FINAL EXAM

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Contents

1.Consider the ARMA(1, 2) model

$$Z_t = 0.5 + 0.8Z_{t-1} + \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2}$$

a. Compute the theoretical ACF and plot it

definition

$$Z_t = c + \phi_1 Z_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

theoretical Autocovariance

$$Z_t - \mu = \phi_1(Z_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$$

$$\begin{aligned}\gamma_0 &= E[Z_t - \mu]^2 \\ &= E[\phi_1(Z_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}]^2 \\ &= \phi_1^2\gamma_0 + \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 + 2\phi_1\theta_1\sigma^2 + 2\phi_1\theta_2(\phi_1 + \theta_1)\sigma^2\end{aligned}$$

$$\gamma_0 = \frac{\sigma^2[1 + 2\phi_1\theta_1 + 2\phi_1\theta_2(\phi_1 + \theta_1) + \theta_1^2 + \theta_2^2]}{1 - \phi_1^2}$$

$$\begin{aligned}\gamma_1 &= E[(Z_t - \mu)(Z_{t-1} - \mu)] \\ &= E\{[\phi_1(Z_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}][Z_{t-1} - \mu]\} \\ &= \phi_1\gamma_0 + \theta_1\sigma^2 + \theta_2(\phi_1 + \theta_1)\sigma^2\end{aligned}$$

$$\begin{aligned}\gamma_2 &= E[(Z_t - \mu)(Z_{t-2} - \mu)] \\ &= E\{[\phi_1(Z_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}][Z_{t-2} - \mu]\} \\ &= \phi_1\gamma_1 + \theta_2\sigma^2\end{aligned}$$

$$for\ j \geqslant 3$$

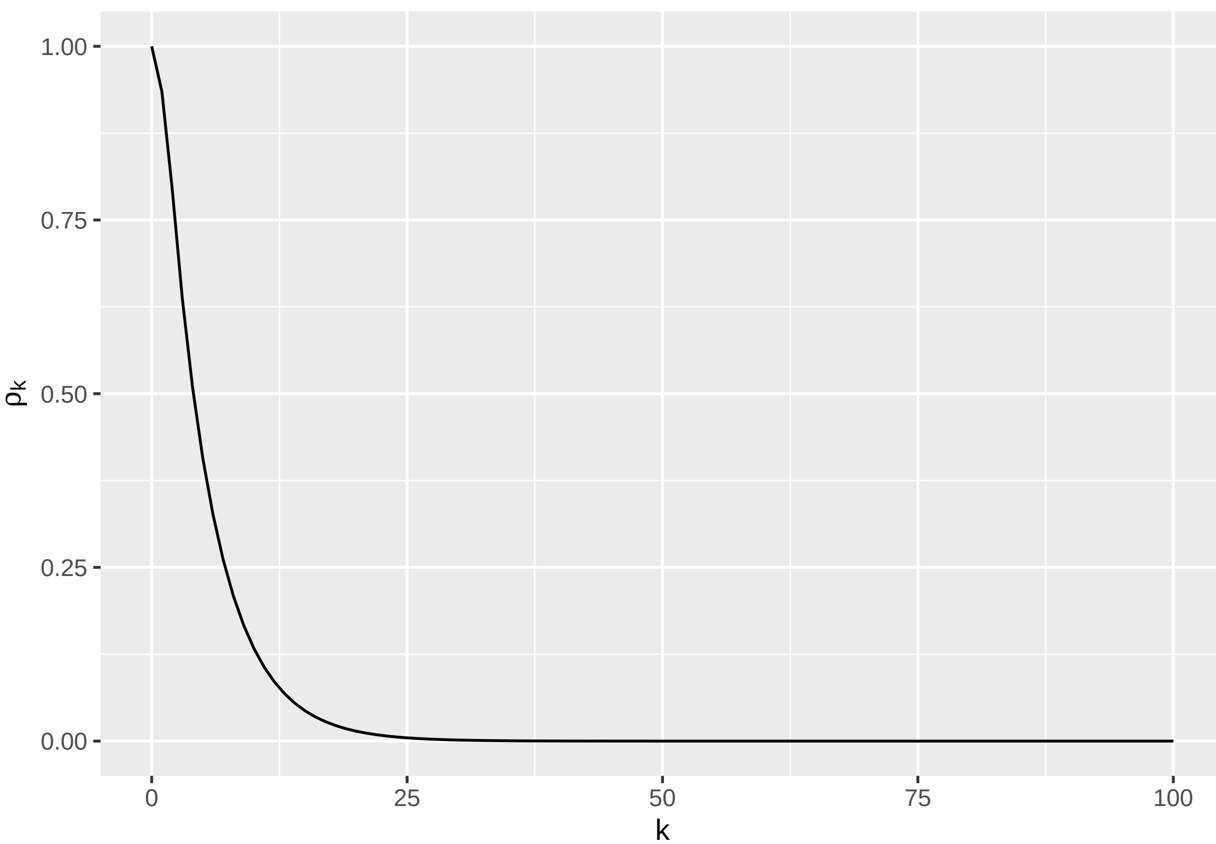
$$\begin{aligned}\gamma_j &= E[(Z_t - \mu)(Z_{t-j} - \mu)] \\ &= E\{[\phi_1(Z_{t-1} - \mu) + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}][Z_{t-j} - \mu]\} \\ &= \phi_1 E[(Z_{t-1} - \mu)(Z_{t-j} - \mu)] \\ &= \phi_1\gamma_{j-1}\end{aligned}$$

theoretical Autocorrelation

$$\begin{aligned}\phi_1 &= 0.8 \\ \theta_1 &= 1 \\ \theta_2 &= 1 \\ \gamma_0 &= \frac{\sigma^2[1 + 2\phi_1\theta_1 + 2\phi_1\theta_2(\phi_1 + \theta_1) + \theta_1^2 + \theta_2^2]}{1 - \phi_1^2} \\ \gamma_1 &== \phi_1\gamma_0 + \theta_1\sigma^2 + \theta_2(\phi_1 + \theta_1)\sigma^2 \\ \gamma_2 &== \phi_1\gamma_1 + \theta_2\sigma^2 \\ \gamma_j &= \phi_1\gamma_{j-1}\ \ for\ j \geqslant 3 \\ \rho_1 &= \frac{\gamma_1}{\gamma_0} \\ \rho_2 &= \frac{\gamma_2}{\gamma_0} \\ \rho_j &= \frac{\gamma_j}{\gamma_0}\end{aligned}$$

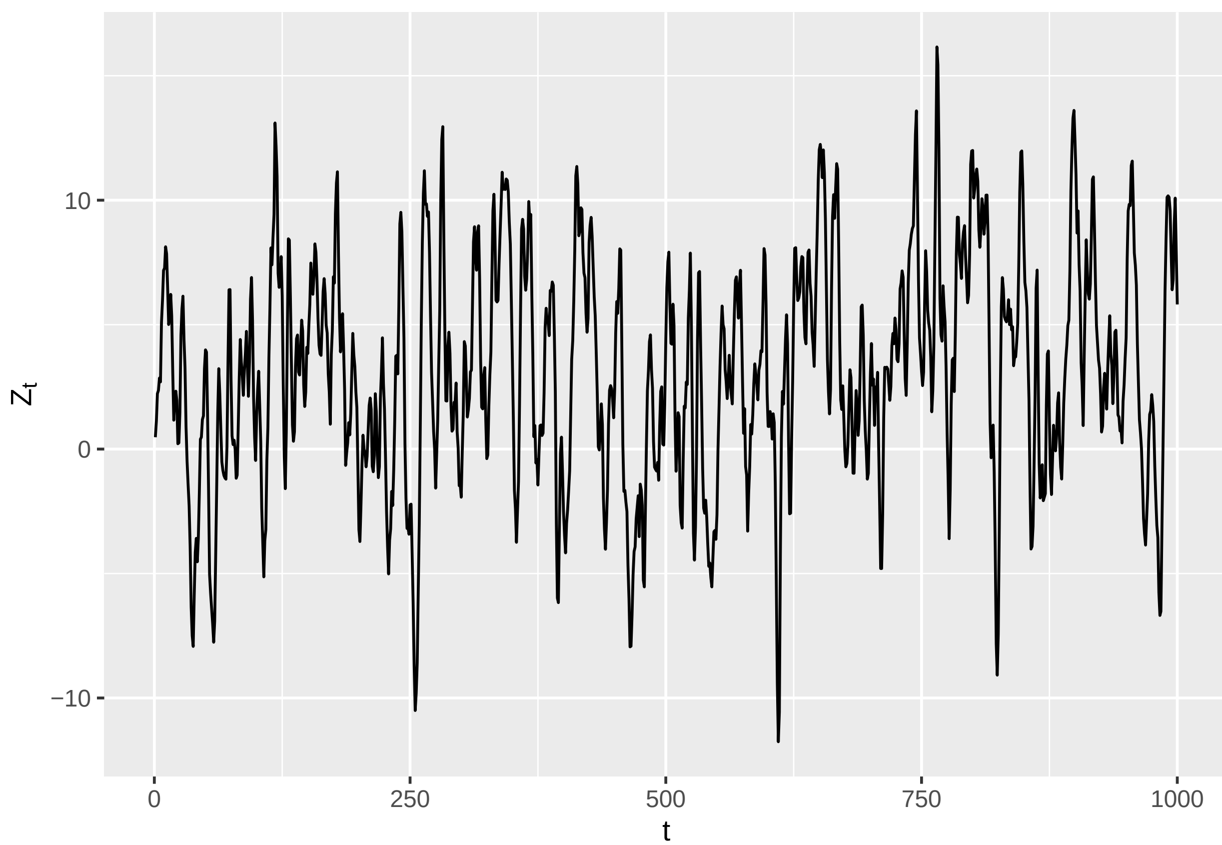
Plot the ACF

```r
library(ggplot2)
sigma <- 1
phi1 <- 0.8
theta1 <- 1
theta2 <- 1
gamma0 <-
  sigma ** 2 * (1 + 2 * phi1 * theta1 + 2 * phi1 * theta2 * (phi1 + theta1) +
                theta1 ** 2 + theta2 ** 2) / (1 - phi1 ** 2)
gamma1 <- phi1 * gamma0 + theta1 * sigma ** 2 + theta2 * (phi1 + theta1) *
  sigma ** 2
gamma2 <- phi1 * gamma1 + theta2 * sigma ** 2
gamma3 <- c(gamma0, gamma1, gamma2)
t <- 3
while (t < 101) {
  gamma3 <- c(gamma3, phi1 * gamma3[t])
  t <- t + 1
}
p <- gamma3 / gamma0
df <- data.frame(x = 0:100, y = p)
p <- ggplot(df, aes(x = x, y = y)) +
  geom_line() +
  xlab("k") +
  ylab(expression(rho[k]))
p
```



b. Generate Z_t for $1 \leqslant t \leqslant 1000$ and plot Z_t

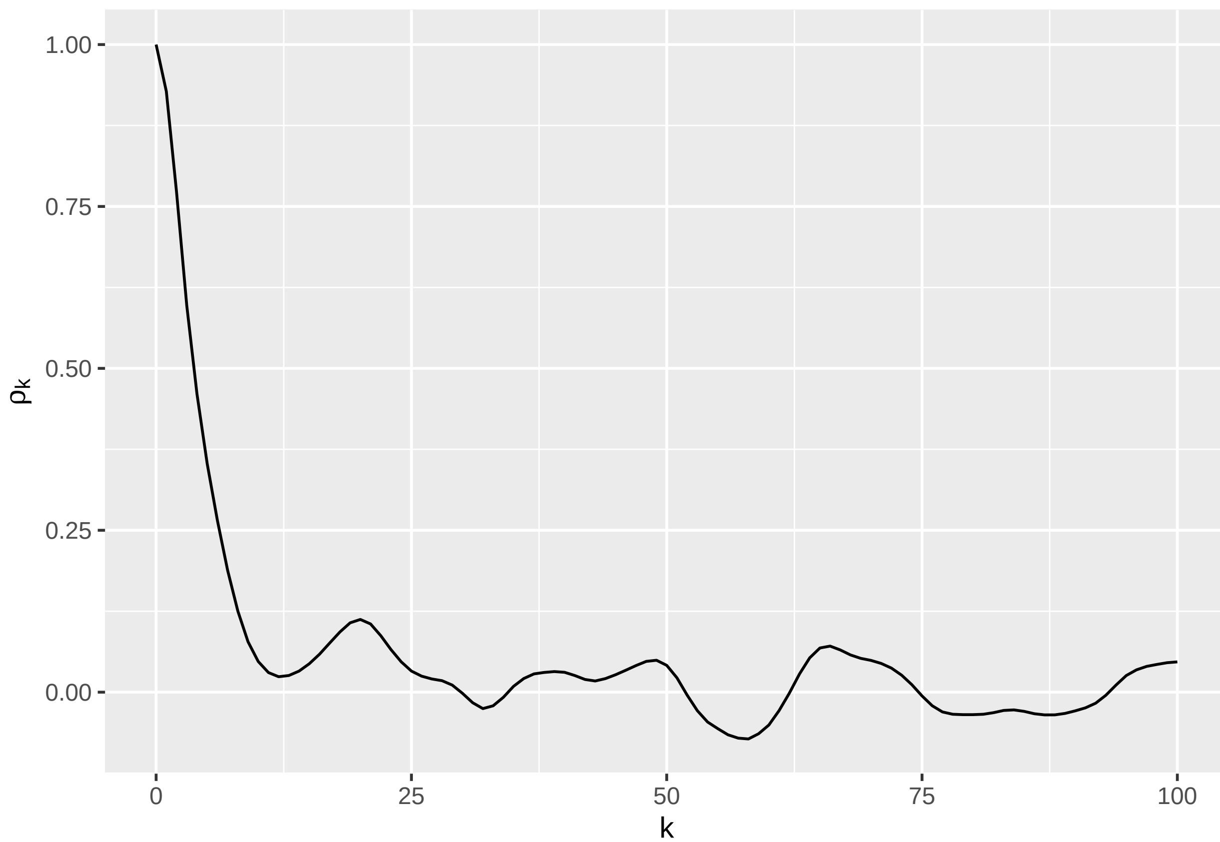
```
set.seed(9527)
e<-rnorm(1000)
z1<-0.5+e[1]
z2<-0.5+0.8*z1+e[2]+e[1]
z<-c(z1,z2)
t<-3
while(t<1001){
  z[t]<-0.5+0.8*z[t-1]+e[t]+e[t-1]+e[t-2]
  t<-t+1
}

df <- data.frame(x = 1:1000, y = z)
p <- ggplot(df, aes(x = x, y = y)) +
  geom_line() +
  xlab("t") +
  ylab(expression(Z[t]))
p
```



c. Compute the empirical ACF using the samples and compare it with your answer from (a)

The trends of answers from (a) and (c) are almost the same. But the ACF of (c) fluctuate around zero in the right tail.

```
p <- acf(z,lag.max = 100,plot = F)$acf
df <- data.frame(x = 0:100, y = p)
p <- ggplot(df, aes(x = x, y = y)) +
  geom_line() +
  xlab("k") +
  ylab(expression(rho[k]))
p
```

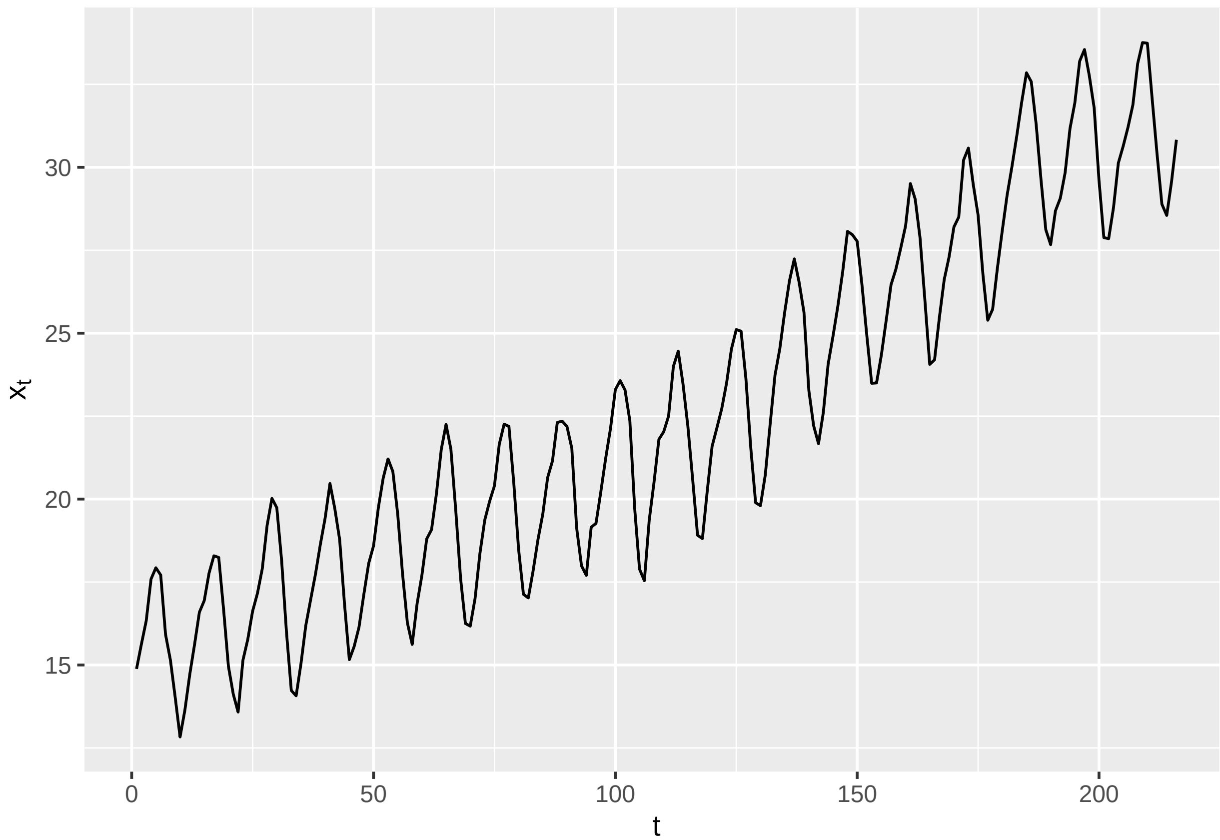


2. Download 1 time series containing trend and seasonality from http://www.statsci.org/datasets.html

Mauna Loa Carbon Dioxide MLCO2.DAT

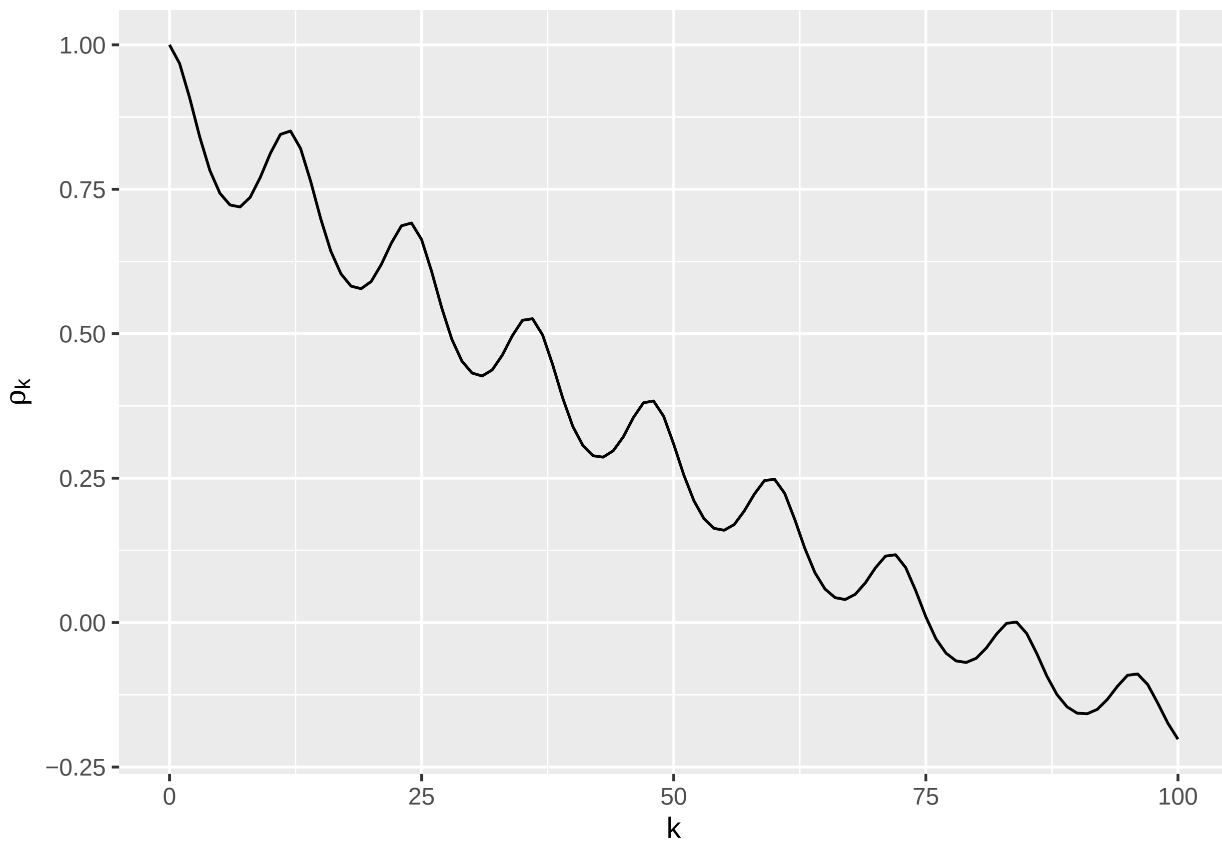
a. Plot the series and compute its ACF and PACF

```
co<-scan("MLCO2.DAT",skip =2)
t<-1:length(co)
df<-data.frame(x=t,y=co)
p<-ggplot(df,aes(x=x,y=y))+
  geom_line()+
  xlab("t")+
  ylab(expression(x[t]))
  # scale_x_continuous(breaks = scales::pretty_breaks(n = 100))
p
```



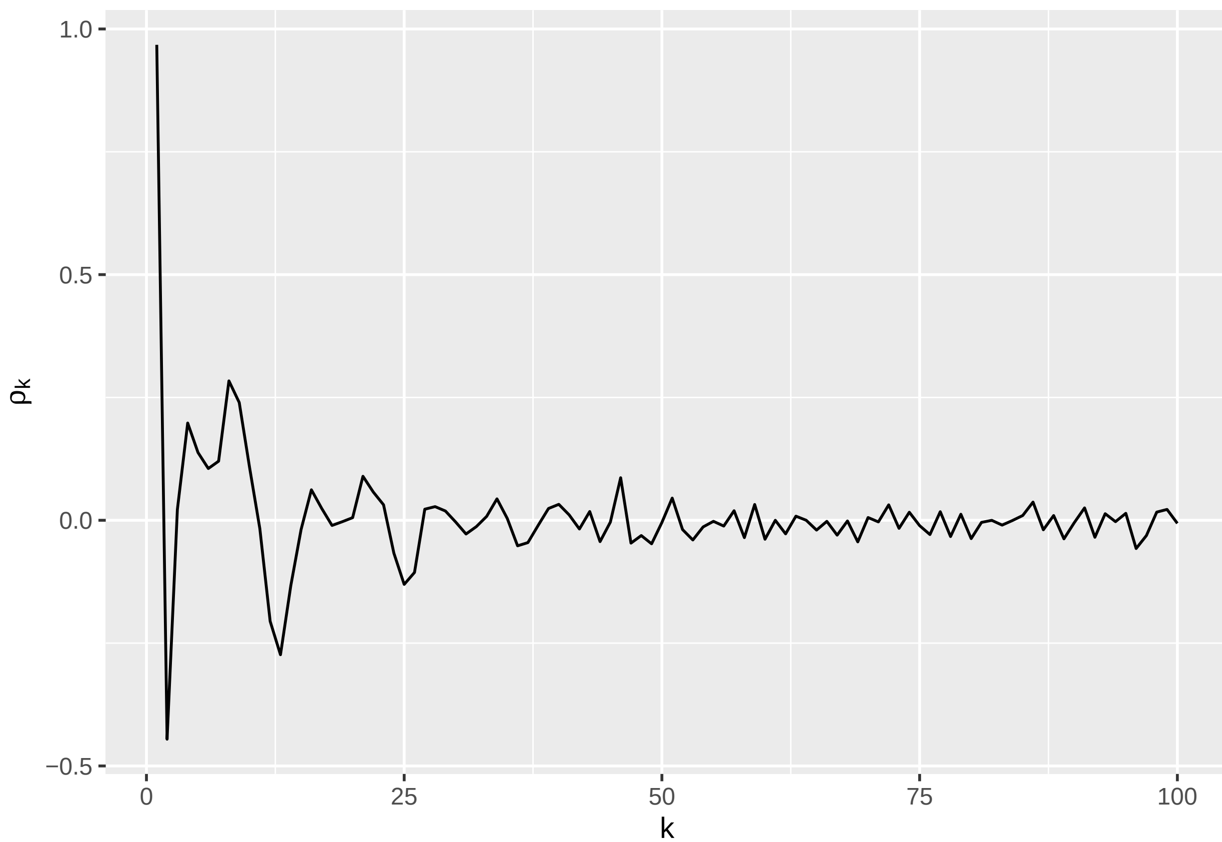
ACF

```
p <- acf(co,lag.max = 100,plot = F)$acf
df <- data.frame(x = 0:100, y = p)
p <- ggplot(df, aes(x = x, y = y)) +
  geom_line() +
  xlab("k") +
  ylab(expression(rho[k]))
p
```



PACF

```
p <- pacf(co,lag.max = 100,plot = F)$acf
df <- data.frame(x = 1:100, y = p)
p <- ggplot(df, aes(x = x, y = y)) +
  geom_line() +
  xlab("k") +
  ylab(expression(rho[k]))
p
```



b. Based on the ACF and PACF, pick 2 distinct models and estimate their parameters

Model 1: estimate by AR1 with season

```
coAR1 <- arima(co , order = c(1, 0, 0))
coAR1
```

```
##
## Call:
## arima(x = co, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.9826    22.6836
## s.e.  0.0124     3.5046
##
## sigma^2 estimated as 1.22:  log likelihood = -329.62,  aic = 665.24
```

```
coAR1Season <-
  arima(co ,
        order = c(1, 0, 0),
        seasonal = list(order = c(1, 0, 0), period = 12))
coAR1Season
```

```
##
## Call:
## arima(x = co, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0), period = 12))
##
## Coefficients:
##          ar1    sar1  intercept
##       0.9954  0.9189    22.7349
## s.e.  0.0014  0.0238    84.1726
##
## sigma^2 estimated as 0.1953:  log likelihood = -141.6,  aic = 291.2
```

Model 2: estimate by AR1 with season and trend

```
trend <- time(co)
coAR1SeasonTrend <-
  arima(co ,
        order = c(1, 0, 0),
        seasonal = list(order = c(1, 0, 0), period = 12),
        xreg = trend)
coAR1SeasonTrend
```

```
##
## Call:
## arima(x = co, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0), period = 12),
##     xreg = trend)
##
## Coefficients:
##          ar1    sar1  intercept   trend
##       0.7470  0.9203    14.7172  0.0765
## s.e.  0.0447  0.0212     1.2026  0.0071
##
## sigma^2 estimated as 0.1716:  log likelihood = -127.85,  aic = 265.69
```

c. Compare the residual for each model

The residual of model2 is less

```
coAR1Season.re<-residuals(coAR1Season)
sum(coAR1Season.re**2)
```

```
## [1] 42.1939
```

```
coAR1SeasonTrend.re<-residuals(coAR1SeasonTrend)
sum(coAR1SeasonTrend.re**2)
```

```
## [1] 37.07029
```

d. Which model is better and why?

The model 2 is better because it has less residual and less AIC. And it considers the trend of the data

3. Derive an expression for the power spectral density, $S_y(\omega)$ for the ARMA(1,1) model and plot $S_y(\omega)$ for $0 \leqslant \omega \leqslant \pi$

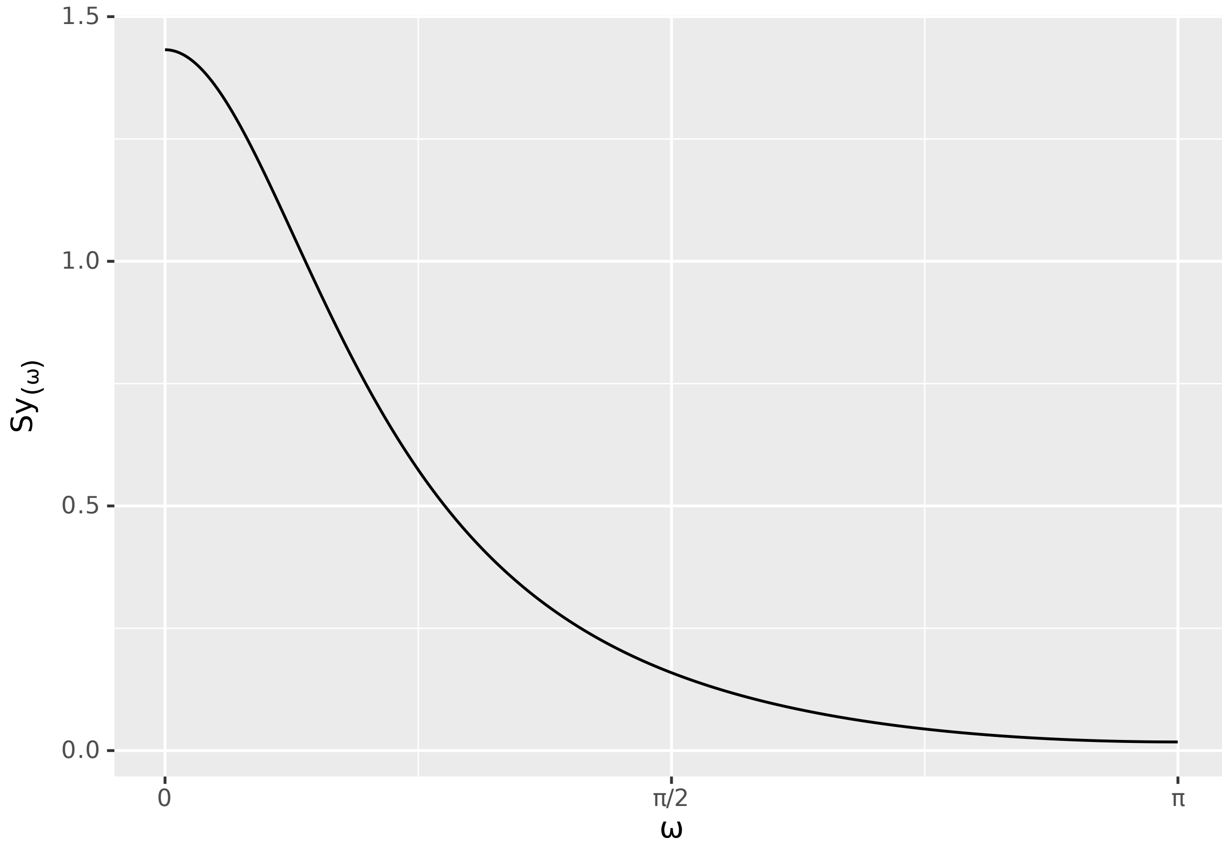
Define ARMA(1,1)

$$x_t = c + \phi_1 x_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}$$

$$\begin{aligned}
S_y &= \frac{1}{2\pi}\left\{\gamma_0 + 2\sum_{j=1}^{\infty}[\gamma_j cos(j\omega)]\right\} \\
S_y &= \frac{\sigma^2}{2\pi}\bar{\psi}(e^{-i\omega})\bar{\psi}(e^{i\omega}) \\
&= \frac{\sigma^2}{2\pi}\frac{\prod_{j=1}^{p}[1+\alpha_j^2 - 2\alpha_j cos\omega]}{\prod_{j=1}^{q}[1+\beta_j^2 - 2\beta_j cos\omega]} \\
&= \frac{\sigma^2}{2\pi}\frac{1+\alpha_1^2 - 2\alpha_1 cos\omega}{1+\beta_1^2 - 2\beta_1 cos\omega}
\end{aligned}$$

$$\begin{gathered}
\alpha_1 \; is \; root \; of \; \phi(L) = 1 + \theta_1 L = 0 \\
\beta_1 \; is \; root \; of \; \psi(L) = 1 - \phi_1 L = 0 \\
\alpha_1 = -\frac{1}{\theta_1} \\
\beta_1 = \frac{1}{\phi_1}
\end{gathered}$$

```
sigma <- 1
phi1 <- 0.5
theta1 <- 0.5
alpha1 <- -1 / theta1
beta1 <- 1 / phi1
omega <- seq(0, pi, 0.001)
sy <- sigma ** 2 / (2 * pi) * (1 + alpha1 ** 2 - 2 * alpha1 * cos(omega)) /
  (1 + beta1 ** 2 - 2 * beta1 * cos(omega))
dfP <- data.frame(x = omega, y = sy)
p <- ggplot(dfP, aes(x = x, y = y)) +
  geom_line() +
  xlab(expression(omega)) +
  ylab(expression(Sy[(omega)])) +
  scale_x_continuous(
    breaks  = c(seq(0, pi, pi / 2)),
    labels = c("0", "\u03c0/2", "\u03c0")
  )
p
```



4. Prove that conditional expectation minimizes the mean square error.

Let

$$
\begin{aligned}
E[x_{k+1}|F_k] &= x_{\widehat{k+1}}|k \\
e_{k+1} &= x_{k+1} - \hat{x}_{k+1}|k
\end{aligned}
$$

$$MSE = E[e_{k+1}^2] = E[(x_{k+1} - \hat{x}_{k+1})^2]$$

Let g_k , an arbitrary function of the available information in F_k , be a candidate forecast

$$
\begin{aligned}
MSE &= E[(x_{k+1} - g_k)^2] \\
&= E[(x_{k+1} - \hat{x}_{k+1}|k + \hat{x}_{k+1}|k - g_k)^2] \\
&= E[(x_{k+1} - \hat{x}_{k+1}|k)^2] + E[(\hat{x}_{k+1}|k - g_k)^2] + 2E[(x_{k+1} - \hat{x}_{k+1}|k)(\hat{x}_{k+1}|k - g_k)] \\
\eta_{k+1} &= (x_{k+1} - \hat{x}_{k+1}|k)(\hat{x}_{k+1}|k - g_k)
\end{aligned}
$$

Since both $\hat{x}_{k+1}|k$ and g_k are functions of F_k

$$
\begin{aligned}
E[\eta_{k+1|F_k}] &= [E[x_{k+1}|F_k] - \hat{x}_{k+1}|k] \cdot [\hat{x}_{k+1}|k - g_k] = 0 \\
E[\eta_{k+1}] &= 0 \\
MSE &= E[(x_{k+1} - \hat{x}_{k+1}|k)^2] + E[(\hat{x}_{k+1}|k - g_k)^2]
\end{aligned}
$$

MSE is minimum when $g_k = \hat{x}_{k+1}|k$, conditional expectation minimizes the mean square error

$$MSE = E[(x_{k+1} - \hat{x}_{k+1}|k)^2]$$

5. Compute expressions for the partial autocorrelation function for AR(1) and MA(1) models.

AR(1)

Define

$$x_t = \phi x_{t-1} + \epsilon_t$$

$$\begin{aligned}
\phi_{11} &= \frac{cov(x_t, x_{t-1})}{var(x_t)} \\
&= \frac{E[(\phi x_{t-1+\epsilon_t})x_{t-1}]}{var(x_t)} \\
&= \phi
\end{aligned}$$

projection of x_{t+2} on x_{t+1}

$$\begin{aligned}
\hat{x}_{t+2} &= P[x_{t+2}|x_{t+1}] = \alpha x_{t+1} \\
e_{t+2} &= x_{t+2} - \alpha x_{t+1} \\
E[e_{t+2}x_{t+1}] &= E[(x_{t+2} - \alpha x_{t+1})x_{t+1}] = 0
\end{aligned}$$

$$\begin{aligned}
\alpha &= \frac{E(x_{t+2}x_{t+1})}{var(x_t)} \\
&= \frac{E[(\phi x_{t+1} + \epsilon_{t+2})x_{t+1}]}{var(x_t)} \\
&= \phi
\end{aligned}$$

projection of x_t on x_{t+1}

$$\begin{aligned}
\hat{x}_t &= P[x_t|x_{t+1}] = \beta x_{t+1} \\
e_t &= x_t - \beta x_{t+1} \\
E[e_t x_{t+1}] &= E[(x_t - \beta x_{t+1})x_{t+1}] = 0
\end{aligned}$$

$$\begin{aligned}
\beta &= \frac{E(x_t x_{t+1})}{var(x_t)} \\
&= \frac{E[(\phi x_t + \epsilon_{t+1})x_t]}{var(x_t)} \\
&= \phi
\end{aligned}$$

$$
\begin{aligned}
\phi_{22} &= \frac{cov(e_t, e_{t+2})}{[var(e_t)var(e_{t+2})]^{\frac{1}{2}}} \\
cov(e_t, e_{t+2}) &= E[(x_t - \phi x_{t+1})(x_{t+2} - \phi x_{t+1})] \\
&= E[(x_t - \phi x_{t+1})\epsilon_{t+2}] \\
&= 0 \\
\phi_{22} &= 0
\end{aligned}
$$

Similarly for $k \geqslant 2$

$$
\phi_{kk} = 0
$$

MA(1)

Define $x_t = \epsilon_t + \theta\epsilon_{t-1}$

$$
\begin{aligned}
var(x_t) &= (1+\theta^2)\sigma^2 \\
E(x_t x_{t+1}) &= \theta\sigma^2 \\
E(x_t x_{t+2}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\phi_{11} &= \frac{E(x_t x_{t+1})}{var(x_t)} \\
&= \frac{\theta}{1+\theta^2}
\end{aligned}
$$

projection of x_{t+2} on x_{t+1}

$$
\begin{aligned}
e_{t+2} &= x_{t+2} - \alpha x_{t+1} \\
E[e_{t+2}x_{t+1}] &= E[(x_{t+2} - \alpha x_{t+1})x_{t+1}] = 0
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \frac{E(x_{t+2}x_{t+1})}{var(x_t)} \\
&= \frac{E[(\phi x_{t+1} + \epsilon_{t+2})x_{t+1}]}{var(x_t)} \\
&= \frac{\theta}{1+\theta^2}
\end{aligned}
$$

projection of x_t on x_{t+1}

$$
\begin{aligned}
e_t &= x_t - \beta x_{t+1} \\
E[e_t x_{t+1}] &= E[(x_t - \beta x_{t+1})x_{t+1}] = 0
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \frac{E(x_t x_{t+1})}{var(x_t)} \\
&= \frac{E[(\phi x_t + \epsilon_{t+1})x_t]}{var(x_t)} \\
&= \frac{\theta}{1+\theta^2}
\end{aligned}
$$

$$\begin{aligned} cov(e_t, e_{t+2}) &= E[(x_t - \alpha x_{t+1})(x_{t+2} - \beta x_{t+1})] \\ &= E[x_t x_{t+2}] - \alpha E[x_t x_{t+1}] - \alpha E[x_{t+1} x_{t+2}] + \alpha^2 E[x_{t+1}^2] \\ &= -\frac{\theta^2 \sigma^2}{(1+\theta^2)} \end{aligned}$$

$$\begin{aligned} Var(e_{t+2}) &= E[x_{t+2} - \alpha x_{t+1}]^2 \\ &= E(x_{t+2}^2) - 2\alpha E(x_{t+2} x_{t+1}) + \alpha^2 E(x_{t+1}^2) \\ &= \frac{\sigma^2}{1+\theta^2}[1 + \theta^2 + \theta^4] \end{aligned}$$

$$\begin{aligned} Var(e_t) &= E[x_t - \beta x_{t+1}]^2 \\ &= E(x_t^2) - 2\beta E(x_t x_{t+1}) + \beta^2 E(x_{t+1}^2) \\ &= \frac{\sigma^2}{1+\theta^2}[1 + \theta^2 + \theta^4] \end{aligned}$$

$$\begin{aligned} \phi_{22} &= \frac{cov(e_t, e_{t+2})}{[var(e_t) var(e_{t+2})]^{\frac{1}{2}}} \\ &= -\frac{\theta^2}{1 + \theta^2 + \theta^4} \end{aligned}$$

for $k \geqslant 2$

$$\begin{aligned} \phi_{kk} &= \frac{(-1)^{k-1}\theta^k}{\sum_{j=0}^{k} (\theta^2)^j} \\ &= \frac{(-1)^{k-1}\theta^k (1-\theta^2)}{1 - \theta^{2(k+1)}} \end{aligned}$$