# Announcements

- No class next Tuesday
- CS Department Research Symposium (10/08, next Tuesday)

# CS6501: Topics in Learning and Game Theory
# (Fall 2019)

# Optimal Auction Design for Single-Item Allocation
# (Part II)

Instructor: Haifeng Xu

# Outline

- Recap: Mechanism Design Basics
- Optimal Auction Design for Independent Bidders
- Optimal Auction for Correlated Bidders

# Single-Item Allocation

# Single-Item Allocation

**Mechanism Design for Single-Item Allocation**

Described by $\langle n, V, X, u, f \rangle$ where:

- $[n] = \{1, \cdots, n\}$ is the set of $n$ buyers
- $V = V_1 \times \cdots \times V_n$ is the set of all possible value profiles
- $X = \{0,1, \cdots, n\}$ is the set of winners
- $u = (u_1, \cdots, u_n)$ where $u_i = v_i x_i - p_i$ is the utility function of $i$ for any (randomized) allocation $x \in \Delta_{n+1}$ and payment $p_i$
- $f$ is the public prior on buyer values $v \in V$

- For convenience, think of $v_i \sim f_i$ independently
- Objective: maximize revenue $\sum_{i \in [n]} p_i$

# The Design Space – Mechanisms

A mechanism (i.e., the game) is specified by $\langle A, g \rangle$ where:

- $\mathrm{A} = A_1 \times \cdots \times A_n$ where $A_i$ is allowable actions for buyer $i$
- $g: A \to [x, p]$ maps an action profile to outcome = [an allocation $x(a)$ + a vector of payments $p(a)$] for any $a = (a_1, \cdots, a_n) \in A$

- That is, we will design $\langle A, g \rangle$
- Players' utility function will be fully determined by $\langle A, g \rangle$
- We want to maximize revenue at the Bayes Nash equilibrium of this resulting game

# The Design Space – Mechanisms

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- $g: A \to [x, p]$ maps an action profile to outcome = [an allocation $x(a)$ + a vector of payments $p(a)$] for any $a = (a_1, \cdots, a_n) \in A$

Example: second-price auction

- $A_i = \mathbb{R}_+$ for all $i$
- $g(a)$ allocates the item to the buyer $i^* = \arg\max_i a_i$ and asks $i^*$ to pay $\max 2_i \, a_i$, and all other buyers pay $0$

- Truthful bidding is a dominant-strategy equilibrium, thus also a BNE
- Thus expect truthful bidding (i.e., $a_i = v_i$); Revenue will be $\mathbb{E}_{v \sim f} \max 2_i \, v_i$

# Incentive Compatible Mechanisms

**Definition**. A mechanism $\langle A, g\rangle$ is a direct revelation mechanism if $A_i = V_i$ for all $i$. In this case, the mechanism is described by $g$.

- In DR mechanism, we only need to design $g$

**Definition**. A direct revelation mechanism $g$ is Bayesian incentive-compatible (a.k.a., truthful or BIC) if truthful bidding forms a Bayes Nash equilibrium in the resulting game

- A stronger notion of IC is dominant-strategy IC (DIC)
- A DIC mechanism is also BIC
- Example: second-price auction is DIC
    - First price auction can be "modified" to be BIC

# The Revelation Principle

**Theorem**. If there is a mechanism that achieves revenue $R$ at a Bayes Nash equilibrium [resp. dominant-strategy equilibrium], then there is a direct revelation, Bayesian incentive-compatible [resp. DIC] mechanism achieving revenue $R$.

- Proof idea: let the auctioneer to simulate the strategic behaviors on behalf of bidders, so they only need to react honestly

**Optimal Mechanism Design for Single-Item Allocation**

Given instance $\langle n, V, X, u, f \rangle$, design the allocation function $x: V \to X$ and payment $p: V \to \mathbb{R}^n$ such that truthful bidding is a BNE in the following Bayesian game:

1. Solicit bid $b_1 \in V_1, \cdots, b_n \in V_n$
2. Select allocation $x(b_1, \cdots, b_n) \in X$ and payment $p(b_1, \cdots, b_n)$

# Optimal Bayesian Mechanism Design

- Previous formulation and simplification leads to the following optimization problem
- If $V$ has finite support, this is an LP with variables $\{x_i(v), p_i(v)\}_{i,v}$

$$\max_{x,p} \ \mathbb{E}_{v\sim f} \ \sum_{i=1}^{n} p_i(v_1,\cdots,v_n)$$

BIC constraints

$$\text{s.t.} \quad \boxed{\begin{aligned}&\mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(v_i,v_{-i}) - p_i(v_i,v_{-i})\right] \\ &\quad \geq \mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(b_i,v_{-i}) - p_i(b_i,v_{-i})\right],\end{aligned}} \qquad \forall i\in[n], v_i, b_i \in V_i$$

$$\boxed{\mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(v_i,v_{-i}) - p_i(v_i,v_{-i})\right] \geq 0,} \qquad \forall i\in[n], v_i \in V_i$$

$$\sum_{i=0}^{n} x_i(v) = 1, \qquad \forall v\in V$$

Individually rational (IR) constraints

$$x_i(v) \geq 0, \qquad \forall v\in V, \ \forall i = 0,1\cdots,n$$

# Optimal Bayesian Mechanism Design

- Previous formulation and simplification leads to the following optimization problem
- If $V$ has finite support, this is an LP with variables $\{x_i(v), p_i(v)\}_{i,v}$
- Drawbacks of this algorithmic approach:
  (1) Support of $V$ may be extremely large in which case LP is large
  (2) Do not reveal any structure about the optimal auction – do not know what it is like except that it is a solution to an LP
- Next, will look at continuous $V$ and solve out for the optimal function $x(v), p(v)$
  - This will also lead to an elegant form of the optimal auction

# Outline

- Recap: Mechanism Design Basics
- Optimal Auction Design for Independent Bidders
  - That is, will assume $v_i \sim f_i$ independently
- Optimal Auction for Correlated Bidders

# The Optimal Auction (Myerson'1981)

**Theorem (informal).** For single-item allocation with prior distribution $v_i \sim f_i$ independently, the following auction is BIC and optimal:

1. Solicit buyer values $v_1, \cdots, v_n$
2. Transform $v_i$ to "virtual value" $\phi_i(v_i)$ where $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
3. If $\phi_i(v_i) < 0$ for all $i$, keep the item and no payments
4. Otherwise, allocate item to $i^* = \arg\max_{i \in [n]} \phi_i(v_i)$ and charge him the minimum bid needed to win, i.e., $\phi_i^{-1}\left( \max\left(\max_{j \neq i^*} \phi_j(v_j), 0\right) \right)$; Other bidders pay $0$.

# Stages of a Bayesian Game

- Stages of a Bayesian game of mechanism design:
  - Ex-ante: Before players learn their types
  - Interim: A player learns his own type, but not the types of others
  - Ex-post: All players types are revealed
- Interim stage is when players make decisions
  - The interim allocation for buyer $i$ tells us what $i$'s probability of winning is as a function of his bid $b_i$, in expectation over others' truthful report

$$\overline{x_i}(b_i) = \mathbb{E}_{v_{-i}\sim f_{-i}} x_i(b_i, v_{-i})$$

  - Similarly, the interim payment is

$$\overline{p_i}(b_i) = \mathbb{E}_{v_{-i}\sim f_{-i}} p_i(b_i, v_{-i})$$

  - Expected bidder utility of bidding $b_i$

$$\mathbb{E}_{v_{-i}\sim f_{-i}}[v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})] = v_i \overline{x_i}(b_i) - \overline{p_i}(b_i)$$

  - If BIC, expected revenue

$$\mathbb{E}_{v\sim f} \sum_{i=1}^{n} p_i(v_1, \cdots, v_n) = \sum_{i=1}^{n} \mathbb{E}_{v\sim f}\, p_i(v_1, \cdots, v_n)$$

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  - If BIC, expected revenue

$$\mathbb{E}_{v\sim f} \sum_{i=1}^{n} p_i(v_1, \cdots, v_n) = \sum_{i=1}^{n} \mathbb{E}_{v\sim f}\, p_i(v_1, \cdots, v_n) = \sum_{i=1}^{n} \mathbb{E}_{v_i\sim f_i}\, \overline{p_i}\,(v_i)$$

# Examples

Assume two buyers, $v_1, v_2 \sim U[0,1]$ independently

Second-price auction

- $\overline{x_1}(b_1) = \mathbb{E}_{v_2 \sim f_2} x_1(b_1, v_2) = b_1$
- $\overline{p_1}(b_1) = \mathbb{E}_{v_2 \sim f_2} p_1(b_1, v_2) = \int_0^{b_1} v_2 \, f_2(v_2)\, dv_2 = (b_1)^2/2$
- $\overline{x_2}(b_2), \overline{p_2}(b_2)$ have the same form

Modified first-price auction (Recall: truthful bidding is an BNE)

- $\overline{x_1}(b_1) = \mathbb{E}_{v_2 \sim f_2} x_1(b_1, v_2) = b_1$
- $\overline{p_1}(b_1) = \mathbb{E}_{v_2 \sim f_2} p_1(b_1, v_2) = \int_0^{b_1} \frac{b_1}{2} \cdot f_2(v_2)\, dv_2 = (b_1)^2/2$
- $\overline{x_2}(b_2), \overline{p_2}(b_2)$ have the same form

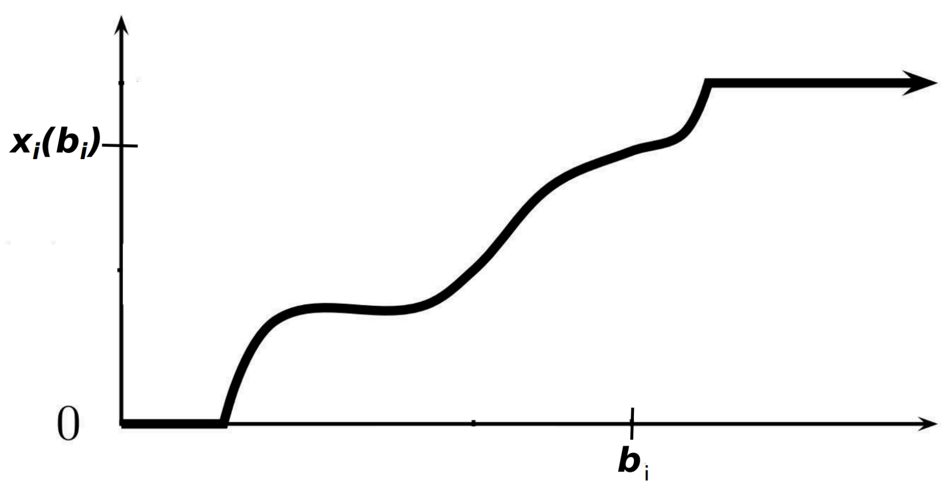
From now on we will write $x_i(b_i) = \overline{x_i}(b_i)$ to avoid cumbersome notation

# Myerson's Monotonicity Lemma

**Lemma.** Consider single-item allocation with prior distribution $v_i \sim f_i$ independently. A direct-revelation mechanism with interim allocation $x$ and interim payment $p$ is BIC if and only if for each buyer $i$:

1. $x_i(b_i)$ is a monotone non-decreasing function of $b_i$
2. $p_i(b_i)$ is uniquely determined as follows, with $p_i(0) = 0$,

$$p_i(b_i) = b_i \cdot x_i(b_i) - \int_{b=0}^{b_i} x_i(b)\, db\,.$$



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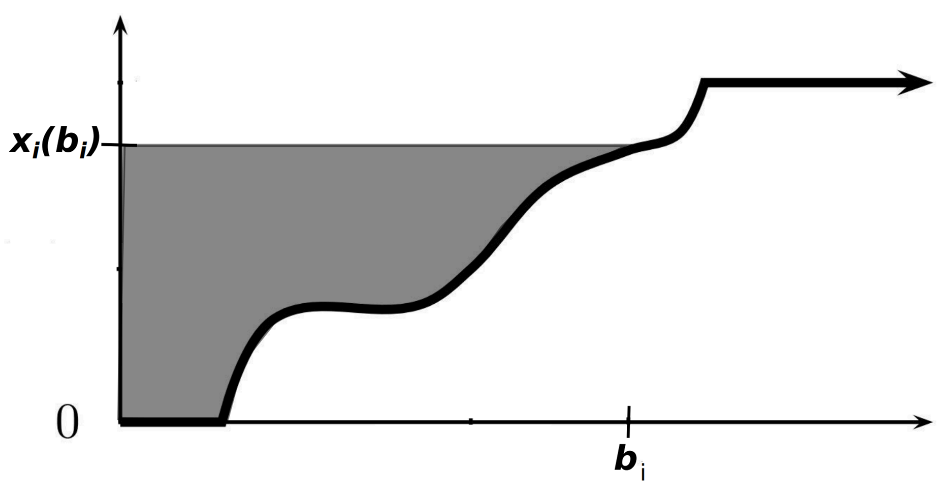
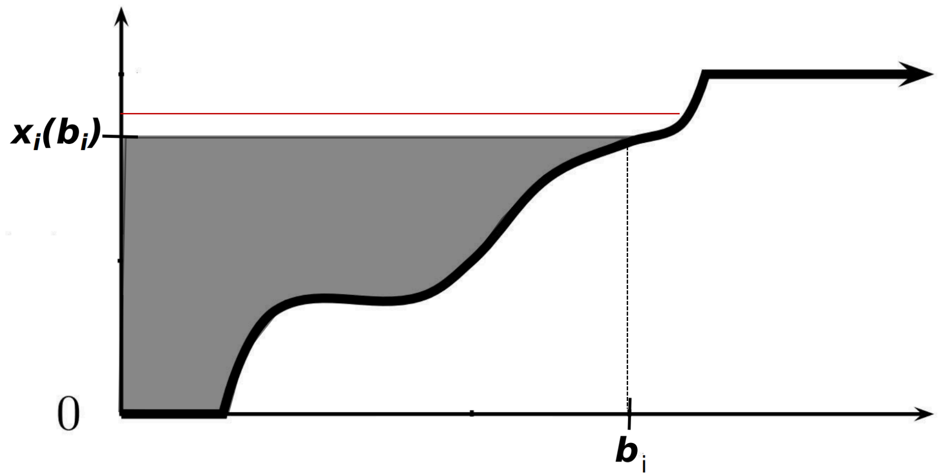


# Interpretation of Myerson's Lemma




- The higher a player bids, the higher the probability of winning
- For each additional $\epsilon$ of winning probability, pay additionally at a rate equal to the current bid

# Corollaries of Myerson's Lemma

**Corollaries.**

1. Interim allocation uniquely determines interim payment
2. Expected revenue depends only on the allocation rule
3. Any two auctions with the same interim allocation rule at BNE have the same expected revenue at the same BNE

Therefore, second-price and first-price auction (and its modified version) all have the same revenue in previous two bidder i.i.d example

# Revenue as Virtual Welfare

- Define the virtual value of player $i$ as a function of his value $v_i$:

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

**Lemma.** Consider a BIC mechanism $M$ with interim allocation $x$ and interim payment $p$, normalized to $p_i(0) = 0$. The expected revenue of $M$ is equal to the expected virtual welfare served

$$\sum_{i=1}^{n} \mathbb{E}_{v_i \sim f_i}[\, \phi_i(v_i) x(v_i)]$$

- This is the expected virtual value of the winning bidder
- Proof is an application of Myerson's monotonicity lemma, plus algebraic calculations
- Recall the expected revenue is $\sum_{i=1}^{n} \mathbb{E}_{v_i \sim f_i} \overline{p_i}\, (v_i)$

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# Proof

$$\mathbb{E}_{v_i \sim f_i}\, \overline{p_i}\,(v_i) = \int_{v_i} \left[ v_i \cdot x_i(v_i) - \int_{b=0}^{v_i} x_i(b)\, db \right] f_i(v_i) dv_i$$

By Myerson's monotonicity lemma

Assumed bidder $i$ bids truthfully

# Proof

$$
\begin{aligned}
\mathbb{E}_{v_i \sim f_i}\, \overline{p_i}\,(v_i) &= \int_{v_i} \left[ v_i \cdot x_i(v_i) - \int_{b=0}^{v_i} x_i(b)\, db \right] f_i(v_i) dv_i \\
&= \int_{v_i} v_i \cdot x_i(v_i) f_i(v_i) dv_i - \int_{v_i} \int_{b=0}^{v_i} x_i(b)\, f_i(v_i) db\, dv_i \\
&= \int_{v_i} v_i \cdot x_i(v_i) f_i(v_i) dv_i - \int_{b} \int_{v_i \geq b} x_i(b)\, f_i(v_i) dv_i db \\
&= \int_{v_i} v_i \cdot x_i(v_i) f_i(v_i) dv_i - \int_{b} x_i(b)(1 - F_i(b))\, db \\
&= \int_{v_i} v_i \cdot x_i(v_i) f_i(v_i) dv_i - \int_{v_i} x_i(v_i)\big(1 - F_i(v_i)\big) dv_i \\
&= \int_{v_i} x_i(v_i) \cdot v_i f_i(v_i) - \big(1 - F_i(v_i)\big)\, dv_i \\
&= \int_{v_i} x_i(v_i) \cdot f_i(v_i) \left[ v_i - \frac{\big(1 - F_i(v_i)\big)}{f_i(v_i)} \right] dv_i
\end{aligned}
$$

# The Optimal Auction

➢Revenue of any BIC mechanism equals $\sum_{i=1}^{n} \mathbb{E}_{v_i \sim f_i}[\ \phi_i(v_i) x(v_i)]$

Q: how to extract the maximum revenue then?

1. Solicit buyer values $v_1, \cdots, v_n$ and calculate virtual values $\phi_i(v_i)$
2. If $\phi_i(v_i) < 0$ for all $i$, keep the item and no payments (why?)
3. Otherwise, allocate item to $i^* = \arg\max_{i \in [n]} \phi_i(v_i)$
4. How much to charge? Myerson's lemma says there is a unique interim payment
   - Charging minimum bid needed to win $\phi_i^{-1}\left(\max\left(\max_{j \neq i^*} \phi_j(v_j), 0\right)\right)$ works.

The optimal auction

# The Optimal Auction

1. Solicit buyer values $v_1, \cdots, v_n$ and calculate virtual values $\phi_i(v_i)$
2. If $\phi_i(v_i) < 0$ for all $i$, keep the item and no payments (why?)
3. Otherwise, allocate item to $i^* = \arg\max_{i \in [n]} \phi_i(v_i)$, charge him the minimum bid needed to win $\phi_i^{-1}\left(\max\left(\max_{j \neq i^*} \phi_j(v_j), 0\right)\right)$; others pay $0$

**Observations.**

- The allocation rule maximizes virtual welfare point-point, thus also maximizes expected virtual welfare
- By previous lemma, this is the maximum possible revenue
- Payment satisfies Myerson's lemma (check it after class)

Are we done?

# A Wrinkle

- One step away – Myerson lemma requires the interim allocation to be monotone
- Weird things may happen if the virtual value (VV) function $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ is not monotonically increasing
- Fortunately, most natural distributions will lead to monotone VV function (e.g., Gaussian, uniform, exp, etc.)
  - Such a distribution is called regular

**Conclusion**. When values are drawn from regular distributions independently, the VV maximizing auction (aka Myerson's optimal auction) is a revenue-optimal BIC mechanism!

Can be extended to non-regular distributions via ironing, but won't cover

# Remark 1

- The optimal auction just so happens to be DIC
  - Think of each bidder's bid as bidding the virtual value instead
  - It is effectively a second-price auction with reserve price 0, but in the virtual value space
- For single-item auction, optimal BIC mechanism achieves the same revenue as optimal DIC mechanism
  - Not true for selling multiple items (even two items to two bidders)

# Remark 2

- When buyers' values are i.i.d., optimal auction has an even simpler format
  - Assume regular distribution, allocate the item to largest $\phi_i(v_i) = \phi(v_i)$
  - Regularity implies monotonicity of $\phi$, so really just allocate to largest $v_i$
  - Payment is the minimum bid to win, which is $\max(\ \max2\, v_i\ ,\ \phi^{-1}(0)\ )$.
  - This is a second price auction with reserve $\phi^{-1}(0)$

# Remark 3

- Applies to “single parameter” problems more generally
  - Intuitively, each bidder’s value can be captured by a single parameter
- For example, sell many copies of the same item to buyers
  - Can even have allocation constraints, e.g., if bidder 1 gets 1 copy then bidder 2 is not allowed to get one

# Thank You

Haifeng Xu

University of Virginia

hx4ad@virginia.edu