

# Announcements

- HW 3 is out, due Nov 5'th
- Project instruction is out
  - Format: proposal (5') + presentation (10') + report (25')
  - Proposal due Nov 7'th -- mainly to check you formed a team and have some ideas about what to do
  - We have some suggested topics, but you are more encouraged to find your own

# CS6501:Topics in Learning and Game Theory (Fall 2019)

## Scoring Rules

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Instructor: Haifeng Xu

# Outline

- Recap: Prediction Markets
- Scoring Rule and its Characterization
- Connection to Prediction Markets

# Prediction Markets

A prediction market is a **financial market** that is designed for **event prediction** via information aggregation

- Payoffs of the traded **contract** are determined by outcomes of future events

\$1 iff  $e_1$     · · ·    \$1 iff  $e_n$

contracts

We design a **market maker** by specifying the payment for bundles of contracts.



# Example: Logarithmic Market Scoring Rule (LMSR [Hanson 03, 06])

\$1 iff  $e_1$     · · ·    \$1 iff  $e_n$

- Define **value function** ( $q = (q_1, \dots, q_n)$  is current sales quantity)

$$V(q) = b \log \sum_{j \in [n]} e^{q_j/b}$$

Parameter  $b$   
adjusts liquidity

- Price function

$$p_i(q) = \frac{e^{q_i/b}}{\sum_{j \in [n]} e^{q_j/b}} = \frac{\partial V(q)}{\partial q_i}$$

- To buy  $x \in \mathbb{R}^n$  amount, a buyer pays:  $V(q + x) - V(q)$ 
  - Negative  $x_i$ 's mean selling contracts to MM
  - Negative payment means market maker pays the buyer
  - Market starts with  $V(0) = b \log n$

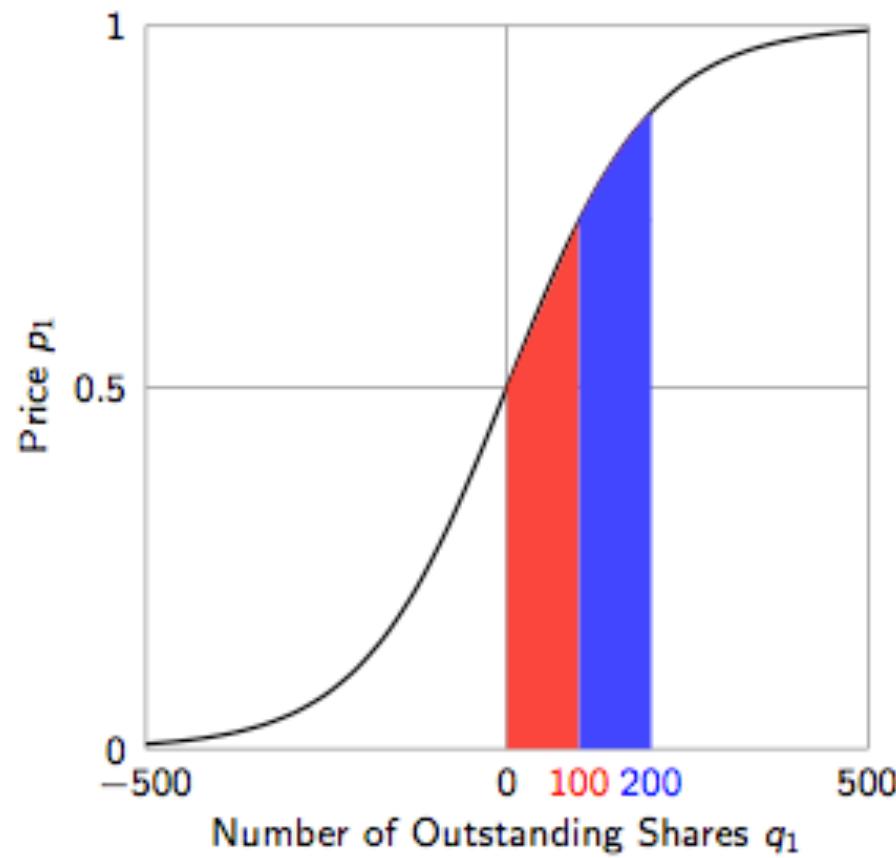
# Properties of LMSR

**Fact.** The optimal amount an expert purchases is the amount that moves the market price to her belief  $\lambda$ . Her expected utility of purchasing this amount is always non-negative.

- I.e., should purchase  $x^*$  such that  $\frac{\partial V(q+x^*)}{\partial x_i^*} = \lambda_i$
- Market efficiency

**Fact.** Worst case market maker loses is  $d \log n$  (i.e., bounded).

# Price Curve as a Function of Share Quantities



# Examples of LMSR in Practice

- Has been implemented by several prediction markets
  - E.g., InklingMarkets, Washington Stock Exchange, BizPredict, Net Exchange, and (reportedly) at YooNew.

SELECTED PREDICTION	CURRENT PRICE
Barack Obama	\$57.02

**TIP:** A price of \$57.02 means there is currently a 57.0% chance this will occur.

Do you think:

- Chances are higher than 57.02% this will occur
- Chances are lower than 57.02% this will occur

**TIP:** A price of \$57.02 means there is currently a 57.0% chance this will occur.

If you think the current odds of 57% are:

Way too low...

Low...

Just below...

Advanced...

Buy 50 shares

your cost  
\$2,971.95

estimated new price  
\$61.84

Buy 20 shares

your cost  
\$1,159.83

estimated new price  
\$58.97

Buy 5 shares

your cost  
\$286.30

estimated new price  
\$57.51

Buy  shares

your cost  
...

estimated new price  
...

Markets can potentially be a very effective forecasting tool

Big on-going project: “replication market” for DARPA SCORE project



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## Systematizing Confidence in Open Research and Evidence (SCORE)

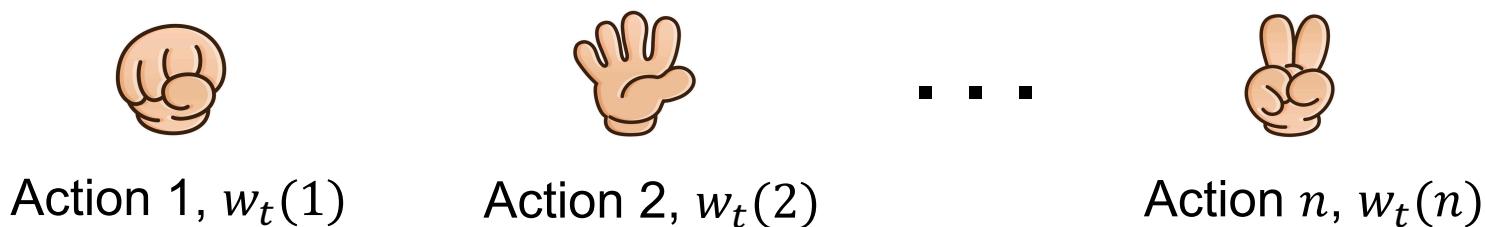
Dr. Adam Russell

The Department of Defense (DoD) often leverages social and behavioral science (SBS) research to design plans, guide investments, assess outcomes, and build models of human social systems and behaviors as they relate to national security challenges in the human domain. However, a number of recent empirical studies and meta-analyses have revealed that many SBS results vary dramatically in terms of their ability to be independently reproduced or replicated, which could have real-world implications for DoD's plans, decisions, and models. To help address this situation, DARPA's Systematizing Confidence in Open Research and Evidence (SCORE) program aims to develop and deploy automated tools to assign "confidence scores" to different SBS research results and claims. Confidence scores are quantitative measures that should enable a DoD consumer of SBS research to understand the degree to which a particular claim or result is likely to be reproducible or replicable. These tools will assign explainable confidence scores with a reliability that is equal to, or better than, the best current human expert methods. If successful, SCORE will enable DoD personnel to quickly calibrate the level of confidence they should have in the reproducibility and replicability of a given SBS result or claim, and thereby increase the effective use of SBS literature and research to address important human domain challenges, such as enhancing deterrence, enabling stability, and reducing extremism.

## Connection between LMSR and Exponential Weight Updates (EWU)

# Recap: Exponential Weight Update

- Played for  $T$  rounds; each round selects an action  $i \in [n]$
- Maintains weights over  $n$  actions:  $w_t(1), \dots, w_t(n)$
- Observe cost vector  $c_t$ , and update  $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon c_t(i)}, \forall i \in [n]$



$$\begin{aligned}w_{t+1}(i) &= w_t(i) \cdot e^{-\epsilon c_t(i)} \\&= [w_{t-1}(i) \cdot e^{-\epsilon c_{t-1}(i)}] \cdot e^{-\epsilon c_t(i)} \\&= \dots = e^{-\epsilon C_t(i)} \text{ where } C_t(i) = \sum_{\tau \leq t} c_\tau(i)\end{aligned}$$

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- Observe cost vector  $c_t$ , and update  $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon c_t(i)}, \forall i \in [n]$
- At round  $t + 1$ , select action  $i$  with probability

$$\frac{w_t(i)}{W_t} = \frac{e^{-\epsilon c_t(i)}}{\sum_{j \in [n]} e^{-\epsilon c_t(j)}}$$

where  $C_t = \sum_{\tau \leq t} c_\tau$  is the accumulated cost vector

This looks very much like the price function in LMSR ( $q$  is the accumulated sales quantity)

$$p_i = \frac{e^{q_i/b}}{\sum_{j \in [n]} e^{q_j/b}}$$

# EWU vs LMSR

## ➤ Exponential Weight Update

- $n$  actions
- Maintain weight  $w_t(i)$
- Total cost  $C_T(i) = \sum_{t \leq T} c_t(i)$
- Select  $i$  with prob

$$p_i = \frac{e^{-\epsilon C_t(i)}}{\sum_{j \in [n]} e^{-\epsilon C_t(j)}}$$

- Weights reflect how good an action is
- Care about worst case regret

$$C_T(\text{Alg}) - \min_i C_T(i)$$

## ➤ LMSR

- $n$  contracts (i.e., outcomes)
- Maintain prices  $p(i)$
- Total shares sold  $q(i)$
- Price of contract  $i$

$$p_i = \frac{e^{q_i/b}}{\sum_{j \in [n]} e^{q_j/b}}$$

- Prices reflect how probable is an event
- Care about worst case MM loss

$$(\$ \text{ received}) - \min_i q(i)$$

- LMSR is just one particular automatic MM
  - Similar relation holds for other market markers and no-regret learning algorithms (see [[Chen and Vaughan 2010](#)])
- Next: will study other possible scoring rules, and see why they work

# Outline

- Recap: Prediction Markets
- Scoring Rule and its Characterization
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# Consider a Simpler Setting

- We (designer) want to learn the distribution of random var  $E \in [n]$ 
  - $E$  will be sampled in the future
- We have no samples from  $E$ ; Instead, we have an expert/predictor who has a predicted distribution  $\lambda \in \Delta_n$
- We want to incentivize the expert to truthfully report  $\lambda$



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## Example

- $E$  is whether UVA will win NCAA title in 2020
- Expert is the UVA coach

- Expert's prediction does not need to be perfect
  - But, better than the designer who knows nothing
- Assume expert will **not** give you truthful info **for free**

# Idea: “Score” Expert’s Report

Will reward the expert certain amount  $S(i; p)$  where:

- (1)  $p$  is the expert’s report (does not have to equal  $\lambda$ );
- (2)  $i \in [n]$  is the event realization

Not like a prediction market yet, but will see later they are related

# Idea: “Score” Expert’s Report

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- (2)  $i \in [n]$  is the event realization

**Q:** what is the expert’s expected utility?

- Expert believes  $i \sim \lambda$
- Expected utility  $\mathbb{E}_{i \sim \lambda} S(i; p) = \sum_{i \in [n]} \lambda_i \cdot S(i; p) = S(\lambda; p)$

**Q:** what  $S(i; p)$  function can elicit truthful report  $\lambda$ ?

- When expert finds that  $\lambda = \arg \max_{p \in \Delta_n} [\sum_{i \in [n]} \lambda_i \cdot S(i; p)]$
- Ideally,  $\lambda$  is the unique maximizer

# Proper Scoring Rules

**Definition.** The score function  $S(i; p)$  is [strictly] proper if truthful report  $p = \lambda$  [uniquely] maximizes expected utility  $S(\lambda; p)$ .

➤ Expert is incentivized to report truthfully iff  $S(e; p)$  is proper

## Observations.

1.  $S(i; p) = 0$  is a trivial proper scoring fnc
2. Proper scores are closed under affine transformation
  - I.e., if  $S(i; p)$  is [strictly] proper, so is  $\alpha \cdot S(i; p) + \beta$  for any constant  $\alpha \neq 0, \beta$

➤ Thus, typically, strict properness is desired

# Examples of Scoring Rules

## Example 1 [Log Scoring Rule]

- $S(i; p) = \log p_i$
- $S(\lambda; p) = \sum_{i \in [n]} \lambda_i \cdot \log p_i$

- Negative, but okay – can always add a constant
- Properness requires  $\lambda = \arg \max_{p \in \Delta_n} S(\lambda; p)$

$$\begin{aligned} S(\lambda; p) &= \sum_{i \in [n]} \lambda_i \cdot \log p_i \\ &= \sum_{i \in [n]} \lambda_i [\log p_i - \log \lambda_i] + \sum_{i \in [n]} \lambda_i \log \lambda_i \end{aligned}$$

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KL-divergence  $KL(\lambda; p)$  (a.k.a. relative entropy)

- Measures the distance between two distributions
- Always non-negative, and equals 0 only when  $p = \lambda$

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- $p$  should minimize distance  $KL(\lambda; p)$ , which is achieved at  $p = \lambda$
- So log scoring rule is strictly proper

# Examples of Scoring Rules

## Example 2 [Quadratic Scoring Rule]

- $S(i; p) = 2p_i - \sum_{j \in [n]} p_j^2$
- $S(\lambda; p) = \sum_{i \in [n]} \lambda_i [2p_i - \sum_{j \in [n]} p_j^2]$

$$\begin{aligned} S(\lambda; p) &= \sum_{i \in [n]} \lambda_i [2p_i - \sum_{j \in [n]} p_j^2] \\ &= \sum_{i \in [n]} 2\lambda_i p_i - (\sum_{i \in [n]} \lambda_i) \cdot \sum_{j \in [n]} p_j^2 \\ &= \sum_{i \in [n]} 2\lambda_i p_i - \sum_{i \in [n]} p_i^2 \\ &= - \sum_{i \in [n]} [p_i - \lambda_i]^2 + \sum_{i \in [n]} \lambda_i^2 \end{aligned}$$

- Prediction  $p$  should minimize  $l_2$ -distance between  $p$  and  $\lambda$
- $p_i = \lambda_i$  is the unique maximizer of  $S(\lambda; p)$
- So quadratic scoring rule is also strictly proper

# Examples of Scoring Rules

## Example 3 [Linear Scoring Rule]

- $S(i; p) = p_i$
- $S(\lambda; p) = \sum_{i \in [n]} \lambda_i p_i$

- Linear scoring rule turns out to be not proper (verify it after class)

# What $S(e; p)$ Are Proper?

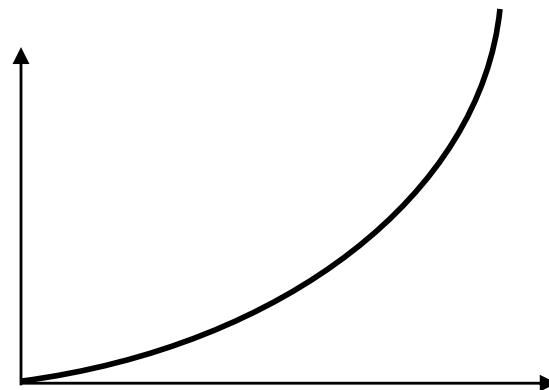
**Theorem.** The scoring rule  $S(e; p)$  is (strictly) proper if and only if there exists a (strictly) convex function  $G: \Delta_n \rightarrow \mathbb{R}$  such that

$$S(i; p) = G(p) + \nabla G(p)(e_i - p)$$

basis vector

Recall  $G(p)$  is convex if for any  $\alpha \in [0,1]$

$$\alpha G(p) + (1 - \alpha)G(q) \geq G(\alpha p + (1 - \alpha)q)$$



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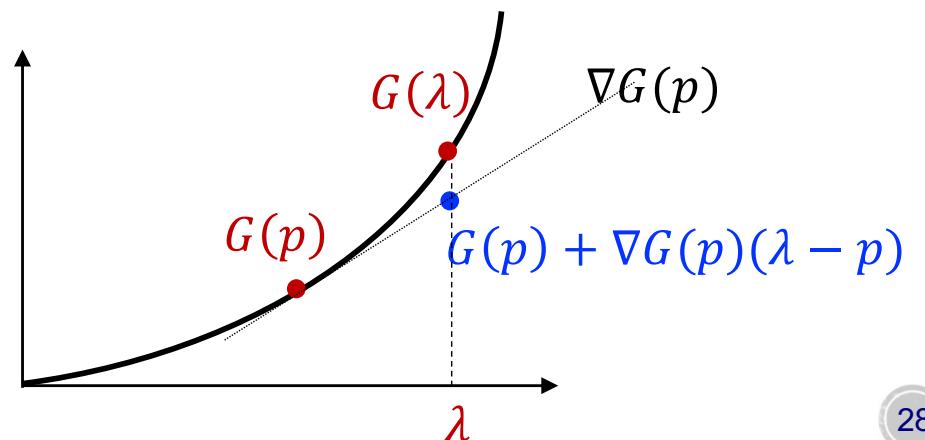
Proof of “ $\Leftarrow$ ”

$$S(\lambda; p) = \mathbb{E}_{i \sim \lambda}[G(p) + \nabla G(p)(e_i - p)]$$

$$= G(p) + \nabla G(p)(\lambda - p)$$

$$\leq G(\lambda) = S(\lambda; \lambda)$$

By convexity



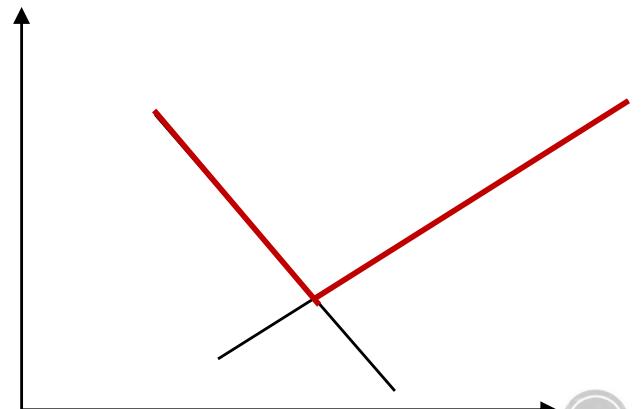
# What $S(e; p)$ Are Proper?

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$$S(i; p) = G(p) + \nabla G(p)(e_i - p)$$

Proof of “ $\Rightarrow$ ”

- $S(\lambda; p) = \sum_{i \in [n]} \lambda_i S(i; p)$  is a linear fnc of  $\lambda$  for any  $p$
- By properness,  $S(\lambda; \lambda) = \max_{p \in \Delta_n} \sum_{i \in [n]} \lambda_i S(i; p)$ , denoted as  $G(\lambda)$ 
  - $G(\lambda)$  is convex in  $\lambda$



# What $S(e; p)$ Are Proper?

**Theorem.** The scoring rule  $S(e; p)$  is (strictly) proper if and only if there exists a (strictly) convex function  $G: \Delta_n \rightarrow \mathbb{R}$  such that

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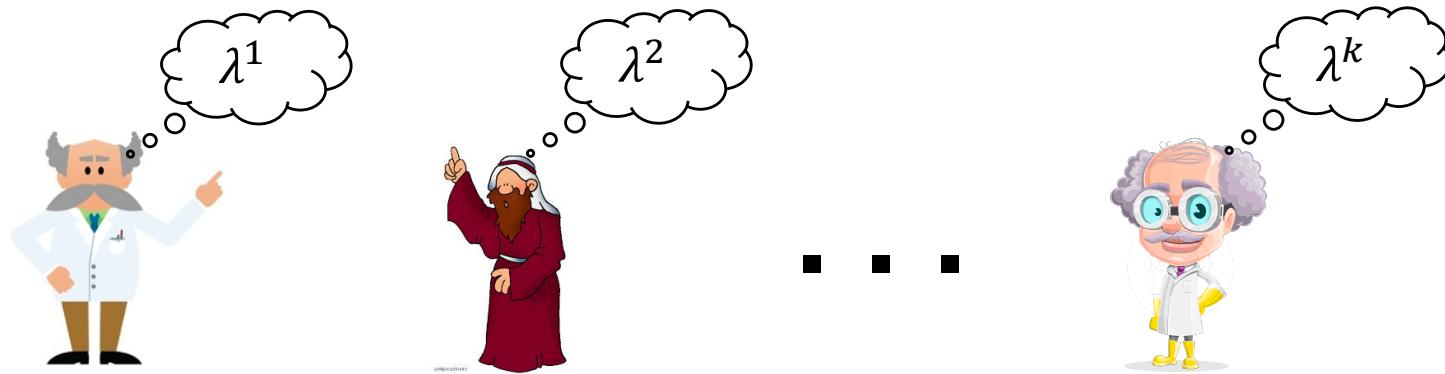
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- By properness,  $S(\lambda; \lambda) = \max_{p \in \Delta_n} \sum_{i \in [n]} \lambda_i S(i; p)$ , denoted as  $G(\lambda)$ 
  - $G(\lambda)$  is convex in  $\lambda$
- The gradient of  $G(\lambda)$  is the gradient of  $\sum_{i \in [n]} \lambda_i S(i; p)$  for the  $p = \lambda$ 
  - I.e.,  $\nabla G(\lambda) = S(\cdot; \lambda)$
- Thus,
$$\begin{aligned} S(i; p) &= S(p; p) + [S(i; p) - S(p; p)] \\ &= G(p) + S(\cdot; p) \cdot [e_i - p] \\ &= G(p) + \nabla G(p)[e_i - p] \end{aligned}$$

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# What If There are Many Experts



- One idea: elicit their predictions privately/separately
- Drawbacks
  - May be expensive or wasteful – if experts all agree, we pay many times for the same prediction
  - Not clear how to aggregate these predictions (average or geometric mean would not work)
  - In fact, it may require experts' knowledge to correctly aggregate predictions

# Sequential Elicitation

- Ask experts to make predictions in sequence
- The reward for expert  $k$ 's prediction  $p^k$  will be

$$S(i; p^k) - S(i; p^{k-1})$$

where  $p^{k-1}$  is the prediction of expert  $k - 1$

- I.e., experts are paid based on how much they improved the prediction

**Theorem.** If  $S$  is a proper scoring rule and each expert can only predict once, then each expert maximizes utility by reporting true belief **given her own knowledge**.

- Proof: since  $S(i; p^{k-1})$  not under  $k$ 's control, she maximizes reward by maximizing  $S(i; p^k)$

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**Theorem.** If  $S$  is a proper scoring rule and each expert can only predict once, then each expert maximizes utility by reporting true belief **given her own knowledge**.

Remarks:

- $k$  may see previous reports and then update his prediction
  - Experts will aggregate predictions automatically

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**Theorem.** If  $S$  is a proper scoring rule and each expert can only predict once, then each expert maximizes utility by reporting true belief **given her own knowledge**.

Remarks:

- Not true if an expert can predict for multiple times
  - She may manipulate her initial report to mislead others' prediction so that she has opportunity to significantly improve her prediction later
  - Will see an example in next lecture

## Equivalence to Prediction Markets Described Previously

- It turns out that sequential elicitation is equivalent (in incentives) to the prediction market (PM) for buying and selling contracts
- Each expert moves the prediction to his own belief
  - Recall in PMs, expert will buy shares until prices hit his own belief
- Any strictly proper scoring rule can be used to implement a PM and any PM correspond to some proper scoring rules

# Remarks

Mechanism design for prediction tasks

- ML is not the only way of making predictions
- In some settings, aggregating predictions from experts is better

# Thank You

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