Q6

(1)

Let $X \in \mathcal{B}(n,p)$ be a random variable that follows binomial distribution. Recall from the definition of a binomial distribution that $X = I_1 + I_2 + \ldots + I_n$ where each of I_i is an independent coin toss with PMF:

$$\begin{cases} P(I_i = 1) = p \\ P(I_i = 0) = 1 - p \end{cases}$$

Let $X_1 \in \mathcal{B}(n_1,p), X_2 \in \mathcal{B}(n_2,p)$ be two independent random variables following binomial distributions, then:

$$X_1 = \sum_{i=1}^{n_1} I_i$$
$$X_2 = \sum_{i=n_1+1}^{n_1+n_2} I_i$$

Therefore $X_1 + X_2 = \sum_{i=1}^{n_1+n_2} I_i$, which is a binomial distribution $\mathcal{B}(n_1 + n_2, p)$.

Recall that centered binomial distribution is defined by subtracting the corresponding binomial distribution by a constant (the expectation of said binomial distribution): $C_i = X_i - E(X_i)$. Therefore, given centered binomial distributions $C_1 = X_1 - E[X_1], C_2 = X_2 - E[X_2]$:

$$\begin{aligned} C_1 + C_2 &= X_1 + X_2 - E[X_1] - E[X_2] \\ &= (X_1 + X_2) - E[X_1 + X_2] \end{aligned}$$

From the results above we know that because X_1, X_2 are independent binomial distributions, $X_1 + X_2$ follows binomial distribution $\mathcal{B}(n_1+n_2,p)$, thus C_1+C_2 follows centered binomial distribution $\mathcal{B}(n_1+n_2,p)$.

(2)

From part (a) we know that the sum of k i.i.d. random variables following binomial $\mathcal{B}(n,p)$ is a random variable following binomial $\mathcal{B}(kn,p)$.

Let $I^m = \{0,1\}^m$ denote the set of bit-masks with length m and $K \subseteq I$ denote the subset of vectors with exactly k entries being 1, then $|K| = \binom{m}{k}$. Thus we iterate through all possible values in $\mathbf{k} \in K$ and compute the inner product $\mathbf{e}^\intercal \mathbf{k}$. Since $\mathbf{k}$ has exactly k entries being 1 and $\mathbf{e}$ contains m independent samples from centered binomial (n,p), $\mathbf{e}^\intercal \mathbf{k}$ is the sum of k i.i.d. centered binomial with parameters (kn,p)