

CMSC 3540I: The Interplay of Economics and ML (Winter 2024)

Introduction to Game Theory (I)

Instructor: Haifeng Xu



Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

(Recall) Example 1: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
 - ❖ No communications between them

	B	B stays silent	B betrays
A		-1	0
A stays silent	-1		-3
A betrays	0	-3	-2

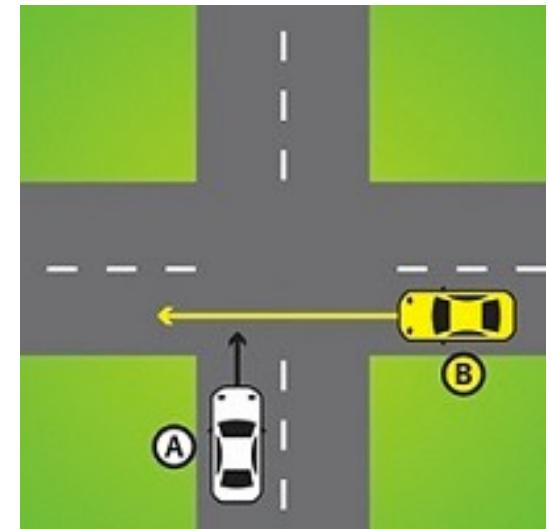
Q: How should each prisoner act?

- Both of them betray, though (-1,-1) is better for both

Example 2: Traffic Light Game

- Two cars heading to orthogonal directions

		B
		STOP
A		GO
STOP	STOP	(-3, -2)
	GO	(0, -2)
GO	STOP	(-3, 0)
GO	GO	(-100, -100)



Q: what are the equilibrium statuses?

Answer: (STOP, GO) and (GO, STOP)

Example 3: Rock-Paper-Scissor

		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	(0, 0)	(-1, 1)	(1, -1)
	Paper	(1, -1)	(0, 0)	(-1, 1)
	Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

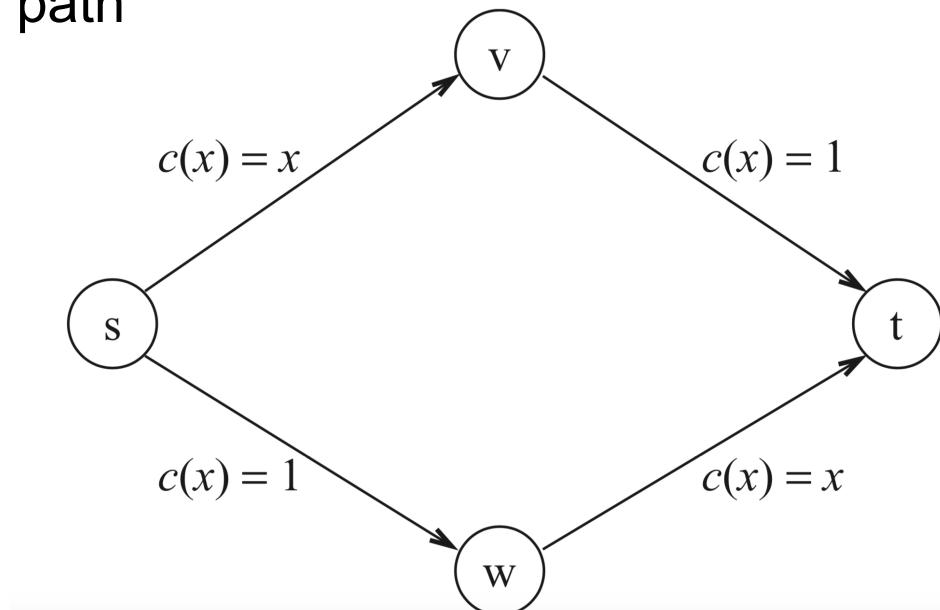
- Need to randomize – any deterministic action pair cannot make both players happy
- Common sense suggests $(1/3, 1/3, 1/3)$

Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status?

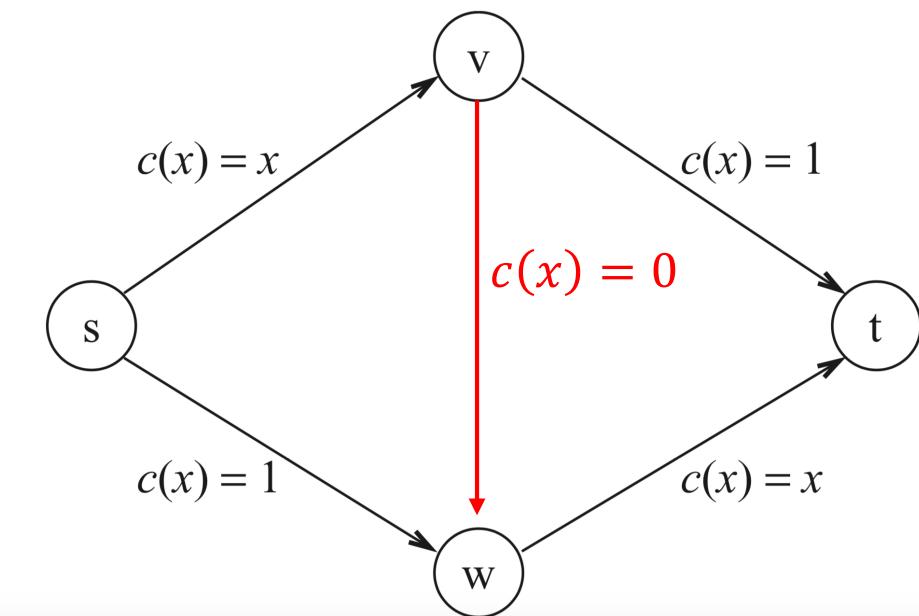
- Half unit flow through each path
- Social cost = $3/2$



Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior highway with 0 traveling cost?

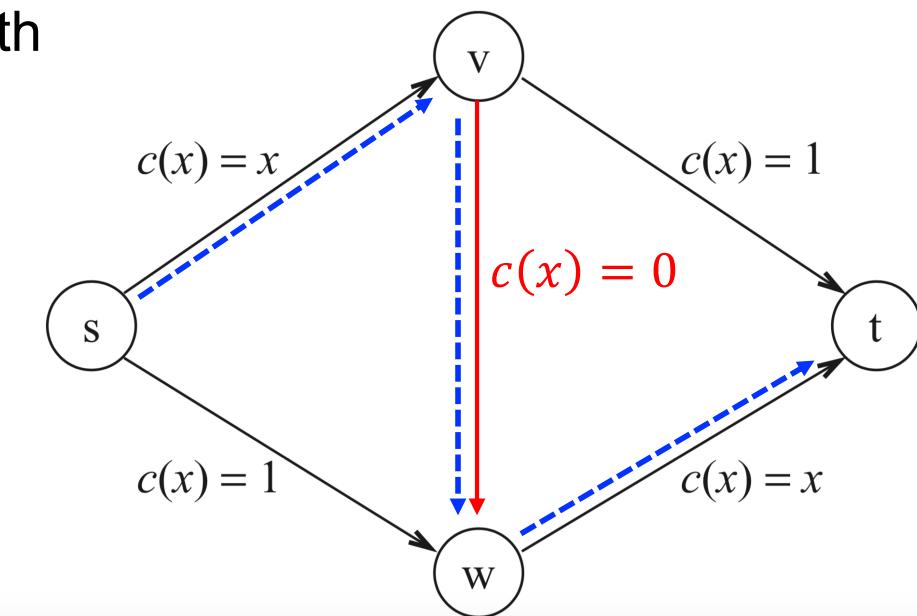


Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior high way with 0 traveling cost?

- Everyone takes the blue path
- Social cost = 2



Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent's payoff depends on other agents' actions
- The interaction stabilizes at a state where no agent can increase his payoff via **unilateral deviation**

Strategic Games Are Ubiquitous

➤ Pricing

<input type="checkbox"/> Spirit Airlines (2) \$438	6:30am - 8:15am United Very Good Flight (8.1/10) Details & baggage fees	2h 45m (Nonstop) BOS - ORD	5 left at \$236 roundtrip	Select
<input type="checkbox"/> Morning (5:00am - 11:59am)				
<input type="checkbox"/> Afternoon (12:00pm - 5:59pm)	9:23am - 11:27am American Airlines Very Good Flight (8.3/10) Details & baggage fees	3h 4m (Nonstop) BOS - ORD	\$236 roundtrip	Select
<input type="checkbox"/> Evening (6:00pm - 11:59pm)				
<input type="checkbox"/> Early Morning (12:00am - 4:59am)	7:01am - 9:10am American Airlines Very Good Flight (8.3/10) Details & baggage fees	3h 9m (Nonstop) BOS - ORD	\$236 roundtrip	Select
<input type="checkbox"/> Morning (5:00am - 11:59am)				
<input type="checkbox"/> Afternoon (12:00pm - 5:59pm)	5:30am - 8:50am Delta Satisfactory Flight (6.4/10) Details & baggage fees	4h 20m (1 stop) BOS - 42m in DTW - ORD	1 left at \$246 roundtrip	Select
<input type="checkbox"/> Evening (6:00pm - 11:59pm)				

[Run this search again](#) ad

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue

Google search results for "where to buy cruise vacation". The results page shows a header with the Google logo, a search bar containing the query, and navigation links for All, Shopping, Images, News, Videos, More, Settings, and Tools. Below the header, it says "About 103,000,000 results (0.63 seconds)".

The first result is a sponsored ad for Carnival Cruise Line, highlighted with a red box. It features a large price of \$1.03 and text about 2-5 Day and 6-9 Day Cruises.

The second result is an ad for Expedia Cruises, showing a price of \$1.02 and text about finding the perfect cruise at the best price.

The third result is an ad for VacationToGo.com, showing a price of \$0.60 and text about 2019 cruise deals.

The fourth result is an ad for KAYAK, showing a price of \$0.21 and text about finding the cheapest cruise deals.

On the right side of the results, there is a sidebar with a "See cruise vac..." link and a "Sponsored" section featuring a 3-D Cruise Ship Centerpiece for \$0.65.

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users

The screenshot shows the top navigation bar of the FCC website. It features the FCC logo and name on the left, followed by two search/filter options: "Browse by CATEGORY" and "Browse by BUREAUS & OFFICES". To the right is a search bar with a magnifying glass icon. Below the header is a blue navigation bar with links: "About the FCC", "Proceedings & Actions", "Licensing & Databases", "Reports & Research", "News & Events", and "For Consumers".

Home / Economics and Analytics /

Auctions

Proceedings & Actions

Proceedings and Actions Overview

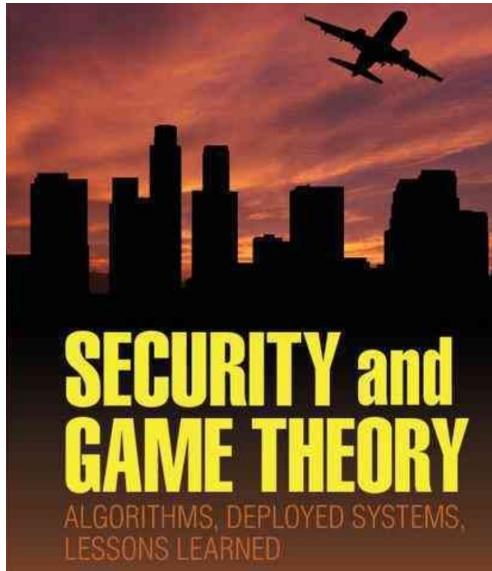
Since 1994, the Federal Communications Commission (FCC) has conducted auctions of licenses for electromagnetic spectrum. These auctions are open to any eligible company or individual that submits an application and upfront payment, and is found to be a qualified bidder by the Commission ([More About Auctions...](#))

Go to an Auction

Select an Au...

Strategic Games Are Ubiquitous

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- FCC's Allocation of spectrum to radio frequency users
- National security, border patrolling, counter-terrorism



Optimize resource allocation against
attackers/adversaries

Strategic Games Are Ubiquitous

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- FCC's Allocation of spectrum to radio frequency users
- National security, border patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when

The screenshot shows the UNOS website with a blue header bar containing the UNOS logo and navigation links: Transplant, Solutions, Technology, Data, Policy, Community, Resources, News, and a magnifying glass icon for search. Below the header, a breadcrumb trail reads "Home > Transplant > Kidney paired donation". The main content area has a blue background and features the text "Kidney paired donation" in white. At the bottom left of this area is a button labeled "Download PDF".

Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched).

[Download PDF](#)

[Learn about kidney paired donation](#)

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Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's
- FCC's Allocation of spectrum
- National security, border control
- Kidney exchange – decisions
- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.



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- Social choice problems such as voting, fair division, etc.

These are just a few example domains *where computer science has made significant impacts*; There are many others.

Main Components of a Game

- **Players**: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- **Strategies**: actions available to each player
- **Outcome**: the profile of player strategies
- **Payoffs**: a function mapping an outcome to a utility for each player

Normal-Form Representation

- n players, denoted by set $[n] = \{1, \dots, n\}$
- Player i takes action $a_i \in A_i$
- An outcome is the **action profile** $a = (a_1, \dots, a_n)$
 - As a convention, $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ denotes all actions excluding a_i
- Player i receives payoff $u_i(a)$ for any outcome $a \in \prod_{i=1}^n A_i$
 - $u_i(a) = u_i(a_i, a_{-i})$ depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$ are public knowledge

This is the most basic game model

- There are game models with richer and more intricate structures

Illustration: Prisoner's Dilemma

- 2 players: 1 and 2
- $A_i = \{\text{silent, betray}\}$ for $i = 1, 2$
- An outcome can be, e.g., $a = (\text{silent, silent})$
- $u_1(a), u_2(a)$ are pre-defined, e.g., $u_1(\text{silent, silent}) = -1$
- The whole game is public knowledge; players take actions simultaneously
 - Equivalently, take actions without knowing the others' actions

Dominant Strategy

An action a_i is a **dominant strategy** for player i if a_i is better than any other action $a'_i \in A_i$, **regardless** what actions other players take.

Formally,

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \quad \forall a'_i \neq a_i \text{ and } \forall a_{-i}$$

Note: “strategy” is just another term for “action”

	B	B stays silent	B betrays
A			
A stays silent	-1	0	
A betrays	-1	-3	

Prisoner's Dilemma

- Betray is a dominant strategy for both
- Dominant strategies do not always exist
 - For example, the traffic light game

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

Equilibrium

- An outcome a^* is an equilibrium if no player has incentive to deviate **unilaterally**. More formally,

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \quad \forall a_i \in A_i$$

- A special case of Nash Equilibrium, a.k.a., *pure strategy NE*
- If each player has a dominant strategy, they form an equilibrium

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Prisoner's Dilemma

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- But, an equilibrium does not need to consist of dominant strategies

Quiz: find equilibrium

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,5

Equilibrium

- An outcome a^* is an equilibrium if no player has incentive to deviate **unilaterally**. More formally,

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- A special case of Nash Equilibrium, a.k.a., *pure strategy NE*

- If each player has a dominant strategy, they form an equilibrium
➤ But, an equilibrium does not need to consist of dominant strategies

What about this?

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

Pure strategy NE does not always exist...

Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

Pure vs Mixed Strategy

- Pure strategy: take an action deterministically
- Mixed strategy: can randomize over actions
 - Described by a distribution x_i where $x_i(a_i) = \text{prob. of taking action } a_i$
 - $|A_i|$ -dimensional simplex $\Delta_{A_i} := \{x_i: \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0\}$ contains all possible mixed strategies for player i
 - Players draw their own actions *independently*
- Given **strategy profile** $x = (x_1, \dots, x_n)$, expected utility of i is

$$\sum_{a \in A} u_i(a) \cdot \prod_{i \in [n]} x_i(a_i)$$

- Often denoted as $u_i(x)$ or $u(x_i, x_{-i})$ or $u_i(x_1, \dots, x_n)$
- When x_i corresponds to some pure strategy a_i , we also write $u_i(a_i, x_{-i})$
- Fix x_{-i} , $u_i(x_i, x_{-i})$ is **linear** in x_i

Best Responses

Fix any x_{-i} , x_i^* is called a best response to x_{-i} if

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \forall x_i \in \Delta_{A_i}.$$

Claim. There always exists a pure best response

Proof: linear program “ $\max u_i(x_i, x_{-i})$ subject to $x_i \in \Delta_{A_i}$ ” has a vertex optimal solution

Remark: If x_i^* is a best response to x_{-i} , then any a_i in the support of x_i^* (i.e., $x_i^*(a_i) > 0$) must be equally good and are all “pure” best responses

Nash Equilibrium (NE)

A mixed strategy profile $x^* = (x_1^*, \dots, x_n^*)$ is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \quad \forall x_i \in \Delta_{A_i}, \forall i \in [n].$$

That is, for any i , x_i^* is a best response to x_{-i}^* .

Remarks

- An equivalent condition: $u_i(x_i^*, x_{-i}^*) \geq u_i(a_i, x_{-i}^*), \forall a_i \in A_i, \forall i \in [n]$
 - Since there always exists a pure best response
- It is not clear yet that such a mixed strategy profile would exist
 - Recall that pure strategy Nash equilibrium may not exist

Nash Equilibrium (NE)

Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- A foundational result in game-theory
- Example: rock-paper-scissor – what is a mixed strategy NE?

- $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		1/3	1/3	1/3
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

Nash Equilibrium (NE)

Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- An equilibrium outcome is not necessarily the best for players
 - Equilibrium only describes where the game stabilizes at
 - Many researches on understanding how self-interested behaviors reduces overall social welfare (recall the selfish routing game)
- A game may have many, even infinitely many, NEs
 - Which equilibrium you think it will stabilize at? → **the issue of equilibrium selection**

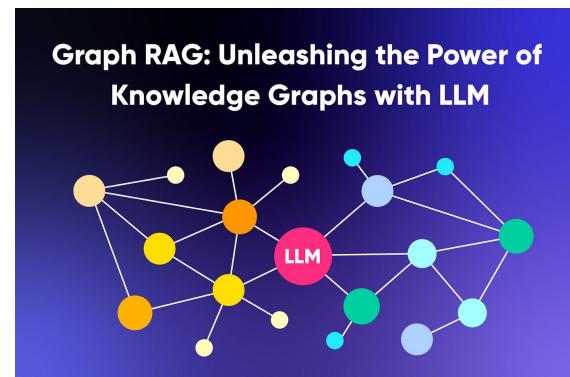
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A			
A stays silent	-1	-1	0
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Nash Equilibrium (NE)

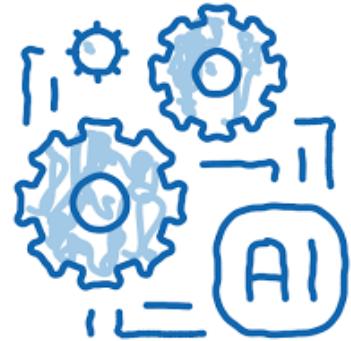
Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

Why do we bother spending so much effort studying equilibrium?

- Answer is just like why we study machine learning – equilibrium is a **prediction** of the behaviors/outcomes of strategic interactions
 - Key difference: ML is data-driven; equilibrium analysis is model-driven
 - However: modern approach is very often a combination (this is what EconCS does)
 - In spirit, not much difference from *ML+Science* or *LLM + Knowledge graph*



Computing a NE



Why we want to compute?

- Reason 1: just like why we want our ML prediction to be efficiently computable
- Reason 2: want to figure out best action to take
 - E.g., want to figure out best GO/Poker agent strategy
 - Just like why we want to solve classic optimization problem

Intractability of Finding a NE

Theorem: Computing a Nash equilibrium for any two-player normal-form game is PPAD-hard.

Note: PPAD-hard problems are believed to not admit poly time algorithm

- A two player game can be described by $2mn$ numbers – $u_1(i, j)$ and $u_2(i, j)$ where $i \in [m]$ is player 1's action and $j \in [n]$ is player 2's.
- Theorem implies no $\text{poly}(mn)$ time algorithm to compute an NE for any input game
- Ok, so what can we hope?
 - If the game has good structures, maybe we can find an NE efficiently
 - For example, zero-sum ($u_1(i, j) + u_2(i, j) = 0$ for all i, j), some resource allocation games

An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE: S_1, S_2 for player 1 and 2?
- The NE can be formulated by a **linear feasibility** problem with variables x_1^*, x_2^*, U_1, U_2

$$\forall j \in S_2: \quad \sum_{i \in S_1} u_2(i, j) x_1^*(i) = U_2$$

$$\forall j \notin S_2: \quad \sum_{i \in S_1} u_2(i, j) x_1^*(i) \leq U_2$$

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$$\sum_{i \in [m]} x_1^*(i) = 1$$

$$\forall i \notin S_1: \quad x_1^*(i) = 0$$

$$\forall i \in [m]: \quad x_1^*(i) \geq 0$$

An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE: S_1, S_2 for player 1 and 2?
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$$\sum_{i \in [m]} x_1^*(i) = 1$$

$$\forall i \notin S_1: \quad x_1^*(i) = 0$$

$$\forall i \in [m]: \quad x_1^*(i) \geq 0$$

Symmetric constraints for player 2

- The challenge of computing a NE is to find the correct supports
 - No general tricks, typically just try all possibilities
 - Some pre-processing may help, e.g., eliminating dominated actions
- This approach does not work for > 2 players games (why?)

An Example

	A	B	C
D	0, 0	-1, 1	1, -1
E	1, -1	0, 0	-1, -2
F	-1, 1	1, -1	0, -2

Step 1: pre-processing

- Column player never wants to play C

An Example

	A	B	C
D	0, 0	-1, 1	1, -1
E	1, -1	0, 0	-1, -2
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Step 1: pre-processing

- Column player never wants to play C

An Example

		q_1	q_2	0
		A	B	C
0	D	0, 0	-1, 1	1, -1
p_2	E	1, -1	0, 0	-1, -2
p_3	F	-1, 1	1, -1	0, -2

Step 2: Guess support and parameterize the equilibrium

$$\begin{aligned} u_1(E, A) \times q_1 + u_1(E, B) \times q_2 &= u \\ u_1(F, A) \times q_1 + u_1(F, B) \times q_2 &= u \end{aligned} \quad \left. \begin{array}{l} \text{Row player indifferent} \\ \text{between } \{E, F\} \end{array} \right\}$$

$$u_1(D, A) \times q_1 + u_1(D, B) \times q_2 \leq u \quad \rightarrow \quad \begin{array}{l} \text{Row player prefers } \{E, \\ F\} \text{ over } \{D\} \end{array}$$

... same for column player

Solve LP for p_2, p_3, q_1, q_2, u, v

An Example

		1/3	2/3	0
		A	B	C
0	D	0, 0	-1, 1	1, -1
2/3	E	1, -1	0, 0	-1, -2
1/3	F	-1, 1	1, -1	0, -2

Turns out our guess of support is correct

- If not, LP will be infeasible;
- In general, try all possibilities of support → Nash's theorem guarantees that one of LP systems must be feasible

Intractability of Finding “Best” NE

Theorem: It is NP-hard to compute the NE that maximizes the sum of players' utilities or any single player's utility even in two-player games.

- Proofs of these results for NEs are beyond the scope of this course

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A Remark: Simultaneous vs Sequential Move

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

A Remark: Simultaneous vs Sequential Move

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

➤ What is an NE?

- (a_2, b_2) is the unique Nash, resulting in utility pair (1,2)

➤ If A moves first; B sees A's move and then best responds, how should A play?

- Play action a_1 deterministically!

		B	
		b_1	b_2
		a_1	$(2, 1)$
		a_2	$(2.01, -2)$
			$(1, 2)$

This sequential game model is called **Stackelberg game**, originally used to model market competition and now adversarial attacks.

Extension I: Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
 - This is typical in Bayesian decision making, but not the only way

	B	B stays silent	B betrays
A			
A stays silent		$\theta - 1$ $+ \theta$	0 $+ \theta$
A betrays		$\theta - 3$	-2

I will give an additional reward θ for whoever staying silent



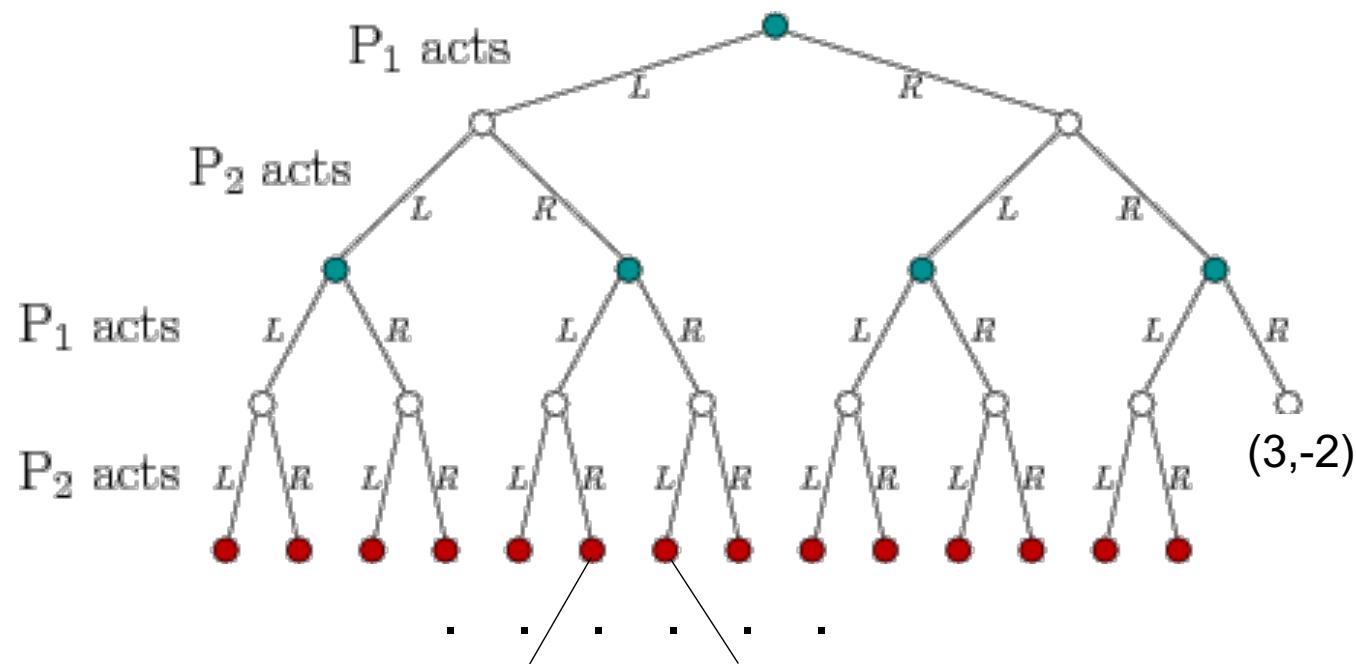
- It is believed that $\theta \in \{0, 2, 4\}$ uniformly at random
- Or maybe the two players have different beliefs about θ

Extension I: Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
 - This is typical in Bayesian decision making, but not the only way
- More generally, can model player i ' payoffs as u_i^θ where θ is a **random** state of the game
- Each player obtains a (random) signal s_i that is correlated with θ
 - A joint prior distribution over $(\theta, s_1, \dots, s_n)$ is assumed the public knowledge
- Can define a similar notion as Nash equilibrium, but expected utility also incorporates the randomness of the state of the game θ
- Applications: poker, blackjack, auction design, etc.

Extension II: Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**



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- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**
- EFGs are extremely general, can represent almost all kinds of games, but of course very difficult to solve

Thank You

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