

# Announcements

➤ HW 1 is out, due 10/18 2 pm (please start early!)

# CMSC 3540I: The Interplay of Learning and Game Theory (Autumn 2022)

## Linear Programming Duality

Instructor: Haifeng Xu



# Outline

- Recap and Weak Duality
- Strong Duality and Its Proof
- Consequence of Strong Duality

# Linear Program (LP)

General form:

minimize (or maximize)	$c^T \cdot x$
subject to	$a_i \cdot x \leq b_i \quad \forall i \in C_1$
	$a_i \cdot x \geq b_i \quad \forall i \in C_2$
	$a_i \cdot x = b_i \quad \forall i \in C_3$

Standard form:

maximize	$c^T \cdot x$
subject to	$a_i \cdot x \leq b_i \quad \forall i = 1, \dots, m$
	$x_j \geq 0 \quad \forall j = 1, \dots, n$

# Application: Optimal Production

- $n$  products,  $m$  raw materials
- Every unit of product  $j$  uses  $a_{ij}$  units of raw material  $i$
- There are  $b_i$  units of material  $i$  available
- Product  $j$  yields profit  $c_j$  per unit
- Factory wants to maximize profit subject to available raw materials

Can be formulated as an LP in *standard form*

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in [m] \\ & x_j \geq 0, \quad \forall j \in [n] \end{aligned}$$

# Primal and Dual Linear Program

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in [m] \\ & x_j \geq 0, \quad \forall j \in [n] \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j \in [n] \\ & y_i \geq 0, \quad \forall i \in [m] \end{aligned}$$

## Economic Interpretation:

Dual LP corresponds to the **buyer's optimization problem**, as follows:

- Buyer wants to directly buy the raw material
- Dual variable  $y_i$  is buyer's proposed **price** per unit of raw material  $i$
- Dual price vector is feasible if factory is incentivized to sell materials
- Buyer wants to spend as little as possible to buy raw materials

# Economic Interpretation

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in [m] \\ & x_j \geq 0, \quad \forall j \in [n] \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j \in [n] \\ & y_i \geq 0, \quad \forall i \in [m] \end{aligned}$$

price of material ←

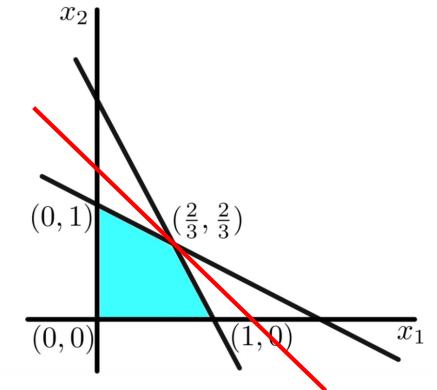
units of products →

	$x_1$	$x_2$	$x_3$	$x_4$	
$y_1$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$b_1$
$y_2$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	$b_2$
$y_3$	$a_{31}$	$a_{32}$	$a_{33}$	$a_{34}$	$b_3$
	$c_1$	$c_2$	$c_3$	$c_4$	

## Interpretation II: Finding Best Upperbound

- Consider the simple LP from previous 2-D example

$$\begin{array}{ll}\text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0\end{array}$$



- We found that the optimal solution was at  $(\frac{2}{3}, \frac{2}{3})$  with an optimal value of  $\frac{4}{3}$ .
- What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by 1 and summing gives  $x_1 + x_2 \leq 4/3$ .

## Interpretation II: Finding Best Upperbound

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

➤ In Primal, multiplying each row  $i$  by  $y_i$  and summing gives inequality

$$y^T A x \leq y^T b \quad (1)$$

(now we see why  $y_i \geq 0$  when  $a_i x \leq b_i$  but  $y_i \in \mathbb{R}$  when  $a_i x = b_i$ )

➤ Under constraint  $c^T \leq y^T A$ , we have

$$c^T x \leq y^T A x \leq y^T b \quad (\text{by Ineq. (1)})$$

that is,  $y^T b$  is an upper bound for  $c^T x$  for every feasible  $x$

➤ The dual LP can be interpreted as finding the best upperbound on the primal that can be achieved this way.

# Properties of Duals

- Duality is an inversion

**Fact:** Given any primal LP, the dual of its dual is itself.

Proof: homework exercise

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & a_i^T x \leq b_i, \quad \forall i \in C_1 \\ & a_i^T x = b_i, \quad \forall i \in C_2 \\ & x_j \geq 0, \quad \forall j \in D_1 \\ & x_j \in \mathbb{R}, \quad \forall j \in D_2 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \bar{a}_j y \geq c_j, \quad \forall j \in D_1 \\ & \bar{a}_j y = c_j, \quad \forall j \in D_2 \\ & y_i \geq 0, \quad \forall i \in C_1 \\ & y_i \in \mathbb{R}, \quad \forall i \in C_2 \end{aligned}$$

- So far, mainly writing the Dual based on syntactic rules
- Next, will show Primal and Dual are inherently related

# Weak Duality

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

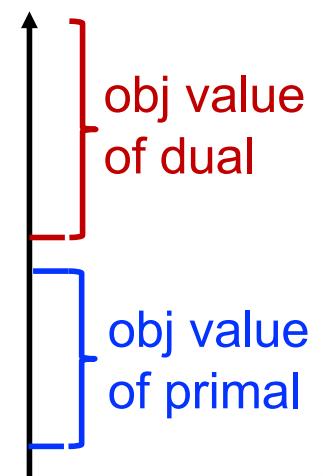
$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

**Theorem [Weak Duality]:** For any primal feasible  $x$  and dual feasible  $y$ , we have  $c^T \cdot x \leq b^T \cdot y$

## Corollary:

- If primal is unbounded, dual is infeasible
- If dual is unbounded, primal is infeasible
- If primal and dual are both feasible, then

$$\text{OPT(primal)} \leq \text{OPT(dual)}$$



# Weak Duality

Primal LP

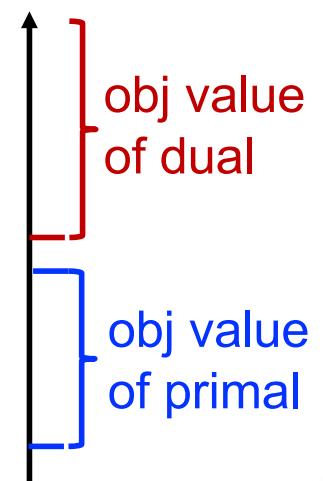
$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

**Theorem [Weak Duality]:** For any primal feasible  $x$  and dual feasible  $y$ , we have  $c^T \cdot x \leq b^T \cdot y$

**Corollary:** If  $x$  is primal feasible and  $y$  is dual feasible, and  $c^T \cdot x = b^T \cdot y$ , then both are optimal.



# Interpretation of Weak Duality

## Economic Interpretation:

If prices of raw materials are set such that there is incentive to sell raw materials directly, then factory's total revenue from sale of raw materials would exceed its profit from any production.

## Upperbound Interpretation:

The method of rescaling and summing rows of the Primal indeed gives an upper bound of the Primal's objective value (well, self-evident...).

# Proof of Weak Duality

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

$$y^T \cdot b \geq y^T \cdot Ax = x^T \cdot A^T y \geq x^T \cdot c$$

# Outline

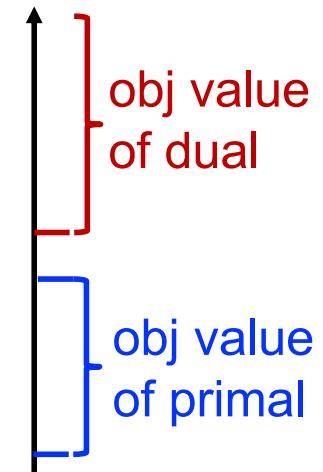
- Recap and Weak Duality
- Strong Duality and Its Proof
- Consequence of Strong Duality

# Strong Duality

**Theorem [Strong Duality]:** If either the primal or dual is feasible and bounded, then so is the other and  $\text{OPT}(\text{primal}) = \text{OPT}(\text{dual})$ .



*... I thought there was nothing worth publishing until the Minimax Theorem was proved.*



John von Neumann

# Interpretation of Strong Duality

## Economic Interpretation:

There exist raw material prices such that the factory is indifferent between selling raw materials or products.

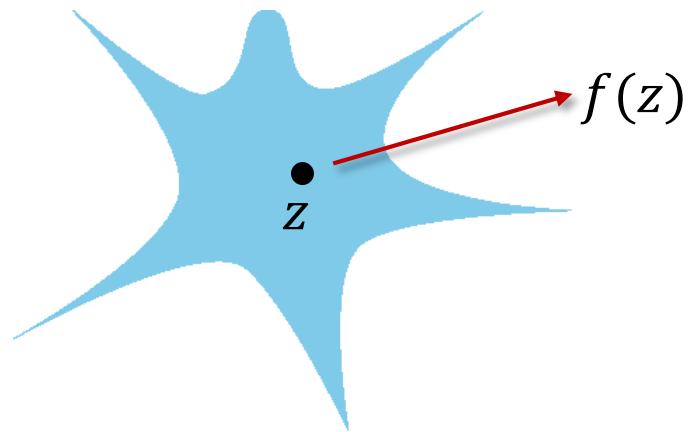
## Upperbound Interpretation:

The method of scaling and summing constraints yields a **tight** upperbound for the primal objective value.

# Proof of Strong Duality

# Projection Lemma

**Weierstrass' Theorem:** Let  $Z$  be a compact set, and let  $f(z)$  be a continuous function on  $z$ . Then  $\min\{ f(z) : z \in Z \}$  exists.

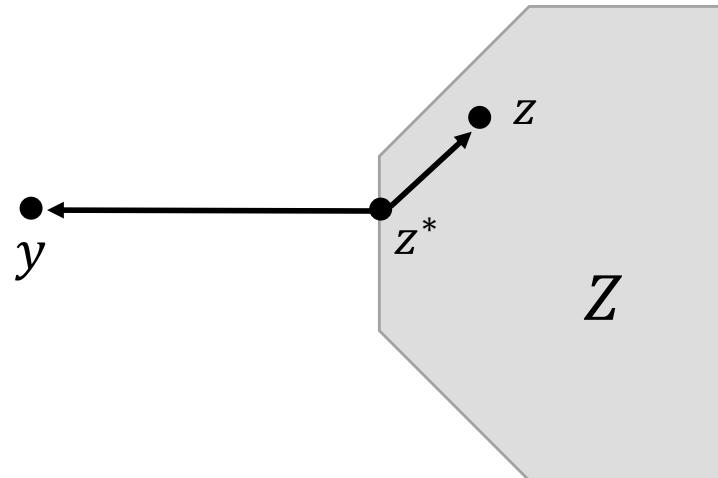


# Projection Lemma

**Weierstrass' Theorem:** Let  $Z$  be a compact set, and let  $f(z)$  be a continuous function on  $z$ . Then  $\min\{ f(z) : z \in Z \}$  exists.

**Projection Lemma:** Let  $Z \subset \mathbb{R}^m$  be a nonempty closed convex set and let  $y \notin Z$ . Then there exists  $z^* \in Z$  with minimum  $l_2$  distance from  $y$ . Moreover,  $\forall z \in Z$  we have  $(y - z^*)^T(z - z^*) \leq 0$ .

Proof: homework exercise

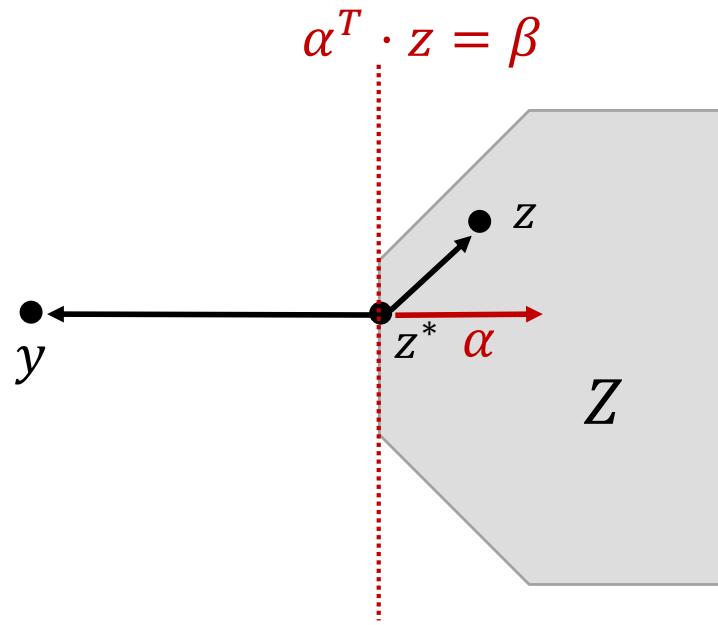


# Separating Hyperplane Theorem

**Theorem:** Let  $Z \subset \mathbb{R}^m$  be a nonempty closed convex set and let  $y \notin Z$ . Then there exists a hyperplane  $\alpha^T \cdot z = \beta$  that **strictly separates**  $y$  from  $Z$ . That is,  $\alpha^T \cdot z \geq \beta, \forall z \in Z$  and  $\alpha^T \cdot y < \beta$ .

Proof: choose  $\alpha = z^* - y$  and  $\beta = \alpha \cdot z^*$  and use projection lemma

- Homework exercise



# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$b_1$
$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	$b_2$
$a_{31}$	$a_{32}$	$a_{33}$	$a_{34}$	$b_3$

# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

Case b):

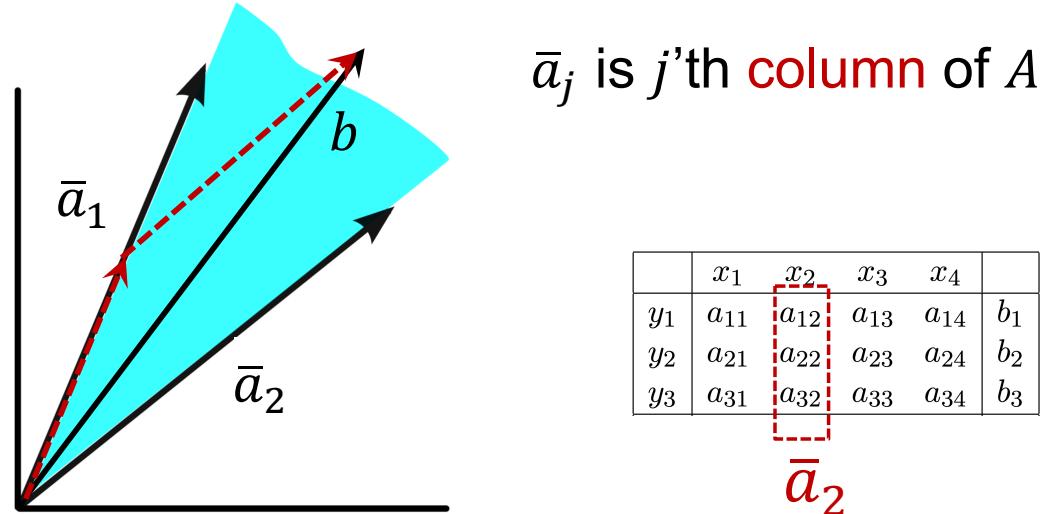
$y_1$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$b_1$
$y_2$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	$b_2$
$y_3$	$a_{31}$	$a_{32}$	$a_{33}$	$a_{34}$	$b_3$

# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

**Geometric interpretation:**



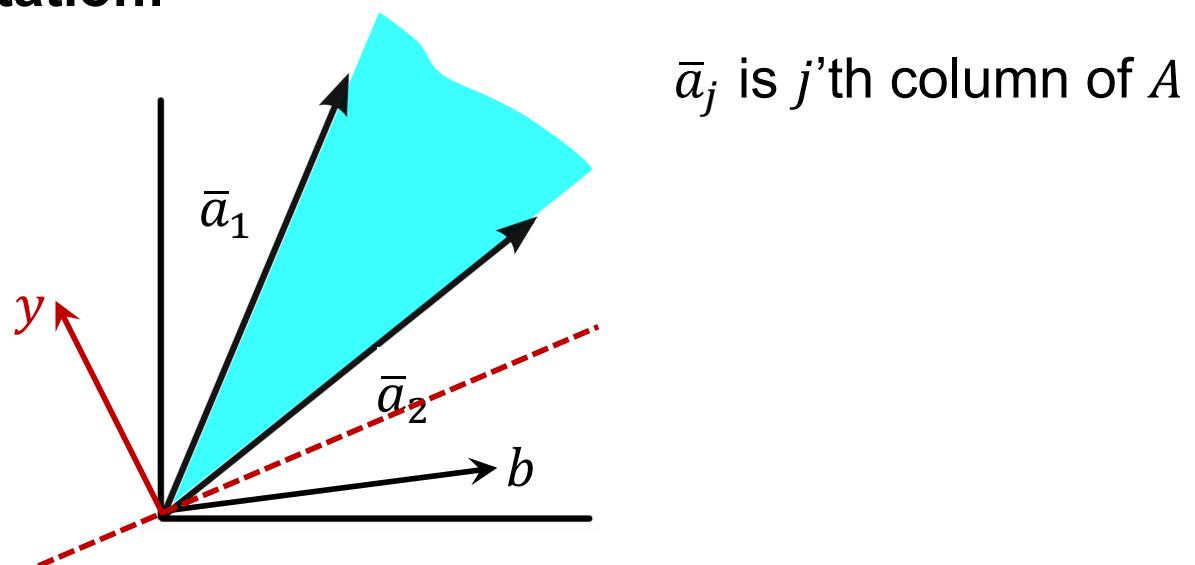
- a)  $b$  is in the cone

# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

**Geometric interpretation:**



- a)  $b$  is in the cone
- b)  $b$  is not in the cone, and there exists a hyperplane with direction  $y$  that separates  $b$  from the cone

# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

Proof:

- Cannot both hold; Otherwise, yields contradiction as follows:

$$0 \leq (A^T y)^T \cdot x = y^T \cdot (Ax) = y^T \cdot b < 0.$$

- Next, we prove if (a) does not hold, then (b) must hold

- This implies the lemma

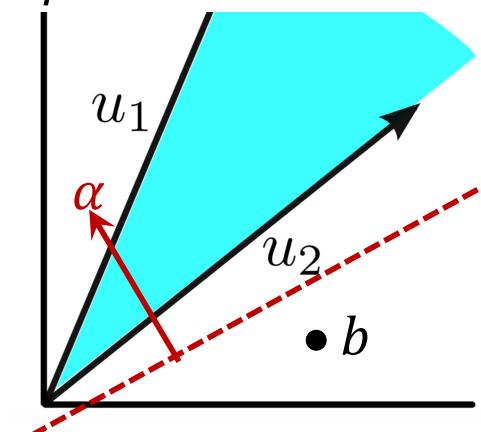
# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

**Claim:** if (a) does not hold, then (b) must hold.

- Consider  $Z = \{Ax: x \geq 0\}$  so that  $Z$  is closed and convex
- (a) does not hold  $\Leftrightarrow b \notin Z$
- By separating hyperplane theorem, there exists hyperplane  $\alpha \cdot z = \beta$  such that  $\alpha^T \cdot z \geq \beta$  for all  $z \in Z$  and  $\alpha^T \cdot b < \beta$



# Farkas' Lemma

**Farkas' Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then exactly one of the following two statements holds:

- a) There exists  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$
- b) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \geq 0$  and  $b^T y < 0$

**Claim:** if (a) does not hold, then (b) must hold.

- Consider  $Z = \{Ax: x \geq 0\}$  so that  $Z$  is closed and convex
- (a) does not hold  $\Leftrightarrow b \notin Z$
- By separating hyperplane theorem, there exists hyperplane  $\alpha \cdot z = \beta$  such that  $\alpha^T \cdot z \geq \beta$  for all  $z \in Z$  and  $\alpha^T \cdot b < \beta$
- Note  $0 \in Z$ , therefore  $\beta \leq \alpha^T \cdot 0 = 0$  and thus  $\alpha^T \cdot b < 0$
- $\alpha^T Ax \geq \beta$  for any  $x \geq 0$  implies  $\alpha^T A \geq 0$  since  $x$  can be arbitrary large
- Letting  $\alpha$  be our  $y$  yields the lemma

# An Alternative of Farkas' Lemma

Following corollary of Farkas' lemma is more convenient for our proof

**Corollary:** Exactly one of the following systems holds:

$$\exists x \in \mathbb{R}^n, \text{ s.t.}$$

$$A \cdot x \leq b$$

$$x \geq 0$$

$$\exists y \in \mathbb{R}^m, \text{ s.t.}$$

$$A^t \cdot y \geq 0$$

$$b^t \cdot y < 0$$

$$y \geq 0$$

Compare to the original version

$$\exists x \in \mathbb{R}^n, \text{ s.t.}$$

$$A \cdot x = b$$

$$x \geq 0$$

$$\exists y \in \mathbb{R}^m, \text{ s.t.}$$

$$A^t \cdot y \geq 0$$

$$b^t \cdot y < 0$$

# An Alternative of Farkas' Lemma

Following corollary of Farkas' lemma is more convenient for our proof

**Corollary:** Exactly one of the following systems holds:

$$\exists x \in \mathbb{R}^n, \text{ s.t.}$$

$$A \cdot x \leq b$$

$$x \geq 0$$

$$\exists y \in \mathbb{R}^m, \text{ s.t.}$$

$$A^t \cdot y \geq 0$$

$$b^t \cdot y < 0$$

$$y \geq 0$$

Proof: Apply Farkas' lemma to the following linear systems

$$\exists x, s \in \mathbb{R}^n, \text{ s.t.}$$

$$A \cdot x + I \cdot s = b$$

$$x, s \geq 0$$

$$\exists y \in \mathbb{R}^m, \text{ s.t.}$$

$$A^t \cdot y \geq 0$$

$$I \cdot y \geq 0$$

$$b^t \cdot y < 0$$

# Proof of Strong Duality

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

**Theorem [Strong Duality]:** If either the primal or dual is feasible and bounded, then so is the other and  $\text{OPT}(\text{primal}) = \text{OPT}(\text{dual})$ .

## Proof

- Dual of the dual is primal; so w.l.o.g assume primal is feasible and bounded
- Weak duality yields  $\text{OPT}(\text{primal}) \leq \text{OPT}(\text{dual})$
- Next we prove the converse, i.e.,  $\text{OPT}(\text{primal}) \geq \text{OPT}(\text{dual})$

# Proof of Strong Duality

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

- We prove if  $\text{OPT}(\text{primal}) < \beta$  for some  $\beta$ , then  $\text{OPT}(\text{dual}) < \beta$
- Apply Farkas' lemma to the following linear system

$$\begin{aligned} \exists x \in \mathbb{R}^n \text{ such that} \\ Ax \leq b \\ -c^t \cdot x \leq -\beta \\ x \geq 0 \end{aligned}$$

$$\begin{aligned} \exists y \in \mathbb{R}^m \text{ and } z \in \mathbb{R} \\ A^t y - cz \geq 0 \\ b^T y - \beta z < 0 \\ y, z \geq 0 \end{aligned}$$

- By assumption, the first system is infeasible, so the second must hold
  - If  $z > 0$ , can rescale  $(y, z)$  to make  $z = 1$ , yielding  $\text{OPT}(\text{dual}) < \beta$
  - If  $z = 0$ , then system  $A^t y \geq 0, b^T y < 0, y \geq 0$  feasible. Farkas' lemma implies that system  $Ax \leq b, x \geq 0$  is infeasible, contradicting theorem assumption.

# Outline

- Recap and Weak Duality
- Strong Duality and Its Proof
- Consequence of Strong Duality

# Complementary Slackness

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

- $s_i = (b - Ax)_i$  is the  $i$ 'th **primal slack variable**
- $t_j = (A^T y - c)_j$  is the  $j$ 'th **dual slack variable**

## Complementary Slackness:

$x$  and  $y$  are optimal if and only if they are feasible and

- $x_j t_j = 0$  for all  $j = 1, \dots, m$
- $y_i s_i = 0$  for all  $i = 1, \dots, n$

Remark: can be used to recover optimal solution of the primal from optimal solution of the dual (very useful in optimization).

## Economic Interpretation of Complementary Slackness:

Given the optimal production and optimal raw material prices

- It only produces products for which profit equals raw material cost
- A raw material is priced greater than 0 only if it is used up in the optimal production

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in [m] \\ & x_j \geq 0, \quad \forall j \in [n] \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j \in [n] \\ & y_i \geq 0, \quad \forall i \in [m] \end{aligned}$$

# Proof of Complementary Slackness

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

# Proof of Complementary Slackness

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax + s = b \\ & x, s \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y - t = c \\ & y, t \geq 0 \end{aligned}$$

- Add slack variables into both LPs

$$y^T b - x^T c = y^T (Ax + s) - x^T (A^T y - t) = y^T s + x^T t$$

- For **any** feasible  $x, y$ , the gap between primal and dual objective value is precisely the “aggregated slackness”  $y^T s + x^T t$
- Strong duality implies  $y^T s + x^T t = 0$  for the optimal  $x, y$ .
- Since  $x, s, y, t \geq 0$ , we have  $x_j t_j = 0$  for all  $j$  and  $y_i s_i = 0$  for all  $i$ .

# Thank You

Haifeng Xu

University of Chicago

[haifengxu@uchicago.edu](mailto:haifengxu@uchicago.edu)