# HW5_Yi_Xiong

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## Contents



# 1.(30 points) Following the developments in modules 4.3 and 4.4, derive the expressions for ACF for:

## a.MA(1) and MA(2) models

### MA(1)

#### definition

$$y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

#### Autocovariance

$$\begin{aligned}\gamma_0 &= E[y_t - E(y_t)]^2 \\ &= E[y_t - \mu] \\ &= E[\epsilon_t + \theta\epsilon_{t-1}]^2 \\ &= E[\epsilon_t^2 + 2\theta\epsilon_t\epsilon_{t-1} + \theta^2\epsilon_{t-1}^2] \\ &= \theta^2 + 0 + \theta^2\sigma^2 \\ &= (1+\theta^2)\sigma^2\end{aligned}$$

$$\begin{aligned}\gamma_j &= E[(y_t - \mu)(y_{t-j} - \mu)] \\ &= E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-j} + \theta\epsilon_{t-j-1})] \\ &= \begin{cases} \theta E[\epsilon_{t-1}^2] = \sigma\theta^2 \ \ if\ j = 1 \\ 0 \ \ otherwise \end{cases}\end{aligned}$$

#### Autocorrelation

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \begin{cases} \frac{\theta}{(1+\theta^2)} \ \ if\ j = 1 \\ 0 \ \ if\ j \neq 1 \end{cases}$$

**MA(2)**

**MA(q) definition**

$$y_t = \mu + \sum_{j=0}^{q} \theta_j \epsilon_{t-j},\ q \geqslant 1\ and\ \theta_0 = 1$$

**MA(2) definition**

$$y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

**Autocovariance of MA(q)**

$$\begin{aligned} \gamma_0 &= E(y_t - \mu)^2 \\ &= [\sum_{j=0}^{q} \theta_j^2]\sigma^2 \end{aligned}$$

$$\gamma_j = E[(y_t - \mu)(y_{t-j} - \mu)]$$
$$\gamma_j = \begin{cases} (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + ... + \theta_q\theta_{q-j})\sigma^2\ for\ 1 \leqslant j \leqslant q \\ 0\ \ for\ j > q \end{cases}$$

**Autocovariance of MA(2)**

$$\begin{aligned} \gamma_0 &= (1 + \theta_1^2 + \theta_2^2)\sigma^2 \\ \gamma_1 &= (\theta_1 + \theta_2\theta_1)\sigma^2 \\ \gamma_2 &= \theta_2\sigma^2 \\ \gamma_j &= 0,\ j \geqslant 3 \end{aligned}$$

**Autocorrelation of MA(2)**

$$\begin{aligned} \rho_1 &= \frac{(\theta_1 + \theta_2\theta_1)\sigma^2}{(1 + \theta_1^2 + \theta_2^2)\sigma^2} \\ \rho_2 &= \frac{\theta_2\sigma^2}{(1 + \theta_1^2 + \theta_2^2)\sigma^2} \\ \rho_j &= 0,\ j \geqslant 3 \end{aligned}$$

## b.AR(1) and AR(2) models

**AR(1)**

**definition**

$$y_t = c + \phi y_{t-1} + \epsilon_t$$

**Autocovariance**

**When $|\phi| < 1$, $\mathbf{y_t}$ is weakly stationary and ergodic**

$$y_t - \mu = \phi(y_{t-1} - \mu) + \epsilon_t$$

$$\begin{aligned}\gamma_0 &= E[y_t - \mu]^2 \\ &= \sigma^2[1 + \phi^2 + \phi^4 + ... + \phi^{2k} + ...] \\ &= \frac{\sigma^2}{1 - \phi^2}\end{aligned}$$

$$\begin{aligned}\gamma_j &= E[(y_t - \mu)(y_{t-j} - \mu)] \\ &= \sigma^2[\phi^j + \phi^{j+2} + \phi^{j+4} + ...] \\ &= \sigma^2\phi^j[1 + \phi^2 + \phi^4 + ...] \\ &= \sigma^2\frac{\phi^j}{1 - \phi^2}\end{aligned}$$

**Autocorrelation**

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j$$

**AR(2)**

**definition**

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$$

**For stationarity:**

$$\begin{aligned}&\lambda^2 - \phi_1\lambda - \phi_2 = 0 \\ &\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi 2}}{2} \\ &|\lambda| < 1\end{aligned}$$

**Autocovariance**

$$y_t - \mu = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \epsilon_t$$

$$\begin{aligned}\gamma_0 &= E[(y_t - \mu)(y_t - \mu)] \\ &= E[\phi_1(y_t - \mu)(y_{t-1} - \mu)] \\ &+ E[\phi_2(y_t - \mu)(y_{t-2} - \mu)] \\ &+ E[(y_t - \mu)\epsilon_t] \\ &= \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2\end{aligned}$$

$$for\ j \geqslant 1$$

$$\begin{aligned}\gamma_j &= E[(y_t-\mu)(y_{t-j}-\mu)] \\ &= E[\phi_1(y_{t-1}-\mu)(y_{t-j}-\mu)] \\ &+ E[\phi_2(y_{t-2}-\mu)(y_{t-j}-\mu)] \\ &+ E[(y_{t-j}-\mu)\epsilon_t] \\ &= \phi_1\gamma_1 + \phi_2\gamma_2\end{aligned}$$

**Autocorrelation**

$$for\ j \geqslant 1$$

$$\begin{aligned}\rho_j &= \frac{\gamma_j}{\gamma_0} \\ &= \theta_1\frac{\gamma_{j-1}}{\gamma_0} + \theta_2\frac{\gamma_{j-2}}{\gamma_0} \\ &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2}\end{aligned}$$

$$\begin{aligned}\rho_1 &= \phi_1\rho_0 + \phi_2\rho_{-1} \\ \rho_1 &= \frac{\phi_1}{1-\phi_2} \\ \rho_2 &= \phi_1\rho_1 + \phi_2\rho_0 \\ &= \phi_1\rho_1 + \phi_2\end{aligned}$$

$$\gamma_0 = \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)[(1-\phi_2)^2-\phi_1^2]}$$

**Summary**

$$\begin{aligned}\rho_1 &= \frac{\phi_1}{1-\phi_2} \\ \rho_2 &= \phi_1\rho_1 + \phi_2 \\ \rho_j &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2} \ \ for\ j \geqslant 3\end{aligned}$$

# c.ARMA(1,1)

**definition**

$$y_t = c + \phi_1 t_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1}$$

**Autocovariance**

$$y_t - \mu = \phi_1(y_{t-1}-\mu) + \epsilon_t + \theta_1\epsilon_{t-1}$$

$$\begin{aligned}\gamma_0 &= E[y_t-\mu]^2 \\ &= E[\phi_1(y_{t-1}-\mu) + \epsilon_t + \phi_1\epsilon_{t-1}]^2 \\ &= \phi_1^2\gamma_0 + \sigma^2 + \theta_1^2\sigma^2 + 2\phi_1\theta_1\sigma^2\end{aligned}$$

$$\gamma_0 = \frac{\sigma^2[1+2\phi_1\theta_1+\theta_1^2]}{1-\theta_1^2}$$

$$
\begin{aligned}
\gamma_1 &= E[(y_t - \mu)(y_{t-1} - \mu)] \\
&= E\{[\phi_1(y_{t-1} - \mu) + \epsilon_t + \phi_1\epsilon_{t-1}][y_{t-1} - \mu]\} \\
&= \phi_1\gamma_0 + \theta_1\sigma^2
\end{aligned}
$$

$for\ j \geqslant 2$

$$
\begin{aligned}
\gamma_j &= E[(y_t - \mu)(y_{t-j} - \mu)] \\
&= E\{[\phi_1(y_{t-1} - \mu) + \epsilon_t + \phi_1\epsilon_{t-1}][y_{t-j} - \mu]\} \\
&= \phi_1 E[(y_{t-1} - \mu)(y_{t-j} - \mu)] \\
&= \phi_1\gamma_{j-1}
\end{aligned}
$$

**Autocorrelation**

$$
\begin{aligned}
\gamma_0 = \frac{\sigma^2[1 + 2\phi_1\theta_1 + \theta_1^2]}{1 - \phi_1^2} \\
\gamma_1 = \phi_1\gamma_0 + \theta_1\sigma^2 \\
\gamma_j = \phi_1\gamma_{j-1} \ \ for\ j \geqslant 2 \\
\rho_1 = \frac{\gamma_1}{\gamma_0} \\
\rho_j = \frac{\gamma_j}{\gamma_0}
\end{aligned}
$$