Homework 1

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1 Distinguish quantum states

(a)

Let $\psi_0 = |1\rangle$ and $\psi_1 = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. I claim without proof that **an optimal distinguisher can be achieved using** $R(\theta)$ for some rotation θ. Without loss of generality, we define the distinguisher to be $\hat{b} \leftarrow D(\phi_b)$ where $b \xleftarrow{\$} \{0,1\}$ and $\hat{b}$ is D's guess at the choice of quantum state.

After the rotation, the quantum states are as follows:

$$
\begin{aligned}
R \cdot \psi_0 &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \\
R \cdot \psi_1 &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta) \\ \frac{1}{\sqrt{2}}\sin(\theta) - \frac{1}{\sqrt{2}}\cos(\theta) \end{bmatrix}
\end{aligned}
$$

The probability of successfully distinguishing the quantum state is as follows:

$$
\begin{aligned}
P[\hat{b} = b] &= P[\hat{b} = b \cap b = 0] + P[\hat{b} = b \cap b = 1] \\
&= P[\hat{b} = b \mid b = 0] \cdot P[b = 0] + P[\hat{b} = b \mid b = 1] \cdot P[b = 1] \\
&= \frac{1}{2}(P[\hat{b} = b \mid b = 0] + P[\hat{b} = b \mid b = 1]) \\
&= \frac{1}{2}(P[R\psi_0 = 0] + P[R\psi_1 = 1]) \\
&= \frac{1}{2}(\sin^2(\theta) + (\frac{1}{\sqrt{2}}\sin(\theta) - \frac{1}{\sqrt{2}}\cos(\theta))^2)
\end{aligned}
$$

To maximize $P[\hat{b} = b]$ with respect to θ, take the derivative:

$$
\begin{aligned}
\frac{dP}{d\theta} &= \frac{1}{2}\left(2\sin\theta\cos\theta + 2(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta) \cdot \frac{d}{d\theta}\left[\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right]\right) \\
&= \frac{1}{2}\left(\sin 2\theta + 2(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta) \cdot \left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right) \\
&= \frac{1}{2}\left(\sin 2\theta - \cos 2\theta\right) \\
\frac{d^2P}{d\theta^2} &= \cos\theta + \sin\theta
\end{aligned}
$$

Solving the first derivative yields $\theta = \frac{1}{8}\pi$ or $\theta = \frac{5}{8}\pi$, but only the second solution makes the second derivative negative and is thus a local maximum.

Thus an optimal distinguisher can be built by rotation $R(\frac{5}{8}\pi)$, the maximal probability is:

$$\frac{1}{2}\left(\sin^2\frac{5}{8}\pi + \left(\frac{1}{\sqrt{2}}\sin\frac{5}{8}\pi - \frac{1}{\sqrt{2}}\cos\frac{5}{8}\pi\right)^2\right) \approx 0.85355$$