Homework–7

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Contents

1.Derive an expression for the optimal forecast using AR(1) and MA(1) models

as described in Module 6.4

AR(1)

define

$$x_t = \phi x_{t-1} + \epsilon_t \qquad (1)$$

$$\gamma_0 = \frac{\sigma^2}{1-\phi^2} \qquad (2)$$

$$\gamma_j = \frac{\sigma^2}{1-\phi^2}\phi^j \qquad (3)$$

The best linear optimal forecast of $x_(n+1)$ given

$$\hat{x}_{n+1} = b_1 x_n + b_2 x_{n-1} + ... + b_n x_1 \qquad (4)$$

where $b = (b_1, b_2, ..., b_n)^T \in R^n$ is by

$$\Gamma_n b = \gamma(1:n) \qquad (5)$$

covariance matrix Γ_n

$$\Gamma_n = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{n-3} \\ \gamma_{n-1} & \gamma_{n-2} & \gamma_1 & \cdots & \gamma_0 \end{bmatrix} \qquad (6)$$

$$\gamma(1:n) = (\gamma_1, \gamma_2, ..., \gamma_n)^T \in R \qquad (7)$$

combine (2) (3) (5) (6) (7) and solve

$$\begin{aligned} b_1 &= \phi \quad and \quad b_j = 0 \quad for \quad 2 \leqslant j \leqslant n \qquad (8) \\ \hat{x}_{n+1} &= \phi x_n \qquad (9) \\ Min\ MSE &= E\left[(x_{n+1} - \phi x_n)^2\right] \qquad (10) \\ &= \gamma_0 - \phi^2\gamma_0 \\ &= \sigma^2 \end{aligned}$$

s-Step Prediction

$$\begin{aligned} x_{n+s} &= \phi x_{n+s-1} + \epsilon_{n+s} \qquad (11) \\ \hat{x}_{n+s} &= \phi^s x_n \qquad (12) \\ Min\ MSE &= E\left[(\sum_{j=1}^{s} \phi^{s-j}\epsilon_{n+j})^2\right] \qquad (13) \\ &= \sigma^2(\frac{1-\phi^{2s}}{1-\phi^2}) \end{aligned}$$

MA(1)

define

$$\begin{aligned} x_t &= \epsilon_t + \theta\epsilon_{t-1} \quad for\ |\theta| < 1 \qquad (14) \\ \gamma_0 &= \sigma^2(1+\theta^2) \qquad (15) \\ \gamma_1 &= \theta\sigma^2 \qquad (16) \\ \gamma_j &= 0 \quad for\ j \geqslant 2 \qquad (17) \end{aligned}$$

covariance matrix Γ_n

$$\Gamma_n = \sigma^2 \begin{bmatrix} (1+\theta^2) & \theta & 0 & ... & 0 & 0 \\ \theta & (1+\theta^2) & \theta & ... & 0 & 0 \\ 0 & \theta & (1+\theta^2) & \theta & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ...\ \theta & (1+\theta^2) & \theta \\ 0 & 0 & 0 & ... & \theta & (1+\theta^2) \end{bmatrix} \qquad (18)$$

$$\begin{aligned} \gamma(1:n) &= \sigma^2((1+\theta^2), \theta, 0, ..., 0)^T \in R \qquad (19) \\ \Gamma_n b &= \gamma(1:n) \qquad (20) \end{aligned}$$

combine (4) (18) (19) (20) and solve, we get vector b

$$Min\ MSE = \gamma(0) - b^T\gamma(1:n) \qquad (21)$$

s - Step Prediction

$$\hat{x}_{n+s} = b_1 x_n + b_2 x_{n-1} + ... + b_n x_1 \qquad (22)$$

$$E\left[(x_{n+s} - \sum_{j=1}^{n} b_j x_{n-j+1}) xi\right] = 0 \qquad (23)$$

$$\Gamma_n b = \gamma(s : n + s - 1) \qquad (24)$$

$$\gamma(s : n + s - 1) = (\gamma_s, \gamma_{s+1}, ..., \gamma_{n+s-1})^T \qquad (25)$$

combine (24) (25) and solve, we get vector b

$$for\ s = 1, \quad \gamma(1 : n) = \sigma^2((1 + \theta^2), 0, .., 0)^T \qquad (26)$$

$$for\ s = 2, \quad \gamma(2 : n + 1) = \sigma^2(\theta, 0, .., 0)^T \qquad (27)$$

$$for\ s = 3, \quad \gamma(3 : n + 2) = \sigma^2(0, 0, .., 0)^T \qquad (28)$$

combining 26-28, we got $b = 0 \quad for\ s \geq 3$