ECE 612, Information Theory

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Preliminares

Definition 0.1. *The normal distribution $N(\mu, \sigma^2)$ has the probability density function*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)$$

Definition 0.2. *The joint normal distribution $N(\mu, K)$ is defined by probability density function:*

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(K)}} \exp(-\frac{1}{2}(\mathbf{x}-\mu)^\intercal K^{-1}(\mathbf{x}-\mu))$$

Theorem 0.1 (Joint normality implies marginal normality). *If $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ follows a joint normal distribution, then any linear combination of $\mathbf{X}$ follows normal distribution.*

1 Entropy, mutual information, divergence

Definition 1.1 (Entropy). *The entropy of a random variable $X \in \mathcal{X}$ is defined by*

$$H(X) = -\sum_{x \in \mathcal{X}} p_X(x) \log p_X(x)$$

Definition 1.2 (KL Divergence). *The **KL Divergence** between two probability distributions p, q on the same support $\mathcal{X}$ is defined by*

$$D(p \| q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$$

Definition 1.3 (Mutual information). *The mutual information between two random variables X, Y is defined by*

$$I(X;Y) = H(X) - H(X \mid Y)$$

It can also be expressed as the KL divergence between the joint distribution $p_{X,Y}$ and the product of marginal distribution $p_X p_Y$:

$$I(X;Y) = D(p_{X,Y} \| p_X \cdot p_Y)$$

Proposition 1.1. *Conditioning does not increase entropy*

$$H(X \mid Y) \leq H(X)$$

A nice trick for reasoning about the entropy of "sum of random variables":

$$\begin{aligned} H(X+Y \mid X) &= \sum_{x\in\mathcal{X}} p_X(x) H(X+Y \mid X=x) \\ &= \sum_{x\in\mathcal{X}} p_X(x) H(x+Y \mid X=x) \\ &= \sum_{x\in\mathcal{X}} p_X(x) H(Y \mid X=x) \\ &= H(Y \mid X) \end{aligned}$$

1.1 Convexity

Definition 1.4. *A function f is convex over the interval $x_1 \le x \le x_2$ if for all $0 \le t \le 1$:*

$$(1-t)f(x_1) + tf(x_2) \ge f((1-t)x_1 + tx_2)$$

Some notable functions' convexity:

1. Entropy is a concave function with respect to the probability distribution
2. Mutual information $I(X;Y)$ is concave with respect to p_X and is convex with respect to $p_{Y|X}$ for some fixed p_X
3. log is a concave function
4. The negative of a convex function is concave
5. The sum or product of a linear function and a concave function is concave
6. The composition of a linear function and a concave function is concave

Theorem 1.1 (Jensen's Inequality)**.** *If f is a convex function and X is a random variable, then*

$$E[f(X)] \ge f(E[X])$$

1.2 Markov chain

Definition 1.5. *Random variables X, Y, Z form a Markove chain (denoted by $X \to Y \to Z$) if $p(z \mid x, y) = p(z \mid y)$*

Theorem 1.2 (Data processing inequality)**.** *If $X \to Y \to Z$, then*

$$I(X;Z) \le I(Y;Z)$$

Proposition 1.2.

$$I(X;Y \mid Z) = 0 \Leftrightarrow X \to Z \to Y$$

Proposition 1.3. *If $X \to Y \to Z$, then $Z \to Y \to X$*

Theorem 1.3. *Let X, Y be random variables. $I(X;Y)$ is concave with respect to the probability distribution of X. For a fixed marignal distribution of X, $I(X;Y)$ is convex with respect to $f_{Y|X}$*

Theorem 1.4 (Fano's inequality)**.** *Let $X \to Y \to \hat{X}$ represent an encode-decode process, where $X, \hat{X} \in \mathcal{X}$ have the same support. Let e denote decoding error $\hat{X} \ne X$, then:*

$$H(X \mid Y) \le h(P_e) + P_e \log(|\mathcal{X}|)$$

2 Entropy rate

Definition 2.1. *A stochastic process is stationary if for all integer $n \geq 1$ and integer time shift (possibly negative) l:*

$$P[X_1^n] = P[X_{1+l}^{n+l}]$$

Definition 2.2. *A Markov chain is time invariant if for all $n \geq 1$:*

$$P[X_{n+1} \mid X_n] = P[X_2 \mid X_1]$$

Definition 2.3. *There are two definitions of entropy rate:*

1. $H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1^n)$
2. $H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n \mid X_1^{n-1})$

Theorem 2.1. *If X_i is a stationary process, then the two entropy rates both exist and are equal*

Lemma 2.1.1. *If X_i is a stationary process, then $H(X_n \mid X_1^{n-1})$ is non-increasing with respect to n and has a limit*

3 Asymptotic equipartition property

Theorem 3.1 (Weak law of large numbers (WLLN))**.** *Let $X \sim p_X$ be a random variable, and let $X_l \sim p_X$ be i.i.d. random variables following the same distributions, then:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X]$$

In other words, for any $\delta, \epsilon > 0$, there exists $N > 0$ such that for $n > N$:

$$P[|\frac{1}{n} \sum_{i=1}^{n} X_i - E[X]| < \delta] < \epsilon$$

Recall that the entropy of a random variable can be defined as $H(X) = -E[\log(P_X)]$, so the weak law of large numbers also applies: the joint distribution of i.i.d. X_i converges to the probability $2^{-nH(X)}$.

Definition 3.1 (Typical sequence)**.** *For random variable $X \sim p_X$, the set of* ***typical sequence*** *A_ϵ^n denotes the set of sequences whose probability is close to the entropy of the random variable*

$$A_\epsilon^{(n)} = \{\mathbf{x} \in \mathcal{X}^n \mid |-\frac{1}{n} \log p_{\mathbf{X}}(\mathbf{x}) - H(X)| < \epsilon\}$$

Theorem 3.2 (Asymptotic equipartition property)**.** *Let $X \sim p_X$ and let $\mathbf{X} \in \mathcal{X}^n$ denote i.i.d. sequence following the same distribution, then:*

1. $\lim_{n\to\infty} P[\mathbf{X} \in A_\epsilon^{(n)}] = 1$
2. $|A_\epsilon^{(n)}| \leq 2^{n(H-\epsilon)}$
3. *For sufficiently large n,* $|A_\epsilon^{(n)}| \geq (1-\epsilon)2^{n(H+\epsilon)}$

Proof. A sketch of the proof (1) is the direct consequence of the weak law of large numbers
(2): begin with **the sum of probability of all possible sequence is 1**
(3): begin with **for sufficiently large n,** $P[\mathbf{x} \in A_\epsilon^{(n)}] \geq 1 - \epsilon$ □

4 Data compressions

Theorem 4.1 (Kraft inequality)**.** *A D-ary prefix code for a finite set of m symbols $\mathcal{X}$ exists if and only if the lengths of the code words $l_1, l_2, \ldots, l_m$ satisfy:*

$$\sum_{x \in \mathcal{X}} D^{-l(x)} \leq 1$$

Using the method of Lagrange multiplier, we can optimize the expected codeword length $E[L]$ under the constraint fo Kraft inequality:

$$l_i = \lceil \log_D \frac{1}{p_i} \rceil$$

Theorem 4.2 (Optimal code length)**.** $H_D(X) \leq E[L] \leq H_D(X) + 1$

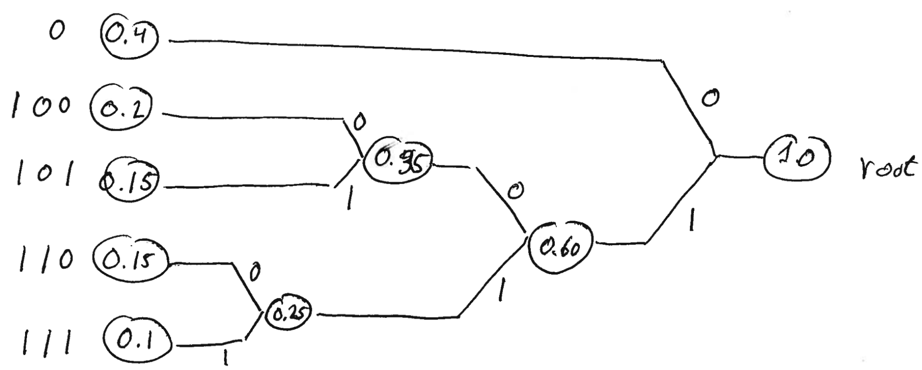
Theorem 4.3 (McMillan inequality)**.** *A D-ary uniquely decodable code exists if and only if*

$$\sum_i D^{-l_i} \leq 1$$

In practice, **Huffman code** offers the optimal noiseless compression:

Requires knowledge of source probabilities.

Ex: $p_1 = 0.4$, $p_2 = 0.2$, $p_3 = 0.15$, $p_4 = 0.15$, $p_5 = 0.1$



Codewords are defined by path from root to corresponding leaf.

5 Channel capacity

In this section we discuss **how much information can be transmitted through a noisy channel**

Definition 5.1. *A discrete channel consists of a discrete set of words $\mathcal{W}$, two sets of symbols $\mathcal{X}, \mathcal{Y}$, an encoder $\mathcal{W} \to \mathcal{X}$, a decoder $\mathcal{Y} \to \mathcal{W}$, and a channel described by the conditional distribution $p_{Y|X}$:*

$$W \to X \to Y \to \hat{W}$$

*For this section we are only concerned with **memoryless** channel: given X_n, Y_n is independent of all other variables:*

$$p_{\mathbf{Y}|\mathbf{X}} = \prod_{l=1}^{n} p_{Y_l|X_l}$$

*For a fixed $p_{Y|X}$, **information channel capacity** is defined by*

$$C^I = \max_{p_X} I(p_X; p_{Y|X})$$

5.1 Channel rate

Definition 5.2 ((M, n) code)**.** *An (M,n)-code consists of a collection of M words $\mathcal{W} : |\mathcal{W}| = M$, two sets of symbols $\mathcal{X}, \mathcal{Y}$, an encoder $f : \mathcal{W} \to \mathcal{X}^n$, and a decoder $g : \mathcal{Y}^n \to \mathcal{W}$*

Definition 5.3 (Decoding error)**.** *For a single word $w \in \mathcal{W}$, the **conditional probability of error** is defined by*

$$\lambda_l = P[g(\mathbf{Y}) \neq l \mid \mathbf{X} = f(w)]$$

*Denote the **maximal probability of error** across all words by*

$$\lambda^(n) = \max_{w \in \mathcal{W}} \lambda_w$$

*Denote the **average probability of error** across all words by*

$$P_e^{(n)} = \frac{1}{M} \sum_{w \in \mathcal{W}} \lambda_w$$

*The **rate of an** (M,n)**-code** denotes the number of bits of information transmitted per channel use:*

$$R = \frac{\log(M)}{n}$$

Definition 5.4 (Achievable rate)**.** *A rate $R > 0$ is **achievable** if there exists a sequence of $(2^{nR}, n)$-code such that:*

$$\lim_{n \to \infty} \lambda^{(n)} = 0$$

*The **capacity** of a channel is the supremum of all achievable rates*

5.2 Jointly typical sequence

Let $(\mathbf{X}, \mathbf{Y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ be i.i.d. sequences drawn according to the joint distribution $P_{X,Y}$, then by the weak law of large numbers, $\lim_{n\to\infty} -\frac{1}{n}\log(p_{X,Y}(\mathbf{x}, \mathbf{y})) = H(X,Y)$. We can define notion of typicality on the joint sequence similar to the notion of typicality on individual sequences

Definition 5.5 (Jointly typical sequence)**.** *The set of jointly typical sequence $A_\epsilon^{(n)}$ is the set of $(\mathbf{x}, \mathbf{y})$ such that:*

1. $|-\frac{1}{n}\log(p_X(\mathbf{x})) - H(X)| \leq \epsilon$
2. $|-\frac{1}{n}\log(p_Y(\mathbf{y})) - H(Y)| \leq \epsilon$
3. $|-\frac{1}{n}\log(p_{X,Y}(\mathbf{x}, \mathbf{y})) - H(X,Y)| \leq \epsilon$

In other words, $(\mathbf{x}, \mathbf{y})$ is jointly typical if $\mathbf{x}$ and $\mathbf{y}$ are each individually typical according to their marginal distribution, and $(\mathbf{x}, \mathbf{y})$ is typical according to the joint distribution

Theorem 5.1 (Joint asymptotic equipartition property)**.** *Let $(\mathbf{X}, \mathbf{Y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ be i.i.d. sequences drawn according to $p_{X,Y}$, then*

1. $\lim_{n\to\infty} P[(\mathbf{X}, \mathbf{Y}) \in A_\epsilon^{(n)}] = 1$
2. $|A_\epsilon^{(n)}| \leq 2^{n(H(X,Y)+\epsilon)}$
3. *For sufficiently large n:*
$$|A_\epsilon^{(n)}| \geq (1-\epsilon)2^{n(H(X,Y)-\epsilon)}$$
4. *Let $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ be drawn independently from p_X and p_Y, then*
$$P[(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \in A_\epsilon^n] \leq 2^{-n(I(X;Y)-3\epsilon)}$$
5. *Let $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ be drawn independently from p_X and p_Y, then for sufficiently large n:*
$$P[(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \in A_\epsilon^n] \geq (1-\epsilon)2^{-n(I(X;Y)+3\epsilon)}$$

Proof. (1) is the direct consequence of the weak law of large numbers

(2):
$$\begin{aligned}1 &= \sum_{(\mathbf{x},\mathbf{y})\in\mathcal{X}^n\times\mathcal{Y}^n} p(\mathbf{x},\mathbf{y}) \\ &\geq \sum_{(\mathbf{x},\mathbf{y})\in A_\epsilon^{(n)}} p(\mathbf{x},\mathbf{y}) \\ &\geq \sum_{(\mathbf{x},\mathbf{y})\in A_\epsilon^{(n)}} 2^{-n(H(X,Y)+\epsilon)} \\ &= |A_\epsilon^{(n)}| 2^{-n(H(X,Y)+\epsilon)}\end{aligned}$$

(3): for sufficiently large n, $P[(\mathbf{X}, \mathbf{Y}) \in A_\epsilon^{(n)}] \geq 1-\epsilon$. On the other hand:
$$\begin{aligned}P[(\mathbf{X}, \mathbf{Y}) \in A_\epsilon^{(n)}] &= \sum_{(\mathbf{x},\mathbf{y})\in A_\epsilon} P[(\mathbf{X}, \mathbf{Y}) = (\mathbf{x},\mathbf{y})] \\ &\leq \sum_{(\mathbf{x},\mathbf{y})\in A_\epsilon} 2^{-n(H_{X,Y}-\epsilon)} \\ &= |A_\epsilon| \cdot 2^{-n(H_{X,Y}-\epsilon)}\end{aligned}$$

Putting the two inequalities above together:
$$|A_\epsilon| \cdot 2^{-n(H_{X,Y}-\epsilon)} \geq 1-\epsilon$$

(4): consider the probability of $(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}})$ falling into the jointly typical set:
$$\begin{aligned}P[(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \in A_\epsilon^{(n)}] &= \sum_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in A_\epsilon^{(n)}} P[\tilde{\mathbf{X}} = \tilde{\mathbf{x}}, \tilde{\mathbf{Y}} = \tilde{\mathbf{y}}] \\ &= \sum_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in A_\epsilon^{(n)}} P[\tilde{\mathbf{X}} = \tilde{\mathbf{x}}] \cdot P[\tilde{\mathbf{Y}} = \tilde{\mathbf{y}}] \\ &\leq \sum_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in A_\epsilon^{(n)}} 2^{-n(H_X-\epsilon)} \cdot 2^{-n(H_Y-\epsilon)} \\ &= |A| \cdot 2^{-n(H_X-\epsilon)} \cdot 2^{-n(H_Y-\epsilon)} \\ &\leq 2^{n(H(X,Y)+\epsilon)} \cdot 2^{-n(H_X-\epsilon)} \cdot 2^{-n(H_Y-\epsilon)}\end{aligned}$$

(5): Using similar logic as shown above:

$$\begin{aligned}
P[(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \in A_\epsilon^{(n)}] &= \sum_{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in A_\epsilon^{(n)}} P[\tilde{\mathbf{X}} = \tilde{\mathbf{x}}, \tilde{\mathbf{Y}} = \tilde{\mathbf{y}}] \\
&= \sum_{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in A_\epsilon^{(n)}} P[\tilde{\mathbf{X}} = \tilde{\mathbf{x}}] \cdot P[\tilde{\mathbf{Y}} = \tilde{\mathbf{y}}] \\
&\geq \sum_{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in A_\epsilon^{(n)}} 2^{-n(H_X + \epsilon)} \cdot 2^{-n(H_Y + \epsilon)} \\
&= |A_e| 2^{-n(H_X + \epsilon)} \cdot 2^{-n(H_Y + \epsilon)} \\
&\geq (1 - \epsilon) 2^{n(H_{X,Y} - \epsilon)} \cdot 2^{-n(H_X + \epsilon)} \cdot 2^{-n(H_Y + \epsilon)}
\end{aligned}$$

□

5.3 Channel coding theorem

Theorem 5.2. *A rate R is achievable if and only if it is below the information channel capacity*

Proof. This is a sketch of proof

First we show that rate below information capacity is achievable: Let R be a rate such that $R < C$, then we can construct a $(2^{nR}, n)$-code through the following procedure:

1. Find some distribution of p_X and $\epsilon > 0$ such that $R = I(p_X; p_{Y|X}) - 4\epsilon$
2. For each $w \in \{1, \ldots, 2^{nR}\}$, the encoding is $f(w) = (x_1(w), x_2(w), \ldots, x_n(w))$ where $x_i(w)$ is i.i.d. sampled from p_X
3. Transmit each symbol $x_i(w)$ according to $p_{Y|X}$
4. Upon receiving $\mathbf{y}$, find some $\hat{w} \in \{1, \ldots, 2^{nR}\}$ such that $(\mathbf{x}, \mathbf{y})$ is jointly typical according to p_X and $p_{Y|X}$ (which can be used to compute $p_{X,Y}$ and p_Y)

For each word w, decoding error occurs if and only if one of two scenarios occurs:

1. $(\mathbf{x}(w), \mathbf{y})$ is not jointly typical, in which case no appropriate $\hat{w}$ can be found in the decoding step
2. there is another $w' \neq w$ such that $(\mathbf{x}(w'), \mathbf{y})$ is also jointly typical

The probability of (1) converges to 0 because $\lim_{n \to \infty} P[(\mathbf{x}, \mathbf{y}) \notin A_\epsilon^{(n)}] = 0$.

The probability of (2) converges to 0 because $\mathbf{y}$ is independent of $\mathbf{x}(w')$. By joint AEP, the probability of independently sampled sequences being jointly typical is at most $2^{-n(I(X;Y) - 3\epsilon)}$, hence:

$$\begin{aligned}
\sum_{w' \neq w} P[(\mathbf{x}(w'), \mathbf{y}) \in A_\epsilon^{(n)}] &= (2^{nR} - 1) P[(\mathbf{x}(w'), \mathbf{y}) \in A_\epsilon^{(n)}] \\
&\leq (2^{nR} - 1) 2^{-n(I(X;Y) - 3\epsilon)} \\
&\leq 2^{nR - n(I_{X;Y} - 3\epsilon)} \\
&= 2^{n(I_{X;Y} - 4\epsilon) - n(I_{X;Y} - 3\epsilon)} \\
&= 2^{-n\epsilon}
\end{aligned}$$

Which also converges to 0 as $n \to \infty$

Converse: achievable rates are below information capacity

Recall that the channel represents a Markov chain $W \to \mathbf{X} \to \mathbf{Y} \to \hat{W}$. By Fano's inequality, we know that:

$$H(W \mid \hat{W}) \leq h(p_e) + p_e \cdot \log(|\mathcal{W}|)$$

Consider the mutual information between W and $\hat{W}$:

$$I(W;\hat{W}) = H(W) - H(W \mid \hat{W})$$

Assuming that W is uniformly distributed, $H(W) = \log(|\mathcal{W}|) = nR$.

By the data-processing inequality $I(W;\hat{W}) \le I(\mathbf{X};\mathbf{Y})$. By the chain rule of mutual information: $I(\mathbf{X};\mathbf{Y}) \le nI(X;Y)$. By the definition of information channel capacity $I(X;Y) \le C$. Putting everything together, we have

$$nR - nC \le p_e$$

Assuming the rate is achievable, then $\lim_{n\to\infty} p_e = 0$, which implies $R \le C$. □

6 Differential entropy

Definition 6.1. *Let $X \in \mathbb{R}$ be a random variable with probability density function f_X. The* ***differential entropy*** *of X is defined by*

$$h(X) = -\int_{x\in\mathbb{R}} f_X(x)\ln(f_X(x))dx = -E[\ln(f_X(X))]$$

Some notable results:

- For uniform distribution over $[0, a]$, $f_X(x) = \frac{1}{a}$, $h(X) = \ln(a)$
- For $X \sim N(0, \sigma^2)$, $h(X) = \frac{1}{2}\ln(2\pi e\sigma^2)$
- For multivariate normal $\mathbf{X} \sim N(\mathbf{0}, K)$, $h(\mathbf{X}) = \frac{1}{2}\log((2\pi e)^n \det K)$
- For some constant a, $h(X + a) = h(X)$
- For some constant a, $h(aX) = \ln(|a|) + h(X)$

Theorem 6.1. *Let X be continuous random variable with 0 mean and σ^2 variance, then*

$$h(X) \le \frac{1}{2}\ln(2\pi e\sigma^2)$$

Equality is reached if and only if X is Gaussian

7 Gaussian channel

Definition 7.1 (Information channel capacity)**.** *Let $Y = X + Z$, where $Z \xleftarrow{\$} N(0, \sigma^2)$ and $E[X^2] \le P$ for some power level constraint P. The* ***information channel capacity*** *is defined by*

$$C^I = \max_{f_X : E[X^2]\le P} I(X;Y)$$

Theorem 7.1. *The information channel capacity of a Gaussian channel is*

$$\max_{f_X : E[X^2]\le P} I(X;Y) = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) \tag{1}$$

Where P is the power constraint, and σ^2 is the variance of the Gaussian noise. The maximum is achieved when X follows Gaussian distribution $X \sim N(0, P)$

7.1 Parallel Gaussian Channel

Suppose for $1 \le l \le n$, $Y_l = X_l + Z_l$, where $Z_l \sim N(0, N_l)$ is independent Gaussian noise, then $\mathbf{Y} = (Y_l)_{l=1}^n$ can be seen as the output of n parallel Gaussian channels combined into a single channel. The capacity of the combined channel is

$$C = \max_{f_{\mathbf{X}}} I(\mathbf{X}; \mathbf{Y})$$

The optimal strategy for maximizing the capacity is to make X_l independent and individually Gaussian. If there is a combined power constraint $\sum_{l=1}^n E[X_l^2] \le P$, then *power should be first given to the channel with the lowest amount of noise until the combined noise + power exceeds the next lowest noise* (**waterfilling**).

8 Rate distortion theory

Definition 8.1. *For some fixed p_X, the **information rate distortion** is defined by*

$$R^I(D) = \min_{p_{\hat{X}|X}: E[d(X,\hat{X})] \le D} I(X; \hat{X})$$

Theorem 8.1. *Let X follow* Bernoulli(p)*, then*

$$R(D) = \begin{cases} h(p) - h(D) & \textit{When } D < \min(p, 1-p) \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 8.2. *For $X \xleftarrow{\$} N(0, \sigma^2)$:*

$$R(D) = \begin{cases} \frac{1}{2}\log(\frac{\sigma^2}{D}) & (D \le \sigma^2) \\ 0 & \textit{otherwise} \end{cases}$$