

# CS6501:Topics in Learning and Game Theory (Spring 2021)

## Introduction to Game Theory (I)

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Instructor: Haifeng Xu

# Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

# (Recall) Example 1: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
  - ❖ No communications between them

	B	B stays silent	B betrays
A		-1	0
A stays silent	-1	-3	
A betrays	0	-3	-2

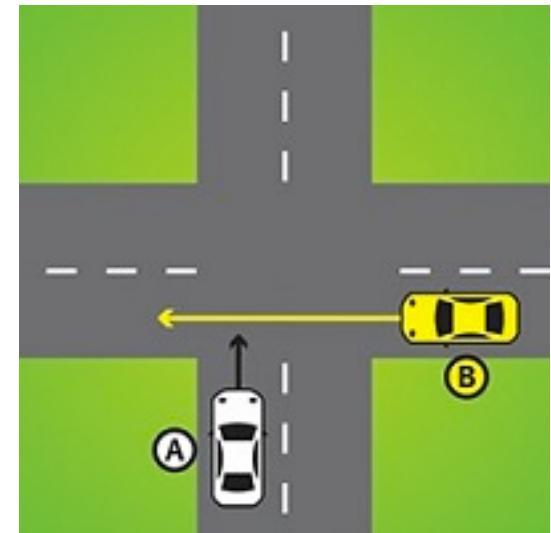
Q: How should each prisoner act?

- Both of them betray, though (-1,-1) is better for both

## Example 2: Traffic Light Game

- Two cars heading to orthogonal directions

		B
		STOP
A		STOP
STOP		(-3, -2)
GO		(0, -2)
		(-3, 0)
		(-100, -100)



Q: what are the equilibrium statuses?

Answer: (STOP, GO) and (GO, STOP)

# Example 3: Rock-Paper-Scissor

		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	(0, 0)	(-1, 1)	(1, -1)
	Paper	(1, -1)	(0, 0)	(-1, 1)
	Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

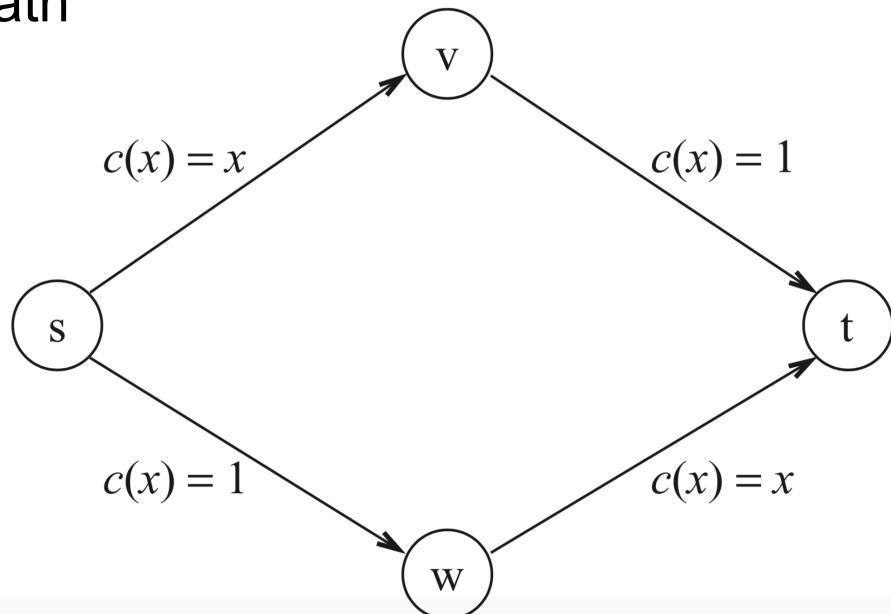
- Need to randomize – any deterministic action pair cannot make both players happy
- Common sense suggests  $(1/3, 1/3, 1/3)$

# Example 4: Selfish Routing

- One unit flow from  $s$  to  $t$  which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status?

- Half unit flow through each path
- Social cost =  $3/2$

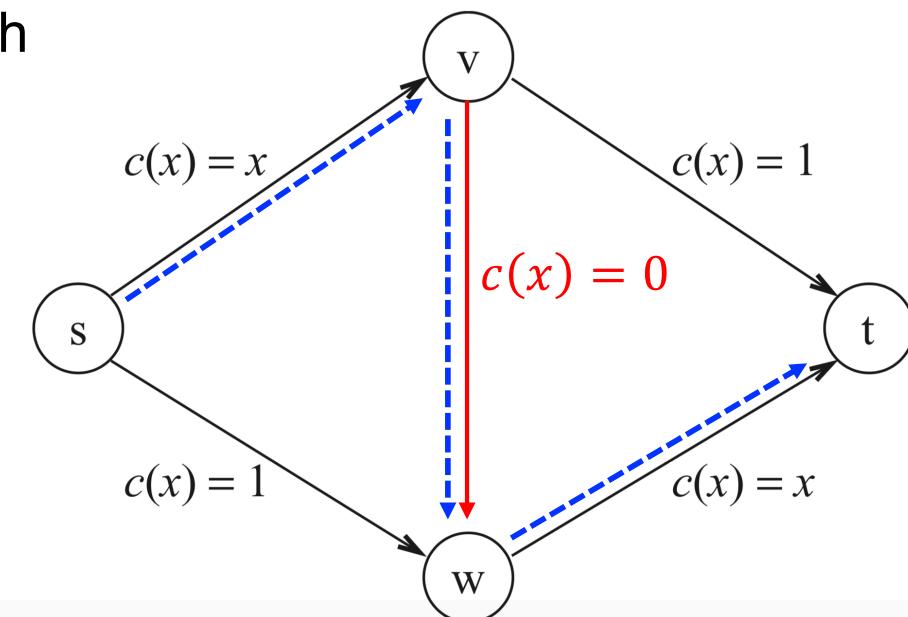


## Example 4: Selfish Routing

- One unit flow from  $s$  to  $t$  which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior high way with 0 traveling cost?

- Everyone takes the blue path
- Social cost = 2



# Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent's payoff depends on other agents' actions
- The interaction stabilizes at a state where no agent can increase his payoff via **unilateral deviation**

# Strategic Games Are Ubiquitous

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<b>Arrival time - Chicago</b>					
<input type="checkbox"/> Early Morning (12:00am - 4:59am)	7:01am - 9:10am American Airlines	3h 9m (Nonstop) BOS - ORD	Very Good Flight (8.3/10)	\$236 roundtrip	<b>Select</b>
<input type="checkbox"/> Morning (5:00am - 11:59am)	<a href="#">Details &amp; baggage fees ▾</a>				
<input type="checkbox"/> Afternoon (12:00pm - 5:59pm)	5:30am - 8:50am Delta	4h 20m (1 stop) BOS - 42m in DTW - ORD	Satisfactory Flight (6.4/10)	1 left at <b>\$246</b> roundtrip	<b>Select</b>
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# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue

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# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users

The screenshot shows the top navigation bar of the FCC website. On the left is the FCC logo and name. To the right are two search/filter options: "Browse by CATEGORY" and "Browse by BUREAUS & OFFICES". Further right is a search bar with a magnifying glass icon. Below the header is a blue navigation bar with links: About the FCC, Proceedings & Actions, Licensing & Databases, Reports & Research, News & Events, and For Consumers.

Home / Economics and Analytics /

## Auctions

### Proceedings & Actions

Proceedings and Actions Overview

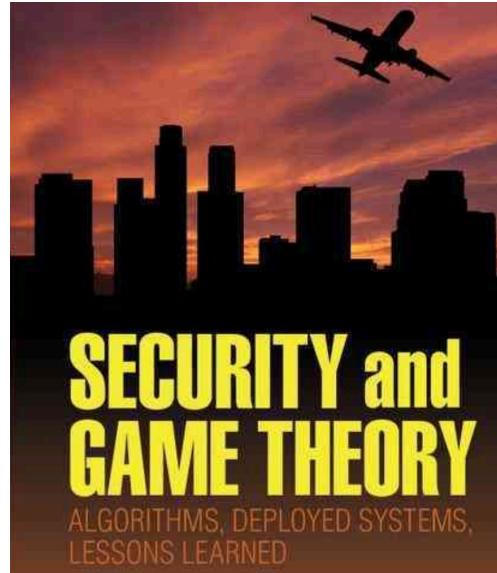
Since 1994, the Federal Communications Commission (FCC) has conducted auctions of licenses for electromagnetic spectrum. These auctions are open to any eligible company or individual that submits an application and upfront payment, and is found to be a qualified bidder by the Commission ([More About Auctions...](#))

#### Go to an Auction

Select an Au...

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- National security, border patrolling, counter-terrorism



Optimize resource allocation against  
attackers/adversaries

# Strategic Games Are Ubiquitous

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- Sponsored search
  - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, border patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when

The screenshot shows the UNOS website with a blue header bar containing the UNOS logo and a navigation menu with links to Transplant, Solutions, Technology, Data, Policy, Community, Resources, News, and a search icon. Below the header, a breadcrumb trail indicates the user is at Home > Transplant > Kidney paired donation. A large blue banner with white text reads "Kidney paired donation". Below the banner, a paragraph of text explains what KPD is, and a "Download PDF" button is available. At the bottom of the page, there is a link to "Learn about kidney paired donation".

Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched).

[Download PDF](#)

[Learn about kidney paired donation](#)

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# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum
- National security, border control
- Kidney exchange – decide who gets organs
- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.



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- National security, border patrolling, counter-terrorism
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- Social choice problems such as voting, fair division, etc.

These are just a few example domains *where computer science has made significant impacts*; There are many others.

# Main Components of a Game

- **Players**: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- **Strategies**: actions available to each player
- **Outcome**: the profile of player strategies
- **Payoffs**: a function mapping an outcome to a utility for each player

# Normal-Form Representation

- $n$  players, denoted by set  $[n] = \{1, \dots, n\}$
- Player  $i$  takes action  $a_i \in A_i$
- An outcome is the **action profile**  $a = (a_1, \dots, a_n)$ 
  - As a convention,  $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$  denotes all actions excluding  $a_i$
- Player  $i$  receives payoff  $u_i(a)$  for any outcome  $a \in \prod_{i=1}^n A_i$ 
  - $u_i(a) = u_i(a_i, a_{-i})$  depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$  are public knowledge

This is the most basic game model

- There are game models with richer and more intricate structures

# Illustration: Prisoner's Dilemma

- 2 players: 1 and 2
- $A_i = \{\text{silent, betray}\}$  for  $i = 1, 2$
- An outcome can be, e.g.,  $a = (\text{silent, silent})$
- $u_1(a), u_2(a)$  are pre-defined, e.g.,  $u_1(\text{silent, silent}) = -1$
- The whole game is public knowledge; players take actions simultaneously
  - Equivalently, take actions without knowing the others' actions

# Dominant Strategy

An action  $a_i$  is a **dominant strategy** for player  $i$  if  $a_i$  is better than any other action  $a'_i \in A_i$ , **regardless** what actions other players take.

Formally,

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \quad \forall a'_i \neq a_i \text{ and } \forall a_{-i}$$

Note: “strategy” is just another term for “action”

	B	B stays silent	B betrays
A			
A stays silent	-1	-3	0
A betrays	-1	-3	-2

Prisoner's Dilemma

- Betray is a dominant strategy for both
- Dominant strategies do not always exist
  - For example, the traffic light game

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

# Equilibrium

- An outcome  $a^*$  is an equilibrium if no player has incentive to deviate **unilaterally**. More formally,

$$u_i(\mathbf{a}_i^*, \mathbf{a}_{-i}^*) \geq u_i(\mathbf{a}_i, \mathbf{a}_{-i}^*), \quad \forall a_i \in A_i$$

- A special case of Nash Equilibrium, a.k.a., *pure strategy NE*
- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

	B	
A	STOP	GO
	STOP	(-3, -2)

A

	STOP	GO
A	STOP	(-3, 0)
	GO	(0, -2)

(-3, 0)

(0, -2)

(-100, -100)

Traffic Light Game

# Equilibrium

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- A special case of Nash Equilibrium, a.k.a., *pure strategy NE*
- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

Pure strategy NE does not always exist...

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

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# Pure vs Mixed Strategy

- Pure strategy: take an action deterministically
- Mixed strategy: can randomize over actions
  - Described by a distribution  $x_i$  where  $x_i(a_i) = \text{prob. of taking action } a_i$
  - $|A_i|$ -dimensional simplex  $\Delta_{A_i} := \{x_i: \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0\}$  contains all possible mixed strategies for player  $i$
  - Players draw their own actions *independently*
- Given **strategy profile**  $x = (x_1, \dots, x_n)$ , expected utility of  $i$  is
$$\sum_{a \in A} u_i(a) \cdot \prod_{i \in [n]} x_i(a_i)$$
  - Often denoted as  $u(x)$  or  $u(x_i, x_{-i})$  or  $u(x_1, \dots, x_n)$
  - When  $x_i$  corresponds to some pure strategy  $a_i$ , we also write  $u(a_i, x_{-i})$
  - Fix  $x_{-i}$ ,  $u(x_i, x_{-i})$  is **linear** in  $x_i$

# Best Responses

Fix any  $x_{-i}$ ,  $x_i^*$  is called a best response to  $x_{-i}$  if

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \forall x_i \in \Delta_{A_i}.$$

Claim. There always exists a pure best response

Proof: linear program “ $\max u_i(x_i, x_{-i})$  subject to  $x_i \in \Delta_{A_i}$ ” has a vertex optimal solution

Remark: If  $x_i^*$  is a best response to  $x_{-i}$ , then any  $a_i$  in the support of  $x_i^*$  (i.e.,  $x_i^*(a_i) > 0$ ) must be equally good and are all “pure” best responses

# Nash Equilibrium (NE)

A mixed strategy profile  $x^* = (x_1^*, \dots, x_n^*)$  is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \quad \forall x_i \in \Delta_{A_i}, \forall i \in [n].$$

That is, for any  $i$ ,  $x_i^*$  is a best response to  $x_{-i}^*$ .

## Remarks

- An equivalent condition:  $u_i(x_i^*, x_{-i}^*) \geq u_i(a_i, x_{-i}^*), \forall a_i \in A_i, \forall i \in [n]$ 
  - Since there always exists a pure best response
- It is not clear yet that such a mixed strategy profile would exist
  - Recall that pure strategy Nash equilibrium may not exist

# Nash Equilibrium (NE)

**Theorem (Nash, 1951):** Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- A foundational result in game-theory
- Example: rock-paper-scissor – what is a mixed strategy NE?

- $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  is a best response to  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		1/3	1/3	1/3
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

# Nash Equilibrium (NE)

**Theorem (Nash, 1951):** Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- An equilibrium outcome is not necessarily the best for players
  - Equilibrium only describes where the game stabilizes at
  - Many researches on understanding how self-interested behaviors reduces overall social welfare (recall the selfish routing game)
- A game may have many, even infinitely many, NEs
  - The issue of **equilibrium selection**

	B	B stays silent	B betrays
A			
A stays silent	-1	-1	0
A betrays	0	-3	-2

# Intractability of Finding a NE

**Theorem:** Computing a Nash equilibrium for any two-player normal-form game is PPAD-hard.

Note: PPAD-hard problems are believed to not admit poly time algorithm

- A two player game can be described by  $2mn$  numbers –  $u_1(i, j)$  and  $u_2(i, j)$  where  $i \in [m]$  is player 1's action and  $j \in [n]$  is player 2's.
- Theorem implies no  $\text{poly}(mn)$  time algorithm to compute an NE for any input game
- Ok, so what can we hope?
  - If the game has good structures, maybe we can find an NE efficiently
  - For example, zero-sum ( $u_1(i, j) + u_2(i, j) = 0$  for all  $i, j$ ), some resource allocation games

# An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE:  $S_1, S_2$  for player 1 and 2?
- The NE can be formulated by a **linear feasibility** problem with variables  $x_1^*, x_2^*, U_1, U_2$

$$\begin{aligned}\forall j \in S_2: \quad & \sum_{i \in S_1} u_2(i, j) x_1^*(i) = U_2 \\ \forall j \notin S_2: \quad & \sum_{i \in S_1} u_2(i, j) x_1^*(i) \leq U_2\end{aligned}$$

- The challenge of computing a NE is to find the correct supports
  - No general tricks, typically just try all possibilities
  - Some pre-processing may help, e.g., eliminating dominated actions
- This approach does not work for  $> 2$  players games (why?)

# Intractability of Finding “Best” NE

**Theorem:** It is NP-hard to compute the NE that maximizes the sum of players' utilities or any single player's utility even in two-player games.

- Proofs of these results for NEs are beyond the scope of this course

# Outline

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- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

# Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
  - This is typical in Bayesian decision making, but not the only way

	B	B stays silent	B betrays
A			
A stays silent		$\theta - 1$ $+ \theta$	0 $+ \theta$
A betrays		$\theta - 3$	-2

*I will give an additional reward  $\theta$  for whoever staying silent*



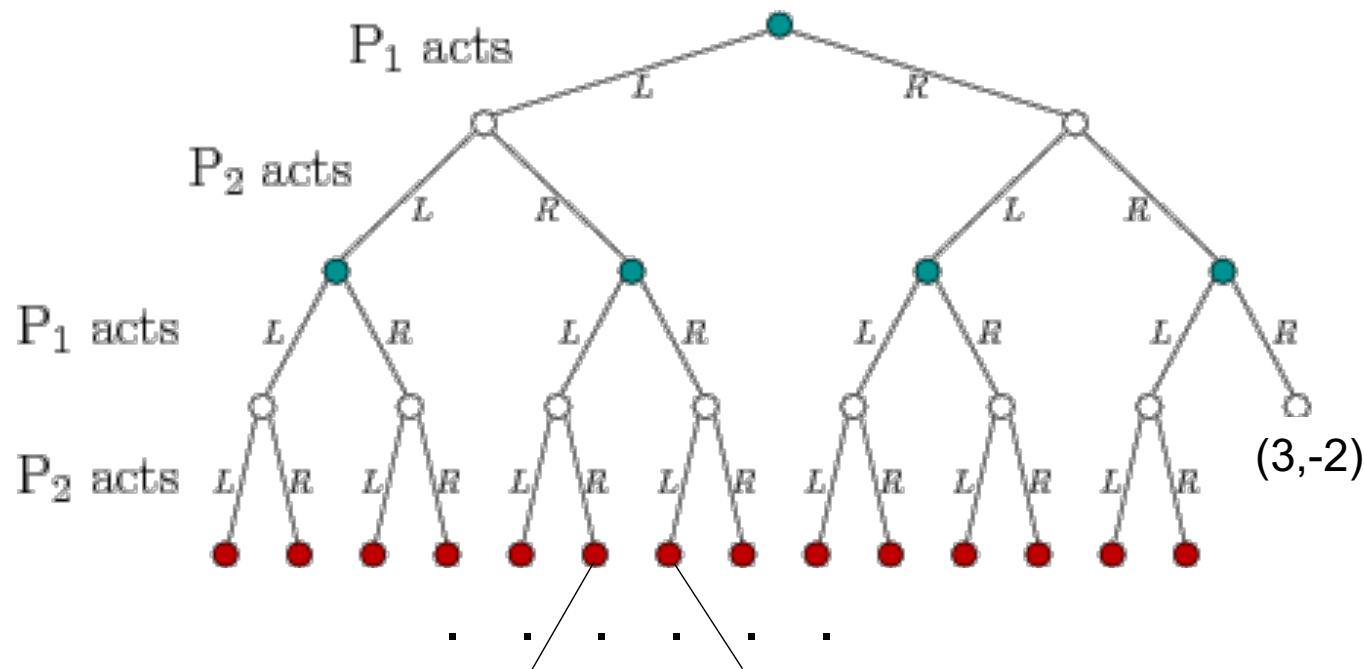
- It is believed that  $\theta \in \{0, 2, 4\}$  uniformly at random
- Or maybe the two players have different beliefs about  $\theta$

# Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
  - This is typical in Bayesian decision making, but not the only way
- More generally, can model player  $i$ ' payoffs as  $u_i^\theta$  where  $\theta$  is a **random** state of the game
- Each player obtains a (random) signal  $s_i$  that is correlated with  $\theta$ 
  - A joint prior distribution over  $(\theta, s_1, \dots, s_n)$  is assumed the public knowledge
- Can define a similar notion as Nash equilibrium, but expected utility also incorporates the randomness of the state of the game  $\theta$
- Applications: poker, blackjack, auction design, etc.

# Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**



# Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**
- EFGs are extremely general, can represent almost all kinds of games, but of course very difficult to solve

# A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

# A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

➤ What is an NE?

- $(a_2, b_2)$  is the unique Nash, resulting in utility pair (1,2)

➤ If A moves first; B sees A's move and then best responds, how should A play?

- Play action  $a_1$  deterministically!

		B	
		$b_1$	$b_2$
		$a_1$	$(2, 1)$
		$a_2$	$(2.01, -2)$
			$(1, 2)$

This sequential game model is called **Stackelberg game**, originally used to model market competition and now adversarial attacks.

# Thank You

Haifeng Xu

University of Virginia

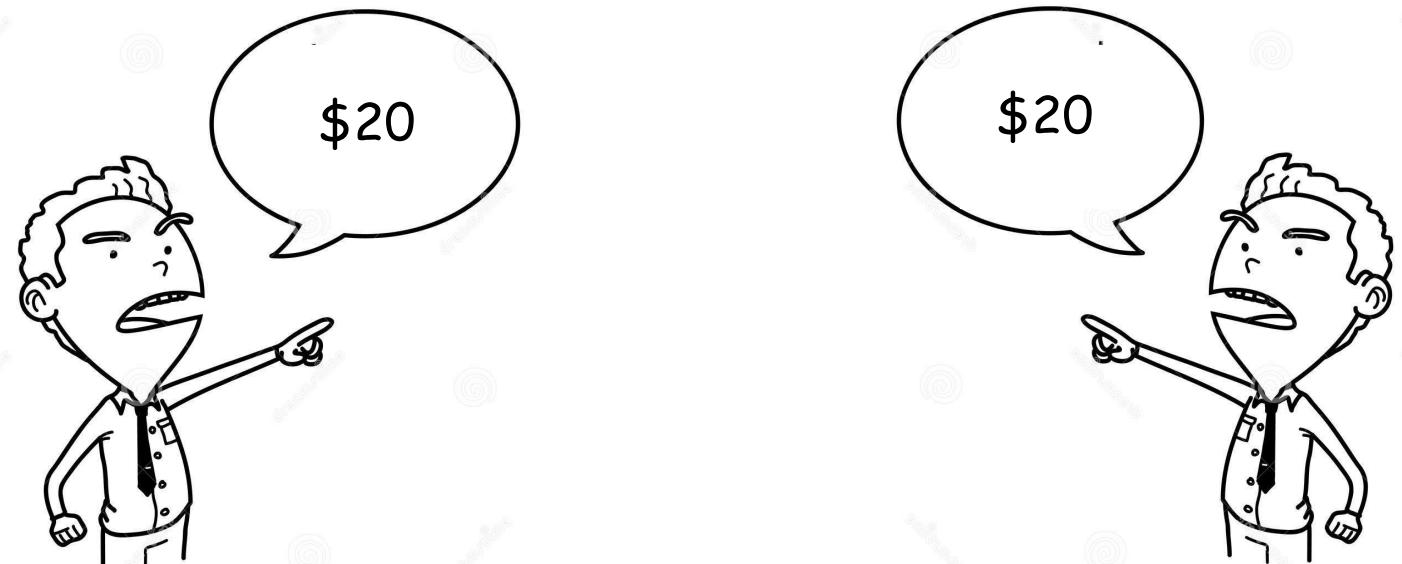
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# Appendix

# Recall: Competing Book Sellers

- Assume people will buy if the book price  $\leq \$200$
- Product cost = \$20
- Two book sellers compete for customers

Q: what price should each seller set?



# Recall: Competing Book Sellers

- Assume people will buy if the book price  $\leq \$200$
- Product cost = \$20
- Two book sellers compete for customers

Q: what price should each seller set?

- The market reaches a “stable status” (a.k.a., equilibrium)
- Nobody can benefit via *unilateral deviation*

