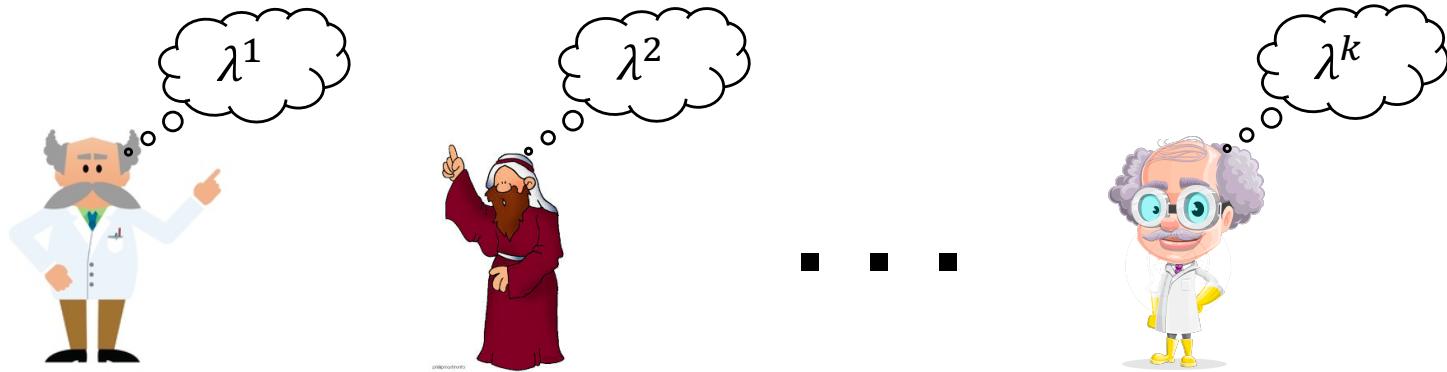


Theorem. If S is a **proper scoring rule** and each expert can **only predict once**, then each expert maximizes utility by reporting true belief given her own knowledge.

Remark

- Each expert is expected to improve the prediction by aggregating previous predictions and then update it
 - Otherwise they will lose money

Information Elicitation from Many Experts



Theorem. If S is a **proper scoring rule** and each expert can **only predict once**, then each expert maximizes utility by reporting true belief given her own knowledge.

Q1: how does sequential elicitation relate to prediction market?

Q2: what happens if an expert can predict for multiple times?

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Equivalence of PMs and Sequential Elicitation

Theorem (informal). Under mild technical assumptions, efficient prediction markets are in one-to-one correspondence to sequential information elicitation using proper scoring rules.

What does it mean?

- Experts will have exactly the same incentives and receive the same return
- Market maker's total loss is the same

Next: will *informally* argue using the LMSR and log-scoring rules

Equivalence of LMSR and Log-Scoring Rules

Recall LMSR

- Value function with current sales quantity q : $V(q) = b \log \sum_{j \in [n]} e^{q_j/b}$
- To buy $x \in \mathbb{R}^n$ amount, a buyer pays: $V(q + x) - V(q)$
- Price function (they sum up to 1)
$$p_i(q) = \frac{e^{q_i/b}}{\sum_{j \in [n]} e^{q_j/b}} = \frac{\partial V(q)}{\partial q_i}$$

Fact. The optimal amount an expert purchases is the amount that moves the market price to her belief λ .

Fact. Worst case market maker loses is $b \log n$.

Equivalence of LMSR and Log-Scoring Rules

Q1: If current market price is p^{k-1} , what is the optimal payoff for an expert with belief $\lambda = p^k$?

- Let q^{k-1} denote the market standing corresponding to price p^{k-1}
 - That is

$$\frac{e^{q_i^{k-1}/b}}{\sum_{j \in [n]} e^{q_j^{k-1}/b}} = p_i^{k-1}$$

Crucial terms:

- Value function $V(q) = b \log \sum_{j \in [n]} e^{q_j/b}$
- Price function $p_i(q) = \frac{e^{q_i/b}}{\sum_{j \in [n]} e^{q_j/b}} = \frac{\partial V(q)}{\partial q_i}$

Equivalence of LMSR and Log-Scoring Rules

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- Let q^{k-1} denote the market standing corresponding to price p^{k-1}
- Optimal purchase for the expert is x^* such that

$$p_i(q^{k-1} + x^*) = \frac{e^{(q_i^{k-1} + x_i^*)/b}}{\sum_{j \in [n]} e^{(q_j^{k-1} + x_j^*)/b}} = p_i^k$$

and pays

$$\begin{aligned} & V(q^{k-1} + x^*) - V(q^{k-1}) \\ &= b \log \sum_{j \in [n]} e^{(q_j^{k-1} + x_j^*)/b} - b \log \sum_{j \in [n]} e^{q_j^{k-1}/b} \end{aligned}$$

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$$\sum_{j \in [n]} e^{(q_j^{k-1} + x_j^*)/b} = \frac{e^{(q_i^{k-1} + x_i^*)/b}}{p_i^k}$$

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Equivalence of LMSR and Log-Scoring Rules

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Equivalence of LMSR and Log-Scoring Rules

Q1: If current market price is p^{k-1} , what is the optimal payoff for an expert with belief $\lambda = p^k$?

- Let q^{k-1} denote the market standing corresponding to price p^{k-1}
- Repeat our finding: expert pays $x_i^* - b(\log p_i^k - \log p_i^{k-1})$
 - x^* is optimal amount for purchase
- What is the expert utility if outcome i is ultimately realized?

$$x_i^* - [x_i^* - b(\log p_i^k - \log p_i^{k-1})]$$



from contracts' return

Equivalence of LMSR and Log-Scoring Rules

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 - x^* is optimal amount for purchase
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$$\begin{aligned}x_i^* - [x_i^* - b(\log p_i^k - \log p_i^{k-1})] \\= b \cdot [\log p_i^k - \log p_i^{k-1}] \\= b \cdot [S^{\log}(i; p^k) - S^{\log}(i; p^{k-1})] \\= \text{payment in the sequential elicitation} \\(\text{constant } b \text{ is a scalar})\end{aligned}$$

Equivalence of LMSR and Log-Scoring Rules

Q1: If current market price is p^{k-1} , what is the optimal payoff for an expert with belief $\lambda = p^k$?

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- Repeat our finding: expert pays $x_i^* - b(\log p_i^k - \log p_i^{k-1})$
 - x^* is optimal amount for purchase
- What is the expert utility if outcome i is ultimately realized?

Expert achieves the same utility in LMSR and log-scoring-rule elicitation *for any event realization*

Equivalence of LMSR and Log-Scoring Rules

Q2: What is the worst case loss (i.e., maximum possible payment) when using log-scoring rule in sequential info elicitation?

- Total payment – if event i realized – is

$$\begin{aligned}\sum_{k=1}^K [\log p_i^k - \log p_i^{k-1}] &= \log p_i^K - \log p_i^0 \\ &\leq 0 - \log p_i^0\end{aligned}$$

- To avoid cases where some p_i^0 is too small (then we need to pay a lot), should choose $p^0 = (\frac{1}{n}, \dots, \frac{1}{n})$ as uniform distribution
- Worst-case loss is thus $\log n$ (same as LMSR, up to constant b)

Back to Our Original Theorem...

Theorem (informal). Under mild technical assumptions, efficient prediction markets are in one-to-one correspondence with sequential information elicitation using proper scoring rules.

- Previous argument generalizes to arbitrary proper scoring rules
- Formal proof employs *duality theory*
 - Recall, any proper scoring rule corresponds to a convex function
 - A prediction market is determined by a value function $V(q)$

The Correspondence

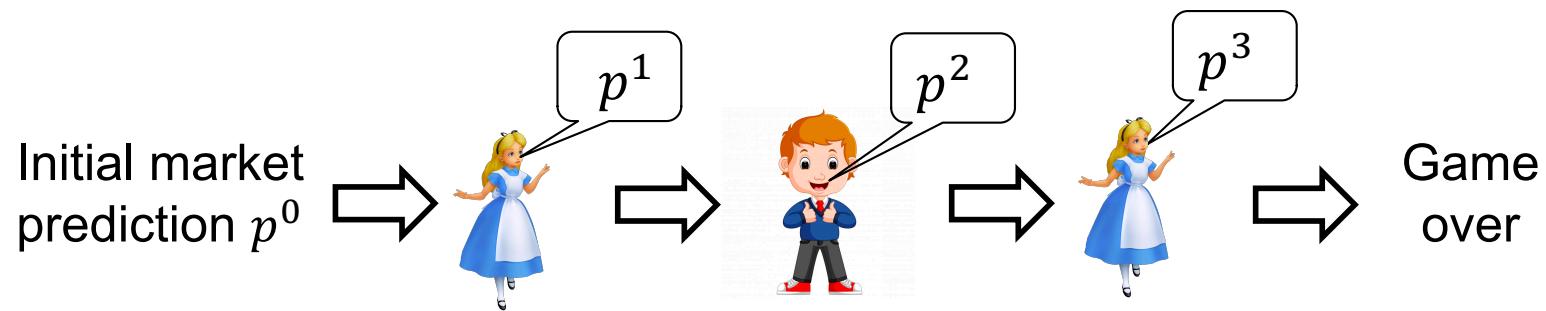
PM with $V(q)$ corresponds to sequential elicitation with scoring rules determined by $V^*(p) = \text{the convex conjugate of } V(q)$

- Convex conjugate is in some sense the “dual” of function $V(q)$
- See paper *Efficient Market Making via Convex Optimization* for details

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- Generally, we cannot force experts to participate just once
 - E.g., in prediction market, cannot force expert to just purchase once
- Manipulations arise when experts can predict multiple times
 - This is the case even **two experts A, B and only A can predict twice**
 - The so-called **A-B-A game** (arguably the most fundamental setting with multiple-round predictions)



An Example of A-B-A Game

- Predict event $E \in \{0,1\}$; Outcome drawn uniformly at random
- Expert Alice observes a signal $A = E$
 - She exactly observes outcome
- Expert Bob also observes the outcome, i.e., signal $B = E$

Q: In A-B-A game, what should Alice predict at stage 1 and 3?

Report her true prediction at stage 1 (which is perfectly correct)

A-B-A Game: Example 2

- Alice observes signal $A \in \{0,1\}$, and $\Pr(A = 0) = 0.51$
- Bob observes signal $B \in \{0,1\}$, and $\Pr(B = 0) = 0.49$
 - A, B are independent
- They are asked to predict event $E = (\text{whether } A + B = 1)$
 - The answer is YES or NO

Q: what is the optimal experts behaviors in A-B-A game?

Market starts with initial prediction $p^0(\text{YES}) = P^0(\text{NO}) = 1/2$

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 - The answer is YES or NO

Q: what is the optimal experts behaviors in A-B-A game?

- At stage 1, what is Alice's probability belief of YES?
 - If Alice's $A = 1$, then $\Pr(\text{YES}) = 0.49$
 - If Alice's $A = 0$, then $\Pr(\text{YES}) = 0.51$
- Should Alice report this at stage 1?
 - No, her truthful report tells B exactly the value of her A
 - Bob can then make a perfect prediction

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Q: what is the optimal experts behaviors in A-B-A game?

- What should Alice do at stage 1 then?
 - Say nothing, or equivalently, predict $p^1 = p^0$

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Q: what is the optimal experts behaviors in A-B-A game?

- What should Bob predict at stage 2?
 - Bob learns nothing from stage 1
 - So If $B = 1$, then $\Pr(\text{YES}) = 0.51$; if $B = 0$, then $\Pr(\text{YES}) = 0.49$
 - Should report truthfully based on the above belief – why?

He only has one chance to predict, and his belief is the best given his current knowledge

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 - Bob learns nothing from stage 1
 - So If $B = 1$, then $\Pr(\text{YES}) = 0.51$; if $B = 0$, then $\Pr(\text{YES}) = 0.49$
 - Should report truthfully based on the above belief – why?
 - Bob's truthful report reveals his signal, but gains little utility

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- Bob observes signal $B \in \{0,1\}$, and $\Pr(B = 0) = 0.49$
 - A, B are independent
- They are asked to predict event $E = (\text{whether } A + B = 1)$
 - The answer is YES or NO

Q: what is the optimal experts behaviors in A-B-A game?

- What should Alice predict at stage 3?
 - She just learned Bob's signal B
 - So can precisely predict "whether $A + B = 1$ " now
 - Alice now moves the prediction from $\Pr(\text{YES}) = 0.51 \text{ or } 0.49$ to $\Pr(\text{YES}) = 1 \text{ or } 0 \rightarrow \text{receiving a lot of credits}$

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 - The answer is YES or NO

Remarks

- Example shows how experts aggregate previous information and update their predictions along the way
- Manipulations arise even if a single expert can predict twice

A-B-A Game: Example 2

- Alice observes signal $A \in \{0,1\}$, and $\Pr(A = 0) = 0.51$
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 - The answer is YES or NO

Remarks

- This is an issue in prediction markets, since experts can buy and sell whenever they want
- Equilibrium of PMs are still poorly understood, even for the fundamental A-B-A games
 - See a recent paper *Computing Equilibria of Prediction Markets via Persuasion* for state-of-the-art results

Thank You

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